Least-Square-Problem

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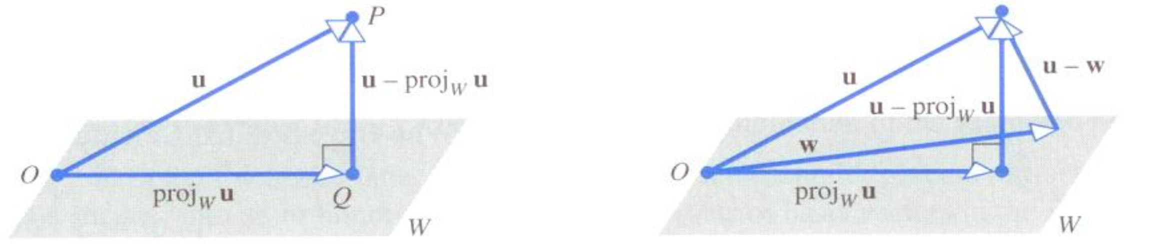
6-4 Orthogonal Projections Viewed as Approximations

- If P is a point in 3-dimensional space and W is a plane through the origin, then the point Q in W closest to P is obtained by dropping a perpendicular from P to W.



(a) Q is the point in W closest to P. (b) $\|\mathbf{u} - \mathbf{w}\|$ is minimized by $\mathbf{w} = \text{proj}_W\, \mathbf{u}$.

Figure 6.4.1

- If we let $\mathbf{u} = OP$, the distance between P and W is given by $|| \mathbf{u} - \text{proj}_W\mathbf{u} ||$.
- In other words, among all vectors $\mathbf{w}$ in W the vector $\mathbf{w} = \text{proj}_W\mathbf{u}$ minimize the distance $|| \mathbf{u} - \mathbf{w} ||$.

6-4 Best Approximation

- Remark
  - Suppose $\mathbf{u}$ is a vector that we would like to approximate by a vector in W.
  - Any approximation $\mathbf{w}$ will result in an "error vector" $\mathbf{u} - \mathbf{w}$ which, unless $\mathbf{u}$ is in W, cannot be made equal to $\mathbf{0}$.
  - However, by choosing $\mathbf{w} = \text{proj}_W\mathbf{u}$ we can make the length of the error vector $\| \mathbf{u} - \mathbf{w} \| = \| \mathbf{u} - \text{proj}_W\mathbf{u} \|$ as small as possible.
  - Thus, we can describe $\text{proj}_W\mathbf{u}$ as the "*best approximation*" to $\mathbf{u}$ by the vectors in W.

Theorem 6.4.1
(Best Approximation Theorem)

- If W is a finite-dimensional subspace of an inner product space V, and if $\mathbf{u}$ is a vector in V,
  - then $\text{proj}_W\mathbf{u}$ is the best approximation to $\mathbf{u}$ form W in the sense that
$$\|\mathbf{u} - \text{proj}_W\mathbf{u}\| < \|\mathbf{u} - \mathbf{w}\|$$
    for every vector $\mathbf{w}$ in W that is different from $\text{proj}_W\mathbf{u}$.

6-4 Least Square Problem

- Given a linear system $A\mathbf{x} = \mathbf{b}$ of m equations in n unknowns
  - find a vector $\mathbf{x}$, if possible, that <u>minimize $\| A\mathbf{x} - \mathbf{b} \|$ with respect to the Euclidean inner product on R^m</u>.
  - Such a vector is called a <u>least squares solution</u> of $A\mathbf{x} = \mathbf{b}$.

