Macroeconomics II Homework 3

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December 11, 2024

Q1

(a)

The recursive formulation of a standard neoclassical growth model studied in class in Lecture 3 is

$$v(k) = \max_{0 \le k' \le f(k)} \{U(f(k) - k') + \beta v(k')\}.$$

Define metricx space $(B(X), d)$, the space of bounded functions on $X = [0, \infty)$ with the sup-norm d, and difine operator T as

$$Tv(k) = \max_{0 \le k' \le f(k)} \{U(f(k) - k') + \beta v(k')\}.$$

Then, by showing T satisfies Blackwerll's condition, we can show T is a contraction mapping. Then, by CMT, we can show convergent k is an unique fixed point.

(b)

Let X be a set. Consider the space $B(X)$ of all bounded functions $f : X \to \mathbb{R}$, equipped with the *supremum norm*:

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

The metric on $B(X)$ is defined as:

$$d(f, g) = \|f - g\|_\infty.$$

We aim to show that the metric space $(B(X), d)$ is complete, i.e., every Cauchy sequence of functions in $B(X)$ converges to a function in $B(X)$.

Let $\{f_n\}$ be a Cauchy sequence in $(B(X), d)$. By definition, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \ge N$:

$$\|f_n - f_m\|_\infty < \epsilon,$$

which implies:

$$\sup_{x \in X} |f_n(x) - f_m(x)| < \epsilon.$$

For each $x \in X$, the sequence $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$ because:

$$|f_n(x) - f_m(x)| < \epsilon \quad \text{for all } m, n \ge N.$$

Since $\mathbb{R}$ is complete, $\{f_n(x)\}$ converges to a limit, say $f(x) \in \mathbb{R}$. Thus, we can define a pointwise limit function $f : X \to \mathbb{R}$ by:

$$f(x) = \lim_{n\to\infty} f_n(x), \quad \text{for each } x \in X.$$

Since $\{f_n\} \subseteq B(X)$, each f_n is bounded, i.e., there exists M_n such that $\|f_n\|_\infty \leq M_n$. Let $M = \sup_n M_n$. Then for all n and $x \in X$:

$$|f_n(x)| \leq M.$$

Taking the limit as $n \to \infty$, we obtain:

$$|f(x)| \leq M, \quad \text{for all } x \in X.$$

Thus, f is bounded, and hence $f \in B(X)$.

For $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$:

$$\|f_n - f_m\|_\infty < \epsilon.$$

Fix $n \geq N$. Then for all $x \in X$:

$$|f_n(x) - f_m(x)| < \epsilon \quad \text{for all } m \geq N.$$

Taking the limit as $m \to \infty$, we get:

$$|f_n(x) - f(x)| \leq \epsilon.$$

Thus:

$$\|f_n - f\|_\infty \leq \epsilon \quad \text{for all } n \geq N.$$

This shows that $\{f_n\}$ converges uniformly to f.

Since $\{f_n\}$ is a Cauchy sequence in $(B(X), d)$ and converges uniformly to $f \in B(X)$, the metric space $(B(X), d)$ is complete.

(c)

Suppose not, there exists feasible allocation $\{\tilde{c}_t^1,\ \tilde{c}_t^2\}_{t=0,\ s^t \in S^t}^{\infty}$ such that

$$u(\hat{c}^i) \leq u(\tilde{c}^i) \text{ for all } i \in \{1, 2\}$$
$$u(\hat{c}^i) < u(\tilde{c}^i) \text{ for some } i \in \{1, 2\}$$

Without loss of generality, assume strict inequality holds for $i = 1$.

