Session 3

$\text{span}\{u_1, \dots, u_k\}$

[ ][ ]

A different way to manufacture a vector space:

$$\underbrace{\text{homogeneous}}_{\substack{\text{all the terms of} \\ \text{the polynom.} \\ \text{have same} \\ \text{degree} = 1}} \ \underbrace{\text{linear}}_{\substack{\text{polynomials} \\ \text{of} \\ \text{degree} = 1}} \ \underbrace{\text{system of equations}}_{\substack{\text{solutions have to satisfy} \\ \text{several equations} \\ \text{at the same}}}$$

degree = 1

degree = 0

$$\left.\begin{aligned} x + 2y + 1 &= 0 \\ x - y + 10 &= 0 \end{aligned}\right\}$$

NOT HOMOGENEOUS

1 1

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$$

$$\left.\begin{aligned} x + 2y &= 0 \\ x - y - z &= 0 \end{aligned}\right\} (*)$$

1

HOMOGENEOUS

$$(*) \quad r\begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & -1 \end{bmatrix} \underset{c}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$A\underset{\text{unknowns}}{u} = \vec{0}$

u can be thought of as a matrix that has just 1 column

$$\begin{bmatrix} * \\ * \\ * \end{bmatrix}$$

The solutions of the system $Au = \vec{0}$ are a vector subspace of $\mathbb{R}^m$

where $m = \#$ unknowns $= \#$ cols. of A

The key idea:

1) $Au = \vec{0}$, $Av = \vec{0}$ $\qquad A(u+v) = \overset{\vec{0}}{\overset{\|}{Au}} + \overset{\vec{0}}{\overset{\|}{Av}} = \vec{0}$

u, v are solutions $\Rightarrow$ u+v a solution

2) $Au = \vec{0}$ $\qquad A(\lambda u) = \lambda(\underset{\vec{0}}{\underset{\|}{Au}}) = \lambda \cdot \vec{0} = \vec{0}$

u solution $\Rightarrow$ λu solution

Def The null space of $A \in \mathbb{R}^{n \times m}$ $m \to$ column; $n \to$ rows

is the vector subspace of $\mathbb{R}^m$ given by

$$N(A) = \{ u \in \mathbb{R}^m \mid Au = \vec{0} \}$$

Example:

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & -1 \end{bmatrix} = A$$

$$\left.\begin{aligned} x + 2y &= 0 \\ x - y - z &= 0 \end{aligned}\right\} \qquad \left.\begin{aligned} x + 2y &= 0 \\ -3y - z &= 0 \end{aligned}\right\}$$

$$z = -3y \qquad y : \text{parameter}$$
$$x = -2y$$

$$N(A) = \{ (-2\lambda, \lambda, -3\lambda) \text{ for any } \lambda \in \mathbb{R} \}$$
$$= \{ \lambda(-2, 1, -3) \mid \lambda \in \mathbb{R} \} = \text{span}\{(-2,1,-3)\}$$

$$\dim N(A) = 1$$

$$\boxed{\begin{gathered} C(A) \subset \mathbb{R}^2, \; R(A) \subset \mathbb{R}^3 \\ N(A) \subset \mathbb{R}^3 \end{gathered}}$$

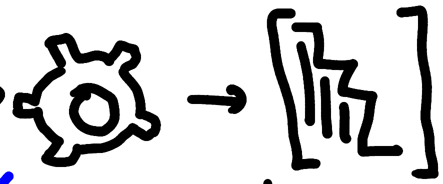
We will look at the computations with vectors from the matrix perspective.

Find a basis for a vector space; or the dimension...

Prove that some vectors are l.i. ...

Phylosophy

$$\{u_1,\dots,u_k\} = \operatorname{span}(u_1,\dots,u_k) \longrightarrow [u_1|\cdots|u_k] \longrightarrow$$



preserve information

echelon form

we can recapitulate properties of $\operatorname{span}\{u_1,\dots,u_k\}$

Example:

$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $u_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$u_4 = u_1 + u_2 + u_3$$

simpler form

Input $\{u_1, \dots, u_k\} \longrightarrow \{u'_1, \dots, u'_k\}$

$\mathrm{span}(u_1, \dots, u_k)$ $\qquad$ $\mathrm{span}(u'_1, \dots, u'_k)$

Example :

$$A = \begin{array}{ccc} u & v & w \\ \hline \end{array}$$
$$\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{ccc} u & v & w \end{array}$$
$$A = \left[\begin{array}{ccc} 1 & 2 & 1 \\ 2 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc} 1 & 0 & 0 \\ 2 & -3 & -1 \\ \hline 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc} 1 & 0 & 0 \\ 2 & 1 & -1 \\ \hline 1 & 2/3 & -1 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \hline 1 & 2/3 & -1/3 \\ 0 & -1/3 & -1/3 \\ 0 & 0 & 1 \end{array}\right] \to \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

non-zero columns

linearly independent

$$\mathrm{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = S$$

$$-\frac{1}{3}u - \frac{1}{3}v + w =$$

$$\begin{bmatrix} -1/3 \\ -2/3 \end{bmatrix} + \begin{bmatrix} -2/3 \\ -1/3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$w = \frac{1}{3}u + \frac{1}{3}v$$

$$N(A) = \mathrm{span}\left\{ \begin{bmatrix} -1/3 \\ -1/3 \\ 1 \end{bmatrix} \right\}$$

Conclusion :

