EXERCISES: SPECIAL ASSIGNMENT

Orthogonalization with Gram-Schmidt process

Orthonormal bases enjoy nice properties that make them suitable to describe and think about vector spaces and linear transformations. This part portrays a step-by-step exercise to understand the so-called Gram-Schmidt process, a well-known algorithm to manufacture an orthonormal basis out of any given basis.

Exercise 1. Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a collection of non-zero vectors in some vector space V, such that $v_i \perp v_j$ for each $i \neq j$, i.e., $\mathcal{B}$ is an orthogonal collection of vectors. Justify that the vectors in $\mathcal{B}$ are linearly independent.

Remark. Exercise 1 conveys the idea that being orthogonal is a stronger notion for irredundancy than linear independence. You can regard orthogonal vectors as vectors having the least possible amount of redundancy between them. Also notice that the converse is clearly not true: you may easily come up with many examples of linearly independent vectors that are not orthogonal.

Exercise 2. Let $v = (1, 0, -2) \in \mathbb{R}^3$.

a) Find two vectors $w_1, w_2 \in \mathbb{R}^3$ that satisfy the following requirements: i) they are linearly independent; ii) both are orthogonal to v.
b) Denote $W = \mathrm{span}\{w_1, w_2\}$. Show that if some vector u is orthogonal to v, then $u \in W$.

Remark. You will find in textbooks and videos that this idea of taking all the vectors that are orthogonal to all the vectors of a given vector space $V \subset \mathbb{R}^n$ has a name: it is called the "orthogonal vector space" of V, denoted $V^\perp$. The sum $V + V^\perp$ fills out $\mathbb{R}^n$ completely, but with minimum overlap, since $V \cap V^\perp = \{0\}$.

Exercise 3. Given the vectors $v = (1, 1, 1)$ and $w = (2, -1, -1)$ of $\mathbb{R}^3$. Let's denote the vector space they span V.

a) Compute length(v) and length(w)
b) Compute the matrix P of the linear transformation of $\pi : \mathbb{R}^3 \to \mathbb{R}^3$ that does the orthogonal projection onto the line $V = \mathrm{span}\{v\}$
c) Compute the transform of w by P, i.e., the orthogonal projection of w onto V which we will denote $\pi(w)$.
d) Why is the vector $e = w - \pi(w)$ orthogonal to v?
e) Justify that both $\{v, w\}$ and $\{v, e\}$ are bases of V.

Remark. Notice that $\mathcal{B} = \{v, e\}$ is an orthogonal basis for V, i.e., we have managed to construct an alternative orthogonal basis for the vector space V. The main idea conveyed by these steps is formalized as the Gram-Schmidt process, which can be applied iteratively to collections of linearly independent vectors of any size.

Exercise 4. Let's consider a set of 3 linearly independent vectors $\{q_1, q_2, u\}$ in $\mathbb{R}^4$, with $q_1 = (0, 1, 0, 1)$, $q_2 = (-2, 1, 0, -1)$ and $u = (1, 1, 1, 5)$.

a) Show that q_1 and q_2 are orthogonal.
b) Compute the matrix P of the linear transformation of $\pi : \mathbb{R}^4 \to \mathbb{R}^4$ that conducts the orthogonal projection onto the plane $V = \operatorname{span}\{q_1, q_2\}$.
c) Is it true that doing the orthogonal projection onto V is the same as taking the orthogonal projection onto $\operatorname{span}\{q_1\}$ and $\operatorname{span}\{q_2\}$ separately, then adding the results? More explicitly, if we denote π_W the orthogonal projection onto some vector space W, is it true that the following relationship holds for any vector v?

$$\pi_{\operatorname{span}\{q_1,q_2\}}(v) = \pi_{\operatorname{span}\{q_1\}}(v) + \pi_{\operatorname{span}\{q_2\}}(v)$$

d) Compute the transform of u by P, i.e., the orthogonal projection of u onto V which we will denote $\pi(u)$.
e) Show that vector $e = u - \pi(u)$ is orthogonal to both q_1 and q_2.
f) Justify that $\{q_1, q_2, e\}$ and $\{q_1, q_2, u\}$ are bases of the same vector space.

Remark. As you can see, we can replace in this way any linearly independent collection of vectors by another which comprises only orthogonal vectors that span the same vector space.