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5. STOCHASTIC PROCESS & MARKOV CHAIN

Many real world applications of prob theory have the particular future that data are collected sequentially in time. A few examples are weather data, stock market indices, air pollution data, demographic data, political tracking polls.

These also have in common successive observations that are typically not independent.

We refer to any such collection of observations as a stochastic process.

Stochastic process: A stochastic process is a collection of random variables. Let $\{X_t \mid t \in T\}$ be a collection of r.v's where 'T' is an index set. Let all the random variables 'X_t' takes the values in a set 'S' called "state space".

the values of r.v
i.e., $X(m,n) = m+n$
eg $X(2,4) = 6$
$X(1,6) = 7$
state space

Note

The two most common index patterns we use in this stochastic process are:

1) $T = \{0, 1, 2, \ldots\}$ and $T = [0, \infty)$ which usually represent discrete time and continuous time respectively.

The 1st index set gives a sequence of random variables i.e., $\{X_0, X_1, X_2 \ldots\}$ and the 2nd a collection of random variables $\{X_t \mid t \in [0,\infty)\}$, one random variable for each time t.

* The state space of the random variables may be finite or infinite.

State space: The values assumed by random variables are called 'states' and collection of all values is called 'state space'.

The state space is discrete if it contains a 'finite' or 'countable' collection. Otherwise it is called 'continuous'.

1) Consider an expt of throwing a dice. Let X_n be the random variable which takes the outcome of n^{th} throw $\forall n \geq 1$

i.e; $\{X_1, X_2, X_3, \dots\}$ be the r.v

X_1 - outcome in 1st throw $= \{1,2,3,4,5,6\}$
X_2 - " " 2nd " = "
⋮

∴ state space of all r.v $= \{1,2,3,4,5,6\}$
which is finite & also discrete.
Here no. of r.v is infinite & discrete.

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2) In the above expt, let X_n be the no. of 3's in the 1st 'n' throws,

X_n = no. of 3's in 1st 'n' throws
then, X_1 = no. of 3's till 1st throw $= \{0,1\}$
X_2 = " " " 2nd " $= \{0,1,2\}$
X_3 = " " " 3rd " $= \{0,1,2,3\}$

$X_n = \{0,1,2,\dots,n\}$

∴ No. of r.v's $= \{1,2,3,\dots\}$
state space of $X_n = \{0,1,2,\dots,n\}$ where $n=1,2,\dots$

3) No. of telephone calls received by a switch board during a particular time $(0,t)$.

X_t = no. of calls received in time $(0,t)$
i.e; X_{30min} = " " " " $(0,30)$

Here, r.v's are continuous
and state space $= \{0,1,2,\dots\}$ i.e; discrete & infinite

4) Temperature

X_t = max temp in Hyd during the time $(0,t)$

X_{3hrs} = " " " ... $(0,3)$

No.of r.v's is continuous

and state space is continuous say $(15°c, 35°c)$

5) Turbulent fluid flow

X_x = velocity of fluid in x-dir'n

X_y = " " " y - "

X_z = " " " z - "

No.of r.vs = $\{1,2,3\}$

State space is continuous

<u>Markov process</u> or <u>Markov chain</u>

Let $\{X_0, X_1, \ldots\}$ be a sequence of discrete random variables taking the values in some state space 'S' such that,

$$P(X_{t+1}=j \mid X_0=i_0, X_1=i_1, \ldots, X_t=i) = P(X_{t+1}=j \mid X_t=i)$$

where, $i_0, i_1, \ldots, i, j$ are states in 'S', for all values of t, then the seq of the r.vs is said to be a "markov chain" and the property is called "markov property."

⇒ So, the markov property can be stated as "future depends only on present but not on the past". In other words, the occurance of a future state depends only on its immediate preseeding state. (i.e., X_t)

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<u>Transition probability</u>

The prob of moving from one state to another state or remaining in the same state is called "transition prob" i.e., it is a conditional probability given by $P(X_{t+1}=j \mid X_t=i)$. It is denoted by "P_{ij}" which is also called "one step transition probability".

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i.e., $\boxed{P_{ij} = P(X_{t+1}=j \mid X_t=i)}$

Ex:- $P_{10} = P(X_{t+1}=0 \mid X_t=1)$

$P_{23} = P(X_{t+1}=3 \mid X_t=2)$

Stationary transition probabilities

If for each $t = 0,1,2,3\ldots$ $P(X_{t+1}=j \mid X_t=i) = P(X_1=j \mid X_0=i)$ then the one step transition prob said to be "stationary."

i.e., $P(X_1=j \mid X_0=i) = P(X_2=j \mid X_1=i) = P(X_3=j \mid X_2=i) = \ldots\ldots$

n-step transition probability

The n-step transition probability is the conditional probability that the system will be in state 'j' after n-steps given that it starts in state 'i' at any time 't'.

It is denoted by,

$$\boxed{P_{ij}^{(n)} = P(X_{t+n}=j \mid X_t=i)}$$

Note: $P_{ij}^{(1)} = P_{ij}$

The n-step transition probabilities always satisfies the following properties:

(i) $P_{ij}^{(n)} \geq 0 \quad \forall\, i,j,\ \forall\, n$

(ii) $\sum_{j=0}^{M} P_{ij}^{(n)} = 1 \quad \forall\, i\ \&\ n$ i.e., $P_{i0} + P_{i1} + \ldots + P_{iM} = 1$

(iii) The markov chains to be considered in this chapter have the following properties:

(a) A finite no. of states

(b) A stationary transition probabilities.

Transition matrix

The transition probabilities can be arranged in a matrix form such that all elements are non-negative and sum of all elements in a row is 1, is said to be a "transition matrix" or "one-step transition probability matrix".

→ A matrix to be a transition matrix

(1) Square

(2) non negative i.e., ≥ 0

(3) Sum in row = 1

$$T = \text{Current state} \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix}_{4\times 4} \end{array} \text{New state}$$

Ex:- $T = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}_{2\times 2}$ is a T.M

$T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$ is a T.M

$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$ not T.M $\because$ not square matrix

$T = \begin{bmatrix} 1 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$ not T.M $\because 1 + 1/2 = \frac{3}{2} \neq 1$

$T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 3/2 \end{bmatrix}$ not T.M $\because$ negative value

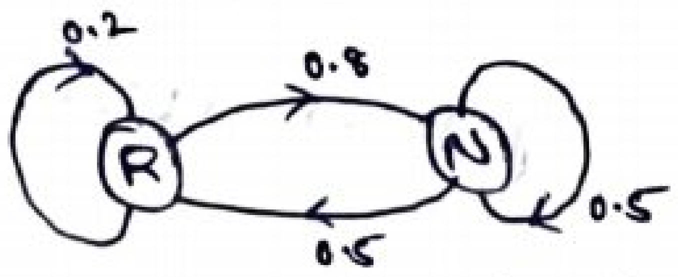
Diagramatic representation of Transition probabilities

The transition probabilities can be represented by two types of diagrams.

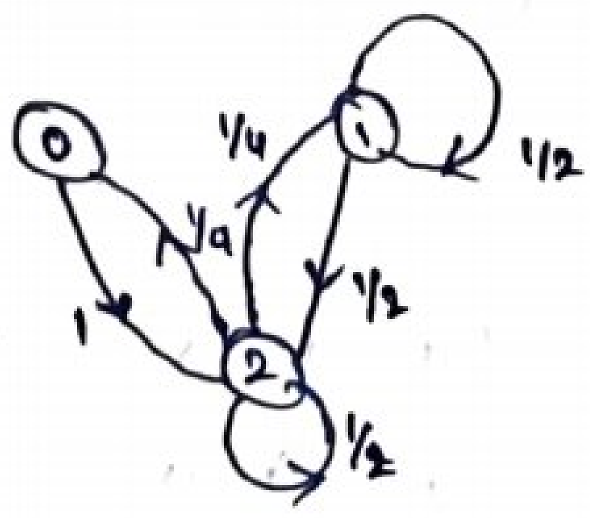
(1) Transition diagram: It shows the transition prob that can occur in any particular situation.

