Session 5: Square matrices, determinant, change of basis

Def: Determinant of a square matrix $\qquad \det : \text{square matrix} \to \mathbb{R}$

recursive recipe

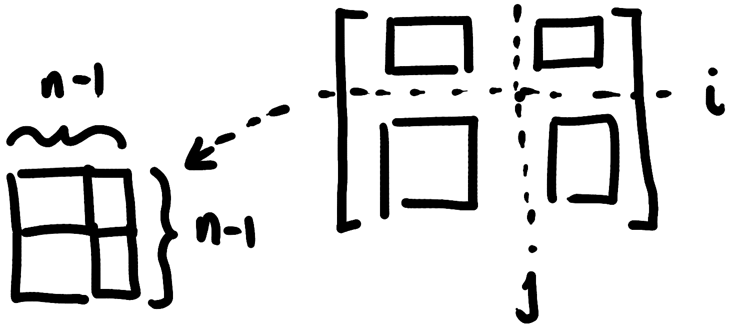
1) det 1×1 matrix $[x]$

$$\det([x]) = x$$

2) $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ $\qquad$ i = row, j = column

M_{ij} = (i,j)-minor of matrix A

M_{ij} =



⋮

$$\dots \quad \det(A) = \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det(M_{i1})$$

Example:

3x3 matrix

$$A = \begin{bmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ \boxed{a_{21}} & a_{22} & a_{23} \\ \boxed{a_{31}} & a_{32} & a_{33} \end{bmatrix}$$

$i=1$

$$\det(A) = a_{11}(-1)^{1+1}\det(M_{11}) +$$

$i=2$

$$a_{21}(-1)^{2+1}\det(M_{21}) +$$

$i=3$

$$a_{31}(-1)^{3+1}\det(M_{31}) =$$

$$= a_{11}\det\left(\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}\right) - a_{21}\det\left(\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}\right) + a_{31}\det\left(\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}\right)$$

Finish this computation

Example:

$$\det\left(\begin{bmatrix} a & b \\ c & \boxed{d} \end{bmatrix}\right) = a\cdot\det([d]) - c\,\det([b])$$

$$= ad - cb$$

base case

"what are determinants useful for"

Properties of determinant

1) $\det(A) \neq 0 \iff$ A is full-rank

$A\in\mathbb{R}^{n\times n}$ $\quad \mathrm{rank}(A) = n$

$\iff$ columns are a basis of $\mathbb{R}^n$

$\iff$ rows are a basis of $\mathbb{R}^n$

$\iff$ A is invertible, i.e., there is some matrix A^{-1} such that

$$A^{-1}A = AA^{-1} = Id_{n\times n} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

2) $\det(AB) = \det(A)\cdot\det(B)$

the determinant is "multiplicative" w.r.t. matrix mult.

$$
\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "A"] \arrow[rrr, bend left, "D"] & \mathbb{R}^3 \arrow[r, "B"] & \mathbb{R}^3 \arrow[r, "C"] & \mathbb{R}^3 \\
x \arrow[r, mapsto] & Ax \arrow[r, mapsto] & B(Ax) \arrow[r, mapsto] & C(B(Ax)) = (CBA)x
\end{tikzcd}
$$

input

$D = CBA$

⚠ the order in which we multiply matrices reverses the order in which we compose linear transformations

3) If $A = \text{diagonal}(\lambda_1, \dots, \lambda_n)$

$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$

$$A = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

4) If A is triangular

$\det(A) =$ product of diagonal entries

$$A = \begin{bmatrix} * & & * \\ & * & \\ 0 & & \ddots \; * \end{bmatrix} \quad \text{upper}$$

$$A = \begin{bmatrix} * & \ddots & 0 \\ * & & \ddots \\ & & * \end{bmatrix} \quad \text{lower}$$

Examples (elementary column operations)

$$\frac{A}{\mathrm{Id}} \xrightarrow[\text{Gauss-Jordan}]{} \cdots \longrightarrow \frac{B}{E} \qquad AE = B$$

"columns of this matrix are linearly independent"

"this matrix is full rank"

elementary column operation

$E_1 E_2 \dots E_K$

$$\mathrm{Id} \to \cdots \to E$$

$$E = E_1 E_2 \dots E_K$$

$$\det(E) = \underbrace{\det(E_1)}_{\neq 0}\,\underbrace{\det(E_2)}_{\neq 0} \cdots \underbrace{\det(E_K)}_{\neq 0}$$

$$\Rightarrow \det(E) \neq 0 \Rightarrow E \text{ full rank}$$

This is related to [Exercise 1 - Session 3] → check it out!

