# Session 2

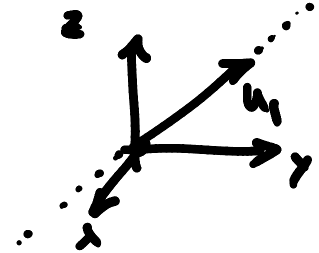
## Linearly independent vectors

$u_1, u_2, \dots, u_K$

$$\underset{u_1}{\begin{bmatrix}1\\2\\1\end{bmatrix}} \quad \underset{u_2}{\begin{bmatrix}0\\0\\1\end{bmatrix}} \quad \underset{u_3 = u_1 + u_2}{\begin{bmatrix}1\\2\\2\end{bmatrix}}$$

**Def**: We say that $u_1, \dots, u_K$ is a linearly independent collection if no vector $u_i$ can be written as a linear combination of the others.

$\text{span}(\underbrace{u_1, \dots, u_K}) =$ all possible linear combinations of $\{u_1, \dots u_K\}$




$\text{span}(u_1)$

$\neq$

$\text{span}(u_1, u_2)$ $\quad u_2 \notin \text{span}(u_1)$

$u_2 \neq \lambda u_1$

$\|$

$\text{span}(u_1, u_2, \overline{u_3})$

$u_3 \in \text{span}(u_1, u_2)$

In practice, $\{u_1, \dots, u_k\}$ are linearly independent, implies ...

$$\lambda_1 u_1 + \dots + \lambda_k u_k = \vec{0}$$

this is only possible if $\lambda_1 = \dots = \lambda_k = 0$

$\downarrow$

the only solution to this equation is $\lambda_1 = \dots = \lambda_k = 0$

$$\underset{u_1}{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}} \quad \underset{u_2}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \quad \underset{u_3}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}} \rightarrow$$

these vector are not linearly independent

$\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = \vec{0}$

$u_1 + u_2 = u_3$

$u_1 + u_2 - u_3 = 0$

$\lambda_1 = 1, \lambda_2 = 1$
$\lambda_3 = -1$

Example :

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are these vectors linearly independent ?

$$\alpha \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} \alpha 1 + \beta 0 + \gamma \cdot 0 &= 0 \\ \alpha 0 + \beta \cdot 1 + \gamma \cdot 0 &= 0 \\ \alpha \cdot 1 + \beta \cdot 0 + \gamma \cdot 1 &= 0 \end{aligned}\right\} \quad \begin{aligned} \alpha &= 0 \\ \beta &= 0 \\ \alpha + \gamma &= 0 \implies \gamma = 0 \end{aligned}$$

The only solution is $\alpha = \beta = \gamma = 0$

These vector are l.i. → linearly independent

## Basis:

$(u_i \in V)$

A collection $u_1, \dots, u_k$ is a **basis** of a vector space $V$ if

1) $u_1, \dots, u_k$ l.i.
2) $V = \text{span}(u_1, \dots, u_k)$

## Example:

$u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad u_1, u_2 \in \mathbb{R}^2$

Is $\{u_1, u_2\}$ a basis of $\mathbb{R}^2$?

1) $\alpha\begin{bmatrix}1\\0\end{bmatrix}+\beta\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow \begin{matrix}\alpha+\beta=0 \Rightarrow \alpha=0\\ \beta=0\end{matrix}$

They are l.i.

2) $\alpha\begin{bmatrix}1\\0\end{bmatrix}+\beta\begin{bmatrix}1\\1\end{bmatrix}=\boxed{\begin{bmatrix}a\\b\end{bmatrix}} \Rightarrow \begin{matrix}\alpha+\beta=a \Rightarrow \boxed{\alpha=a-b}\\ \boxed{\beta=b}\end{matrix}$

There is a solution for any $\begin{bmatrix}a\\b\end{bmatrix}$

$$\operatorname{span}\left(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right)=\mathbb{R}^2$$

$\Rightarrow \left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right\}$ is a basis of $\mathbb{R}^2$

**Remark:**

If $\{u_1, \dots, u_k\} =: \mathcal{B}$ is a basis of V vector space

$w \in V$ any vector

coordinates of w in the basis $\{u_1, u_2, \dots, u_k\} = \mathcal{B}$

$$w = \lambda_1 u_1 + \dots + \lambda_k u_k$$

$\equiv$

$\downarrow$

has a solution in $\lambda_1, \dots, \lambda_k$
and this solution is <u>unique</u>.

