Q1(1):

Note that: $\log_2 3 \cdot \log_3 4 \cdot ... \cdot \log_{2019} 2020$

$= \frac{\log 3}{\log 2} \cdot \frac{\log 4}{\log 3} \cdot ... \cdot \frac{\log 2020}{\log 2019}$

$= \frac{\log 2020}{\log 2}$

$= \log_2 2020.$

As $2^{10} = 1024 < 2020 < 2048 = 2^{11}$, we have $10 < \log_2 2020 < 11$.

Therefore, the largest natural numbers less than the expression is $\boxed{10}$.

Q1(2):

Note that $f(x)$ is a bijective function. Then, $f(f(x)) = f(x) \iff f(x) = x$.

Solving, we have

$$1 + \frac{1}{x-1} = x$$

$$x^2 - 2x = 0$$

$$x = \boxed{0}, \boxed{2}$$

Alternative

$$f(f(x)) = f(x)$$

$$1 + \frac{1}{1 + \frac{1}{x-1} - 1} = 1 + \frac{1}{x-1}$$

$$\frac{1}{x-1} = x - 1$$

$$x^2 - 2x = 0$$

$$x = \boxed{0}, \boxed{2}$$

Q1(3):

Solving $x^4 + ax^2 + b = 0$, we have $x^2 = \frac{-a \pm \sqrt{a^2-4b}}{2}$, where $a^2 - 4b \geq 0$.

Suppose $a^2 - 4b > 0$, as $\sqrt{a^2 - 4b} \leq \sqrt{a^2} = |a|$, we have

$-a \pm \sqrt{a^2 - 4b} \leq 0$. Therefore, the equation has either no solution or one solution $(x = 0)$ if $a^2 - 4b > 0$.

Therefore, the equation has exactly two solutions if and only if $a^2 - 4b = 0$ and $a < 0$.

Then, $a + 2b = a + \frac{a^2}{2} = \frac{1}{2}(a+1)^2 - \frac{1}{2}$ by completing the square.

As $\frac{1}{2}(a+1)^2 \geq 0$, we have $a + 2b \geq -\frac{1}{2}$. Hence the minimum value of $a + 2b$ is $\boxed{-\dfrac{1}{2}}$.

On the other hand, $a - b = a - \frac{a^2}{4} = -\frac{1}{4}(a-2)^2 + 1$ by completing the square.

As $a < 0$, we have $-\frac{1}{4}(a-2)^2 < -1$ and hence $a - b < 0$. Then, $\lceil a - b \rceil \leq 0$ and the required maximum is $\boxed{0}$.

Q1(4):

By the provided information, we have

$f(x) = (x-1)^2 Q_1(x) + (x-1) = x^2 Q_2(x) + (2x+3)$, where $Q_1(x)$ and $Q_2(x)$ are two polynomials.

As $f(1) = 1^2 Q_2(1) + 5 = 2$, we have $Q_2(1) = -3$. By the reminder theorem, the reminder when $Q_2(x)$ is divised by $x - 1$ is -3. Then, $Q_2(x)$ can be written as $(x-1)Q_3(x) - 3$, where $Q_3(x)$ is a polynomial.

Therefore,

$f(x) = x^2((x-1)Q_3(x) - 3) + (2x+3) = x^2(x-1)Q_3(x) + (-3x^2 + 2x + 3)$.

Hence, the required remainder is $\boxed{-3x^2 + 2x + 3}$.

Q1(5):

Note that the two lines are the reflection of each other with respect to the x-axis.

Therefore, the angle θ equals to twice the angle between the line $y = (2-\sqrt{3})x$ and the positive x-axis.

Then, $\tan\frac{\theta}{2}$ =the slope of the line=$2-\sqrt{3}$.

Consider $\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

$= \frac{2(2-\sqrt{3})}{1-(2-\sqrt{3})^2}$

$= \frac{2-\sqrt{3}}{2\sqrt{3}-3}$

$= \frac{\sqrt{3}}{3}$,

we have $\theta = \boxed{\dfrac{\pi}{6}}$.

(Note: One can also consider $\sin\theta$ or $\cos\theta$ by using the tangent half-angle formula.)

Alternative Use a similar logic, the angle can also be calculated as

$\theta = \arctan(2-\sqrt{3}) - \arctan(\sqrt{3}-2)$

$= \arctan\frac{(2-\sqrt{3})-(\sqrt{3}-2)}{1+(2-\sqrt{3})(\sqrt{3}-2)}$

$= \arctan\frac{\sqrt{3}}{3}$

$= \boxed{\dfrac{\pi}{6}}$.

Alternative A vector of the direction of the line $y = (2-\sqrt{3})x$ is $< 1, 2-\sqrt{3} >$ and a vector of the direction of the line $y = (\sqrt{3}-2)x$ is $< 1, \sqrt{3}-2 >$.

The angle between the two lines=the angle between the two vectors

$=\arccos \frac{<1,2-\sqrt{3}>\cdot<1,\sqrt{3}-2>}{|<1,2-\sqrt{3}>|\cdot|<1,\sqrt{3}-2>|}$

$= \arccos \frac{1-(7-4\sqrt{3})}{(1+(7-4\sqrt{3}))}$

$= \arccos \frac{\sqrt{3}}{2}$

$= \boxed{\dfrac{\pi}{6}}$.

