Kernel And Range

Dr. Biswambhar Rakshit

- A function T that maps a vector space V into a vector space W:

V: the domain of T $\qquad$ W: the codomain of T

$$T : V \xrightarrow{\text{mapping}} W, \quad V, W : \text{vector spaces}$$

- Image of **v** under T:

If **v** is a vector in V and **w** is a vector in W such that

$$T(\mathbf{v}) = \mathbf{w},$$

then **w** is called the image of **v** under T

(For each **v**, there is only one **w**)

- The range of T:

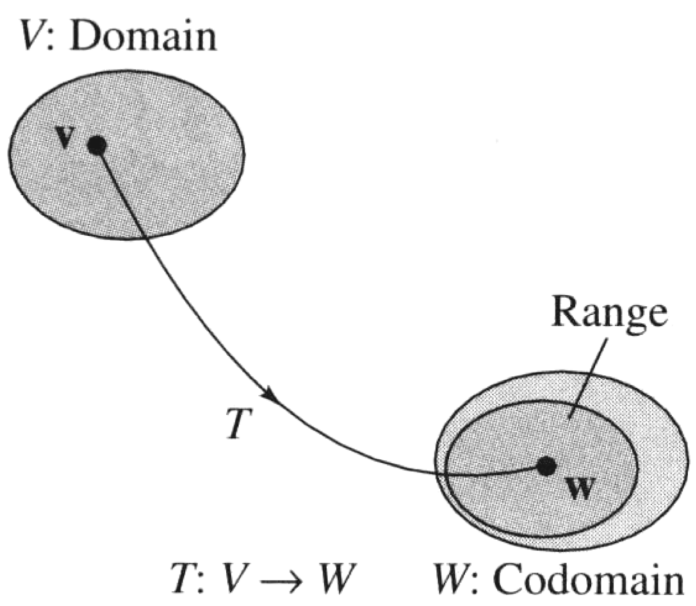
The set of all images of vectors in V (see the figure on the next slide)

- The preimage of **w** :

  The set of all **v** in V such that $T(\mathbf{v})=\mathbf{w}$

  (For each **w**, **v** may not be unique)

- The graphical representations of the domain, codomain, and range



※ For example, V is R^3, W is R^3, and T is the orthogonal projection of any vector (x, y, z) onto the xy-plane, i.e. $T(x, y, z) = (x, y, 0)$

※ Then the domain is R^3, the codomain is R^3, and the range is xy-plane (a subspace of the codomian R^3)

※ $(2, 1, 0)$ is the image of $(2, 1, 3)$

※ The preimage of $(2, 1, 0)$ is $(2, 1, s)$, where s is any real number

Ex 1: A function from R^2 into R^2

$$T : R^2 \rightarrow R^2 \qquad \mathbf{v} = (v_1, v_2) \in R^2$$

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

(a) Find the image of **v**=(-1,2) (b) Find the preimage of **w**=(-1,11)

Sol:

(a) $\mathbf{v} = (-1,\ 2)$

$$\Rightarrow T(\mathbf{v}) = T(-1,\ 2) = (-1-2,\ \ -1+2(2)) = (-3,\ 3)$$

(b) $T(\mathbf{v}) = \mathbf{w} = (-1,\ 11)$

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2) = (-1, 11)$$

$$\Rightarrow v_1 - v_2 = -1$$

$$v_1 + 2v_2 = 11$$

$\Rightarrow v_1 = 3,\ v_2 = 4$ Thus {(3, 4)} is the preimage of **w**=(-1, 11)

Definition

If $T: V \rightarrow W$ is a function from a vector space V into a vector space W, then T is called a ***linear transformation*** from V to W if for all vectors **u** and **v** in V and all scalors c

- $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
- $T(c\mathbf{u}) = cT(\mathbf{u})$

In the special case where $V=W$, the linear transformation $T:V \rightarrow V$ is called a ***linear operator*** on V.

