Functions in one variable $f : \mathbb{R} \to \mathbb{R}$

Session - 11

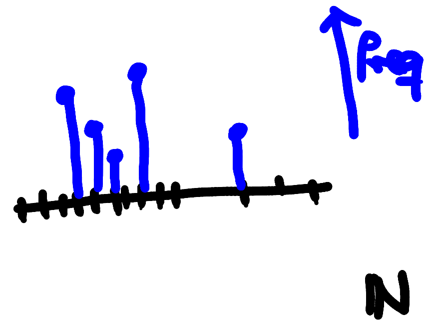
The derivative

- motivation
- concepts you might come across
- derivative (definition & basics)
- affine functions
- affine approximation of diff. functions
- master rules

statistical modeling

classical theory
- random variable
- distribution
- likelihood

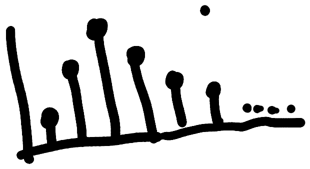
Data $\equiv$ a collection of values or points



$P(\lambda)$ family of random processes

parameter governing the process

$$P(n;\lambda) = \frac{\lambda^n}{n!} e^{-\lambda}$$

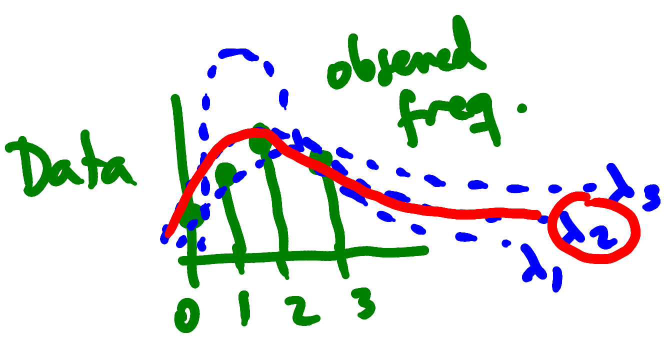


Typical Problem

What is the λ that best explains the data?

$\mathcal{L}(\lambda)$ = "how well $P(\lambda)$ explains my data"

