19CSE 212: Data Structures and Algorithms

Lecture 7:Trees

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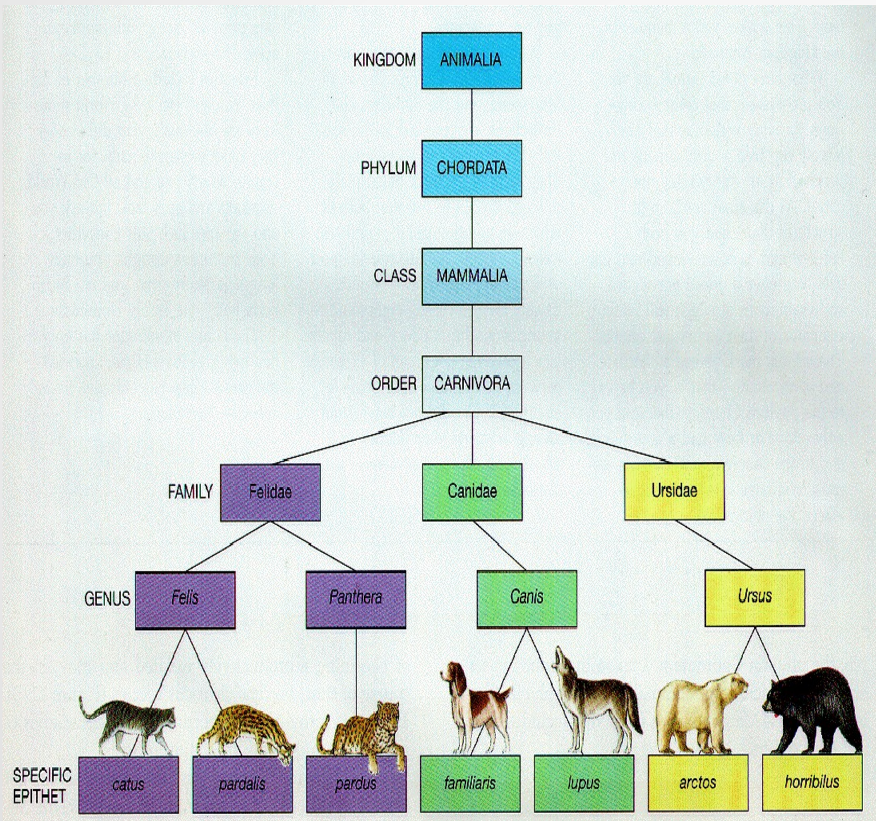
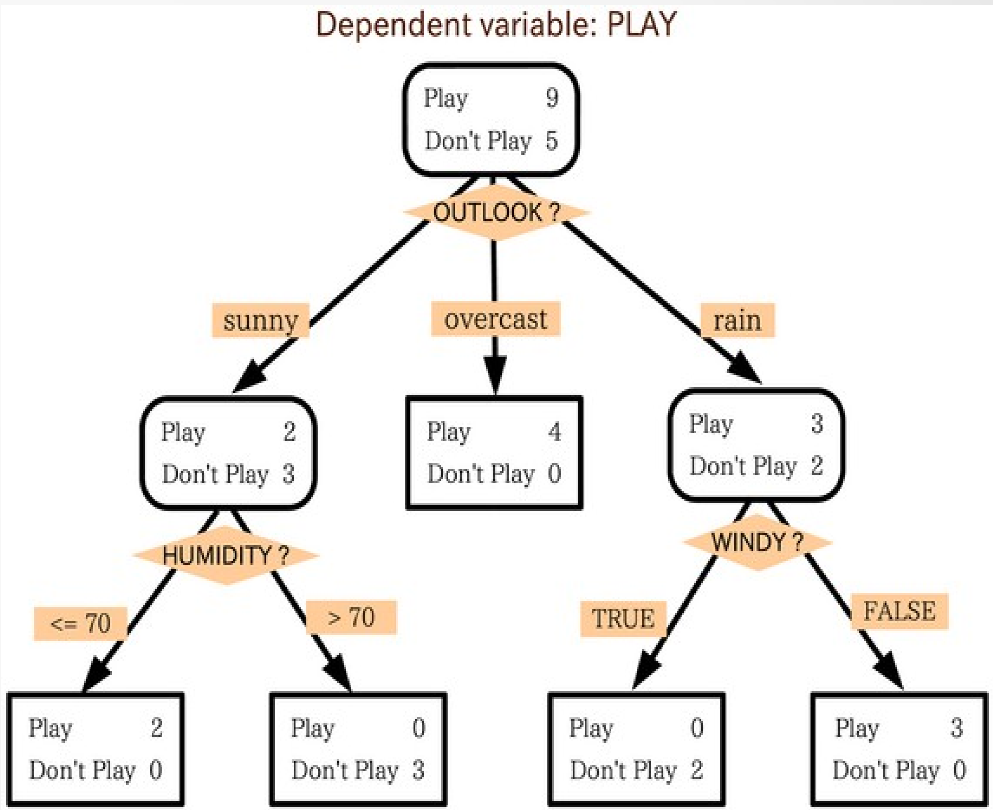
Non Linear Data Structures

- Represent relationships more richer than “before” and “after” in sequences
- Examples
  - Trees
  - Hash Table
  - Dictionaries
  - Skip Lists

Basic Concept

- Root : Primary category, starting point etc
- Children: Subcategory, descendants etc





http://ridge.icu.ac.jp/gen-ed/classif-gifs/animal-class-example.gif

Applications/Scenarios Trees Model

- Trees capture situations where branching happens
- Hierarchies
  - Captures types subtype relationships
  - e.g Family trees or hierarchies
  - Animal or Plant Taxonomies
- Decisions
  - Root denotes the primary start point, and every branch leads to an alternate decision
- Organization into ranges
  - Each child represents one range

Trees: Basic Definitions

- Tree is an abstract model of a hierarchical structure
- A tree *T* is a set of ***nodes*** storing elements in a ***parent-child*** relationship with the following properties
  - *T* has special node *r* called the "***root***" which has no parent node
  - Each node *v* of *T* different from *r* has a unique parent node *u*
  - If u is the parent node, *v* is the ***child*** of *u*
  - Two nodes that are children of the same parent are ***siblings***

