# Tutorial 2 - Orthogonal Projection

① We say that the vector spaces $V, W \subset \mathbb{R}^n$ are orthogonal (denoted $V \perp W$)

if for each choice $v \in V$ and $w \in W$ we have $v \cdot w = 0$

**Examples**

null space, row space

$A \in \mathbb{R}^{n \times m}$ $\quad$ $N(A), R(A) \subset \mathbb{R}^m$

$N(A) \perp R(A) \left(= C(A^t)\right)$

By taking transposes

rows of $A^t$ are columns of $A$

$N(A^t) \perp R(A^t)$ $\quad$ but $R(A^t) = C(A)$

$(= C(A))$

(2)

$$\overset{A}{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}} \overset{x}{\begin{bmatrix} x \\ y \end{bmatrix}} = \overset{b}{\begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}}$$

You can think of $\hat{x} = (A^tA)^{-1}Ab$ in a series of steps to make the solution managable:

i) get a new square and invertible matrix by multiplying with $A^t$ in both members of the equation

ii) solve the resulting well-defined system

i)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}}_{A^tA} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \Longrightarrow$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\frac{1}{2} \begin{bmatrix} \frac{5}{3} & -1 \\ -1 & 1 \end{bmatrix}}_{(A^tA)^{-1}} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 10 \\ -6 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

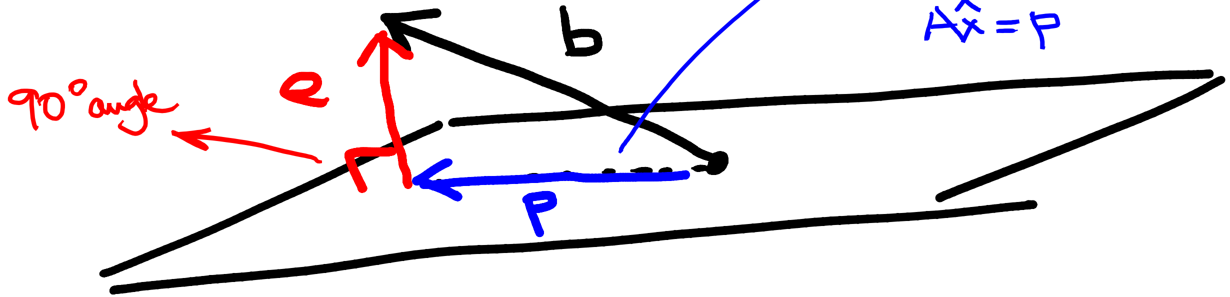
ii)

③

$A=[a_1|a_2]$

Since $p \in C(A)$ then there are $\hat{x}_1, \hat{x}_2$ scalars such that $\hat{x}_1 a_1 + \hat{x}_2 a_2 = p$ or equivalently $A\hat{x} = p$

$S = \mathrm{span}(a_1, a_2) = C(A)$




$b = p + e$

$e = b - p$

$$e \perp S \iff \left.\begin{array}{l} e \cdot a_1 = 0 \\ e \cdot a_2 = 0 \end{array}\right\} \iff$$

$$\iff \underbrace{\begin{bmatrix} a_1 \\ \hline a_2 \end{bmatrix}}_{= A^t} e = \vec{0}$$

## ④ Projection onto a vector subspace $S$

**$\dim S = 1$ :**

$$S = \text{span}\{w\}, \quad \text{length}(w) = 1$$

$$P = ww^t \quad \text{(projection matrix)}$$

**General case:**

$$S = \text{span}\{w_1, \dots, w_d\}$$

**step 1** : $A = [w_1 | \dots | w_d]$

**step 2** : $P = A(A^tA)^{-1}A^t$

**Remark** : for $\dim S = 1$, what if $\text{length}(w) \neq 1$ ?

$$A = [w] = w$$

just 1 column $\equiv$ vector

$$P = w \underbrace{(w^tw)^{-1}}_{\text{scalar}} w^t = \frac{1}{\text{length}(w)^2} ww^t$$

$$\text{length}(w) = 1 \Rightarrow P = ww^t$$