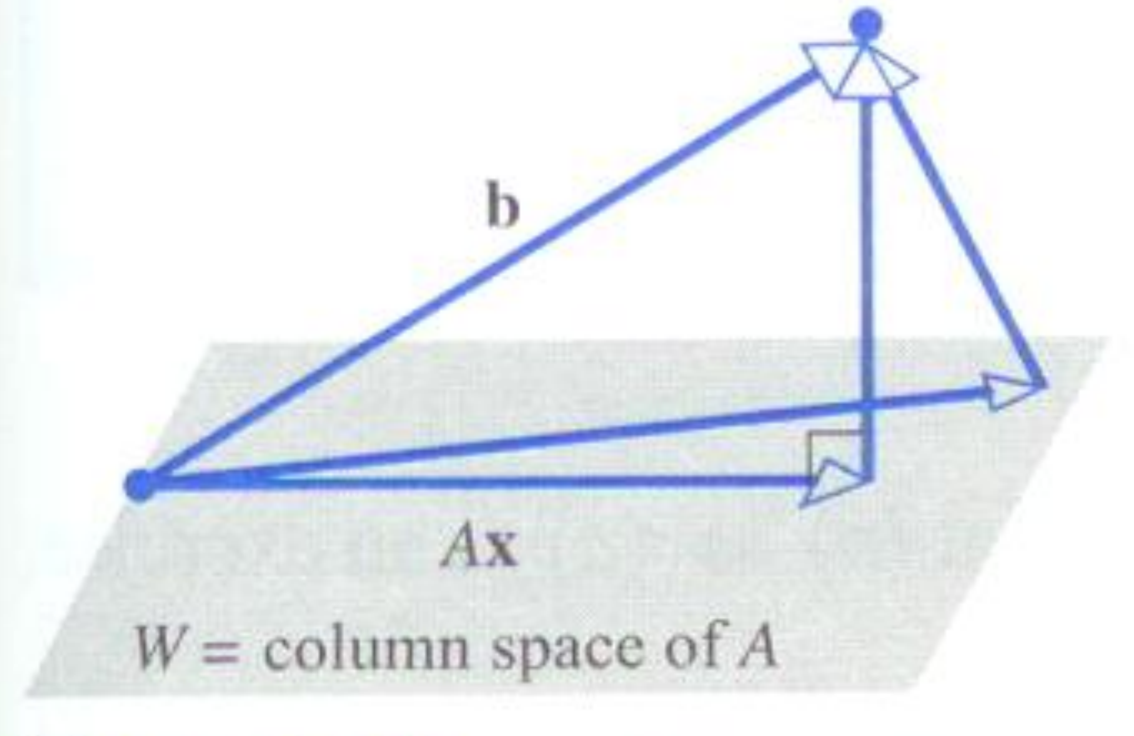


Figure 6.4.2 A least squares solution **x** produces the

One way to find a least squares solution of $A\mathbf{x} = \mathbf{b}$ is to calculate the orthogonal projection $\mathrm{proj}_W \mathbf{b}$ on the column space W of A and then solve the equation

$$A\mathbf{x} = \mathrm{proj}_W \mathbf{b} \tag{2}$$

However, we can avoid calculating the projection by rewriting (2) as

$$\mathbf{b} - A\mathbf{x} = \mathbf{b} - \mathrm{proj}_W \mathbf{b}$$

and then multiplying both sides of this equation by A^T to obtain

$$A^T(\mathbf{b} - A\mathbf{x}) = A^T(\mathbf{b} - \mathrm{proj}_W \mathbf{b}) \tag{3}$$

Since $\mathbf{b} - \text{proj}_W \mathbf{b}$ is the component of $\mathbf{b}$ that is orthogonal to the column space of A, it follows from Theorem 4.8.7(b) that this vector lies in the null space of A^T, and hence that

$$A^T(\mathbf{b} - \text{proj}_W \mathbf{b}) = \mathbf{0}$$

Thus, (3) simplifies to

$$A^T(\mathbf{b} - A\mathbf{x}) = \mathbf{0}$$

which we can rewrite as

$$A^T A\mathbf{x} = A^T\mathbf{b} \tag{4}$$

This is called the ***normal equation*** or the ***normal system*** associated with $A\mathbf{x} = \mathbf{b}$. When viewed as a linear system, the individual equations are called the ***normal equations*** associated with $A\mathbf{x} = \mathbf{b}$.

Theorem 6.4.2

- For *any* linear system $A\mathbf{x} = \mathbf{b}$, the associated normal system

$$A^T A\mathbf{x} = A^T\mathbf{b}$$

is *consistent*, and all solutions of the normal system are least squares solutions of $A\mathbf{x} = \mathbf{b}$.

