Session 12

Derivatives +

Taylor's polynomial (local polynomial approximation)

Optimization in 1 variable

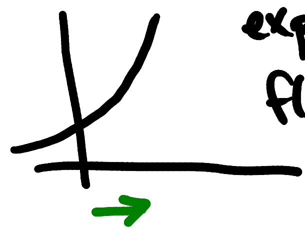
if $f'(x_0)$ exists

$$f(x) \approx \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{\text{the best linear approximation locally around } x_0}$$

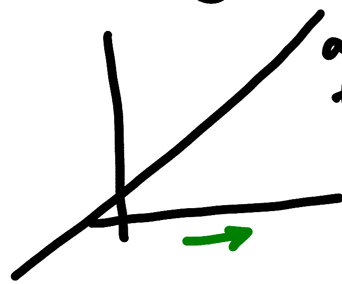
approx.

locally around x_0

the best linear approximation locally around x_0



exponential $f(x)=e^x$



affine function

polynomial of high degree

Master rules of derivation

1) product rule: $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$

2) chain rule: $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$

Product rule:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & \bullet \end{tikzcd}$$

$$\bullet \xrightarrow{fg} \bullet$$
$$x \longmapsto f(x) \cdot g(x)$$

$$f(x) \approx f(a) + f'(a)(x-a)$$
$$g(x) \approx g(a) + g'(a)(x-a)$$

product

$$\underbrace{f(x)g(x)}_{} \approx \boxed{f(a)g(a) + \boxed{\left[f'(a)g(a) + f(a)g'(a)\right]}(x-a)} + f'(a)g'(a)(x-a)^2$$

Best linear approximation

the best linear approximation of $f(x)g(x)$ has slope $f'(a)g(a) + f(a)g'(a)$

Remark:

We can follow the same rationale by using little-o formalism

$$f(x) = f(a) + f'(a)(x-a) + o(x-a)$$

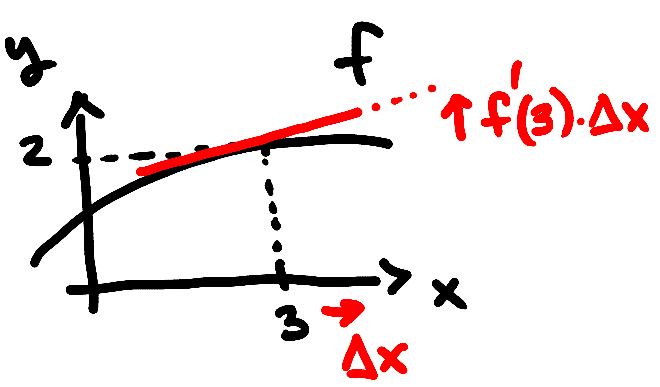
$\varepsilon(x)$ s.t. $\displaystyle\lim_{x \to a} \frac{\varepsilon(x)}{x-a} = 0$

Chain rule

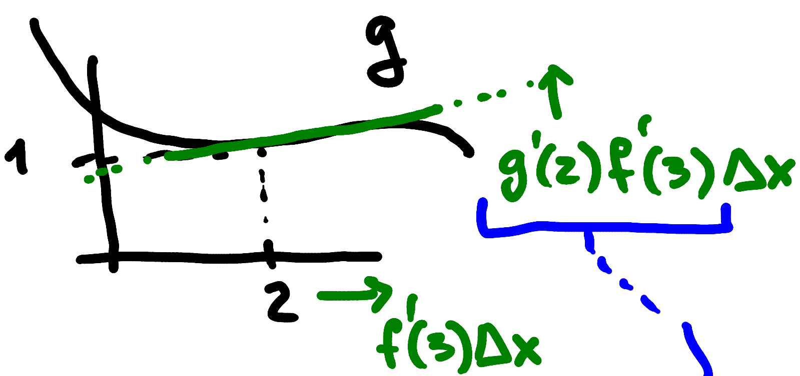
$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & \cdot \arrow[r, "g"] & \cdot \end{tikzcd}$$

$$x \longmapsto f(x) \mapsto g(f(x))$$
$$y \longmapsto g(y)$$

$$\left.\begin{aligned} &f(x) \approx f(a) + f'(a)(x-a) \quad \text{about the point } a \\ &g(x) \approx g(f(a)) + g'(f(a))(x - f(a)) \quad \text{about the point } f(a) \end{aligned}\right\} (*)$$



