8.1 General Linear Transformation

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Definition

If $T: V \rightarrow W$ is a function from a vector space V into a vector space W, then T is called a ***linear transformation*** from V to W if for all vectors **u** and **v** in V and all scalors c

- $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
- $T(c\mathbf{u}) = cT(\mathbf{u})$

In the special case where $V=W$, the linear transformation $T: V \rightarrow V$ is called a ***linear operator*** on V.

Example 1
Zero Transformation

The mapping $T{:}V \rightarrow W$ such that $T(\mathbf{v})=0$ for every $\mathbf{v}$ in V is a linear transformation called the ***zero transformation***. To see that T is linear, observe that

$$T(\mathbf{u}+\mathbf{v}) = 0. \quad T(\mathbf{u}) = 0,\ T(\mathbf{v}) = 0. \text{ And } T(k\,\mathbf{u}) = 0$$

Therefore,

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(k\,\mathbf{u}) = kT(\mathbf{u})$$

Example 2
Identity Operator

The mapping $I: V \longrightarrow V$ defined by $I(\mathbf{v}) = \mathbf{v}$ is called the identify operator on V.

Example 3
Dilation and Contraction operators

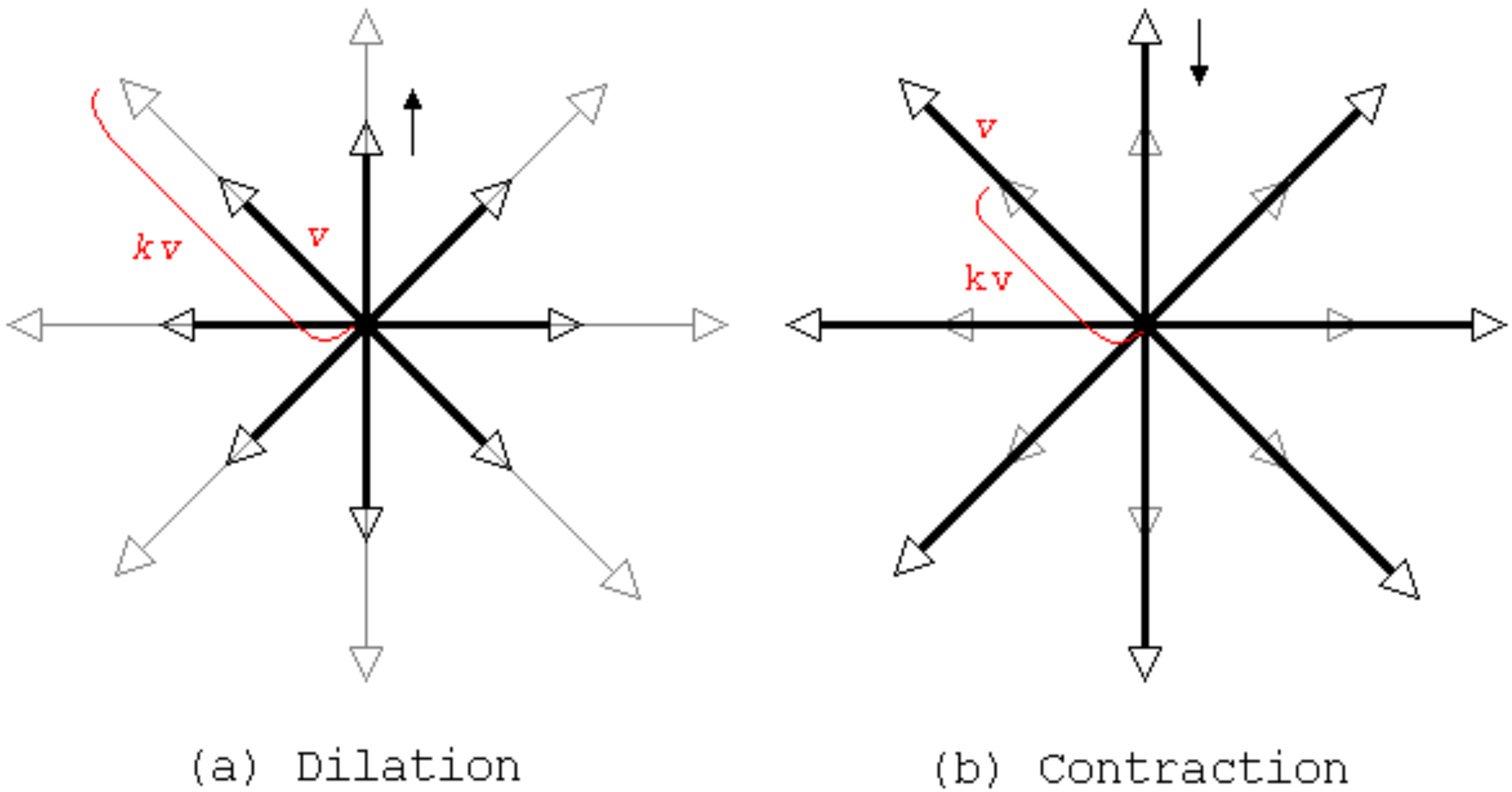
Let V be any vector space and k any fixed scalar. The function $T: V \to V$ defined by

