

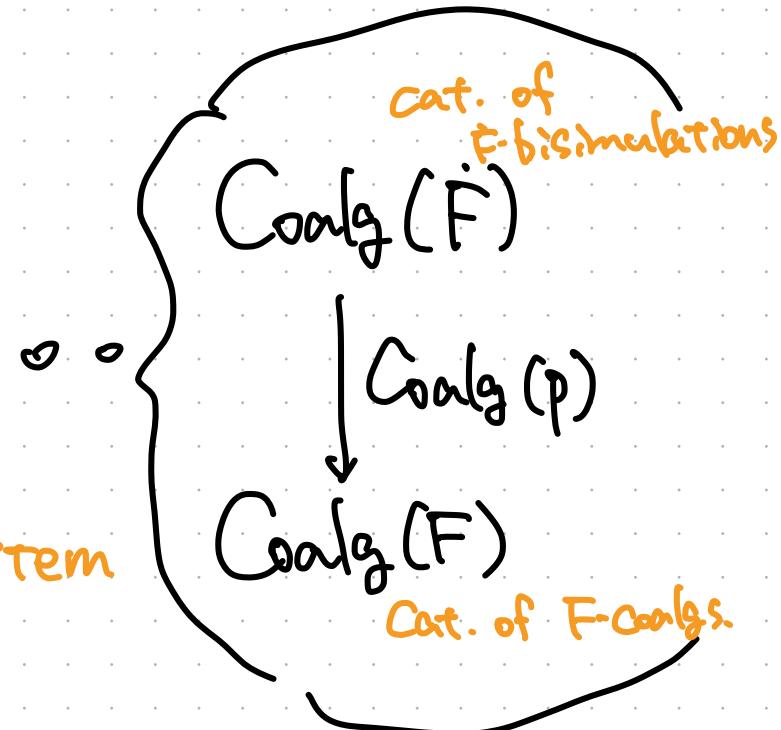
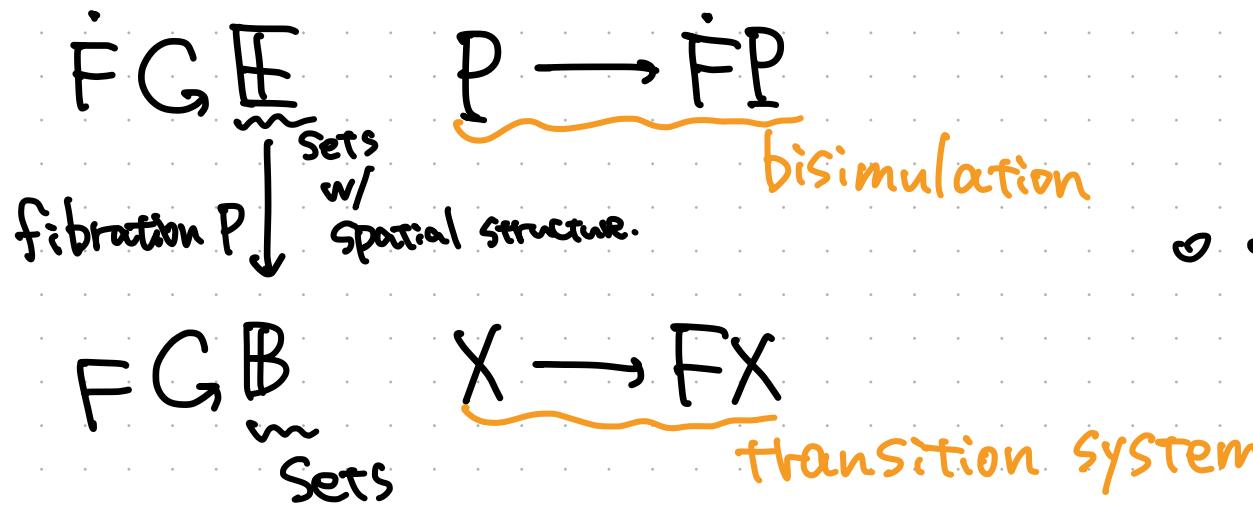
Composing Codensity Bisimulations

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Coalgebraic bisimulation

[Hermida & Jacobs, Inf. Comput. '98]



e.g.

