# PROBLEM SOLVING-2

Problem-1

**Exercise 6.3.13 (Ex. 36, p. 397)** Let

$$T(x, y, z) = (3x - 2y + z, 2x - 3y, y - 4z).$$

1. Write down the standard matrix of $T$.

**Solution:** with $\mathbf{e_1} = (1,0,0)^T, \mathbf{e_2} = (0,1,0)^T, \mathbf{e_3} = (0,0,1)^T$ we have (write/think everything as columns):

$$T(\mathbf{e_1}) = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \quad T(\mathbf{e_2}) = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}, \quad T(\mathbf{e_2}) = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}.$$

So, the standard matrix is

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 2 & -3 & 0 \\ 0 & 1 & -4 \end{bmatrix}.$$

Problem-2

$$T_1 : \mathbb{R}^2 \to \mathbb{R}^2, \quad T_1(x, y) = (x - 2y, 2x + 3y)$$

and

$$T_2 : \mathbb{R}^2 \to \mathbb{R}^2, \quad T_2(x, y) = (y, 0).$$

Compute the standard matrices of $T = T_2 o T_1$ and $T' = T_1 T_2$.

Solution :

First, compute the standard matrix of $T_1$. With $\mathbf{e_1} = (1,0)^T, \mathbf{e_2} = (0,1)^T$ we have (write/think everything as columns):

$$T_1(\mathbf{e_1}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad T_1(\mathbf{e_2}) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

So, the standard matrix of $T_1$ is

$$A_1 = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}.$$

Now, compute the standard matrix of $T_2$. With $\mathbf{e_1} = (1,0)^T, \mathbf{e_2} = (0,1)^T$ we have :

$$T_2(\mathbf{e_1}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad T_2(\mathbf{e_2}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So, the standard matrix of $T_2$ is

$$A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

$$A_2A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}. \quad So, \quad T(x,y) = (2x+3y, 0).$$

Similarly, the standard matrix of $T' = T_1T_2$ is

$$A_1A_2 = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}. \quad So, \quad T'(x,y) = (y, 2y).$$

Problem-3

**Exercise 6.3.15 (Ex. 46, p. 397)** Determine whether

$$T(x, y) = (x + 2y, x - 2y).$$

is invertible or not.

With $\mathbf{e_1} = (1,0)^T, \mathbf{e_2} = (0,1)^T$ we have :

$$T(\mathbf{e_1}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad T(\mathbf{e_2}) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

So, the standard matrix of $T$ is

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}.$$

Note, $\det A = -4 \neq 0$. So, $T$ is invertible and hence $T$ is invertible.

Problem-4

$$T(x, y) = (x - y, 0, x + y).$$

Use $B = \{\mathbf{v_1} = (1,2), \mathbf{v_2} = (1,1)\}$ as basis of the domain $\mathbb{R}^2$ and $B' = \{\mathbf{w_1} = (1,1,1), \mathbf{w_2} = (1,1,0), \mathbf{w_3} = (0,1,1)\}$ as basis of codomain $\mathbb{R}^3$. Compute matrix of $T$ with respect to $B, B'$.

$$T(\mathbf{u_1}) = T(1,2) = (-1,0,3), \quad T(\mathbf{u_2}) = T(1,1) = (0,0,2).$$

We solve the equation:

$$(-1,0,3) = a\mathbf{w_1} + b\mathbf{w_2} + c\mathbf{w_3} = a(1,1,1) + b(1,1,0) + c(0,1,1)$$

and we have

$$(-1,0,3) = 2(1,1,1) - 3(1,1,0) + 1(0,1,1) = \begin{bmatrix} \mathbf{w_1} & \mathbf{w_2} & \mathbf{w_3} \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}.$$

Similarly, we solve

$$(0,0,2) = a\mathbf{w_1} + b\mathbf{w_2} + c\mathbf{w_3} = a(1,1,1) + b(1,1,0) + c(0,1,1)$$

and we have

$$(0,0,2) = 2(1,1,1) - 2(1,1,0) + 0(0,1,1) = \left[\begin{array}{ccc} \mathbf{w_1} & \mathbf{w_2} & \mathbf{w_3} \end{array}\right] \left[\begin{array}{r} 2 \\ -2 \\ 0 \end{array}\right].$$

So, the matrix of $T$ with respect to the bases $B, B'$ is

$$A = \left[\begin{array}{rr} 2 & 2 \\ -3 & -2 \\ 1 & 0 \end{array}\right].$$