Note for Infinite Relational Model

Koji Kosugi

This document explains the derivation of equations in Gibbs sampling for latent classes and latent ranks, pertaining to the Infinite Relational Model found in Chapter 7, Section 7.8 of Shojima (2022). This is based on Chapter 13 by Ishii & Ueda (2014), with formulas and symbols modified to align with Shojima's notation. First, I present a summary of the notation rules, followed by the expansion of the model equations, leading to the derivation of the formula found on page 341 of Shojima (2022).

表 1　Table of Notations

Term	**Description**
Number of Subjects	S
Number of Items	J
Response	$\boldsymbol{U} = \{u_{ij}\}$
Correct answer	$u_{sj} = 1$
Incorrect answer	$u_{sj} = 0$
Missing Indicator matrix	$\boldsymbol{Z} = \{z_{sj}\}$
item j presented to Student s	$z_{sj} = 1$
item j didn't presented to Student s	$z_{sj} = 0$
Response vector	$\boldsymbol{U}_s = (u_{s1}, \dots, u_s)$
Reminder after removing s	$\boldsymbol{U}^{-s}$
Classes	$\boldsymbol{C} = \{c_s, = 1, \dots, S\}$
Fields	$\boldsymbol{F} = \{f_j = 1, \dots, J\}$
Class membership matrix	$\boldsymbol{M}_C = \{m_{sc}\}$
Field membership matrix	$\boldsymbol{M}_F = \{m_{jf}\}$
Reference matrix	$\boldsymbol{\Pi}_B = \{\pi_{fc}\}$
Parameter set for (c, f) block, where f ranges from 1 to F.	$\boldsymbol{\Pi}_B^{c,+} = (\pi_{1c}, \cdots, \pi_{Fc})$
Binary class membership matrix	$\tilde{\boldsymbol{M}}_C = \{\tilde{m}_{sc}\}$
Class labels vector	$\boldsymbol{n}_C = (1, 2, \cdots, C)'$
Latent Class vector	$\boldsymbol{l}_C = \tilde{\boldsymbol{M}}_C \boldsymbol{n}_C$
Set of elements in $\boldsymbol{l}_C$, excluding s^*	$\boldsymbol{l}_C^{-s^*}$
Binary field membership matrix	$\tilde{\boldsymbol{M}}_F = \{\tilde{m}_{jf}\}$
Field labels vector	$\boldsymbol{n}_F = (1, 2, \cdots, F)'$
Latent Field vector	$\boldsymbol{l}_F = \tilde{\boldsymbol{M}}_F \boldsymbol{n}_F$
Set of elements in $\boldsymbol{l}_F$, excluding j^*	$\boldsymbol{l}_F^{-j^*}$

1 Model

Suppose the current number of classes is C. Student s^* may belong to one of the existing C classes or to a new class. The probability that s^* belongs to class c is denoted by p_{s^*c} and is represented as follows:

$$\begin{aligned} p_{s^*c} &= P(l_{Cs^*} = c \mid \boldsymbol{U}, \boldsymbol{l}_C^{-s^*}, \boldsymbol{l}_F, \gamma_C, \beta_0, \beta_1) \\ &= P(l_{Cs^*} = c \mid \boldsymbol{U}^{s^*}, , \boldsymbol{U}^{-s^*} \boldsymbol{l}_C^{-s^*}, \boldsymbol{l}_F, \gamma_C, \beta_0, \beta_1) \\ &\propto P(l_{Cs^*} = c \mid \boldsymbol{l}_C^{-s^*}, \gamma_C) P(\boldsymbol{U}_{s^*} \mid l_{Cs^*} = c, \boldsymbol{l}_F, \boldsymbol{U}^{-s^*}, \beta_0, \beta_1) \end{aligned} \tag{1}$$

The first term can be transformed as follows(by Hoppe's urn model; Ishii and Ueda(2014),Ch11 and Ch12.).

