19MAT 115 Discrete Mathematics

Relations

Ordered n-tuples

- An ordered n-tuple is an ordered sequence of n objects
- $(x_1, x_2, \ldots, x_n)$
  - First coordinate (or component) is x_1
  - …
  - n-th coordinate (or component) is x_n
- An ordered pair: An ordered 2-tuple
  - (x, y)
- An ordered triple: an ordered 3-tuple
  - (x, y, z)

Equality of tuples vs sets

- Two tuples are equal iff they are equal coodinate-wise
  - $(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$ iff
    $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$
- $(2, 1) \neq (1, 2)$, but $\{2, 1\} = \{1, 2\}$
- $(1, 2, 1) \neq (2, 1)$, but $\{1, 2, 1\} = \{2, 1\}$
- $(1, 2\text{-}2, a) = (1, 0, a)$
- $(1, 2, 3) \neq (1, 2, 4)$ and $\{1, 2, 3\} \neq \{1, 2, 4\}$

Cartesian products

- Let A_1, A_2, …A_n be sets
- The cartesian products of A_1, A_2, …A_n is
  - $A_1 \times A_2 \times \ldots \times A_n$
    $= \{ (x_1, x_2, \ldots, x_n) \mid x_1 \in A_1$ and $x_2 \in A_2$ and …
    and $x_n \in A_n)$
- Examples: A = {x, y}, B = {1, 2, 3}, C = {a, b}
- AxB={(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)}
- AxBxC = {(x, 1, a), (x, 1, b), …, (y, 3, a), (y, 3, b)}
- Ax(BxC) = {(x, (1, a)), (x, (1, b)), …, (y, (3, a)), (y, (3, b))}

Relations

- A relation R from the a set A to the set B is a set of ordered pairs such that $R \subseteq AxB$
  - Let x R y mean x is R-related to y
  - Let A be a set containing all possible x
  - Let B be a set containing all possible y

  Relation R can be treated as a set of ordered pairs

  $R = \{(x, y) \in AxB \mid x R y\}$

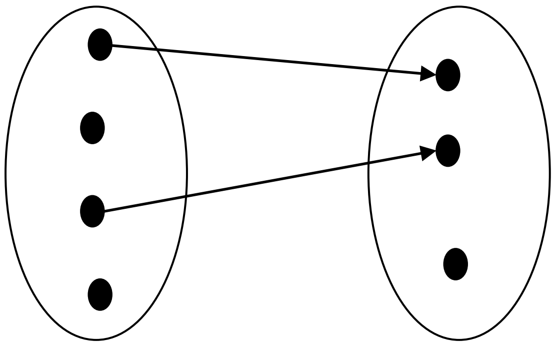
Relations are sets

- $R \subseteq AxB$ as a relation from A to B
- R is a relation from A to B iff $R \subseteq AxB$
  - Furthermore, x R y iff $(x, y) \in R$.
- If the relation R only involves two sets, we say it is a **binary relation**.
- We can also have an n-ary relation, which involves n sets.

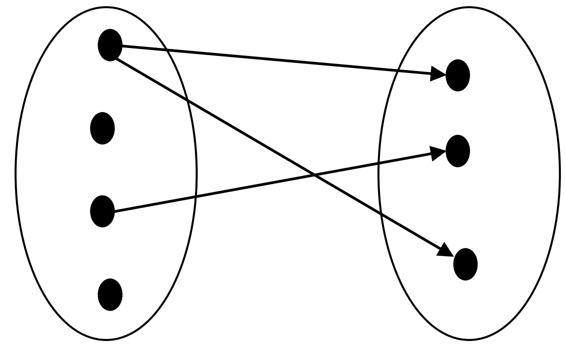
Various kinds of binary relations

- One-to-one relation: each first component and each second component appear only once in the relation.
- One-to-many relation: if some first component s_1 appear more than once.
- Many-to-one relation: if some second component s_2 is paired with more than one first component.
- Many-to-many relation: if at least one s_1 is paired with more than one second component and at least one s_2 is paired with more than one first component.