Ex 2: Verifying a linear transformation T from R^2 into R^2

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

Pf:

$$\mathbf{u} = (u_1, u_2),\ \ \mathbf{v} = (v_1, v_2) : \text{vector in } R^2,\ \ c : \text{any real number}$$

$$\text{(1) Vector addition :}$$

$$\mathbf{u} + \mathbf{v} = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$$

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= T(u_1 + v_1, u_2 + v_2) \\ &= ((u_1 + v_1) - (u_2 + v_2), (u_1 + v_1) + 2(u_2 + v_2)) \\ &= ((u_1 - u_2) + (v_1 - v_2), (u_1 + 2u_2) + (v_1 + 2v_2)) \\ &= (u_1 - u_2, u_1 + 2u_2) + (v_1 - v_2, v_1 + 2v_2) \\ &= T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

(2) Scalar multiplication

$$c\mathbf{u} = c(u_1, u_2) = (cu_1, cu_2)$$

$$\begin{aligned} T(c\mathbf{u}) &= T(cu_1, cu_2) = (cu_1 - cu_2, cu_1 + 2cu_2) \\ &= c(u_1 - u_2, u_1 + 2u_2) \\ &= cT(\mathbf{u}) \end{aligned}$$

Therefore, T is a linear transformation

- Zero transformation :

$$T: V \to W \qquad T(\mathbf{v}) = \mathbf{0}, \ \ \forall \mathbf{v} \in V$$

- Identity transformation :

$$T: V \to V \qquad T(\mathbf{v}) = \mathbf{v}, \ \ \forall \mathbf{v} \in V$$

- Theorem 6.1: Properties of linear transformations

$$T: V \to W, \quad \mathbf{u}, \mathbf{v} \in V$$

(1) $T(\mathbf{0}) = \mathbf{0}$ ($T(c\mathbf{v}) = cT(\mathbf{v})$ for c=0)

(2) $T(-\mathbf{v}) = -T(\mathbf{v})$ ($T(c\mathbf{v}) = cT(\mathbf{v})$ for c=-1)

(3) $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$ ($T(\mathbf{u}+(-\mathbf{v}))=T(\mathbf{u})+T(-\mathbf{v})$ and property (2))

(4) If $\mathbf{v} = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$,

then $T(\mathbf{v}) = T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n)$

$$= c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n)$$

(Iteratively using $T(\mathbf{u}+\mathbf{v})=T(\mathbf{u})+T(\mathbf{v})$ and $T(c\mathbf{v}) = cT(\mathbf{v})$)

Ex 3: Linear transformations and bases

Let $T : R^3 \rightarrow R^3$ be a linear transformation such that

$$T(1,0,0) = (2,-1,4)$$

$$T(0,1,0) = (1,5,-2)$$

$$T(0,0,1) = (0,3,1)$$

Find $T(2, 3, -2)$

Sol:

$$(2,3,-2) = 2(1,0,0) + 3(0,1,0) - 2(0,0,1)$$

$$\left(\begin{array}{l}\text{According to the fourth property on the previous slide that} \\ T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n)\end{array}\right)$$

$$\begin{aligned} T(2,3,-2) &= 2T(1,0,0) + 3T(0,1,0) - 2T(0,0,1) \\ &= 2(2,-1,4) + 3(1,5,-2) - 2T(0,3,1) \\ &= (7,7,0) \end{aligned}$$

Ex 4: A linear transformation defined by a matrix

The function $T: R^2 \rightarrow R^3$ is defined as $T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

(a) Find $T(\mathbf{v})$, where $\mathbf{v} = (2, -1)$

(b) Show that T is a linear transformation form R^2 into R^3

Sol:

(a) $\mathbf{v} = (2, -1)$

R^2 vector R^3 vector

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$$

$$\therefore T(2, -1) = (6, 3, 0)$$

(b) $T(\mathbf{u} + \mathbf{v}) = A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v})$ (vector addition)

$T(c\mathbf{u}) = A(c\mathbf{u}) = c(A\mathbf{u}) = cT(\mathbf{u})$ (scalar multiplication)