$$T(\mathbf{v}) = k\,\mathbf{v}$$

is linear operator on V.

- **Dilation:** $k > 1$
- **Contraction:** $0 < k < 1$

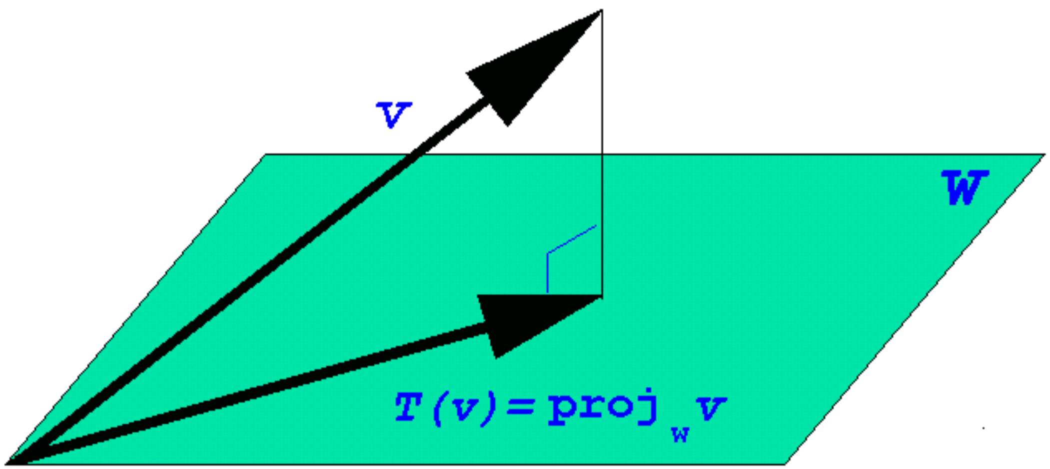
Dilation and Contraction operators



Example 4
Orthogonal Projections

Suppose that W is a finite-dimensional subspace of an inner product space V; then the ***orthogonal projection*** of V onto W is the transformation defined by

$$T(\boldsymbol{v}) = \text{proj}_W \boldsymbol{v}$$



Example 5

A Linear Transformation from a space V to R^n

Let $S = \{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_n}\}$ be a basis for an n-dimensional vector space V, and let

$$(\mathbf{v})_s = (k_1, k_2, \ldots, k_n)$$

Be the coordinate vector relative to S of a vector **v** in V thus

$$\mathbf{v} = k_1\mathbf{w} + k_2\mathbf{w_2} + \ldots + k_n\mathbf{w_n}$$

Define $T: V \rightarrow R^n$ to be the function that maps **v** into its coordinate vector relative to S; that is,

$$T(\mathbf{v}) = (\mathbf{v})_s = (k_1, k_2, \ldots, k_n)$$

The function T is linear transformation. To see that this is so, suppose that u and **v** are vectors in V and that

$$\mathbf{u} = c_1\mathbf{w_1} + c_2\mathbf{w_2} + \ldots + c_n\mathbf{w_n} \quad \text{and}$$

$$\mathbf{v} = d_1\mathbf{w_1} + d_2\mathbf{w_2} + \ldots + d_n\mathbf{w_n}$$

Thus,

$$(\mathbf{u})_s = (c_1, c_2, \ldots, c_n) \text{ and } (\mathbf{v})_s = (d_1, d_2, \ldots, d_n)$$