* Each stage is represented by a node.
* The arrows from each node to some other node indicate the possible states to which a process can move from the given state.

Ex:- $\begin{array}{c|cc} & R & N \\ \hline R & 0.2 & 0.8 \\ N & 0.5 & 0.5 \end{array} \rightarrow$



$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} \end{array} \rightarrow$



n-step transition matrix

The n-step transition probability matrix $P^{(n)}$ can be obtained by computing the n^{th} power of one-step transition matrix P.

i.e., $P^{(n)} = P^n$ (P.P....(n times))

Ex: $P = \begin{array}{c|c} & \begin{array}{cc} 0 & 1 \end{array} \\ \hline \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix} \end{array}$

$$P^{(2)} = P^2 = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$$

2-step transition matrix

1) A Gambling: Suppose that a player has \$1 and with each play of the game wins \$1 with prob $p>0$ or losses \$1 with prob $1-p$. The game ends when the player either accumulates \$3 or goes broke. Write the Transition matrix.

The states of the player are related to the amount of money with the player, 0\$, 1\$, 2\$, 3\$

$$\begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \hline \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$P_{00} = P(\text{next is } 0 \mid \text{current is } 0) = 1$

$P_{01} = P(\text{next is } 1 \mid \text{current is } 0) = 0$

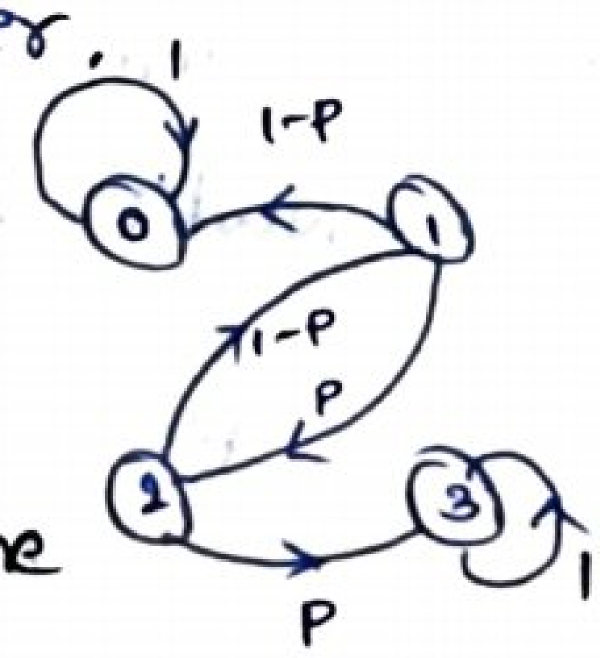
$P_{02} = P(\text{next is } 2 \mid \text{current is } 0) = 0$

⋮

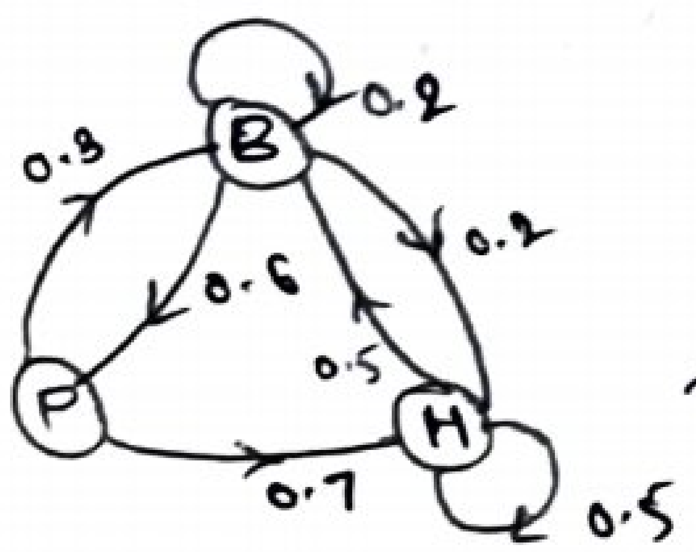
$P_{ij} = P(\text{next is } j \mid \text{current is } i)$

At 00 the probability is 1 as the game is over.

At 33 the prob is 1 as the game is over.



In a restaurant three special items Burger, Pizza, Hotdog are being served such that only one given by the following diagram. Write the transition matrix.



$$\begin{array}{c|c} & \text{tomorrow} \\ & \begin{array}{ccc} B & P & H \end{array} \\ \hline \text{today}\ \begin{array}{c} B \\ P \\ H \end{array} & \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ 0.3 & 0 & 0.7 \\ 0.5 & 0 & 0.5 \end{bmatrix}_{3\times 3} \end{array}$$

P(TP|CP) = P(Tomorrow pizza | current pizza) = 0

P(TB|CB) = 0.5

P(TB|CP) = 0.3

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3) Which of the following matrices are stochastic.

(i) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ Not a square so not stochastic

(ii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ square, sum = 1, non-negative
So stochastic matrix

(iii) $\begin{bmatrix} 0 & 1 \\ 1/3 & 1/4 \end{bmatrix}$ square, sum $\neq 1$ ×, non--ve
Not stochastic

(iv) $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ square, sum = 1, non-negative
So, stochastic

(v) $\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$ square, sum = 1, non-negative ×
So, not stochastic.

4) Prof symons either walks to school, or he rides his bicycle. If he walks to school one day, then the next day, he will walk or cycle with equal prob. But if he bycycles one day, then the prob that he will walk the next day is 1/4.

(a) Express this infor in transition matrix.

(b) If it is assumed that the initial day is Monday, write a matrix that gives prob of a transition from Monday to Wednesday.

states are walk, cycle i.e; {w, c}

(a) The transition matrix is of order 2x2

$P(NW|TW) = P(\text{next day walk} \mid \text{today walk})$

$$P = \begin{array}{c c} & \begin{array}{cc} w & c \end{array} \text{ tomorrow} \\ \begin{array}{c} w \\ \text{today } c \end{array} & \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix} \end{array}$$

(b) $P^{(2)}$ i.e., transition matrix from monday to wednesday is a 2-step T.M.

$$P^{(2)} = P \cdot P$$

$$= \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$$

$$= \begin{bmatrix} 3/8 & 5/8 \\ 5/16 & 11/16 \end{bmatrix}$$

4) Two manufacturers A and B are competing with each other in a restricted market. Over the years exhibited a high degree of loyalty as measured by the fact that customers using A's product 80% of time. Also, former customers purchasing the product from B have switched back to A's 60% of time.

(a) Construct and interpret the state transition matrix in terms of retention and loss.

(b) Draw the transition diagram.

(c) Calculate the prob of a customer purchasing A's product at the end of the second period.

