i.e., $3k^4 - 7k \ldots 2 \ldots - 14k^3 + 10k^2 - 1 = 0$ $\ldots \frac{\;}{2}$

$\therefore \; k = 0.452$

Example 9 : A continuous random variable X has the distribution function

$$F(x) = \begin{cases} 0, \text{ if } \; x \le 1 \\ k(x-1)^4, \text{ if } \; 1 < x \le 3 \\ 1, \text{if } \; x > 3 \end{cases}$$

Determine (*i*) $f(x)$ (*ii*) k (*iii*) Mean **[JNTU 2004S, 2007S, (A) Apr. 2012 (Set No. 2)]**

<u>**Solution**</u> : (*i*) We know that $f(x) = \dfrac{d}{dx}[F(x)]$

$$\therefore f(x) = \begin{cases} 0, \text{if } x \le 1 \\ 4k(x-1)^3, \text{if } 1 < x \le 3 \\ 0, \text{if } x > 3 \end{cases}$$

(*ii*) Since total probability is unity, we have

$$\int_1^3 f(x)\,dx = 1 \text{ i.e., } 4k \int_1^3 (x-1)^3\,dx = 1$$

$$\text{i.e., } 4k\left[\frac{(x-1)^4}{4}\right]_1^3 = 1 \quad \text{i.e., } k\,(16-0) = 1 \quad \text{or} \quad k = \frac{1}{16}$$

$$\text{Hence } f(x) = \begin{cases} 0, \text{ if } x \le 1 \\ \frac{1}{4}(x-1)^3, \text{ if } 1 < x \le 3 \\ 0, \text{ if } x > 3 \end{cases}$$

(*iii*) Mean of X, $\mu = \int_{-\infty}^{\infty} x\,f(x)\,dx$

$$= \int_{-\infty}^{1} x.f(x)\,dx + \int_1^3 x.f(x)\,dx + \int_3^{\infty} x.f(x)\,dx$$

$$= 0 + \int_1^3 x.\frac{1}{4}(x-1)^3\,dx + 0 = \frac{1}{4}\int_1^3 x(x-1)^3\,dx$$

$$= \frac{1}{4}\int_0^2 (t+1)\,t^3\,dt \quad (\text{Putting } x - 1 = t)$$

$$= \frac{1}{4}\int_0^2 (t^4 + t^3)\,dt = \frac{1}{4}\left(\frac{t^5}{5} + \frac{t^4}{4}\right)_0^2$$

$$= \frac{1}{4}\left(\frac{2^5}{5} + \frac{2^4}{4}\right) = \frac{2^4}{4}\left(\frac{2}{5} + \frac{1}{4}\right)$$

$$= 4\left(\frac{13}{20}\right) = \frac{13}{5} = 2.6$$

Example 5 : A continuous random vaiable has the probability density function

$$f(x) = \begin{cases} k\,x\,e^{-\lambda x}, \text{ for } x \geq 0, \lambda > 0 \\ 0, \text{ otherwise} \end{cases}$$

Determine (*i*) k (*ii*) Mean (*iii*) Variance

[JNTU (A) Dec. 2009, Nov. 2010, Dec. 2011, (H) May 2011, Nov. 2012, (K) May 2013 (Set No. 1)]

Solution : (*i*) Since the total probability is unity, we have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$i.e., \int_{-\infty}^{0} 0.\,dx + \int_{0}^{\infty} kx\,e^{-\lambda x} dx = 1 \quad i.e., \; k\int_{0}^{\infty} x\,e^{-\lambda x} dx = 1$$

$$i.e., \; k\left[x\left(\frac{e^{-\lambda x}}{-\lambda}\right) - 1.\left(\frac{e^{-\lambda x}}{\lambda^2}\right)\right]_0^{\infty} = 1$$

$$i.e., \; k\left[(0-0) - \left(0 - \frac{1}{\lambda^2}\right)\right] = 1 \text{ or } k = \lambda^2$$

Now $f(x)$ becomes

$$f(x) = \begin{cases} \lambda^2 x e^{-\lambda x}, \text{ for } x \geq 0, \lambda > 0 \\ 0, \text{ otherwise} \end{cases}$$

(*ii*) Mean of the distribution, $\mu = \int_{-\infty}^{\infty} x\,f(x)\,dx$

$$i.e., \mu = \int_{-\infty}^{0} 0.\,dx + \int_{0}^{\infty} x.\lambda^2 x\,e^{-\lambda x} dx = \lambda^2 \int_{0}^{\infty} x^2\,e^{-\lambda x} dx$$

$$= \lambda^2\left[x^2\left(\frac{e^{-\lambda x}}{-\lambda}\right) - 2x\left(\frac{e^{-\lambda x}}{\lambda^2}\right) + 2\left(\frac{e^{-\lambda x}}{-\lambda^3}\right)\right]_0^{\infty}, \text{ using Bernoulli's Rule}$$

$$= \lambda^2\left[(0-0+0) - (0-0-\frac{2}{\lambda^3})\right] = \frac{2}{\lambda}$$

(*iii*) Variance of the distribution, $\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$

$$i.e., \sigma^2 = \int_{0}^{\infty} x^2 f(x)\,dx - \left(\frac{2}{\lambda}\right)^2 = \lambda^2 \int_{0}^{\infty} x^3 e^{-\lambda x} dx - \frac{4}{\lambda^2} \text{ (Apply Bernoulli's Rule)}$$

$$= \lambda^2 \left[x^3 \left(\frac{e^{-\lambda x}}{-\lambda} \right) - 3x^2 \left(\frac{e^{-\lambda x}}{\lambda^2} \right) + 6x \left(\frac{e^{-\lambda x}}{-\lambda^3} \right) - 6 \left(\frac{e^{-\lambda x}}{\lambda^4} \right) \right]_0^\infty - \frac{4}{\lambda^2}$$

$$= \lambda^2 \left[(0 - 0 + 0 - 0) - \left(0 - 0 + 0 - \frac{6}{\lambda^4} \right) \right] - \frac{4}{\lambda^2}$$

$$= \frac{6}{\lambda^2} - \frac{4}{\lambda^2} = \frac{2}{\lambda^2}$$

Example 1 : If a random variable has the probability density $f(x)$ as

$$f(x) = \begin{cases} 2e^{-2x}, \text{ for } x > 0 \\ 0, \text{ for } x \leq 0 \end{cases}$$ find the probabilities that it will take on a value

(*i*) between 1 and 3 (*ii*) greater than 0.5.

[JNTU 2001, 2006S (Set No. 4), (H) III yr. Nov. 2015]

Solution :

(*i*) The probability that a variate takes a value between 1 and 3 is given by

$$P(1 \leq X \leq 3) = \int_1^3 f(x)\,dx = \int_1^3 2e^{-2x}\,dx$$

$$= 2\left(\frac{e^{-2x}}{-2}\right)_1^3 = -(e^{-6} - e^{-2}) = e^{-2} - e^{-6}$$

(*ii*) The probability that a variable takes a value greater than 0.5 is

$$P(X \geq 0.5) = \int_{0.5}^{\infty} f(x)\,dx = \int_{0.5}^{\infty} 2e^{-2x}\,dx$$

$$= 2\left(\frac{e^{-2x}}{-2}\right)_{0.5}^{\infty} = -\left(e^{-\infty} - e^{-1}\right) = -(0 - e^{-1}) = e^{-1} = \frac{1}{e}$$

Example 2 : If the probability density of a random variable is given b

$$f(x) = \begin{cases} k(1 - x^2), \text{ for } 0 < x < 1 \\ 0, \text{ otherwise} \end{cases}$$