Matrix Representation of Linear Transformation

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Matrices for Linear Transformations

- Two representations of the linear transformation $T: R^3 \to R^3$:

$$(1)\ T(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, -x_1 + 3x_2 - 2x_3, 3x_2 + 4x_3)$$

$$(2)\ T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- Three reasons for matrix representation of a linear transformation:
  - It is simpler to write
  - It is simpler to read
  - It is more easily adapted for computer use

Standard matrix for a linear transformation

Let $T : R^n \to R^m$ be a linear trtansformation such that

$$T(\mathbf{e}_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad T(\mathbf{e}_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \cdots, \quad T(\mathbf{e}_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix},$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_\mathrm{n}\}$ is a standard basis for R^n. Then the $m \times n$ matrix whose n columns correspond to $T(\mathbf{e}_i)$, is the standard matrix representation of T

$$A = \left[T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ \cdots\ T(\mathbf{e}_n) \right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + v_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n$$

$$\begin{aligned} T \text{ is a linear transformation} \Rightarrow T(\mathbf{v}) &= T(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n) \\ &= T(v_1\mathbf{e}_1) + T(v_2\mathbf{e}_2) + \cdots + T(v_n\mathbf{e}_n) \\ &= v_1T(\mathbf{e}_1) + v_2T(\mathbf{e}_2) + \cdots + v_nT(\mathbf{e}_n) \end{aligned}$$

If $A = \left[T(\mathbf{e}_1) \; T(\mathbf{e}_2) \; \cdots \; T(\mathbf{e}_n) \right]$, then

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}$$

$$= v_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + v_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= v_1 T(\mathbf{e}_1) + v_2 T(\mathbf{e}_2) + \cdots + v_n T(\mathbf{e}_n)$$

Therefore, $T(\mathbf{v}) = A\mathbf{v}$ for each $\mathbf{v}$ in R^n

- Ex 1: Finding the standard matrix of a linear transformation

Find the standard matrix for the L.T. $T: R^3 \to R^2$ defined by

$$T(x, y, z) = (x - 2y, 2x + y)$$

Sol:

Vector Notation

$$T(\mathbf{e}_1) = T(1,\ 0,\ 0) = (1,\ 2)$$

$$T(\mathbf{e}_2) = T(0,\ 1,\ 0) = (-2,\ 1)$$

$$T(\mathbf{e}_3) = T(0,\ 0,\ 1) = (0,\ 0)$$

Matrix Notation

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$T(\mathbf{e}_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A = \left[T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ T(\mathbf{e}_3) \right]$$

$$= \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

- Check:

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-2y \\ 2x+y \end{bmatrix}$$

i.e., $T(x,y,z) = (x-2y, 2x+y)$

- Note: a more direct way to construct the standard matrix

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{matrix} \leftarrow & 1x-2y+0z \\ \leftarrow & 2x+1y+0z \end{matrix}$$

※ The first (second) row actually represents the linear transformation function to generate the first (second) component of the target vector

- Ex 2: Finding the standard matrix of a linear transformation

  The linear transformation $T: R^2 \to R^2$ is given by projecting each point in R^2 onto the x-axis. Find the standard matrix for T

  Sol:

$$T(x, y) = (x, 0)$$

$$A = \left[T(\mathbf{e}_1) \; T(\mathbf{e}_2) \right] = \left[T(1,\ 0) \; T(0,\ 1) \right] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

- Notes:

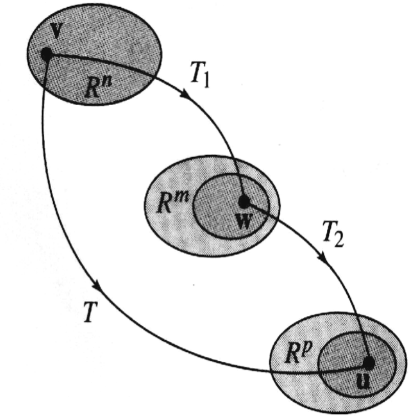
  (1) The standard matrix for the zero transformation from R^n into R^m is the $m \times n$ zero matrix