Suppose

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} P_t(s^t)\hat{c}_t^1(s^t) \geq \sum_{t=0}^{\infty} \sum_{s^t \in S^t} P_t(s^t)\tilde{c}_t^1(s^t).$$

Then as $\hat{c}^1$ is CE, $u(\hat{c}^1) \geq u(\tilde{c}^1)$. Therefore,

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} P_t(s^t)\hat{c}_t^1(s^t) < \sum_{t=0}^{\infty} \sum_{s^t \in S^t} P_t(s^t)\tilde{c}_t^1(s^t).$$

Suppose

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} P_t(s^t)\hat{c}_t^2(s^t) > \sum_{t=0}^{\infty} \sum_{s^t \in S^t} P_t(s^t)\tilde{c}_t^2(s^t).$$

Then there exists $\delta > 0$ such that

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} P_t(s^t)\hat{c}_t^2(s^t) \geq \sum_{t=0}^{\infty} \sum_{s^t \in S^t} P_t(s^t)\tilde{c}_t^2(s^t) + \delta.$$

Define $\bar{c}^2$ as

$$\begin{aligned} \bar{c}_t^2(s^t) &= \tilde{c}_t^2 && \text{for } t \neq 0 \\ \bar{c}_0^2(s^0) &= \tilde{c}_0^2 + \pi(s_0)\delta && \text{for } t = 0 \end{aligned}$$

Then

$$u(\bar{c}^2) \geq u(\tilde{c}^2) \geq u(\hat{c}^2).$$

This contradicts that $\hat{c}^2$ is CE. Hence,

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} P_t(s^t)\hat{c}_t^2(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t \in S^t} P_t(s^t)\tilde{c}_t^2(s^t).$$

Then

$$\sum_{i \in \{1,2\}} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} P_t(s^t)\hat{c}_t^i(s^t) < \sum_{i \in \{1,2\}} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} P_t(s^t)\tilde{c}_t^i(s^t).$$

As $(\hat{c}^1, \hat{c}^2)$ and $(\tilde{c}^1, \tilde{c}^2)$ are feasible,

$$\forall t \ \forall s^t \in S^t \qquad \hat{c}_t^1(s^t) + \hat{c}_t^2(s^t) = \tilde{c}_t^1(s^t) + \tilde{c}_t^2(s^t) \tag{1}$$

Hence

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} P_t(s^t) < \sum_{t=0}^{\infty} \sum_{s^t \in S^t} P_t(s^t)$$

This is a contradiction. This shows $(\hat{c}^1, \hat{c}^2)$ is a Pareto efficient allocation.

Q2

(a)

Given k_0 and z_0, the recursive formulation of the problem is

$$\begin{aligned} w(k_0, z_0) &= \max_{\{c_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \log(c_t) \\ &= \max_{\{k_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \log(e^{z_t} k_t^{\alpha} + (1-\delta)k_t - k_{t+1}) \\ &= \max_{k_1} \left[\log(e^{z_0} k_0^{\alpha} + (1-\delta)k_0 - k_1) + \beta \max_{\{k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \sum_{z_t} \beta^{t-1} \pi_t(z_t) \log(e^{z_t} k_t^{\alpha} + (1-\delta)k_t - k_{t+1}) \right] \\ &= \max_{k_1} \left[\log(e^{z_0} k_0^{\alpha} + (1-\delta)k_0 - k_1) + \beta \sum_{z_1} \pi_1(z_1|z_0) w(k_1, z_1) \right] \end{aligned}$$

(b)

The grid Z, the transition matrix P, and the stationary distribution π gained by Tauchen's method are as follows:

