# Macroeconomics II Homework 3

Graduate School of Economics, The University of Tokyo

29–246029 Rin NITTA
29-246033 Rei HANARI
29–246004 Kosuke IGARASHI

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## Q1

### (a)

The recursive formulation of a standard neoclassical growth model studied in class in Lecture 3 is

$$v(k) = \max_{0 \leq k' \leq f(k)} \{U(f(k) - k') + \beta v(k')\}.$$

Define metricx space $(B(X), d)$, the space of bounded functions on $X = [0, \infty)$ with the sup-norm $d$, and difine operator $T$ as

$$Tv(k) = \max_{0 \leq k' \leq f(k)} \{U(f(k) - k') + \beta v(k')\}.$$

Then, by showing $T$ satisfies Blackwerll's condition, we can show $T$ is a contraction mapping. Then, by CMT, we can show convergent $k$ is an unique fixed point.

### (b)

Let $X$ be a set. Consider the space $B(X)$ of all bounded functions $f : X \to \mathbb{R}$, equipped with the *supremum norm*:

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

The metric on $B(X)$ is defined as:

$$d(f, g) = \|f - g\|_\infty.$$

We aim to show that the metric space $(B(X), d)$ is complete, i.e., every Cauchy sequence of functions in $B(X)$ converges to a function in $B(X)$.

Let $\{f_n\}$ be a Cauchy sequence in $(B(X), d)$. By definition, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$:

$$\|f_n - f_m\|_\infty < \epsilon,$$

which implies:

$$\sup_{x \in X} |f_n(x) - f_m(x)| < \epsilon.$$

For each $x \in X$, the sequence $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$ because:

$$|f_n(x) - f_m(x)| < \epsilon \quad \text{for all } m, n \geq N.$$

Since $\mathbb{R}$ is complete, $\{f_n(x)\}$ converges to a limit, say $f(x) \in \mathbb{R}$. Thus, we can define a pointwise limit function $f : X \to \mathbb{R}$ by:

$$f(x) = \lim_{n\to\infty} f_n(x), \quad \text{for each } x \in X.$$

Since $\{f_n\} \subseteq B(X)$, each $f_n$ is bounded, i.e., there exists $M_n$ such that $\|f_n\|_\infty \leq M_n$. Let $M = \sup_n M_n$. Then for all $n$ and $x \in X$:

$$|f_n(x)| \leq M.$$

Taking the limit as $n \to \infty$, we obtain:

$$|f(x)| \leq M, \quad \text{for all } x \in X.$$

Thus, $f$ is bounded, and hence $f \in B(X)$.

For $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$:

$$\|f_n - f_m\|_\infty < \epsilon.$$

Fix $n \geq N$. Then for all $x \in X$:

$$|f_n(x) - f_m(x)| < \epsilon \quad \text{for all } m \geq N.$$

Taking the limit as $m \to \infty$, we get:

$$|f_n(x) - f(x)| \leq \epsilon.$$

Thus:

$$\|f_n - f\|_\infty \leq \epsilon \quad \text{for all } n \geq N.$$

This shows that $\{f_n\}$ converges uniformly to $f$.

Since $\{f_n\}$ is a Cauchy sequence in $(B(X), d)$ and converges uniformly to $f \in B(X)$, the metric space $(B(X), d)$ is complete.

## (c)

Suppose not, there exists feasible allocation $\{\tilde{c}_t^1,\ \tilde{c}_t^2\}_{t=0,\ s^t\in S^t}^{\infty}$ such that

$$u(\hat{c}^i) \leq u(\tilde{c}^i) \text{ for all } i \in \{1,2\}$$
$$u(\hat{c}^i) < u(\tilde{c}^i) \text{ for some } i \in \{1,2\}$$

Without loss of generality, assume strict inequality holds for $i = 1$.

Suppose

$$\sum_{t=0}^{\infty}\sum_{s^t\in S^t} P_t(s^t)\hat{c}_t^1(s^t) \geq \sum_{t=0}^{\infty}\sum_{s^t\in S^t} P_t(s^t)\tilde{c}_t^1(s^t).$$

Then as $\hat{c}^1$ is CE, $u(\hat{c}^1) \geq u(\tilde{c}^1)$. Therefore,

$$\sum_{t=0}^{\infty}\sum_{s^t\in S^t} P_t(s^t)\hat{c}_t^1(s^t) < \sum_{t=0}^{\infty}\sum_{s^t\in S^t} P_t(s^t)\tilde{c}_t^1(s^t).$$