- Standard bisimulation on a Kripke frame \mathcal{S}

$$\begin{array}{ccc} \dot{P}G \in \text{ERel} & R \xrightarrow{\quad} \dot{P}R & \xrightarrow{\text{equiv.}} \\ \downarrow & & \\ PG \text{ Set} & S \xrightarrow{\delta} PS & \left(\forall (x,y) \in R. \left\{ \begin{array}{l} x' \in S_x \Rightarrow \exists y' \in S_y. (x'y') \in R \\ y' \in S_y \Rightarrow \exists x' \in S_x. (x'y') \in R \end{array} \right. \right) \end{array}$$

$R \subseteq S \times S$ is a bisimulation on \mathcal{S}

- language equivalence of det. automata
- bisimilarity metric for MDP.

Composing bisimulations via lifting of distributive law

[Bonchi et al. Act. info.'17]

by liftings $\dot{F}, \dot{T}, \dot{\lambda}$.

of F, T, λ along p .

by $F: B \rightarrow B, T: B^N \rightarrow B$,

distributive law $\lambda: TF^N \Rightarrow FT$

Compose bisimulations

$$\text{Coalg}(\dot{F})^N \xrightarrow{!} \text{Coalg}(\dot{F})$$

$$\downarrow \quad \quad \quad \downarrow \text{Coalg}(p)$$

$$\text{Coalg}(F)^N \xrightarrow{!} \text{Coalg}(F)$$

Compose Systems

where

$$\dot{T}\dot{\lambda} = \dot{\lambda} \circ \dot{T}(-)$$

$$T\lambda = \lambda \circ T(-)$$

e.g. product composition of standard bisimulations.

$T\lambda$... product of Kripke frames. δ_1, δ_2

$\dot{T}\dot{\lambda}$... product of bisimulations R_1, R_2

Prop. $\begin{cases} R_1 \in \text{Coalg}(\dot{F})_{\delta_1} \\ R_2 \in \text{Coalg}(\dot{F})_{\delta_2} \end{cases}$

bisim on δ_i

$$\Rightarrow \dot{T}\dot{\lambda}(R_1, R_2) \in \text{Coalg}(\dot{F})_{T\lambda(\delta_1, \delta_2)} = R_1 \times R_2$$

bisim on $T\lambda(\delta_1, \delta_2)$

Overview of our work

$$\begin{array}{c} \text{Coalg}(\dot{F})^N \xrightarrow{\dot{T}, \dot{\lambda}} \text{Coalg}(\ddot{F}) \\ \downarrow \qquad \qquad \downarrow \\ \text{Coalg}(\dot{F})^N \xrightarrow{T, \lambda} \text{Coalg}(F) \end{array}] \text{ given by } \begin{array}{l} \text{liftings } \dot{F}, \dot{T}, \dot{\lambda} \\ \text{defined by } \alpha \text{ Suf. Cond.} \\ \text{Codensity liftings.} \end{array}$$

ensured by

We adopt { Codensity lifting for \dot{F} [Sprunger et al. '21]
generalized Codensity lifting for \ddot{T} new!

and provide a sufficient condition of liftability of λ
that gives various composition of bisimulations.

Results are based on 2-categorical extension of Beohart's decomposition.

[Beohart, Gürke, König, Messing, Förster, Schröder, Wild, STACS '24]

Comparison to related works.

1. Abstract GSOS rules. [Turi & Plotkin, LICS'99] et al.
not directly apply to similarity or behavioural metric

Our results cover.

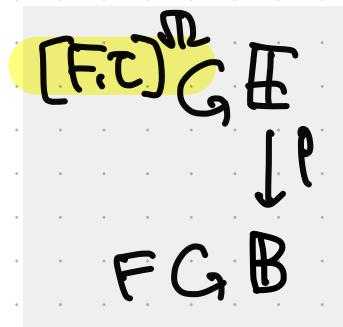
2. Liftability of λ in [Bonchi et al., Act.info.'17]

	[§6, Bonchi et al.'17]	[§8, Bonchi et al.'17]	Ours.
fibration P	$\text{Rel} \rightarrow \text{Set}$	✓ general various	Sweet spot! various
F, T	$\text{Rel}(F), \text{Rfl},$ Sym, \dots	various	Codensity liftings
proof of liftability	✓ good cond. all distributive laws lift	case-by-case	by sufficient cond.