(a) states : A's product, B's product

$$P = \begin{array}{c c} & \begin{array}{cc} A & B \ \text{next} \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} \\ \text{current} & \end{array}$$

$P(NA|CA) = 0.8$ (retention)

$P(NB|CA) = 0.2$ (loss)

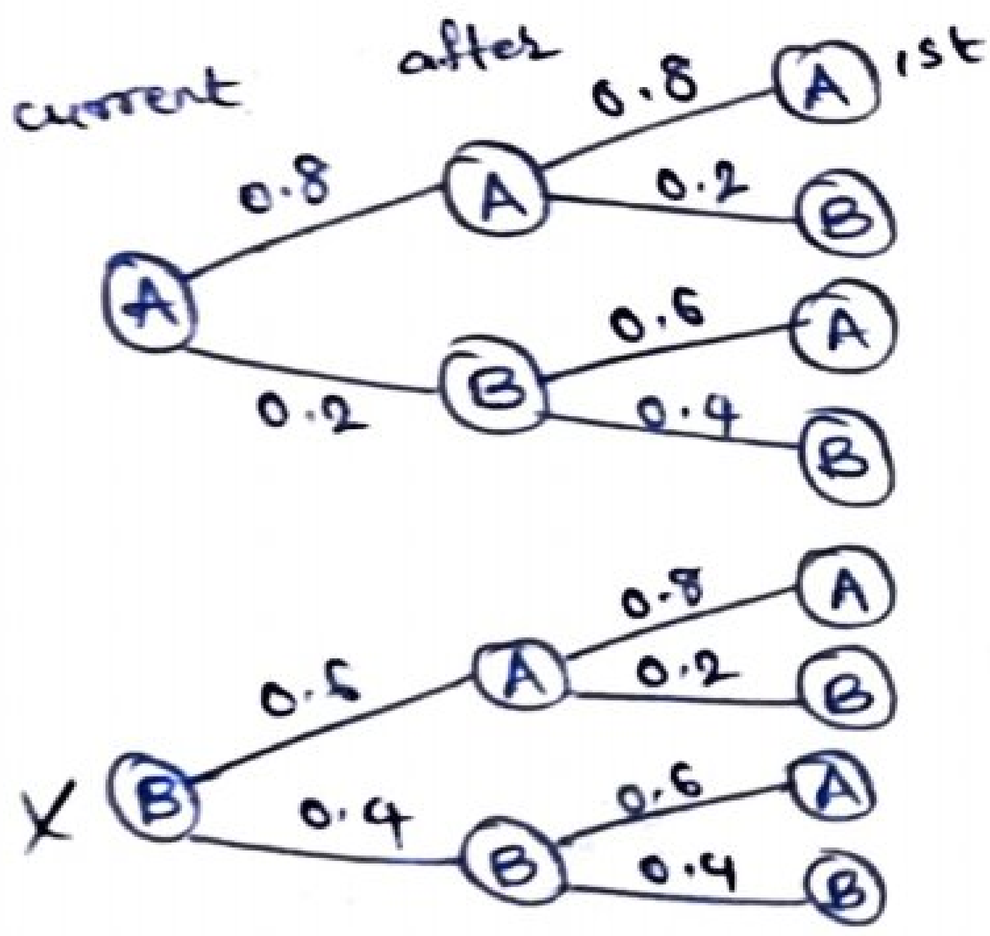
$P(NA|CB) = 0.6$ (loss)

$P(NB|CB) = 0.4$ (retention)

A's product retention is more than B's product.

As more no. of customers of B are switching to A (0.6) than from A to B, B's product has more loss.

(b)



After 2nd period

(c) P(a customer purchase a product at the end of 2nd year)

$= 0.8 \times 0.8 + 0.2 \times 0.6$

$= 0.76$

$$P^2 = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.76 & 0.24 \\ 0.72 & 0.28 \end{bmatrix}$$

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Types of Markov chains

Markov chains are categorised into two types:

(i) Ergodic (ii) Regular

(i) Ergodic Markov chain:

If it is possible to pass from one state to another in a finite no. of steps, regardless the present state then the markov chain is said to be Ergodic.

Ex:-

(1) $$P = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix} \end{array}$$

We can move from one state to another state in one step (∵ all entries are positive)

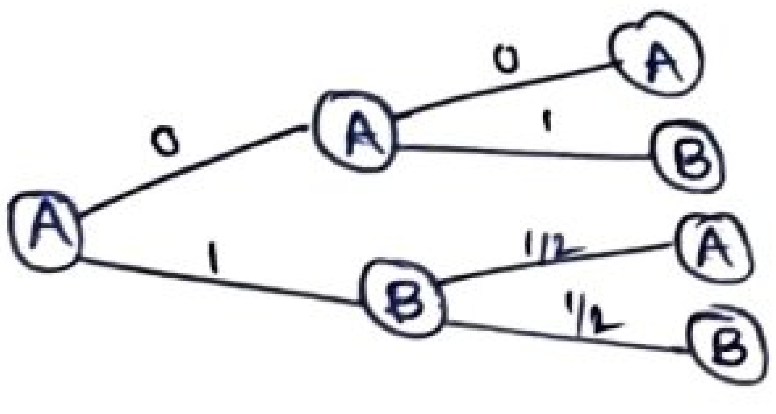
∴ Markov chain is Ergodic.

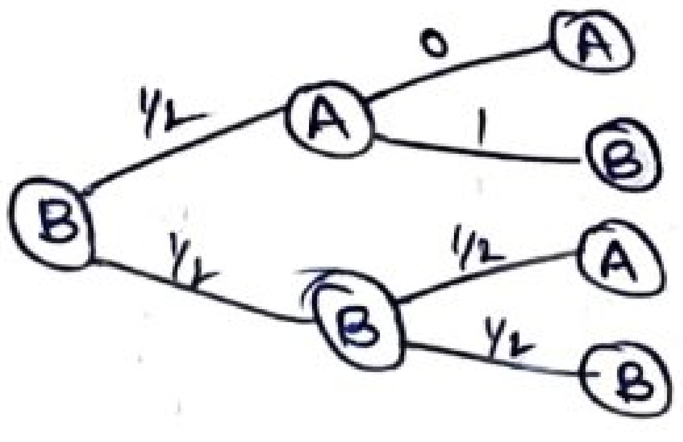
(2) $$P = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

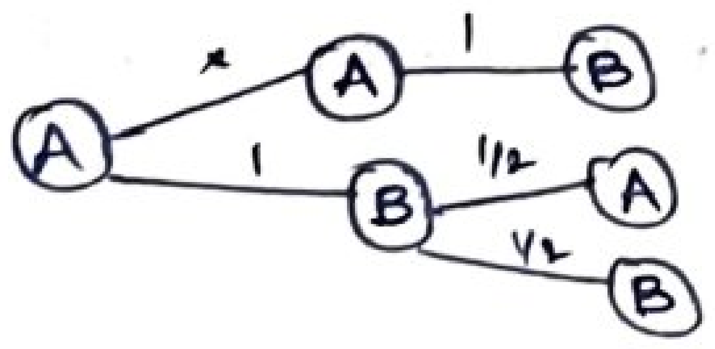
$$P^{(2)} = P^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We can't move from B to A and A to B in any no. of steps. So it is not Ergodic.

(3) $$P = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \end{array}$$







$A \to A = A \text{ to } B + B \text{ to } A$

$= 1 \times \frac{1}{2}$

$= \frac{1}{2}$

We can move from A to B in 1 step, B to A in 1 step, B to B in 1 step and A to A in 2 steps.

We can travel from any state to next state in finite no. of steps.

So, Ergodic

Determine the given transition matrix is an ergodic markov chain.

