Final test (Wednesday 13 December 2023, 15:00-17:00 CET)
Elements of Mathematics – Bioinformatics for Health Sciences

1. Consider the following matrix:

$$M = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 0 & 4 & 4 \end{bmatrix}.$$

   (a) **(1 point)** Provide a basis of the column space of M. Justify your answer.

   (b) **(0.5 points)** Provide a basis of the null space of M. Justify your answer.

2. In this exercise we are going to find step-by-step the matrix of the symmetry of $\mathbb{R}^2$ with respect to the line L that goes through the origin forming an angle of $\pi/6$ radians with respect to the x-axis.

   (a) **(0.5 points)** Find a basis of $\mathbb{R}^2$ with respect to which the symmetry sought has the following matrix:
$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

   (b) **(0.5 points)** Compute the inverse of the decoding matrix B which has as columns the vectors of the basis you just found.

   (c) **(0.5 points)** Using all of the information above, compute the matrix of the symmetry with respect to L in coordinates of the canonical basis.

3. **(2 points)** Consider the set H of all possible solutions of the equation $x + y - z = 0$. H is a vector subspace of $\mathbb{R}^3$. Find an orthonormal basis of H.

4. **(1 point)** Find the Taylor approximation of order 2 of the function $f(x) = \sqrt{x^2+1}$ at $a = 0$.

5. If b is a positive real number, the *logarithm of x to base b*, denoted $\log_b(x)$, is a function that satisfies the following identity for all $x > 0$:
$$x = b^{\log_b(x)}.$$

   The natural logarithm, which we simply denote $\log(x)$, is the logarithm to base e, where e is Euler's number.

   For example, $\log_2(2^{100}) = 100$, $\log_5(25) = 2$ and $\log(e^3) = 3$.

   (a) **(0.5 points)** Explain why the following identity holds for all $x > 0$:
$$b^x = e^{x\log(b)}$$

   (b) **(0.5 point)** Using the fact that $x = b^{\log_b(x)}$, compute the derivative of the function $\log_b(x)$?

6. In this exercise we are going to study the critical points of the function
$$f(x,y,z) = x^2 + y^3 + z^2 - 2x - 4y - 6z.$$

   (a) **(1 point)** Compute the critical points of f.

   (b) **(1 point)** Compute the Hessian matrix of the only critical point of f that has all three coordinates positive (let's denote it P_1).

   (c) **(0.5 points)** What is the nature of the critical point P_1?

   (d) **(0.5 points)** What is the nature of the other critical point?