Macroeconomics II Homework 3

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Q1

(a)

The recursive formulation of a standard neoclassical growth model studied in class in Lecture 3 is

$$v(k) = \max_{0 \leq k' \leq f(k)} \{U(f(k) - k') + \beta v(k')\}.$$

Define metricx space $(B(X), d)$, the space of bounded functions on $X = [0, \infty)$ with the sup-norm d, and difine operator T as

$$Tv(k) = \max_{0 \leq k' \leq f(k)} \{U(f(k) - k') + \beta v(k')\}.$$

Then, by showing T satisfies Blackwerll's condition, we can show T is a contraction mapping. Then, by CMT, we can show convergent k is an unique fixed point.

(b)

Let X be a set. Consider the space $B(X)$ of all bounded functions $f : X \to \mathbb{R}$, equipped with the *supremum norm*:

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

The metric on $B(X)$ is defined as:

$$d(f, g) = \|f - g\|_\infty.$$

We aim to show that the metric space $(B(X), d)$ is complete, i.e., every Cauchy sequence of functions in $B(X)$ converges to a function in $B(X)$.

Let $\{f_n\}$ be a Cauchy sequence in $(B(X), d)$. By definition, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$:

$$\|f_n - f_m\|_\infty < \epsilon,$$

which implies:

$$\sup_{x \in X} |f_n(x) - f_m(x)| < \epsilon.$$

For each $x \in X$, the sequence $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$ because:

$$|f_n(x) - f_m(x)| < \epsilon \quad \text{for all } m, n \geq N.$$

Since $\mathbb{R}$ is complete, $\{f_n(x)\}$ converges to a limit, say $f(x) \in \mathbb{R}$. Thus, we can define a pointwise limit function $f : X \to \mathbb{R}$ by:

$$f(x) = \lim_{n\to\infty} f_n(x), \quad \text{for each } x \in X.$$

Since $\{f_n\} \subseteq B(X)$, each f_n is bounded, i.e., there exists M_n such that $\|f_n\|_\infty \leq M_n$. Let $M = \sup_n M_n$. Then for all n and $x \in X$:

$$|f_n(x)| \leq M.$$

Taking the limit as $n \to \infty$, we obtain:

$$|f(x)| \leq M, \quad \text{for all } x \in X.$$

Thus, f is bounded, and hence $f \in B(X)$.

For $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$:

$$\|f_n - f_m\|_\infty < \epsilon.$$

Fix $n \geq N$. Then for all $x \in X$:

$$|f_n(x) - f_m(x)| < \epsilon \quad \text{for all } m \geq N.$$

Taking the limit as $m \to \infty$, we get:

$$|f_n(x) - f(x)| \leq \epsilon.$$

Thus:

$$\|f_n - f\|_\infty \leq \epsilon \quad \text{for all } n \geq N.$$

This shows that $\{f_n\}$ converges uniformly to f.

Since $\{f_n\}$ is a Cauchy sequence in $(B(X), d)$ and converges uniformly to $f \in B(X)$, the metric space $(B(X), d)$ is complete.

(c)

Suppose not, there exists feasible allocation $\{\tilde{c}^1_t,\ \tilde{c}^2_t\}^\infty_{t=0,\ s^t\in S^t}$ such that

$$u(\hat{c}^i) \leq u(\tilde{c}^i) \text{ for all } i \in \{1,2\}$$
$$u(\hat{c}^i) < u(\tilde{c}^i) \text{ for some } i \in \{1,2\}$$

Without loss of generality, assume strict inequality holds for $i = 1$.

Suppose

$$\sum_{t=0}^{\infty}\sum_{s^t\in S^t} P_t(s^t)\hat{c}^1_t(s^t) \geq \sum_{t=0}^{\infty}\sum_{s^t\in S^t} P_t(s^t)\tilde{c}^1_t(s^t).$$

Then as $\hat{c}^1$ is CE, $u(\hat{c}^1) \geq u(\tilde{c}^1)$. Therefore,

$$\sum_{t=0}^{\infty}\sum_{s^t\in S^t} P_t(s^t)\hat{c}^1_t(s^t) < \sum_{t=0}^{\infty}\sum_{s^t\in S^t} P_t(s^t)\tilde{c}^1_t(s^t).$$

Suppose

$$\sum_{t=0}^{\infty}\sum_{s^t\in S^t} P_t(s^t)\hat{c}^2_t(s^t) > \sum_{t=0}^{\infty}\sum_{s^t\in S^t} P_t(s^t)\tilde{c}^2_t(s^t).$$

Then there exists $\delta > 0$ such that

$$\sum_{t=0}^{\infty}\sum_{s^t\in S^t} P_t(s^t)\hat{c}^2_t(s^t) \geq \sum_{t=0}^{\infty}\sum_{s^t\in S^t} P_t(s^t)\tilde{c}^2_t(s^t) + \delta.$$

