Macroeconomics II Homework 5

Graduate School of Economics, The University of Tokyo

29–246029 Rin NITTA
29-246033 Rei HANARI
29–246004 Kosuke IGARASHI

January 19, 2025

1

2

3

(a)

Suppose the market structure is sequential markets. Then the competitive equilibrium is given as a sequential market equilibrium. Let $\bar{k}_1$ be the capital stock of the initial old. Suppose $g_0 = 0$.

Given $\bar{k}_1$, sequential markets equilibrium is allocations for households $\hat{c}_1^0, \{\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t\}_{t=0}^{\infty}$, allocations fot the firm $\{\hat{K}_t, \hat{L}_t\}_{t=0}^{\infty}$, and prices $\{\hat{r}_t, \hat{w}_t\}_{t=0}^{\infty}$ such that

1. For all $t \geq 0$, given $(\hat{w}_t, \hat{r}_{t+1})$, $(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)$ solve the household's optimization problem:

$$\begin{aligned} \max_{(c_t^t, c_{t+1}^t) \geq 0,\, s_t^t} \quad & \log c_t^t + \beta \log c_{t+1}^t \\ \text{s.t.} \quad & c_t^t + s_t^t + \tau_t \leq \hat{w}_t \\ & c_{t+1}^t \leq (1 + \hat{r}_{t+1} - \delta) s_t^t - b_{t+1} \end{aligned}$$

2. Given $\bar{k}_1$ and $\hat{r}_1$, $\hat{c}_1^0$ solves the household's optimization problem:

$$\begin{aligned} & \max_{c_1^0 \geq 0} \log c_1^0 \\ \text{s.t.} \quad & c_1^0 \leq (1 + \hat{r}_1 - \delta) \bar{k}_1 \end{aligned}$$

3. For all $t \geq 1$, given $(\hat{r}_t, \hat{w}_t)$, $(\hat{K}_t, \hat{L}_t)$ solves the firm's optimization problem:

$$\max_{(K_t, L_t) \geq 0} K_t^{\alpha} L_t^{1-\alpha} - \hat{r}_t K_t - \hat{w}_t L_t$$

4. For all $t \geq 1$,
   - Goods market: $N_t \hat{c}_t^t + N_{t-1} \hat{c}_t^{t-1} + \hat{K}_{t+1} - (1 - \delta) \hat{K}_t = \hat{K}_t^{\alpha} \hat{L}_t^{1-\alpha}$
   - Asset market: $N_t \hat{s}_t^t + N_t g_t = \hat{K}_{t+1}$
   - Labor market: $N_t = \hat{L}_t$

where $\tau_t = g_t$ and $b_t = -(1 + r_t - \delta) g_{t-1}$ for all $t \geq 1$.

(b)

Using $\tau_t = g_t$ and $b_t = -(1 + r_t - \delta)g_{t-1}$, the budget constraint of the household is rewritten as

$$
\begin{gathered}
c_t^t + s_t^t + g_t \leq \hat{w}_t \\
c_{t+1}^t \leq (1 + \hat{r}_{t+1} - \delta)(s_t^t + g_t)
\end{gathered}
$$

and the market clearing condition of the asset market is rewritten as

$$
N_t(s_t^t + g_t) = \hat{K}_{t+1}
$$

These are the same as the budget constraint and the market clearing condition in the model without the government, except that s_t^t is replaced by $s_t^t + g_t$. Then, if the household doesnt' care about the borrowing constraint, the competitive equilibrium is the same as the competitive equilibrium in the model without the government.

(c)

Given $\bar{k}_1$, sequential markets equilibrium is allocations for households $\hat{c}_1^0, \{\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t\}_{t=0}^\infty$, allocations fot the firm $\{\hat{K}_t, \hat{L}_t\}_{t=0}^\infty$, and prices $\{\hat{r}_t, \hat{w}_t\}_{t=0}^\infty$ such that

1. For all $t \geq 0$, given $(\hat{w}_t, \hat{r}_{t+1})$, $(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)$ solve the household's optimization problem:

$$
\begin{aligned}
\max_{(c_t^t, c_{t+1}^t) \geq 0,\ s_t^t} \quad & \log c_t^t + \beta \log c_{t+1}^t \\
\text{s.t.} \quad & c_t^t + s_t^t + \tau_t \leq \hat{w}_t \\
& c_{t+1}^t \leq (1 + \hat{r}_{t+1} - \delta)s_t^t + b_{t+1}
\end{aligned}
$$

2. Given $\bar{k}_1$ and $\hat{r}_1$, $\hat{c}_1^0$ solves the household's optimization problem:

$$
\begin{aligned}
&\max_{c_1^0 \geq 0} \log c_1^0 \\
\text{s.t.} \quad & c_1^0 \leq (1 + \hat{r}_1 - \delta)\bar{k}_1
\end{aligned}
$$

3. For all $t \geq 1$, given $(\hat{r}_t, \hat{w}_t)$, $(\hat{K}_t, \hat{L}_t)$ solves the firm's optimization problem:

$$
\max_{(K_t, L_t) \geq 0} K_t^\alpha L_t^{1-\alpha} - \hat{r}_t K_t - \hat{w}_t L_t
$$

4. For all $t \geq 1$,
   - Goods market: $N_t \hat{c}_t^t + N_{t-1}\hat{c}_t^{t-1} + \hat{K}_{t+1} - (1 - \delta)\hat{K}_t = \hat{K}_t^\alpha \hat{L}_t^{1-\alpha}$
   - Asset market: $N_t \hat{s}_t^t = \hat{K}_{t+1}$
   - Labor market: $N_t = \hat{L}_t$

where $(1 + n)\tau_t = (1 + n)\tau_y = b_t$ for all $t \geq 1$.

