Solutions to final test (Monday 15 December 2025, 16:00-18:00 CET)
Elements of Mathematics – Master in Bioinformatics for Health Sciences

1. **(1 point)** Consider the matrix:

$$A = \begin{bmatrix} 1 & -5 & 4 \\ 2 & 4 & 1 \\ 0 & -8 & 4 \end{bmatrix}.$$

   (a) **(0.5 points)** Provide a basis of the column space of A.
   Answer: Note that the first and third columns are linearly independent and the second column can be expressed as a linear combination of the first and third columns like this: $c_2 = 3c_1 - 2c_3$. This implies that $\dim C(A) = 2$ and $\{c_1, c_3\}$ is a basis of the column space of A.

   (b) **(0.5 points)** Provide a basis of the null space of A.
   Answer: Because $\dim C(A) = 2$ it follows that $\dim N(A) = 3 - \dim C(A) = 1$ by the Fundamental Theorem of Linear Algebra. Since we have that $c_2 = 3c_1 - 2c_3$, it follows that the vector $(3, -1, -2)$ is a non-zero vector in the null space of A. Therefore, a basis of the null space of A is given by the set: $\{(3, -1, -2)\}$.

2. **(1 point)** Let $v = (1, 2, -1) \in \mathbb{R}^3$. Give an example of a non-zero matrix $A \in \mathbb{R}^{3\times 3}$ such that $Av = \vec{0}$.
   Answer: Any matrix with the following form $A = [c_1|c_2|c_3]$ such that the columns satisfy $c_1 + 2c_2 - c_3 = \vec{0}$ will work. For example:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

3. **(1 point)** Let $H \subset \mathbb{R}^3$ be the vector space of solutions of the equation $x - y + z = 0$. Compute an orthonormal basis of H.

   Answer: We could follow different paths to solve this problem. Here, because we are only interested in an orthonormal basis of H, we will first find some basis of H in a somewhat rudimentary fashion and then we will use the Gram-Schmidt process to orthonormalize it.

   Because $x = y - z$, we can express any vector $v \in H$ as a linear combination of two vectors:

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y - z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

   Therefore, $\mathcal{B} = \{v_1, v_2\}$ is a basis of H, with $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

   Now we use the Gram-Schmidt method to find an orthonormal basis $\{u_1, u_2\}$ that spans H.

   (a) First step:

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

   (b) Second step:

$$\tilde{u}_2 = v_2 - (v_2 \cdot u_1)u_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{-1}{\sqrt{2}}\right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}$$

$$u_2 = \frac{\tilde{u}_2}{\|\tilde{u}_2\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

4. **(1.5 points)** Consider the matrix:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$$

a) **(0.5 points)** Find the eigenvalues of $B = A^t A$.
Answer: First, compute B:

$$B = A^t A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

To find the eigenvalues λ, we compute the roots of the characteristic polynomial

$$Q(t) = \det(B - tI) = \det \begin{bmatrix} 2-t & -1 \\ -1 & 2-t \end{bmatrix} = t^2 - 4t + 3 = (t-3)(t-1)$$

Therefore, the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 1$.

b) **(0.5 points)** Find a basis of $\mathbb{R}^2$ of eigenvectors of B.
Answer:

(a) For $\lambda_1 = 3$, find a basis of the eigenspace $N(B - \lambda_1 I) = N(B - 3I)$.

$$\begin{bmatrix} 2-3 & -1 \\ -1 & 2-3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives the equation $-x - y = 0 \Leftrightarrow y = -x$. The eigenvectors of λ_1 are of the form

$$\begin{bmatrix} x \\ -x \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

so we can pick $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as a single generator of the eigenspace.

(b) For $\lambda_2 = 1$, find a basis of the eigenspace$N(B - \lambda_2 I) = N(B - I)$.

$$\begin{bmatrix} 2-1 & -1 \\ -1 & 2-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives the equation $x - y = 0 = x$. The eigenvectors of λ_2 are of the form

$$\begin{bmatrix} x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

so we can pick $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as a single generator of the eigenspace.

An eigenbasis is given by

$$\{v_1, v_2\} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

c) **(0.5 points)** Write B as a matrix product $Q\Lambda Q^{-1}$, where Q is invertible and Λ is diagonal.
Answer:
So far we have found an orthogonal eigenbasis $\{v_1, v_2\}$, so setting $Q = [v_1|v_2]$ will give us the decoding matrix that we can use to express the matrix B in the new basis, which would give

$$\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = Q^{-1} B Q \Leftrightarrow B = Q \Lambda Q^{-1}.$$

Note that the eigenbasis is already orthogonal, so we can make it orthonormal by rescaling its vectors, which would render Q orthogonal, and thus $Q^{-1} = Q^t$. Since both v_1 and v_2 have length

$\sqrt{2}$, we have that $\tilde{v}_1 = v_1/\|v_1\|$ and $\tilde{v}_2 = v_2/\|v_2\|$ form an orthonormal basis. Therefore, we can set $Q = [\tilde{v}_1|\tilde{v}_2]$ so that:

$$B = Q\Lambda Q^{-1} = Q\Lambda Q^t = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

5. **(1.5 points)** Suppose that we want to conduct PCA on a gene expression dataset represented by a matrix $A \in \mathbb{R}^{100\times 5}$ that comprises the experimental transcript counts readouts for 5 genes (columns) and 100 cells (rows). To do so, we can compute the eigendecomposition of $\Omega \in \mathbb{R}^{5\times 5}$, the covariance matrix of the dataset.