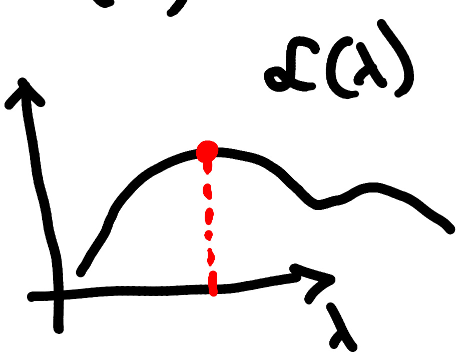
likelihood function



optimization

find λ which gives max $\mathcal{L}(\lambda)$

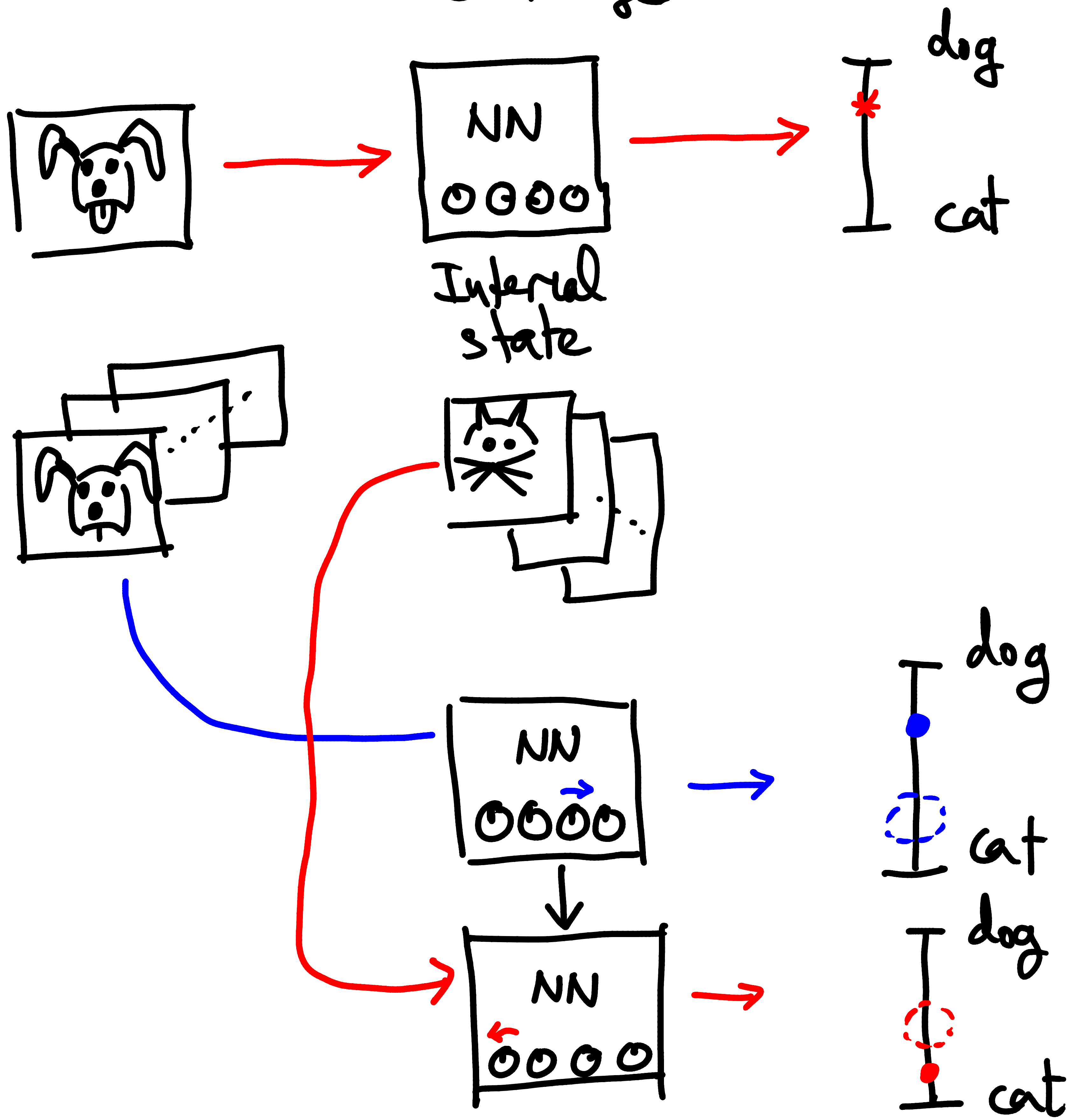
check local maxima



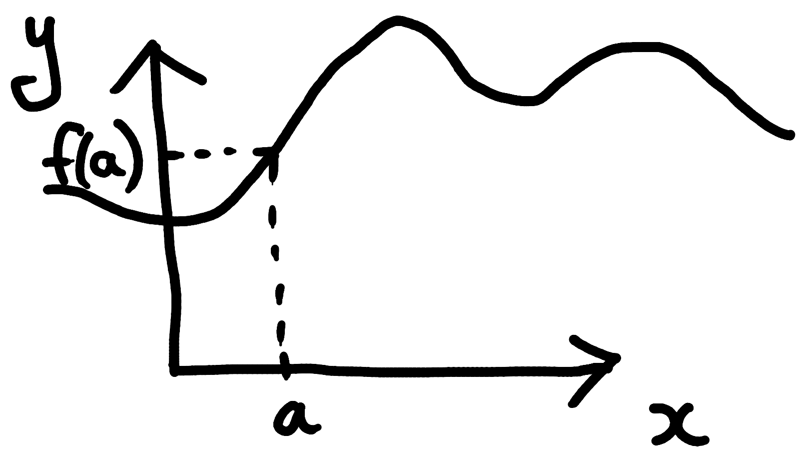
Maximum Likelihood Estimation (MLE)

→ Neural Nets (machine learning)

cat or dog challenge



Derivative



functions $\rightsquigarrow$ graph

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

input $\quad$ output

$$x \longmapsto f(x)$$

"f of x"
"image of x by f"
"output"

