Final Test (Wednesday 15th Dec 2021)
Elements of Mathematics – Bioinformatics for Health Sciences

1. **(1 point)** Find a basis of the column space $C(A)$ of the following matrix A:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & -1 \\ -2 & -2 & 2 \end{bmatrix}$$

Answer:

Via Gauss-Jordan elimination we can verify that $\text{rank}(A) = 2$. In particular, if u, v and w are the columns of A, it turns out that $w = 2v - 3u$. In this case, any two columns of A constitutes a basis of $C(A)$.

2. **(1 point)** Find an example of a 3×3 matrix A such that $\dim C(A) = 2$ and $\dim N(A^2) = 2$.

Answer:

Using the fundamental theorem of linear algebra, we know that $\dim C(A) + \dim N(A) = 3$, so the first assumption implies $\dim N(A) = 1$. A typical way to think about this kind of examples is to consider the image of each vector of the canonical basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ by the linear transformation encoded by A. Note that if the said linear transformation f does $f(e_1) = 0$, $f(e_2) = e_1$ and $f(e_3) = e_3$, when we do the composition $f^2 = f \circ f$ of f with itself, we get $f^2(e_1) = f(f(e_1)) = f(0) = 0$, $f^2(e_2) = f(f(e_2)) = f(e_1) = 0$ and $f^2(e_3) = f(f(e_3)) = f(e_3) = e_3$. The matrix A that realizes the linear transformation f is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In other words, $N(A) = \text{span}\{e_1\}$ and $N(A^2) = \text{span}\{e_1, e_2\}$.

3. **(2 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

a) **(1 point)** Find a basis of $N(A)$ the null space of A.
b) **(1 point)** Find an orthonormal basis of $N(A)$.

Answer:

a) Let's apply Gauss-Jordan elimination

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{i}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 3 \\ 1 & 0 & 0 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{ii}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 1/2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 1 \end{bmatrix},$$

From this calculation we see that $N(A) = \text{span}\{(1, -3, 2)\}$.

b) Note we just need to find a unit length representative of $N(A)$. If $v = (1, -3, 2)$ we can pick $u = v/\text{length}(v) = \frac{1}{\sqrt{15}}(1, -3, 2)$. Then $\{u\}$ is an orthonormal basis of $N(A)$.

4. **(1 point)** Given the 2×2 matrices:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Are the eigenvalues of AB equal to the eigenvalues of BA?

Answer:

Let's compute AB and BA.

$$AB = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

Note that $\det(AB) = \det(BA) = 1$ and $\text{trace}(AB) = \text{trace}(BA) = 4$. But the trace and determinant define the eigenvalues, as the eigenvalues of a given 2×2 matrix Ω must fullfill $\lambda_1 + \lambda_2 = \text{trace}(\Omega)$ and $\lambda_1 \lambda_2 = \det(\Omega)$. Therefore the eigenvalues in both cases must coincide.

5. **(1 point)** Provide the Taylor approximation of order 2 at $a = 0$ of the function $f(x) = 1/(1 + e^{-x})$.

Answer:

Let's compute the first and second-order derivatives of $f(x)$:

$$f'(x) = \frac{-1}{(1+e^{-x})^2} \cdot (-e^{-x}) = \frac{e^{-x}}{(1+e^{-x})^2}$$

Applying directly the product rule,

$$\begin{aligned} f''(x) &= -e^{-x} \frac{1}{(1+e^{-x})^2} + e^{-x} \frac{-2}{(1+e^{-x})^3} \cdot (-e^{-x}) = \frac{2e^{-2x}}{(1+e^{-x})^3} - \frac{e^{-x}}{(1+e^{-x})^2} = \\ &= \frac{2e^{-2x}}{(1+e^{-x})^3} - \frac{e^{-x}(1+e^{-x})}{(1+e^{-x})^3} = \frac{e^{-2x} - e^{-x}}{(1+e^{-x})^3} \end{aligned}$$

Alternatively, we could also observe that $f'(x) = f(x)(1 - f(x))$, then $f''(x) = f'(x)(1 - f(x)) - f(x)f'(x) = f'(x)(1 - 2f(x))$. Note that both expressions coincide, since

$$f'(x)(1 - 2f(x)) = \frac{e^{-x}}{(1+e^{-x})^2} \frac{e^{-x} - 1}{(1+e^{-x})} = \frac{e^{-2x} - e^{-x}}{(1+e^{-x})^3}$$

Evaluating f and its first and second-order derivatives at $a = 0$ gives

$$f(0) = 1/2, \quad f'(0) = 1/4, \quad f''(0) = 0$$

The Taylor approximation up to order 2 is given by a linear polynomial $T(x) = \frac{1}{2} + \frac{1}{4}x$, in other words, $f(x) = \frac{1}{2} + \frac{1}{4}x + o(x^2)$ about the point $a = 0$.

6. **(2 points)** Consider the following function:

$$f(x, y) = e^{-(ax^2 + by^2)}$$

where $a, b \in \mathbb{R}$ are parameters of the function. Compute $\nabla f(1, 1)$, i.e., the gradient vector of f at the point (1,1).

Answer:

Let's compute partial derivatives:

$$\frac{\partial f}{\partial x} = e^{-(ax^2+by^2)}(-2ax) = -2axe^{-(ax^2+by^2)}$$
$$\frac{\partial f}{\partial y} = e^{-(ax^2+by^2)}(-2by) = -2bye^{-(ax^2+by^2)}$$

Whence $\nabla f(1,1) = (-2ae^{-(a+b)}, -2be^{-(a+b)})$

7. **(2 points)** Determine the nature of all the critical points of the function

$$f(x,y) = x^3 - x + y^3 - y.$$

Answer:

Let's compute partial derivatives:

$$\frac{\partial f}{\partial x} = 3x^2 - 1, \quad \frac{\partial f}{\partial y} = 3y^2 - 1$$
$$\frac{\partial^2 f}{\partial x^2} = 6x, \quad \frac{\partial^2 f}{\partial y^2} = 6y, \quad \frac{\partial^2 f}{\partial x \partial y} = 0$$

The function has four critical points, namely all the combinations $(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}})$. To determine their nature we must study the spectral decomposition of the Hessian matrix

$$Hf(x,y) = \begin{bmatrix} 6x & 0 \\ 0 & 6y \end{bmatrix}$$

The Hessian has trivial eigenvalues $6x$ and $6y$. Accordingly, the points $(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$ and $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ are saddle points, $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$ is a local maximum and $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ is a local minimum.