$$P(l_{Cs^*} = c \mid \boldsymbol{l}_C^{-s^*}, \gamma_C) = \begin{cases} \dfrac{S^{-s^*}}{S - 1 - \gamma_c} & \text{existing class } c \\ \\ \dfrac{\gamma_c}{S - 1 - \gamma - c} & \text{new class} \end{cases} \tag{2}$$

The second term is as follow:

$$P(\boldsymbol{U}_{s^*} \mid l_{C^{s^*}} = c, \boldsymbol{l}_F, \boldsymbol{U}^{-s^*}, \beta_0, \beta_1)$$

$$\propto \begin{cases} \displaystyle\int \prod_j P(u_{s^*j} \mid l_{C^{s^*}} = c, \boldsymbol{l}_F, \boldsymbol{\Pi}_B^{c,+}) p(\boldsymbol{\Pi}_B^{c,+} \mid l_{C^{s^*}} = c, \boldsymbol{l}_C^{-s^*}, \boldsymbol{l}_F, \boldsymbol{U}^{-s^*}, \beta_0, \beta_1) d\boldsymbol{\Pi}_B^{c,+} \\ \qquad\qquad \text{exsisting class } c \\ \\ \displaystyle\int \prod_j P(u_{s^*j} \mid l_{C^{s^*}} = c, \boldsymbol{l}_F, \boldsymbol{\Pi}_B^{c,+}) P(\boldsymbol{\Pi}_B^{c,+} \mid \beta_0, \beta_1) d\boldsymbol{\Pi}_B^{c,+} \\ \qquad\qquad \text{new class} \end{cases} \tag{3}$$

The likelihood part of eq.(3) is

$$\begin{aligned} \textstyle\prod_j P(u_{ij} \mid l_{C^{s^*}} = c, \boldsymbol{l}_F, \boldsymbol{\Pi}_B^{c,+}) &= \prod_j \prod_f P(u_{ij} \mid \boldsymbol{\Pi}_{\boldsymbol{B}}^{c,+})^{\tilde{m}_{sc}\tilde{m}_{jf}} \\ &= \prod_f \pi_{fc}^{\tilde{U}_{1fc}^{s^*}} (1 - \pi_{fc})^{\tilde{U}_{0fc}^{s^*}} \end{aligned} . \tag{4}$$

In (4),$\tilde{U}_{1fc}^{s^*}$ and $\tilde{U}_{0fc}^{s^*}$ is defined as follows:

$$\tilde{U}_{1fc}^{s^*} = \sum_{j=1}^{J} \tilde{m}_{s^*c} \tilde{m}_{jf} z_{sj} u_{s^*j}. \tag{5}$$

$$\tilde{U}_{0fc}^{s^*} = \sum_{j=1}^{J} \tilde{m}_{s^*c} \tilde{m}_{jf} z_{sj} (1 - u_{s^*j}). \tag{6}$$

Next, the posterior distribution of the parametes in the second term on the right hand side of eq(3) can be transformed as follows, according to Bayes' theorem.

$$
\begin{aligned}
p(\boldsymbol{\Pi}_B^{s^*} \mid l_{C^{s^*}} = c, \boldsymbol{l}_C^{-s^*}, \boldsymbol{l}_F, \boldsymbol{U}^{-s^*}, \beta_0, \beta_1) &= \prod_f p(\pi_{fc} \mid l_{C^{s^*}} = c, \boldsymbol{l}_C^{-s^*}, \boldsymbol{l}_F, \boldsymbol{U}^{-s^*}, \beta_0, \beta_1) \\
&= \prod_f \frac{P(\boldsymbol{U}^{-s^*} \mid \boldsymbol{l}_C^{-s^*}, \boldsymbol{l}_F, \pi_{fc}) p(\pi_{fc} \mid \beta_0, \beta_1)}{\int P(\boldsymbol{U}^{-s^*} \mid \boldsymbol{l}_C^{-s^*}, \boldsymbol{l}_F, \pi_{fc}) p(\pi_{fc} \mid \beta_0, \beta_1) d\pi_{fc}} \\
&= \prod_f \frac{\left(\prod_j \prod_{s' \neq s^*} P(u_{s'j} \mid \pi_{fc})^{\tilde{m}_{s'c}\tilde{m}_{jf}}\right) p(\pi_{fc} \mid \beta_0, \beta_1)}{\int \left(\prod_j \prod_{s' \neq s^*} P(u_{s'j} \mid \pi_{fc})^{\tilde{m}_{s'c}\tilde{m}_{jf}}\right) p(\pi_{fc} \mid \beta_0, \beta_1) d\pi_{fc}}
\end{aligned} \tag{7}
$$

