Linear Transformation

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Jordan form of matrices

Defn: A Jorden block is a square matrix whose diagonal elements are all equal, whose superdiagonal elements (those immediately above the main diagonal) all equal 1, and whose other elements are all zero. It has the form

$$\begin{bmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \dots & 0 & 0 \\ 0 & 0 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{bmatrix}$$

A Jordan block is completely determined by its order and the value of its diagonal elements.

Jordan canonical form

Defn: A matrix is in Jordan canonical form if it is a diagonal matrix or if it has one of the following two partitioned forms:

$$\begin{bmatrix} D & 0 & 0 & 0 & 0 \\ 0 & J_1 & 0 & 0 & 0 \\ 0 & 0 & J_2 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & J_k \end{bmatrix} \qquad \begin{bmatrix} J_1 & 0 & 0 & 0 & 0 \\ 0 & J_2 & 0 & 0 & 0 \\ 0 & 0 & J_3 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & J_k \end{bmatrix}$$

Where D denotes diagonal matrix and J_i denotes Jordan blocks.

Examples:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} & 0 \\ 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 1 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 8 & 1 \\ 0 & 8 \end{bmatrix} & 0 & 0 \\ 0 & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 & [0] \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 8 & 1 \\ 0 & 8 \end{bmatrix} & & 0 \\ & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \\ 0 & & [0] \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} [1] & & & & \\ & [2] & & & \\ & & [3] & & \\ & & & [4] & \\ & & & & [5] \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \end{bmatrix}$$

How to find Jordan canonical form:

- For a transformation T : V $\longrightarrow$ V , use a basis B to get a matrix $A = [T]_B$ for the transformation. If you are just given a matrix, use that matrix.

- Compute the characteristic polynomial of the transformation and find the Eigen values & linearly independent Eigen vectors of the transformation. If the characteristic polynomial not factorizable, we will not be able to put the transformation into Jordan canonical form.

- For each Eigen value λ, compute the dimensions of $N((A-\lambda I)^r)$ for r = 1, 2, 3, … k until $\dim(N((A-\lambda I)^r)) = k$, say $\dim(N((A-\lambda I)^s)) = k$, k is the algebraic multiplicity of λ. Letting $d_i = \dim(N((A-\lambda I)^r))$ this will give us a sequence $0 = d_0 < d_1 < d_2 < ... < d_s$

- The sequence $d_0, d_1 \ldots d_s$ determines how many Jordan blocks corresponding to λ we have and their respective sizes. Here is one way to interpret the sequence:
  - $d_1 - d_0 = d_1$ tells us how many blocks of size at least one there are.
  - Then, $d_2 - d_1$ tells how many of those blocks are actually of size at least two.
  - Then, $d_3 - d_2$ tells us how many of those blocks are actually of size at least three.
  - Repeating this, we can see exactly how many blocks of each size we have.

Examples:

Find a Jordan canonical form of the matrix $\begin{bmatrix} 4 & 1 \\ -1 & 6 \end{bmatrix}$

Soln:

$$A-\lambda I=\begin{bmatrix} 4 & 1 \\ -1 & 6 \end{bmatrix}-\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$|A-\lambda I|=\begin{vmatrix} 4-\lambda & 1 \\ -1 & 6-\lambda \end{vmatrix}=(\lambda-5)^2$$

Hence 5 is the Eigen value with algebraic multiplicity 2.

$$A-5I=\begin{bmatrix} 4-5 & 1 \\ -1 & 6-5 \end{bmatrix}=\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

Therefore the $\dim(N(A-\lambda I))=1$. Hence the canonical form having 1 block. So that the canonical form of the matrix is $\begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$.

Examples:

Find a Jordan canonical form of the matrix $\begin{bmatrix} -3 & -1 & 1 \\ -1 & -3 & 1 \\ -2 & -2 & 0 \end{bmatrix}$

Soln:

$$A - \lambda I = \begin{bmatrix} -3 & -1 & 1 \\ -1 & -3 & 1 \\ -2 & -2 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -3-\lambda & -1 & 1 \\ -1 & -3-\lambda & 1 \\ -2 & -2 & -\lambda \end{vmatrix} = (\lambda + 2)^3$$

Hence -2 is the Eigen value with algebraic multiplicity 3.

$$A + 2I = \begin{bmatrix} -3+2 & -1 & 1 \\ -1 & -3+2 & 1 \\ -2 & -2 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ -2 & -2 & 2 \end{bmatrix}$$

$Rank(A-\lambda I)=1$

Therefore the $\dim(N(A-\lambda I))=2$. Hence the canonical form having 2 block.

$$(A+2I)^2=\begin{bmatrix}-1 & -1 & 1\\ -1 & -1 & 1\\ -2 & -2 & 2\end{bmatrix}^2=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$Rank(A-\lambda I)^2=0$

Therefore the $\dim((N(A-\lambda I)^2))=3$. Hence the canonical form having 3-2=1 block of size at least two.

Hence the canonical form is

$$\begin{bmatrix}\begin{bmatrix}-2 & 1\\ 0 & -2\end{bmatrix} & 0\\ 0 & \begin{bmatrix}-2\end{bmatrix}\end{bmatrix}$$

Examples:

Find a Jordan canonical form of the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix}$

Soln:

$$A-\lambda I=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix}-\begin{bmatrix} \lambda & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix}$$

$$|A-\lambda I|=\begin{vmatrix} 1-\lambda & 0 & 0 & 0 & 0 \\ 1 & -1-\lambda & 0 & 0 & -1 \\ 1 & -1 & -\lambda & 0 & -1 \\ 0 & 0 & 0 & -\lambda & -1 \\ -1 & 1 & 0 & 0 & 1-\lambda \end{vmatrix}=x^4(x-1)$$

Hence 0 is the Eigen value with algebraic multiplicity 4 & 1 is another Eigen value with algebraic multiplicity 1.

Case 1: $\lambda = 0$.

$$A+0I=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$Rank(A-\lambda I)=3$$

Therefore the $\dim(N(A-\lambda I))=2$. Hence the canonical form having 2 block.

$$(A+0I)^2=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix}^2=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$Rank(A-\lambda I)^2 = 2$

Therefore the $\dim(N(A-\lambda I)^2) = 3$. Hence the canonical form having 3-2=1 block of size at least two.

If the canonical form having Jordan block of size 2, then we will get 2 Jordan block with one of them having size 1 another one of size two, total sum is 3. But in Jordan canonical form the size allotted for Eigen value zero is 4.

$$(A+0I)^3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$Rank(A-\lambda I)^3 = 1$

Therefore the $\dim(N(A-\lambda I)^3) = 4$. Hence the canonical form having 4-3=1 block of size at least 3. So that the Jordan canonical form having one Jordan block of size 1 and one Jordan block of size 3 corresponding to Eigen value zero.

Case 2: $\lambda = 1$.

$$A - I = \begin{bmatrix} 1-1 & 0 & 0 & 0 & 0 \\ 1 & -1-1 & 0 & 0 & -1 \\ 1 & -1 & 0-1 & 0 & -1 \\ 0 & 0 & 0 & 0-1 & -1 \\ -1 & 1 & 0 & 0 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & -1 \\ 1 & -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 \\ -1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$Rank(A - \lambda I) = 4$

Therefore the $\dim(N(A - \lambda I)) = 1$. Hence the canonical form having 1 block.

Hence the canonical form is

$$\begin{bmatrix} [1] & & 0 \\ & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} & \\ 0 & & [0] \end{bmatrix}$$