HW2

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1 Exercise 22.1

The model is

$$Y = X'\theta + e,$$

where e is independent of X and density function $f(e)$ is known and continuously differentiable .

(a)

Because $Y = X'\theta + e$, so $e = Y - X'\theta$.
e is independent of X and has density $f(e)$, the conditional density of Y given $X = x$ is:

$$f_{Y|X}(y|x) = f(y - x'\theta).$$

(b)

The log-likelihood for a single observation is $\ell(\theta) = \log f(y - x'\theta)$.
Let me define the M-estimator objective function as $\rho(Y, X, \theta) = -\log f(Y - X'\theta)$. The score function is:

$$\psi(Y, X, \theta) = \frac{\partial}{\partial\theta} \log f(Y - X'\theta).$$

By the chain rule, we have:

$$\psi(Y, X, \theta) = -\frac{f'(Y - X'\theta)}{f(Y - X'\theta)} \cdot (-X) = -X \cdot \frac{f'(Y - X'\theta)}{f(Y - X'\theta)}.$$

(c)

We assume some regularity conditions for the m-estimator.

The asymptotic covariance matrix of the M-estimator is

$$\Sigma = H^{-1} S H^{-1}$$

$$\text{where } H = E\left[-\frac{\partial}{\partial\theta'}\psi(Y, X, \theta_0)\right], \ S = E\left[\psi(Y, X, \theta_0)\psi(Y, X, \theta_0)'\right].$$

To compute H, we differentiate $\psi(Y, X, \theta) = -X \cdot \frac{f'(Y - X'\theta)}{f(Y - X'\theta)}$, with respect to θ',

$$-\frac{\partial}{\partial\theta'}\psi(Y, X, \theta) = -XX' \cdot \left[\frac{f''(Y - X'\theta) f(Y - X'\theta) - (f'(Y - X'\theta))^2}{f(Y - X'\theta)^2}\right]$$

We evaluate this at the true parameter θ_0 and take the expectation to compute H. Thus, we have

$$H = E\left[-XX' \cdot \left[\frac{f''(Y - X'\theta_0)f(Y - X'\theta_0) - (f'(Y - X'\theta_0))^2}{f(Y - X'\theta_0)^2}\right]\right].$$

Then,

$$S = E\left[XX' \cdot \left(\frac{f'(Y - X'\theta_0)}{f(Y - X'\theta_0)}\right)^2\right].$$

Finally, the asymptotic covariance matrix is:

$$\Sigma = H^{-1}SH^{-1},$$

which can be explicitly evaluated depending on the known density f and the distribution of X.

2 Exercise 22.3

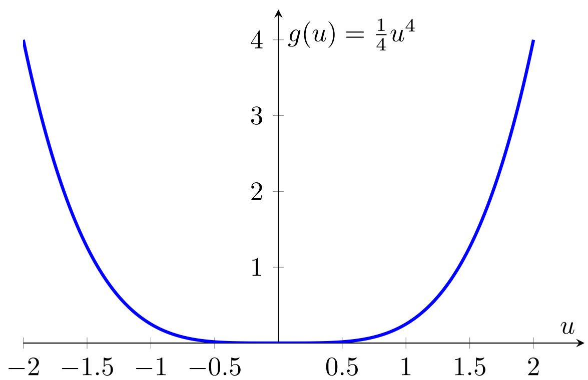


Figure 1: Graph of $g(u) = \frac{1}{4}u^4$ on $[-2, 2]$

(a) **(i) Continuity.** Fix $u_0 \in \mathbb{R}$ and $\varepsilon > 0$. Choose

$$\delta = \min\left\{1, \frac{\varepsilon}{(1 + |u_0|)^3}\right\}.$$

Whenever $|u - u_0| < \delta$, write $u = u_0 + h$ with $|h| < \delta \leq 1$. A straightforward expansion gives

$$|g(u) - g(u_0)| = \frac{1}{4}|(u_0 + h)^4 - u_0^4| = \frac{1}{4}|4u_0^3h + 6u_0^2h^2 + 4u_0h^3 + h^4|.$$

Using $|h| < 1$ and $|h| < \delta$,

$$\begin{aligned}|g(u) - g(u_0)| &\leq |u_0|^3|h| + \tfrac{3}{2}|u_0|^2|h|^2 + |u_0||h|^3 + \tfrac{1}{4}|h|^4 \\ &< (|u_0|^3 + \tfrac{3}{2}|u_0|^2 + |u_0| + \tfrac{1}{4})|h| \\ &< (1 + |u_0|)^3|h| < \varepsilon.\end{aligned}$$

Hence g is continuous at u_0; since u_0 was arbitrary, g is continuous on $\mathbb{R}$.

