Session 6 Orthogonality

Exercise 1 - session 5

$\mathcal{B} = \{ \underset{u_1}{(1,0,0)}, \underset{u_2}{(0,1,1)}, \underset{u_3}{(1,0,1)} \} \leftarrow$ new basis of $\mathbb{R}^3$

v has coordinates $(1,1,1)$ w.r.t. $\mathcal{B}$ $(*)$

$\mathcal{B}$ $\xrightarrow{\text{Decoding}}$ Canonical Basis

new basis

$$(*) \Rightarrow v = 1u_1 + 1u_2 + 1u_3$$

$$= 1\begin{bmatrix}1\\0\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\\1\end{bmatrix} + 1\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$= \begin{bmatrix}2\\1\\2\end{bmatrix} \checkmark$$

Remark :

$C = [u_1 | u_2 | u_3]$

decoding is equivalent to multiplying by C

$\mathcal{B} \xrightarrow{C}$ Canonical

$\begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{bmatrix} \mapsto C\begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{bmatrix}$

Exercise 2 - session 5

$v = (1, -2, 1) \longrightarrow$ Canonical basis

$\mathcal{B} = \{ \underset{u_1}{(1,0,0)}, \underset{u_2}{(0,1,1)}, \underset{u_3}{(1,0,1)} \}$

$\mathcal{B} \xleftarrow{\text{encoding}}$ Canonical basis

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$\underset{?}{\lambda_1} u_1 + \underset{?}{\lambda_2} u_2 + \underset{?}{\lambda_3} u_3$

$$C = \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \\ \begin{matrix} u_1 & u_2 & u_3 \end{matrix} \end{array}$$

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = C^{-1} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$\rightarrow$ compute!

$$\frac{C}{Id} \to \cdots \to \frac{Id}{C^{-1}}$$

Gauss-Jordan

Exercise 3 - session 5

$$\mathcal{B}_1 = \{ \underset{u_1}{(1,0,0)}, \underset{u_2}{(0,1,1)}, \underset{u_2}{(1,0,1)} \}$$

$$\mathcal{B}_2 = \{ \underset{w_1}{(1,1,0)}, \underset{w_2}{(1,0,1)}, \underset{w_3}{(0,1,1)} \}$$

⚠ order of vectors in the basis matters

$v = (0,-1,1)$ in coordinates of $\mathcal{B}_2$ $\left(\Rightarrow v = -w_2 + w_3 \right)$

$$v = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3$$

??

$$\begin{tikzcd}
\mathcal{B}_1 & & \text{Canonical basis} \arrow[ll, "\text{encoding } D^{-1}"'] \\
& \mathcal{B}_2 \arrow[ul, dashed] \arrow[ur, "\text{decoding } C"'] &
\end{tikzcd}$$

$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$$C = [w_1 | w_2 | w_3]$$

$$D = [u_1 | u_2 | u_3]$$

1st step

$$C \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}$$

$$\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

2nd step

$$D^{-1} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = D^{-1} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

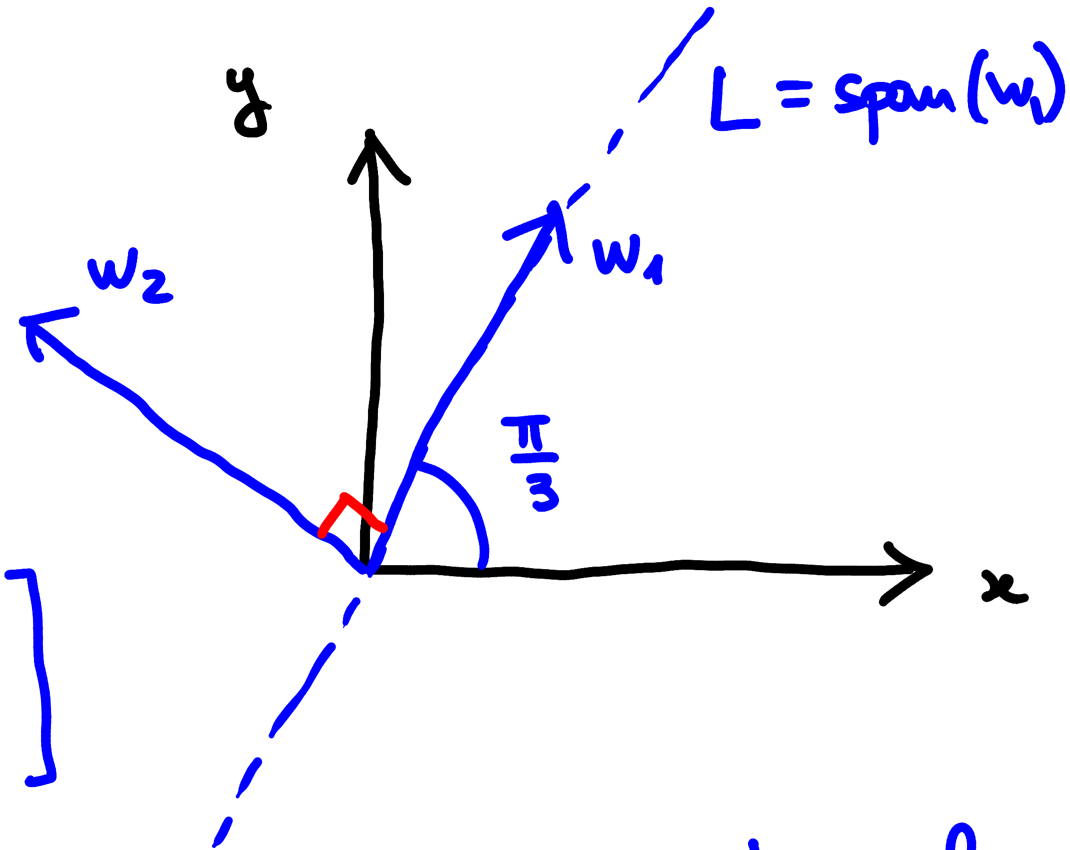
you can use Gauss-Jordan to compute this inverse.

