Q1 Fill in the blanks with the correct numbers.

(1) The line $y = \sqrt{3}x + 2$ is rotaed by $60°$ anti-clockwise about the origin to the line $y = \boxed{}x + \boxed{}$.

(2) The equations of the angle bisector of the two lines $y = (2 - \sqrt{3})x$ and $y = (\sqrt{3} - 2)x$ are $x = \boxed{}$ and $y = \boxed{}$.

(3) If a line intersect a circle at $(0, 0)$ and $(6, 8)$, and the shortest distance between the centre of the circle and the line is 2, then the equation of the circle is $(x - \boxed{})^2 + (y - \boxed{})^2 = \boxed{}$.

(4) The point $(2, 3)$ is moved by $\sqrt{2}$ units in the direction of $45°$ with the positive x axis and reflected along the line $y = 3x + 4$ to the point $(\boxed{}, \boxed{})$.

(5) If $x^2 + xy + y^2 = 1$, then the maximum value of $x^2 + y^2$ is $\boxed{}$.

(Warning: MEXT has never included any questions with "coordinate geometry" only in Q2 or Q3. Skip the following questions if one has limited time.)

Q2 Given a circle with centre O and radius R, the inversion of a point P with respect to the circle, denoted as P', is a point such that O, P, P' are collinear and $OP \cdot OP' = R^2$. Fill in the blanks with the answers to the following questions.

(1) Let $P = (3, 4)$, when it is inversed with respect to $x^2 + y^2 = 50$, find the coordinates of P'.

(2) If the line $l : y = 2x + 1$ is inversed with respect to $x^2 + y^2 = 9$, write down the equation of l'.

(3) Given a circle Γ with radius R, if a circle C with radius r, with the distance between the centre of Γ and the centre of C d units, express the radius of C' in terms of d, r, R.

(1) $P'=$ $\boxed{}$

(2) C' : $\boxed{}$

(3) $\boxed{}$

Q3 Let C be the parabola $y = x^2 + ax + b$, which has two tangents l_1, l_2 that are perpendicular to each other and intersect at the origin. The x-coordinate of the point of tangency between C and l_1, P_1, is smaller than that between C and l_2, P_2. Find in the blanks with the answers to the following questions.

(1) Express b in terms of a.

(2) Find the range of a.

(3) For $i = 1, 2$, circle D_i is centred at the axis of symmetry of C, and tangent to l_i at P_i. If the radius of D_2 is twice that of D_1, find a.

(1) $b = \boxed{}$

(2) $\boxed{}$

(3) $a = \boxed{}$

Brief Solutions and Comments

Q1(1) Ref: Not specific

Question related to rotation.

We put $x' = \cos 60^\circ x + \sin 60^\circ y = \frac{1}{2}x + \frac{\sqrt{3}}{2}y$ and

$y' = -\sin 60^\circ x + \cos 60^\circ y = -\frac{\sqrt{3}}{2}x + \frac{1}{2}y$.

The equation becomes $-\frac{\sqrt{3}}{2}x + \frac{1}{2}y = \frac{\sqrt{3}}{2}x + \frac{3}{2}y + 2$, i.e. $y = \boxed{-\sqrt{3}}\,x - \boxed{2}$.

This kind of question has appeared several times in past papers. It can be solved using so many methods (e.g. trigonometry, vector, polar coordinate, etc.) that it won't be a real problem. It is recommended that one should master at least one method so that one will not waste too much time on such kind of question.

Q1(2) Ref: 2020 Math B Q1(5)

Question related to symmetry.

Note the symmetry. The two angle bisectors are $x = \boxed{0}$ and $y = \boxed{0}$.

MEXT has asked about angle bisector for several times. Although similar to Q1(1), there are so many methods to solve this kind of questions, I personally recommend one use the angle bisector theorem from plane geometry as it is usually the fastest way (but not in this question). It is recommended that one should master at least one method so that one will not waste too much time on such kind of question.

Q1(3) Ref: 2008 Math B Q1(4)

Question related to circle.

The slope of the line is $\frac{4}{3}$ and the mid-point of the two interception points is $A(3,4)$. Therefore, the centre is lie on the the line $y=-\frac{3}{4}x+\frac{25}{4}$. Another point lying on it is $B(7,1)$, where $AB=5$. Therefore, the coordinate of centre is $(3,4)+\frac{2}{5}(7-3,1-4)=(\frac{23}{5},\frac{14}{5})$ and squared radius$=(\frac{23}{5})^2+(\frac{14}{5})^2=29$. The equation of the circle is $(x-\boxed{\frac{23}{5}})^2+(y-\boxed{\frac{14}{5}})^2=\boxed{29}$.

A fake coordinate geometry question. Although coordinate is introduced, using equation of line and circle will only complicate the problem. Identify quickly what exactly the question is testing candidates before start woring on it.

Q1(4) Ref: Not specific

Question related to transformation.

The point is first moved by 1 unit in the x and y directions to $(3,4)$.