(ii) Differentiability. For $u_0 \in \mathbb{R}$, consider the difference quotient

$$\frac{g(u_0 + h) - g(u_0)}{h} = \frac{(u_0 + h)^4 - u_0^4}{4h} = u_0^3 + \frac{3}{2}u_0^2h + u_0h^2 + \frac{1}{4}h^3.$$

Taking the limit $h \to 0$,

$$\lim_{h\to 0} \frac{g(u_0+h)-g(u_0)}{h} = u_0^3,$$

so the derivative exists and equals u_0^3.

(iii) Continuity of the derivative. For $u_0 \in \mathbb{R}$, consider the difference quotient

$$\frac{g'(u_0+h)-g'(u_0)}{h} = \frac{(u_0+h)^3-u_0^3}{h} = 3u_0^2 + 3u_0 h + h^2.$$

Taking the limit $h \to 0$,

$$\lim_{h\to 0} \frac{g'(u_0+h)-g'(u_0)}{h} = 3u_0^2,$$

so the second derivative exists and equals $3u_0^2$.

(b)

$$\rho(Y,X,\theta) = \tfrac{1}{4}\big(Y - X'\theta\big)^4,\ \psi(Y,X,\theta) = \frac{\partial \rho(Y,X,\theta)}{\partial \theta} = -X\big(Y-X'\theta\big)^3.$$

(c) The asymptotic covariance matrix of the m-estimator is

$$\Sigma = H^{-1}S[H^{-1}]$$

$$\text{where } H = E\left[-\frac{\partial}{\partial\theta'}\psi(Y,X,\theta_0)\right],\ S = E\left[\psi(Y,X,\theta_0)\psi(Y,X,\theta_0)'\right]$$

To compute H, we differentiate $\psi(Y,X,\theta)$ with respect to θ'

$$-\frac{\partial}{\partial\theta'}\psi(Y,X,\theta) = -3XX'(Y-X'\theta)^2$$

We evaluate this at the true parameter θ_0 and take the expectation to compute H.
we have S from (b), $S = E\left[XX'(Y-X'\theta_0)^6\right]$.
Finally, the asymptotic covariance matrix is

$$\frac{1}{9}\big(E[XX'(Y-X'\theta_0)^2]\big)^{-1}\ E\left[XX'(Y-X'\theta_0)^6\right]\ \big(E[XX'(Y-X'\theta_0)^2]\big)^{-1}.$$

3 Exercise 23.8

Attached at the end of this document.

4 Exercise 24.6

$$\begin{aligned}
\mathbb{E}[\psi_\tau(e)|X] &= \mathbb{E}[\tau - \mathbb{I}(e<0)|X] \\
&= \tau - \mathbb{P}[Y < q_\tau(X)|X] \\
&= \tau - \mathbb{P}[Y \le q_\tau(X)|X] \quad (\because \mathbb{P}[Y = q_\tau(X)] = 0) \\
&= \tau - \tau \quad (\because \text{Definition of conditional quantile}) \\
&= 0
\end{aligned}$$

5 Exercise 24.13

Attached at the end of this document.

6 Exercise 13.4

Assume W is a positive definite symmetric matrix.

(a)

$$\begin{aligned} V_0 &= (Q'\Omega^{-1}Q)^{-1}Q'\Omega^{-1}\Omega\Omega^{-1}Q(Q'\Omega^{-1}Q)^{-1} \\ &= (Q'\Omega^{-1}Q)^{-1}Q'\Omega^{-1}Q(Q'\Omega^{-1}Q)^{-1} \\ &= (Q'\Omega^{-1}Q)^{-1}. \end{aligned}$$

(b) $A = WQ(Q'WQ)^{-1}$ and $B = \Omega^{-1}Q(Q'\Omega^{-1}Q)^{-1}$ yield $V = A'\Omega A, V_0 = B'\Omega B$.