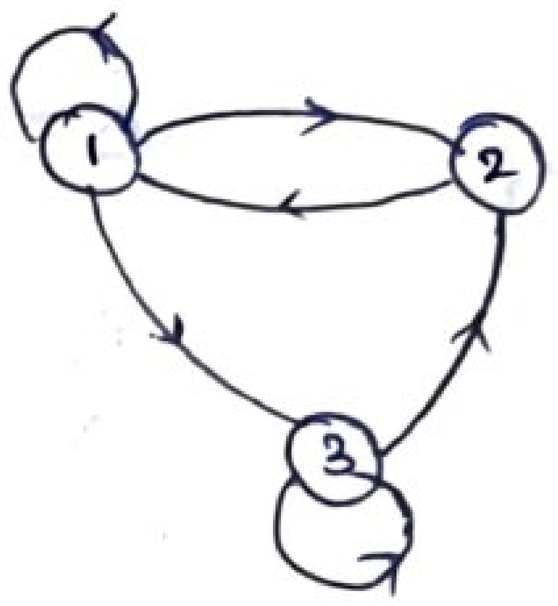
$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1/3 & 1/3 & 3 & 1/2 \\ 0 & 1/2 & 0 & 1/4 \\ 1/4 & 0 & 1/4 & 1/4 \\ 0 & 0 & 1/3 & 2/3 \end{bmatrix} \end{array}$$

1) 2 — 4 — 3 — 1 ∴ 2 — 1 ✓
×
2) 2 — 4 — 3 ✓ ∴ 2 — 3 ✓
3) 3 — 1 — 2 ∴ 3 — 2 ✓
4) 4 — 3 — 1 ∴ 4 — 1 ✓
5) 4 — 3 — 1 — 2 ∴ 4 — 2 ✓

∴ We can pass from one state to another state, so it is ergodic.

Which of the following matrix are ergodic on Irreducible

1) $\begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}$



We can move from one state to another state in one step for the following states.

1-1 2-1 3-2
1-2 3-3
1-3

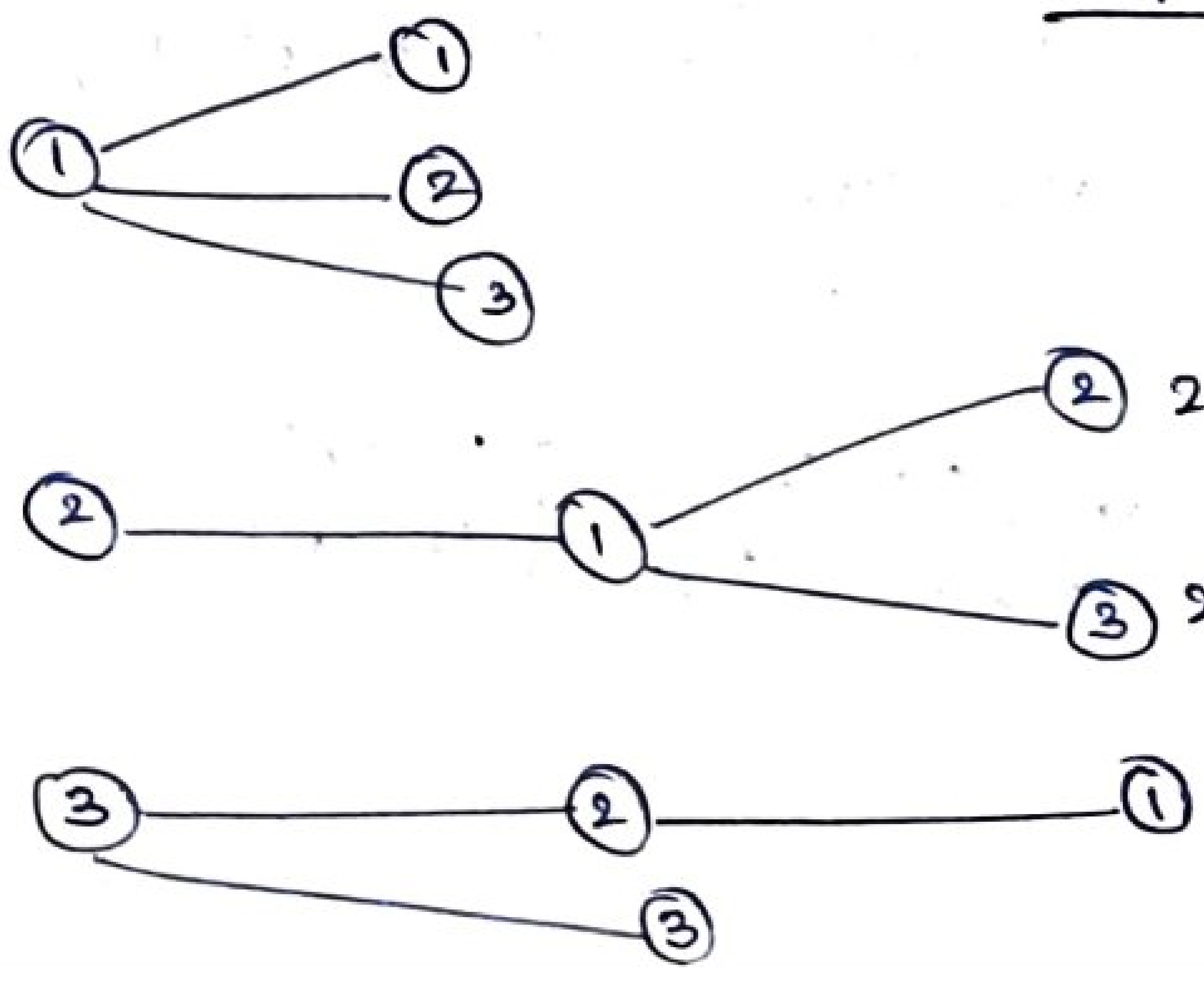
So, movement from one state to another state in 2 steps:

2-1-2 i.e., 2-2 in 2 steps
2-1-3 i.e., 2-3 in 2 steps
3-2-1 i.e., 3-1 in 2 steps

∴ From one state to another state movement can be done in finite no.of steps. Therefore given markov chain is ergodic.

(or)

<u>Initial</u> <u>step-I</u> <u>step-II</u>

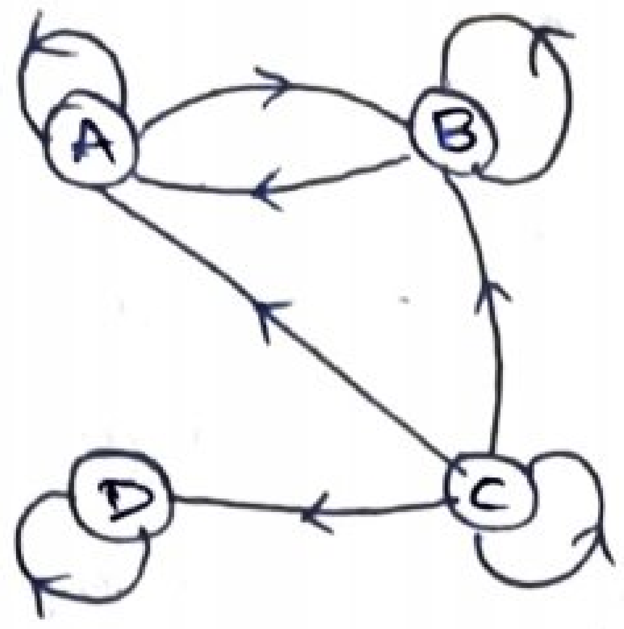


2-1-2 i.e., 2-2 in 2 steps

2-1-3 i.e., 2-3 in 2 steps

3-2-1 i.e., 3-1 in 2 steps

2)
$$\begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0.4 & 0.6 & 0 & 0 \\ B & 0.3 & 0.7 & 0 & 0 \\ C & 0.2 & 0.4 & 0.1 & 0.3 \\ D & 0 & 0 & 0 & 1 \end{array}$$



$A \xrightarrow{\times} C$

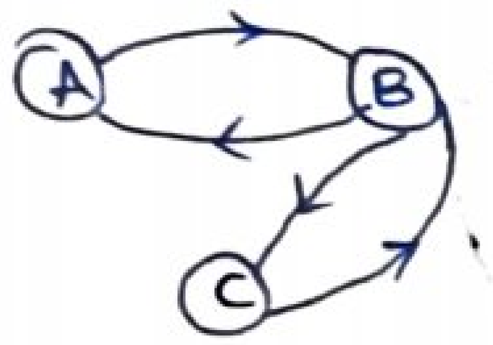
$A \xrightarrow{\checkmark} B \xrightarrow{\times} C$

$\searrow D$

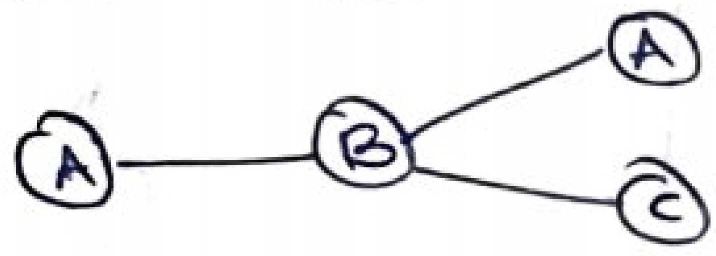
$A \xrightarrow{\times} D$

We cannot move from A to C and A to D in finite no. of chain and there is no movement from D to A, D to B, D to C. So it is not Ergodic.

