Fixed-point iteration method

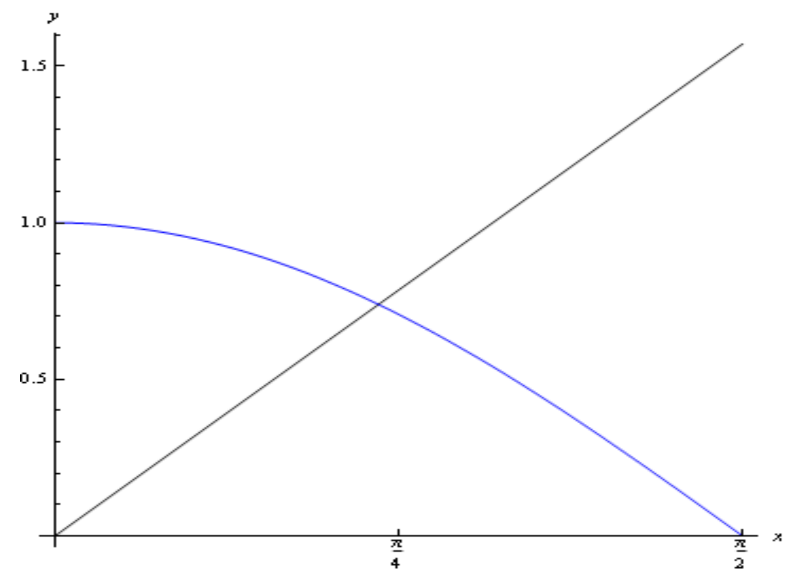
The idea of the fixed point iteration methods is to first reformulate a equation to an equivalent fixed point problem:

$$f(x) = 0 \Longleftrightarrow x = g(x)$$

and then to use the iteration: with an initial guess x_0 chosen, compute a sequence

$$x_{n+1} = g(x_n),\ n \geq 0$$

in the hope that x_n converges to root of f.



Fixed point iteration method

Algorithm:

Given an equation $f(x) = 0$

Step1: Convert $f(x) = 0$ into the form $x = g(x)$

Step2: Let the initial guess be x_0

Calculate

$$x_{n+1} = g(x_n) \quad n=0,1,2\ldots$$

Convert the given equation in the form $x = g(x)$

Examples

$x^2 - 1 = 0$ can be written as

$x = 1 / x$ (i.e., $g(x) = 1 / x$)

$x = x^2 + x - 1$ (i.e., $g(x) = x^2 + x - 1$)

$x^2 + x - 2 = 0$ can be written as

$x = 2 - x^2$ (i.e., $g(x) = 2 - x^2$)

$x = \mathrm{sqrt}(2 - x)$ (i.e., $g(x) = \mathrm{sqrt}(2 - x)$)

Condition for Convergence :

If $g(x)$ and $g'(x)$ are continuous on an interval J about their root s of the equation $x = g(x)$, and if $|g'(x)|<1$ for all x in the interval J then the fixed point iterative process $x_{n+1}=g(x_n)$, $n = 0, 1, 2, \ldots$, will converge to the root $x = s$ for any initial approximation x_0 belongs to the interval J .

Example 1: Use fixed point method to find a root of the equation $\mathbf{x^4-x-10 = 0}$ (with initial approximations $x_0 = 2$).

Solution: $f(x) = x^4-x-10 = 0$

$x = 10 / (x^3-1)$

$g(x) = 10 / (x^3-1)$

Calculate:

$x_{n+1} = g(x_n) = 10 / (x_n^3 - 1)$

Iteration	x_n
0	2
1	1.429
2	5.214
3	0.071
4	-10.004
5	-9.978E-3
6	-10
7	-9.978E-3
8	-10

So the iterative process with **g(x)** gone into an infinite loop without converging.

Example 2: Use fixed point method to find a root of the equation $x^4 - x - 10 = 0$ (with initial approximations $x_0 = 2$). (correct to 5 decimal places)

Solution: $f(x) = x^4 - x - 10 = 0$

$x = (x + 10)^{1/4}$

$g(x) = (x + 10)^{1/4}$

Calculate:

$x_{n+1} = g(x_n) = (x_n + 10)^{1/4}$

Iteration	x_n
0	2
1	1.861
2	1.8558
3	1.85559
4	1.85558
5	1.85558

First five decimal places have been stabilized; hence 1.85558 is the real root correct to five decimal places.

Example 3: Use fixed point method to find a root of the equation x^4-x-10 = 0 (with initial approximations x_0= 1.8). (Do 6 iteration)

Solution: $f(x) = x^4 - x - 10 = 0$

$x = (x+10)^{1/2}/x$

$g(x) = (x+10)^{1/2}/x$

Calculate:

$x_{n+1} = g(x_n) = (x_n+10)^{1/2}/x_n$

Iteration	x_n
0	1.8
1	1.9084
2	1.80825
3	1.90035
4	1.81529
5	1.89355
6	1.82129
	.
	.
98	1.8555

It's clear from the example 1, 2 and 3:

Ex1, the iterative process does not converge for any initial approximation

Ex2, the iterative process converges very quickly to the root

Ex3, the iterative process converges but very slowly.

Exercise Problems:

1. Find the root of the equation $\cos(x)$ - $xe^x = 0$ using fixed point iteration method. (with initial guess x_0= 2) (Hint: Do 38 iteration with g(x)= $\cos(x)$ /e^x)

2. Apply fixed point iteration method to find a root of the equation $xe^x = 1$ correct to three decimal places.

3. Apply fixed point iteration method to find a root of the equation x- $\sin(x)$-1/2 =0 correct to three decimal places. (with initial guess x_0= 2)

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If $f(x) = x^2 - x - 2$, then fixed points of each of functions

- $g(x) = x^2 - 2$
- $g(x) = \sqrt{x+2}$
- $g(x) = 1 + 2/x$
- $g(x) = \dfrac{x^2+2}{2x-1}$

are solutions to equation $f(x) = 0$

Apply fixed point iteration method to the above function for each g with initial guess x_0= 1.8. (Do 5 iteration)