And this condition is sufficient to ensure that $u_1, \dots, u_k$ is a basis.

# Exercice 4 (session 1)

$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$ are a basis of $\mathbb{R}^3$

Alternatives: 1) Prove separately conditions 1) & 2) def. basis.

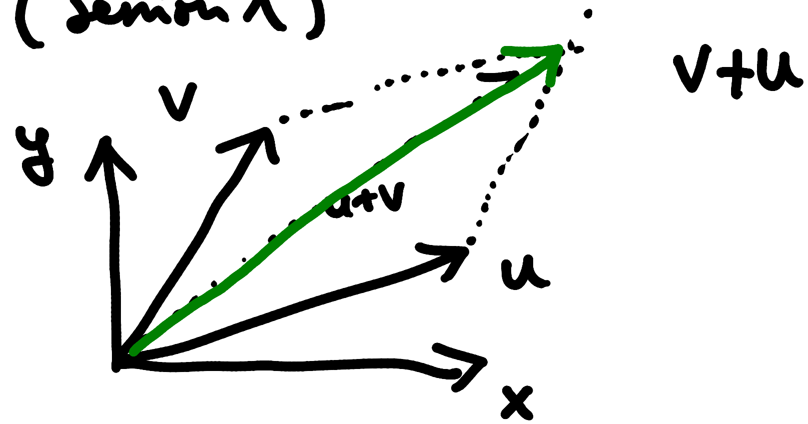
2) Prove that there is a unique solution

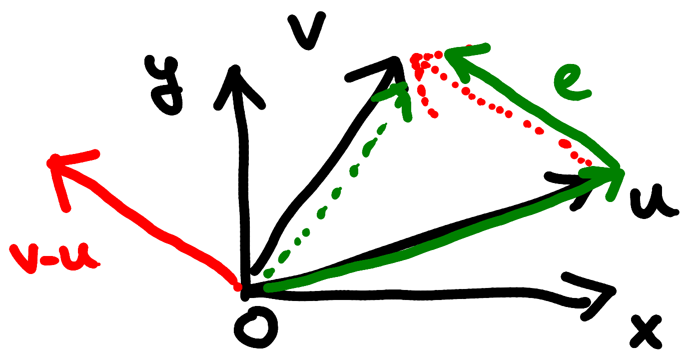
$$\alpha\begin{bmatrix}1\\0\\0\end{bmatrix}+\beta\begin{bmatrix}0\\1\\1\end{bmatrix}+\gamma\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}a\\b\\c\end{bmatrix}$$

$\alpha, \beta, \gamma$ as a function of $a, b, c$ ✓

Exercice → Forum?

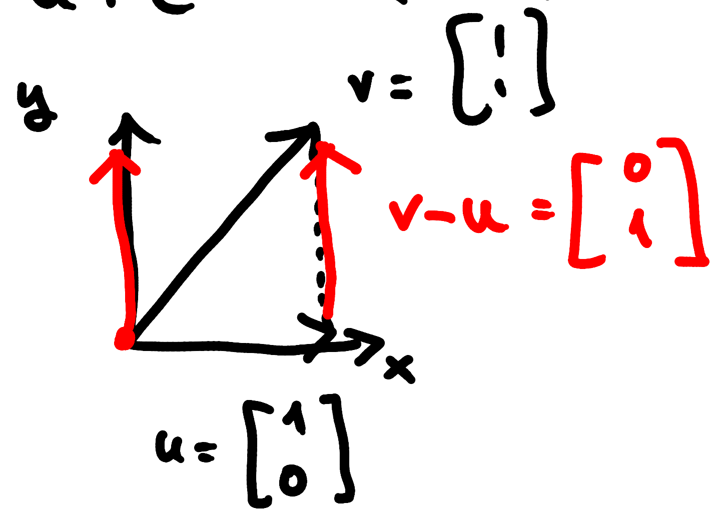
# Exercice 1 (session 1)




