Macroeconomics II Homework 2

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(a)

At the optimal consumption, the budget constraint holds with equality:

$$c_t = Ak_t^\alpha - k_{t+1}$$

Define

$$w(k_0) = \max_{\{k_{t+1}\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t \log c_t$$

Then, this is equivalent to

$$\begin{aligned}
w(k_0) &= \max_{\{k_{t+1}\}_{t=0}^\infty ; k_0 \text{ given}} \sum_{t=0}^\infty \beta^t \log c_t \\
&= \max_{k_1 ; k_0 \text{ given}} \log c_0 + \beta \left[\max_{\{k_{t+1}\}_{t=1}^\infty ; k_1 \text{ given}} \sum_{t=1}^\infty \beta^{t-1} \log c_t \right] \\
&= \max_{k_1 ; k_0 \text{ given}} \log(Ak_0^\alpha - k_1) + \beta w(k_1)
\end{aligned}$$

with conditions $0 \leq k_1 \leq Ak_0^\alpha$ for all $t = 0, \cdots, \infty$. This is the recursive problem of the social planner.

(b)

(i): Guess and Verify

Guess $V(k) = X + Y \log Ak^\alpha$ where X and Y are coefficients to be determined. Then, the Bellman equation is

$$V(k) = \max_{0 \leq k' \leq Ak^\alpha ; k \text{ given}} \log(Ak^\alpha - k') + \beta(X + Y \log Ak'^\alpha)$$

The first-order condition is

$$\frac{\partial V(k)}{\partial k'} = -\frac{1}{Ak^\alpha - k'} + \alpha\beta Y \frac{1}{k'} = 0$$
$$\therefore \quad k' = \frac{\alpha\beta Y}{1+\alpha\beta Y} A k^\alpha$$

Evaluating the value function at the optimal k', we have

$$\begin{aligned} V(k') &= \log\left(Ak^\alpha - \frac{\alpha\beta Y}{1+\alpha\beta Y}Ak^\alpha\right) + \beta(X + Y\log A\left(\frac{\alpha\beta Y}{1+\alpha\beta Y}Ak^\alpha\right)^\alpha) \\ &= \beta X + \log\left(\frac{1}{1+\alpha\beta Y}\right) + \alpha\beta Y \log\left(\frac{\alpha\beta Y}{1+\alpha\beta Y}\right) + \alpha(1+\beta Y)\log\left(Ak^\alpha\right) \end{aligned}$$

Solving this, we have

$$\begin{aligned} X &= \frac{1}{1-\beta}\left[\log\left(\frac{1}{1+\alpha\beta Y}\right) + \alpha\beta Y \log\left(\frac{\alpha\beta Y}{1+\alpha\beta Y}\right)\right] \\ Y &= \frac{1}{\alpha(1-\beta)} \end{aligned}$$

(ii): Value Function Iteration

(iii): Policy Function Iteration

Using the result in (i), we have the true policy function:

$$k' = \frac{\alpha\beta Y}{1+\alpha\beta Y}Ak^\alpha = \frac{\alpha\beta}{1-\beta+\alpha\beta}Ak^\alpha$$

Using this policy function, we can caluculate the value function.

$$\begin{aligned} V_{h_j}(k_0) &= \sum_{t=0}^{\infty} \beta^t \log\left(Ak_t^\alpha - \frac{\alpha\beta}{1-\beta+\alpha\beta}Ak^\alpha\right) \\ &= \frac{1}{1-\beta}\log\left(\frac{1-\beta}{1-\beta+\alpha\beta}A\right) + \sum_{t=0}^{\infty}\beta^t \log k_t^\alpha \end{aligned}$$

Since the second term is caluculated as

$$\begin{aligned} \sum_{t=0}^{\infty}\beta^t \log k_t^\alpha &= \sum_{t=o}^{\infty}\beta^t\alpha^{t+1}\log k_0 + \sum_{t=o}^{\infty}\beta^t(\alpha+\cdots+\alpha^t)\log\left(\frac{\alpha\beta}{1-\beta+\alpha\beta}A\right) \\ &= \frac{\alpha}{1-\alpha\beta}\log k_0 + \frac{\alpha\beta}{(1-\beta)(1-\alpha\beta)}\log\left(\frac{\alpha\beta}{1-\beta+\alpha\beta}A\right) \end{aligned}$$

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(a)

The agent's life-time budget constraint is

$$\sum_{t=0}^{\infty} p_t\left(c_t + \frac{b_{t+1}}{1+r_{t+1}} - b_t - e\right)$$

(b)

Define the Lagrangian as

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\log c_t + \lambda_t \left(p_t \left(c_t + \frac{b_{t+1}}{1 + r_{t+1}} - b_t - e \right) \right) \right]$$

The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial c_t} = U'(c_t) - \lambda_t p_t = 0$$
$$\frac{\partial \mathcal{L}}{\partial b_t + 1} = -\beta^t \frac{\lambda_t p_t}{1 + r_{t+1}} + \beta^{t+1} \lambda_{t+1} p_{t+1} = 0$$

Thus, the Euler equation is

$$U'(c_t) = \beta U'(c_{t+1})(1 + r_{t+1})$$

The transversality condition is

$$\lim_{t \to \infty} \beta^t (1 + r_{t+1}) U'(c_t) b_t = 0$$

with $c_t = b_t - \frac{b_{t+1}}{1+r_{t+1}} + e$.

(c)

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