N players. (tx cost = c) CEX price P.

(X, Y, γ)

$$P_c = \frac{Y + 2\sum y_i}{X + 2\sum x_i}, \quad P_{-i} = \frac{Y + 2\sum_{j \neq i} y_j}{X + 2\sum_{j \neq i} x_j}$$

Arbitrageur's Profit

$$U_i = -(1+\gamma)(P x_i + y_i) + (1-\gamma)\left(P \cdot \frac{y_i}{P_c} + P_c x_i\right)$$

Lemma)

Optimal Response is either $(x, 0)$ or $(0, y)$

$$U_i\Big|_{x_i = 0} = -(1+\gamma) y_i + (1-\gamma) \cdot \frac{P}{P_c} y_i$$

$$= y_i \left[(1-\gamma) \cdot \frac{X_{-i}}{Y_{-i} + 2y_i} \cdot P - (1+\gamma) \right]$$

By the first order condition,

$$\frac{\partial}{\partial y_i}\left[y_i \left((1-\gamma) \cdot \frac{X_{-i}}{Y_{-i}+2y_i} \cdot P - (1+\gamma) \right) \right]$$

$$= (1-\gamma) \cdot \frac{X_{-i}}{Y_{-i}+2y_i} \cdot P - (1+\gamma)$$

$$+ y_i \cdot \frac{X_{-i}}{(Y_{-i}+2y_i)^2} \cdot (-1) \cdot 2 \cdot (1-\gamma) \cdot P$$

$$= (1-\gamma) P \cdot X_{-i} \cdot \frac{Y_{-i}}{(Y_{-i}+2y_i)^2} - (1+\gamma)$$

First order condition implies:

$$\frac{(1-\gamma)}{(1+\gamma)} \cdot P \cdot \frac{Y_{-i}}{X_{-i}} = \left(\frac{Y_{-i}+2y_i}{X_{-i}} \right)^2,$$

i.e.,

$$\sqrt{\frac{1-\gamma}{1+\gamma} \cdot P \cdot P_{-i}} = P_c .$$

U_i should be positive in each case.

Similarly,

$$U_i\Big|_{y_i=0}$$

$$= (1-\gamma)\cdot\frac{Y_{-i}}{X_{-i}+2x_i}\cdot x_i - (1+\gamma)\cdot P\cdot x_i$$

$$\frac{\partial}{\partial x_i}\left(U_i\Big|_{y_i=0}\right)$$

$$= -(1+\gamma)P + (1-\gamma)\cdot\frac{Y_{-i}}{X_{-i}+2x_i}$$

$$+ (1-\gamma)\cdot x_i\cdot\frac{Y_{-i}}{(\quad)^2}\times(-1)\times 2$$

$$= -(1+\gamma)P + (1-\gamma)Y_{-i}\cdot\frac{X_{-i}}{(X_{-i}+2x_i)^2} = 0$$

$$\Rightarrow \frac{(1+\gamma)}{(1-\gamma)}\cdot P\cdot\frac{Y_{-i}}{X_{-i}} = \frac{Y_{-i}^2}{(X_{-i}+2x_{-i})^2}$$

$\therefore$ Best response results in:

$$P_c = \sqrt{\frac{1+\gamma}{1-\gamma}\cdot P\cdot P_{-i}}$$

In short, The best response will be

$$B_i(P, P_{-i}, \gamma) = \begin{cases} P_c = \sqrt{\frac{1-\gamma}{1+\gamma} P \cdot P_{-i}}, & P \geq u \\ \text{None}, & P \in (l, u) \\ P_c = \sqrt{\frac{1+\gamma}{1-\gamma} P \cdot P_{-i}}, & P \leq l \end{cases}$$

where $l = \frac{1-\gamma}{1+\gamma} P_{-i}$, $u = \frac{1+\gamma}{1-\gamma} P_{-i}$.