$$v - u = e$$

$$u + e = u + (v-u) = v$$



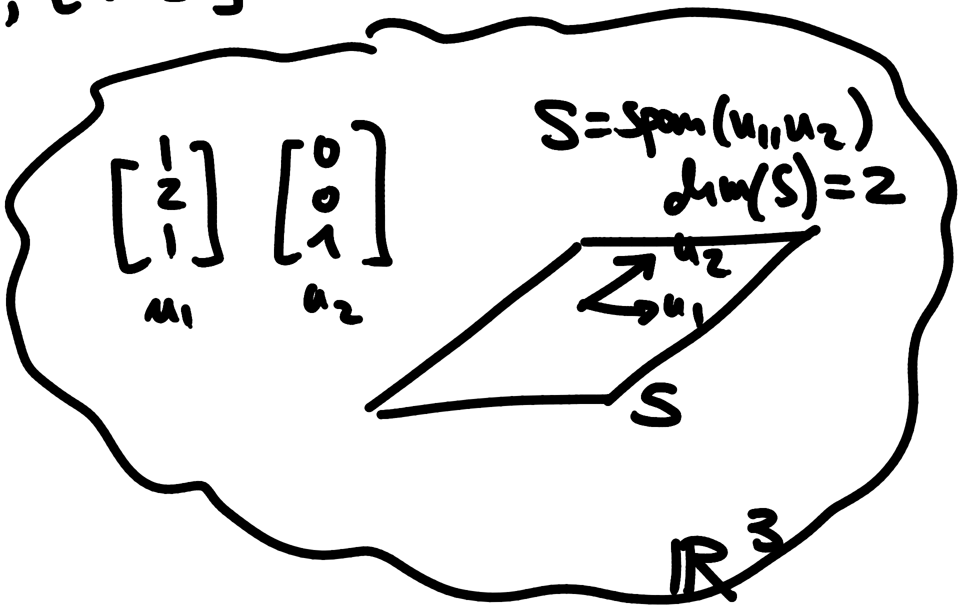
## Definition:

The dimension of a vector space $V$ is the number of vectors in any basis of $V$.

$$\dim(\mathbb{R}^3) = \#\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

$$= 3$$

$$\dim(\mathbb{R}^n) = n$$



# Session 2 - Part 2

## Matrices

$$
\begin{array}{cc}
r & \left[\begin{array}{cccc} * & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & & \ddots & \\ * & * & \cdots & * \end{array}\right] \\
 & c
\end{array}
$$

real numbers $\mathbb{R}$

row vector

column vector

$r = c \Rightarrow$ square matrix

$r \neq c \Rightarrow$ rectangular matrix

$$3 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 5 & 2 \end{bmatrix}$$

same size

$$
\underbrace{r\left\{\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}\right.}_{A}^{\overline{k}}
\underbrace{\begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}}_{c}\Big|\, k \quad B
$$

$$= \quad r\left\{\underbrace{\begin{bmatrix} * & & \\ & & \\ & & \end{bmatrix}}_{c}\right. \quad C$$

$$C_{ij} = \sum_{k=1}^{2} A_{ik} B_{kj}$$

# How can we think about matrix multiplication?

$A = (a_{ij}) \in \mathbb{R}^{r \times k}$ $\quad B = (b_{ij}) \in \mathbb{R}^{k \times c}$

## 1) component-wise perspective

$$AB = C = (c_{ij}) \in \mathbb{R}^{r \times c} \quad \text{with} \quad c_{ij} = \sum_{t=1}^{k} a_{it} \cdot b_{tj}$$