  (2) The standard matrix for the identity transformation from R^n into R^n is the $n \times n$ identity matrix I_n

- Composition of T_1:$R^n \rightarrow R^m$ with T_2:$R^m \rightarrow R^p$

$$T(\mathbf{v}) = T_2(T_1(\mathbf{v})), \quad \mathbf{v} \in R^n$$

This composition is denoted by $T = T_2 \circ T_1$



Composition of Transformations

Theorem : Composition of linear transformations

Let $T_1 : R^n \rightarrow R^m$ and $T_2 : R^m \rightarrow R^p$ be linear transformations with standard matrices A_1 and A_2 ,then

(1) The composition $T : R^n \rightarrow R^p$, defined by $T(\mathbf{v}) = T_2(T_1(\mathbf{v}))$, is still a linear transformation

(2) The standard matrix A for T is given by the matrix product $A = A_2 A_1$

Pf:

(1) (T is a linear transformation)

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in R^n and let c be any scalar. Then

$$T(\mathbf{u}+\mathbf{v}) = T_2(T_1(\mathbf{u}+\mathbf{v})) = T_2(T_1(\mathbf{u})+T_1(\mathbf{v}))$$
$$= T_2(T_1(\mathbf{u})) + T_2(T_1(\mathbf{v})) = T(\mathbf{u}) + T(\mathbf{v})$$
$$T(c\mathbf{v}) = T_2(T_1(c\mathbf{v})) = T_2(cT_1(\mathbf{v})) = cT_2(T_1(\mathbf{v})) = cT(\mathbf{v})$$

(2) (A_2A_1 is the standard matrix for T)

$$T(\mathbf{v}) = T_2(T_1(\mathbf{v})) = T_2(A_1\mathbf{v}) = A_2A_1\mathbf{v} = (A_2A_1)\mathbf{v}$$

- Note:

$$T_1 \circ T_2 \neq T_2 \circ T_1$$

- Ex 3: The standard matrix of a composition

Let T_1 and T_2 be linear transformations from R^3 into R^3 s.t.

$$T_1(x, y, z) = (2x + y, 0, x + z)$$

$$T_2(x, y, z) = (x - y, \text{z}, y)$$

Find the standard matrices for the compositions $T = T_2 \circ T_1$ and $T' = T_1 \circ T_2$

Sol:

$$A_1 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{(standard matrix for } T_1\text{)}$$

$$A_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{(standard matrix for } T_2\text{)}$$

The standard matrix for $T = T_2 \circ T_1$

$$A = A_2 A_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The standard matrix for $T' = T_1 \circ T_2$

$$A' = A_1 A_2 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

- Transformation matrix for nonstandard bases

Let V and W be finite-dimensional vector spaces with bases B and B', respectively, where $B=\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$

If $T: V \rightarrow W$ is a linear transformation s.t.

$$\left[T(\mathbf{v}_1)\right]_{B'}=\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \left[T(\mathbf{v}_2)\right]_{B'}=\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \cdots, \quad \left[T(\mathbf{v}_n)\right]_{B'}=\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

then the $m \times n$ matrix whose n columns correspond to $\left[T(v_i)\right]_{B'}$

$$A = \left[[T(\mathbf{v}_1)]_{B'} \ [T(\mathbf{v}_2)]_{B'} \ \cdots \ [T(\mathbf{v}_n)]_{B'} \right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is such that $\left[T(\mathbf{v})\right]_{B'} = A[\mathbf{v}]_B$ for every $\mathbf{v}$ in V

※The above result state that the coordinate of $T(\mathbf{v})$ relative to the basis B' equals the multiplication of A defined above and the coordinate of $\mathbf{v}$ relative to the basis B.

