- Eigenvectors & Eigenvalues
- Singular Value Decomposition

$N(A-\lambda \mathrm{Id})$ ← eigenspaces

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$N(P) \rightsquigarrow \lambda_1 = 0$

$N(P-\mathrm{Id}) \rightsquigarrow \lambda_2 = 1$

P diagonalizes & the diagonal form $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

1) eigenvalues ✓ $\lambda_1 = 0, \lambda_2 = 1$

2) eigenvectors

2.1) $\lambda_1 = 0 \rightarrow N(P)$ by Gauss-Jordan elimination

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{array}{ccc} w_1 & u_1 & u_2 \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}$$

$C(P) = \mathrm{span}(w_1)$

$N(P) = \mathrm{span}(u_1, u_2)$

$\{u_1, u_2\}$ is a basis of $N(P)$

eigenvectors of P of eigenvalue $\lambda_1 = 0$ $= N(P) = \mathrm{span}(u_1, u_2)$

$u_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $u_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Remark: $u_1 \perp u_2$

2.2) $\lambda_2 = 1 \longrightarrow N(P-\mathrm{Id})$ by G-J elim.

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{C_2 \leftrightarrow C_3}{\sim} \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \underset{C_3 \leftarrow C_3 + C_2}{\sim} \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$N(P-\mathrm{Id}) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)$

Conclusion:

$N(P) = \mathrm{span}\left(\underset{u_1}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}, \underset{u_2}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}\right)$

$N(P-\mathrm{Id}) = \mathrm{span}\left(\underset{w_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}\right)$

Remark: $w_1 \perp u_1$

$w_1 \perp u_2$

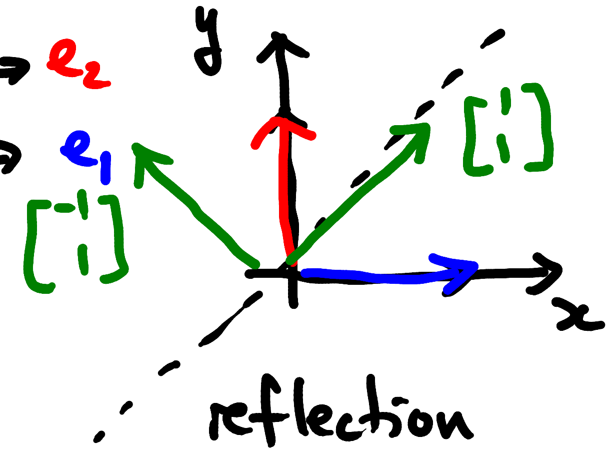
Remark: $\{u_1, u_2, w_1\}$ is a basis of $\mathbb{R}^3$ of eigenvectors of P

↳ "eigenbasis"

③ $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$e_1 \longmapsto e_2$

$e_2 \longmapsto e_1$



1) eigenvalues = roots of $Q(\lambda)$

$$Q(\lambda) = \det(A - \lambda \, Id)$$
$$= \det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right)$$
$$= \lambda^2 - 1$$
$$= (\lambda - 1)(\lambda + 1) \implies \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = -1 \end{matrix} \quad \text{eigenvalues of } A$$

2) eigenvectors

2.1) $\lambda_1 = 1$: $N(A - Id)$

$$\begin{array}{cc} -1 & 1 \\ 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{array} \sim \begin{array}{cc} -1 & 0 \\ 1 & 0 \\ 1 & \boxed{1} \\ 0 & \boxed{1} \end{array}$$

$N(A - Id) = \text{span}\left(\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{u_1}\right)$

2.2) $\lambda_2 = -1$: $N(A + Id)$

$$\begin{array}{cc} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{array} \sim \begin{array}{cc} 1 & 0 \\ 1 & 0 \\ 1 & \boxed{-1} \\ 0 & \boxed{1} \end{array}$$

$N(A + Id) = \text{span}\left(\underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{u_2}\right)$

$\{u_1, u_2\}$ is an orthogonal basis of $\mathbb{R}^2$ of eigenvectors of A

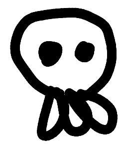
$\{u_1, u_2\}$ is an orthogonal eigenbasis of $\mathbb{R}^2$