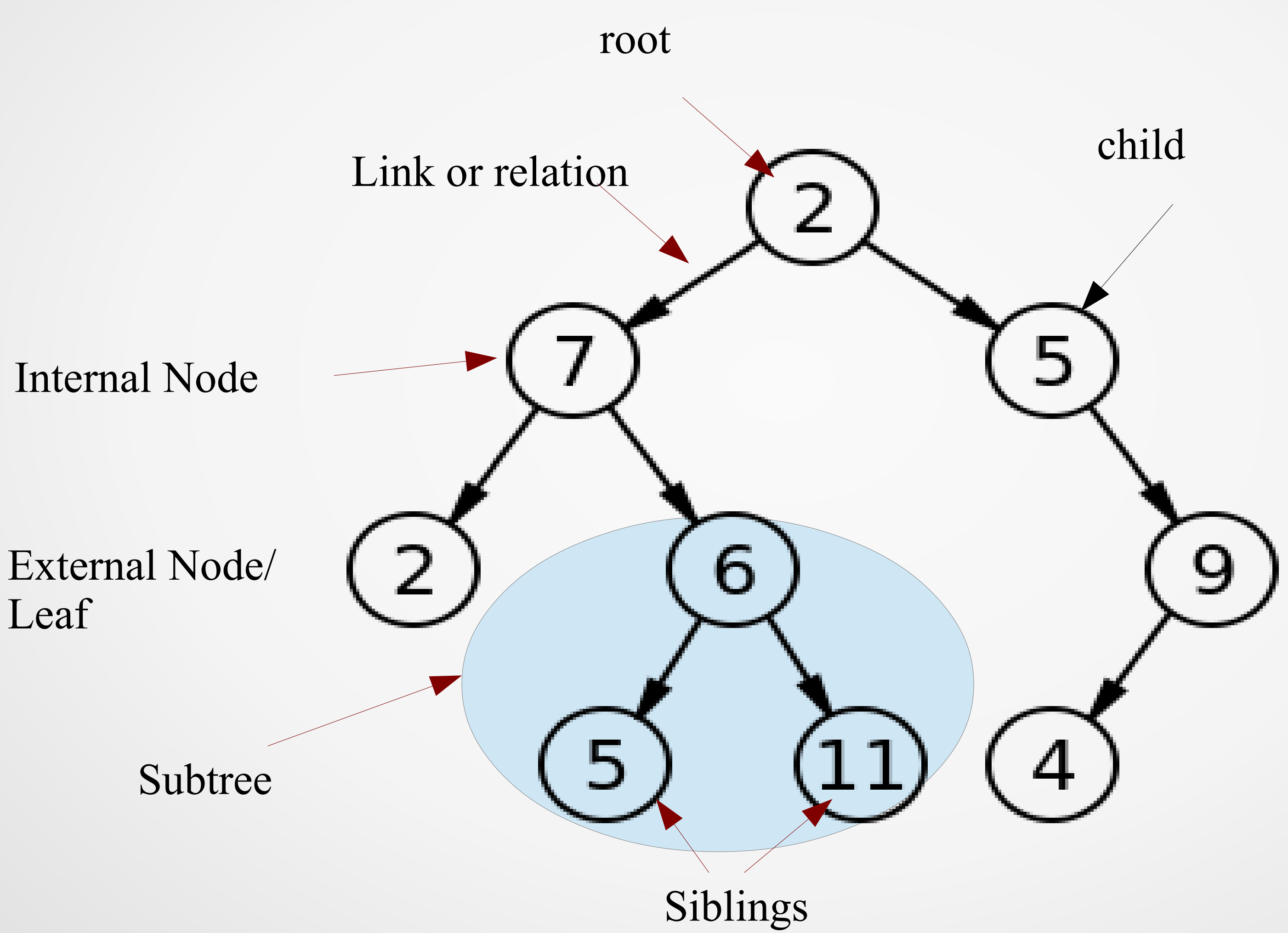
Trees: Basic Definitions

- ***External node***: node that has no children
  - Also called ***leaf*** node
- ***Internal node*** has one or more children
- ***Subtree*** of *T* rooted at a node *v* is the tree consisting of all the descendents of *v* in *T* (including *v*)
- ***Ancestor*** of a node is
  - Node itself or
  - ancestor of the parent of the node
  - *v* is ***descendent*** of *u* if *u* is ancestor of *v*

Trees: Basic Definition

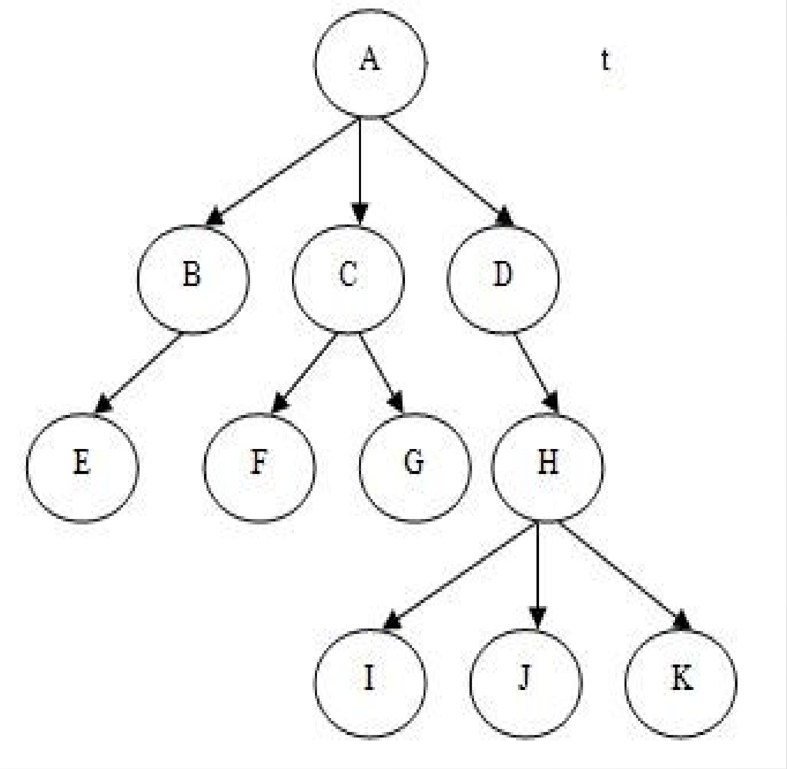
- ***Depth*** of a node
  - length of the path to the root
    - Indicates the level of the node
  - Depth of the root is 0, and is at level 0
- ***Height*** of a node
  - length of the longest downward path to a leaf from that node
  - Height of the tree is the height of the root
    - Equal to the depth of the deepest node in the tree
      - If depth of the deepest node is 3, height is 3

Trees: Basic Definitions



Exercise

- Consider the tree given in figure
  - Which node is the root
  - List the internal nodes
  - How many descendents does node C have
  - How many ancestors does node J and node G have
  - Which nodes are in the subtree rooted at D
  - What is the height of the tree
  - What is the depth of node F



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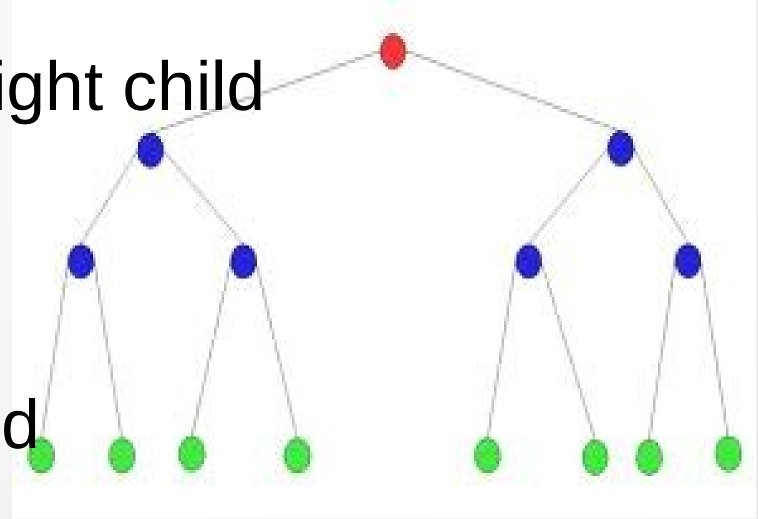
Ordered Trees

- A tree is ***ordered*** if there is a linear ordering defined for the children of each node
  - Each child of a node can be identified as the first, second, third etc
  - Usually done by arranging siblings left to right
- Ordered trees represent the linear order relationship between siblings
  - e.g in exercise : Children of node H can be ordered as follows
    - I, J, K

