Newton-Raphson’s Method

Newton-Raphson method named after Isaac Newton and Joseph Raphson is a powerful technique for solving equations numerically. The Newton-Raphson method in one variable is implemented as follows:

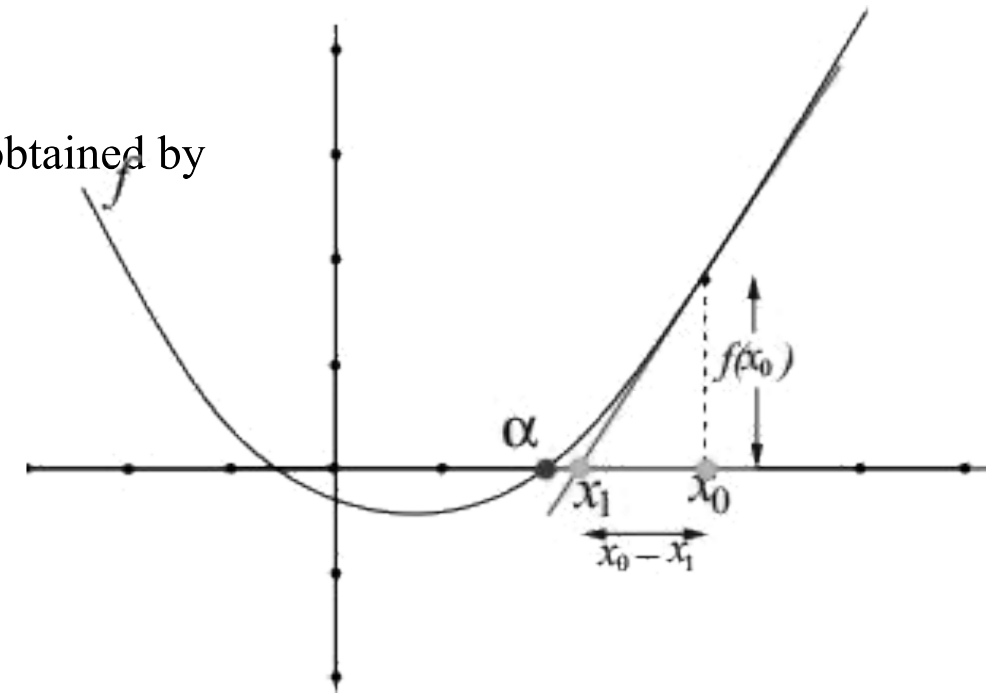
Let α be an exact root and x_0 be the initial approximate root of the equation $f(x) = 0$. First approximation x_1 is taken by drawing a tangent to curve $y = f(x)$ at the point $(x_0, f(x_0))$. If θ is the angle which tangent through the point $(x_0, f(x_0))$ makes with x- axis, then slope of the tangent is given by $\tan(\theta) = f(x_0)/ (x_0 - x_1) = f'(x_0)$

Hence, $x_1 = x_0 - (f(x_0) / f'(x_0))$

Similarly $x_2 = x_1 - (f(x_1) / f'(x_1))$

The required root to desired accuracy is obtained by drawing tangents to the curve at points $(x_n, f(x_n))$ successively.

Hence, $\boldsymbol{x_{n+1} = x_n - (f(x_n) / f'(x_n))}$



Remarks:

Newton-Raphson method can be used for solving both algebraic and transcendental equations.

Initial approximation x_0 can be taken randomly in the interval $[a, b]$, such that $f(a)f(b) < 0$.

Newton-Raphson method has **quadratic convergence**, but in case of bad choice of x_0 (the initial guess), Newton- Raphson method may fail to converge.

This method is useful in case of large value of $f'(x_n)$ i.e. when graph of $f(x)$ while crossing x -axis is nearly vertical

Newton-Raphson method

Algorithm:

Given an equation $f(x) = 0$

Step1. Take the initial approximation x_0 randomly in the domain of function $f(x)$.

Step2. Calculate $\boldsymbol{x_{n+1} = x_n - (f(x_n) / f'(x_n))}$ **, for n=0,1,2,3…**

Example 1: Use Newton-Raphson method to find a root of the equation $x^3 - 5x + 3 = 0$ correct to three decimal places.

Solution: $f(x) = x^3 - 5x + 3$ and $f'(x) = 3x^2 - 5$

Here $f(0) = 3$ and $f(1) = -1 \Rightarrow f(0)f(1) < 0$

Also fx is continuous on $[0,1]$, $\therefore$ atleast one root exists in $[0,1]$

Let initial approximation x_0 in the interval $[0,1]$ be 0.8 .

Iteration	x_n	$f(x_n)$	$f'(x_n)$
0	0.8	-0.488	-3.08
1	0.6416	0.0561	-3.7650
2	0.6565	0.0004	-3.7070
3	0.6566	0.00008	-3.7066

First three decimal places have been stabilized; hence **0.6566** is the real root correct to three decimal places.

Example 2: Find the approximate value of square root of 28 correct to 3 decimal places using Newton Raphson method .

Solution: $x^2 -28 = 0$

i.e., $f(x) = x^2 -28$ and $f'(x) = 2x$

Here $f(5) = -3$ and $f(6) = 8 \Rightarrow f(5)f(6) < 0$

Also $f(x)$ is continuous on [5,6], $\therefore$ atleast one root exists in [5,6]

Let initial approximation x_0 in the interval [5,6] be 5.5 .

Iteration	x_n	$f(x_n)$	$f'(x_n)$
0	5.5	2.25	11
1	5.2955	0.0423	10.519
2	5.2915	-0.00003	10.583
3	5.2915	-0.00002	10.583

First three decimal places have been stabilized; hence **5.2915** is the real root correct to three decimal places.

Exercise Problems:

1. Calculate the first 3 iterations of Newton Raphson method on $x^3 - 7x^2 + 14x - 6 = 0$ with the following initial approximation
i) $x_0 = 0.5$
ii) $x_0 = 2.8$
iii) $x_0 = 3.5$

2. Apply Newton Raphson to find a root of the equation $xe^x = 1$ correct to three decimal places (with initial approximation ($x_0 = 0.5$))

3. Calculate the first 5 iterations of the Newton Raphson method on $f(x) = xe^x - 2 = 0$ (with initial approximation ($x_0 = 3.5$))

4. Do 3 iterations of Newton Raphson method on $f(x) = x^2 - 5$ with initial approximation ($x_0 = 2.5$).
Or
Find the approximate value of square root of 5 using Newton Raphson method .
(Do 3 iterations with initial approximation ($x_0 = 2.5$)]).