   - **(0.5 points)** How do we obtain Ω from A?

     Answer: The covariance matrix Ω is computed from the data matrix A, as $\Omega = \tilde{A}^t\tilde{A}$, where $\tilde{A}$ is the matrix obtained after centering and normalizing A.
   - **(0.5 points)** How do we get the principal components from the eigendecomposition of Ω?

     Answer: The eigendecomposition of the covariance matrix Ω yields a set of eigenvalues λ_j and their corresponding eigenvectors v_j, i.e., $\Omega v_j = \lambda_j v_j$. The collection of eigenvectors v_j, sorted by their respective eigenvalues, are the principal components.
   - **(0.5 points)** If the eigenvalues of Ω are $\lambda_1 = 5$, $\lambda_2 = 2$, $\lambda_3 = 1.5$, $\lambda_4 = 1$, $\lambda_5 = 0.5$, what is the proportion of variance explained by the first 3 principal components?

     Answer: The total variance of the centered and scaled dataset is the trace of Ω, which is the sum of the eigenvalues of Ω:

     $$\mathcal{V} = \sum_{i=1}^{5} \lambda_i = 5 + 2 + 1.5 + 1 + 0.5 = 10$$

     The variance explained by the first 3 principal components (commonly denoted PC1, PC2, PC3) is the sum of their corresponding eigenvalues:

     $$\mathcal{V}_3 = \lambda_1 + \lambda_2 + \lambda_3 = 5 + 2 + 1.5 = 8.5$$

     Therefore the proportion of variance represented by the first 3 principal components is:

     $$\mathcal{R}_3 = \frac{\mathcal{V}_3}{\mathcal{V}} = \frac{8.5}{10} = 0.85$$

     The proportion of variance explained by the first 3 principal components is 0.85 or 85%.

6. **(1.5 points)** Find the Taylor approximation of degree 2 of the function $f(x) = e^{x^2}$ at $a = 0$.

   Answer: The Taylor polynomial of f of degree 2 is:

   $$T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$$

   We need to compute the values of $f(0)$, $f'(0)$ and $f''(0)$:

   - $f(x) = e^{x^2} \implies f(0) = e^0 = 1$
   - $f'(x) = 2xe^{x^2} \implies f'(0) = 0$
   - $f''(x) = 2e^{x^2} + 2x(2xe^{x^2}) \implies f''(0) = 2$

   Substituting into the Taylor formula:

   $$T_2(x) = 1 + 0x + \frac{2}{2!}x^2 = 1 + x^2$$

7. **(2.5 points)** Consider the function $f(x,y) = x^2y - 2xy^2 + 3xy + 4$.

(a) **(0.2 points)** Compute the first order partial derivatives of f.
Answer:

$$f_x = \frac{\partial f}{\partial x} = 2xy - 2y^2 + 3y$$
$$f_y = \frac{\partial f}{\partial y} = x^2 - 4xy + 3x$$

(b) **(1 point)** Find all the critical points of f.
Answer: To find the critical points of f, we need to solve the system of equations given by setting the first order partial derivatives to zero:

$$\begin{cases} f_x = 2xy - 2y^2 + 3y = 0 \\ f_y = x^2 - 4xy + 3x = 0 \end{cases}$$

From the first equation, we can factor out y:

$$y(2x - 2y + 3) = 0$$

- Case $y = 0$:
  Substituting $y = 0$ into the second equation:
  $$x(x - 4(0) + 3) = 0 \implies x(x+3) = 0$$
  This gives two possible solutions $x = 0$, $x = -3$, whence the critical points $P_1 = (0,0)$, $P_2 = (-3, 0)$.
- Case $2x - 2y + 3 = 0$:
  Substituting $y = x + 3/2$ into the second equation:
  $$x(x - 4(x + 3/2) + 3) = 0 \Leftrightarrow x(x - 4x - 6 + 3) = 0 \Leftrightarrow x(-3x - 3) = 0$$
  This gives two possible solutions $x = 0$, $x = -1$, whence the critical points $P_3 = (0, 3/2)$, $P_4 = (-1, 1/2)$.

In sum, the critical points of f are $(0,0)$, $(-3,0)$, $(0,3/2)$, and $(-1,1/2)$.

(c) **(0.3 points)** Compute the second order partial derivatives of f.
Answer:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(2xy - 2y^2 + 3y) = 2y$$
$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(x^2 - 4xy + 3x) = -4x$$
$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}(2xy - 2y^2 + 3y) = 2x - 4y + 3$$

(d) **(1 point)** What kind of critical point is $P = (-1, 1/2)$? Justify your answer.
Answer:
Let's compute the Hessian matrix of f:

$$Hf(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 2y & 2x - 4y + 3 \\ 2x - 4y + 3 & -4x \end{bmatrix}$$

Evaluating at the point $P = (-1, 1/2)$:

$$Hf(P) = \begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}$$

The eigenvalues of $Hf(P)$ are:

$$\lambda_1 = \frac{5 - \sqrt{13}}{2}, \quad \lambda_2 = \frac{5 + \sqrt{13}}{2}$$

Since both $\lambda_1, \lambda_2 > 0$, the Hessian matrix is positive definite and P is a local minimum.