The likelihood in eq(7) can be calculated as follows:

$$
\begin{aligned}
\prod_j \prod_{s' \neq s^*} P(u_{s'j} \mid \pi_{fc})^{\tilde{m}_{s'c}\tilde{m}_{jf}} &= \prod_j \prod_{s' \neq s^*} \pi_{fc}^{u_{s'j}\tilde{m}_{sc}\tilde{m}_{jf}} (1 - \pi_{fc})^{(1-u_{s'j})\tilde{m}_{sc}\tilde{m}_{jf}} \\
&= \pi_{fc}^{\sum_j \sum_{s' \neq s^*} u_{s'j}\tilde{m}_{sc}\tilde{m}_{jf}} (1 - \pi_{fc})^{\sum_j \sum_{s' \neq s^*} (1-u_{s'j})\tilde{m}_{sc}\tilde{m}_{jf}} \\
&= \pi_{fc}^{\tilde{U}_{1fc}^{-s^*}} (1 - \pi_{fc})^{\tilde{U}_{0fc}^{-s^*}}
\end{aligned} \tag{8}
$$

Meanwhile, $\tilde{U}_{1fc}^{-s^*}$ and $\tilde{U}_{0fc}^{-s^*}$ do not include the data of Student s^*. That is,

$$
\tilde{U}_{1fc}^{-s^*} = \sum_{s(\neq s^*)}^{S} \sum_{j=1}^{J} \tilde{m}_{sc}\tilde{m}_{jf} z_{sj} u_{sj}. \tag{9}
$$

$$
\tilde{U}_{0fc}^{-s^*} = \sum_{s(\neq s^*)}^{S} \sum_{j=1}^{J} \tilde{m}_{sc}\tilde{m}_{jf} z_{sj} (1 - u_{sj}). \tag{10}
$$

Note that $p(\pi_{fc} \mid \beta_0, \beta_1) = Beta(\pi_{fc}; \beta_0, \beta_1)$. The definition of the Beta distribution is as follow:

$$
Beta(\theta; \beta_0, \beta_1) = \frac{\theta^{\beta_0 - 1}(1-\theta)^{\beta_1 - 1}}{B(\beta_0, \beta_1)}
$$

.

The dominator $B(\beta_0, \beta_1)$ is the Beta function. It acts as a normalization constant for the Beta distribution.

Considering this, Eq.(7) can be written as follows:

$$
\begin{aligned}
&= \prod_f \frac{\pi_{fc}^{\tilde{U}_{1fc}^{-s^*}} (1-\pi_{fc})^{\tilde{U}_{0fc}^{-s^*}} \dfrac{\pi_{fc}^{\beta_1-1}(1-\pi_{fc})^{\beta_0-1}}{B(\beta_1,\beta_0)}}{\int \pi_{fc}^{\tilde{U}_{1fc}^{-s^*}} (1-\pi_{fc})^{\tilde{U}_{0fc}^{-s^*}} \dfrac{\pi_{fc}^{\beta_1-1}(1-\pi_{fc})^{\beta_0-1}}{B(\beta_1,\beta_0)} d\pi_{fc}} \\
&= \prod_f \frac{\pi_{fc}^{\tilde{U}_{1fc}^{-s^*}+\beta_1-1}(1-\pi_{fc})^{\tilde{U}_{0fc}^{-s^*}+\beta_0-1}}{\int \pi_{fc}^{\tilde{U}_{1fc}^{-s^*}+\beta_1-1}(1-\pi_{fc})^{\tilde{U}_{0fc}^{-s^*}+\beta_0-1} d\pi_{fc}} \\
&= \prod_f \frac{\pi_{fc}^{\tilde{U}_{1fc}^{-s^*}+\beta_1-1}(1-\pi_{fc})^{\tilde{U}_{0fc}^{-s^*}+\beta_0-1}}{B(\tilde{U}_{1fc}^{-s^*}+\beta_1, \tilde{U}_{0fc}^{-s^*}+\beta_0)} \\
&= \prod_f Beta(\pi_{fc}; \tilde{U}_{1fc}^{-s^*}+\beta_1, \tilde{U}_{0fc}^{-s^*}+\beta_0)
\end{aligned} \tag{11}
$$