$f(3)=2$



$g(2)=1$

this is the candidate for $(g\circ f)'(3) = g'(f(3))\,f'(3)$

$$(g\circ f)(3) = g(f(3)) = g(2) = 1$$

$$\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot$$

Remark: If we compose two affine functions f and g, the resulting function is also affine and has slope = the product of slopes of f and g.

↓

Exercise
$$f(x) = m_1 x + n_1 \qquad m_i, n_i \in \mathbb{R}$$
$$g(x) = m_2 x + n_2$$

By applying this remark to $(*)$

$$(g\circ f)'(a) = g'(f(a))\cdot f'(a)$$

Exercise: $(h\circ g\circ f)'(a) = ??$

$$\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot \xrightarrow{h} \cdot$$

Main rules:

$$(f+g)'(a) = f'(a) + g'(a) \quad \rightarrow \text{check it out!}$$

$$(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a)$$

$$(g \circ h)'(a) = g'(f(a))f'(a)$$

Other rules:

1) $(cf)'(a) \overset{\text{Product}}{=} c'(a)f(a) + c(a)f'(a) = cf'(a)$

where c is constant

2) $\mathrm{Id}(x) = x \Rightarrow \mathrm{Id}'(a) = 1$ for all a

3) $f(x) = x^2 = x \cdot x$

$\Rightarrow f'(x) = 1 \cdot x + x \cdot 1 = 2x$

4) $f(x) = x^3 = x^2 \cdot x$

$\Rightarrow f'(x) = (2x)x + x^2 \cdot 1 = 3x^2$

5) $f(x) = x^n = x^{n-1} \cdot x$ $\qquad (x^n)' = nx^{n-1}$

Reasoning by integer induction

Assume (Induction hypothesis): $(x^{n-1})' = (n-1)x^{n-2}$

(product rule)

$$f'(x) = (x^{n-1})'x + x^{n-1}(x)'$$

$$= \left[(n-1)x^{n-2}\right]x + x^{n-1} \cdot 1$$

$$= nx^{n-1}$$

$\longrightarrow$ this is valid for all natural numbers.

6) $f(x) = \sqrt{x}$

$g(x) = x^2 \qquad g'(x) = 2x$

$(g \circ f)(x) = x$ $\qquad$ (assume $x > 0$)

$$\begin{tikzcd} \cdot \arrow[r, "\sqrt{x}"] \arrow[rr, bend right, "\text{Identity}"'] & \cdot \arrow[r, "x^2"] & \cdot \end{tikzcd}$$

