Inverse Linear Transformations

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Rotation in the plane

Show that the L.T. $T: R^2 \rightarrow R^2$ given by the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

has the property that it rotates every vector in R^2 counterclockwise about the origin through the angle θ

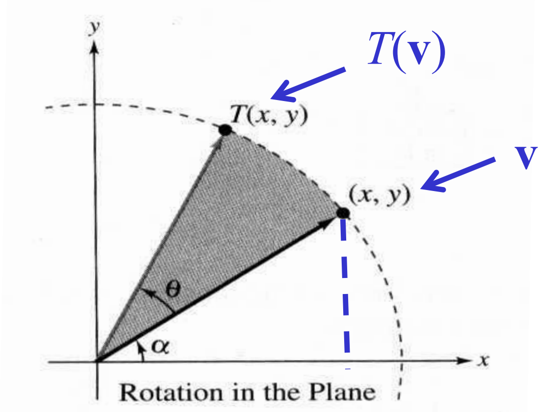
Sol:

$$\mathbf{v} = (x, y) = (r\cos\alpha, r\sin\alpha)$$

(Polar coordinates: for every point on the *xy*-plane, it can be represented by a set of (r, α))

r : the length of $\mathbf{v}$ ($= \sqrt{x^2 + y^2}$)

α : the angle from the positive x-axis counterclockwise to the vector $\mathbf{v}$



$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} r\cos\theta\cos\alpha - r\sin\theta\sin\alpha \\ r\sin\theta\cos\alpha + r\cos\theta\sin\alpha \end{bmatrix}$$

according to the addition formula of trigonometric identities

$$= \begin{bmatrix} r\cos(\theta+\alpha) \\ r\sin(\theta+\alpha) \end{bmatrix}$$

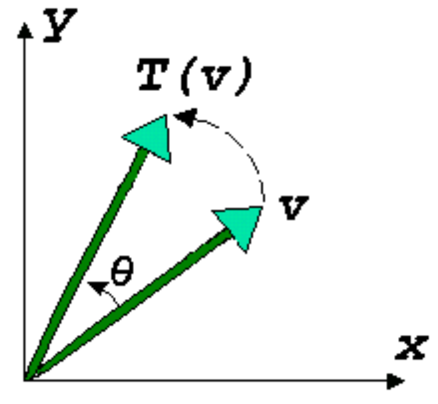
r ： remain the same, that means the length of $T(\mathbf{v})$ equals the length of $\mathbf{v}$

$\theta+\alpha$ ： the angle from the positive x-axis counterclockwise to the vector $T(\mathbf{v})$

Thus, $T(\mathbf{v})$ is the vector that results from rotating the vector $\mathbf{v}$ counterclockwise through the angle θ

- Let $T: R^2 \to R^2$ be the linear operator that rotates each vector in the *xy*-plane through the angle θ.

- Since every vector in the *xy*-plane can be obtained by rotating through some vector through angle θ, we have $R(T) = R^2$.
- The only vector that rotates into **0** is **0**, so $\ker(T) = \{\mathbf{0}\}$.



- Ex : Finding a basis for the kernel

Let $T : R^5 \to R^4$ be defined by $T(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x}$ is in R^5 and

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}$$

Find a basis for ker(T) as a subspace of R^5

Sol:

To find ker(T) means to find all $\mathbf{x}$ satisfying $T(\mathbf{x}) = A\mathbf{x} = \mathbf{0}$.

Thus we need to form the augmented matrix $[A \mid \mathbf{0}]$ first

$$[A \mid \mathbf{0}] = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 & 0 \\ 2 & 1 & 3 & 1 & 0 & 0 \\ -1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 8 & 0 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{array}{c} \begin{bmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} & & s & & t & \end{matrix} \end{array}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s + t \\ s + 2t \\ s \\ -4t \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

$B = \{(-2, 1, 1, 0, 0), (1, 2, 0, -4, 1)\}$: one basis for the kernel of T

Kernel and Range of a Matrix Transformation

If $T_A : R^n \rightarrow R^m$ is multiplication by the $m \times n$ matrix A, then,

- *the kernel of T_A* is the null space of A
- *the range of T_A* is the column space of A

Let $T: R^n \to R^m$ be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$.

The range of T is equal to the column space of A, i.e. $\text{range}(T) = CS(A)$

(1) According to the definition of the range of $T(\mathbf{x}) = A\mathbf{x}$, we know that the range of T consists of all vectors $\mathbf{b}$ satisfying $A\mathbf{x}=\mathbf{b}$, which is equivalent to find all vectors $\mathbf{b}$ such that the system $A\mathbf{x}=\mathbf{b}$ is consistent

(2) $A\mathbf{x}=\mathbf{b}$ can be rewritten as

$$A\mathbf{x} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \mathbf{b}$$

Therefore, the system $A\mathbf{x}=\mathbf{b}$ is consistent iff we can find $(x_1, x_2, \ldots, x_n)$ such that $\mathbf{b}$ is a linear combination of the column vectors of A, i.e. $\mathbf{b} \in CS(A)$.