<u>Remark</u>: $f : S \subset \mathbb{R} \longrightarrow \mathbb{R}$

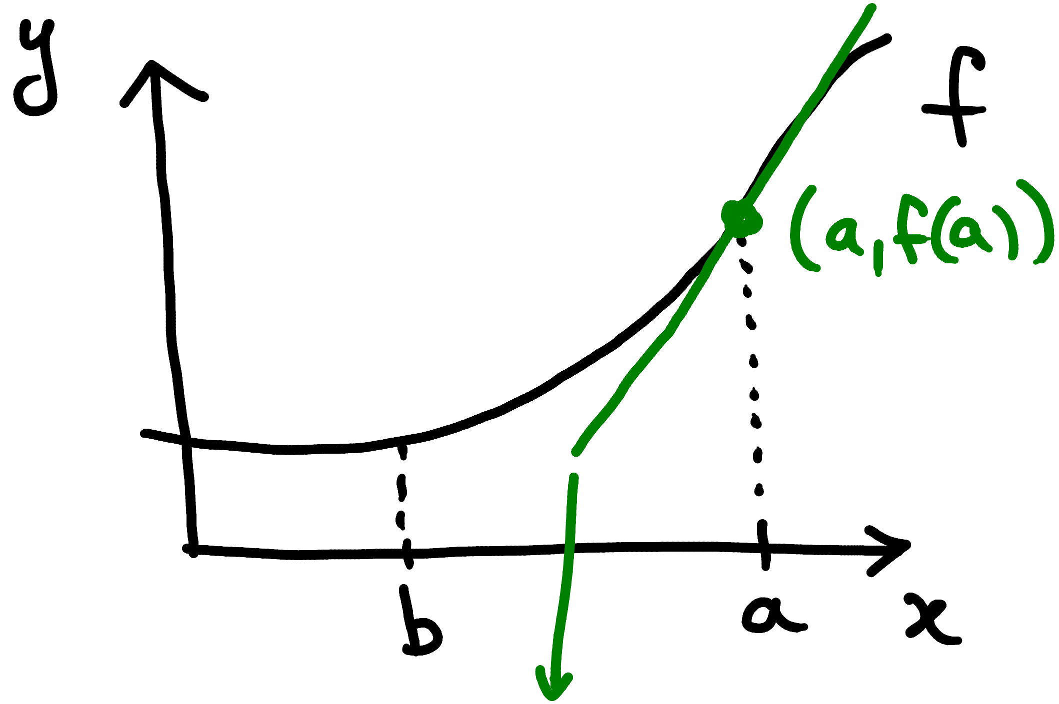
$g(x) = \sqrt{x} \dashrightarrow S = [0, +\infty)$

<u>Remark</u>: Composition of functions

$(f \circ g)(x) =$ "composition of f and g"

$$\mathbb{R} \xrightarrow{g} \mathbb{R} \xrightarrow{f} \mathbb{R}$$
$$x \longmapsto g(x) \longmapsto f(g(x))$$