1) $\det(\text{Permutation}) = \text{sign}(\text{Permutation})$

$= \pm 1 \; (\neq 0)$

$$\underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}}_{n} \xrightarrow[\text{columns}]{\text{rearrange}} \begin{bmatrix} 0 & 1 & 0 & \\ 1 & 0 & \vdots & \cdots \\ 0 & \vdots & 0 & \\ \vdots & 0 & 1 & \\ 0 & & & \end{bmatrix}$$

2) $$\det\left(\begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \lambda & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix} \right) = \lambda \; (\neq 0)$$

3) $$\det\left(\begin{bmatrix} 1 & & \lambda \\ & \ddots & \vdots \\ & & 1 \end{bmatrix} \right) = 1 \quad (\neq 0)$$

Change of basis

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$f(e_1)$

$f(e_2)$

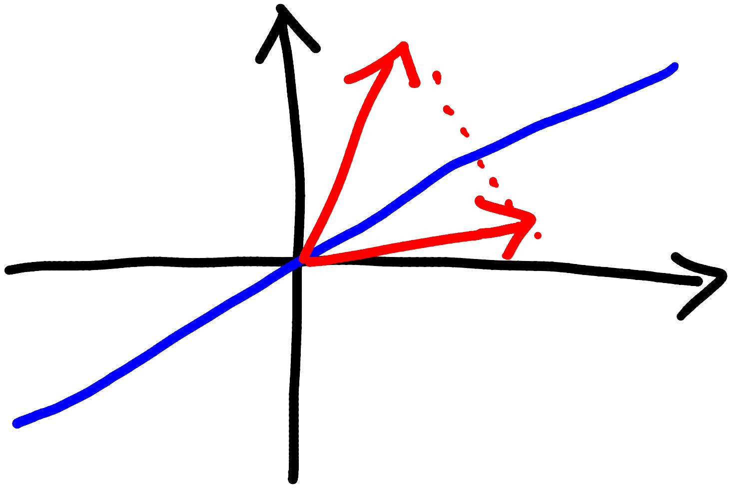
$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$f : \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Matrices define linear transf.

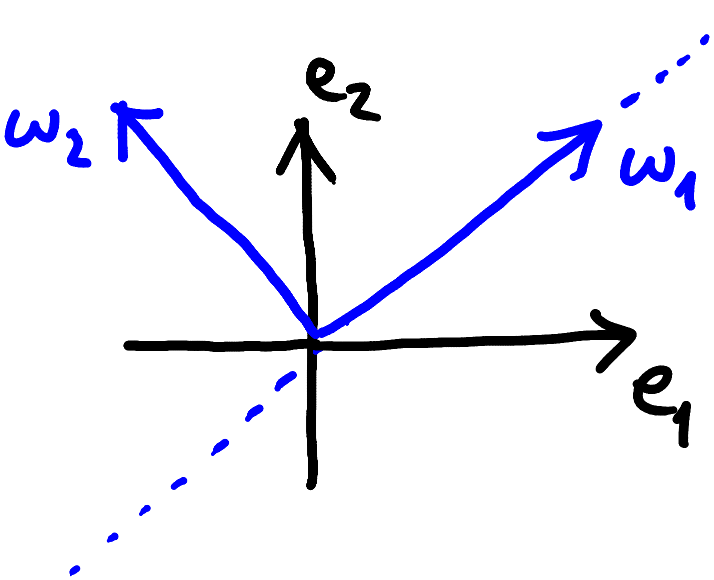
outputs of the canonical basis



imagine a linear transf. that <u>reflects</u> vectors wrt (L) I described

$$\begin{bmatrix} * & * \\ * & * \end{bmatrix}$$

Sometimes matrices of a linear transformation are not intuitive enough



w_1 — is over the line (L)

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

if we define the reflection using $\{w_1, w_2\}$ then the matrix is easier to understand.