$$\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix} = D^{-1}\left(C\begin{bmatrix}0\\-1\\1\end{bmatrix}\right) = \underbrace{\left(D^{-1}C\right)}\begin{bmatrix}0\\-1\\1\end{bmatrix}$$

Exercise 5 - session 5

$$w_1 = \begin{bmatrix}\cos\left(\frac{\pi}{3}\right)\\ \sin\left(\frac{\pi}{3}\right)\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{bmatrix}$$

$$w_2 = \begin{bmatrix}-\sin\left(\frac{\pi}{3}\right)\\ \cos\left(\frac{\pi}{3}\right)\end{bmatrix} = \begin{bmatrix}-\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{bmatrix}$$



$$\mathcal{B} = \{w_1, w_2\}$$

P is the matrix of the orthogonal projection onto L in coords. of basis $\mathcal{B}$

$$w_1 = 1 \cdot w_1 + 0 \cdot w_2$$

$$P = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$$

$\uparrow$ Proj (w_1) $\quad$ $\uparrow$ Proj (w_2)

$$
\begin{tikzcd}
\mathcal{B} \arrow[r, "P"] & \mathcal{B} \arrow[d, "\text{decoding } C"] \\
\text{Canonical basis} \arrow[u, "C^{-1} \text{ encoding}"] \arrow[r, "\Pi"'] \arrow[ur, bend left] & \text{Canonical basis}
\end{tikzcd}
$$

input given in coords. canonical basis

output given in coords. canonical basis

what does this projection look like in the canonical basis?

$$C = [w_1 | w_2] = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

orthogonal matrix

$$
\begin{aligned}
\Pi &= C P C^{-1} \\
&= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{\sqrt{3}}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix} = \\
&= \frac{1}{4} \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{bmatrix}
\end{aligned}
$$

$$\Pi w_1 = \frac{1}{4} \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 \\ 2\sqrt{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = w_1 \checkmark$$

$$\Pi w_2 = \frac{1}{4} \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0} \checkmark$$

Orthogonality

Dot Product :

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad w = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{R}^n$$

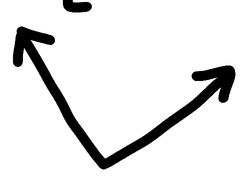
[Definition] $\quad v \cdot w = \sum_{i=1}^{n} v_i w_i \quad$ "dot product"

Example :

1) $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \qquad v \cdot e_i = v_i$

i-th vector of canonical basis

2) $w_2 = \begin{bmatrix} -s \\ c \end{bmatrix}$



$w_1 = \begin{bmatrix} c \\ s \end{bmatrix}$

$\mathbb{R}^2 \qquad w_1 \cdot w_2 = (-s)c + cs = 0$

3) $v, w \in \mathbb{R}^n$

$$(\alpha v) \cdot w = \sum_i (\alpha v_i) w_i = \alpha \sum_i v_i w_i$$
$$= \alpha \, (v \cdot w)$$
$$v \cdot (\alpha w) = \alpha \, (v \cdot w)$$

4) $v \cdot \vec{0} = 0$

5) $u, v, w \in \mathbb{R}^n \qquad (u+v) \cdot w = \boxed{(u \cdot v) + (v \cdot w)}$

$u \cdot (v + w) =$

6) $v \cdot v = \sum v_i^2$

$$\text{length}(v) = \sqrt{v \cdot v}$$

check the abuse of notation

7) $u, v \in \mathbb{R}^n \qquad u^t v = u \cdot v$

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

ej. $u^t v = u \cdot v$

(1x1 matrices $\longleftrightarrow \mathbb{R}$)

Result:

90° angle

$$u \cdot v = 0 \iff \begin{cases} u, v \text{ non-zero and } u \perp v \\ \text{or} \\ \text{some of them is } \vec{0} \end{cases}$$