Thus, we can conclude that the range consists of all vectors **b**, which is a linear combination of the column vectors of *A* or said $\mathbf{b} \in CS(A)$. So, the column space of the matrix *A* is the same as the range of *T*, i.e. range(*T*) = *CS*(*A*)

※ For the orthogonal projection of any vector (*x*, *y*, *z*) onto the *xy*-plane, i.e. *T*(*x*, *y*, *z*) = (*x*, *y*, 0)

※ According to the above analysis, we already knew that the range of *T* is the *xy*-plane, i.e. range(*T*)={(*x*, *y*, 0)| *x* and *y* are real numbers}

※ *T* can be defined by a matrix *A* as follows

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ such that } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

※ The column space of *A* is as follows, which is just the *xy*-plane

$$CS(A) = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}, \text{ where } x_1, x_2 \in R$$

Ex : Finding a basis for the range of a linear transformation

Let $T : R^5 \rightarrow R^4$ be defined by $T(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x}$ is R^5 and

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}$$

Find a basis for the range of T

Sol: Since range(T) = $CS(A)$, finding a basis for the range of T is equivalent to fining a basis for the column space of A. We can easily find that the basis for the column space is $\{(1,2,-1,0),(2,1,0,0),(1,1,0,2)\}$.

- Ex : Finding the rank and nullity of a linear transformation

Let $T: R^5 \rightarrow R^7$ be a linear transformation

(a) Find the dimension of the kernel of T if the dimension of the range of T is 2

(b) Find the rank of T if the nullity of T is 4

(c) Find the rank of T if $\ker(T) = \{\mathbf{0}\}$

Sol:

(a) $\dim(\text{domain of } T) = n = 5$

$\dim(\text{kernel of } T) = n - \dim(\text{range of } T) = 5 - 2 = 3$

(b) $\text{rank}(T) = n - \text{nullity}(T) = 5 - 4 = 1$

(c) $\text{rank}(T) = n - \text{nullity}(T) = 5 - 0 = 5$

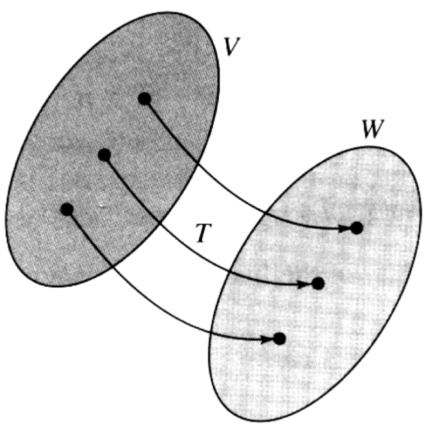
Definition

- *one-to-one*

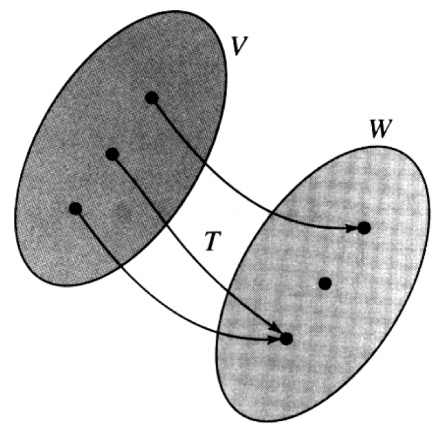
A linear transformation $T:V\rightarrow W$ is said to be ***one-to-one*** if T maps distinct vectors in V into distinct vectors in W.

One-to-one :

A function $T: V \rightarrow W$ is called one-to-one if the preimage of every $\mathbf{w}$ in the range consists of a single vector. This is equivalent to saying that T is one-to-one iff for all $\mathbf{u}$ and $\mathbf{v}$ in V, $T(\mathbf{u}) = T(\mathbf{v})$ implies that $\mathbf{u} = \mathbf{v}$



one-to-one



not one-to-one

Example
A One-to-One Linear Transformation

Let $T: P_n \rightarrow P_{n+1}$ be the linear transformation

$$T(\mathbf{p}) = T(p(x)) = xp(x)$$

If $\quad \mathbf{p} = p(x) = c_0 + c_1 x + \ldots + c_n x^n$ and

$$\mathbf{q} = q(x) = d_0 + d_1 x + \ldots + d_n x^n$$

are distinct polynomials, then they differ in at least one coefficient. Thus,

$$\mathrm{T}(\mathbf{p}) = c_0 x + c_1 x^2 + \ldots + c_n x^{n+1} \quad \text{and}$$

$$\mathrm{T}(\mathbf{q}) = d_0 x + d_1 x^2 + \ldots + d_n x^{n+1}$$

Also differ in at least one coefficient. Thus, since it maps distinct polynomials **p** and **q** into distinct polynomials $T(\mathbf{p})$ and $T(\mathbf{q})$.