Since this holds for every $i \in [N]$,

$$P_c = \sqrt{P P_{-1}}$$

$$= \sqrt{P P_{-2}}$$

$$= \cdots$$

$$= \sqrt{P P_{-N}}$$

$\Rightarrow$ The equilibrium is symmetric.

$$y = \frac{1}{4N^2}\Bigg[(N-1)\cdot\frac{1-r}{1+r}\cdot PX - 2NY$$

$$+\frac{1-r}{1+r}PX\sqrt{(N-1)^2 + 4N\cdot\frac{Y}{X}\cdot\frac{1+r}{1-r}\cdot\frac{1}{P}}\,\Bigg]$$

let $P_0 := \frac{Y}{X}$, and $\frac{1-r}{1+r}\cdot P\cdot\frac{1}{P_0} =: 1+\varepsilon$.

since

$$\frac{\partial}{\partial\varepsilon}\left(\sqrt{(N-1)^2 + 4N\cdot\frac{1}{1+\varepsilon}}\,\right)\Bigg|_{\varepsilon=0}$$

$$= \frac{1}{(\quad)^{\frac{1}{2}}}\cdot\left(\frac{1}{2}\right)\cdot 4N\cdot\frac{-1}{(1+\varepsilon)^2}\Bigg|_{\varepsilon=0}$$

$$= \frac{-1}{N+1}\cdot\frac{1}{2}\cdot 4N = -\frac{2N}{N+1},$$

$$y \approx \frac{Y}{4N^2}\left[(N-1)(1+\varepsilon) - 2N + (1+\varepsilon)\left(N+1-\frac{2N}{N+1}\varepsilon\right)\right]$$

$$= \frac{Y}{4N^2}\Bigg[N-1-2N+N+1$$

$$+\varepsilon\left(N-1+N+1-\frac{2N}{N+1}\right) + O(\varepsilon^2)\Bigg]$$

$$= \frac{\Upsilon}{4N^2} \cdot \left[\frac{2N^2}{N+1} \varepsilon + o(\varepsilon^2) \right]$$

$$= \frac{\Upsilon}{2(N+1)} \varepsilon + o(\varepsilon^2).$$

ARB

$$= -(1+\gamma)\, y_{eq} + (1-\gamma) \cdot P \cdot \frac{y_{eq}}{P_c}$$

$$= y_{eq}(1+\gamma) \left[\frac{1-\gamma}{1+\gamma} \cdot \frac{P}{P_c} - 1 \right]$$

$$= y_{eq}(1+\gamma) \left[\frac{P_c}{P_{-i}} - 1 \right]$$

$$= y_{eq}(1+\gamma) \cdot \frac{2 y_{eq}}{\Upsilon + 2(N-1)\, y_{eq}}$$

$$= y_{eq}(1+\gamma) \cdot 2 y_{eq} \cdot \frac{1-\gamma}{1+\gamma} \cdot P \cdot \frac{1}{X} \cdot \frac{1}{P_c^2}$$

$$= (1-\gamma) \cdot P \cdot 2y^2 \cdot \frac{1}{X} \cdot \frac{X^2}{(\Upsilon + 2Ny)^2}$$

$$= (1+\gamma) \cdot \frac{1-\gamma}{1+\gamma} P \cdot 2y^2 \cdot \frac{X}{(\Upsilon + 2Ny)^2}$$