Q2:

(1): The condition that there exists a triangle with the side lengths $\sqrt{a}, \sqrt{b}, \sqrt{c}$ is:

$\sqrt{a}+\sqrt{b} > \sqrt{c}$ and $\sqrt{b}+\sqrt{c} > \sqrt{a}$ and $\sqrt{c}+\sqrt{a} > \sqrt{b}$.

Consider the inverse statement, we have

$P_1 = 1 - P(\ \sqrt{a}+\sqrt{b} \leq \sqrt{c}$ or $\sqrt{b}+\sqrt{c} \leq \sqrt{a}$ or $\sqrt{c}+\sqrt{a} \leq \sqrt{b})$.

Note that the events $\sqrt{a}+\sqrt{b} \leq \sqrt{c}$, $\sqrt{b}+\sqrt{c} \leq \sqrt{a}$ and $\sqrt{c}+\sqrt{a} \leq \sqrt{b}$ are mutually exclusive. Therefore, the probability$=1-3P(\sqrt{a}+\sqrt{b} \leq \sqrt{c})$.

Exhausting all the possibilities: $(a, b, c) = (1, 1, 4), (1, 1, 5), (1, 1, 6), (1, 2, 6), (2, 1, 6)$

We have $P(\sqrt{a}+\sqrt{b} \leq \sqrt{c}) = \frac{5}{6^3} = \frac{5}{216}$.

Therefore, $P_1 = 1 - \frac{15}{216} = \boxed{\dfrac{67}{72}}$.

(2): By Pythagoras' theorem, the condition for there to exist such a triangle is:

$a+b = c$ or $b+c = a$ or $c+a = b$

Note that the three events are mutually exclusive.

Therefore, $P_2 = \frac{3 \cdot P(a+b=c \text{ and } a,b,c \text{ are mutually different})}{3 \cdot P(a+b=c)} = \frac{P(a+b=c \text{ and } a,b,c \text{ are mutually different})}{P(a+b=c)}$.

Exhausting all the possibilities:

$(a, b, c) = (1, 1, 2), (2, 2, 4), (3, 3, 6)$

$(a, b, c) = (1, 2, 3), (1, 3, 4), (1, 4, 5), (1, 5, 6), (2, 3, 5), (2, 4, 6)$ (order of a, b omitted)

Hence, $P_2 = \frac{6 \cdot 2}{3+6 \cdot 2} = \boxed{\frac{4}{5}}$.

(3): We separate the situation into three cases:

1: The triangle is an equilateral triangle. Then, the maximum value of $\frac{bc}{a}$ will be 6.

2: The $60°$ angle is the angle between the side with length $\sqrt{b}$ (same for $\sqrt{c}$ as the value of $\frac{bc}{a}$ is independent on the order of b, c) and the side with length $\sqrt{a}$.

Then, by cosine formula, we have $c = a + b - \sqrt{ab}$.

As c is an integer, we have ab a perfect square. Take $(a, b) = (1, 4)$, a maximum value of $\frac{bc}{a} = 12$ can be obtained.

3: The $60°$ angle is the angle between the side with length $\sqrt{b}$ and the side with length $\sqrt{c}$.

Similar to case 2, a maximum value of $\frac{4}{3}$ can be obtained when $(b, c) = (1, 4), (4, 1)$.

Given the above, the maximum value of $\frac{bc}{a}$ is $\boxed{12}$.

Q3:

(1): Using the chain rule, we have $\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$

$= \frac{dy}{d\theta} \cdot \frac{1}{\frac{dx}{d\theta}}$

$= (\sin\theta) \cdot \frac{1}{1-\cos\theta}$

$= \boxed{\dfrac{\sin\theta}{1-\cos\theta}}.$

(2): Let $x = f(\theta) = \theta - \sin\theta$, then $\theta = f^{-1}(x)$, the inverse function of f.

The required area=$\int_{f(0)}^{f(2\pi)} (1 - \cos(f^{-1}(x)))dx$.

Substitue $x = f(\theta)$, then $dx = f'(\theta)d\theta$ and the range of integral becomes $(0, 2\pi)$.

The area=$\int_0^{2\pi} (1-\cos\theta)(1-\cos\theta)d\theta$

$= \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)d\theta$

$= [\frac{3}{2}\theta - 2\sin\theta - \frac{1}{4}\sin 2\theta]_0^{2\pi}$

$= \boxed{3\pi}.$

(3): With the same setting as (2), the volume=$\pi \int_{f(0)}^{f(2\pi)} (1 - \cos(f^{-1}(x)))^2 dx$

$= \pi \int_0^{2\pi} (1-\cos\theta)^2(1-\cos\theta)d\theta$

$= \pi \int_0^{2\pi} (1 - 3\cos\theta + 3\cos^2\theta - \cos^3\theta)d\theta$

$= \pi[\frac{5}{2}\theta - 3\sin\theta - \frac{3}{4}\sin 2\theta]_0^{2\pi} - \pi\int_0^{2\pi} (1-\sin^2\theta)d\sin\theta$

$= \pi(5\pi) - \pi[\sin\theta - \frac{1}{3}\sin^3\theta]_0^{2\pi}$

$= \boxed{5\pi^2}.$