Define $\bar{c}^2$ as

$$\begin{aligned} \bar{c}_t^2(s^t) &= \tilde{c}_t^2 && \text{for } t \neq 0 \\ \bar{c}_0^2(s^0) &= \tilde{c}_0^2 + \pi(s_0)\delta && \text{for } t = 0 \end{aligned}$$

Then

$$u(\bar{c}^2) \geq u(\tilde{c}^2) \geq u(\hat{c}^2).$$

This contradicts that $\hat{c}^2$ is CE. Hence,

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} P_t(s^t)\hat{c}_t^2(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t \in S^t} P_t(s^t)\tilde{c}_t^2(s^t).$$

Then

$$\sum_{i \in \{1,2\}} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} P_t(s^t)\hat{c}_t^i(s^t) < \sum_{i \in \{1,2\}} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} P_t(s^t)\tilde{c}_t^i(s^t).$$

As $(\hat{c}^1, \hat{c}^2)$ and $(\tilde{c}^1, \tilde{c}^2)$ are feasible,

$$\forall t \ \forall s^t \in S^t \quad \hat{c}_t^1(s^t) + \hat{c}_t^2(s^t) = \tilde{c}_t^1(s^t) + \tilde{c}_t^2(s^t) \tag{1}$$

Hence

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} P_t(s^t) < \sum_{t=0}^{\infty} \sum_{s^t \in S^t} P_t(s^t)$$

This is a contradiction. This shows $(\hat{c}^1, \hat{c}^2)$ is a Pareto efficient allocation.

Q2

(a)

(b)

(c)

Q3

(a)

$$P^2 = \begin{bmatrix} 0.83 & 0.15 & 0.01 & 0.01 \\ 0.31 & 0.41 & 0.15 & 0.13 \\ 0.05 & 0.27 & 0.53 & 0.15 \\ 0.17 & 0.17 & 0.27 & 0.39 \end{bmatrix}.$$

For all $i, j \in \{1, 2, 3, 4, \}$, $P_{ij}^2 > 0$. Therefore, by the LS theorem 2.2.2, P has a unique stationary distribution and the process is asymptotically stationary.

(b)

Stationary distribution $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ of P satisfies

$$\pi^T = \pi^T P$$
$$\begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix} = \begin{bmatrix} 0.9\pi_1 + 0.2\pi_2 + 0.0\pi_3 + 0.1\pi_4 \\ 0.1\pi_1 + 0.6\pi_2 + 0.2\pi_3 + 0.1\pi_4 \\ 0.0\pi_1 + 0.1\pi_2 + 0.7\pi_3 + 0.2\pi_4 \\ 0.0\pi_1 + 0.1\pi_2 + 0.1\pi_3 + 0.6\pi_4 \end{bmatrix}$$

Solving this, we have

$$\pi = (\frac{6}{11}, \frac{5}{22}, \frac{3}{22}, \frac{1}{11})$$

(c)

(i)

$$Pr(e^1_{t+1}|e^1_t) = 0.9$$
$$Pr(e^1_{t+1}|e^2_t) = 0.2$$
$$Pr(e^1_{t+1}|e^3_t) = 0.0$$
$$Pr(e^1_{t+1}|e^4_t) = 0.1$$
$$Pr(e^2_{t+1}|e^1_t) = 0.1$$
$$Pr(e^3_{t+1}|e^2_t) = 0.1$$
$$Pr(e^4_{t+1}|e^3_t) = 0.1$$

Then, the likelihood is

$$\left(\frac{6}{11} * 0.9 + \frac{5}{22} * 0.2 + \frac{3}{22} * 0 + \frac{1}{11} * 0.1\right) * 0.1^3 = \frac{6}{11000}$$

(ii)

$$Pr(e^1_{t+1}|e^1_t) = 0.9$$
$$Pr(e^1_{t+1}|e^2_t) = 0.2$$
$$Pr(e^1_{t+1}|e^3_t) = 0.0$$
$$Pr(e^1_{t+1}|e^4_t) = 0.1$$

Then, the likelihood is

$$\left(\frac{6}{11} * 0.9 + \frac{5}{22} * 0.2 + \frac{3}{22} * 0 + \frac{1}{11} * 0.1\right) * 0.9^3 = \frac{4374}{11000}$$

(d)

$$E[y_1|e_0 = e^1] = 0.9 * 0 + 0.1 * 1 = 0.1$$

$$P^5 = \begin{bmatrix} 0.70524 & 0.19908 & 0.05284 & 0.04284 \\ 0.44100 & 0.25652 & 0.17908 & 0.12340 \\ 0.23420 & 0.27964 & 0.31708 & 0.16908 \\ 0.31476 & 0.24476 & 0.25964 & 0.18084 \end{bmatrix}.$$

$$E[y_5|e_0 = e^1] = 0.70524 * 0 + 0.19908 * 1 + 0.05284 * 2 + 0.04284 * 4 = 0.47612.$$