3)
$$\begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & 1/2 & 0 & 1/2 \\ C & 0 & 1 & 0 \end{array}$$

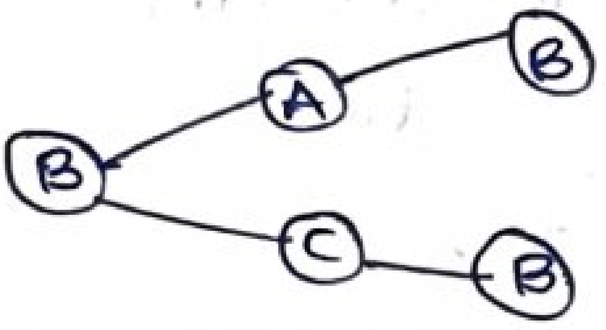


Initial step I step II

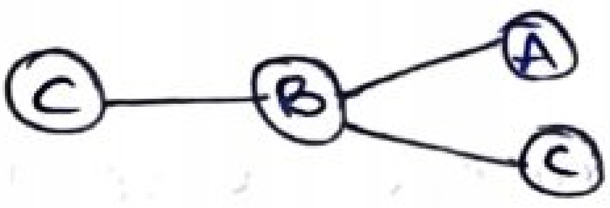


A – B – A i.e., A – A in 2 steps

A – B – C i.e., A – C in 2 steps



B – B in 2 steps



C – B – A i.e., C – A in 2 steps

C – B – C i.e., C – C in 2 steps

1 - step moments are

A – B
B – A
B – C
C – B

2 - step moments are

A – A C – A
A – C C – C
B – B

∴ We can move from one step to another step in finite no. of steps. So, Ergodic.

(ii) Regular Markov Chain

A Markov chain having a transition matrix P is said to be a regular markov chain if P^k has only non-zero positive prob values for some k.

i) $P = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix}$

$$P^2 = P \cdot P = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3/4 & 1/4 \end{bmatrix}$$

$$p^3 = p^2 \cdot p = \begin{bmatrix} 1 & 0 \\ 3/4 & 1/4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 7/8 & 1/8 \end{bmatrix}$$

Every power of P i.e; P^k has '0'

∴ The given markov chain is not regular.

2) $\begin{bmatrix} 0.6 & 0.4 \\ 0 & 1 \end{bmatrix}$

Let $P = \begin{bmatrix} 0.6 & 0.4 \\ 0 & 1 \end{bmatrix}$

$$P^2 = P.P = \begin{bmatrix} 0.6 & 0.4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0.6 & 0.4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.36 & 0.64 \\ 0 & 1 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0.216 & 0.784 \\ 0 & 1 \end{bmatrix}$$

Every power of P i.e; P^k has '0'

∴ The given markov chain is not regular.

3) $\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$

$$P^2 = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$$

∵ All elements of P^2 are positive, the given markov chain is regular.

$\Rightarrow \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}\begin{bmatrix} x & x \\ 0 & x \end{bmatrix} \Rightarrow$ not regular, diagonal element is 1

$\begin{bmatrix} 0 & x \\ x & x \end{bmatrix}\begin{bmatrix} x & x \\ x & 0 \end{bmatrix} \Rightarrow$ regular.

* Every regular markov chain is ergodic but an ergodic may or may not be regular.

4) $P = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$

$$P^2 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}; \quad P^3 = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}, \quad P^4 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

∴ P is not regular since no P^k with +ve values.

* Every regular markov chain is ergodic but an ergodic chain may or may not be regular.

* Example of ergodic chain which is not regular,

$P = \begin{bmatrix} 0 & 1 & 0 \\ & & 1/2 \end{bmatrix}$

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State probability, probability vector, steady state (equilibrium) condition:

State probability: It is the probability that the system is being in state i at time, t=n. It is denoted by P_i

Ex:- $P_1(0) \Rightarrow$ Prob that the system is at state 1 at time 0

$P_A(1) \Rightarrow$ Prob that the system is in state A at time 1

Probability vector: It is a row vector representing the state probabilities of all states at time t=n. It is denoted by R(n).

Ex:-
$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} P_{AA} & P_{AB} \\ P_{BA} & P_{BB} \end{bmatrix} \end{array}$$

current / next

$R(0) = [P_A(0) \;\; P_B(0)]$

$R(1) = [P_A(1) \;\; P_B(1)]$

$\vdots$

$R(n) = [P_A(n) \;\; P_B(n)]$

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} \end{array}$$

$R(0) = [P_1(0) \;\; P_2(0) \;\; P_3(0)]$

$R(1) = [P_1(1) \;\; P_2(1) \;\; P_3(1)]$

$\vdots$

$R(n) = [P_1(n) \;\; P_2(n) \;\; P_3(n)]$

* The sum of the prob of all states is 1

(i) i.e., $\sum_{i=1}^{m} P_i(n) = 1, \quad n = 0, 1, 2, \ldots$

(ii) If all the state prob are known at time t=n then the state prob at time t=n+1 can be determined by the eq'n

$$P_j(n+1) = \sum_{i=1}^{m} P_i(n) \cdot P_{ij}, \quad n = 0, 1, 2, 3, \ldots$$

i.e., $P_j(n+1) = P_1(n) \cdot P_{1j} + P_2(n) \cdot P_{2j} + \cdots + P_m(n) \cdot P_{mj}$

More clearly we can write above eq'ns as $R(n) = [P_A(n) \;\; P_B(n)]$

$R_1(n) = [P_1(n) \;\; P_2(n) \; \ldots \; P_m(n)]$

$$[P_1(n) \;\; P_2(n) \; \ldots \; P_m(n)] \begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & \ldots & j & \ldots m \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ m \end{array} & \begin{bmatrix} & & & P_{1j} & \\ & & & P_{2j} & \\ & & & \vdots & \\ & & & P_{mj} & \end{bmatrix} \end{array}$$

$= [P_1(n+1) \;\; P_2(n+1) \; \ldots \; P_j(n+1) \; \ldots \; P_m(n+1)]$.

Precisely we can write as,

$$\boxed{R(n) \cdot P = R(n+1)}$$

P = transition matrix

R(n) = Prob vector at t=n

R(n+1) = Prob vector at t=n+1

→ for $n=0$, $R(1) = R(0).P$

$n=1$, $R(2) = R(1).P = (R(0).P).P = R(0).P^2$

$n=2$, $R(3) = R(2).P = R(0).P^3$

⋮

$$\boxed{\begin{aligned} R(n+1) &= R(n).P \\ R(n+1) &= R(0).P^{n+1} \end{aligned}}$$

1) A city is served by 2 cable TV companies, Best TV & Cable Cast. Due to their aggressive sales tactics, each yr 40% of Best TV customers switch to cable cast; the other 60% of Best TV customers stay with Best TV. On the other hand, 30% of the cable cast customers switch to Best TV.