But

$$\mathbf{u}+\mathbf{v} = (c_1+d_1)\,\mathbf{w_1} + (c_2+d_2)\,\mathbf{w_2} + \ldots + (c_n+d_n)\,\mathbf{w_n}$$

$$k\,\mathbf{u} = (kc_1)\,\mathbf{w_1} + (kc_2)\,\mathbf{w_2} + \ldots + (kc_n)\,\mathbf{w_n}$$

So that

$$(\mathbf{u}+\mathbf{v})_s = (c_1+d_1, c_2+d_2 \ldots, c_n+d_n)$$

$$(k\,\mathbf{u})_s = (kc_1, kc_2, \ldots, kc_n)$$

Therefore,

$$(\mathbf{u}+\mathbf{v})_s = (\mathbf{u})_s + (\mathbf{v})_s \quad \text{and} \quad (k\,\mathbf{u})_s = k\,(\mathbf{u})_s$$

Expressing these equations of T, we obtain

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \quad \text{and} \quad T(k\,\mathbf{u}) = kT(\mathbf{u})$$

Which shows that T is a linear transformation.

REMARK. The computations in preceding example could just as well have been performed using coordinate matrices rather than coordinate vectors; that is ,

$$[\mathbf{u}+\mathbf{v}] = [\mathbf{u}]_s + [\mathbf{v}]_s \quad \text{and} \quad [k\,\mathbf{u}]_s = k\,[\mathbf{u}]_s$$

Example 6
A Linear Transformation from $\boldsymbol{p_n}$ to $\boldsymbol{p_{n+1}}$

Let $\mathbf{p} = p(x) = C_0 X + C_1 X^2 + \ldots + C_n X^n$ be a polynomial in P_n, and define the function $T: P_n \rightarrow P_{n+1}$ by

$$T(\mathrm{p}) = T(p(x)) = xp(x) = C_0 X + C_1 X^2 + \ldots + C_n X^{n+1}$$

The function T is a linear transformation, since for any scalar k and any polynomials $\mathbf{p_1}$ and $\mathbf{p_2}$ in P_n we have

$$T(\mathbf{p_1}+\mathbf{p_2}) = T(p_1(x) + p_2(x)) = x(p_1(x)+p_2(x))$$
$$= x\,p_1(x) + x\,p_2(x) = T(\mathbf{p_1}) + T(\mathbf{p_2})$$

and

$$T(k\,\mathbf{p}) = T(k\,p(x)) = x\,(k\,p(x)) = k\,(x\,p(x)) = k\,T(\mathbf{p})$$

Example 7
A linear Operator on P_n

Let $\mathbf{p} = p(x) = c_0 X + c_1 X^2 + \ldots + c_n X^n$ be a polynomial in P_n, and let a and b be any scalars. We leave it as an exercise to show that the function T defined by

$$T(\mathbf{p}) = T(p(x)) = p(ax+b) = c_0 + c_1(ax+b) + \ldots + c_n(ax+b)^n$$

is a linear operator. For example, if $ax+b = 3x - 5$, then $T: P_2 \rightarrow P_2$ would be the linear operator given by the formula

$$T(c_0 + c_1 x + c_2 x^2) = c_0 + c_1(3x-5) + c_2(3x-5)^2$$

Example 8
A Transformation That Is Not Linear

Let $T{:}M_{nn} \rightarrow R$ be the transformation that maps an $n \times n$ matrix into its determinant; that is,

$$T(A) = \det(A)$$

If n>1, then this transformation does not satisfy either of the properties required of a linear transformation. For example, we saw Example 1 of Section 2.3 that

$$\det(A_1+A_2) \neq \det(A_1) + \det(A_2)$$

in general. Moreover, $\det(cA) = C^n \det(A)$, so

$$\det(cA) \neq c\det(A)$$

in general. Thus, T is not linear transformation.