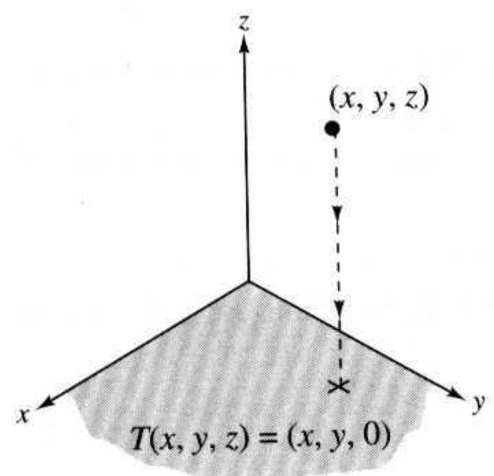
Ex 6: A projection in R^3

The linear transformation $T : R^3 \to R^3$ is given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is called a projection in R^3

$$\text{If } \mathbf{v} \text{ is } (x, y, z),\ A\mathbf{v} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$



Projection onto *xy*-plane

※ In other words, T maps every vector in R^3 to its orthogonal projection in the xy-plane, as shown in the right figure

- Ex 7: The transpose function is a linear transformation from $M_{m\times n}$ into $M_{n\times m}$

$$T(A) = A^T \quad (T : M_{m\times n} \rightarrow M_{n\times m})$$

Show that T is a linear transformation

Sol:

$$A, B \in M_{m\times n}$$

$$T(A+B) = (A+B)^T = A^T + B^T = T(A) + T(B)$$

$$T(cA) = (cA)^T = cA^T = cT(A)$$

Therefore, T (the transpose function) is a linear transformation from $M_{m\times n}$ into $M_{n\times m}$

Definition

- ker(*T*): *the kernel of T*

  If *T:V→W* is a linear transformation, then the ker(*T*) is the set of all vectors in *V* that

  T maps into **0.** $\ker(T) = \{\mathbf{v} \mid T(\mathbf{v}) = \mathbf{0}, \forall \mathbf{v} \in V\}$

- *R* (*T*): *the range of T*

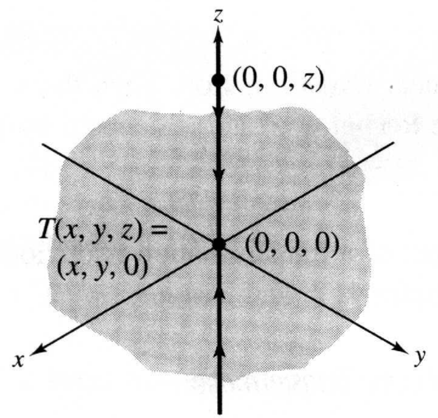
  The set of all vectors in *W* that are images under *T* of at least one vector in *V.*

  $\text{Range}(T) = \{T(u),: \forall u \in V\}$

※ For example, V is R^3, W is R^3, and T is the orthogonal projection of any vector (x, y, z) onto the xy-plane, i.e. $T(x, y, z) = (x, y, 0)$

※ Then the kernel of T is the set consisting of $(0, 0, s)$, where s is a real number, i.e.

$$\ker(T) = \{(0,0,s) \mid s \text{ is a real number}\}$$



The kernel of T is the set of all vectors on the z-axis.

※ For example, V is R^3, W is R^3, and T is the orthogonal projection of any vector (x, y, z) onto the xy-plane, i.e. $T(x, y, z) = (x, y, 0)$

※ Then the range of T is xy plane

Kernel and Range of a Matrix Transformation

If $T_A : R^n \rightarrow R^m$ is multiplication by the $m \times n$ matrix A, then from the discussion preceding the definition above,

- *the kernel of T_A* is the nullspace of A
- *the range of T_A* is the column space of A

Kernel and Range of the Zero Transformation

Let $T:V \rightarrow W$ be the zero transformation. Since T maps *every* vector in V into $\mathbf{0}$, it follows that $\ker(T) = V$.

Moreover, since $\mathbf{0}$ is the *only* image under T of vectors in V, we have $R(T) = \{\mathbf{0}\}$.

Kernel and Range of the Identity Operator

Let $I{:}V \to V$ be the identity operator. Since $I(\mathbf{v}) = \mathbf{v}$ for all vectors in V, *every* vector in V is the image of some vector; thus, $R(I) = V$.

