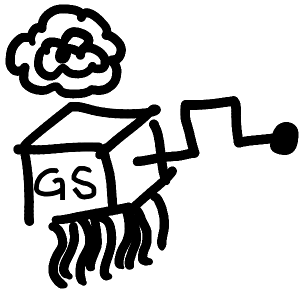
Session 8

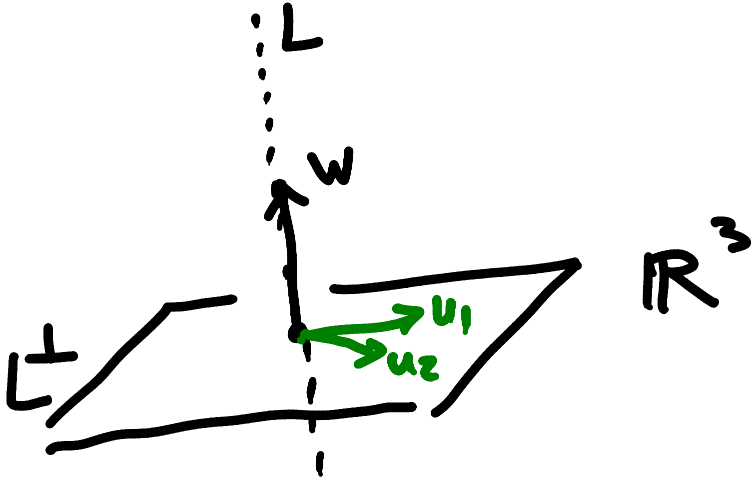
- Announcements
- Gram-Schmidt
- Eigenvalues and eigenvectors

Gram-Schmidt

$u_1, \dots, u_n$ basis of V



$q_1, \dots, q_n$ orthogonal set that spans V

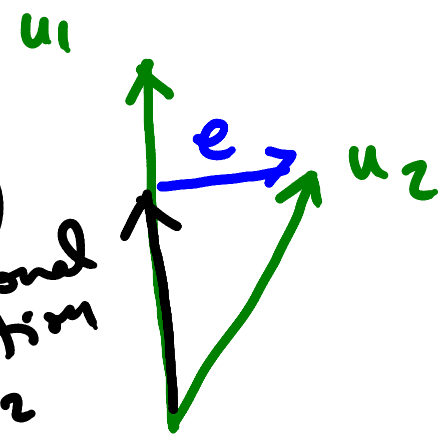


$\text{span}(w) = L$

$$L^\perp = \{ v \mid v \cdot w = 0 \}$$
$$= \text{span}\{u_1, u_2\}$$

all the vector that are orthogonal to w

$L^{\perp}$



$\pi(u_2)$ orthogonal projection of u_2 onto $\mathrm{span}(u_1)$

$$\pi(u_2) + e = u_2$$

$$e = u_2 - \pi(u_2)$$

linear comb of u_1, u_2

$$\text{matrix of } \pi = \tilde{u}_1 \tilde{u}_1^{t}$$

$$\tilde{u}_1 = \frac{1}{\mathrm{length}(u_1)} u_1$$

$$e \cdot u_1 = 0$$

check the last tutorial

<u>Conclusion</u>

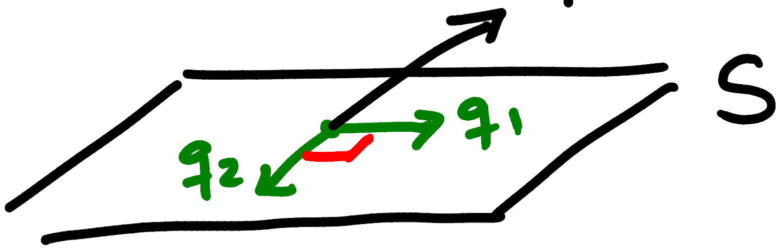
1) $\mathrm{span}(u_1, e) = L^{\perp}$

2) $u_1 \perp e$

$\{u_1, e\}$ orthogonal basis of $L^{\perp}$

<u>Remark</u>: The idea is to keep removing "shadows" cast by each vector onto the vector space spanned by the preceding vectors.

$w \notin S$ $\qquad \mathbb{R}^3$



q_1, q_2 already orthogonal

$\{q_1, q_2, w\}$ basis of $\mathbb{R}^3$

$$e = w - \pi_S(w)$$

π_S : orthogonal projection onto S

$\{q_1, q_2, e\}$ orthogonal basis of $\mathbb{R}^3$

Gram-Schmidt via Elimination

$$\mathcal{B} = \left\{ \underset{u_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \underset{u_2}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}, \underset{u_3}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} \right\}$$

