Macroeconomics II Homework 2

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November 4, 2024

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(a)

At the optimal consumption, the budget constraint holds with equality:

$$c_t = Ak_t^\alpha - k_{t+1}$$

Define

$$w(k_0) = \max_{\{k_{t+1}\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t \log c_t$$

Then, this is equivalent to

$$\begin{aligned}
w(k_0) &= \max_{\{k_{t+1}\}_{t=0}^\infty ; k_0 \text{ given}} \sum_{t=0}^\infty \beta^t \log c_t \\
&= \max_{k_1 ; k_0 \text{ given}} \log c_0 + \beta \left[\max_{\{k_{t+1}\}_{t=1}^\infty ; k_1 \text{ given}} \sum_{t=1}^\infty \beta^{t-1} \log c_t \right] \\
&= \max_{k_1 ; k_0 \text{ given}} \log(Ak_0^\alpha - k_1) + \beta w(k_1)
\end{aligned}$$

with conditions $0 \leq k_1 \leq Ak_0^\alpha$ for all $t = 0, \cdots, \infty$. This is the recursive problem of the social planner.

(b)

(i): Guess and Verify

Guess $V(k) = X + Y \log Ak^\alpha$ where X and Y are coefficients to be determined. Then, the Bellman equation is

$$V(k) = \max_{0 \leq k' \leq Ak^\alpha ; k \text{ given}} \log(Ak^\alpha - k') + \beta(X + Y \log Ak'^\alpha)$$

The first-order condition is

$$\frac{\partial V(k)}{\partial k'} = -\frac{1}{Ak^\alpha - k'} + \alpha\beta Y \frac{1}{k'} = 0$$
$$\therefore \quad k' = \frac{\alpha\beta Y}{1+\alpha\beta Y} A k^\alpha$$

Evaluating the value function at the optimal k', we have

$$\begin{aligned} V(k') &= \log\left(Ak^\alpha - \frac{\alpha\beta Y}{1+\alpha\beta Y}Ak^\alpha\right) + \beta(X + Y\log A\left(\frac{\alpha\beta Y}{1+\alpha\beta Y}Ak^\alpha\right)^\alpha) \\ &= \beta X + \log\left(\frac{1}{1+\alpha\beta Y}\right) + \alpha\beta Y \log\left(\frac{\alpha\beta Y}{1+\alpha\beta Y}\right) + \alpha(1+\beta Y)\log\left(Ak^\alpha\right) \end{aligned}$$

Solving this, we have

$$\begin{aligned} X &= \frac{1}{1-\beta}\left[\log\left(\frac{1}{1+\alpha\beta Y}\right) + \alpha\beta Y\log\left(\frac{\alpha\beta Y}{1+\alpha\beta Y}\right)\right] \\ Y &= \frac{1}{\alpha(1-\beta)} \end{aligned}$$

(ii): Value Function Iteration

(iii): Policy Function Iteration

Using the result in (i), we have the true policy function:

$$k' = \frac{\alpha\beta Y}{1+\alpha\beta Y}Ak^\alpha = \frac{\alpha\beta}{1-\beta+\alpha\beta}Ak^\alpha$$

Using this policy function, we can caluculate the value function.

$$\begin{aligned} V_{h_j}(k_0) &= \sum_{t=0}^{\infty}\beta^t \log\left(Ak_t^\alpha - \frac{\alpha\beta}{1-\beta+\alpha\beta}Ak^\alpha\right) \\ &= \frac{1}{1-\beta}\log\left(\frac{1-\beta}{1-\beta+\alpha\beta}A\right) + \sum_{t=0}^{\infty}\beta^t \log k_t^\alpha \end{aligned}$$

Since the second term is caluculated as

$$\begin{aligned} \sum_{t=0}^{\infty}\beta^t\log k_t^\alpha &= \sum_{t=o}^{\infty}\beta^t\alpha^{t+1}\log k_0 + \sum_{t=o}^{\infty}\beta^t(\alpha+\cdots+\alpha^t)\log\left(\frac{\alpha\beta}{1-\beta+\alpha\beta}A\right) \\ &= \frac{\alpha}{1-\alpha\beta}\log k_0 + \frac{\alpha\beta}{(1-\beta)(1-\alpha\beta)}\log\left(\frac{\alpha\beta}{1-\beta+\alpha\beta}A\right) \end{aligned}$$

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(a)

The agent's life-time budget constraint is

$$\sum_{t=0}^{\infty} p_t\left(c_t + \frac{b_{t+1}}{1+r_{t+1}} - b_t - e\right)$$

(b)

Define the Lagrangian as

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\log c_t + \lambda_t \left(p_t \left(c_t + \frac{b_{t+1}}{1 + r_{t+1}} - b_t - e \right) \right) \right]$$

The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial c_t} = U'(c_t) - \lambda_t p_t = 0$$
$$\frac{\partial \mathcal{L}}{\partial b_t + 1} = -\beta^t \frac{\lambda_t p_t}{1 + r_{t+1}} + \beta^{t+1} \lambda_{t+1} p_{t+1} = 0$$

Thus, the Euler equation is

$$U'(c_t) = \beta U'(c_{t+1})(1 + r_{t+1})$$

The transversality condition is

$$\lim_{t \to \infty} \beta^t (1 + r_{t+1}) U'(c_t) b_t = 0$$

with $c_t = b_t - \frac{b_{t+1}}{1+r_{t+1}} + e$.