Properties of Linear Transformation

If $T:V \rightarrow W$ is a linear transformation, then for any vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in V and any scalars c_1 and c_2, we have

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = T(c_1\mathbf{v}_1) + T(c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)$$

and more generally, if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are vectors in V and $c_1, c_2, \ldots, c_n$ are scalars, then

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \ldots + c_nT(\mathbf{v}_n) \qquad (1)$$

Formula (1) is sometimes described by saying that ***linear transformations preserve linear combinations***.

Theorem 8.1.1

If $T: V \rightarrow W$ is a linear transformation, then:

(a) $T(0) = 0$

(b) $T(-\mathbf{v}) = -T(\mathbf{v})$ *for all* $\mathbf{v}$ *in* V

(c) $T(\mathbf{v}-\mathbf{w}) = T(\mathbf{v}) - T(\mathbf{w})$ *for all* $\mathbf{v}$ *and* $\mathbf{w}$ *in* V

Proof.

(a) Let $\mathbf{v}$ be any vector in V. Since $0\mathbf{v}=0$, we have

$$T(\mathbf{0})=T(0\mathbf{v})=0T(\mathbf{v})=\mathbf{0}$$

(b) $T(-\mathbf{v}) = T((-1)\mathbf{v}) = (-1)T(\mathbf{v})=-T(\mathbf{v})$

(c) $\mathbf{v}-\mathbf{w}=\mathbf{v}+(-1)\mathbf{w}$; thus,

$$\begin{aligned} T(\mathbf{v}-\mathbf{w}) &= T(\mathbf{v}+(-1)\mathbf{w}) = T(\mathbf{v})+(-1)T(\mathbf{w}) \\ &= T(\mathbf{v})-T(\mathbf{w}) \end{aligned}$$

Finding Linear Transformations from Images of Basis

If $T: V \rightarrow W$ is a linear transformation, and if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is any basis for V, then the image $T(\mathbf{v})$ of any vector $\mathbf{v}$ in V can be calculated from images

$$T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)$$

of the basis vectors. This can be done by first expressing $\mathbf{v}$ as a linear combination of the basis vectors, say

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

and then using Formula(1) to write

$$T(\mathbf{v}) = c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2) + \ldots + c_n T(\mathbf{v}_n)$$

In words, ***a linear transformation is completely determined by its images of any basis vectors.***

Computing with Images of Basis Vectors

Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for R^3, where $\mathbf{v}_1 = (1,1,1)$, $\mathbf{v}_2 = (1,1,0)$, and $\mathbf{v}_3 = (1,0,0)$. Let $T: R^3 \rightarrow R^2$ be the linear transformation such that

$$T(\mathbf{v}_1)=(1,0), \quad T(\mathbf{v}_2)=(2,-1), \quad T(\mathbf{v}_3)=(4,3)$$

Find a formula for $T(x_1, x_2, x_3)$; then use this formula to compute $T(2,-3,5)$.

Solution.

We first express x = (x_1, x_2, x_3) as a linear combination of $\mathbf{v}_1$ =(1,1,1), $\mathbf{v}_2$ =(1,1,0), and $\mathbf{v}_3$ = (1,0,0). If we write

$$(x_1, x_2, x_3) = c_1(1,1,1) + c_2(1,1,0) + c_3(1,0,0)$$

then on equating corresponding components we obtain

$$\begin{aligned} c_1 + c_2 + c_3 &= x_1 \\ c_1 + c_2 \qquad &= x_2 \\ c_1 \qquad\qquad &= x_3 \end{aligned}$$

which yields $c_1 = x_3$, $c_2 = x_2 - x_3$, $c_3 = x_1 - x_2$, so that

$$(x_1, x_2, x_3) = x_3(1,1,1) + (x_2 - x_3)(1,1,0) + (x_1 - x_2)(1,0,0)$$
$$= x_3\mathbf{v}_1 + (x_2 - x_3)\mathbf{v}_2 + (x_1 - x_2)\mathbf{v}_3$$

Thus,

$$T(x_1, x_2, x_3) = x_3 T(\mathbf{v}_1) + (x_2 - x_3) T(\mathbf{v}_2) + (x_1 - x_2) T(\mathbf{v}_3)$$
$$= x_3(1,0) + (x_2 - x_3)(2,-1) + (x_1 - x_2)(4,3)$$
$$= (4x_1 - 2x_2 - x_3, 3x_1 - 4x_2 + x_3)$$