Suppose

$$\sum_{t=0}^{\infty}\sum_{s^t\in S^t} P_t(s^t)\hat{c}_t^2(s^t) > \sum_{t=0}^{\infty}\sum_{s^t\in S^t} P_t(s^t)\tilde{c}_t^2(s^t).$$

Then there exists $\delta > 0$ such that

$$\sum_{t=0}^{\infty}\sum_{s^t\in S^t} P_t(s^t)\hat{c}_t^2(s^t) \geq \sum_{t=0}^{\infty}\sum_{s^t\in S^t} P_t(s^t)\tilde{c}_t^2(s^t) + \delta.$$

Define $\bar{c}^2$ as

$$\begin{aligned}
\bar{c}_t^2(s^t) &= \tilde{c}_t^2 && \text{for } t \neq 0 \\
\bar{c}_0^2(s^0) &= \tilde{c}_0^2 + \pi(s_0)\delta && \text{for } t = 0
\end{aligned}$$

Then

$$u(\bar{c}^2) \geq u(\tilde{c}^2) \geq u(\hat{c}^2).$$

This contradicts that $\hat{c}^2$ is CE. Hence,

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} P_t(s^t)\hat{c}_t^2(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t \in S^t} P_t(s^t)\tilde{c}_t^2(s^t).$$

Then

$$\sum_{i \in \{1,2\}} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} P_t(s^t)\hat{c}_t^i(s^t) < \sum_{i \in \{1,2\}} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} P_t(s^t)\tilde{c}_t^i(s^t).$$

As $(\hat{c}^1, \hat{c}^2)$ and $(\tilde{c}^1, \tilde{c}^2)$ are feasible,

$$\forall t \ \forall s^t \in S^t \qquad \hat{c}_t^1(s^t) + \hat{c}_t^2(s^t) = \tilde{c}_t^1(s^t) + \tilde{c}_t^2(s^t) \tag{1}$$

Hence

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} P_t(s^t) < \sum_{t=0}^{\infty} \sum_{s^t \in S^t} P_t(s^t)$$

This is a contradiction. This shows $(\hat{c}^1, \hat{c}^2)$ is a Pareto efficient allocation.

# Q2

## (a)

Given $k_0$ and $z_0$, the recursive formulation of the problem is

$$\begin{aligned}
w(k_0, z_0) &= \max_{\{c_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \log(c_t) \\
&= \max_{\{k_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \log(e^{z_t} k_t^{\alpha} + (1-\delta)k_t - k_{t+1}) \\
&= \max_{k_1} \left[ \log(e^{z_0} k_0^{\alpha} + (1-\delta)k_0 - k_1) + \beta \max_{\{k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \sum_{z_t} \beta^{t-1} \pi_t(z_t) \log(e^{z_t} k_t^{\alpha} + (1-\delta)k_t - k_{t+1}) \right] \\
&= \max_{k_1} \left[ \log(e^{z_0} k_0^{\alpha} + (1-\delta)k_0 - k_1) + \beta \sum_{z_1} \pi_1(z_1|z_0) w(k_1, z_1) \right]
\end{aligned}$$

## (b)

The grid $Z$, the transition matrix $P$, and the stationary distribution $\pi$ gained by Tauchen's method are as follows:

$$Z = \begin{pmatrix} -0.6822 \\ -0.3441 \\ 0 \\ 0.3441 \\ 0.6822 \end{pmatrix}, \quad P = \begin{pmatrix} 0.8491 & 0.1509 & 0.0000 & 0.0000 & 0 \\ 0.0195 & 0.8962 & 0.0843 & 0.0000 & 0 \\ 0.0000 & 0.0427 & 0.9147 & 0.0427 & 0.0000 \\ 0.0000 & 0.0000 & 0.0843 & 0.8962 & 0.0195 \\ 0 & 0.0000 & 0.0000 & 0.1509 & 0.8491 \end{pmatrix}, \quad \pi = \begin{pmatrix} 0.0305 \\ 0.2361 \\ 0.4668 \\ 0.2361 \\ 0.0305 \end{pmatrix}$$