(c)

$$\begin{aligned} B'\Omega A &= (Q'\Omega^{-1}Q)^{-1}Q'\Omega^{-1}WQ(Q'WQ)^{-1} \\ &= (Q'\Omega^{-1}Q)^{-1} \\ &= B'\Omega B. \end{aligned}$$

Therefore, $B'\Omega(A - B) = 0$.

(d) We have

$$\begin{aligned} V &= A'\Omega A \\ &= (B + A - B)'\Omega(B + A - B) \\ &= B'\Omega B + (A - B)'\Omega(A - B) \\ &= V_0 + (A - B)'\Omega(A - B) \\ &\geq V_0 \end{aligned}$$

23.8

```r
library(haven)
data <- read_dta("PSS2017.dta")
```

```r
Y <- log(data$EG_total)
X1 <- data$EC_c_alt
X2 <- data$EC_d_alt

ces_model <- nls(
Y~ beta + (nu / rho) * log(alpha * X1^rho + (1- alpha) * X2^rho),
start = list(rho = 0.36, nu = 1.05, alpha = 0.39, beta = 1.66),
data = data
)
```

```
## Warning in min(x): no non-missing arguments to min; returning Inf

## Warning in max(x): no non-missing arguments to max; returning -Inf
```

```r
summary(ces_model)
```

```
##
## Formula: Y ~ beta + (nu/rho) * log(alpha * X1^rho + (1 - alpha) * X2^rho)
##
## Parameters:
##         Estimate Std. Error t value Pr(>|t|)
## rho     0.411489   0.058496   7.035 1.14e-11 ***
## nu      1.042580   0.007826 133.220  < 2e-16 ***
## alpha   0.319427   0.011581  27.581  < 2e-16 ***
## beta  -12.582163   0.184574 -68.169  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.1951 on 334 degrees of freedom
##
## Number of iterations to convergence: 5
## Achieved convergence tolerance: 8.564e-06
##   (52 observations deleted due to missingness)
```

Interpretation

Comparing the estimates with Table 23.1, we find that rho, alpha, and nu show similar values. Therefore, apart from beta, the estimates can be considered comparable.

24.13

```r
library(pacman)
p_load(haven, quantreg)

data <- read_dta("cps09mar.dta")
data <- subset(data, education >= 11, female = 0)
```

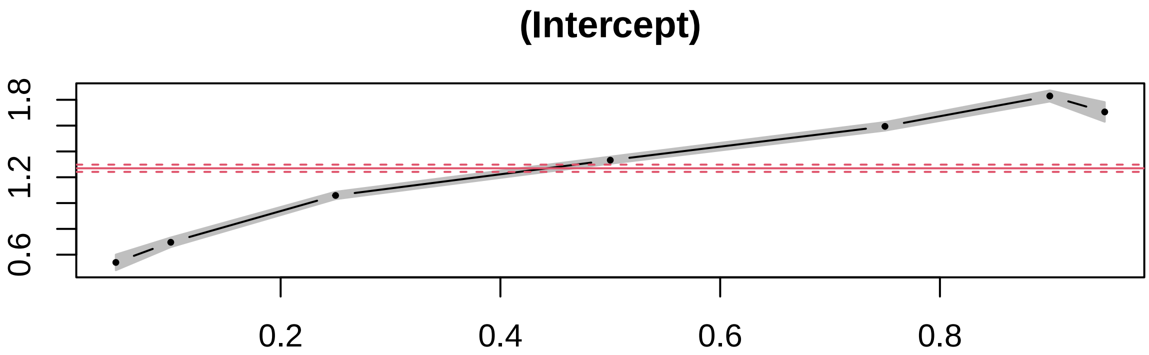
```
## Warning: In subset.data.frame(data, education >= 11, female = 0) :
##  extra argument 'female' will be disregarded
```

