Q1(1)(10 pts):

As x_6 has 100 digits, we have

$$6^{99} \leq \log_2 x < 6^{100}$$

$$99 \log_{10} 6 \leq \log_{10} x < 100 \log_{10} 6$$

$$99(0.3010 + 0.4771) \leq \log_{10} x < 100(0.3010 + 0.4771)$$

$$77.0319 \leq \log_{10} x < 77.81$$

Therefore, x_{10} has $\boxed{78}$ digits.

Question Design:

Difficulty: Average.

Topics involved: Properties of integers, Logarithm.

The calculation of number of digits of a number has appeared in MEXT for several times. Moreover, in 2018 Math (A) Q1(11), MEXT has asked about the number system. This question is a standard textbook exercise when combined the two topics. Be especially careful with the relationship between $[\log_{10} x]$ and the number of digits of x. Although MEXT has once awarded 3 points to the common wrong answer $[\log_{10} x]$, no points have been given if the candidate cannot give the correct answer $[\log_{10} x] + 1$ in recent years.

Q1(2)(10 pts):

By the AM-GM inequality, we have $\frac{1}{\sin x} + \frac{1}{\cos x} \geq \frac{2}{\sqrt{\sin x \cos x}} = \frac{2\sqrt{2}}{\sqrt{\sin 2x}}$, where the equality holds when $\frac{1}{\sin x} = \frac{1}{\cos x}$, i.e. $x = \frac{\pi}{4}$.

As for $0 < x < \frac{\pi}{2}$, $1 \geq \sin 2x > 0$, $\frac{1}{\sqrt{\sin 2x}} \geq 1$, where the equality holds also when $x = \frac{\pi}{4}$.

Combine the above, we have $\frac{1}{\sin x} + \frac{1}{\cos x} \geq \boxed{2\sqrt{2}}$, where the equality holds when $x = \boxed{\dfrac{\pi}{4}}$.

Alternative $(\frac{1}{\sin x} + \frac{1}{\cos x})' = \frac{\sin^3 x - \cos^3 x}{\sin^2 x \cos^2 x}$. To find the extremum, we set $(\frac{1}{\sin x} + \frac{1}{\cos x})' = 0$, i.e. $x = \frac{\pi}{2}$.

The table of signs is given as:

x	$(0, \frac{\pi}{4})$	$(\frac{\pi}{4}, \frac{\pi}{2})$
$(\frac{1}{\sin x} + \frac{1}{\cos x})'$	$-$	$+$
$\frac{1}{\sin x} + \frac{1}{\cos x}$	$\searrow$	$\nearrow$

Therefore, $\frac{1}{\sin x} + \frac{1}{\cos x}$ attains to its minimum when $x = \boxed{\dfrac{\pi}{4}}$, and the corresponding value is $\boxed{2\sqrt{2}}$.

Question Design:

Difficulty: Easy.

Topics involved: Trigonometry, Algebra (Math II).

The model of this question is Math B 2018 Q1(1), which tested the AM-GM inequality. Here the question requires candidates to apply the inequality to find the range of a function. Although the methodology requires some thinkings, all the questions related to finding extremum can be solved by differentiation, which is expected to be mastered by a Math B candidate. Therefore, the overall difficulty of such kind of questions is fair.

Q1(3)(10 pts):

If $|\bar{A} \cap B| = 9$ and $|A \cap \bar{B}| = 11$, then $|B \backslash (A \cap B)| = 9$ and $|A \backslash (A \cap B)| = 11$.

Therefore, $|(A \cup B) \backslash (A \cap B)| = 9 + 11 = 20 = |A \cup B|$, which implies $A \cap B = \emptyset$ and $|A \cap B| = \boxed{0}$.

Question Design:

Difficulty: Very easy.

Topics involved: Set and logics.

This question is a simplified version of Math A 2020 Q1(1), which can be solved immediately if the candidate has some basic concepts of sets.

Q1(4)(10 pts):

The perpendicular distance from the origin (the centre of the circle) to the line $y = x + a$ is $\frac{|0+0-a|}{\sqrt{1^2+1^2}} = \frac{a}{\sqrt{2}}$.

By Pythagoras' theorem, the length of the chord=$2\sqrt{1^2 - (\frac{a}{\sqrt{2}})^2} = \sqrt{2(2-a^2)}$ as the radius of the circle is 1.