$u_i \cdot u_j \neq 0$

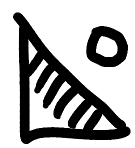
$i \neq j$

$\Downarrow$

$\mathcal{B}$ not orthogonal

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

lower triangular



$$\frac{A^t A}{A} \sim \cdots \sim \frac{\quad}{Q}$$

$Q \longrightarrow$ columns are an orthogonal basis spanning $C(A)$

we are only allowed to use $c_i \leftarrow c_i + \lambda c_j$ $(j < i)$

Gauss-Jordan

$$\frac{A}{\color{red}{Id}} \sim \cdots \sim \frac{B}{E}$$

$B \rightarrow$ reduced col. echelon form

$AE = B$

$$\frac{A^tA}{A} \qquad A^tA = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{check it!}$$

step 1

step 2

$$\begin{array}{ccc} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \\ \hline 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ \downarrow & \downarrow & \downarrow \\ u_1 & u_2 & u_3 \end{array}
\ \underset{\text{step 1}}{\sim}\
\begin{array}{ccc} 2 & 0 & 0 \\ 1 & 3/2 & 1/2 \\ 1 & 1/2 & 3/2 \\ \hline 1 & -1/2 & 1/2 \\ 1 & 1/2 & -1/2 \\ 0 & 1 & 1 \end{array}
\ \underset{\text{step 2}}{\sim}\
\begin{array}{cccc} 2 & 0 & 0 & \\ 1 & 3/2 & 0 & \\ 1 & 1/2 & 3/2 - 1/6 & \\ \hline 1 & -1/2 & 1/2 + 1/6 & = \frac{2}{3} \\ 1 & 1/2 & -1/2 - 1/6 & = -\frac{2}{3} \\ 0 & 1 & 1 - 1/3 & = \frac{2}{3} \\ \downarrow & \downarrow & & \downarrow \\ w_1 & w_2 & & w_3 \end{array}$$

Mantra: ... project then substact ...

$\{w_1, w_2, w_3\}$ is an orthogonal basis

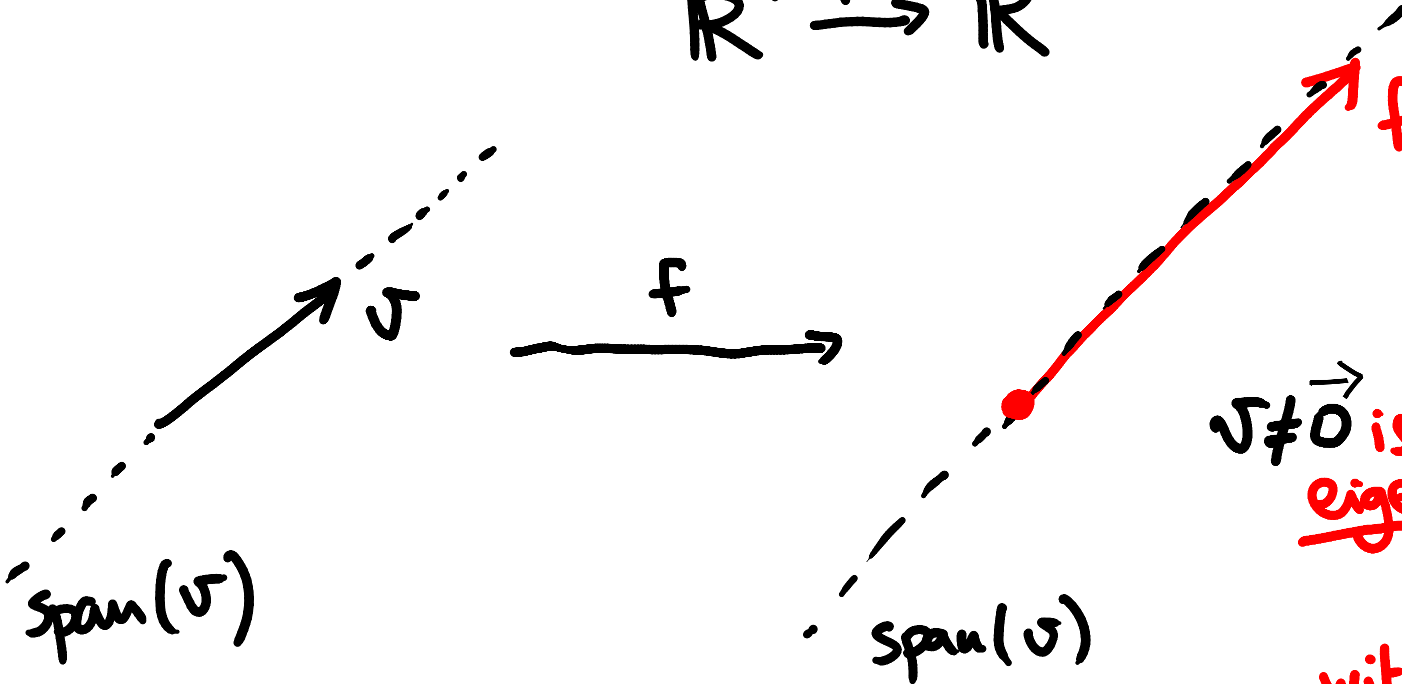
$$w_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad w_2 = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix} \quad w_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