From this formula we obtain

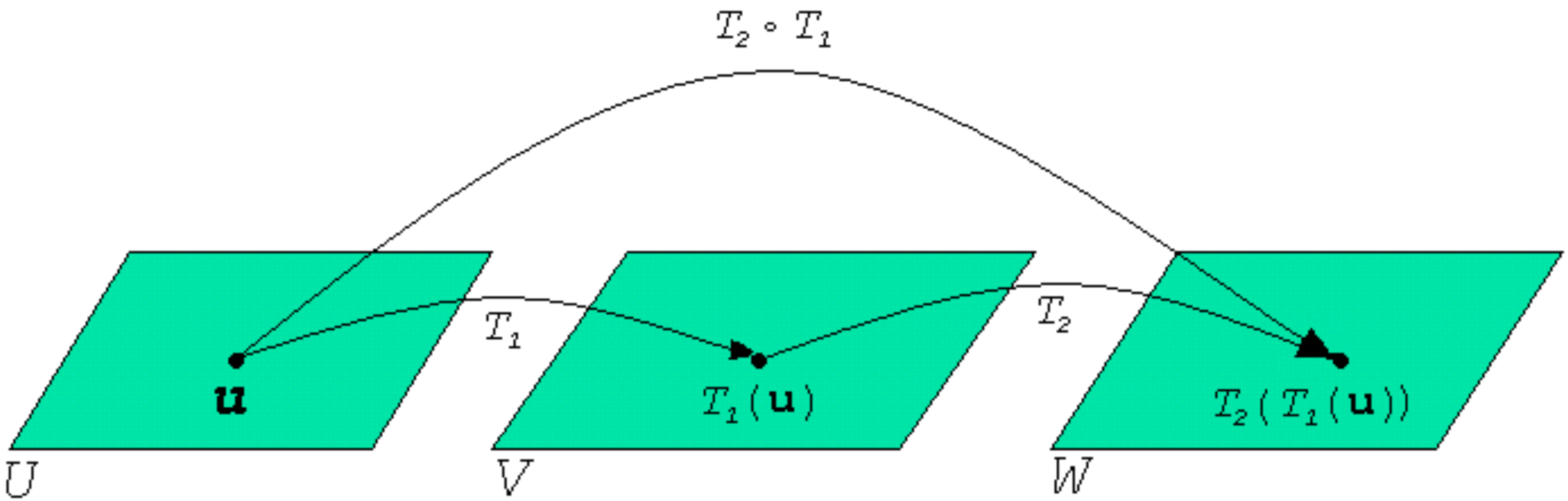
$$T(2, -3, 5) = (9,23)$$

Composition of T_2 with T_1

If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, *the composition of T_2 with T_1*, denoted by $T_2 \circ T_1$ (read "T_2 circle T_1"), is the function defined by the formula

$$(T_2 \circ T_1)(\mathbf{u}) = T_2(T_1(\mathbf{u})) \qquad (2)$$

where $\mathbf{u}$ is a vector in U



The composition of T_2 with T_1

- Theorem 8.1.2

If $T_1 : U \to V$ *and* $T_2 : V \to W$ *are linear transformations, then* $(T_2 \circ T_1) : U \to W$ *is also a linear transformation.*

Proof. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in U and c is a scalar, then it follows from (2) and the linearity of T_1 and T_2 that

$$
\begin{aligned}
(T_2 \circ T_1)(\mathbf{u}+\mathbf{v}) &= T_2(T_1(\mathbf{u}+\mathbf{v})) = T_2(T_1(\mathbf{u})+T_1(\mathbf{v})) \\
&= T_2(T_1(\mathbf{u})) + T_2(T_1(\mathbf{v})) \\
&= (T_2 \circ T_1)(\mathbf{u}) + (T_2 \circ T_1)(\mathbf{v})
\end{aligned}
$$

and

$$
\begin{aligned}
(T_2 \circ T_1)(c\,\mathbf{u}) &= T_2(T_1(c\,\mathbf{u})) = T_2(cT_1(\mathbf{u})) \\
&= cT_2(T_1(\mathbf{u})) = c(T_2 \circ T_1)(\mathbf{u})
\end{aligned}
$$

Thus, $T_2 \circ T_1$ satisfies the two requirements of a linear transformation.

