Similar Matrices

- Transformation matrix for nonstandard bases

Let V and W be finite-dimensional vector spaces with bases B and B', respectively, where $B=\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$

If $T: V \rightarrow W$ is a linear transformation s.t.

$$\left[T(\mathbf{v}_1)\right]_{B'}=\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \left[T(\mathbf{v}_2)\right]_{B'}=\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \cdots, \quad \left[T(\mathbf{v}_n)\right]_{B'}=\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

then the $m \times n$ matrix whose n columns correspond to $\left[T(v_i)\right]_{B'}$

$$A = \left[[T(\mathbf{v}_1)]_{B'} \ [T(\mathbf{v}_2)]_{B'} \ \cdots \ [T(\mathbf{v}_n)]_{B'} \right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is such that $\left[T(\mathbf{v})\right]_{B'} = A[\mathbf{v}]_B$ for every $\mathbf{v}$ in V

A is denoted by $[T\}_{B,B'}$

※The above result state that the coordinate of $T(\mathbf{v})$ relative to the basis B' equals the multiplication of A defined above and the coordinate of $\mathbf{v}$ relative to the basis B.

※ Comparing to the result in Thm. 6.10 ($T(\mathbf{v}) = A\mathbf{v}$), it can infer that the linear transformation and the basis change can be achieved in one step through multiplying the matrix A defined above (see the figure on 6.74 for illustration)

Matrices of Linear Operators

In the special case where $V = W$, it is usual to take $B = B'$ when constructing a matrix for T. In this case the resulting matrix is called the ***matrix for T with respect to the basis B and is denoted by $[T]_B$***

Simple Matrices for Linear Operators

- Consider the linear operator $T : R^2 \to R^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ -2x_1 + 4x_2 \end{bmatrix}$ and the standard basis $B = \{\mathbf{e}_1, \mathbf{e}_2\}$ for R^2.
- The matrix for T with respect to this basis is the standard matrix for T;
  that is, $[T]_B = [T] = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2)]$.
- Since $T(\mathbf{e}_1) = [1\ {-2}]^T$, $T(\mathbf{e}_2) = [1\ 4]^T$, we have

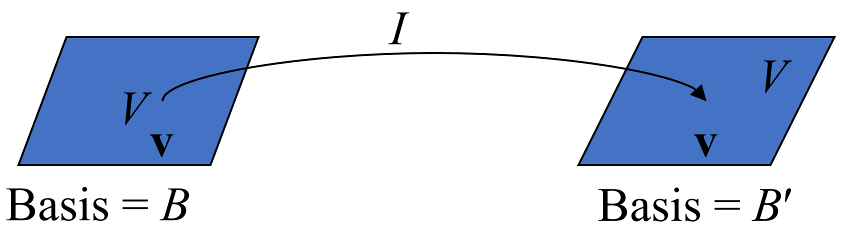
$$[T]_B = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$

- However, if $\mathbf{u}_1 = [1\ 1]^T$, $\mathbf{u}_2 = [1\ 2]^T$, then the matrix for T with respect to the basis $B' = \{\mathbf{u}_1, \mathbf{u}_2\}$ is the diagonal matrix

$$[T]_{B'} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Theorem :

- If B and B' are bases for a finite-dimensional vector space V, and if $I : V \to V$ is the identity operator, then $[I]_{B',B}$ is the transition matrix from B to B'.



$[I]_{B',B}$ is the transition matrix from B to B'.

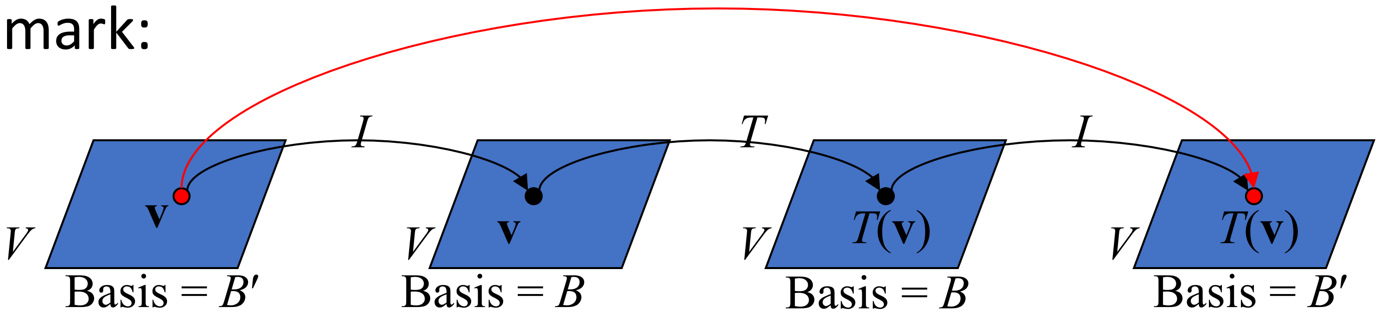
Theorem:

- Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V, and let B and B' be bases for V. Then

$$[T]_{B'} = P^{-1}[T]_B P$$

where P is the transition matrix from B' to B.