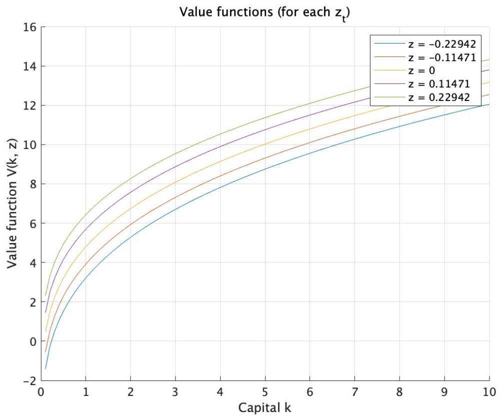
$$Z = \begin{pmatrix} -0.2294 \\ -0.1147 \\ 0 \\ 0.1147 \\ 0.2294 \end{pmatrix}, \quad P = \begin{pmatrix} 0.6346 & 0.2974 & 0.0638 & 0.0041 & 0.0001 \\ 0.2456 & 0.4312 & 0.2690 & 0.0512 & 0.0030 \\ 0.0427 & 0.2405 & 0.4337 & 0.2405 & 0.0427 \\ 0.0030 & 0.0512 & 0.2690 & 0.4312 & 0.2456 \\ 0.0001 & 0.0041 & 0.0638 & 0.2974 & 0.6346 \end{pmatrix}, \quad \pi = \begin{pmatrix} 0.1708 \\ 0.2101 \\ 0.2381 \\ 0.2101 \\ 0.1708 \end{pmatrix}$$

The Matlab code is below.

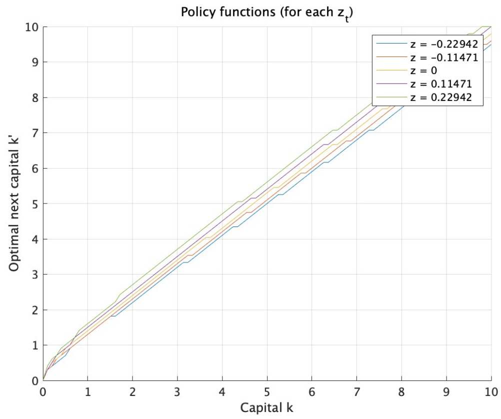
```
function [Z,Zprob] = tauchen(N,mu,rho,sigma,m)

    Zprob = zeros(N,N); % Transition Matrix
    c = (1-rho)*mu; % Constant

    % Define Grids
    zmax  = m*sqrt(sigma^2/(1-rho^2));
    zmin  = -zmax;
    w = (zmax-zmin)/(N-1);
    Z = linspace(zmin,zmax,N)';

    % Stationary value, mu
    Z = Z + mu;

    % Create Transition Matrix
    for j = 1:N
        for k = 1:N
            if k == 1
                Zprob(j,k) = normcdf((Z(1)-c-rho*Z(j)+w/2)/sigma);
            elseif k == N
                Zprob(j,k) = 1 - normcdf((Z(N)-c-rho*Z(j)-w/2)/sigma);
            else
                Zprob(j,k) = normcdf((Z(k)-c-rho*Z(j)+w/2)/sigma) - ...
                             normcdf((Z(k)-c-rho*Z(j)-w/2)/sigma);
            end
        end
    end
end

N = 5;
mu = 0;
rho = 0.9;
sigma = 0.1;
m = 1;

[Z,Zprob] = tauchen(N,mu,rho,sigma,m);

disp('Grid (Z):');
disp(Z);

disp('Transition matrix (P):');
disp(Zprob);

[V, D] = eig(Zprob');
[~, idx] = max(abs(diag(D)));
pi = V(:, idx);
pi = pi / sum(pi);

disp('stationary distribution:');
disp(pi);
```

(c)

Using the results from (b), we can solve the model by value function iteration. The value function and the policy function are as follows:



(a) Value function



(b) Policy function

The Matlab code is below.

```
beta = 0.95;
alpha = 0.4;
delta = 0.06;
u = @(c) log(c);
k_min = 0;
k_max = 10;
Nk = 100;
k_grid = linspace(k_min, k_max, Nk)';

Nz = 5;
mu = 0;
rho = 0.9;
sigma = 0.1;
m = 1;
[Z,Zprob] = tauchen(Nz,mu,rho,sigma,m);

V = zeros(Nk, Nz);
policy_k = zeros(Nk, Nz);

max_iter = 1000;
tol = 1e-6;
for iter = 1:max_iter
    V_new = zeros(Nk, Nz);
    for i = 1:Nk
        for j = 1:Nz
            z0 = Z(j);
            k0 = k_grid(i);

            c = exp(z0) * k0^alpha + (1 - delta) * k0 - k_grid;
            U = u(c);
            U(c <= 0) = -inf;

            EV = V * Zprob(j,:)';

            total_value = U + beta * EV;

            [V_new(i,j), policy_index] = max(total_value);
            policy_k(i,j) = k_grid(policy_index);
        end
    end

    if max(abs(V_new(:) - V(:))) < tol
        disp(['Converged (number of iterations: ', num2str(iter), ')']);
        break;
    end
```

```
    V = V_new;
end

figure;
hold on;
for i_z = 1:Nz
    plot(k_grid, V(:, i_z), 'DisplayName', ['z = ', num2str(Z(i_z))]);
end
xlabel('Capital k');
ylabel('Value function V(k, z)');
title('Value functions (for each z_t)');
legend show;
grid on;

figure;
hold on;
for i_z = 1:Nz
    plot(k_grid, policy_k(:, i_z), 'DisplayName', ['z = ', num2str(Z(i_z))]);
end
xlabel('Capital k');
ylabel('Optimal next capital k''');
title('Policy functions (for each z_t)');
legend show;
grid on;
```

Q3

(a)

$$P^2 = \begin{bmatrix} 0.83 & 0.15 & 0.01 & 0.01 \\ 0.31 & 0.41 & 0.15 & 0.13 \\ 0.05 & 0.27 & 0.53 & 0.15 \\ 0.17 & 0.17 & 0.27 & 0.39 \end{bmatrix}.$$

For all $i, j \in \{1, 2, 3, 4, \}$, $P^2_{ij} > 0$. Therefore, by the LS theorem 2.2.2, P has a unique stationary distribution and the process is asymptotically stationary.

(b)

Stationary distribution $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ of P satisfies

$$\pi^T = \pi^T P$$

$$\begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix} = \begin{bmatrix} 0.9\pi_1 + 0.2\pi_2 + 0.0\pi_3 + 0.1\pi_4 \\ 0.1\pi_1 + 0.6\pi_2 + 0.2\pi_3 + 0.1\pi_4 \\ 0.0\pi_1 + 0.1\pi_2 + 0.7\pi_3 + 0.2\pi_4 \\ 0.0\pi_1 + 0.1\pi_2 + 0.1\pi_3 + 0.6\pi_4 \end{bmatrix}$$

Solving this, we have

$$\pi = (\frac{6}{11}, \frac{5}{22}, \frac{3}{22}, \frac{1}{11})$$

(c)

(i)

$$
\begin{aligned}
Pr(e^1_{t+1}|e^1_t) &= 0.9\\
Pr(e^1_{t+1}|e^2_t) &= 0.2\\
Pr(e^1_{t+1}|e^3_t) &= 0.0\\
Pr(e^1_{t+1}|e^4_t) &= 0.1\\
Pr(e^2_{t+1}|e^1_t) &= 0.1\\
Pr(e^3_{t+1}|e^2_t) &= 0.1\\
Pr(e^4_{t+1}|e^3_t) &= 0.1
\end{aligned}
$$

Then, the likelihood is

$$\left(\frac{6}{11} * 0.9 + \frac{5}{22} * 0.2 + \frac{3}{22} * 0 + \frac{1}{11} * 0.1\right) * 0.1^3 = \frac{6}{11000}$$

(ii)

$$
\begin{aligned}
Pr(e^1_{t+1}|e^1_t) &= 0.9\\
Pr(e^1_{t+1}|e^2_t) &= 0.2\\
Pr(e^1_{t+1}|e^3_t) &= 0.0\\
Pr(e^1_{t+1}|e^4_t) &= 0.1
\end{aligned}
$$

Then, the likelihood is

$$\left(\frac{6}{11} * 0.9 + \frac{5}{22} * 0.2 + \frac{3}{22} * 0 + \frac{1}{11} * 0.1\right) * 0.9^3 = \frac{4374}{11000}$$