Outline

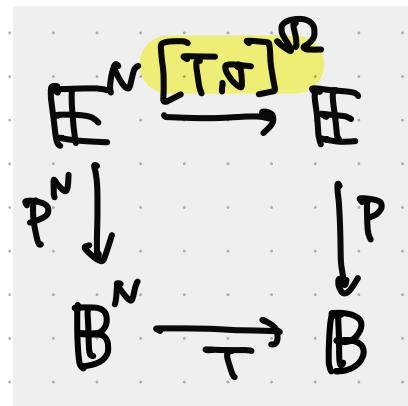
1. Codensity lifting.

- Ordinary one.



... for system's behavior

- Generalizing
Codensity lifting.



... for Composing coalg carriers

2. Sufficient condition of liftability of distributive laws.

$$[T, \tau]^R ([F, \tau]^R)^N \xrightarrow{\quad} [F, \tau]^R [T, \tau]^R$$

⋮

$$TF^N \xrightarrow{\quad} FT$$

Codensity lifting $[F, \tau]^{\Omega}$

[Sprunger +, JLC'21]

Def.

Given $\begin{bmatrix} \mathbb{E} & \text{truth value} \\ \downarrow \text{clm-fib. } P & \Omega \in \mathbb{E}_{\Omega} \\ FGC & F\Omega \xrightarrow{\tau} \Omega \\ \text{modality} \end{bmatrix}$,

$[F, \tau]^{\Omega} : \mathbb{E} \rightarrow \mathbb{E}$ is the largest lifting of F s.t. τ is liftable.

$$\begin{array}{ccc} [F, \tau]^{\Omega} G \mathbb{E} & & [F, \tau]^{\Omega} \Omega \xrightarrow{\tau} \Omega \\ \downarrow P & & \\ FGC & F\Omega \xrightarrow{\tau} \Omega & \end{array}$$

It can be written as $[F, \tau]^{\Omega}(P) = \bigwedge_{R \in \mathbb{E}(P, \Omega)} (\tau \circ F_R)^* \Omega$

e.g. Lifting for standard bisim, Kantorovich metric, ...

Generalizing Codensity lifting $[F, \tau]^{\Omega, \Pi}$

original Beohart's decomposition [Beohart, STACS'14]

$$\begin{array}{c} \mathbb{E} \supseteq [F, \tau]^{\Omega} \\ \downarrow P \\ B \supseteq F \end{array} = R^{\Omega, \Omega} \circ S_p(F, \tau) \circ L^{\Omega, \Omega} \left(= \bigwedge_{\ell \in P} (\tau \circ F \ell)^* \Omega \right)$$

\Downarrow \Downarrow \Downarrow
 $\wedge(-)^* \Omega$ $\tau \circ F(-)$ $\ell \in P$

Def.

$$\mathbb{E} \xrightarrow{\quad \quad \quad} \mathbb{F}$$

$$\begin{array}{ccc} \downarrow P & & \downarrow g \\ B & \xrightarrow{F} & C \end{array}$$

Given $\Omega \in \mathbb{E}_\Omega$, $\Pi \in \mathbb{F}_\Pi$.

$\tau: \mathbb{F}\Omega \rightarrow \Pi$ in \mathbb{C} ,

truth values

modality

2-functor
 $S_p: 1/\text{CAT} \rightarrow \text{CAT}^{\text{Latn}}$

$$[F, \tau]^{\Omega, \Pi} := R^{\Omega, \Pi} \circ S_p(F, \tau) \circ L^{\Omega, \Pi} \left(= \bigwedge_{\ell \in P} (\tau \circ F \ell)^* \Pi \right)$$

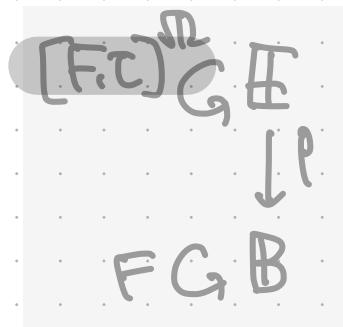
\Downarrow \Downarrow \Downarrow
 $\wedge(-)^* \Pi$ $\tau \circ F(-)$ $\ell \in P$

Thm. $[F, \tau]^{\Omega, \Pi}$ is the largest lifting of F s.t. τ is liftable.