graph of f



tangent line to the graph of f at (a,f(a)) : L

derivative of f at a denoted $f'(a)$ is the slope of L

Def: The derivative of a function f at a point a as $f'(a)$

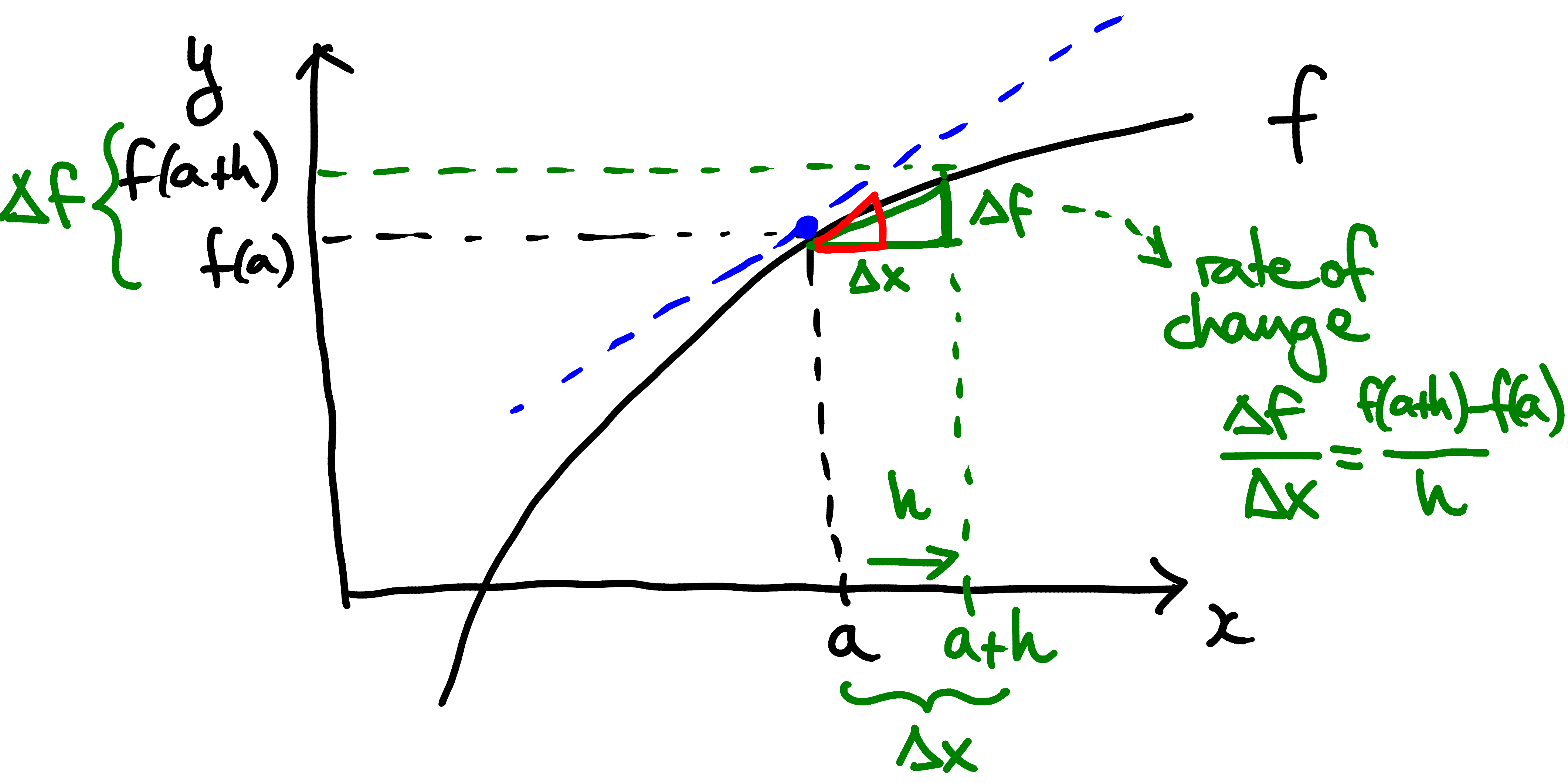
$$f'(a) = \lim_{h \to 0} \boxed{\frac{f(a+h) - f(a)}{h}}$$

whenever this exists.

in some textbooks

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Remark: Depending on f and a, $f'(a)$ may not be defined.



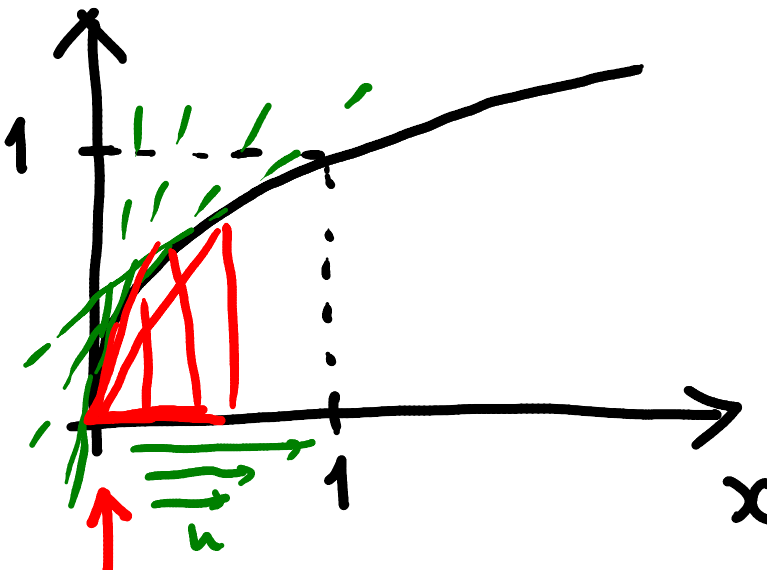
Def: We will say f is differentiable at a if $f'(a)$ exists.