Square
Full-rank
matrices $\longrightarrow$ dictionaries

$\boxed{\text{how to change basis}}$

$\{w_1, \dots, w_n\}$ basis of $\mathbb{R}^n$

$v \in \mathbb{R}^n$ $\qquad v = \lambda_1 w_1 + \dots + \lambda_n w_n$

coordinates of v w.r.t. basis $\{w_1, \dots, w_n\}$

DECODER

Given v with coordinates $(\lambda_1, \dots, \lambda_n)$
wrt basis $\{u_1, \dots, u_n\}$

What are the coords. of v in canonical basis?

$\mathbb{R}^2$

$e_1 = (1,0)$
$e_2 = (0,1)$

$\mathbb{R}^3$

$e_1 = (1,0,0)$
$e_2 = (0,1,0)$
$e_3 = (0,0,1)$

Example:

$\{ \overbrace{(1,0)}^{u_1}, \overbrace{(1,1)}^{u_2} \}$ basis of $\mathbb{R}^2$

$u_1 = (1,0)$
$u_2 = (1,1)$

$$v = 2u_1 + 3u_2$$

$$= \mu_1 e_1 + \mu_2 e_2$$

$\rightarrow$ the DECODER will find μ_1, μ_2

$$v = 2u_1 + 3u_2 \longrightarrow 2(1,0)+3(1,1) = (2,0)+(3,3) = (5,3) = \boxed{5}\begin{bmatrix}1\\0\end{bmatrix} + \boxed{3}\begin{bmatrix}0\\1\end{bmatrix}$$

coordinates of v in the basis $\{u_1, u_2\}$

INPUT

$5 = \mu_1$, $3 = \mu_2$

OUTPUT

You may simply write (μ_1, μ_2)

Another example

$v = 5u_1 + 3u_2$ ← input

$u_1 = (-1, 0)$
$u_2 = (1, 1)$
$\{u_1, u_2\}$ is a basis of $\mathbb{R}^2$

¿? how is v written in the canonical basis
$\equiv$
component-wise

$$v = 5(-1,0) + 3(1,1)$$
$$= (-5,0) + (3,3)$$
$$= (-2,3) \quad \text{output}$$

matrix :

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = 5 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$\{v_1, v_2\}$ basis of $\mathbb{R}^2$

$\{e_1, e_2\}$ canonical basis of $\mathbb{R}^2$

$e_1 = (1,0)$

$e_2 = (0,1)$

$$u = \begin{array}{c} \lambda_1 v_1 + \lambda_2 v_2 \\ \shortparallel \\ \mu_1 e_1 + \mu_2 e_2 \end{array}$$

<u>DECODER</u> :

input : λ_1, λ_2

output : μ_1, μ_2

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \xrightarrow[\text{DECODER}]{} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$

$\equiv$

multiplying by matrix

$$A = \left[v_1 \middle| v_2 \right]$$

$$\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = A \cdot \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

ENCODER

$\{v_1, v_2\}$ basis of $\mathbb{R}^2$

input : μ_1, μ_2

output : λ_1, λ_2

??

$$\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = A \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$$A = [v_1 | v_2]$$

$\Downarrow$

A is invertible because A is full rank

$\Downarrow$

A^{-1} satisfies

$$A^{-1} A = Id$$

$$\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = A \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$$A^{-1} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = A^{-1} A \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \mathrm{Id} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

In particular I have solved the <u>ENCODER problem</u>

$$\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \rightsquigarrow A^{-1} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \checkmark$$

we know how to compute this, i.e. Gauss-Jordan

$$\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \xrightarrow[\text{ENCODER}]{} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$\text{ENCODER} \equiv$ multiplying by matrix A^{-1}

Final Summary

$$
\begin{tikzcd}
\text{Coords in new basis } \{u_1,\dots,u_n\} \arrow[r, shift left, "A"', "\text{``decoding''}"] & \text{Coords in canonical basis} \arrow[l, shift left, "A^{-1}", "\text{``encoding''}"']
\end{tikzcd}
$$

$$v \longmapsto Av$$

$$A^{-1}w \longmapsfrom w$$

where

$$A = \left[\, u_1 \,\middle|\, \cdots \,\middle|\, u_n \,\right]$$