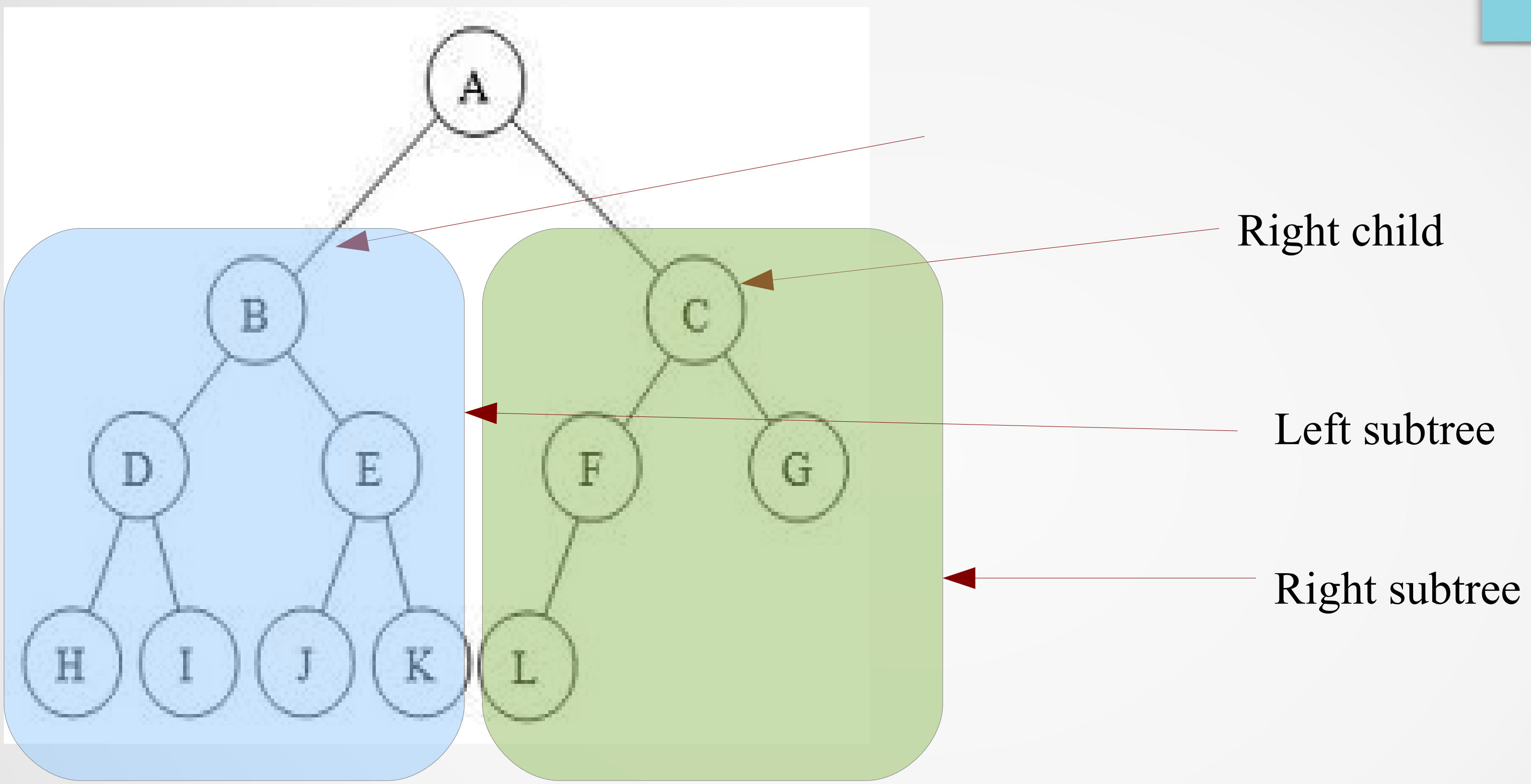
Binary Trees

- Ordered tree in which every node atmost two children
- Tree is proper if
  - Each node has either zero or two children
  - ie every internal node has exactly two children
  - Also known as full binary tree
- Each child is labeled as left child or right child
- Subtrees
  - Left subtree rooted at a left child
  - Right subtree rooted at a right child



Src: commons.wikimedia.org

Binary trees



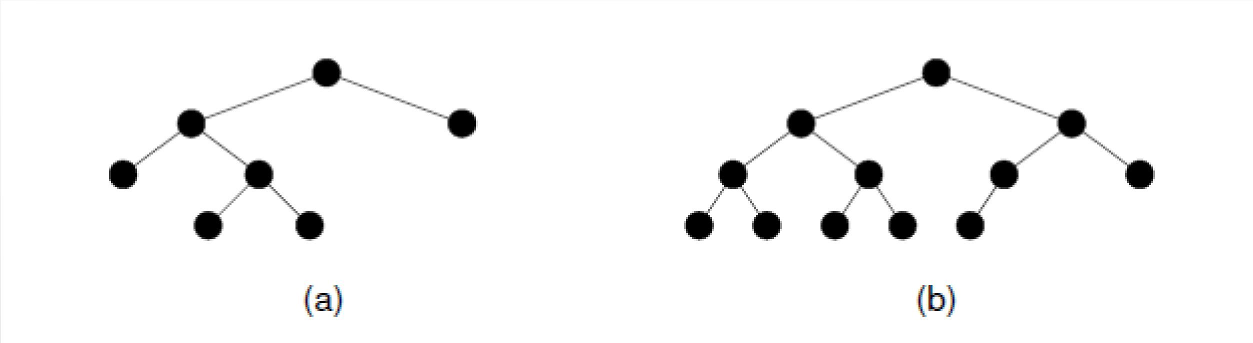
Src: web.cecs.pdx.edu

Binary Trees: Types

- Rooted Binary Tree
  - tree with a root node in which every node has at most two children.
- Full Binary Tree (Proper or strictly binary tree)
- Perfect Binary tree
  - full binary tree in which all leaves are at the same depth and in which every parent has two children
- Balanced Binary tree
  - depth of the two subtrees of every node differ by 1 or less

Binary Trees

- Complete Binary Trees
  - A complete binary tree has a restricted shape obtained by starting at the root and filling the tree by levels from left to right
    - all levels except possibly level d−1 are completely full



Src : Clifford A. Shaffer, A Practical Introduction to Data Structures and Algorithm Analysis

Binary Tree ADT

- Stores values at nodes, and are defined relative to the neighboring node
  - Relationships satisfy the parent child relationship
- Accessor Functions
  - root(): returns the root of the tree
  - parent(v): returns the parent of the node v; returns error if v is root.
  - children(v): returns an iterator of the children of node v
    - children can be stored in an array or list
      - For instance left child is at index 0, right at index 1

Binary Tree ADT

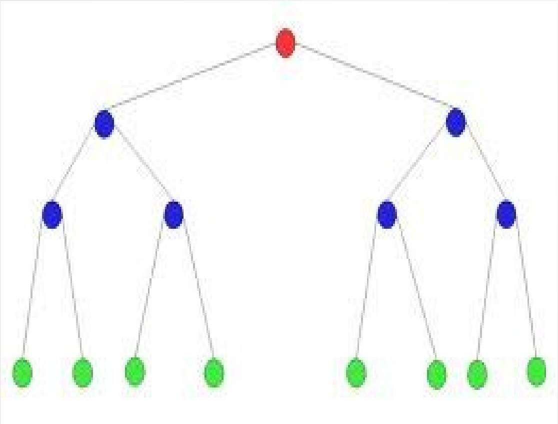
- Query Functions
  - isInternal(*v*): Test whether node *v* is internal; output is Boolean
  - isExternal(*v*): Test whether node *v* is external; output is Boolean
  - isRoot(*v*): Test whether node *v* is a root: output is Boolean
- Update Methods
  - swapElements(*v*, *w*): swap elements stored at nodes *v* and *w*
  - replaceElement(*v*, *e*): replace the element at node *v* with element *e*

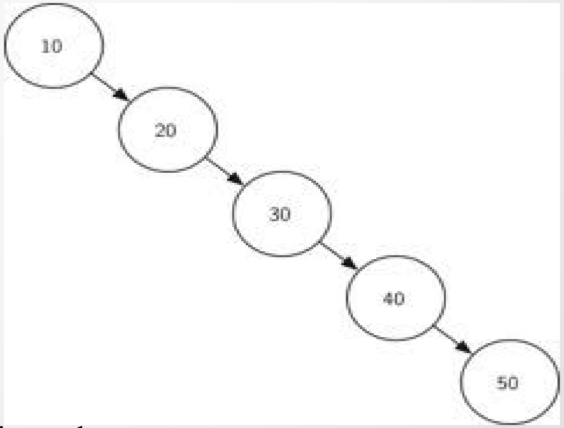
Binary Tree ADT

- Generic methods:
  - Size()
  - isEmpty()
  - elements(): returns an iterator of all elements stored at the nodes of the tree
  - nodes(): return an iterator of all nodes of the tree

Properties of a Binary Tree

- The number of nodes n in a perfect binary tree can be found as follows
  - $n = 2^{h+1} - 1$ where h is height of the tree
  - In figure $n = 2^{3+1} - 1 = 16-1 = 15$
- The number of nodes n in a binary tree of height h is
  - at least $n=h+1$ (2h+1 for a proper binary tree) and
    - See figure
  - atmost $n = 2^{h+1} - 1$ (in a perfect binary tree)





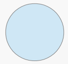
https://interactivepython.org

Properties of a Binary tree

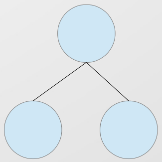
- Number of leaves in a perfect binary tree
  - $L = 2^h$, where h is height of the tree
  - Therefore $n = 2L-1$
- Number of internal nodes in a complete binary tree of n nodes
  - $\lfloor n/2 \rfloor$
- For any non-empty binary tree with n_0 leaf nodes and n_2 nodes of degree 2,
  - $n_0 = n_2 + 1$
- $h \le (n-1)/2$, and $\le$ number of internal nodes
- $h \ge \log_2 L$, and $h \ge \log_2 (n + 1) - 1$

Properties of Binary Trees

- Full Binary Tree Theorem
  - The number of leaves in a non-empty full binary tree is one more than the number of internal nodes
  - Proof: Use induction
    - Base Cases:
      - The non-empty tree with zero internal nodes has one leaf node.