Therefore, we have

$$\sqrt{2(2-a^2)} = \sqrt{2}$$

$$a^2 = 1$$

$$a = \boxed{\pm 1}$$

Alternative Substitute $y = x + a$ into $x^2 + y^2 = 1$, we have

$$x^2 + (x + a)^2 = 1$$

$$2x^2 + 2ax + (a^2 - 1) = 0$$

Let (x_1, y_1) and (x_2, y_2) be the coordinates of the intersection points, then the length of the chord

$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(x_2 - x_1)^2 + (x_2 + a - x_1 - a)^2}$

$= \sqrt{2(x_2 - x_1)^2}$

$= \sqrt{2(x_1 + x_2)^2 - 8x_1x_2}.$

By the quadratic equation, we have $x_1 + x_2 = -a$ and $x_1x_2 = \frac{a^2-1}{2}$.

Therefore, we have

$$\sqrt{2a^2 - 4a^2 + 4} = \sqrt{2}$$

$$a^2 = 1$$

$$a = \boxed{\pm 1}$$

Question Design:

Difficulty: Easy.

Topics involved: Coordinates geometry.

This question is a standard textbook exercise in the topic coordinates geometry, which frequetly appears in Q1 of Math B. Note that although only one blank is provided, all the possible answers should be included or not points will be awarded (refer to 2015 Q3(2)).

Q1(5)(10 pts):

Let the numbers be x_1, x_2, x_3 respectively.

Note that we have $(x_1 + x_2 + x_3)^2 = (x_1^2 + x_2^2 + x_3^2) + 2(x_1x_2 + x_1x_3 + x_2x_3)^*$.

As given, $x_1^2 + x_2^2 + x_3^2 = 39$ and $x_1x_2 + x_1x_3 + x_2x_3 = 21$.

Therefore, we have $x_1 + x_2 + x_3 = \sqrt{39 + 2 \cdot 21} = \sqrt{81} = 9$.

Hence, $\bar{x} = \frac{x_1+x_2+x_3}{3} = \boxed{3}$.

Moreover, $Var(X) = \bar{x^2} - \bar{x}^2 = \frac{39}{3} - 3^2 = 4$.

$\sigma = \sqrt{Var(X)} = \sqrt{4} = \boxed{2}$.

*: Generally, $(\sum_{i=1}^{10} x_i)^2 = \sum_{i=1}^{10} x_i^2 + 2\sum_{i \neq j} x_i x_j$.

Question Design:

Difficulty: Average.

Topics involved: Data analysis, Algebra (Math I).

From the past papers of recent years (2016-2020), it can be seen that MEXT seems to be tending to ask some concepts that are new and rare to encounter but not difficult to come up with in the second-last question of Q1. As data analysis has been asked in 2023, it is possible for it to reappear in Q1 in 2024.

Therefore, I designed this question combining the concept of data analysis and the methodology in algebra. The trap of this question will be the number of numbers "3", which lures one to think about solving them out directly, which is impossible. Try to adapt a different train of thought when getting stuck.

Q1(6)(10 pts):

Let $x = 3s + 1 = 5t + 2$. Note that $(s, t) = (2, 1)$ is a possible solution. Therefore, we have

$$\begin{cases} 3s + 1 = 5t + 2 ......(1) \\ 3(2) + 1 = 5(1) + 2 ......(2) \end{cases}$$

By (1)-(2), we have $3(s - 2) = 5(t - 1)$, i.e. $(s, t) = (5k + 2, 3k + 1)$.

Then, $x = 3(5k+2)+1 = 15k+7$. When x is divided by 15, the remainder is $\boxed{7}$.

Note: One may also get the answer directly from the specific case $(s, t) = (2, 1)$.

Alternative To solve the equation $\begin{cases} x = 1 \mod 3 \\ x = 2 \mod 5 \end{cases}$, we set $M = 3 \cdot 5 = 15$, $M_1 = 5$, $M_2 = 3$. Then, $M_1^{-1} = 2$ and $M_2^{-1} = 2$.

Therefore, $x = 1 \cdot 2 \cdot 5 + 2 \cdot 2 \cdot 3 = 22 = \boxed{7} \mod 15$.

Question Design:

Difficulty: Easy.

Topics involved: Properties of integers.

The model of this question is Math B 2020 Q1(4), where the function is replaced by a natural number. In 2020 Q1(4), some people solved the question by long division, which is quite far-fetched and made it very clumsy. Similarly, at first glance, some may use the Chinese remainder theorem to solve this question. However, in fact the question can be solved easily using some trick. Think very carefully before starting to solve a question.