Moreover, if W is the column space of A, and $\mathbf{x}$ is any least squares solution of $A\mathbf{x} = \mathbf{b}$, then the orthogonal projection of $\mathbf{b}$ on W is

$$\text{proj}_W\mathbf{b} = A\mathbf{x}$$

(or you can treat it as $A\mathbf{x} - \text{proj}_W\mathbf{b} = \mathbf{0}$)

Theorem 6.4.3

- If A is an $m \times n$ matrix, then the following are equivalent.
  - A has linearly independent column vectors.
  - $A^T A$ is invertible.

Theorem 6.4.4

- If A is an $m \times n$ matrix with linearly independent column vectors,
  - then for every $m \times 1$ matrix $\mathbf{b}$, the linear system $A\mathbf{x} = \mathbf{b}$ has a unique least squares solution.
  - This solution is given by

$$\mathbf{x} = (A^T A)^{-1} A^T \mathbf{b}$$

  - Moreover, if W is the column space of A, then the orthogonal projection of $\mathbf{b}$ on W is

$$\text{proj}_W \mathbf{b} = A\mathbf{x} = A(A^T A)^{-1} A^T \mathbf{b}$$

6-4 Example 1 (Least Squares Solution)

- Find the least squares solution of the linear system $A\mathbf{x} = \mathbf{b}$ given by

$$x_1 - x_2 = 4$$
$$3x_1 + 2x_2 = 1$$
$$-2x_1 + 4x_2 = 3$$

and find the orthogonal projection of **b** on the column space of A.

- Solution:

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

  - Observe that A has linearly independent column vectors, so we know in advance that there is a unique least squares solution.

Example 1

- We have

$$A^T A = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix}$$

$$A^T \mathbf{b} = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$$

so the normal system $A^{\mathrm{T}}A\mathbf{x} = A^T\mathbf{b}$ in this case is $\begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$

- Solving this system yields the least squares solution

$$x_1 = 17/95,\ x_2 = 143/285$$

- The orthogonal projection of **b** on the column space of A is

$$A\mathbf{x} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 17/95 \\ 143/285 \end{bmatrix} = \begin{bmatrix} -92/285 \\ 439/285 \\ 94/57 \end{bmatrix}$$

Example 2
(Orthogonal Projection on a Subspace)

- Find the orthogonal projection of the vector **u** = (-3,-3,8,9) on the subspace of R^4 spanned by the vectors

$$\mathbf{u}_1 = (3,1,0,1),\ \mathbf{u}_2 = (1,2,1,1),\ \mathbf{u}_3 = (-1,0,2,-1)$$

- Solution:
  - The subspace spanned by $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, is the column space of

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

  - If **u** is expressed as a column vector, we can find the orthogonal projection of **u** on W by finding a least squares solution of the system $A\mathbf{x} = \mathbf{u}$.
  - $\text{proj}_W\mathbf{u} = A\mathbf{x}$ from the least square solution.

6-4 Example 2

- From Theorem 6.4.4, the least squares solution is given by

$$\mathbf{x} = (A^TA)^{-1}A^T\mathbf{u}$$

- That is,

$$\mathbf{x} = \left(\begin{bmatrix} 3 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 3 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ -3 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

- Thus, $\text{proj}_W\mathbf{u} = A\mathbf{x} = [-2\ 3\ 4\ 0]^T$

Assignement-1

In Exercises 3–6, find the least squares solution of the equation $A\mathbf{x} = \mathbf{b}$.

3. $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$

4. $A = \begin{bmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 6 \\ 0 \\ 9 \\ 3 \end{bmatrix}$

6. $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & -2 & 2 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 0 \\ 6 \\ 0 \\ 6 \end{bmatrix}$

17. Find the orthogonal projection of $\mathbf{u}$ on the subspace of R^3 spanned by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$\mathbf{u} = (1, -6, 1); \ \ \mathbf{v}_1 = (-1, 2, 1), \ \ \mathbf{v}_2 = (2, 2, 4)$$

18. Find the orthogonal projection of $\mathbf{u}$ on the subspace of R^4 spanned by the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

$$\mathbf{u} = (6, 3, 9, 6); \ \ \mathbf{v}_1 = (2, 1, 1, 1), \ \ \mathbf{v}_2 = (1, 0, 1, 1),$$
$$\mathbf{v}_3 = (-2, -1, 0, -1)$$