④ $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = A$

1) eigenvalues of A = roots of $Q(\lambda)$

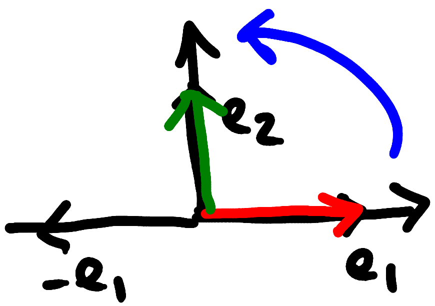
$$Q(\lambda) = \det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} \right)$$

$$= \lambda^2 + 1 \dashrightarrow$$



does not have roots in $\mathbb{R}$

$\Rightarrow$ there are no eigenvalues nor eigenvectors



$\frac{\pi}{2}$ rotation

<u>Important Result</u>: Spectral theorem for symmetric matrices

If $A \in \mathbb{R}^{n\times n}$ symmetric $(A^t = A)$ then the following hold:

1) there are $\lambda_1, \dots, \lambda_r \in \mathbb{R}$ such that

$$Q(\lambda) = \prod_{i=1}^{r} (\lambda - \lambda_i)^{k_i}$$

k_i is known as the "multiplicity" of the root λ_i

$$\sum_{i=1}^{r} k_i = n$$

doing a product with indices

$\sum_{i=1}^{r}$

2) v_i eigenvector of eigenvalue λ_i
v_j eigenvector of eigenvalue λ_j
with $\lambda_i \neq \lambda_j$

then $v_i \perp v_j$

3) $\dim N(A - \lambda_i I) = k_i$ for $i = 1 \dots r$

In particular, using 2) + 3)
there exists an orthogonal <u>eigen</u>basis of $\mathbb{R}^n$

$$\Downarrow$$

A is diagonalizable

"there is a basis in which the linear transformation encoded by A has diagonal matrix"

$$
\begin{tikzcd}
\underset{\text{eigenbasis}}{\mathbb{R}^n} \arrow[r, "D"] & \underset{\text{eigenbasis}}{\mathbb{R}^n} \arrow[d, "\substack{\text{decoding}\\ Q}"] \\
\underset{\text{canonical}}{\mathbb{R}^n} \arrow[u, "\substack{\text{encoding}\\ Q^t}"] \arrow[r, "A"] & \underset{\text{canonical}}{\mathbb{R}^n}
\end{tikzcd}
$$

$$Q = \left[\, q_1 \,\middle|\, \dots \,\middle|\, q_n \,\right]$$

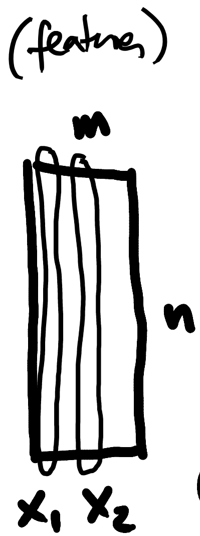
where the q_i are eigenvectors

$$q_1 \longmapsto \lambda_1 q_1$$
$$\vdots$$

$$A = QDQ^t$$
$$\Updownarrow$$
$$\underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \end{bmatrix}}_{\text{eigenvalues of } A} = D = Q^t A Q$$

Special case of symmetric matrices

$$\Omega = A^t A \qquad \Omega \text{ is symmetric}$$



$$A \in \mathbb{R}^{n \times m}$$

think about
tall-and-thin matrices

sample covariance

$$\operatorname{cov}(X_i, X_j) = \frac{1}{n-1} \sum_{t=1}^{n} (X_{ti} - \bar{X}_i)(X_{tj} - \bar{X}_j)$$

$$\bar{X}_i = \frac{1}{n} \sum_{t=1}^{n} X_{ti}$$

A data $\longrightarrow$ A centered data

$\longrightarrow \tilde{A} = \frac{1}{\sqrt{n-1}} A \dashrightarrow \Omega = \tilde{A}^t \tilde{A}$ covariance matrix