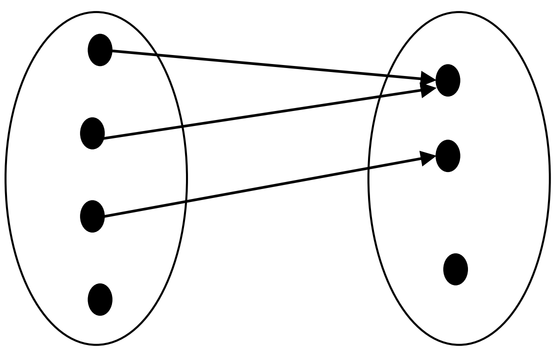
Visualizing the relations



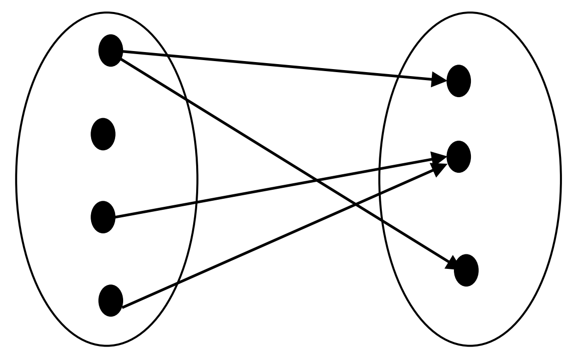
One-to-one



One-to-many



Many-to-one



Many-to-many

Binary relation on a set

- Given a set A, a binary relation R on A is a subset of AxA ($R \subseteq AxA$).
- An example:
  - A = {1, 2}. Then AxA={(1,1), (1,2), (2,1), (2,2)}. Let R on A be given by $x \, R \, y \leftrightarrow x+y$ is odd.
  - then, $(1, 2) \in R$, and $(2, 1) \in R$

Properties of Relations: Reflexive

- Let R be a binary relation on a set A.
  - R is reflexive: iff for all $x \in A$, $(x, x) \in R$.
- Reflexive means that every member is related to itself.
- Example: Let A = {2, 4, a, b}
  - R = {(2, 2), (4, 4), (a, a), (b, b)}
  - S = {(2, b), (2, 2), (4, 4), (a, a), (2, a), (b, b)}
- R, S are reflexive relations on A.
- Another example: the relation $\leq$ is reflexive on the set Z_+.

Symmetric relations

- A relation R on a set A is symmetric iff for all $x, y \in A$, if $(x, y) \in R$ then $(y, x) \in R$.
- Example: $A = \{1, 2, b\}$
  - $R = \{(1, 1), (b, b)\}$
  - $S = \{(1, 2)\}$
  - $T = \{(2, b), (b, 2), (1, 1)\}$
- R, T are symmetric relations on A.
- S is not a symmetric relation on A.
- The relation $\leq$ is reflexive on the set Z_{+}, but not symmetric. E.g., $3 \leq 4$ is in, but not $4 \leq 3$

Anti-symmetric relations

- A relation R on a set A is anti-symmetric iff for all x, y $\in$ A. if (x, y) $\in$ R and (y, x) $\in$ R then x = y.
- Example: A = {1, 2, b}
  - R = {(1, 1), (b, b)}
  - S = {(1, 2)}
  - T = {(2, b), (b, 2), (1, 1)}
- R, S are anti-symmetric relations on A.
- T is not an anti-symmetric relation on A.
- The relation $\leq$ is reflexive on the set $Z_{+,}$ but not symmetric. It is anti-symmetric.

Transitive relations

- A relation R on a set A is transitive iff for all $x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.
- Example: $A = \{1, 2, b\}$
  - $R = \{(1, 1), (b, b)\}$
  - $S = \{(1, 2), (2, b), (1, b)\}$
  - $T = \{(2, b), (b, 2), (1, 1)\}$
- R, S are transitive relations on A.
- T is not a transitive relation on A.
- The relation $\leq$ is reflexive on the set $Z_{+,}$ but not symmetric. It is also anti-symmetric, and transitive (why?).