(d)

$$E[y_1|e_0 = e^1] = 0.9 * 0 + 0.1 * 1 = 0.1$$

$$P^5 = \begin{bmatrix} 0.70524 & 0.19908 & 0.05284 & 0.04284 \\ 0.44100 & 0.25652 & 0.17908 & 0.12340 \\ 0.23420 & 0.27964 & 0.31708 & 0.16908 \\ 0.31476 & 0.24476 & 0.25964 & 0.18084 \end{bmatrix}.$$

$$E[y_5|e_0 = e^1] = 0.70524 * 0 + 0.19908 * 1 + 0.05284 * 2 + 0.04284 * 4 = 0.47612.$$

Q4

(a)

A competitive Arrow-Debreu equilibrium is prices $\{\hat{p}_t(s^t)\}_{t=0,s^t\in S^t}^{\infty}$ and allocations $\{\hat{c_t^i}(s^t)\}_{t=0,s^t\in S^t}^{\infty}$ for $i = 1, 2$ such that

$$\max_{\{c_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty}} \quad \Sigma_{t=0}^{\infty}\Sigma_{s^t\in S^t}\beta^t\pi(s^t)\log c_t^i(s^t)$$

$$\text{s.t. } \Sigma_{t=0}^{\infty}\Sigma_{s^t\in S^t}\hat{p}_t(s^t)c_t^i(s^t) \leq \Sigma_{t=0}^{\infty}\Sigma_{s^t\in S^t}\hat{p}_t(s^t)e_t^i(s^t)$$
$$c_t^i \geq 0 \text{ for all } t, \text{ all } s^t \in S^t$$

and goods market clear, i.e.,

$$\hat{c_t^1}(s^t) + \hat{c_t^2}(s^t) = e_t^1(s^t) + e_t^2(s^t) \text{ for all } t, \text{ all } s^t \in S^t$$

(b)

Let μ^i be the Lagrange multiplier on the budget constraint.
The FOCs w.r.t. $c_t^i(s^t)$ and $c_o^i(s^0)$ are as below.

$$\frac{\beta^t\pi(s^t)}{c_t^i(s^t)} = \mu^i p_t(s^t)$$
$$\frac{\pi(s^0)}{c_0^i(s^0)} = \mu^i p_o(s^0)$$

From these equations,

$$\frac{p_t(s^t)}{p_0(s^0)} = \beta^t\frac{\pi(s^t)}{\pi(s^0)}\frac{c_0^i(s^0)}{c_t^i(s^t)}$$

Thus,

$$\frac{c_0^2(s^0)}{c_0^1(s^0)} = \frac{c_t^2(s^t)}{c_t^1(s^t)}$$

holds. Then, there exist $\theta^i \geq 0$ with $\Sigma_i\theta^i = 1$ such that

$$c_t^i(s^t) = \theta^i e_t(s^t)$$
$$\text{where } \theta^i = \frac{\Sigma_{t=0}^{\infty}\Sigma_{s^t\in S^t}p^t(s^t)e_t^i(s^t)}{\Sigma_{t=0}^{\infty}\Sigma_{s^t\in S^t}p^t(s^t)e_t(s^t)}$$
$$\text{s.t. } \theta^1 + \theta^2 = 1$$

with $e_t(s^t) = \Sigma_i e_t^i(s^t)$. By using this and normalizing $p_0(s^0) = 1$,

$$p_t(s^t) = \beta^t\frac{\pi(s^t)}{\pi(s^0)}\frac{e_0(s^0)}{e_t(s^t)}$$

(c)

Let $q_t(s^t, s_{t+1})$ denote price at period t of a contract that pays one unit of consumption in period $t+1$ if $t+1$ event is s_{t+1}.
Let $a_{t+1}^i(s^t, s_{t+1})$ be the quantities of Arrow securities bought at period t by agent i.