※ Comparing to the result in Thm. 6.10 ($T(\mathbf{v}) = A\mathbf{v}$), it can infer that the linear transformation and the basis change can be achieved in one step through multiplying the matrix A defined above (see the figure on 6.74 for illustration)

- Ex : Finding a matrix relative to nonstandard bases

Let $T: R^2 \to R^2$ be a linear transformation defined by

$$T(x_1, x_2) = (x_1 + x_2, 2x_1 - x_2)$$

Find the matrix of T relative to the basis $B = \{(1,\ 2),\ (-1,\ 1)\}$ and $B' = \{(1,\ 0),\ (0,\ 1)\}$

Sol:

$$T(1,2) = (3,0) = 3(1,0) + 0(0,1)$$

$$T(-1,1) = (0,-3) = 0(1,0) - 3(0,1)$$

$$\left[T(1,2)\right]_{B'} = \begin{bmatrix} 3 \\ 0 \end{bmatrix},\quad \left[T(-1,1)\right]_{B'} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

the matrix for T relative to B and B'

$$A = \left[\left[T(1,2)\right]_{B'} \quad \left[T(1,2)\right]_{B'}\right] = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$$

- Ex :

For the L.T. $T: R^2 \to R^2$ given in Example 5, use the matrix A to find $T(\mathbf{v})$, where $\mathbf{v} = (2,\ 1)$

Sol:

$$\mathbf{v} = (2, 1) = 1(1, 2) - 1(-1, 1) \qquad B = \{(1, 2), (-1, 1)\}$$

$$\Rightarrow [\mathbf{v}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow [T(\mathbf{v})]_{B'} = A[\mathbf{v}]_B = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\Rightarrow T(\mathbf{v}) = 3(1, 0) + 3(0, 1) = (3, 3) \qquad B' = \{(1, 0), (0, 1)\}$$

- Check:

$$T(2, 1) = (2 + 1, 2(2) - 1) = (3, 3)$$

- Note:

(2) If $T: V \rightarrow V$ is the identity transformation

$B=\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$: a basis for V

$\Rightarrow$ the matrix of T relative to the basis B

$$A=\Big[\big[T(\mathbf{v}_1)\big]_B \;\; \big[T(\mathbf{v}_2)\big]_B \;\; \cdots \;\; \big[T(\mathbf{v}_n)\big]_B\Big]=\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}=I_n$$

Example

- Let $T : P_1 \to P_2$ be the transformations defined by

$$T\,(p(x)) = xp(x).$$

Find the matrix for T with respect to the standard bases,

$$B = \{\mathbf{u}_1, \mathbf{u}_2\} \text{ and } B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\},$$

where $\mathbf{u}_1 = 1$, $\mathbf{u}_2 = x$; $\mathbf{v}_1 = 1$, $\mathbf{v}_2 = x$, $\mathbf{v}_3 = x^2$

- Solution:
  - $T(\mathbf{u}_1) = T(1) = (x)(1) = x$ and $T(\mathbf{u}_2) = T(x) = (x)(x) = x^2$
  - $[T\,(\mathbf{u}_1)]_{B'} = [0\ 1\ 0]^T$ $[T\,(\mathbf{u}_2)]_{B'} = [0\ 0\ 1]^T$
  - Thus, the matrix for T w.r.t. B and B' is

$$[T]_{B',B} = [[T(\mathbf{u}_1)]_{B'} \,|\, [T(\mathbf{u}_2)]_{B'}] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example

- Let $T : R^2 \to R^3$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{bmatrix}$$

- Find the matrix for the transformation T with respect to the bases
$B = \{\mathbf{u}_1,\mathbf{u}_2\}$ for R^2 and $B' = \{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ for R^3, where

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

- Solution:

$$T(\mathbf{u}_1) = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} = \mathbf{v}_1 - 2\mathbf{v}_3, \quad T(\mathbf{u}_2) = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = 3\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3$$

Example

$$T(\mathbf{u}_1) = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} = \mathbf{v}_1 - 2\mathbf{v}_3, \quad T(\mathbf{u}_2) = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = 3\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3$$

$$\Rightarrow \quad [T(\mathbf{u}_1)]_{B'} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad [T(\mathbf{u}_2)]_{B'} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \quad [T]_{B',B} = [[T(\mathbf{u}_1)]_{B'} \mid [T(\mathbf{u}_2)]_{B'}] = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$$