Eigenvalues & Eigenvectors

$A \in \mathbb{R}^{n \times n}$

linear transformation

$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^n$



$f(v) = \lambda v$

$[\lambda \in \mathbb{R}]$

$v \neq \vec{0}$ is an eigenvector (of f) with eigenvalue λ

$$\boxed{Av = \lambda v}$$

look for non-zero v's and λ's that satisfy this equation

they could be zero

$$Av - \lambda v = \vec{0}$$

what is the matrix of the linear trans. $v \longmapsto \lambda v$

$$\begin{bmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{bmatrix} = \lambda \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

$$Av - \lambda \mathrm{Id}\, v = \vec{0}$$

$$(*) \quad \boxed{[A - \lambda \mathrm{Id}]}\, v = \vec{0}$$

$A = (a_{ij})$

$$A - \lambda \mathrm{Id} = \begin{bmatrix} a_{11}-\lambda & & & a_{ij} \\ & a_{22}-\lambda & & \\ & & \ddots & \\ a_{ij} & & & a_{nn}-\lambda \end{bmatrix}$$

Looking for $v \neq \vec{0}$ that fulfills $(*)$

$$\Updownarrow$$

looking for $v \neq \vec{0}$ that belongs to $N(A - \lambda \mathrm{Id})$

Conclusion :

λ will be an eigenvalue of A
if and only if

$$\dim N(A - \lambda \mathrm{Id}) > 0$$

Fundamental theorem of linear algebra

$$n = \dim C(A - \lambda \mathrm{Id}) + \dim N(A - \lambda \mathrm{Id})$$

$$\dim N(A - \lambda \mathrm{Id}) > 0 \iff \dim C(A - \lambda \mathrm{Id}) \leq n-1$$

$\iff$ $A - \lambda \mathrm{Id}$ not full rank

$$\iff \det(A - \lambda \mathrm{Id}) = 0$$

Criterion to find out all eigenvalues :

Find those $\lambda \in \mathbb{R}$ such that

$$\det(A - \lambda \mathrm{Id}) = 0$$

$$\downarrow$$

polynomial equation

These solutions λ are known as "roots" of the polynomial $\det(A - \lambda \mathrm{Id})$

Strategy to find eigenvalues & eigenvectors

1. compute the roots $\det(A - \lambda \, Id) = Q(\lambda)$ — "characteristic polynomial of A"

   $\downarrow$

   $\lambda_1, \dots, \lambda_r$

2. What are the eigenvectors for each eigenvalue λ_i

   compute $N(A - \lambda_i Id) = \left\{ \text{all the eigenvect of eigenval } \lambda_i \right\}$

Examples

$$f(e_1) = e_1$$
$$f(e_2) = 2e_2$$
$$f(e_3) = 3e_3$$

① $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$$Q(\lambda) = \det(D - \lambda Id) = \det\left(\begin{bmatrix} 1-\lambda & & 0 \\ & 2-\lambda & \\ 0 & & 3-\lambda \end{bmatrix}\right)$$
$$= (1-\lambda)(2-\lambda)(3-\lambda)$$

what are the roots of Q?

$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$ eigenvalues

eigenvectors of D of eigenvalue $\lambda_1 = 1$

$$N(D - Id) = \left\{ v \;\middle|\; \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} v = \vec{0} \right\}$$
$$= \operatorname{span}\{e_1\}$$

② Projection onto span(w) where $w = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$P = \tilde{w}\tilde{w}^t = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \qquad \tilde{w} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

unit length generator

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenvalues : $Q(\lambda) = \det(P - \lambda \mathrm{Id})$

$$= \det\left(\begin{bmatrix} \frac{1}{2}-\lambda & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2}-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} \right)$$

$$Q(\lambda) = (-\lambda)\det\left(\begin{bmatrix} \frac{1}{2}-\lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}-\lambda \end{bmatrix} \right)$$

$$= (-\lambda)\left[\left(\frac{1}{2}-\lambda\right)^2 - \frac{1}{4} \right] \ldots$$

$a^2-b^2 = (a-b)(a+b)$

$$= (-\lambda)\left(\underbrace{\frac{1}{2}-\lambda}_{a} - \underbrace{\frac{1}{2}}_{b}\right)\left(\underbrace{\frac{1}{2}-\lambda}_{a} + \underbrace{\frac{1}{2}}_{b}\right)$$

$$= (-\lambda)(-\lambda)(1-\lambda) = \lambda^2(1-\lambda)$$

what are the roots of Q?

$\lambda_1 = 0, \ \lambda_2 = 1$

Compute

$\underline{\lambda_1 = 0}$: $N(A - 0\,Id) = N(A)$

$\underline{\lambda_2 = 1}$: $N(A - Id)$