Q2:

(1)(5 pts): As $\vec{OP}$ is a unit vector, we have

$$\vec{OP} = 1$$

$$\sqrt{a^2 + b^2 + \cos^2\theta} = 1$$

$$a^2 + b^2 = 1 - \cos^2\theta = \boxed{\sin^2\theta}$$

(2)(5 pts for 2 correct answers, 3 pts for 1 correct answer): When the value of θ is fixed, by (1), the locus of P is a circle with the equation $x^2 + y^2 = \sin^2\theta$, which has a radius of $\sin\theta$. Therefore, $L = \boxed{2\pi\sin\theta}$.

When θ varies from 0 to π, the area of the region sketched=S

$=\int_0^\pi 2\pi\sin\theta d\theta$

$=2\pi[-\cos\theta]_0^\pi$

$=\boxed{4\pi}$.

Alternative We can also find S by the following way:

When θ varies, the locus of P is a sphere, where the radius of it$=|\vec{OP}| = 1$.

Therefore, $S = 4\pi(1) = \boxed{4\pi}$.

(3)(5 pts): The condition $t < -\frac{l}{2}$ ensured that the plane $z = t$ does no intersect with Π.

Note that as long as the angle between the z-axis and $\vec{OP}$ (which is exactly θ) remains the same, the area of Ω will remain the same. Therefore, we can consider the case that two sides of the square are parallel to the xy-plane and the other two sides make an angle θ with the positive xy-plane.

In this case, Ω is a rectangle with length $l|\cos\theta|$ and width l.

The area of $\Omega = \boxed{l^2|\cos\theta|}$.

Alternative The area of a parallelogram is porportional to the magnitude of its normal vector as the normal vector is parallel to the cross product of its sides.

As the area of Π is l^2 while the magnitude of the normal vector $\vec{OP}$ is 1, the constant of proportionality is l^2.

For Ω, its normal vector is in the z direction. Consider the component of $\vec{OP}$ in the z direction, we have the magnitude of the normal vector of Ω is $|\vec{OP}||\cos\theta| = |\cos\theta|$.

Given the above, the area of $\Omega = \boxed{l^2|\cos\theta|}$.

(4)(5 pts): As the range of (a, b) such that θ can take value depends on the value of θ, the probability that θ can take a certain value depends.

We divide π into n equal pieces, and call the locus of P for $\frac{(i-1)\pi}{n} \leq \theta \leq \frac{i\pi}{n}$ $(1 \leq i \in \mathbb{N} \leq n)$ the $i-$th "belt".

Then, the area of the $i-$th "belt" $S(i)$ can be estimated as the following way:

Take the values when $\theta = \frac{(i-1)\pi}{n}$, by (2), the circumference is $2\pi \sin \frac{(i-1)\pi}{n}$.

We approximate the lower boundary of $S(i)$ as the surface area of a cylinder with circumference $2\pi \sin \frac{(i-1)\pi}{n}$ and height $|\vec{OP}|(\frac{i\pi}{n} - \frac{(i-1)\pi}{n}) = \frac{\pi}{n}*$, we have $S(i) \geq (2\pi \sin \frac{(i-1)\pi}{n})(\frac{\pi}{n})$.

Similarly, we approximate the upper boundary by take the values when $\theta = \frac{i\pi}{n}$, we have $S(i) \leq (2\pi \sin \frac{i\pi}{n})(\frac{\pi}{n})$.

As the magnitude of the sample space (the area of the sphere) is calculated as 4π in (2), the probability that $\frac{(i-1)\pi}{n} \leq \theta \leq \frac{i\pi}{n}$ has the range $[\frac{(2\pi \sin \frac{(i-1)\pi}{n})(\frac{\pi}{n})}{4\pi}, \frac{(2\pi \sin \frac{i\pi}{n})(\frac{\pi}{n})}{4\pi}]$.

As $E(\Omega) = \sum(\Omega(k)P(\theta = k))$, we have

$$\sum_{i=1}^{n} l^2 |\cos(\frac{(i-1)\pi}{n})| \frac{(2\pi \sin \frac{(i-1)\pi}{n})(\frac{\pi}{n})}{4\pi} \leq E(\Omega) \leq \sum_{i=1}^{n} l^2 |\cos(\frac{i\pi}{n})| \frac{(2\pi \sin \frac{i\pi}{n})(\frac{\pi}{n})}{4\pi}$$

Taking $n \to \infty$, by the Rieman sum, we have $E(\Omega) = \int_0^\pi l^2 |\cos\theta| \frac{2\pi \sin\theta}{4\pi} d\theta$

$= \frac{l^2}{2} \int_0^\pi \sin\theta |\cos\theta| d\theta$

$= \frac{l^2}{2}(2\int_0^{\frac{\pi}{2}} \sin\theta \cos\theta d\theta)$

$= \frac{l^2}{2} \int_0^{\frac{\pi}{2}} \sin 2\theta d\theta$

$= \frac{l^2}{2}[-\frac{1}{2}\cos 2\theta]_0^{\frac{\pi}{2}}$

$= \boxed{\frac{l^2}{2}}$

*: When n is large enough, the arc is approximately a stright line.