(d)

If τ_y rises, then $(1 + n)\tau_y = b_t$ rises. Then, the households receive more transfer from the government when they are old. For consumption smoothing, the households save less when they are young, that is, s_t^t decreases. Then, by the asset market clearing, $N_t s_t^t = K_{t+1}$ decreases. Therefore, he steady-state capital-labor ratio $k_t = \frac{K_t}{N_t}$ decreases.

4

(a)

Let $V(a_{i,0}, n_{i,0}) = E_0 \left[\max_{\{c_{i,t}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_{i,t})\right]$ be the value function of the household. The Bellman equation is given by

$$
\begin{aligned}
V(a_{i,0}, n_{i,0}) &= E_0 \left[\max_{\{c_{i,t}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_{i,t})\right] \\
&= \max_{c_{i,0}, a_{i,1}} u(c_{i,0}) + E_0 \left[\max_{\{c_{i,t}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^t u(c_{i,t})\right] \\
&= \max_{c_{i,0}, a_{i,1}} u(c_{i,0}) + \beta E_0 \left[E_1 \left[\max_{\{c_{i,t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_{i,t+1})\right]\right] \\
&= \max_{c_{i,0}, a_{i,1}} u(c_{i,0}) + \beta E_0 [V(a_{i,1}, n_{i,1})]
\end{aligned}
$$

Then, the recursive problem is given by

$$V(a, n) = \max_{c, a'} u(c) + \beta E[V(a', n')]$$

where a and n are state variables, and a' and c are control variables.

(b)

Let $m(a, n)$ be a measure of households in state (a, n).

A stationary competitive equilibrium is a policy function $a' = g(a, n)$, a distribution $m(a, n)$, and real numbers K, r and w such that

1. prices are determined competitively, that is, $r = F_K(K, N) - \delta = \alpha K_t^{\alpha-1} N_t^{1-\alpha} - \delta$ and $w = F_L(K, N) = (1-\alpha) K_t^{\alpha} N_t^{-\alpha}$.
2. The policy function $a' = g(a, n)$ solves the household's optimization problem.
3. The probability distribution $m(a, n)$ is stationary and associated with the policy function $g(a, n)$ and the distribution of n.
4. The capital K equals the sum of households' savings:

   $$K = \sum_a \sum_n g(a, n) \cdot m(a, n)$$

   The labor N equals the sum of labor supplied by each household (exogenous):

   $$L = \sum_a \sum_n n \cdot m(a, n)$$

(c)

The grid Z, the transition matrix P, and the stationary distribution π gained by Tauchen's method are as follows:

$$Z = \begin{pmatrix} -0.8000 \\ -0.5333 \\ -0.26670 \\ 0.2667 \\ 0.5333 \\ 0.8000 \end{pmatrix}, \quad P = \begin{pmatrix} 0.5530 & 0.2351 & 0.1406 & 0.0548 & 0.0139 & 0.0023 & 0.0003 \\ 0.3204 & 0.2589 & 0.2277 & 0.1305 & 0.0487 & 0.0118 & 0.0021 \\ 0.1431 & 0.2015 & 0.2606 & 0.2195 & 0.1205 & 0.0431 & 0.0117 \\ 0.0478 & 0.1109 & 0.2108 & 0.2611 & 0.2108 & 0.1109 & 0.0478 \\ 0.0117 & 0.0431 & 0.1205 & 0.2195 & 0.2606 & 0.2015 & 0.1431 \\ 0.0021 & 0.0118 & 0.0487 & 0.1305 & 0.2277 & 0.2589 & 0.3204 \\ 0.0003 & 0.0023 & 0.0139 & 0.0548 & 0.1406 & 0.2351 & 0.5530 \end{pmatrix}, \quad \pi = \begin{pmatrix} 0.1553 \\ 0.1226 \\ 0.1456 \\ 0.1529 \\ 0.1456 \\ 0.1226 \\ 0.1553 \end{pmatrix}$$

The Matlab code is below.

Code: tauchen.m

```
function [Z,Zprob] = tauchen(N,mu,rho,sigma)

  Zprob = zeros(N,N); % Transition Matrix
  c = (1-rho)*mu; % Constant

  % Define Grids
  zmax  = 2*sigma;
  zmin  = -zmax;
  w = (zmax-zmin)/(N-1);
  Z = linspace(zmin,zmax,N)';

  % Stationary value, mu
  Z = Z + mu;

  % Create Transition Matrix
  for j = 1:N
      for k = 1:N
          if k == 1
              Zprob(j,k) = normcdf((Z(1)-c-rho*Z(j)+w/2)/sigma);
          elseif k == N
              Zprob(j,k) = 1 - normcdf((Z(N)-c-rho*Z(j)-w/2)/sigma);
          else
              Zprob(j,k) = normcdf((Z(k)-c-rho*Z(j)+w/2)/sigma) - normcdf((Z(k)-c-rho*Z(j)-w
                  /2)/sigma);
          end
      end
  end
end
```

Code: tauchen_result.m

```
N = 7; % Number of Grid Points, the number of potential realizations of z.
mu = 0; % Mean
rho = 0.9; % AR(1) Coefficient
sigma = 0.4; % Standard Deviation

[Z,Zprob] = tauchen(N,mu,rho,sigma);

disp('Grid (Z):');
disp(Z);

disp('Transition matrix (P):');
disp(Zprob);

[V, D] = eig(Zprob');
[~, idx] = max(abs(diag(D)));
pi = V(:, idx);
pi = pi / sum(pi);
```

```

disp('stationary distribution:');
disp(pi);
```