(c)

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(a)

$$\begin{aligned} v(k, K) = \max_{c,k \geq 0} U(c) + v(k', K') & \\ & s.t.\ c + k' = w(K) + (1 + r(K) - \delta - \tau)k + T \\ & K' = H(K) \end{aligned}$$

(b)

RCE is a value function $v : \mathbb{R}_+^2 \to \mathbb{R}$ and policy function $C, G :\ \mathbb{R}_+^2 \to \mathbb{R}_+$ for the representative household, pricing function $w, r :\ \mathbb{R} \to \mathbb{R}_+$ and an aggregate law of motion $H :\ \mathbb{R}_+ \to \mathbb{R}_+$ such that

1 Given w, r, H, v solves the Bellman equation and C, G are the associated policy function.
2 Pricing function satisfies firm's FOC.
3 Consistency for all $K \in \mathbb{R}_+$: $H(K) = G(K, K)$
4 Market clearing: for all $K \in \mathbb{R}_+$

$$C(K, K) + G(K, K) = F(K, 1) + (1 - \delta)K$$

.

(c)

From market clear,

$$
\begin{aligned}
k_t^d &= k_t^s \\
c_t + k_{t+1} &= f(k_t).
\end{aligned}
$$

Thus law of motion for aggregate capital stock is

$$K' = H(K) = f(K) - C(K, K).$$

(d)

In the steady state, $K_{t+1} = K_t = K^*$. Thus $I = \delta K^*$ From market clear

$$
\begin{aligned}
I = \delta K^* &= F(K^*, 1) - C(K^*, K^*) \\
K^* &= \frac{1}{\delta}\{F(K^*, 1) - C(K^*, K^*)\}
\end{aligned}
$$

(e)

Law of motion for aggregate variables coincide with CE allocation for representative household and firm.

In CE, Firm solves

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} p_t\big(F(K_t^d, 1) - w_t - r_t k_t^d\big)$$

From FOC and assumption of F(·),

$$
\begin{aligned}
F_K(k_t^d, 1) &= r_t \\
F_N(k_t^d, 1) &= w_t \\
\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} p_t\big(F(K_t^d, 1) - w_t - r_t k_t^d\big) &= 0.
\end{aligned}
$$

Representative household solves

$$
\begin{aligned}
\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} U(c_t) \\
s.t. \\
\sum_{t=0}^{\infty} P_t(c_t + i_t) \leq \sum_{t=0}^{\infty} P_t(w_t + k_t r_t). c_t, k_{t+1} \geq 0 c_t + i_t = k_{t+1} + (1-\delta)k_t
\end{aligned}
$$

Note that we do not have to consider the tax because government pay back all taxes.

Define Lagrangian as

$$\mathcal{L} = \sum_{t=0}^{\infty} U(c_t) + \mu\left[\sum_{t=0}^{\infty} P_t(w_t + k_t r_t) - \sum_{t=0}^{\infty} P_t(c_t + i_t)\right].$$

From FOCs

$$\begin{aligned} \beta^t U'(c_t) &= \mu P_t \\ P_{t+1} r_{t+1} - P_t + (1-\delta) P_{t+1} &= 0. \end{aligned}$$

Combining, we have Euler equation

$$U'(c_t) = \beta U'(c_{t+1})(1 + r_{t+1} - \delta).$$

From $F(K,N) = AK^\alpha L^{1-\alpha}$, and firm's FOC $F_K(K,1) = r_t$

$$\begin{aligned} U'(C(K^*,K^*)) &= \beta U'(C(K^*,K^*))(1 + F_K^*(K^*,1) - \delta) \\ 1 &= \beta(1 + A\alpha(K^*)^{\alpha-1} - \delta) \\ K^* &= \left\{ \frac{1-\beta(1-\delta)}{\alpha A} \right\}^{\frac{1}{\alpha-1}} \end{aligned}$$

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For any $\{k_{t+1}\}_{t=0}^\infty$, denote log difference as $\dot{x}_{t+1} = \log(x_{t+1}) - \log(x_t)$. Under BGP, for all t, $\dot{y}_t = \dot{k}_t = 3$

$$\begin{aligned} \dot{y}_t &= \alpha \dot{k}_t + (1-\alpha)(\dot{z}_t + \dot{l}_t) \\ 3 &= 0.4 * 0.3 + (1-0.4)(\dot{z}_t + \dot{l}_t) \\ \dot{z}_t &= 2. \end{aligned}$$

Thus growth rate of TFP along the balanced growth path is 2%.