Note: Here I provided the precise way to solve the question with Rieman sum. In the examination, it can be replaced by a simple observation and the complexity will be largely reduced.

Question Desgin:

Difficulty: Very difficult.

Topics involved: Integration (Math III), Probability (Math A,B). (Although the concept of vector did appear, the process of problem solving has nothing to deal with vectors.)

The model of this question is 2016 Math B Q2, which is a question that deeply impressed me. The general background of the question is the mean shadow problem. This question is the train of throught of solving the difficult question with a relatively simplier specific situation. The detail of the question setting of each sub-question is explained separately as below:

(1): Similar to 2016 Q2, the first part of this question is just a guidance to the following questions, which is expected to be solved by candidate easily.

(2): Although the methodology to solve this part is similar to 2016 Q2: evaluate an integration following the insight from part (1), the importance of the part is entirely different. Unlike 2016 Q2, which has nothing to deal with the following sub-questions, if one didn't deliberate the situation in this part, he may have difficulty solving part (4).

(3): Similar to 2016 Q2, this part is the key to solve part (4). One has to understand the situation thoroughly (or sketch a highly readable graph, which is even more difficult) so as to get the insight of the question. Although it is difficult to prove, but the answer can be given intuitively if one can do so. Common wrong answers should be $l^2 \cos\theta$ (due to the misunderstand or overlooking of the role of θ) and $l^2 \cos^2\theta$ (due to the misunderstand of the defination of Π). A trap of this part is the introducton of t. If one is clear enough, the question has not thing to deal with t in fact. However, one may be anxious if he got an answer without t and doubt whether the answer is correct or not (inspired by 2016 Q3).

(4): Similar to 2016 Q2, I didn't expect one can solve this part effortlessly. Although the guidances from part (1) to (3) ensure that it is not impossible to solve this question, one may be misled by part (3) and give a wrong answer (the defination of θ in 2016 Q2 is also such confusing). A common wrong answer should be $\frac{\int_0^\pi l^2|\cos\theta|d\theta}{\pi}$ (calculating the mean of the expression got in part (3) directly) as the fact that the probability distribution of θ is not uniform is being overlooked. Although the precise way to solve this question requires the Rieman sum, one can intuitively write down the integral directly if he is clear enough about the situation (similar to 2016 Q2). Leave it to the last if you encountered such a question in the exam.

Q3:

(1)(5 pts): $(f(x))^2 = (\sum_{i=0}^{\infty} C_i x^i)^2$
$= (\sum_{i=0}^{\infty} C_i x^i)(\sum_{j=0}^{\infty} C_j x^j)$
$= \sum_{i+j=n} C_i C_j x^n$
Therefore, the coefficient of $x^n = \sum_{i+j=n} C_i C_j$
$= C_0 C_n + C_1 C_{n-1} + C_2 C_{n-2} + ... + C_n C_0$
$= \sum_{k=0}^{n} C_k C_{n-k}$
$= \boxed{C_{n+1}}$.

(2)(5 pts):

$$(f(x))^2 = \sum_{i=0}^{\infty} C_{i+1} x^i$$

$$x(f(x))^2 = \sum_{i=0}^{\infty} C_{i+1} x^{i+1}$$

$$x(f(x))^2 = \sum_{i=1}^{\infty} C_i x^i$$

$$x(f(x))^2 = \sum_{i=0}^{\infty} C_i x^i - C_0$$

$$x(f(x))^2 = f(x) - 1$$

$f(x) = \boxed{\dfrac{1+\sqrt{1-4x}}{2x}}$ ($\dfrac{1-\sqrt{1-4x}}{2x}$ is rejected by the contradiction $0f(0) = 2$)

(3)(5 pts):

$$g^{(1)}(x) = a_1 + 2a_2 x + 3a_3 x^2 + ...$$

$$g^{(2)}(x) = 2a_2 + 3 \cdot 2x + ...$$

$$g^{(n)}(x) = n \cdot (n-1) \cdot ... \cdot 1 \cdot a_n + n \cdot (n-1) \cdot ... \cdot 2 \cdot a_{n+1} x + ...$$

$$g^{(n)}(0) = n!a_n$$

$$a_n = \boxed{\frac{g^{(n)}(0)}{n!}}$$

(4)(5 pts): We have the n-th derivative of $\sqrt{1-4x}$ ($n = 1, 2, 3, ...$):