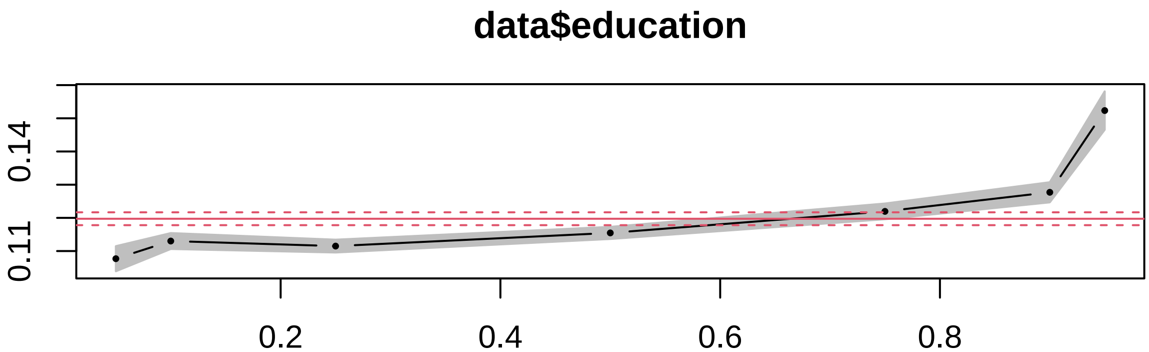
```r
data$wage <- data$earnings / (data$hours * data$week)
data$lwage <- log(data$wage)

taus <- c(.05, .1, .25, .5, .75, .9, .95)
res <- rq(data$lwage~data$education, tau=taus)
summary(res)
```

```
##
## Call: rq(formula = data$lwage ~ data$education, tau = taus)
##
## tau: [1] 0.05
##
## Coefficients:
##                 Value    Std. Error t value  Pr(>|t|)
## (Intercept)      0.54023  0.03645   14.82086  0.00000
## data$education  0.10770  0.00230   46.86972  0.00000
##
## Call: rq(formula = data$lwage ~ data$education, tau = taus)
##
## tau: [1] 0.1
##
## Coefficients:
##                 Value    Std. Error t value  Pr(>|t|)
## (Intercept)      0.69615  0.02224   31.30338  0.00000
## data$education  0.11302  0.00146   77.44038  0.00000
##
## Call: rq(formula = data$lwage ~ data$education, tau = taus)
##
## tau: [1] 0.25
##
## Coefficients:
##                 Value    Std. Error t value  Pr(>|t|)
## (Intercept)      1.05896  0.01715   61.74269  0.00000
## data$education  0.11149  0.00117   94.99647  0.00000
##
## Call: rq(formula = data$lwage ~ data$education, tau = taus)
##
## tau: [1] 0.5
```