<u>Result</u> : $A \in \mathbb{R}^{n \times m}$ $\quad$ $\text{rank}(A) = m$

$A^t A$ satisfies two important properties

rank$(A^t A) = m$

$A^t A \in \mathbb{R}^{m \times m}$

full rank

1) $A^t A$ symmetric $\quad (A^t A)^t = A^t (A^t)^t = A^t A$

2) $A^t A$ has only positive eigenvalues

$\lambda_1, \dots, \lambda_r$ are eigenvalues of $A^t A$

$\Rightarrow \lambda_1, \dots, \lambda_r > 0$

positive-definite

<u>Remark</u> :

rank$(A) < m$

then there are eigenvalues $= 0$

Remark:

A diagonable $A \in \mathbb{R}^{n \times n}$ $\quad A = (a_{ij})$

1) $Q(\lambda) = \lambda^n + \alpha_1 \lambda^{n-1} + \dots + \alpha_n \qquad a_i \in \mathbb{R}$

$$\alpha_1 = \text{trace}(A) = \sum_{i=1} a_{ii}$$

2) trace(A) does not depend on the basis of the linear transformation

$$\text{trace}(C^{-1}AC) = \text{trace}(A)$$

since A diagonalizes,
there is a C (matrix of change of basis)

$$C^{-1}AC = D \Rightarrow \text{trace}(A) = \text{sum of all the eigenvalues with multiplicity}$$

"trace is invariant by change of basis"

diagonal entries are the eigenvalues (possibly repeated)

Singular Value Decomposition

so far $A \in \mathbb{R}^{n\times n}$ symmetric
$A = QDQ^t$
↳ D: diagonal

$$\begin{tikzcd} \text{eigenbasis} \arrow[r, "D"] & \text{eigenbasis} \arrow[d, "Q"] \\ \text{canonical} \arrow[u, "Q^t"] \arrow[r, "A"] & \text{canonical} \end{tikzcd}$$

$A \in \mathbb{R}^{n\times m}$ (n rows, m columns)
$\text{rank}(A) = m$

$$A = \underbrace{U}_{\text{orthogonal},\ U \in \mathbb{R}^{n\times n}} \ \Sigma \ \underbrace{V^t}_{\text{orthogonal},\ V \in \mathbb{R}^{m\times m}}$$

Σ : only diagonal entries are non-zero.
↓
$\sigma_1, \dots, \sigma_m$ are know as "singular values"

$$= U \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ & & \sigma_m \\ & 0 & \end{bmatrix} V^t$$

$$\begin{tikzcd} \mathbb{R}^m \arrow[r, "A"] & \mathbb{R}^n \arrow[d, "\text{encoding}"] \\ \mathbb{R}^m \arrow[u, "\text{decoding}"] \arrow[r, "\Sigma"] & \mathbb{R}^n \end{tikzcd}$$

canonical basis (top $\mathbb{R}^m$) canonical basis (top $\mathbb{R}^n$)

new basis = $\{v_1, \dots, v_m\}$ (bottom $\mathbb{R}^m$) new basis = $\{u_1, \dots, u_n\}$ (bottom $\mathbb{R}^n$)

$V = [v_1 | \cdots | v_m]$

$U = [u_1 | \cdots | u_n]$

$$\Sigma = U^t A V \iff U \Sigma V^t = \underbrace{U\,(U^t}_{Id} A \underbrace{V)\,V^t}_{Id} = A$$

Important result :

$A \in \mathbb{R}^{n\times m}$ $\text{rank}(A) = m$
There are orthogonal matrices $U \in \mathbb{R}^{n\times n}$, $V \in \mathbb{R}^{m\times m}$
and scalars $\sigma_1, \dots, \sigma_m > 0$
such that

$$A = U \Sigma V^t$$

where
$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ & & \sigma_m \\ & 0 & \end{bmatrix} \in \mathbb{R}^{n\times m}$$

1) $V = [v_1 | \cdots | v_m]$
v_i are eigenvectors of A^tA

2) $U = [u_1 | \cdots | u_n]$
u_i are eigenvectors of AA^t

3) $\sigma_i = \sqrt{\lambda_i}$ where λ_i are the eigenvalues of A^tA