chain rule $\downarrow$ $\qquad$ derivatives $\downarrow$

$$g'(f(x)) \cdot f'(x) = 1$$

$$2\sqrt{x} \cdot f'(x) = 1$$

$$\Downarrow$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

7) $f(x) = x^{\alpha} \qquad \alpha \in \mathbb{R}$

$\Rightarrow f'(x) = \alpha x^{\alpha - 1}$

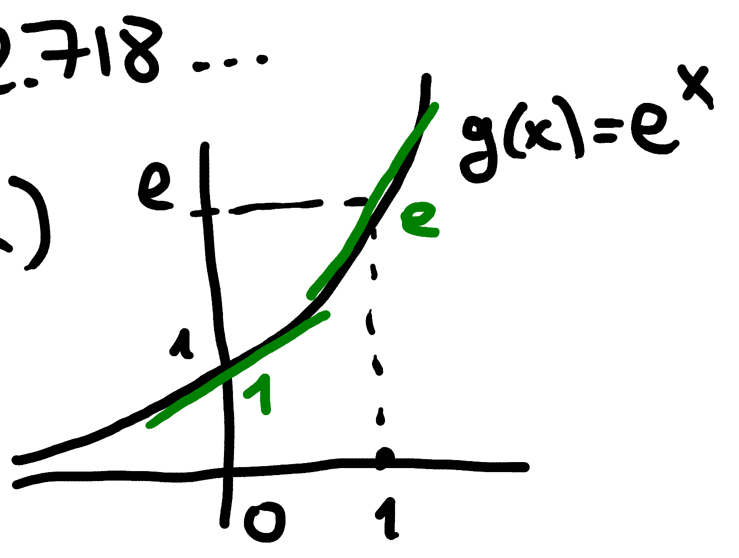
8) $f(x) = \log(x)$

$g(x) = e^x$

$\hookrightarrow$ Euler constant $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$

$\approx 2.718\ldots$

$g'(x) = g(x)$



$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[rr, bend right, "\text{Identity}"'] & \cdot \arrow[r, "g"] & \cdot \end{tikzcd}$$

chain rule

$$g(f(x)) = x$$

$$g'(f(x)) \cdot f'(x) \qquad\qquad 1$$

$$e^{f(x)} \cdot f'(x)$$

$$e^{\log(x)} \cdot f'(x)$$

$$x \cdot f'(x)$$

$$x f'(x) = 1$$

$$f'(x) = \frac{1}{x}$$

5*)

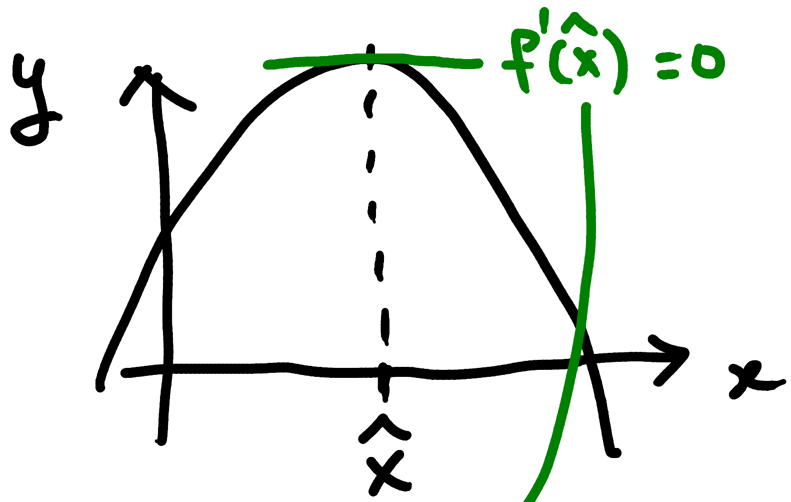
$$P(x) = \sum_{K=0}^{n} a_n x^n = a_0 + a_1 x + \dots + a_n x^n$$

$$P'(x) = \sum_{K=1}^{n} n a_n x^{n-1} = a_1 + 2a_2 x + \dots + n a_n x^{n-1}$$

derivative of a polynomial

Taylor's polynomial

$f(x)$



critical point of f: $\hat{x}$ where $f'(\hat{x}) = 0$

local approx.	degree
$f(a)$	0
$f(a) + f'(a)(x-a)$	1
$f(a) + f'(a)(x-a) + c(x-a)^2$	2
⋮	⋮
$P_{a,n}(x)$	n

If higher order derivatives of f exist at a, we can build a degree n polynomial that renders the best possible local approximation up to degree n.