      - A full binary tree with one internal node has two leaf nodes.
      - Thus, the theorem holds for the base cases ie for n = 0 and n = 1



Full Binary Tree Theorem Proof

- Induction Hypothesis
  - Assume that, any full binary tree *T* containing n − 1 internal nodes has n leaves
- Induction Step
  - Given *T* with *n* internal nodes, select an internal node *u* whose children are both leaf nodes
  - Remove both children of *u*, making it a leaf node, and let the resulting tree be *T'*
    - *T'* has *n*-1 internal nodes, and by the theorem, *n* leaves
  - Restore the children of , hence resulting in T with n internal nodes
    - Since *T'* has *n,* leaves, adding 2 makes it n+2
    - However node u which was earlier counted as leaf, now is an internal node
- Thus, tree T has n + 1 leaf nodes and n internal nodes. Hence proved

More Properties

- Theorem:
  - The number of empty subtrees in a non-empty binary tree is one more than the number of nodes in the tree.
- Proof
  - every node in binary tree T has two children, for a total of 2n children in a tree of n nodes
  - Every node except the root node has one parent, for a total of n − 1 nodes with parents
    - there are n − 1 non-empty children
    - Of the 2n children, the remaining n+1 children must be empty
  - Can also be proved using the Full Binary Tree Theorem

Algorithms on Trees

- Depth is recursively defined
  - If v is the root, depth of v is 0
  - Otherwise, depth of v is one plus depth of parent of v
- **Algorithm** depth(T,v):

  if T.isRoot(v) **then**

      return 0

  else

      return 1+depth(T,T.parent(v))
- Running time:
  - O(1+d_v), where is d_v depth of node v in T
  - Worst case O(n)

Algorithms on Trees

- Height is also recursively defined
  - If v is an external node, height of v is 0
  - Otherwise, height of v is one plus height of child of v
- **Algorithm** height(T,v): //for tree

  for each v in T.nodes() **do**

  if T.isExternal(v) **then**

  h = max(h,depth(T,v))

  return h
- Running time: $O(n + \sum_{e \in E} (1 + d_v))$

  ,where is d_v depth of node v in T
  - Worst case O(n^2)

Algorithms on Trees

- **Algorithm** height2(*T*,*v*):

  if *T*.isExternal(*v*) **then**

      return 0

  else

      h = 0

      for each *w* in *T*.children(*v*) **do**

          h = max(*h*,height2(*T*,*w*))

      return 1+*h*

- Running time:

  $O(\sum_{v \in T}(1+c_v))$, where is c_v is order of computation of function children(*v*)

  Worst case O(*n*) ie each node, except root is a child of another node $\sum_{v \in T}(c_v)=n-1$

Exercise: Properties of Binary trees

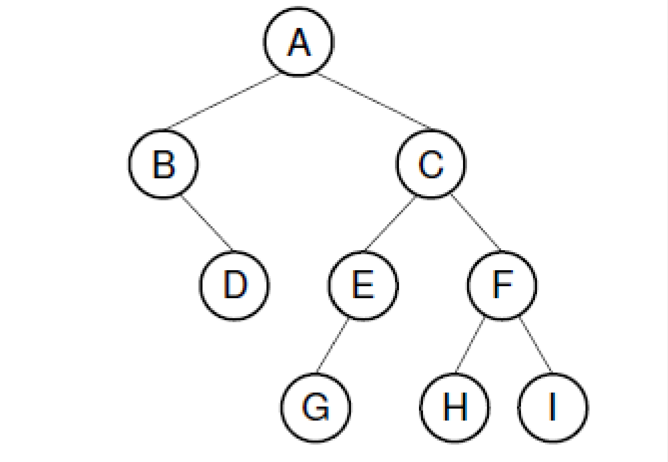
- Draw a binary tree with height 4 and maximum number of external nodes
  - What is the minimum number of external nodes for a binary tree with height h. Justify
  - What is the maximum number of external nodes for a binary tree with height h. Justify
  - Let T be a binary tree with height h and n nodes. Show that
    - $\log(n+1)-1 \leq h \leq (n-1)/2$
  - For which values of n and h can the above lower and upper bounds on h be attained with equality

Tree Traversal

- A traversal of a tree *T* is a systematic way of visiting or accessing all the nodes of *T*
- Types of Traversals
  - Preorder traversal
    - Node is visited before its descendents
  - Postorder traversal
    - Node is visited after its descendents
  - Inorder Traversal
    - node is visited after its left subtree and before its right subtree

Preorder Traversal

- First node accessed/visited is the root or parent
- Then nodes of the left subtree are visited (in preorder) before any node of the right subtree



- **Algorithm** preorder(*T*,v)

  visit(*v*)

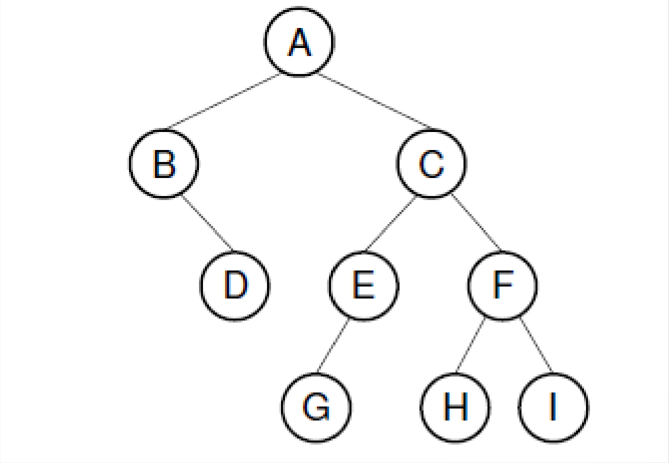
  for each child *w* of *v* **do**

  preorder(*T*,*w*)
- For example shown
  - A B D C E G F H I

Postorder Traversal

- First node is visited after its descendants
- Used commonly compute space used by files in a directory and

  its subdirectories



- **Algorithm** postorder(*T*,v)

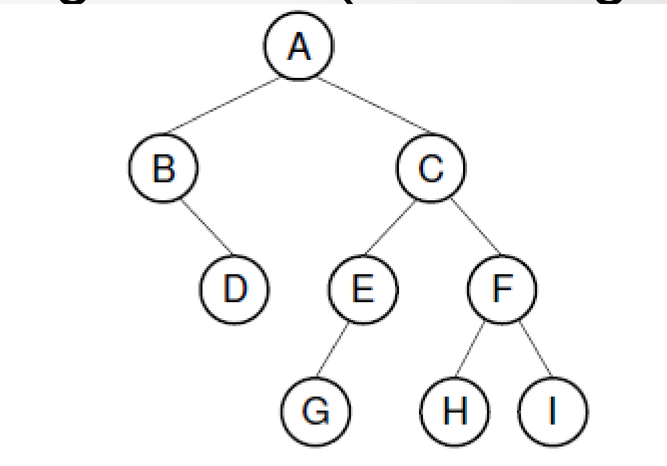
  for each child *w* of *v* **do**

  postorder(*T*,*w*)

  visit(*v*)
- For example shown
  - D B G E H I F C A

Inorder Traversal

- First visit the left child (including its entire subtree)
- Then visit the node, and finally visit the right child (including its entire subtree)



- **Algorithm** inorder(*T*,v)

  if isInternal (*v*)

  inOrder (leftChild (*v*))

  visit(*v*)