The Matlab code is below.

```matlab
function [Z,Zprob] = tauchen(N,mu,rho,sigma,m)

    Z = zeros(N,1); % Grid
    Zprob = zeros(N,N); % Transition Matrix
    c = (1-rho)*mu; % Constant

    % Define Grids
    zmax  = m*sqrt(sigma^2/(1-rho^2));
    zmin  = -zmax;
    w = (zmax-zmin)/(N-1);
    Z = linspace(zmin,zmax,N)';

    % Stationary value, mu
    Z = Z + mu;

    % Create Transition Matrix
    for j = 1:N
        for k = 1:N
            if k == 1
                Zprob(j,k) = normcdf((Z(1)-c-rho*Z(j)+w/2)/sigma);
            elseif k == N
                Zprob(j,k) = 1 - normcdf((Z(N)-c-rho*Z(j)-w/2)/sigma);
            else
                Zprob(j,k) = normcdf((Z(k)-c-rho*Z(j)+w/2)/sigma) - ...
                             normcdf((Z(k)-c-rho*Z(j)-w/2)/sigma);
            end
        end
    end
end

N = 5;
mu = 0;
rho = 0.9;
sigma = 0.1;
m = 3;

[Z,Zprob] = tauchen(N,mu,rho,sigma,m);

disp('Grid (Z):');
disp(Z);

disp('Transition matrix (P):');
disp(Zprob);

[V, D] = eig(Zprob');
[~, idx] = max(abs(diag(D)));
pi = V(:, idx);
pi = pi / sum(pi);

disp('stationary distribution:');
disp(pi);

figure;
bar(Z, pi, 'FaceColor', [0.2, 0.6, 0.8]);
xlabel('State space z_t');
ylabel('Stationary distribution pi');
title('Stationary distribution of AR(1) process by Tauchen's method');
grid on;
```

### (c)

## Q3

### (a)

$$P^2 = \begin{bmatrix} 0.83 & 0.15 & 0.01 & 0.01 \\ 0.31 & 0.41 & 0.15 & 0.13 \\ 0.05 & 0.27 & 0.53 & 0.15 \\ 0.17 & 0.17 & 0.27 & 0.39 \end{bmatrix}.$$

For all $i, j \in \{1, 2, 3, 4, \}$, $P^2_{ij} > 0$. Therefore, by the LS theorem 2.2.2, P has a unique stationary distribution and the process is asymptotically stationary.

### (b)

Stationary distribution $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ of $P$ satisfies

$$\pi^T = \pi^T P$$

$$\begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix} = \begin{bmatrix} 0.9\pi_1 + 0.2\pi_2 + 0.0\pi_3 + 0.1\pi_4 \\ 0.1\pi_1 + 0.6\pi_2 + 0.2\pi_3 + 0.1\pi_4 \\ 0.0\pi_1 + 0.1\pi_2 + 0.7\pi_3 + 0.2\pi_4 \\ 0.0\pi_1 + 0.1\pi_2 + 0.1\pi_3 + 0.6\pi_4 \end{bmatrix}$$

Solving this, we have

$$\pi = (\frac{6}{11}, \frac{5}{22}, \frac{3}{22}, \frac{1}{11})$$

### (c)

#### (i)

$$Pr(e^1_{t+1}|e^1_t) = 0.9$$
$$Pr(e^1_{t+1}|e^2_t) = 0.2$$
$$Pr(e^1_{t+1}|e^3_t) = 0.0$$
$$Pr(e^1_{t+1}|e^4_t) = 0.1$$
$$Pr(e^2_{t+1}|e^1_t) = 0.1$$
$$Pr(e^3_{t+1}|e^2_t) = 0.1$$
$$Pr(e^4_{t+1}|e^3_t) = 0.1$$

Then, the likelihood is

$$\left(\frac{6}{11} * 0.9 + \frac{5}{22} * 0.2 + \frac{3}{22} * 0 + \frac{1}{11} * 0.1\right) * 0.1^3 = \frac{6}{11000}$$

#### (ii)

$$Pr(e^1_{t+1}|e^1_t) = 0.9$$
$$Pr(e^1_{t+1}|e^2_t) = 0.2$$
$$Pr(e^1_{t+1}|e^3_t) = 0.0$$
$$Pr(e^1_{t+1}|e^4_t) = 0.1$$

Then, the likelihood is

$$\left(\frac{6}{11} * 0.9 + \frac{5}{22} * 0.2 + \frac{3}{22} * 0 + \frac{1}{11} * 0.1\right) * 0.9^3 = \frac{4374}{11000}$$

(d)

$$E[y_1|e_0 = e^1] = 0.9 * 0 + 0.1 * 1 = 0.1$$

$$P^5 = \begin{bmatrix} 0.70524 & 0.19908 & 0.05284 & 0.04284 \\ 0.44100 & 0.25652 & 0.17908 & 0.12340 \\ 0.23420 & 0.27964 & 0.31708 & 0.16908 \\ 0.31476 & 0.24476 & 0.25964 & 0.18084 \end{bmatrix}.$$

$$E[y_5|e_0 = e^1] = 0.70524 * 0 + 0.19908 * 1 + 0.05284 * 2 + 0.04284 * 4 = 0.47612.$$