Transitive closure

- Let R be a relation on A
- The smallest transitive relation on A that includes R is called the transitive closure of R.
- Example: $A = \{1, 2, b\}$
  - $R = \{(1, 1), (b, b)\}$
  - $S = \{(1, 2), (2, b), (1, b)\}$
  - $T = \{(2, b), (b, 2), (1, 1)\}$
- The transitive closures of R and S are themselves
- The transitive closure of T is $T \cup \{(2, 2), (b, b)\}$

Equivalence relations

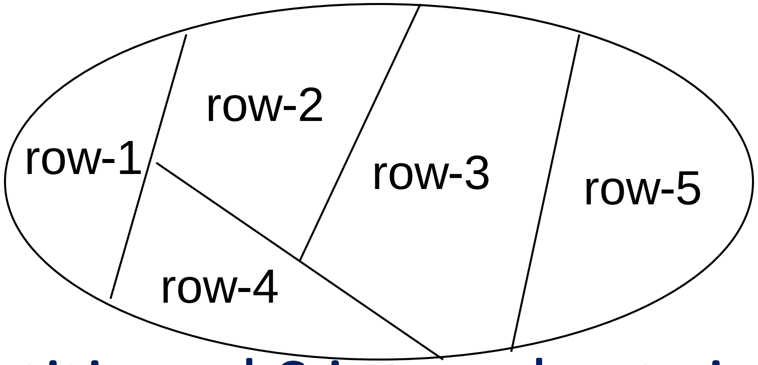
- A relation on a set A is an equivalence relation if it is
  - Reflexive.
  - Symmetric
  - Transitive.
- Examples of equivalence relations
  - On any set S, $x \mathrel{R} y \leftrightarrow x = y$
  - On integers ≥ 0, $x \mathrel{R} y \leftrightarrow x+y$ is even
  - On the set of lines in the plane, $x \mathrel{R} y \leftrightarrow x$ is parallel to y.
  - On $\{0, 1\}$, $x \mathrel{R} y \leftrightarrow x = y^2$
  - On $\{1, 2, 3\}$, $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$.

Congruence relations are equivalence relations

- We say x is congruent modulo m to y
  - That is, x C y iff m divides x-y, or x-y is an integral multiple of m.
  - We also write $x \equiv y \pmod{m}$ iff x is congruent to y modulo m.
- Congruence modulo m is an equivalent relation on the set Z.
  - Reflexive: m divides x-x = 0
  - Symmetry: if m divides x-y, then m divides y-x
  - Transitive: if m divides x-y and y-z,

    then m divides (x-y)+(y-z) = x-z

- Let us look at the equivalence relation:
  - S = {x | x is a student in our class}
  - x R y ↔ "x sits in the same row as y"
- We group all students that are related to one another. We can see this figure:



- We have partitioned S into subsets in such a way that everyone in the class belongs to one and only one subset.

Partition of a set

- A partition of a set S is a collection of nonempty disjoint subsets (S_1, S_2, .., S_n) of S whose union equals S.
  - $S_1 \cup S_2 \cup \ldots \cup S_n = S$
  - If $i \neq j$ then $S_i \cap S_j = \varnothing$ ($S_i \cap S_j$ are disjoint)
- Examples, let A = {1, 2, 3, 4}
  - {{1}, {2}, {3}, {4}} a partition of A
  - {{1, 2}, {3, 4}} a partition of A
  - {{1, 2, 3}, {4}} a partition of A
  - {{}, {1, 2, 3}, {4}} not a partition of A
  - {{1, 2}, {3, 4}, {1, 4}} not a partition of A

Equivalent classes

- Let R be an equivalence relation on a set A.
  - Let $x \in A$
- The equivalent class of x with respect to R is:
  - $R[x] = \{y \in A \mid (x, y) \in R\}$
  - If R is understood, we write [x] instead of R[x].
- Intuitively, [x] is the set of all elements of A to which x is related.

Theorems on equivalent relations and partitions

Theorem 1: An equivalence relation R on a set A determines a partition of A.

- i.e., the distinctive equivalence classes of R form a partition of A.

Theorem 2: a partition of a set A determines an equivalence relation on A.

- i.e., there is an equivalence relation R on A such that the set of equivalence classes with respect to R is the partition.