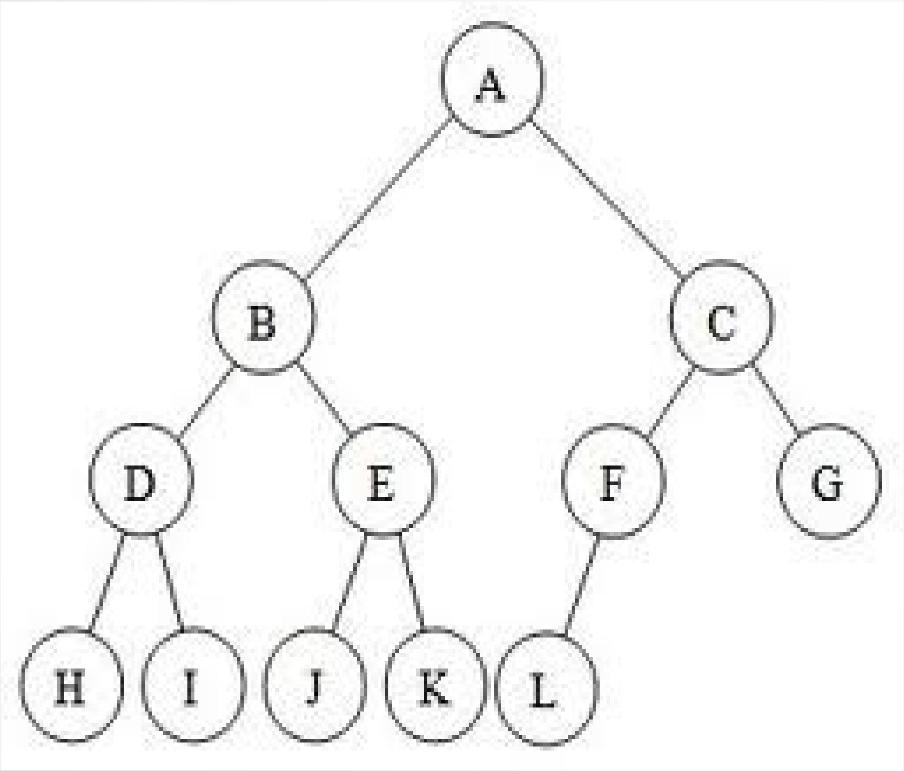
  if isInternal (*v*)

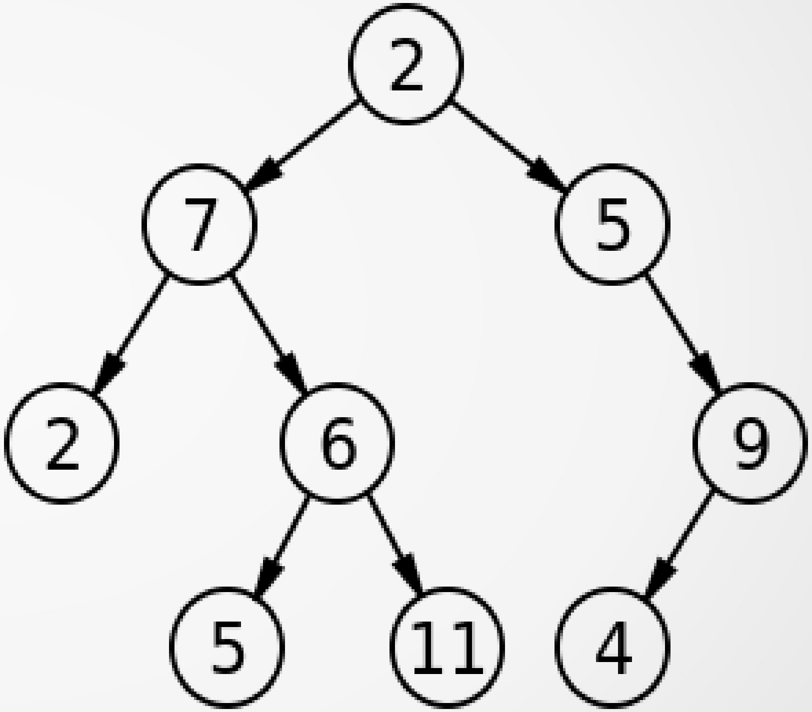
  inOrder (rightChild (*v*))

- For example shown
  - B D A G E C H F I

Exercise

- Show the different traversals on the following trees



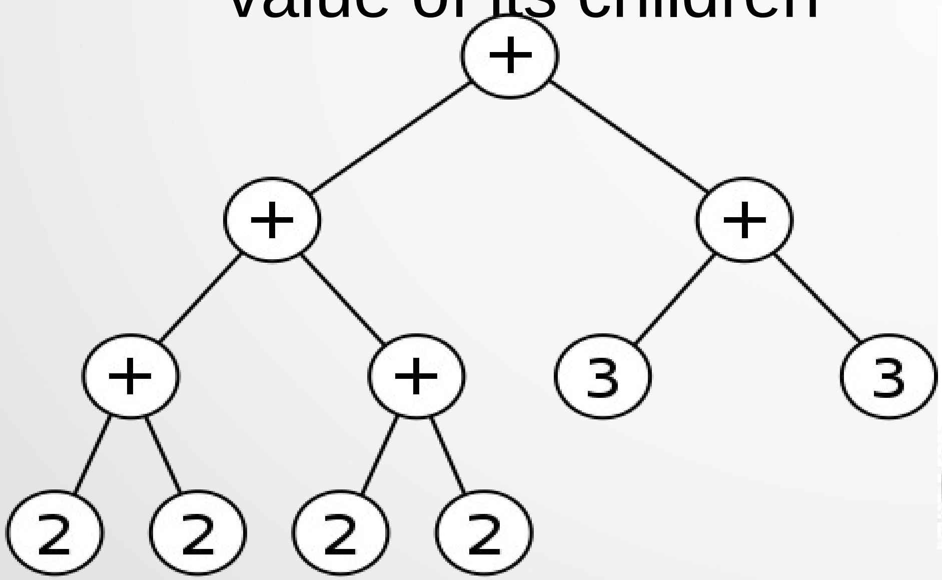


Exercise

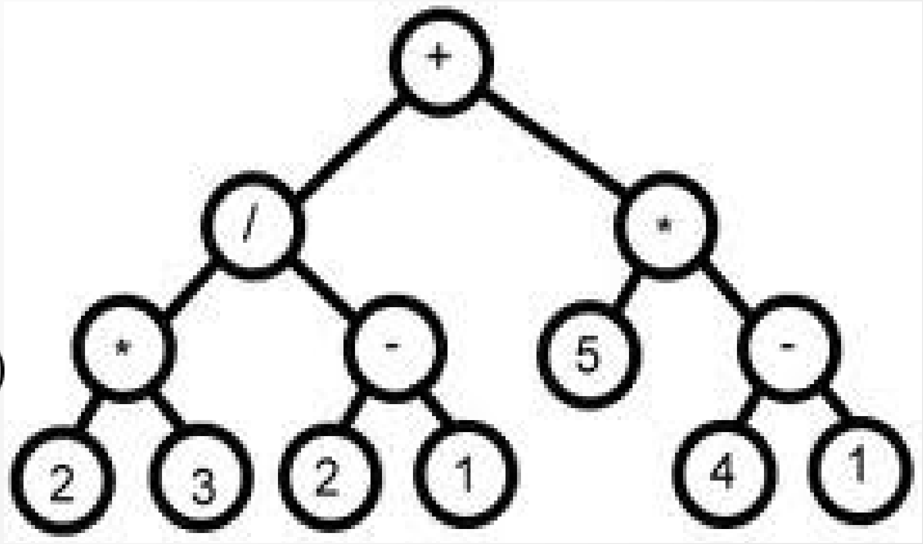
- Draw a binary tree T such that
  - Each internal node of T stores a single character
  - A preorder traversal of T yields EXAMFUN
  - An inorder traversal of T yields MAFXUEN

Application: Binary Expression Tree

- Arithmetic expression can be represented by a binary tree
  - External nodes are variables or constants
  - Internal nodes are operators
    - Its value is defined by applying its operation to the value of its children