$(\sqrt{1-4x})^{(1)} = (\frac{1}{2})(-4)(1-4x)^{-\frac{1}{2}} = (-2)(1-4x)^{-\frac{1}{2}}$

$(\sqrt{1-4x})^{(2)} = (-2)(-\frac{1}{2})(-4)(1-4x)^{-\frac{3}{2}} = (-2)(2)(1-4x)^{-\frac{3}{2}}$

$(\sqrt{1-4x})^{(3)} = (-2)(2)(-\frac{3}{2})(-4)(1-4x)^{-\frac{5}{2}} = (-2)(3)(4)(1-4x)^{-\frac{5}{2}}$

$(\sqrt{1-4x})^{(4)} = (-2)(3)(4)(-\frac{5}{2})(-4)(1-4x)^{-\frac{7}{2}} = (-2)(4)(5)(6)(1-4x)^{-\frac{5}{2}}$

$\vdots$

$(\sqrt{1-4x})^{(n)} = (-2)(\frac{(2n-2)!}{(n-1)!})(1-4x)^{-\frac{2n-1}{2}}$

Therefore, using (3), we have $a_0 = 1$, $a_n = (-2)(\frac{(2n-2)!}{(n-1)!})$.

$\sqrt{1-4x} = 1 + \sum_{n=1}^{\infty} \frac{(-2)(\frac{(2n-2)!}{(n-1)!})}{n!} x^n.$

$f(x) = \frac{1-\sqrt{1-4x}}{2x}$

$= \frac{\sum_{n=1}^{\infty} \frac{(2)(\frac{(2n-2)!}{(n-1)!})}{n!} x^n}{2x}$

$= \sum_{n=1}^{\infty} \frac{(2n-2)!}{n!(n-1)!} x^{n-1}$

$= \sum_{n=0}^{\infty} \frac{(2n)!}{(n+1)!n!} x^n$

By comparing the coefficients, we have $C_n = \frac{(2n)!}{(n+1)!n!} = \frac{1}{n-1}\frac{(2n)!}{n!n!} = \boxed{\frac{1}{2n-1}\binom{2n}{n}}$.

Question Desgin:

Difficulty: Average.

Topics involved: Recurrence, Differentiation (Math III).

The format of the question is referred to a few past questions. The background of this question is finding a closed-form expression for the Catalan number using the method of generating function. Although the methodology itself is difficult to be discovered by oneself, the difficulty is lowered in a significant extend as the sub-questions guided one to solve the problem step by step (and in fact the question setting allowed one to guess the answer). However, as the linkage between sub-questions is very close, if one get any of them wrong, it will be a disaster. Therefore, it is recommended to spend more time to do such a question especially careful in the exam. The detail of the question setting of each sub-question is explained separately as below:

(1): This question is a straight-forward one that guides candidate for the following sub-questions.

(2): This question is a direct application of the result of (1), where MEXT sometimes include such sub-questions in some difficult question. We should definitely get hold of such points. MEXT did require us to solve a functional equation by regarding it as a ordinary equation (Math A 2017 Q3). One do not have to worry about the precision when he encounter such a question. However, this equation does require us to reject one solution. No mark will be awarded to the answer if one didn't do so (refer to Math B 2015 Q2).

(3): The background of this question is the Taylor expansion, which is quite common in university entrance exam in Japan. MEXT has tested the underlying concepts for a few times as well. It is better to have an insight in this topic.

(4): This sub-question is the conclusion part of the whole question. Although it is just a direct application, the calculation is very clumsy. Such kind of question did appear in the exam for a few times. It is better to leave sufficient time of such questions and do it extremely carefully. Moreover, there are so many equivalent forms of the answer. I am not sure about whether MEXT will accept some very complex answers (though it did accept once at Math B 2016 Q3). It is recommended to find out the simplest form of answer so as to play safe.

Review The mean mark for Math B in 2019 is 68.7/100. When comparing this paper to 2019's, the difficulties of Q1 are so-so. While 2019's paper included more cumbersome calculations, this paper focus more on the conceptual aspect, which might be easier for some candidates. However, for Q2 and Q3, although Q3 of this paper is comparable with Q2 of 2019, Q2 of this paper is much more difficult than Q3 of 2019. After considered the above-mentioned, the mean mark for this paper is estimated to be 60/100. It is safer if one can get a 70/100. If one is aiming to ace the scholarship, at least a 85/100 in this paper will be more favourable.