An equivalent relations induces a partition

- Let A = {0, 1, 2, 3, 4, 5}
- Let R be the congruence modulo 3 relation on A
- The set of equivalence classes is:
  - {[0], [1], [2], [3], [4], [5]} =
    {{0, 3}, {1, 4}, {2, 5}, {3, 0}, {4, 1}, {5, 2}} =
    {{0, 3}, {1, 4}, {2, 5}}
- Clearly, {{0, 3}, {1, 4}, {2, 5}} is a partition of A.

An partition induces an equivalent relation

- Let $A = \{0, 1, 2, 3, 4, 5\}$
- Let a partition $P = \{\{0, 5\}, \{1, 2, 3\}, \{4\}\}$
- Let R =

  $\{\{0, 5\} \times \{0, 5\} \cup \{1, 2, 3\} \times \{1, 2, 3\} \cup \{4\} \times \{4\}\}$

  $= \{(0, 0), (0, 5), (5, 0), (5, 5), (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 4)\}$
- It is easy to verify that R is an equivalent relation.

Partial order

- A binary relation R on a set S is a partial order on S iff R is
  - Reflexive
  - Anti-symmetric
  - Transitive
- We usually use $\leq$ to indicate a partial order.
- If R is a partial order on S, then the ordered pair (S, R) is called a **partially ordered set** (also known as **poset**).
- We denote an arbitrary partially ordered set by (S, $\leq$).

Examples

- On a set of integers, $x \; R \; y \leftrightarrow x \leq y$ is a partial order ($\leq$ is a partial order).
- for integers, a, b, c.
  - $a \leq a$ (reflexive)
  - $a \leq b$, and $b \leq a$ implies $a = b$ (anti-symmetric)
  - $a \leq b$ and $b \leq c$ implies $a \leq c$ (transitive)
- Other partial order examples:
  - On the power set P of a set, $A \; R \; B \leftrightarrow A \subseteq B$
  - On Z_+, $x \; R \; y \leftrightarrow x$ divides y.
  - On $\{0, 1\}$, $x \; R \; y \leftrightarrow x = y^2$

Partially ordered sets

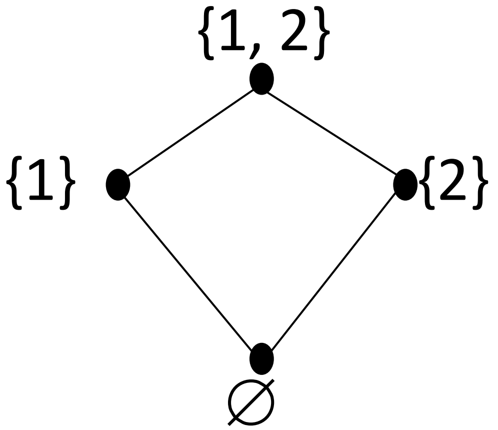
- Let $(S, \leq)$ be a partially ordered set
- If $x \leq y$, then either $x = y$ or $x \neq y$.
- If $x \leq y$, but $x \neq y$, we write $x < y$ and say that x is a **predecessor** of y, or y is a **successor** of x.
- A given y may have many predecessors, but if $x < y$ and there is no z with $x < z < y$, then x is an immediate predecessor of y.

Hasse diagram

- Let S be a finite set.
- Each of the element of S is represented as a dot (called a **node**, or **vertex**).
- If x is an immediate predecessor of y, then the node for y is placed above node x, and the two nodes are connected by a straight-line segment.
- The Hasse diagram of a partially ordered set conveys all the information about the partial order.
- We can reconstruct the partial order just by looking at the diagram

Example:

- $\subseteq$ on the power set P({1, 2}):
  - Poset: (P({1, 2}), $\subseteq$)
- P({1, 2}) = {∅, {1}, {2}, {1, 2}}
- $\subseteq$ consists of the following ordered pairs
  (∅, ∅), ({1}, {1}), ({2}, {2}), ({1, 2}, {1, 2}),
  (∅, {1}), (∅, {2}), (∅, {1, 2}), ({1}, {1, 2}),
  ({2}, {1, 2})