Src: commons.wikimedia.org

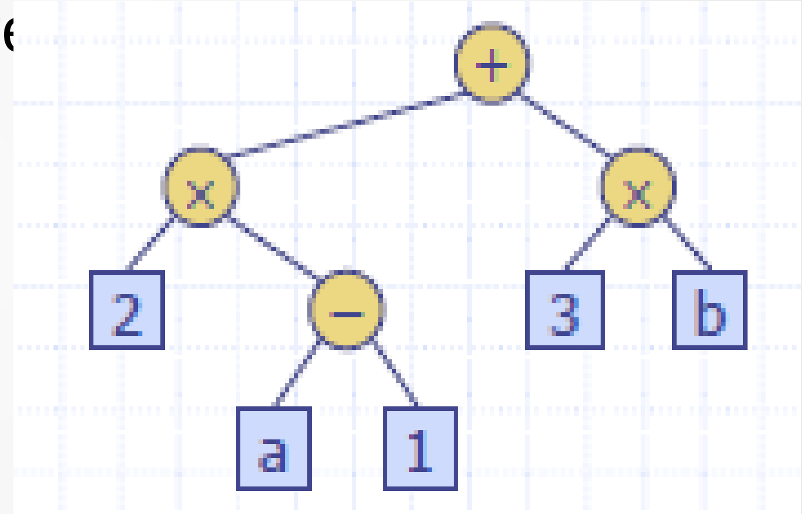


Src: public.arnau-sanchez.com

Printing Arithmetic Expressions

- Uses specialization of inorder traversal
  - print operand or operator when visiting node
  - print “(“ before traversing left subtree
  - print “)“ after traversing right subtree
- Algorithm printExpression(v)

    if isInternal (v)
        print(“(”)
        inOrder (leftChild (v))
    print(v.element ())
    if isInternal (v)
        inOrder (rightChild (v))
        print (“)”)



((2*(a-1))+(3*b))
Src: goodrich notes

Evaluate Arithmetic Expressions

- Specialization of a postorder traversal
  - recursive method returning the value of a subtree
  - if an internal node is visited, combine the values of the subtrees
- Algorithm evalExpr(v)

  if isExternal (v) return v.elemen

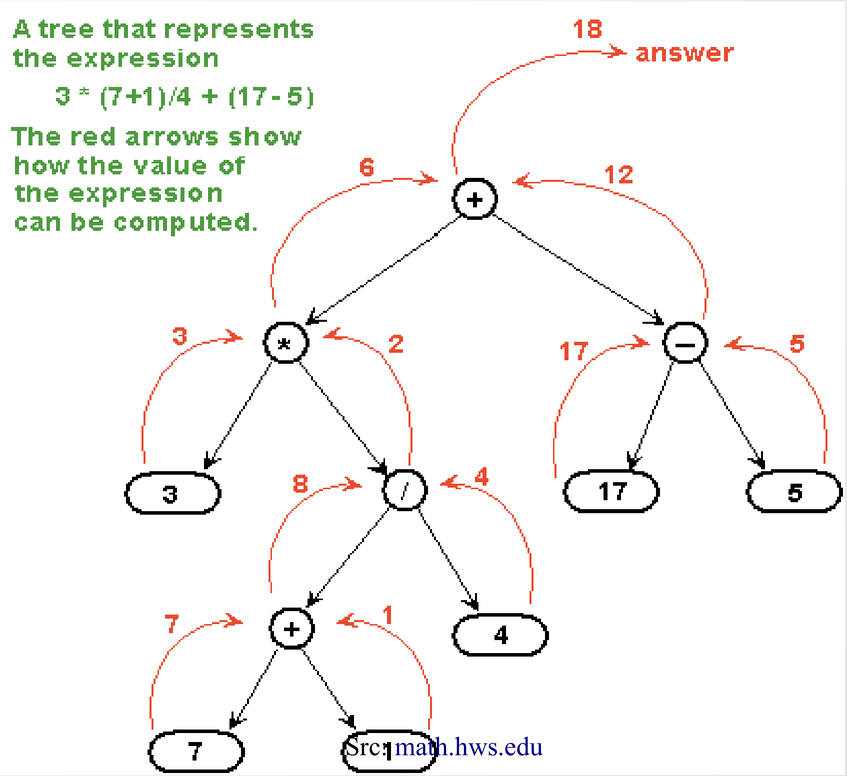
  else

    x ← evalExpr(leftChild (v))

    y ← evalExpr(rightChild (v))

    ◊ ← operator stored at v

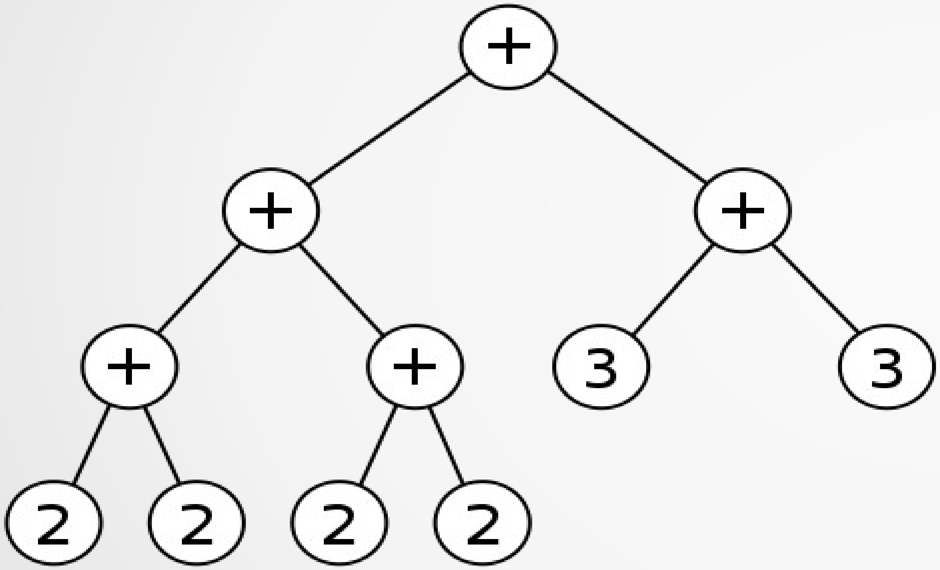
    return x ◊ y

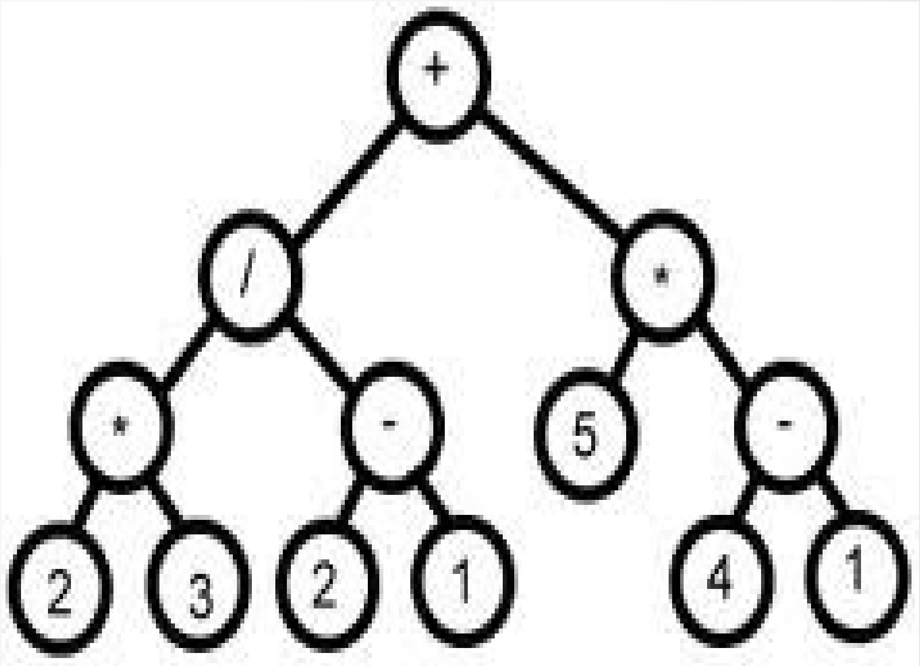


Src: math.hws.edu

Exercise

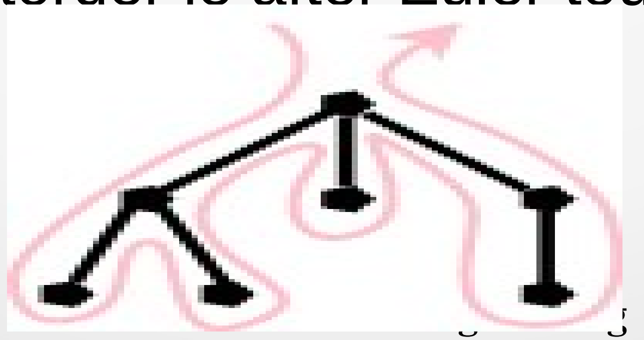
- Print and evaluate the expressions represented by the following trees