Taylor's polynomial

$f(x) \to f(a)$
$\hookrightarrow f'(a)$

the best affine approximation at a takes same value and derivative at a as f

$$F(x) = f(a) + f'(a)(x-a)$$

$$f(a) = F(a)$$

$$f'(a) = F'(a)$$

what if we want the best quadratic approximation?

$$G(x) = f(a) + f'(a)(x-a) + c(x-a)^2$$

$\hookrightarrow G(a) = f(a)$

$$G'(a) = f'(a) + \boxed{\underbrace{2c(x-a)}_{x=a}} = f'(a)$$

$$\boxed{\underbrace{2c(x-a)}_{x=a}} = 0$$

derivative of the derivative

$$G''(a) = 2c$$

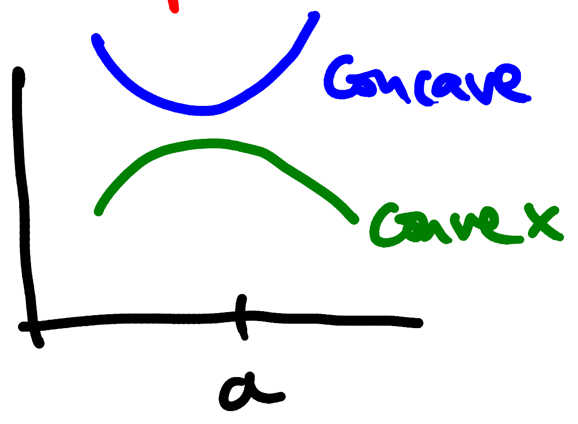
What is c if I want to ensure

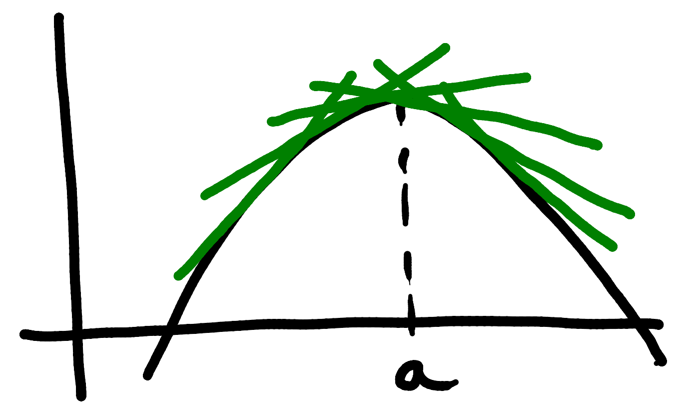
$$G''(a) = f''(a) \ ?$$

<u>Answer</u>: $2c = f''(a)$

$$\Rightarrow c = \frac{1}{2} f''(a)$$

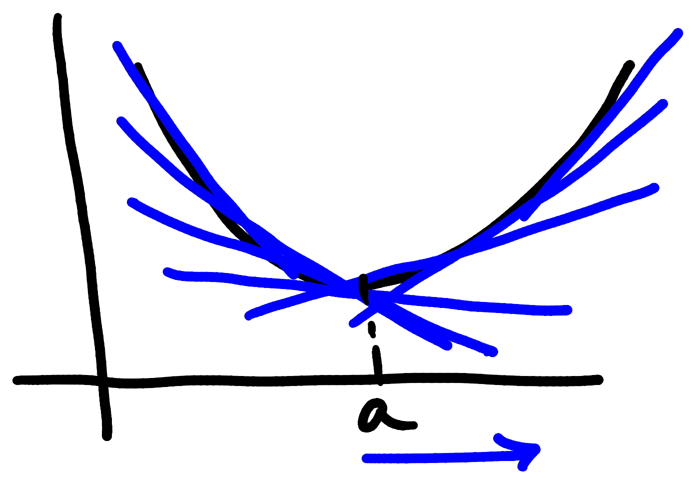
$$G(x) = f(a) + \underbrace{f'(a)(x-a)}_{\text{whether } a \text{ is a critical point}} + \underbrace{\frac{1}{2} f''(a)(x-a)^2}_{\text{local convexity of } f \text{ at } a}$$





$\to$ $f''(x) \to$ growth of $f'(x)$

$f''(a) < 0$



$f''(a) > 0$

a

$\downarrow$

is critical point?

$f'(a) = 0$

YES $\to$ what is the sign of $f''(a)$?

- > 0 $\to$ local MINIMUM
- $= 0$ $\to$ need higher order
- < 0 $\to$ local MAXIMUM

NO $\to$ STOP not interested