(i) Express the info above as T.M which displays the prob of going from one state to another state.

(ii)

(iii)

Two states, Best TV, cable cast i.e, B, C

(i) $$P = \begin{array}{c c} & \begin{array}{cc} B & C \end{array} \\ \begin{array}{c} B \\ C \end{array} & \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix} \end{array}$$

(ii) $R(0) = [P_B(0) \;\; P_C(0)] = [0.25 \;\; 0.75]$

After 1 year, $R(1) = [P_B(1) \;\; P_C(1)]$

WKT, $R(1) = R(0).P$

$$= [P_B(0) \;\; P_C(0)].\begin{bmatrix} P_{BB} & P_{BC} \\ P_{CB} & P_{CC} \end{bmatrix}$$

$$= [1/4 \;\; 3/4].\begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}$$

$$= [3/8 \;\; 5/8]$$

$$= [P_B(1) \;\; P_C(1)]$$

$P_B(1) = \frac{3}{8}$ i.e, % of customers with B TV next year

$= \frac{3}{8} \times 100 = 37.5\%$

$P_C(1) = 5/8$ i.e, % of customers with C TV next year

2) Consider a bike share program with only 3 stations A, B, C. Suppose that all bicycles must be returned to the station at the end of the day, so that each day there is a time, let's say midnight, that all bikes are at some station, and we can examine all the stations at this time of day, every day. We want to model the movement of bikes from midnight of a given day to midnight of the next day. We find that over a 1 day period,

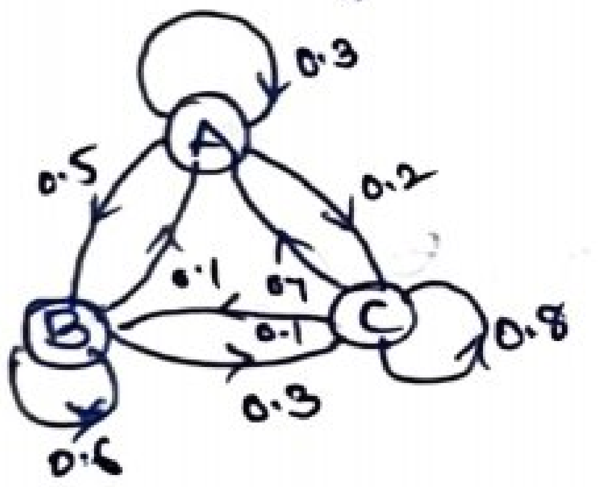
(1) Given,

$P_{AA} = 0.3$, $P_{AB} = 0.5$, $P_{AC} = 0.2$

$P_{BA} = 0.1$, $P_{BB} = 0.6$, $P_{BC} = 0.3$

$P_{CA} = 0.1$, $P_{CB} = 0.1$, $P_{CC} = 0.8$

(1) Arrow diagram,



(2) Transition matrix

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.1 & 0.6 & 0.3 \\ 0.1 & 0.1 & 0.8 \end{bmatrix} \end{array}$$

(3) $R(0) = [P_A(0) \quad P_B(0) \quad P_C(0)]$

$= [0.3 \quad 0.45 \quad 0.25]$

$R(1) = R(0) \cdot P$

$= [0.3 \quad 0.45 \quad 0.25] \cdot \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.1 & 0.6 & 0.3 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}$

$= [0.16 \quad 0.445 \quad 0.395]$

$R(2) = R(1) \cdot P$

$= [0.16 \quad 0.445 \quad 0.395] \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.1 & 0.6 & 0.3 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}$

$= [0.132 \quad 0.3865 \quad 0.4815]$

5) Prof symons either walks to school, or he rides his bicycle. If he walks to school one day, then the next day, he will walk or cycle with equal prob. But if he bicycles one day, then the prob that he will walk next day is 1/4. Express this info in a transition matrix.

(a) If it is assumed that the initial day is Monday, write a matrix that gives prob of a transition from monday to wednesday.

(b) In the long run, how often will he walk to school, and how often will he bycycle?

Given two states are {walk, cycle} i.e., {w, c}

The transition matrix is

$$P = \begin{array}{c|cc} & w & c \\ \hline w & \frac{1}{2} & \frac{1}{2} \\ c & \frac{1}{4} & \frac{3}{4} \end{array}$$

(a) $P_{ww} = \frac{1}{2}$, $P_{wc} = \frac{1}{2}$, $P_{cw} = \frac{1}{4}$, $P_{cc} = \frac{3}{4}$

$$P^2 = P.P = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix} = \begin{bmatrix} 3/8 & 5/8 \\ 5/16 & 11/16 \end{bmatrix}$$

$$\therefore P^2 = \begin{array}{c|cc} & w & c \\ \hline w & P_{ww}^{(2)} & P_{wc}^{(2)} \\ c & P_{cw}^{(2)} & P_{cc}^{(2)} \end{array} = \begin{array}{c|cc} & w & c \\ \hline w & 3/8 & 5/8 \\ c & 5/16 & 11/16 \end{array}$$

⇒ For the above problem,

(b) Let $R = [x \ \ y]$

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}, \quad P^2 = \begin{bmatrix} 3/8 & 5/8 \\ 5/16 & 11/16 \end{bmatrix}$$

$$P^5 = \begin{bmatrix} 0.3339 & 0.666 \\ 0.333 & 0.666 \end{bmatrix}, \quad P^{10} = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{bmatrix}$$

$$P^{20} = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{bmatrix} \quad \text{i.e., } R = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

(or)

Let $R = [x \ \ y]$

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$$

WKT, $RP = R$

$$[x \quad y] \cdot \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix} = [x \quad y]$$

$$\Rightarrow \left[\frac{1}{2}x + \frac{1}{4}y \quad \frac{1}{2}x + \frac{3}{4}y\right] = [x \quad y]$$

$$\Rightarrow \frac{1}{2}x + \frac{1}{4}y = x$$

$$\frac{1}{2}x + \frac{3}{4}y = y$$

$$\Rightarrow 2x + y = 4x \Rightarrow 2x = y$$

$$\Rightarrow 2x + 3y = y \Rightarrow 2x = y$$

WKT, $x + y = 1$

By solving,

$$x = \frac{1}{3} \Rightarrow y = \frac{2}{3}$$

$$\therefore R[x \quad y] = [1/3 \quad 2/3]$$

∴ In long run, prof symons prefers walk by 33%

" " " " " cycle by 66%

Note

1. For a regular matrix P, the powers of P are same for larger values of n i.e; $P^{10}, P^{4}, \ldots P^{20}$
2. In such situations, the system is said to be in steady state (or) state of eq'm.
3. Also note that in powers of P for larger values of n, all row vectors are equal.
4. We call this vector as fixed prob vector or eq'm vector. It is denoted by "R."
5. Also note that for any prob vector R(K), we have $\boxed{R(K).P^{n} = R}$ (for larger values of 'n')
6. Also for larger values of n, R(n) = R i.e;

$$\boxed{R(n) \rightarrow R \text{ as } n \rightarrow \infty}$$

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*. Further more, if the eq'm vector R is multiplied by the original vector P, the result is the eq'm vector R.

$$\boxed{RP = R}$$

→ In a regular markov chain, for larger values of n, the state prob P_{ij} → fixed limits and each prob vector R(n) approaches to a constant vector R, also for this constant prob vector R we have $RP = R$ then the system is said to be in "steady state" or "eq'm state."

25/03/23

1) Consider a three-state Markov chain with the T.M. If the initial prob $P_0 = [0.2 \ \ 0.3 \ \ 0.5]$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2/3 & 1/3 \\ 1/16 & 15/16 & 0 \end{bmatrix}$$

(a) Find the probabilities after two transitions.

(b) Find the limiting probabilities.