- Theorem: One-to-one linear transformation

Let $T: V \to W$ be a linear transformation. Then

T is one-to-one iff $\ker(T) = \{\mathbf{0}\}$

Pf:

$(\Rightarrow)$ Suppose T is one-to-one

Then $T(\mathbf{v}) = \mathbf{0}$ can have only one solution : $\mathbf{v} = \mathbf{0}$

Due to the fact that $T(\mathbf{0}) = \mathbf{0}$ in

i.e. $\ker(T) = \{\mathbf{0}\}$

$(\Leftarrow)$ Suppose $\ker(T) = \{\mathbf{0}\}$ and $T(\mathbf{u}) = T(\mathbf{v})$

$T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v}) = \mathbf{0}$

T is a linear transformation, see Property of linear transform

$\therefore \mathbf{u} - \mathbf{v} \in \ker(T) \Rightarrow \mathbf{u} - \mathbf{v} = \mathbf{0} \Rightarrow \mathbf{u} = \mathbf{v}$

$\Rightarrow T$ is one-to-one (because $T(\mathbf{u}) = T(\mathbf{v})$ implies that $\mathbf{u} = \mathbf{v}$)

- Example: One-to-one and not one-to-one linear transformation

(a) The linear transformation $T: M_{m\times n} \rightarrow M_{n\times m}$ given by $T(A) = A^T$ is one-to-one

because its kernel consists of only the $m \times n$ zero matrix

(b) The zero transformation $T: R^3 \rightarrow R^3$ is not one-to-one

because its kernel is all of R^3

Equivalent Statements

- Theorem 8.3.1

If $T:V \to W$ is a linear transformation, then the following are equivalent.

(a) *T is one-to-one*

(b) *The kernel of T contains only zero vector; that is , $\ker(T) = \{0\}$*

(c) *Nullity $(T) = 0$*

Theorem

If *V* is a finite-dimensional vector space and T:*V* ->V is a linear operator then the following are equivalent.

(a)T is one to one

(b) ker(T) = {0}

(c)nullity(T) = 0

(d)The range of T is V;that is ,R(T) =V

- ## Onto:

A function $T: V \to W$ is said to be onto if every element in W has a preimage in V

(T is onto W when W is equal to the range of T)

- ## Theorem: Onto linear transformations

Let T: $V \to W$ be a linear transformation, where W is finite dimensional. Then T is onto if and only if the rank of T is equal to the dimension of W

$$\text{rank}(T) = \dim(\text{range of } T) = \dim(W)$$

The definition of the rank of a linear transformation

The definition of onto linear transformations

The linear transformation $T: R^n \to R^m$ is given by $T(\mathbf{x}) = A\mathbf{x}$. Find the nullity and rank of T and determine whether T is one-to-one, onto, or neither

$$\text{(a) } A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(b) } A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{(c) } A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \quad \text{(d) } A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Sol:

$T:R^n \to R^m$	dim(domain of T) (1)	rank(T) (2)	nullity(T)	1-1	onto
	$= \dim(R^n) = n$	$= \dim(\text{range of } T) = \#$ of leading 1's	$= (1) - (2) = \dim(\ker(T))$	If nullity(T) $= \dim(\ker(T)) = 0$	If rank(T) $= \dim(R^m) = m$
(a) $T:R^3 \to R^3$	3	3	0	Yes	Yes
(b) $T:R^2 \to R^3$	2	2	0	Yes	No
(c) $T:R^3 \to R^2$	3	2	1	No	Yes
(d) $T:R^3 \to R^3$	3	2	1	No	No

Inverse Linear Transformations

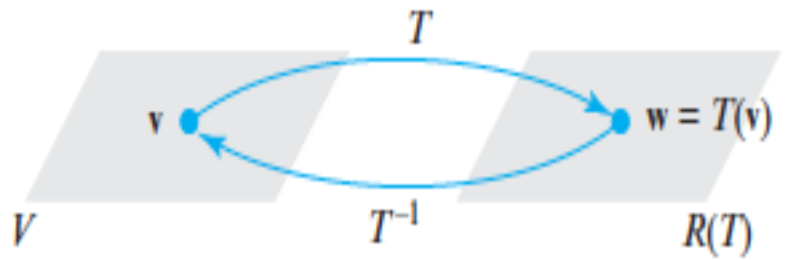
If $T: V \rightarrow W$ is a linear transformation, then the range of T denoted by $R(T)$, is the subspace of W consisting of all images under T of vectors in V.

If T is one-to-one, then each vector $\mathbf{w}$ in $R(T)$ is the image of a unique vector $\mathbf{v}$ in V.

This uniqueness allows us to define a new function, call the inverse of T, denoted by T^{-1}, which maps **w** back into **v**.

The mapping $T^{-1}: R(T) \to V$ is a <u>linear transformation</u>. Moreover,

$T^{-1}(T(\mathbf{v})) = T^{-1}(\mathbf{w}) = \mathbf{v}$
$T(T^{-1}(\mathbf{w})) = T(\mathbf{v}) = \mathbf{w}$



Inverse Linear Transformations

- If $T: V \to W$ is a one-to-one linear transformation, then the domain of T^{-1} is the range of T.

- The range may or may not be all of W (one-to-one but *not* onto).

- For the special case that $T: V \to V$, then the linear transformation is one-to-one and onto.

Theorem

- If T_1:U->V and T_2:V->W are one to one linear transformation then:

(a)$T_2 \circ T_1$ is one to one

(b) $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$