Since the *only* vector that I maps into $\mathbf{0}$ is $\mathbf{0}$, it follows that $\ker(I) = \{\mathbf{0}\}$.

Finding the kernel of a linear transformation

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (T: R^3 \to R^2)$$

$$\ker(T) = ?$$

Sol:

$$\ker(T) = \{(x_1, x_2, x_3) \mid T(x_1, x_2, x_3) = (0,0), \text{ and } (x_1, x_2, x_3) \in R^3\}$$

$$T(x_1, x_2, x_3) = (0,0)$$

$$\begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -2 & 0 \\ -1 & 2 & 3 & 0 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \ker(T) = \{t(1,-1,1) \mid t \text{ is a real number}\}$$
$$= \operatorname{span}\{(1,-1,1)\}$$

- Theorem 8.2.1

If $T: V \rightarrow W$ is linear transformation, then:

(a) *The kernel of T is a subspace of V.*

(b) *The range of T is a subspace of W.*

Proof (a).

Let $\mathbf{v_1}$ and $\mathbf{v_2}$ be vectors in ker(T), and let k be any scalar. Then

$$T(\mathbf{v_1} + \mathbf{v_2}) = T(\mathbf{v_1}) + T(\mathbf{v_2}) = \mathbf{0}+\mathbf{0} = \mathbf{0}$$

so that $\mathbf{v_1} + \mathbf{v_2}$ is in ker(T).

Also,

$$T(k\mathbf{v_1}) = kT(\mathbf{v_1}) = k\mathbf{0} = \mathbf{0}$$

so that $k\mathbf{v_1}$ is in ker(T).

Proof (b).

Let $\mathbf{w_1}$ and $\mathbf{w_2}$ be vectors in the range of T, and let k be any scalar. There are vectors $\mathbf{a_1}$ and $\mathbf{a_2}$ in V such that $T(\mathbf{a_1}) = \mathbf{w_1}$ and $\mathrm{T}(\mathbf{a_2}) = \mathbf{w_2}$. Let $\mathbf{a} = \mathbf{a_1} + \mathbf{a_2}$ and $\mathbf{b} = k\,\mathbf{a_1}$.

Then

$$T(\mathbf{a}) = T(\mathbf{a_1} + \mathbf{a_2}) = T(\mathbf{a_1}) + T(\mathbf{a_2}) = \mathbf{w_1} + \mathbf{w_2}$$

and

$$T(\mathbf{b}) = T(k\,\mathbf{a_1}) = kT(\mathbf{a_1}) = k\,\mathbf{w_1}$$

Definition

- *rank (T): the rank of T*

If $T: V \rightarrow W$ is a linear transformation, then the dimension of the range of T is the rank of T.

- *nullity (T): the nullity of T*

the dimension of the kernel is the nullity of T.

Theorem 8.2.2

If A is an $m \times n$ matrix and $T_A : R^n \to R^m$ is multiplication by A, then:

(a) $\text{nullity}(T_A) = \text{nullity}(A)$

(b) $\text{rank}(T_A) = \text{rank}(A)$

Dimension Theorem for Linear Transformations

- Theorem 8.2.3

If $T: V \to W$ is a linear transformation from an n-dimensional vector space V to a vector space W, then

$$rank\,(T) + nullity\,(T) = n$$

In words, this theorem states that for linear transformations the rank plus the nullity is equal to the dimension of the domain.

Example
Finding Rank and Nullity

Let T: $R^3 \rightarrow R^3$ given by T(x,y,z)=(x,y,0) the kernel of T is the z-axis,

That is ker(T)={(0,0,z):z∈R} which is one-dimensional; and the range of T is the xy-plane, which is two-dimensional. Thus,

$$\text{nullity}(T) = 1 \quad \text{and} \quad \text{rank}(T) = 2$$

Example 9
Using the Dimension Theorem

Let T: $R^2 \to R^2$ be the linear operator that rotates each vector in the xy-plane through an angle θ. We showed in Example 5 that ker(T) = {**0**} and $R(T) = R^2$ .Thus,

$$\text{rank}\,(T) + \text{nullity}\,(T) = 2 + 0 = 2$$

Which is consistent with the fact thar the domain of T is two-dimensional.