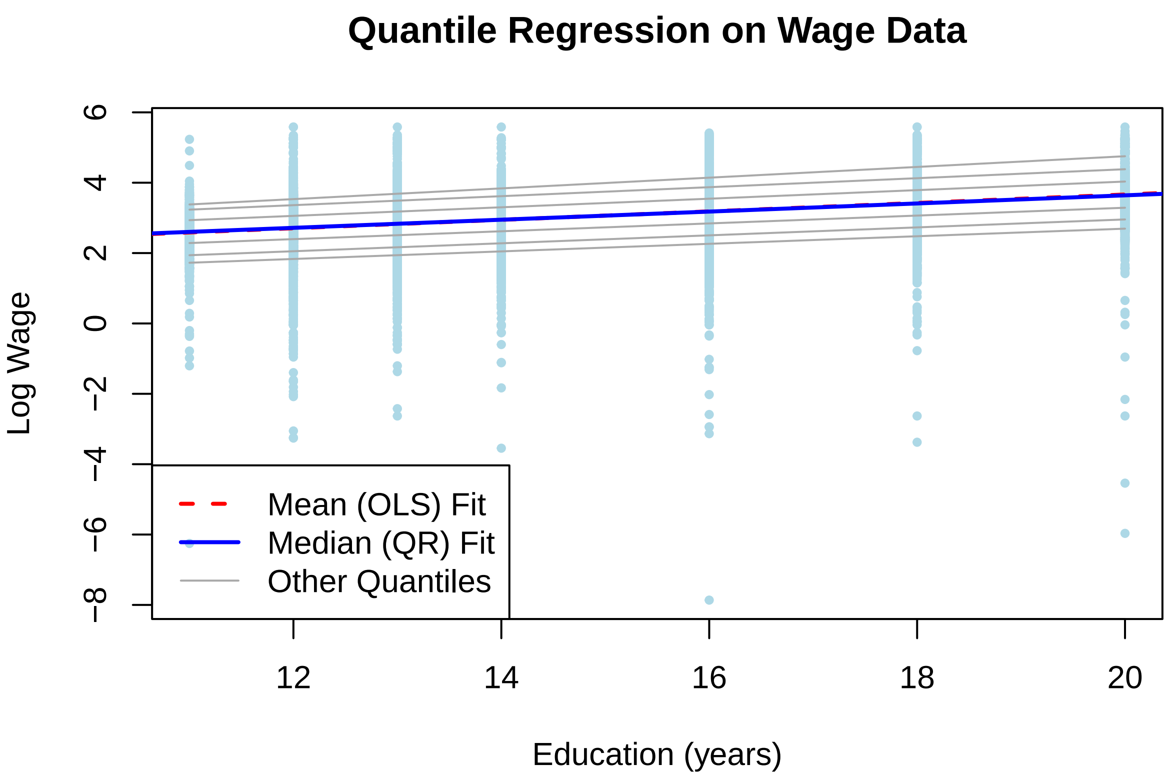
```
##
## Coefficients:
##                  Value     Std. Error t value   Pr(>|t|)
## (Intercept)        1.33225   0.01634   81.51439   0.00000
## data$education   0.11546   0.00111  104.10786   0.00000
##
## Call: rq(formula = data$lwage ~ data$education, tau = taus)
##
## tau: [1] 0.75
##
## Coefficients:
##                  Value    Std. Error t value  Pr(>|t|)
## (Intercept)      1.59426  0.01983   80.38574  0.00000
## data$education  0.12195  0.00144   84.69803  0.00000
##
## Call: rq(formula = data$lwage ~ data$education, tau = taus)
##
## tau: [1] 0.9
##
## Coefficients:
##                  Value    Std. Error t value  Pr(>|t|)
## (Intercept)      1.82950  0.02578   70.97848  0.00000
## data$education  0.12771  0.00184   69.28891  0.00000
##
## Call: rq(formula = data$lwage ~ data$education, tau = taus)
##
## tau: [1] 0.95
##
## Coefficients:
##                  Value    Std. Error t value  Pr(>|t|)
## (Intercept)      1.70607  0.04588   37.18189  0.00000
## data$education  0.15234  0.00352   43.32065  0.00000
```

```r
plot(summary(rq(data$lwage~data$education, tau=taus)))
```






```
plot(data$education, data$lwage, xlab = "Education (years)", ylab = "Log Wage",
main = "Quantile Regression on Wage Data", col = "lightblue", pch = 19, cex = 0.5)

xx <- seq(min(data$education), max(data$education), length.out = 100)
q_fits <- rq(lwage~ education, tau = taus, data = data)
q_coefs <- coef(q_fits)
yy <- cbind(1, xx) %*% q_coefs
for (i in 1:length(taus)) {
lines(xx, yy[, i], col = "darkgray")
}
ols_fit <- lm(lwage~ education, data = data)
abline(ols_fit, col = "red", lty = 2, lwd = 2)
abline(rq(lwage~ education, tau = .5, data = data), col = "blue", lwd = 2)
legend("bottomleft",
legend = c("Mean (OLS) Fit", "Median (QR) Fit", "Other Quantiles"),
col = c("red", "blue", "darkgray"),
lty = c(2, 1, 1),
lwd = c(2, 2, 1))
```


Quantile Regression on Wage Data
Log Wage
Education (years)
6
4
2
0
−2
−4
−6
−8
12
14
16
18
20
Mean (OLS) Fit
Median (QR) Fit
Other Quantiles