$$(1+r)\cdot\frac{1-r}{1+r}P\cdot 2y^2\cdot X\cdot\frac{1}{(Y+2Ny)^2}$$

$$y=\frac{Y}{2(N+1)}\varepsilon$$

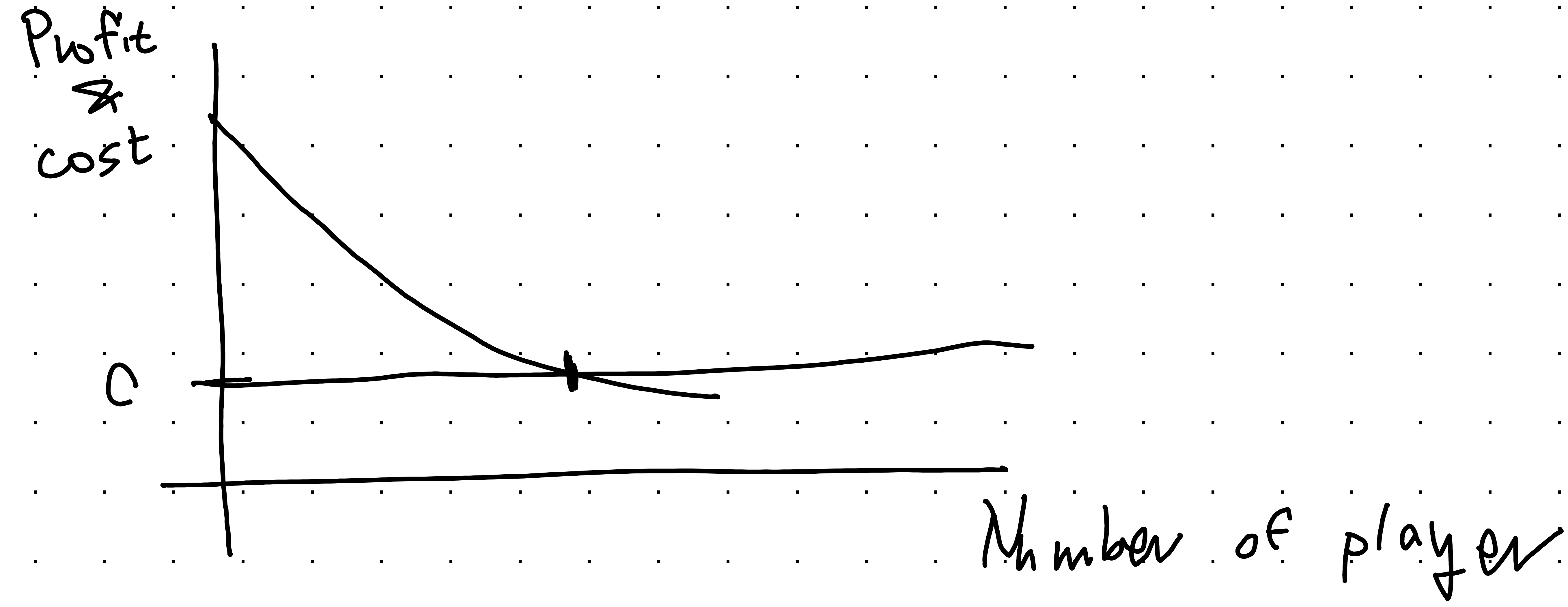
$$Y+2Ny=Y\left(1+\frac{2N}{2N+2}\varepsilon\right)$$

$$\frac{\cancel{Y^2}\cdot\frac{1}{4(N+1)^2}\cdot\varepsilon^2}{\cancel{Y^2}\left(1+\frac{2N}{2N+2}\varepsilon\right)^2}\cdot 2\cdot X\cdot(1-r)P\cdot\frac{Y}{Y}\cdot\frac{1+r}{1+r}$$

$$=\left(1-\frac{2N}{N+1}\varepsilon\right)\cdot\frac{\varepsilon^2}{4(N+1)^2}\cdot 2\cdot(1+\varepsilon)\cdot(1+r)Y$$

$$=(1+r)Y\cdot\frac{1}{2(N+1)^2}\cdot\varepsilon^2+O(\varepsilon^3)$$

$$=L\sqrt{P_0}\cdot\left(\frac{1+r}{2(N+1)^2}\right)\cdot\varepsilon^2\quad(\geq c)$$



Then

$$LVR = N \cdot ARB$$

$$= (1+\gamma) \cdot L\sqrt{P_0} \cdot \varepsilon^2 \cdot \frac{N}{2(N+1)^2}$$

Whereas in CPMM

$$LVR = \left(L\sqrt{P} - L\sqrt{P_0} + \left(\sqrt{P} - \frac{P}{\sqrt{P_0}} \right) \cdot L \right) \cdot (-1)$$

$$= L\sqrt{P_0} \left(\sqrt{\frac{P}{P_0}} - 1 \right)^2$$

$$\approx L\sqrt{P_0} \cdot \frac{\varepsilon^2}{4} \qquad L\sqrt{P_0} \cdot \frac{(\varepsilon-\gamma)^2}{4}$$