Outline

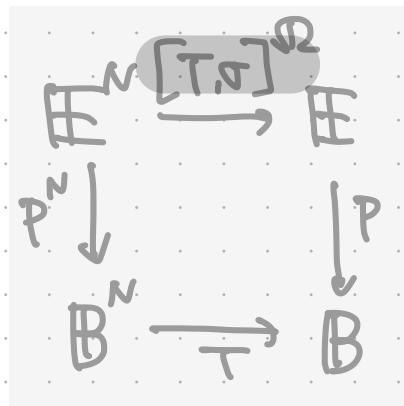
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⋮

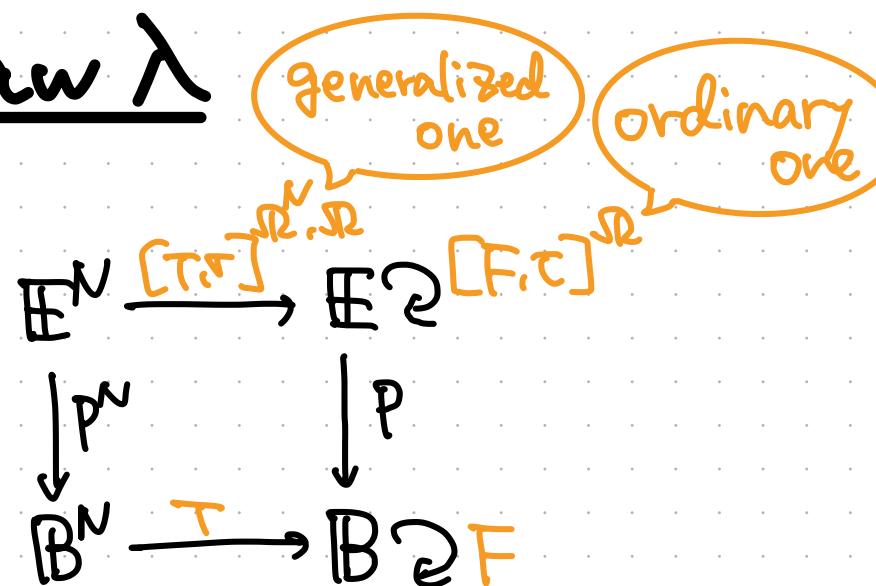
$$TF^N \xrightarrow{\quad ? \quad} FT$$

Liftability of distributive law λ

Consider $\Omega \in E_S$.

$$\tau : F\Omega \rightarrow \Omega \text{ in } B,$$

$$\sigma : T(\Omega, \dots, \Omega) \rightarrow \Omega \text{ in } B,$$



We tackle the problem:

when $\lambda : TF \Rightarrow FT$ is liftable to

$$\tilde{\lambda} : [T, \sigma]^{\Omega} ([F, \tau]^{\Omega})^N \Rightarrow [F, \tau]^{\Omega} [T, \sigma]^{\Omega} ?$$

It gives

$$\text{Coalg}([F, \tau])^N \xrightarrow{[T, \sigma]} \text{Coalg}([F, \tau])$$



$$\downarrow \text{Coalg}(p)$$

Compose bisimulations

$$\text{Coalg}(F)^N \xrightarrow{T\lambda} \text{Coalg}(F)$$

compose systems

Our Sufficient Condition

How do we lift λ to $[\tau, \tau]^{\Omega} ([F, \tau]^{\Omega})^N \Rightarrow [F, \tau]^{\Omega} [\tau, \tau]^{\Omega}$?

$$\begin{aligned} & [\tau, \tau]^{\Omega} ([F, \tau]^{\Omega})^N \\ &= [\tau, \tau]^{\Omega} [F^N, \tau^N]^{\Omega} \\ &= R \circ Sp(\tau, \tau) \circ L \circ R \circ Sp(F^N, \tau^N) \circ L \\ &\Rightarrow R \circ \underline{Sp(\tau, \tau)} \circ \underline{Sp(F^N, \tau^N)} \circ L \quad \text{by } L \dashv R \\ &\stackrel{(1)}{\Rightarrow} R \circ \underline{Sp(F, \tau)} \circ \underline{Sp(\tau, \tau)} \circ L \\ &\stackrel{(2)}{=} [F, \tau]^{\Omega} [\tau, \tau]^{\Omega} \end{aligned}$$

Ihm. $\lambda : TF \Rightarrow FT$ is liftable if (1) and (2) hold.