Composition of Linear Transformations

Let $T_1 : P_1 \rightarrow P_2$ and $T_2 : P_2 \rightarrow P_2$ be the linear transformations given by the formulas

$$T_1(p(x)) = xp(x) \quad \text{and} \quad T_2(p(x)) = p(2x+4)$$

Then the composition is $(T_2 \circ T_1): P_1 \rightarrow P_2$ is given by the formula

$$(T_2 \circ T_1)(p(x)) = (T_2)(T_1(p(x))) = T_2(xp(x)) = (2x+4)p(2x+4)$$

In particular, if $p(x) = c_0 + c_1 x$, then

$$\begin{aligned}(T_2 \circ T_1)(p(x)) &= (T_2 \circ T_1)(c_0 + c_1 x) \\ &= (2x+4)(c_0 + c_1(2x+4)) \\ &= c_0(2x+4) + c_1(2x+4)^2\end{aligned}$$

Composition with the Identify Operator

If $T:V\to V$ is any linear operator, and if $I:V\to V$ is the identity operator, then for all vectors **v** in V we have

$$(T\circ I)(\mathbf{v}) = T(I(\mathbf{v})) = T(\mathbf{v})$$

$$(I\circ T)(\mathbf{v}) = I(T(\mathbf{v})) = T(\mathbf{v})$$

It follows that $T\circ I$ and $I\circ T$ are the same as T; that is,

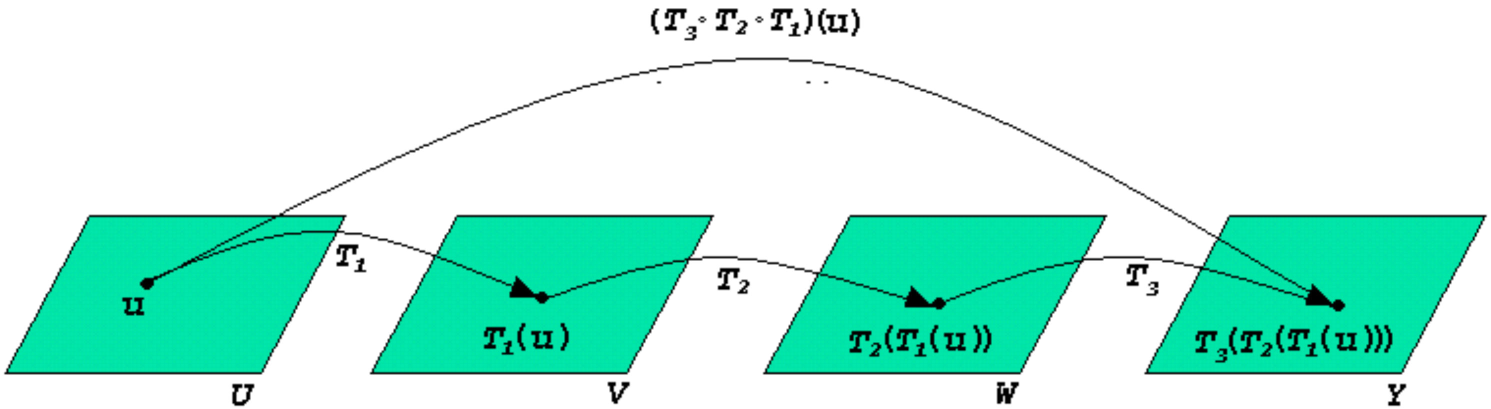
$$T\circ I = T \quad \text{and} \quad I\circ T = T \qquad (3)$$

We conclude this section by noting that compositions can be defined for more than two linear transformations. For example, if

$$T_1 : U\to V \text{ and } T_2 : V\to W \text{ ,and } T_3 : W\to Y$$

are linear transformations, then the composition $T_3 \circ T_2 \circ T_1$ is defined by

$$(T_3 \circ T_2 \circ T_1)(\mathbf{u}) = T_3(T_2(T_1(\mathbf{u}))) \tag{4}$$



The composition of three linear transformations