PROBLEM SOLVING-1

Problem-1

Suppose the mapping $F:\mathbf{R}^2 \to \mathbf{R}^2$ is defined by $F(x,y) = (x+y,\ x)$. Show that F is linear.

Solution :

We need to show that $F(v+w) = F(v) + F(w)$ and $F(kv) = kF(v)$, where u and v are any elements of $\mathbf{R}^2$ and k is any scalar. Let $v = (a, b)$ and $w = (a', b')$. Then

$$v + w = (a + a',\ b + b') \qquad \text{and} \qquad kv = (ka, kb)$$

We have $F(v) = (a + b, a)$ and $F(w) = (a' + b', a')$. Thus,

$$\begin{aligned} F(v+w) &= F(a + a',\ b + b') = (a + a' + b + b',\ a + a') \\ &= (a + b,\ a) + (a' + b',\ a') = F(v) + F(w) \end{aligned}$$

and

$$F(kv) = F(ka, kb) = (ka + kb,\ ka) = k(a + b,\ a) = kF(v)$$

Because v, w, k were arbitrary, F is linear.

Problem-2

Show that the following mappings are not linear:

(a) $F : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $F(x, y) = (xy, x)$

(b) $F : \mathbf{R}^2 \to \mathbf{R}^3$ defined by $F(x, y) = (x + 3,\ 2y,\ x + y)$

(c) $F : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $F(x, y, z) = (|x|,\ y + z)$

Solution :

(a) Let $v = (1,2)$ and $w = (3,4)$; then $v + w = (4,6)$. Also,

$$F(v) = (1(2), 1) = (2,1) \qquad \text{and} \qquad F(w) = (3(4), 3) = (12,3)$$

Hence,

$$F(v+w) = (4(6), 4) = (24, 6) \neq F(v) + F(w)$$

(b) Because $F(0,0) = (3,0,0) \neq (0,0,0)$, F cannot be linear.

(c) Let $v = (1,2,3)$ and $k = -3$. Then $kv = (-3, -6, -9)$. We have

$$F(v) = (1,5) \quad \text{and} \quad kF(v) = -3(1,5) = (-3, -15).$$

Thus,

$$F(kv) = F(-3, -6, -9) = (3, -15) \neq kF(v)$$

Accordingly, F is not linear.

Problem-3

Let $F : \mathbf{R}^4 \to \mathbf{R}^3$ be the linear mapping defined by

$$F(x, y, z, t) = (x - y + z + t, \quad x + 2z - t, \quad x + y + 3z - 3t)$$

Find a basis and the dimension of (a) the image of F, (b) the kernel of F.

Solution :

(a) Find the images of the usual basis of $\mathbf{R}^4$:

$$F(1,0,0,0) = (1,1,1), \qquad F(0,0,1,0) = (1,2,3)$$
$$F(0,1,0,0) = (-1,0,1), \qquad F(0,0,0,1) = (1,-1,-3)$$

Let A is the matrix representation of F.

$$A^t = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $(1,1,1)$ and $(0,1,2)$ form a basis for $\operatorname{Im} F$; hence, $\dim(\operatorname{Im} F) = 2$.

(b) Set $F(v) = 0$, where $v = (x, y, z, t)$; that is, set

$$F(x,y,z,t) = (x - y + z + t,\ \ x + 2z - t,\ \ x + y + 3z - 3t) = (0, 0, 0)$$

Set corresponding entries equal to each other to form the following homogeneous system whose solution space is Ker F:

$$\begin{aligned} x - y + \ z + \ t &= 0 \\ x \qquad + 2z - \ t &= 0 \\ x + y + 3z - 3t &= 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x - y + \ z + \ t &= 0 \\ y + \ z - 2t &= 0 \\ 2y + 2z - 4t &= 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x - y + z + \ t &= 0 \\ y + z - 2t &= 0 \end{aligned}$$

Thus, $(2, 1, -1, 0)$ and $(1, 2, 0, 1)$ form a basis of Ker F.
[As expected, $\dim(\operatorname{Im} F) + \dim(\operatorname{Ker} F) = 2 + 2 = 4 = \dim \mathbf{R}^4$, the domain of F.]

Problem-4

Consider the matrix mapping $A : \mathbf{R}^4 \to \mathbf{R}^3$, where $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix}$. Find a basis and the dimension of (a) the image of A, (b) the kernel of A.

Solution :

(a) The column space of A is equal to Im A. Now reduce A^T to echelon form:

$$A^T = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 8 \\ 3 & 5 & 13 \\ 1 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & -3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $\{(1,1,3),(0,1,2)\}$ is a basis of Im A, and $\dim(\text{Im } A) = 2$.

(b) Here Ker A is the solution space of the homogeneous system $AX = 0$, where $X = \{x, y, z, t)^T$. Thus, reduce the matrix A of coefficients to echelon form:

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 2 & 4 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{aligned} x + 2y + 3z + \ \ t &= 0 \\ y + 2z - 3t &= 0 \end{aligned}$$

Thus, $(1, -2, 1, 0)$ and $(-7, 3, 0, 1)$ form a basis for Ker A.

Problem-5

Exercise 6.1.9 (Ex. 54 (edited), p. 372) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that $T(1,1) = (0,2)$ and $T(1,-1) = (2,0)$.

1. Compute $T(1,4)$.

Solution: We have to write

$(1,4) = a(1,1)+b(1,-1)$. *Solving* $(1,4) = 2.5(1,1)-1.5(1,-1)$.

So,

$T(1,4) = 2.5T(1,1)-1.5T(1,-1) = 2.5(0,2)-1.5(2,0) = (-3,5)$.