By the way, $\tilde{U}_{1fc}$ and $\tilde{U}_{0fc}$ are denoted as in eq.(12) and eq.(13).

$$
\begin{aligned}
\tilde{U}_{1fc}^{s^*} + \tilde{U}_{1fc}^{-s^*} &= \sum_{s(\neq s^*)}^{S} \sum_{j=1}^{J} \tilde{m}_{sc}\tilde{m}_{jf} z_{sj} u_{sj} + \sum_{j=1}^{J} \tilde{m}_{s^*c}\tilde{m}_{jf} z_{sj} u_{s^*j} \\
&= \sum_{s=1}^{S} \sum_{j=1}^{J} \tilde{m}_{sc}\tilde{m}_{jf} z_{sj} u_{sj} \\
&= \tilde{U}_{1fc}.
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
\tilde{U}_{0fc}^{s^*} + \tilde{U}_{0fc}^{-s^*} &= \sum_{s(\neq s^*)}^{S} \sum_{j=1}^{J} \tilde{m}_{sc}\tilde{m}_{jf} z_{sj} (1-u_{sj}) + \sum_{j=1}^{J} \tilde{m}_{s^*c}\tilde{m}_{jf} z_{sj} (1-u_{s^*j}) \\
&= \sum_{s=1}^{S} \sum_{j=1}^{J} \tilde{m}_{sc}\tilde{m}_{jf} z_{sj} (1-u_{sj}) \\
&= \tilde{U}_{0fc}.
\end{aligned} \tag{13}
$$

Returning to Equation eq.(3), the probability of belonging to an existing class c is as follows:

$$
\begin{aligned}
&\int \prod_j P(u_{s^*j} \mid l_{C^{s^*}} = c, \boldsymbol{l}_F, \boldsymbol{\Pi}_B^{c,+}) p(\boldsymbol{\Pi}_B^{c,+} \mid l_{C^{s^*}} = c, \boldsymbol{l}_C^{-s^*}, \boldsymbol{l}_F, \boldsymbol{U}^{-s^*}, \beta_0, \beta_1) d\boldsymbol{\Pi}_B^{c,+} \\
&= \int \left\{ \prod_f \pi_{fc}^{\tilde{U}_{1fc}^{s^*}} (1-\pi_{fc})^{\tilde{U}_{0fc}^{s^*}} \right\} \times \left\{ \prod_f \frac{\pi_{fc}^{\tilde{U}_{1fc}^{-s^*}+\beta_1-1} (1-\pi_{fc})^{\tilde{U}_{0fc}^{-s^*}+\beta_0-1}}{B(\tilde{U}_{1fc}^{-s^*}+\beta_1, \tilde{U}_{0fc}^{-s^*}+\beta_0)} \right\} d\pi_{fc} \\
&= \prod_f \frac{\int (\pi_{fc}^{\tilde{U}_{1fc}+\beta_1-1} (1-\pi_{fc}^{\tilde{U}_{0fc}+\beta_0-1})) d\pi_{fc}}{B(\tilde{U}_{1fc}^{-s^*}+\beta_1, \tilde{U}_{0fc}^{-s^*}+\beta_0)} \\
&= \prod_f \frac{B(\tilde{U}_{1fc}+\beta_1, \tilde{U}_{0fc}+\beta_0)}{B(\tilde{U}_{1fc}^{-s^*}+\beta_1, \tilde{U}_{0fc}^{-s^*}+\beta_0)}.
\end{aligned} \tag{14}
$$

Given its definition[*1], the Beta function exhibits symmetry with respect to its two parameters, making them interchangeable:$B(a,b) = B(b,a)$. Therefore, the above equation is consistent with Shojima(2022),Pp.341.