(1) λ is a 2-cell $(\tau, \tau) \circ (F^N, \tau^N) \Rightarrow (F, \tau) \circ (\tau, \tau)$ in $1//CAT$.

(2) The last equality holds.

(iff $Sp(\tau, \tau) \circ LP$ is approximating to $[F, \tau]^{\Omega}$ for each P)
in the sense of [Komoridat, LICS'21]

Examples of Composition of bisimulations

	Composition of systems. (T_A)	Composition of bisim. ($[T, \sigma]_A^Q$)
Standard bisim.		product
Lang. equivalence of det. automata		
Bisimilarity pseudometric for det. automata.	parallel Composition	max
Similarity " "		
Bisimulation metric for MDP		given by $\sigma(a,b) = 1 - (1-a)(1-b)$

Satisfy our sufficient cond.

unsolved if it satisfies.

Conclusions

$$\text{Coalg}([F, \tau])^N \xrightarrow{[\tau, \tau]^{\Omega}} \text{Coalg}([F, \tau])$$



$$\downarrow \text{Coalg}(p)$$

$$\text{Coalg}(F)^N \xrightarrow{T\lambda} \text{Coalg}(F)$$

compose bisimulations
compose systems

- Generalizing codensity liftings for $[\tau, \tau]^{\Omega} : E^N \rightarrow E$

- A sufficient condition of liftability
 - commutation b/w τ and F .
 - approximation property.

$$\begin{aligned} [\tau, \tau]([F, \tau])^N &\xrightarrow{\lambda} [F, \tau]^{\Omega} [\tau, \tau]^{\Omega} \\ &\vdots \\ TF^N &\xrightarrow{\lambda} FT \end{aligned}$$

- (omitted in this talk) A composition of codensity games via modalities. It preserves invariants under our sufficient condition.

Appendix

Example: Composing bisimulations for Kripke frames

Def. A bisimulation on a Kripke frame $\delta: S \rightarrow \mathcal{P}S$ is $R \subseteq S \times S$ s.t.

$$\left(\begin{array}{c} x \rightarrow x' \\ ;R \\ y \end{array} \right) \Rightarrow \left(\begin{array}{c} x \rightarrow x' \\ ;R \\ y \rightarrow y' \end{array} \right) \text{ and } \left(\begin{array}{c} x \\ ;R \\ y \rightarrow y' \end{array} \right) \Rightarrow \left(\begin{array}{c} x \rightarrow x' \\ ;R \\ y \rightarrow y' \end{array} \right)$$

Prop. R_1 and R_2 are bisim. on δ_1 and δ_2 , respectively.

$\Rightarrow R_1 \times R_2$ is a bisim. on the product of δ_1 and δ_2 .

Coalgebraic reformulation

- Kripke frames $\delta_1, \delta_2 \leftrightarrow$ Coalgebras C_1, C_2
Compose by a product $\qquad\qquad\qquad$ Compose by a distributive law λ
- Bisimulations $R, R_2 \leftrightarrow$ liftings of C_1, C_2 along a fibration.
Compose by a product $\qquad\qquad\qquad$ Compose by a lifting of λ .
 $R_1 \times R_2$ our focus

Composing coalgebras via distributive law

Given functors $F: \mathcal{B} \rightarrow \mathcal{B}$, $T: \mathcal{B}^N \rightarrow \mathcal{B}$,

a distributive law $\lambda: TF^N \Rightarrow FT$,

We have $T\lambda: \text{Coalg}(F)^N \rightarrow \text{Coalg}(F)$

$$\left\{ X_i \xrightarrow{C_i} FX_i \right\}_{i \in \{1, \dots, N\}} \xrightarrow{T\lambda} \underbrace{\overrightarrow{TX} \xrightarrow{\overrightarrow{TC}} \overrightarrow{TFX} \xrightarrow{\lambda \overrightarrow{x}} \overrightarrow{FTX}}_{\text{Composite System.}}$$

Component System

e.g. product of Kripke frames $\{C_i: X_i \rightarrow \mathcal{P}X_i\}_{i=1,2}$

$$T := X: \text{Set}^2 \rightarrow \text{Set}, \quad \lambda_{X,Y}(A, B) := \{(a, b) \mid a \in A, b \in B\}$$

$$\text{Then } T\lambda(C_1, C_2): X_1 \times X_2 \rightarrow \mathcal{P}(X_1 \times X_2)$$

$$a, b \mapsto \{(a', b') \mid a' \in C_1, a, b' \in C_2\}$$

How do we construct a bisimulation on $\underbrace{T\lambda(\vec{C})}_{\text{Composite System}}$?