Def: A set of non-zero vectors $q_1, \dots, q_r \in \mathbb{R}^n$ is said to be **orthogonal** if

$$q_i \cdot q_j = 0 \text{ if } i \neq j$$

unit length

... orthonormal

perpendicular

$$q_i \cdot q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

all q's have unit length

if $r = n$ we say that $q_1, \dots, q_n$ is an **orthonormal basis**

$$Q = \left[\, q_1 \,\middle|\, \cdots \,\middle|\, q_n \,\right] \qquad Q^t = \begin{bmatrix} \text{---}\; q_1 \;\text{---} \\ \vdots \\ \text{---}\; q_n \;\text{---} \end{bmatrix}$$

$\{q_1, \dots, q_n\}$ orthonormal basis of $\mathbb{R}^n$

transpose of Q

$$Q^t Q = \begin{bmatrix} \text{---}\; q_1 \;\text{---} \\ \vdots \\ \text{---}\; q_n \;\text{---} \end{bmatrix} \left[\, q_1 \,\middle|\, \cdots \,\middle|\, q_n \,\right]$$

$$= \begin{bmatrix} q_1 \cdot q_1 & q_1 \cdot q_2 & \cdots & q_1 \cdot q_n \\ q_2 \cdot q_1 & q_2 \cdot q_2 & q_i \cdot q_j & \\ \vdots & & \ddots & \\ q_n \cdot q_1 & & & q_n \cdot q_n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} = \mathrm{Id}$$

$$Q^t = Q^{-1}$$

Def: **orthogonal matrix** is a matrix that has as columns an orthonormal set of vectors.

orthonormal basis $\{q_i\}$

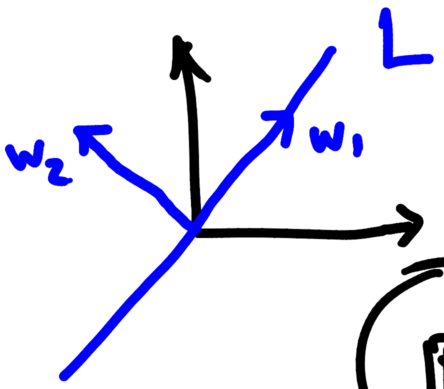
$$\longrightarrow Q = \left[q_1 \middle| \cdots \middle| q_n \right]$$

$$Q^t = Q^{-1}$$

What is the geometric interpretation of the dot product

projection onto a line L

$\{w_1, w_2\}$ orthonormal basis



$\mathbb{R}^2$

$L = \text{span}(w_1)$

$w_2 \perp w_1$

$Q = [w_1 | w_2]$

$$\begin{tikzcd} \cdot \arrow[r, "I"] & \cdot \arrow[d, "Q"] \\ \cdot \arrow[u, "Q^{-1}"] \arrow[r, phantom, "\text{canonical}"'] & \cdot \end{tikzcd}$$

$$\Pi = Q\, P\, Q^{-1}$$

$$= [w_1 | w_2] \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ \hline w_2 \end{bmatrix}$$

$$= [w_1 | 0] \begin{bmatrix} w_1 \\ \hline w_2 \end{bmatrix}$$

$$= w_1 w_1^t + 0 w_2$$

$$= \boxed{w_1 w_1^t}$$

check ex.5 - session5

size 2x2

The matrix of the projection onto $\text{span}(w_1) = L$
where $\text{length}(w_1) = 1$

Suppose there is some vector $u \in \mathbb{R}^2$

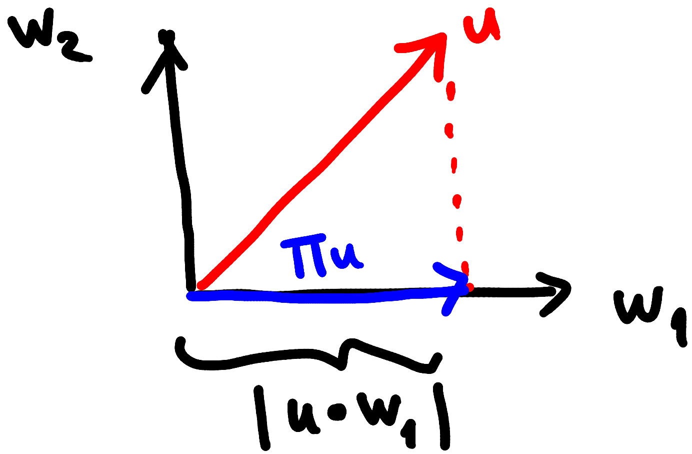
what is the effect of the orthogonal projection Π on u.

$$\Pi u = \underbrace{w_1 w_1^t u}_{\text{"looks like the dot product"}} = \underbrace{(u \cdot w_1)}_{\text{scalar}} \overset{\text{vector}}{w_1}$$

<u>Remark</u>:

length(αw) = $|\alpha|$ length(w)

$\Downarrow$



length(Πu) =

$= |u \cdot w_1|$ length(w_1)

$= |u \cdot w_1|$