Given, $P_0 = [0.2 \ \ 0.3 \ \ 0.5] = R(0)$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2/3 & 1/3 \\ 1/16 & 15/16 & 0 \end{bmatrix}$$

i.e; $P_{11} = P(X_{n+1} = 1 \mid X_n = 1) = 0$

$P_{12} = P(X_{n+1} = 2 \mid X_n = 1) = 1$

⋮

Let the states be 1, 2, 3

(a) $R(1) = R(0) \cdot P$

$$= [0.2 \ \ 0.3 \ \ 0.5] \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2/3 & 1/3 \\ 1/16 & 15/16 & 0 \end{bmatrix}$$

$$= [0.03125 \ \ 0.86875 \ \ 0.1]$$

Now, $R(2) = R(1) \cdot P$

$$= [0.3125 \ \ 0.86875 \ \ 0.1] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2/3 & 1/3 \\ 1/16 & 5/16 & 0 \end{bmatrix}$$

$$= [0.00625 \ \ 0.7041 \ \ 0.2895]$$

(b) Let $R = [x \ \ y \ \ z]$

WKT, $RP = R$

$$\Rightarrow [x \ y \ z] \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2/3 & 1/3 \\ 1/16 & 15/16 & 0 \end{bmatrix} = [x \ y \ z]$$

$$\Rightarrow \left[\tfrac{1}{16}z \quad x+\tfrac{2}{3}y+\tfrac{15}{16}z \quad \tfrac{1}{3}y\right] = [x \ y \ z]$$

$$\Rightarrow \tfrac{1}{16}z = x \ ; \ x+\tfrac{2}{3}y+\tfrac{15}{16}z = y \ ; \ \tfrac{1}{3}y = z$$

$$\Rightarrow z = 16x \ ; \ 48x+32y+45z = 48y \ ; \ y = 3z$$

$$\Rightarrow 16x - z = 0 \to (1) \ ; \ 48x \overset{-16}{\cancel{+32}}y + 45z = 0 \to (2) \ ; \ y - 3z = 0 \to (3)$$

$$-3\times(1) + (2) \Rightarrow \begin{array}{r} 48x - 3z = 0 \\ 48x - 16y + 45z = 0 \\ \hline -16y + 48z = 0 \end{array}$$

$$\Rightarrow y - 3z = 0$$

i.e., $(3) = (2) - 3\times(1)$

Also, WKT $x+y+z=1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 16 & 0 & -1 & 0 \\ 48 & -16 & 45 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right]$$

$R_2 - 16R_1$
$R_3 - 48R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -16 & -17 & -16 \\ 0 & -64 & -3 & -48 \\ 0 & 1 & -3 & 0 \end{array}\right]$$

$R_2 \leftrightarrow R_4$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & -64 & -3 & -48 \\ 0 & -16 & -17 & -16 \end{array}\right]$$

$R_3 + 64R_2$
$R_4 + 16R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -195 & -48 \\ 0 & 0 & -65 & -16 \end{array}\right]$$

$$x+y+z=1$$
$$y-3z=0$$
$$-65z=-16$$
$$\Rightarrow z = \frac{16}{65}$$
$$y - 3\left(\frac{16}{65}\right) = 0$$
$$\Rightarrow y = \frac{48}{65}$$
$$x = 1-y-z$$
$$= 1 - \frac{48}{65} - \frac{16}{65}$$
$$\Rightarrow x = \frac{1}{65}$$

$$\therefore R = \left[\frac{1}{65} \quad \frac{48}{65} \quad \frac{16}{65}\right]$$

2) A prof has 3 pet qsns, one of which occurs on every test he gives. He never uses the same qsn twice in successive examinations. If he used the qsn no.1, he tosses a coin and uses the qsn no.2 if Head appears. If he uses the qsn no.2, he tosses ~~two~~ two coins and use qsn no.3 if both are heads. If he uses the qsn no.3, he tosses three coins and use qsn no.1, if all are heads. In long run which qsn does he use most often and with how much frequency is it used.

States are $\{1, 2, 3\}$

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1/2 & 1/2 \\ 3/4 & 0 & 1/4 \\ 1/8 & 7/8 & \cancel{7/8}\ 0 \end{bmatrix} \end{array}$$

		H	T
1 − 1 coin	⇒	1/2	1/2
2 − 2 coins	⇒	1/4	3/4
3 − 3 coins	⇒	1/8	7/8

Let, $R = [x \quad y \quad z]$

WKT, $RP = R$

$$\Rightarrow [x \quad y \quad z] \cdot \begin{bmatrix} 0 & 1/2 & 1/2 \\ 3/4 & 0 & 1/4 \\ 1/8 & 7/8 & 0 \end{bmatrix} = [x \quad y \quad z]$$

$$\Rightarrow \left[\tfrac{3}{4}y + \tfrac{1}{8}z \quad \tfrac{1}{2}x + \tfrac{7}{8}z \quad \tfrac{1}{2}x + \tfrac{1}{4}y\right] = [x \quad y \quad z]$$

$\Rightarrow \frac{3}{4}y + \frac{1}{8}z = x$ $\quad\Rightarrow \frac{1}{2}x + \frac{7}{8}z = y$ $\quad\Rightarrow \frac{1}{2}x + \frac{1}{4}y = z$

$\Rightarrow 6y + z = 8x$ $\quad\Rightarrow 4x + 7z = 8y$ $\quad\Rightarrow 2x + y = 4z$

$\Rightarrow 8x - 6y - z = 0$ $\quad\Rightarrow 4x - 8y + 7z = 0$ $\quad\Rightarrow 2x + y - 4z = 0$

WKT, $x + y + z = 1$

Now,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 8 & -6 & -1 & 0 \\ 4 & -8 & 7 & 0 \\ 2 & 1 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -14 & -9 & -8 \\ 0 & -12 & 3 & -4 \\ 0 & -1 & -6 & -2 \end{array}\right]$$

$R_2 - 8R_1$

$$\left[\begin{array}{ccc|c} 0 & -1 & -6 & -2 \\ 0 & -12 & 3 & -4 \\ 0 & -14 & -9 & -8 \end{array}\right]$$

$R_3 - 12R_2$

$R_4 - 14R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & -6 & -2 \\ 0 & 0 & 75 & 20 \\ 0 & 0 & 75 & 20 \end{array}\right]$$

$\Rightarrow x + y + z = 1,$

$-y - 6z = -2$

$75z = 20$

$\Rightarrow z = \frac{20}{75} \Rightarrow z = \frac{4}{15}$

$\Rightarrow y = 2 - 6z = 2 - 6\left(\frac{4}{15}\right) = \frac{2}{5}$

$\Rightarrow x = 1 - y - z$

$= 1 - \frac{2}{5} - \frac{4}{15}$

$= \frac{1}{3}$

$\therefore R = \left[\frac{1}{3} \quad \frac{2}{5} \quad \frac{4}{15}\right] = [0.3333 \quad 0.4 \quad 0.2666]$

R(0)	0.5	0.5
R(1)	0.7	0.3
R(2)	0.78	0.22
R(3)	0.812	0.188
R(4)	0.8248	0.1752
R(5)	0.82992	0.17
R(6)	0.83196	0.168
R(7)	0.8327	0.1672
R(8)	0.8331	0.1668
R(9)	0.8332	0.1667
R(10)	0.83329	0.1667
R(11)	0.8333	0.1667
R(12)	0.8333	0.1667

Hence,

$\bar{R} = [0.8333 \quad 0.16667]$

$= [5/6 \quad 1/6]$

3) i.e., In long run prof. uses 2nd qsn most often, and with a freq of 40%.

3) Suppose there are two market products of brand A and B respectively. Let each of these two brands have exactly 50% the total market in same period and let the market be of a fixed size. The transition matrix is given as follows:

If the initial market share breakdown is 50% for each brand, then determine their market shares in the steady state.