## 2) column combination perspective

$$\underbrace{\begin{bmatrix} c_1 & c_2 & \cdots & c_k \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} b_{11} & b_{12} & \cdots \\ b_{21} & b_{22} & \\ \vdots & \vdots & \\ b_{k1} & b_{k2} & \end{bmatrix}}_{B} = \overset{C}{\begin{bmatrix} \quad & \quad & \cdots \end{bmatrix}}$$

$$b_{11} c_1 + b_{21} c_2 + \dots + b_{k1} c_k$$

$$b_{12} c_1 + b_{22} c_2 + \dots + b_{k2} c_k$$

cols are linear combinations of cols of A

3) row combination perspective

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ & \vdots & & \end{bmatrix}} \underset{B}{\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_k \end{bmatrix}} = \overset{C}{\begin{bmatrix} \quad \\ \quad \\ \vdots \\ \quad \end{bmatrix}}$$

$$a_{11} r_1 + a_{12} r_2 + \cdots + a_{1k} r_k$$

$$a_{21} r_1 + a_{22} r_2 + \cdots + a_{2k} r_k$$

rows are linear combinations of rows of B

## Example

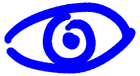
$$\begin{array}{c} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} \\ \begin{array}{cc} c_1 & c_2 \end{array} \end{array} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} = \begin{array}{c} \left[\begin{array}{c|c|c} 3c_1 + 2c_2 & 2c_1 + c_2 & c_1 + c_2 \end{array}\right] \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{1st col.} & \text{2nd col.} & \text{3rd col.} \end{array} \end{array}$$

column perspective

$$= \begin{bmatrix} 3+4 & 2+2 & 1+2 \\ 6+2 & 4+1 & 2+1 \\ 3+2 & 2+1 & 1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 4 & 3 \\ 8 & 5 & 3 \\ 5 & 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{array}{c} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \end{array}\begin{array}{c} r_1 \\ r_2 \end{array} = \begin{bmatrix} r_1 + 2r_2 \\ \hline 2r_1 + r_2 \\ \hline r_1 + r_2 \end{bmatrix}$$



row perspective

$$= \begin{bmatrix} [3\ 2\ 1] + 2\,[2\ 1\ 1] \\ 2\,[3\ 2\ 1] + [2\ 1\ 1] \\ [3\ 2\ 1] + [2\ 1\ 1] \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 4 & 3 \\ 8 & 5 & 3 \\ 5 & 3 & 2 \end{bmatrix}$$

# Column & Row Spaces of a matrix

$$A = \begin{array}{cc} \left[\begin{array}{ccc} & & \\ & & \\ & & \end{array}\right] & \begin{array}{c} r_1 \\ \vdots \\ r_n \end{array} \\ c_1 \cdots c_m & \end{array} \in \mathbb{R}^{n \times m}$$

$\rightarrow m \hat{=} \#$ columns

$\downarrow n \hat{=} \#$ rows

**Def:** The column space of $A$: $\quad C(A) = \operatorname{span}\{c_1, \dots, c_m\}$

The row space of $A$: $\quad R(A) = \operatorname{span}\{r_1, \dots, r_n\}$

**Result:** $\underbrace{\dim C(A)}_{\text{col-rank of } A} = \underbrace{\dim R(A)}_{\text{row-rank of } A} =: \operatorname{rank}(A)$

In practice

$$\operatorname{rank}(A) = \text{max no. of l.i. col. vectors of } A$$
$$= \text{max no. of l.i. row vectors of } A$$

Example:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \Rightarrow \operatorname{rank}(A) = 1$$

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

"we have essentially one row, of which we have multiples"

## Remark:

lower ... more

1) The higher the rank, the less compressible is the matrix.

2) $A = L R$

$$A \in \mathbb{R}^{n \times m}$$
$$L \in \mathbb{R}^{n \times r}$$
$$R \in \mathbb{R}^{r \times m}$$

$$A = \overbrace{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}}^{r} \left.\begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}\right\} r$$

$$\Rightarrow \operatorname{rank}(A) \leq r$$

## Teaser:

A has rank r $\Rightarrow$ $A = LR$

r cols. r. rows