- Remark:



$$[T]_{B'} = [I]_{B',B}[T]_B[I]_{B,B'} = P^{-1}[T]_B P$$

Example:

- Let $T : R^2 \to R^2$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ -2x_1 + 4x_2 \end{bmatrix}$$

  Find the matrix T with respect to the standard basis $B = \{\mathbf{e}_1, \mathbf{e}_2\}$ for R^2, then use Theorem 8.5.2 to find the matrix of T with respect to the basis $B' = \{\mathbf{u}_1', \mathbf{u}_2'\}$, where $\mathbf{u}_1' = [1\ 1]^T$ and $\mathbf{u}_2' = [1\ 2]^T$.

- Solution:

$$[T]_B = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \qquad P = [I]_{B.B'} = [[\mathbf{u}_1']_B \mid [\mathbf{u}_2']_B]$$

$$\Rightarrow \quad P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\Rightarrow \quad [T]_{B'} = P^{-1}[T]_B P = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Definitions

- Definition
  - If A and B are square matrices, we say that B is similar to A if there is an invertible matrix P such that $B = P^{-1}AP$

- Definition
  - A property of square matrices is said to be a similarity invariant or invariant under similarity if that property is shared by any two similar matrices.

Similarity Invariants

Property	Description
Determinant	A and $P^{-1}AP$ have the same determinant.
Invertibility	A is invertible if and only if $P^{-1}AP$ is invertible.
Rank	A and $P^{-1}AP$ have the same rank.
Nullity	A and $P^{-1}AP$ have the same nullity.
Trace	A and $P^{-1}AP$ have the same trace.
Characteristic polynomial	A and $P^{-1}AP$ have the same characteristic polynomial.
Eigenvalues	A and $P^{-1}AP$ have the same eigenvalues
Eigenspace dimension	If λ is an eigenvalue of A and $P^{-1}AP$ then the eigenspace of A corresponding to λ and the eigenspace of $P^{-1}AP$ corresponding to λ have the same dimension.

Determinant of A Linear Operator

- Two matrices representing the same linear operator $T : V \to V$ with respect to different bases are similar.
- For any two bases B and B' we must have

$$\det([T]_B) = \det([T]_{B'})$$

- Thus we define the determinant of the linear operator T to be

$$\det(T) = \det([T]_B)$$

where B is any basis for V.

- Example
  - Let $T : R^2 \to R^2$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ -2x_1 + 4x_2 \end{bmatrix}$$

$$[T]_B = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \Rightarrow \det(T) = 6$$

$$[T]_{B'} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow \det(T) = 6$$

Eigenvalues of a Linear Operator

- A scalar λ is called an eigenvalue of a linear operator $T : V \rightarrow V$ if there is a nonzero vector $\mathbf{x}$ in V such that $T\mathbf{x} = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is called an eigenvector of T corresponding to λ.

- Equivalently, the eigenvectors of T corresponding to λ are the nonzero vectors in the kernel of $\lambda I - T$. This kernel is called the eigenspace of T corresponding to λ.

Eigenvalues of a Linear Operator

- If V is a finite-dimensional vector space, and B is any basis for V, then

  - The eigenvalues of T are the same as the eigenvalues of $[T]_B$.

  - A vector **x** is an eigenvector of T corresponding to $[T]_B$ if and only if its coordinate matrix $[\mathbf{x}]_B$ is an eigenvector of $[T]_B$ corresponding to λ.

Example

- Find the eigenvalues and bases for the eigenvalues of the linear operator $T : P_2 \rightarrow P_2$ defined by

$$T\,(a + bx + cx^2) = -2c + (a + 2b + c)x + (a + 3c)x^2$$

- Solution:

$$[T]_B = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

  - The eigenvalues of T are $\lambda = 1$ and $\lambda = 2$
  - The eigenvectors of $[T]_B$ are:
    - $\lambda = 2$: $\mathbf{u}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\quad \lambda = 1$: $\mathbf{u}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

Example

- Let $T : R^3 \to R^3$ be the linear operator given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -2x_3 \\ x_1 + 2x_2 + x_3 \\ x_1 + 3x_3 \end{bmatrix}$$

  Find a basis for R^3 relative to which the matrix for T is diagonal.

- Solution:
  - $\det(\lambda I - A) = \lambda^3 - 5\lambda^2 + 8\lambda - 4 = (\lambda-2)(\lambda-2)(\lambda-1)$

$$\Rightarrow \quad [T]_{B'} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$