We will say f is differentiable if $f'(a)$ exists for all $a \in S \subset \mathbb{R}$.

Example:

1) $f(x) = \sqrt{x}$

defined in $[0, +\infty)$

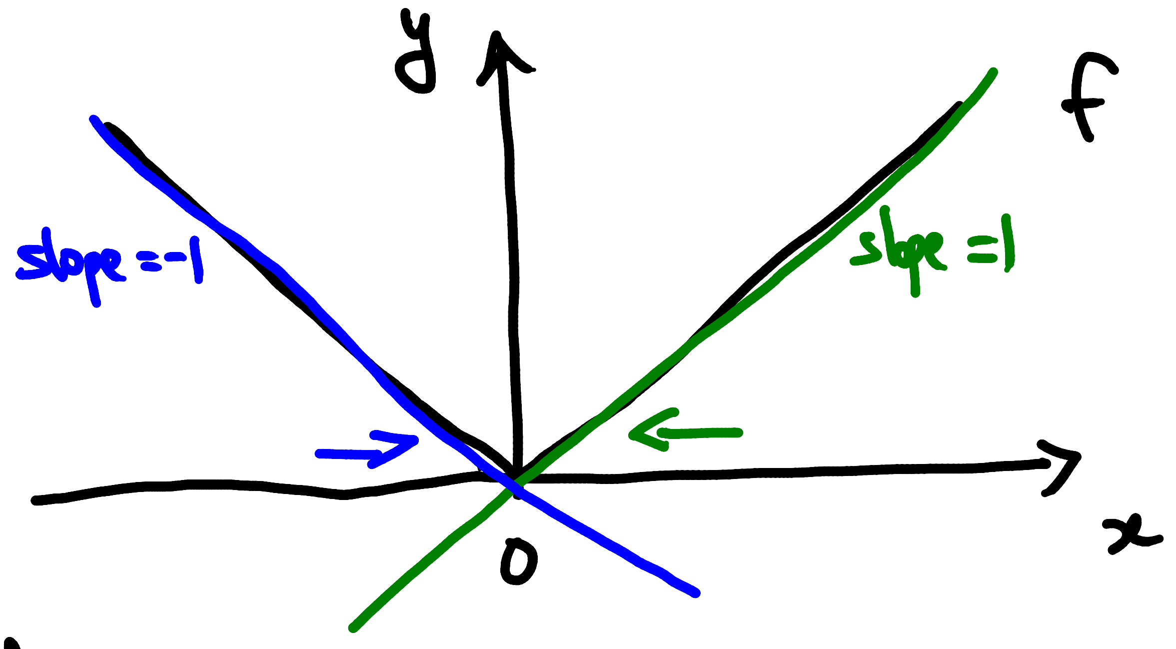


f is differentiable in $(0, +\infty)$

we have some problem at $x=0$

$f'(0)$ does not exist

2) $f(x) = |x|$



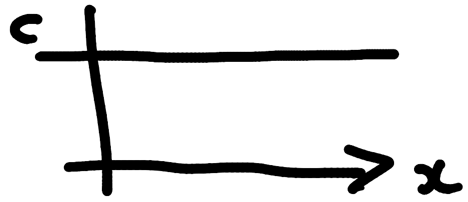
$$\begin{cases} \lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = 1 \\ \lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = -1 \end{cases}$$

$\Rightarrow \lim_{h\to 0} \frac{f(h)-f(0)}{h}$ does not exist!