Beohar et al.'s decomposition, Fibrationally

[Beohar + STACS'24]

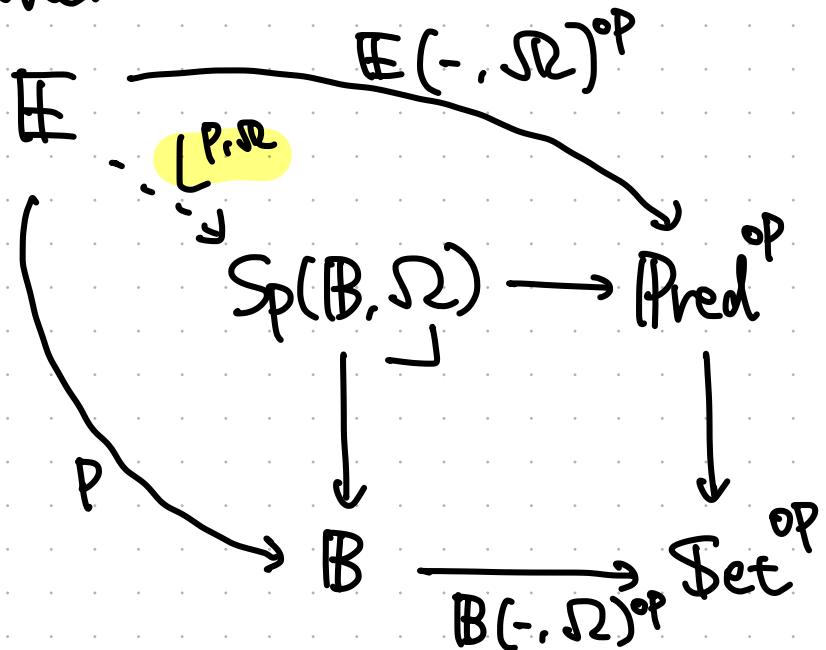
decomposition of $[F, \tau]^{\Omega}$ as being sandwiched b/w adjoints.

$$\text{Thm. } [F, \tau]^{\Omega} = R^{P, \Omega} \circ Sp(F, \tau) \circ L^{P, \Omega} \left(= \bigwedge_{P \in \mathbb{E}(-, \Omega)} (\tau \circ F P)^* \Omega \right)$$

" " "

$\wedge (-)^* \Omega$ $\tau \circ F(-)$ $P \in \mathbb{E}(-, \Omega)$

where



$R^{P, \Omega}$ is the right adjoint of $L^{P, \Omega}$.

fibred lifting of F

$$Sp(B, \Omega) \supseteq Sp(F, \tau)(S \subseteq BC(X, \Omega))$$

\vdash Clat_{-fib.}

$$:= \{\tau \circ F \eta \mid \eta \in S\}$$

$$\subseteq BC(FX, \Omega)$$

$B \supseteq F$

essential part
given by F, τ

Codensity liftings of T. (esp. product functors.)

Example Pseudo-metric lifting by a modality τ .

$$\text{PMet}^2 \xrightarrow{\boxed{X, \tau} \text{ d}_{\mathbb{II}}} \text{PMet}$$

↓ ↓

$$\text{Set}^2 \xrightarrow{X} \text{Set}$$

Euclidean distance.

$$\Omega := d_{\mathbb{II}} \in \text{PMet}[\text{oc}]$$

$\tau : [0,1] \times [0,1] \rightarrow [0,1]$ is a function.

Prop. $\boxed{X, \tau} \text{ d}_{\mathbb{II}} (d_X, d_Y) ((x, y), (x', y')) = \underline{\tau(d_X(x, x'), d_Y(y, y'))}$

if $\begin{cases} \tau \text{ is monotone,} \\ \tau(0, 0) = 0, \\ \tau(a, b) - \tau(c, d) \leq \tau(|a - c|, |b - d|) \text{ for each } a, b, c, d \in [0, 1] \end{cases}$

e.g. $\tau(a, b) := \frac{a+b}{2}, \max(a, b), 1 - (1-a)(1-b)$

They give various non-trivial liftings.