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ basis of S

$\dim S = 2$

$S \subset \mathbb{R}^2 \Rightarrow S = \mathbb{R}^2$

Fundamental Theorem of Linear Algebra

$$\mathbb{R}^{n\times m} \ni A = \overset{m}{\begin{bmatrix} & & \end{bmatrix}} \xrightarrow{\text{elimination}} \begin{bmatrix} \overset{c}{\boxed{\neq 0}} & \overset{m-c}{0} \end{bmatrix}$$

$$\begin{bmatrix} \mathrm{Id}_{m\times m} \end{bmatrix} \longrightarrow \begin{bmatrix} & \boxed{} \end{bmatrix}$$

(reduced) echelon form matrix

basis of C(A)

these cols are basis of N(A)

$\dim N(A) = m - c$

$$m = \underset{\overset{\shortparallel}{c}}{\dim C(A)} + \underset{\overset{\shortparallel}{m-c}}{\dim N(A)}$$

Remarks :

1) $A \xrightarrow[\text{elimination}]{} \underset{E}{B}$ then $AE = B$

2) In particular,

if A square and full-rank (rank = size of A)

then $A \xrightarrow[\text{elim.}]{} \underset{E}{Id}$ and $AE = Id$

$$\boxed{E = A^{-1}}$$

3) $A \xrightarrow[\text{elim.}]{} \underset{E}{B} \to$ echelon form $\dashrightarrow \begin{cases} \text{basis of } C(A) \\ \text{basis of } N(A) \end{cases}$

4)

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{r \text{ cols}} \cdot \underbrace{\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \end{bmatrix}}_{r \text{ row}}$$

$r = \text{rank}(A)$

rank = 1

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

give A such $\text{rank}(A) = r$

$A = CR$ with # cols. of C = # rows of $R = r$

factorization

decomposition

4th perspective of matrix multiplication

$$AB = C_1 + \dots + C_m \qquad A \in \mathbb{R}^{n \times m}$$

$$B \in \mathbb{R}^{m \times t}$$

$$\operatorname{rank}(C_i) = 1$$

Example:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix}}_{\substack{2 \times 2 \\ \operatorname{rank} = 1}} + \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 3 \end{bmatrix}}_{\substack{2 \times 2 \\ \operatorname{rank} = 1}}$$

In future chapters ...

$$\Omega = C_1 + C_2 + \dots + C_r \longrightarrow r = \operatorname{rank} \Omega$$

$$\operatorname{rank}(C_i) = 1$$

Where we are?

vectors
vector spaces

$$\begin{cases} \mathbb{R}^n & \text{"euclidean space"} \\ S \subset \mathbb{R}^n & \begin{cases} \operatorname{span}(u_1, \dots, u_k) \longleftrightarrow \operatorname{col}(A), \quad A = [u_1 | \dots | u_k] \\ N(A) \end{cases} \end{cases}$$

matrices
- how to multiply them
- elimination
- rank $\approx$ how compressible the matrix is

linear transformations

Linear transformations

are functions built ...

$$F : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

$$u \longmapsto Au$$

$$A \in \mathbb{R}^{n \times m}$$

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

"F is a linear transformation between $\mathbb{R}^m$ and $\mathbb{R}^n$"

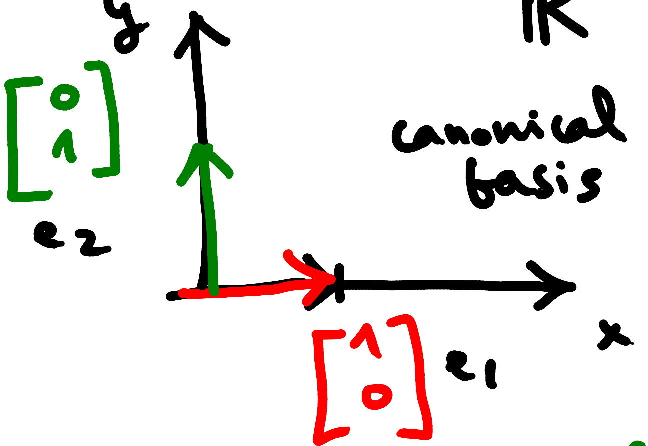
$$\underbrace{F(u)}_{\text{output of } F \text{ when input is } u} = Au$$

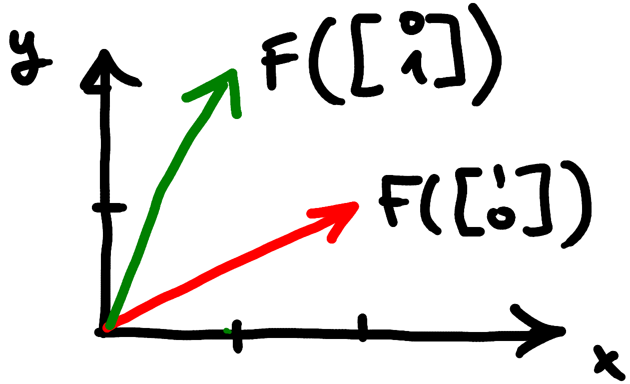
Two fund. properties

1) $F(u+v) = F(u) + F(v)$
2) $F(\lambda u) = \lambda F(u)$

$\mathbb{R}^2$

canonical basis





$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

2×2

geometrically speaking, what is the linear transformation associated to A?

$$F_A : \mathbb{R}^2 \to \mathbb{R}^2$$
$$u \mapsto Au$$

$$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$F(\underbrace{\tfrac{1}{2}e_1 + \tfrac{1}{2}e_2}_{w}) = \tfrac{1}{2}F(e_1) + \tfrac{1}{2}F(e_2)$$