Euler Tour Traversal

- An Euler tour is a trail in a tree which visits every edge exactly once
  - If it is undirected, it visits once is each direction
  - Walk around T from root towards left child viewing edges of T as being walls that we always keep to our left
- Walk around the tree and visit each node three times:
  - on the left (preorder ie before Euler tour of v's left subtree)
  - from below (inorder ie between Euler's tour of both subtrees)
  - on the right (postorder ie after Euler tour of v's right subtree)



http://cs.nyu.edu/~gottlieb/courses/2008-09-fall/compilers/lectures/diagrams/euler-tour.png

Euler Tour Traversal

- Algorithm EulerTour(*T*, *v*)

  visitLeft(*T*,*v*)

  if *T*.hasLeft(*v*)

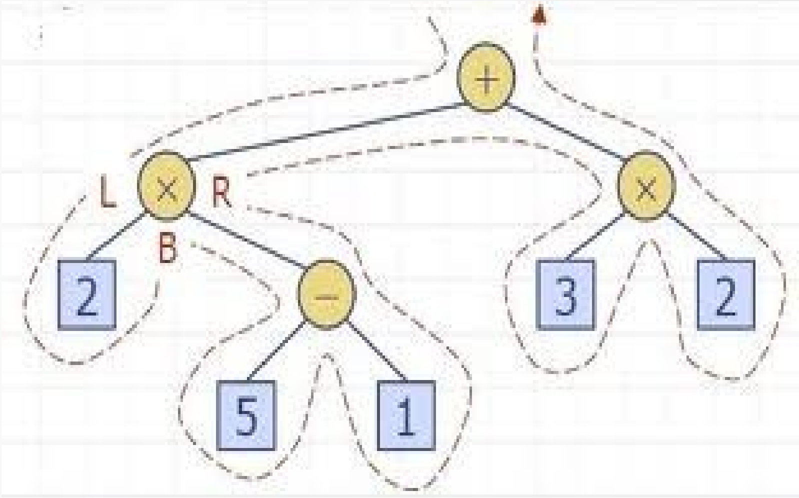
  eulerTour(*T*, leftChild(v))

  visitBelow(*T*,*v*)

  if *T*.hasRight(*v*)

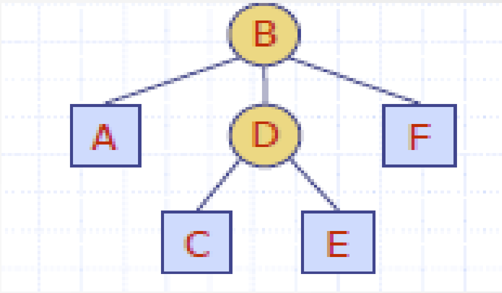
  eulerTour(*T*, rightChild(*v*))

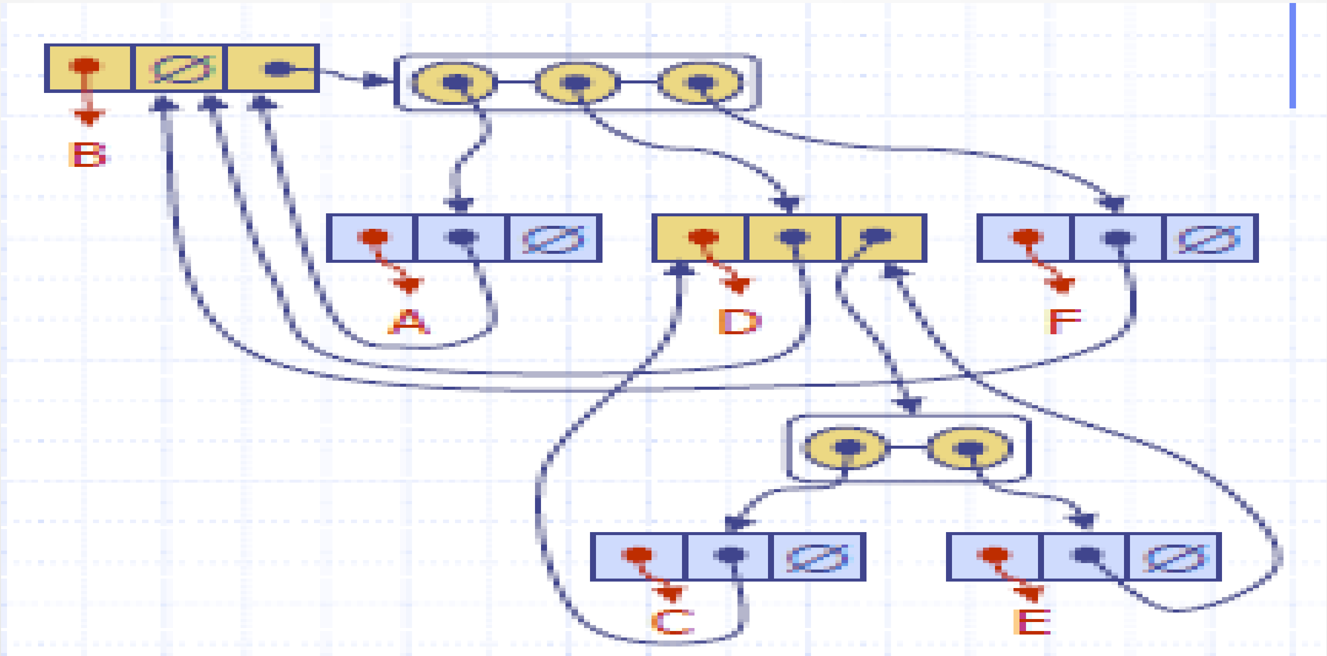
  visitRight(*T*,*v*)



Data Structure for Trees: Linked Representation

- Node represented as follows (Method 1)
  - Element
  - Parent node
  - Sequence of children nodes