Regarding the probability that s^* belongs to a new class, eq.(7) can similarly be expanded as follows. Note $P(\boldsymbol{\Pi}_B^{c,+} \mid \beta_0, \beta_1) = \prod_f Beta(\pi_{fc}; \beta_0, \beta_1)$,

$$
\begin{aligned}
&\int \prod_j P(u_{s^*j} \mid l_{C^{s^*}} = c, \boldsymbol{l}_F, \boldsymbol{\Pi}_B^{c,+}) P(\boldsymbol{\Pi}_B^{c,+} \mid \beta_0, \beta_1) d\boldsymbol{\Pi}_B^{c,+} \\
&= \int \prod_f \pi_{fc}^{\tilde{U}_{1fc}^{s^*}} (1-\pi_{fc}^{\tilde{U}_{0fc}^{s^*}}) \prod_f Beta(\pi_{fc}; \beta_1, \beta_0) d\pi_{fc} \\
&= \prod_f \frac{\int (\pi_{fc}^{\tilde{U}_{1fc}^{s^*}} (1-\pi_{fc})^{\tilde{U}_{0fc}^{s^*}}) (\pi_{fc}^{\beta_1-1} (1-\pi_{fc})^{\beta_0-1})) d\pi_{fc}}{B(\beta_1, \beta_0)} \\
&= \prod_f \frac{\int (\pi_{fc}^{\tilde{U}_{1fc}^{s^*}+\beta_1-1} (1-\pi_{fc})^{\tilde{U}_{0fc}^{s^*}+\beta_0-1}) d\pi_{fc}}{B(\beta_1, \beta_0)} \\
&= \prod_f \frac{B(\tilde{U}_{1fc}^{s^*}+\beta_1, \tilde{U}_{0fc}^{s^*}+\beta_0)}{B(\beta_1, \beta_0)}
\end{aligned} \tag{15}
$$

[*1] The definition of Beta function is $B(\beta_0, \beta_1) = \int_0^1 x^{\beta_0} (1-x)^{\beta_1} dx$

Finally, we arrive at the following equation by Shojima (2022):

$$
\begin{aligned}
p_{s^*c} &= P(l_{C^{s^*}} = c \mid \boldsymbol{U}, \boldsymbol{l}_C^{-s^*}, \boldsymbol{l}_F, \gamma_C, \beta_0, \beta_1) \\
&\propto \begin{cases} \dfrac{S_c^{-s^*}}{S-1+\gamma_C} \dfrac{\prod_{f=1}^{F} B(\tilde{U}_{0fc}+\beta_0, \tilde{U}_{1fc}+\beta_1)}{\prod_{f=1}^{F} B(\tilde{U}_{0fc}^{-s^*}+\beta_0, \tilde{U}_{1fc}^{-s^*}+\beta_1)}, \text{existing class } c \in \mathbb{N}_{C^*} \\ \\ \dfrac{\gamma_C}{S-1+\gamma_C} \dfrac{\prod_{f=1}^{F} B(\tilde{U}_{0fc}^{s^*}, \tilde{U}_{1fc}^{s^*})}{\prod_{f=1}^{F} B(\beta_0, \beta_1)}, \text{new class } c = C^*+1 \end{cases} .
\end{aligned}
\tag{16}
$$

参考文献

[1] Shojima,K. (2022) *Test Data Engineering*, Springer.

[2] Ishii,K and Ueda,N. (2014)Zoku: Wakariyasui pattern ninshiki: Kyoushi nashi gakushuu nyuumon [続・わかりやすいパターン認識: 教師なし学習入門]. Ohmsha.