$\Rightarrow f(x)$ is not differentiable at $x = 0$.

Examples (differentiable function)

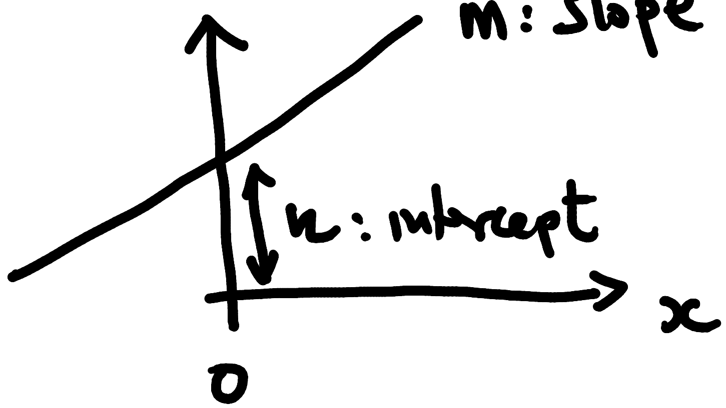
1) $f(x) = c$ for some constant $c \in \mathbb{R}$



"constant function"

$f'(a) = 0$ for all $a \in \mathbb{R}$

2) $g(x) = mx + n$ for $m, n \in \mathbb{R}$



$g'(a) = m$ for all $a \in \mathbb{R}$

"affine functions"

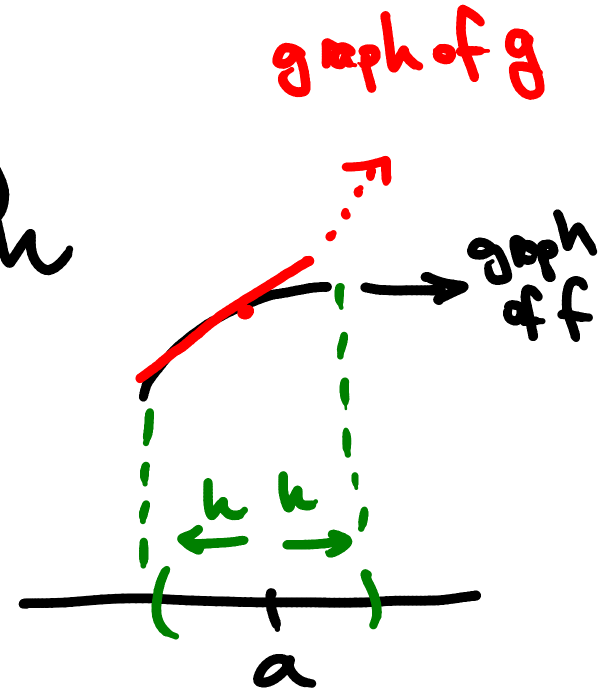
Approximation by affine functions

If f is diff. in $a \in \mathbb{R}$, then

$$f(\underline{x}) \approx f(a) + f'(\underline{a})(x-a)$$

approximation

$$f(\underline{a+h}) \approx f(a) + \underline{f'(a)}\,h$$



In this interval f is very close to the affine function

$$g(x) = f(a) + \underbrace{f'(a)}_{m}(x-a)$$

f can be approximated by an affine function <u>locally</u> in $a \in \mathbb{R}$

$$n = f(a) - f'(a)\cdot a$$

$$m = f'(a)$$

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$$

$$\lim_{h\to 0} \underbrace{\frac{f(a+h)-f(a)}{h} - f'(a)}_{=\,\xi(h)} = 0$$

$$\xi(h) = \frac{f(a+h)-f(a)}{h} - f'(a) \iff$$

$$h\xi(h) = f(a+h) - f(a) - f'(a)h \iff$$

$$f(a+h) = f(a) + f'(a)h + \underbrace{h\xi(h)}_{\text{tends to } 0 \text{ as } h\to 0}$$

$$f(a+h) = f(a) + f'(a)h + o(h)$$

"little-o of h"

this is useful to properly describe of fast $f(a+h)$ and $f(a)+f'(a)h$ get close as $h \to 0$.