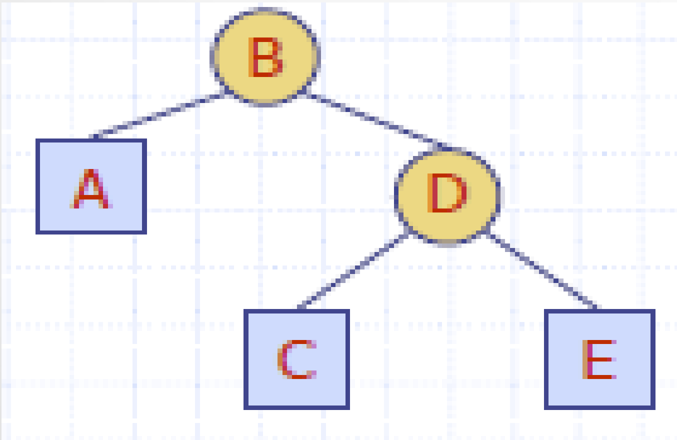


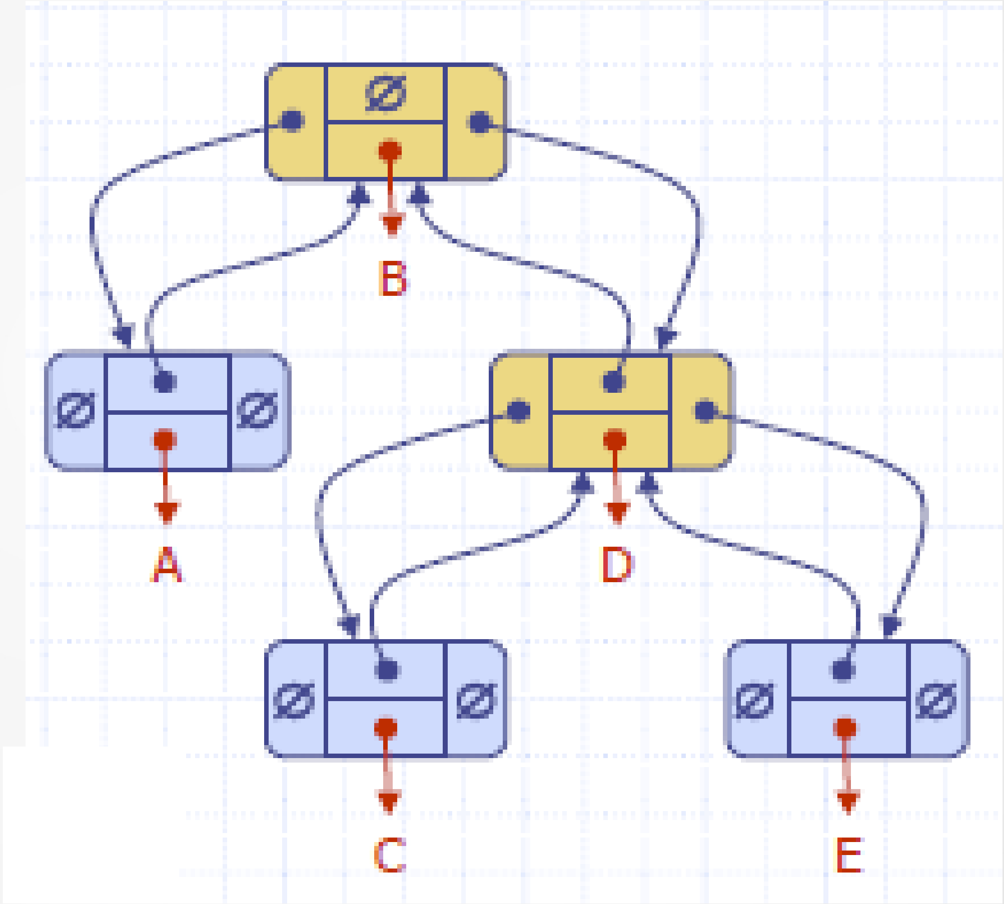


Src: Goodrich notes

Linked Representation

- Node represented as follows (Method 2)
  - Element
  - Parent node
  - Left Child Node
  - Right Child Node





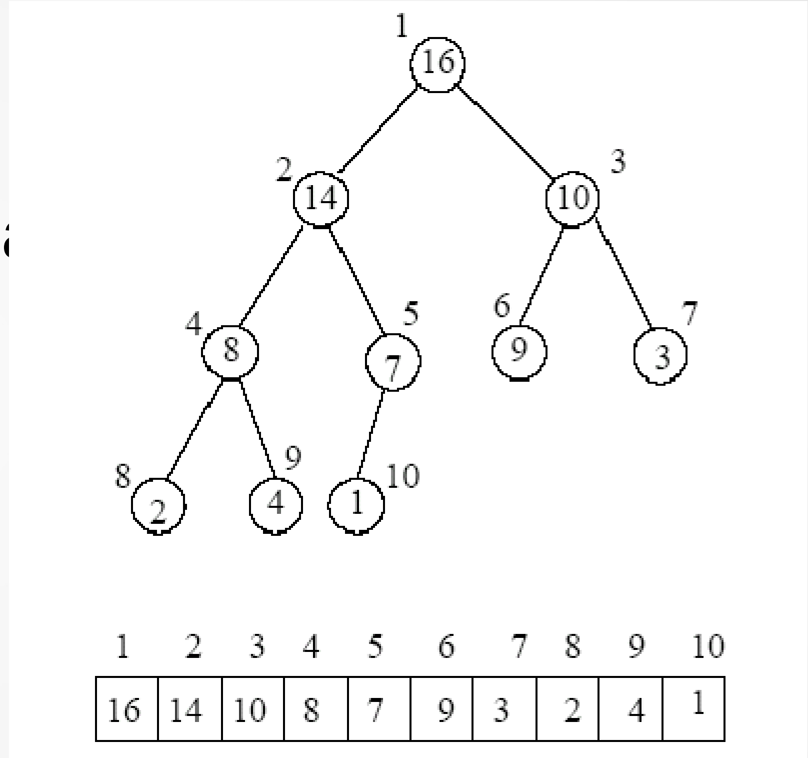
Src: Goodrich notes

Vector based representation

- Based on number nodes
- For each node *v*, $p(v)$ defined as follows
  - If v is the root, $p(v) = 1$
  - If v is left child of node u, then $p(v) = 2p(u)$
  - If v is right child of node u, then $p(v) = 2p(u) + 1$
- Numbering function p is the “level numbering” of the nodes in a binary T
  - Numbers in each level increase from left to right
  - Numbers can be skipped

Vector based representation

- The different functions can be computed using arithmetic operations
- Extendible array representations ca be used to manage the size



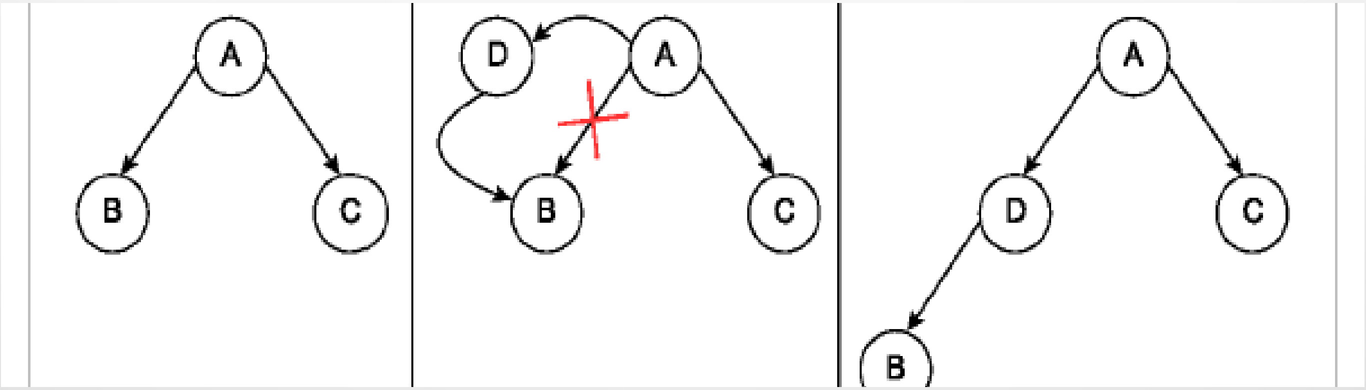
Src: www.cse.hut.fi

Binary Tree ADT

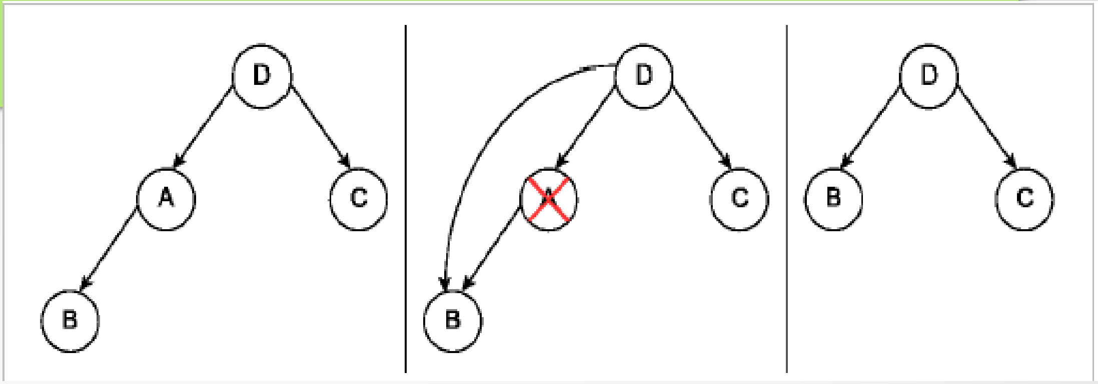
- leftChild(v):
  - Return left child of v
  - Error if v is external node
- rightChild(v):
  - Return right child of v
  - Error if v is external node
- sibling(v)
  - Return sibling of node v
  - Error if v is the root

Insertion and Deletion

- Insertion: InsertAfter(node A)
  - A is an internal node
    - Let node B be child of A
    - A assigns its child to the new node and the new node assigns its parent to A.
    - New node assigns its child to B and B assigns its parent as the new node
  - A is an external node
    - A assigns the new node as one of its children
    - The new node assigns node A as its parent.



Deletion



- Delete(node A)
  - A is an external node
    - set the corresponding child of A's parent to null
  - A has one child
    - set the parent of A's child to A's parent
    - set the child of A's parent to A's child.
  - A has two children
    - Usually not recommended
    - Involves more complex operations