A sequential market equilibrium is allocations $(\hat{c^i}, \hat{a^i})_{i=1,2}$ and prices $\hat{q}$ such that

$$\begin{aligned}
&\text{For } i = 1, 2, \text{ given } \hat{q}, (\hat{c^i}, \hat{a^i})_{i=1,2} \text{ solves} \\
&\max_{c^i, a^i} \log c_t^i(s^t) \\
&\text{s.t. } \log c_t^i(s^t) + \Sigma_{s_{t+1} \in S} \hat{q}_t(s^t, s_{t+1}) \leq e_t^i(s^t) + a_t^i(s^t) \\
&c_t^i(s^t) \geq 0 \\
&a_{t+1}^i(s^t, s_{t+1}) \geq -\bar{A}^i \\
&\text{for all } t, s^t \in S^t \text{ and } s_{t+1} \in S.
\end{aligned}$$

and goods markets clear, i.e.,

$$\begin{aligned}
&\text{For all } t \geq 0, \\
&\Sigma_{i=1,2} \hat{c_t^i}(s^t) = \Sigma_{i=1,2} e_t^i(s^t) \text{ for all } t, s^t \in S^t \\
&\Sigma_{i=1,2} \hat{a}_{t+1}^i(s^t, s_{t+1}) = 0 \text{ for all } t, s^t \in S^t \text{ and } s_{t+1} \in S
\end{aligned}$$

In addition, the natural borrowing limit for each consumer in each state is

$$\bar{A}^i = \Sigma_{t=0}^{\infty} \Sigma_{s^t \in S^t} p_t(s^t) e_t^i(s^t)$$

(d)

Let $q_t(s^t, s_{t+1})$ denote the price of the price of one-period state-contingent bonds. From (b),

$$p_t(s^t) = \beta^t \frac{\pi(s^t)}{\pi(s^0)} \frac{e_0(s^0)}{e_t(s^t)}$$

Then,

$$\begin{aligned}
q_t(s^t, s_{t+1}) &= \frac{p_{t+1}(s^{t+1})}{p_t(s^t)} \\
&= \beta \frac{\pi(s^{t+1})}{\pi(s^t)} \frac{e_t(s^t)}{e_{t+1}(s^{t+1})}
\end{aligned}$$

For this economy, 4 price patterns exist.

(i) $q_t(s^1, s^2) = \beta \times 0.7 \times \frac{1}{3} \fallingdotseq 0.222$
(ii) $q_t(s^1, s^1) = \beta \times 0.3 = 0.285$
(iii) $q_t(s^2, s^1) = \beta \times 0.1 \times 3 = 0.285$
(iv) $q_t(s^2, s^2) = \beta \times 0.9 = 0.855$

Let $R_{t+1}(s^{t+1})$ be the gross return.
The gross returns from each bond purchased at time t are

(i) $q_t(s^1, s^1) = \beta \times 0.3 = 0.285$
$R_{t+1}(s^1) = 1/0.285 \fallingdotseq 3.509$
(ii) $q_t(s^2, s^1) = \beta \times 0.1 \times 3 = 0.285$
$R_{t+1}(s^1) = 1/0.285 \fallingdotseq 3.509$

(e)

Let $P_t^B(d; s^t)$ denote the price of one-period risk-free bond.

$$\begin{aligned} P_t^B(d; s^t) &= \frac{\Sigma_{s^{t+1}} p_{t+1}(s^{t+1})}{p_t(s^t)} \\ &= \Sigma_{s^{t+1}} q_t(s^t, s_{t+1}) \end{aligned}$$

Therefore, the fross return of the bond $R_{t+1}^B(s^{t+1})$ is

(i) $s^t = s^1$
$P_t^B(d; s^1) = 0.285 + 0.222 = 0.507$
$\therefore R_{t+1}(s^{t+1}) = 1.972$

(ii) $s^t = s^2$
$P_t^B(d; s^2) = 0.285 + 0.855 = 1.14$
$\therefore R_{t+1}(s^{t+1}) = 0.877$