Now consider nonzero tx cost c.

$$ARB = (1+\gamma) \cdot L\sqrt{P_0} \cdot \varepsilon^2 \cdot \frac{1}{2(N+1)^2} \geq c$$

$$\Rightarrow (N+1)^2 \leq \frac{1+\gamma}{2c} \cdot L\sqrt{P_0} \cdot \varepsilon^2$$

$$\Leftrightarrow N+1 \leq \varepsilon \cdot \sqrt{\frac{1+\gamma}{2c} \cdot L\sqrt{P_0}}$$

$$\Rightarrow N^* = \left\lfloor \varepsilon \cdot \sqrt{\frac{1+\gamma}{2c} \cdot L\sqrt{P_0}} - 1 \right\rfloor$$

and

$$LVR = (1+r) \cdot L\sqrt{P_0} \cdot \varepsilon^2 \cdot \frac{N^*}{2(N^*+1)^2}$$

$$\approx c \cdot N^*$$

$$= c \left\lfloor \varepsilon \sqrt{\frac{1+r}{2c} \cdot L\sqrt{P_0}} - 1 \right\rfloor$$

$$\leq \varepsilon \sqrt{\frac{1+r}{2} \cdot cL\sqrt{P_0}}$$

Comparison

\$100M pool, 0.5% Jump, \$10 Tx cost

$$LVR_{FM\text{-}AMM} \leq \frac{1}{200}\sqrt{\frac{1}{2} \cdot 10 \cdot 50 \times 10^6}$$

$$= \frac{1}{200} \cdot 5 \cdot 10^3 \cdot \sqrt{10}$$

$$= 25\sqrt{10}$$

$$LVR_{CPMM} = 50 \cdot \cancel{10^6}^{\,10^2} \cdot \frac{1}{4} \cdot \frac{1}{\cancel{200}} \cdot \frac{1}{\cancel{200}}$$

$$= \frac{50}{16} \cdot 10^2$$

$$= 25 \times \frac{100}{8}$$

$$\frac{L}{\sqrt{P_0}} \qquad L\sqrt{P_0}$$

$$\boxed{\text{CPMM: } \frac{L\sqrt{P_0}}{K} \cdot (\varepsilon - \gamma)^2}$$

$$\downarrow$$

$$L\sqrt{P_0}\ \cancel{(1+\gamma)}$$

$$\frac{L}{\sqrt{P}} \qquad L\sqrt{}$$

$$\left(\frac{1+\frac{\varepsilon}{2}}{\sqrt{1+\gamma}} - 1\right)^2$$

$$-P_{ext}\left(\frac{L}{\sqrt{P}} - \frac{L}{\sqrt{P_0}}\right)$$

$$\frac{1}{1+\gamma}\left(1+\frac{\varepsilon}{2} - \sqrt{1+\gamma}\right)^2$$

$$-(1+\gamma)\left(L\sqrt{P} - L\sqrt{P_0}\right)$$

$$\frac{1}{\cancel{1+\gamma}}\left(\frac{\varepsilon}{2} - \frac{\gamma}{2}\right)^2$$

$$+\frac{L P_{ext}}{\sqrt{P}}\left(-1 + \sqrt{\frac{P}{P_0}}\right)$$

$$-(1+\gamma)\, L\sqrt{P_0}\left(\sqrt{\frac{P}{P_0}} - 1\right)$$

$$\left(\sqrt{\frac{P}{P_0}} - 1\right)\left(\sqrt{(1+\gamma) P_{ext}} - (1+\gamma)\sqrt{P_0}\right) L$$

$$\left(\sqrt{\frac{P_{ext}}{(1+\gamma)P_0}} - 1\right)\left(\sqrt{\frac{P_{ext}}{(1+\gamma)P_0}} - 1\right) L\sqrt{P_0}(1+\gamma)$$

TODO: Python Simulation

i) Poisson Jump

ii) Brownian Motion

Article Setup

I. Introduction

II. What is FM-AMM

II.I. How to settle batch

III. Model Setup

IV. Solve

V. Simulations

VI. Discussion

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