Total orders

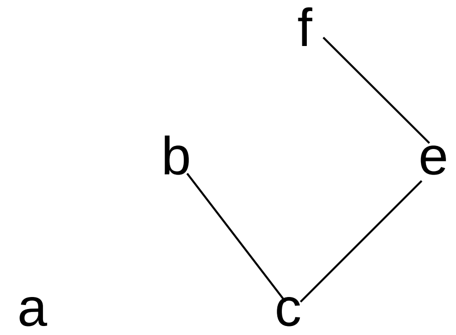
- A partial order on a set is a **total order** (also called **linear order**) iff any two members of the set are related.
- The relation $\leq$ on the set of integers is a total order.
- The Hasse diagram for a total order is on the right

Least element and minimal element

- Let $(S, \leq)$ be a poset. If there is a $y \in S$ with $y \leq x$ for all $x \in S$, then y is a **least element** of the poset. If it exists, is unique.
- An element $y \in S$ is **minimal** if there is no $x \in S$ with $x < y$.
- In the Hasse diagram, a least element is below all orders.
- A minimal element has no element below it.
- Likewise we can define **greatest element** and **maximal element**

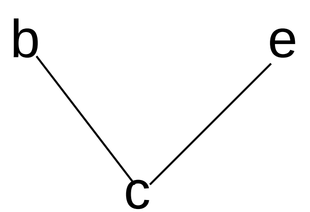
Examples:

- Consider the poset:

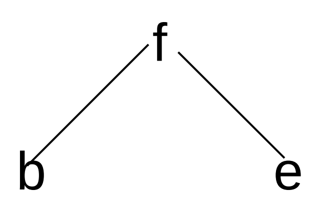


- The maximal elements are a, b, f
- The minimal elements are a, c.

A least element but no greatest element



A greatest element but no least element



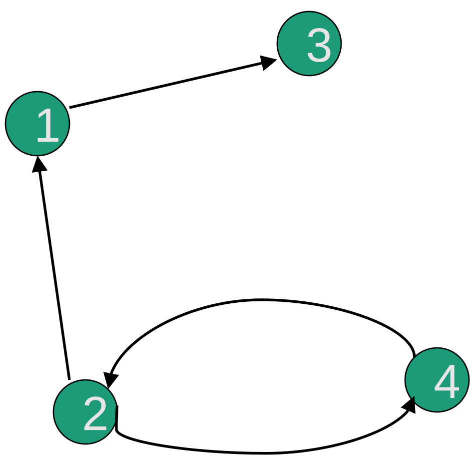
Warshall's Algorithm for Transitive Closure

Algorithm Warshall(A[1..n,1..n])
$R^{(0)} \leftarrow A$
for $k \leftarrow 1$ to n do
    for $i \leftarrow 1$ to n do
        for $j \leftarrow 1$ to n do
            $R^{(k)}[i,j] \leftarrow R^{(k-1)}[i,j]$ or $R^{(k-1)}[i,k]$ and $R^{(k-1)}[k,j]$
return $R^{(k)}$

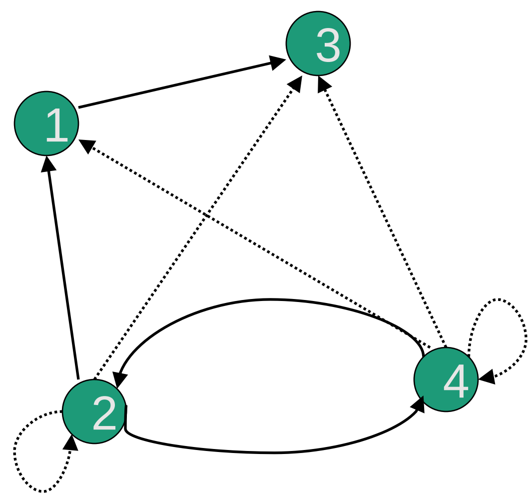
Space and Time efficiency

Warshall's Algorithm: Transitive Closure

- **Computes the transitive closure of a relation**
- **Alternatively: all paths in a directed graph**
- **Example of transitive closure:**