The perpendicular foot of $(3,4)$ on $y = 3x + 4$ is given by solving it with $y = -\frac{1}{3}(x-3)+4$, the perpendicular line of $y = 3x+4$ pass throught $(3,4)$, and it gives $(\frac{3}{10}, \frac{49}{10})$. Therefore, the point after reflection is $2(\frac{3}{10}, \frac{49}{10}) - (3,4) = \boxed{(-\frac{12}{5}, \frac{29}{5})}$.

This type of questions have appeared in Math A for several times, but not Math B. Although it is not difficult if it appears independently, it may appear implicitly as 2018 Math A Q1(7).

Q1(5) Ref: I think a similar question has appeared in past paper, but I forgot which one exactly.

Question related to optimisation.

As $x^2 + y^2 = k$ represent a circle with radius $\sqrt{k}$, we are to find the largest radius that the circle touches the curve (in fact it is an rotated ellipse).

We put $y = mx$, then $(m^2 + m + 1)x^2 - 1 = 0$, i.e. $|x| = \frac{1}{\sqrt{m^2+m+1}}$.

Then, the distance from a certain point on the curve to the origin is given by $\sqrt{\frac{m^2+1}{m^2+m+1}}$.

Let $n = \sqrt{\frac{m^2+1}{m^2+m+1}}$, then $(n^2 - 1)m^2 + n^2 m + (n^2 - 1) = 0$.

We have $\Delta = n^4 - 4(n^2 - 1)^2 \geq 0$, i.e. $\frac{\sqrt{6}}{3} \leq n \leq \sqrt{2}$.

Therefore, the largest radius we can take is $\sqrt{2}$, i.e. the maximum value of $x^2 + y^2 = \boxed{2}$.

Linear optimisation has appeared in Math A for several times. And the concept of it can be adapted in Math B to solve some non-linear problems. However, it is not often to appear. To simplify the calculation, one can simply sketch a fairly accurate graph and use the geometric method in linear optimisation. (If one knows Lagrange's multiplier, one can also solve it by brutal force.)

Q2 Ref: Not specific

Question related to locus.

(1) We have $OP = \sqrt{3^2+4^2} = 5$, hence $OP' = \frac{50}{5} = 10$. As O, P, P' are collinear, $P' = \frac{10}{5}(3,4) = \boxed{(6,8)}$.

(2) By (1), for every point $A(k, 2k+1)$ on l, we have

$A'(u,v) = \frac{9}{k^2+(2k+1)^2}(k, 2k+1)$, i.e. $v = 2u + \frac{9}{k^2+(2k+1)^2}$. However, as $u^2+v^2 = \frac{81}{k^2+(2k+1)^2}$, we have $v = 2u + \frac{u^2+v^2}{9}$, i.e. $u^2+v^2+18u-9v = 0$.

Now replace (u,v) by (x,y), the equation of l' is $\boxed{x^2+y^2+18x-9y=0}$.

(3) Let $\Gamma : x^2+y^2 = R^2$ and $C : (x-d)^2+y^2 = r^2$. Then, C has the x-intercepts $A(d-r,0)$ and $B(d+r,0)$, where $A' = (\frac{R^2}{d-r}, 0)$ and $B' = (\frac{R^2}{d+r}, 0)$. The radius of $C' = \frac{1}{2}A'B' = \boxed{\dfrac{R^2 r}{d^2-r^2}}$.

I don't know how can MEXT set a "coordinate geometry" question in Math B without calculus involved. Just for fun.

Q3 Ref: Not specific

Question related to quadratic curve.

(1) Let the tangent be $y = mx$. We have $\Delta = (a-m)^2 - 4b = 0$, i.e. $m = a \pm 2\sqrt{b}$. As the two tangents are perpendicular to each other, $(a+2\sqrt{b})(a-2\sqrt{b}) = -1$, i.e. $b = \boxed{\dfrac{a^2+1}{4}}$.

(2) As $m = a \pm 2\sqrt{b}$, we have $b > 0$. However, $b = \frac{a^2+1}{4} > 0$. Therefore, $\boxed{a \in (-\infty, \infty)}$.

(3) Solving $x^2 \pm 2\sqrt{b}x + b = 0$, we have $x = \pm\sqrt{b}$, i.e. $p_1 = \frac{m_2 - a}{2}$ and vice versa. The equation of the two radii are $y = -\frac{1}{m_i}(x - p_i) + (p_i^2 + ap_i + b)$. As the centres are on the axis of symmetry, $x = -\frac{a}{2}$, the coordinates of centres are $(-\frac{a}{2}, \frac{1}{2} + (p_i^2 + ap_i + b))$. Consider the radii, we have

$$4((\frac{1}{2})^2 + (p_1 + \frac{a}{2})^2) = (\frac{1}{2})^2 + (p_2 + \frac{a}{2})^2.$$

Simplifying,

$$3a^2 - 20a\sqrt{b} + 12b + 3 = 0$$

$$3\sqrt{a^2+1} = 5a$$

$$a = \boxed{\frac{3}{4}}$$

Unlike Q2, the concept involved in this question did appear in past paper (but that question involved calculus and "coordinate geometry" occupied only a very small partition). However, it is unlikely that MEXT will set such a difficult question for an independent concept.