From \ To	A	B
A	0.9	0.1
B	0.5	0.5

Given, $P = \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}$

Let, $R = [x \ y]$

WKT, $RP = R$

$\Rightarrow [x \ y] \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} = [x \ y]$

$\Rightarrow [0.9x + 0.5y \quad 0.1x + 0.5y] = [x \ y]$

$\Rightarrow 0.9x + 0.5y = x \qquad 0.1x + 0.5y = y$

$\Rightarrow 0.1x = 0.5y \qquad \Rightarrow 0.1x = 0.5y$

$\Rightarrow x = 5y \qquad \Rightarrow x = 5y$

WKT, $x + y = 1$

By solving,

$x + y = 1$
$6y = 1$
$y = \frac{1}{6}$
$\Rightarrow x = \frac{5}{6}$

$x = \frac{5}{6}, \ y = \frac{1}{6}$

$R = \left[\frac{5}{6} \ \frac{1}{6}\right]$

5) A raining process is considered as a two state Markov chain. If it rains it is considered to be state 0 and if it does not rain, the chain is state 1. The transition probability of the Markov chain is defined as $P = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$. Find the prob that it will rain for 3 days from today assuming that it is raining today. Also find the unconditional prob that it will rain after 3 days with the initial prob of state 0 and state 1 as 0.4 and 0.6 respectively.

$x \quad y$
$R(0) \quad 1/2 \quad 1/2$
$R(1)$
$R(2)$

States $= \{\text{rain}, \text{no rain}\} = \{0, 1\}$

Given, $P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix} \end{array}$

$R(0) = [1 \ 0]$

raining today so prob is 1

Prob that it rains for 3 days $= P(\overset{\text{rain}}{X_2 = 0}, \overset{\text{rain}}{X_1 = 0}, \overset{\text{rain}}{X_0 = 0})$

i.e. $P_0(0)$

$= P(X_2 = 0 | X_1 = 0) * P(X_1 = 0 | X_0 = 0) * P(X_0 = 0)$

$= 0.6 \times 0.6 \times 1$

$= 0.36$

(b) Given, $R(0) = [0.4 \quad 0.6]$

$$P = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$$

$R(3) = R(0) \cdot P^3$

$$= [0.4 \quad 0.6] \cdot \begin{bmatrix} 0.376 & 0.624 \\ 0.312 & 0.688 \end{bmatrix}$$

$$= [0.3376 \quad 0.6624]$$

Classification of states and chains

1) A state j is accessible from a state i if $P_{ij}^{(n)} > 0$.

2) Two states are said to be communicate if each state is accessible from other i.e; $P_{ij}^{(n)} > 0$ and $P_{ji}^{(n)} > 0$.

3) A state is called essential state if it communicates with every other state.

4) A state i is said to be absorbing state if no state i is ~~said to be~~ accessible from 'i' and 'i' absorbing if $P_{ii} = 1$.

5) A markov chain is absorbing if

(i) Chain has atleast one absorbing state.

(ii) It is possible to go from a non-absorbing state to absorbing state in one or more steps.

6) A state 'i' of a markov chain is called a return state if $P_{ii}^{(n)} > 0$ for some 'n'.

7) The period of a return state 'i' is defined as

$$d_i = GCD\{m \mid P_{ii}^{(m)} > 0\}.$$

8) A state 'i' is said to be periodic with period d_i, if $d_i > 1$ and 'aperiodic' if $d_i = 1$.

9) If the prob that a chain starting from 'i' return to 'i' in less than 1 i.e; return to state 'i' is uncertain then state 'i' is said to be 'transient'.

10) If the prob that a chain starting from state 'i' returns to 'i' is '1' i.e; return to state 'i' is certain then state 'i' is said to be '<u>recurrent</u>'.

4) The transition prob matrix is given by $P = \begin{bmatrix} 0.1 & 0.4 & 0.5 \\ 0.2 & 0.2 & 0.6 \\ 0.7 & 0.2 & 0.1 \end{bmatrix}$

and $P_0 = [0.4 \ 0.4 \ 0.2]$

i) Find the distribution after three transitions

ii) Find the limiting prob.

Given, $P = \begin{bmatrix} 0.1 & 0.4 & 0.5 \\ 0.2 & 0.2 & 0.6 \\ 0.7 & 0.2 & 0.1 \end{bmatrix}$ and $P_0 = R(0) = [0.4 \ 0.4 \ 0.2]$

(i) Distribution after 3 transitions,

$$R(3) = R(0) \cdot P^3$$

$$= [0.4 \ 0.4 \ 0.2] \begin{bmatrix} 0.326 & 0.288 & 0.386 \\ 0.292 & 0.296 & 0.412 \\ 0.422 & 0.236 & 0.342 \end{bmatrix}$$

$$= [0.3316 \ 0.2808 \ 0.3876]$$

(ii) Limiting prob,

Let $R = [x \ y \ z]$

WKT, $RP = R$

$$\Rightarrow [x \ y \ z] \begin{bmatrix} 0.1 & 0.4 & 0.5 \\ 0.2 & 0.2 & 0.6 \\ 0.7 & 0.2 & 0.1 \end{bmatrix} = [x \ y \ z]$$

$$\Rightarrow [0.1x + 0.2y + 0.7z \quad 0.4x + 0.2y + 0.2z \quad 0.5x + 0.6y + 0.1z] = [x \ y \ z]$$

① $\Rightarrow 0.1x + 0.2y + 0.7z = x$

$\Rightarrow -0.9x + 0.2y + 0.7z = 0$

② $\Rightarrow 0.4x + 0.2y + 0.2z = y$

$\Rightarrow 0.4x - 0.8y + 0.2z = 0$

③ $0.5x + 0.6y + 0.1z = z$

$\Rightarrow 0.5x + 0.6y - 0.9z = 0$

also, WKT, $x + y + z = 1$

By solving, $x = \frac{6}{17}$, $y = \frac{23}{85}$, $z = \frac{32}{85}$

$$\therefore R = \left[\frac{6}{17} \ \frac{23}{85} \ \frac{32}{85}\right]$$

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6) A house-wife buys three kinds of cereals: A, B, C. She never buys the same cereal on successive weeks. If she buys cereal A, then the next week she buys cereal B. However, if she buys B or C, then next week she is 3 times as likely to buy A as the other brand. Find the transition matrix, in the long run, how often she buys each of three brands?

By given,

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 3/4 & 0 & 1/4 \\ 3/4 & 1/4 & 0 \end{bmatrix} \end{array}$$

If $B \Rightarrow P(A) = 3P(C)$

If $C \Rightarrow P(A) = 3P(B)$

Now, in long run

Let $R = [x \ y \ z]$

WKT, $RP = R$

$$\Rightarrow [x \ y \ z] \begin{bmatrix} 0 & 1 & 0 \\ 3/4 & 0 & 1/4 \\ 3/4 & 1/4 & 0 \end{bmatrix} = [x \ y \ z]$$

$\Rightarrow \frac{3}{4}y + \frac{3}{4}z = x$ $\quad \Rightarrow x + \frac{1}{4}z = y$ $\quad \Rightarrow \frac{1}{4}y = z$

$\Rightarrow 4x - 3y - 3z = 0$ $\quad \Rightarrow 4x - 4y + z = 0$ $\quad \Rightarrow y - 4z = 0$

Also wkt, $x + y + z = 1$

By solving, $x = \frac{3}{7}$, $y = \frac{16}{35}$, $z = \frac{4}{35}$

$\therefore R = [0.4285 \quad 0.4571 \quad 0.1142]$

In long run she buys,

cereal A – 42.85% of time

" B – 45.71% "

" C – 11.42% "