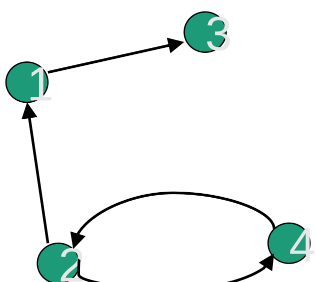
$$\begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$$



$$\begin{matrix} 0 & 0 & 1 & 0 \\ 1 & \mathbf{1} & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \\ \mathbf{1} & 1 & \mathbf{1} & \mathbf{1} \end{matrix}$$

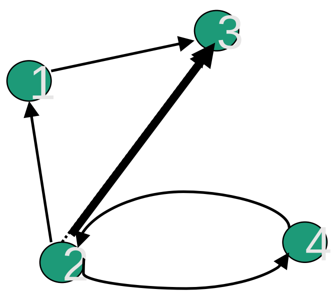
Warshall's Algorithm

- Main idea: a path exists between two vertices i, j, iff
  - there is an edge from i to j; or
  - there is a path from i to j going through vertex 1; or
  - there is a path from i to j going through vertex 1 and/or 2; or
  - ...
  - there is a path from i to j going through any of the other vertices



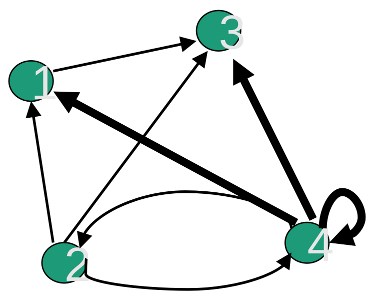
R_0

$$\begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$$



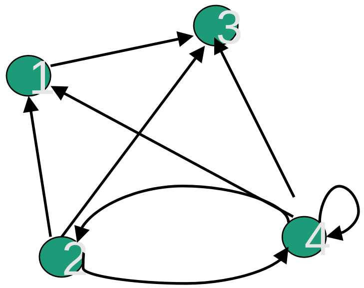
R_1

$$\begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$$



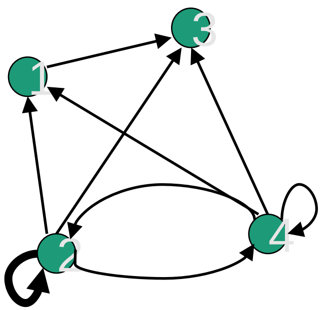
R_2

$$\begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ \mathbf{1} & 1 & \mathbf{1} & \mathbf{1} \end{matrix}$$



R_3

$$\begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{matrix}$$



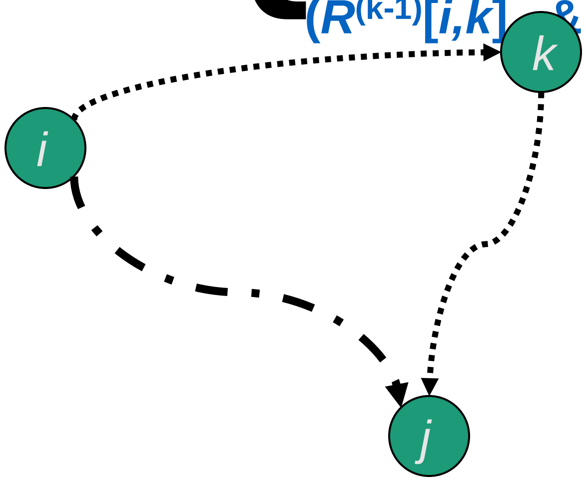
R_4

$$\begin{matrix} 0 & 0 & 1 & 0 \\ 1 & \mathbf{1} & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{matrix}$$

Warshall's Algorithm

In the k^{th} stage find if a path exists between two vertices i, j using just vertices among 1,…,k

$$R^{(k)}[i,j] = \begin{cases} R^{(k-1)}[i,j] & \text{(path using just } 1,\ldots,k\text{-}1) \\ \text{or} & \\ (R^{(k-1)}[i,k] \;\&\; R^{(k-1)}[k,j]) & \text{(path from } i \text{ to } k \text{ and from } k \text{ to } i \text{ using just } 1,\ldots,k\text{-}1) \end{cases}$$



k^{th} stage