

Q1(1):

By testing the potential rational roots $\pm 1, \pm 2$ given by the rational root theorem, we have $x = 1$ is a root.

Then, we can do the factorisation by the long division:

$$x^3 - 2x^2 - x + 2 = 0$$

$$(x - 1)(x^2 - x - 2) = 0$$

$$(x - 1)(x + 1)(x - 2) = 0$$

$$x = \boxed{\pm 1, 2}$$

Q1(2):

$$\cos x - 2 \cos^2 x = 0$$

$$\cos x(2 \cos x - 1) = 0$$

$$\cos x = 0, \frac{1}{2}$$

$$x = \boxed{\frac{\pi}{3}, \frac{\pi}{2}}$$

Q1(3):

$$|\sqrt{8} - 3| + |2 - \sqrt{2}|$$

$$= 3 - \sqrt{8} + 2 - \sqrt{2}$$

$$= 3 - 2\sqrt{2} + 2 - \sqrt{2}$$

$$= \boxed{5 - 3\sqrt{2}}$$

Q1(4):

$$\log_2(x-1) = \log_4(x-1)$$

$$(x-1)^2 = x-1$$

$$x-1 = 1 \quad (x \neq 1 \text{ as } x > 1 \text{ for the logarithms to be defined})$$

$$x = \boxed{2}$$

Q1(5):

$$f(x) = \cos x + \frac{1}{2} \cos x - \frac{\sqrt{3}}{2} \sin x$$

$$= \sqrt{3} \left(\frac{\sqrt{3}}{2} \cos x - \frac{1}{2} \sin x \right)$$

$$= \sqrt{3} \left(\sin \frac{\pi}{3} \cos x - \cos \frac{\pi}{3} \sin x \right)$$

$$= -\sqrt{3} \sin \left(x - \frac{\pi}{3} \right).$$

Therefore, $m = \boxed{\sqrt{3}}$ and it takes when $x - \frac{\pi}{3} = \frac{3\pi}{2}$, i.e. $x = \boxed{\frac{11\pi}{6}}$.

Alternative $f'(x) = -\frac{3}{2} \sin x - \frac{\sqrt{3}}{2} \cos x$.

To find the extremum, we set $f'(x) = 0$, then $x = \frac{5\pi}{6}, \frac{11\pi}{6}$.

$$f''(x) = -\frac{3}{2} \cos x + \frac{\sqrt{3}}{2} \sin x.$$

As $f''(\frac{11\pi}{6}) < 0$, $f(x)$ takes the maximum when $x = \boxed{\frac{11\pi}{6}}$ and the value $m = f(\frac{11\pi}{6}) = \boxed{\sqrt{3}}$.

Q1(6):

$$\begin{aligned} & \lim_{h \rightarrow 0} (1 + 2h)^{\frac{1}{h}} \\ &= \lim_{t \rightarrow 0} (1 + t)^{\frac{2}{t}} \\ &= \boxed{e^2} \end{aligned}$$

Q1(7):

Solving:

$$\begin{cases} 2(x - 1) = 6(y - 1) \dots\dots (1) \\ 3(x - 1) = 6(z - 2) \dots\dots (2) \\ x + 2y - 4z + 1 = 0 \dots\dots (3) \end{cases}$$

By $2 \times (2) - 3 \times (1)$, we have $12(z - 2) - 18(y - 1) = 0$, i.e. $z = \frac{3}{2}y + \frac{1}{2}$.

Substitute it into (3), we have $(3y - 2) + 2y - 4(\frac{3}{2}y + \frac{1}{2}) + 1 = 0$, i.e. $y = \boxed{-3}$.

Substituting back, we have $x = \boxed{-11}$ and $z = \boxed{-4}$.

Q1(8):

$$\frac{dy}{dx} = \frac{1}{x}.$$

Let the point of tangency be $(p, \ln p)$, then the equation of the tangent is $y -$

$$\ln p = \frac{1}{p}(x - p).$$

As it passes through (0,0), we have $-\ln p = \frac{1}{p}(-p)$, i.e. $p = e$.

Therefore, the equation is $y - 1 = \frac{1}{e}(x - e)$, i.e. $y = \boxed{\frac{1}{e}x}$.

Q1(9):

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{n(n+2)} \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{2n} - \frac{1}{2(n+2)} \right) \\ &= \frac{1}{2} + \frac{1}{4} \\ &= \boxed{\frac{3}{4}} \end{aligned}$$

Q1(10):

$$\begin{aligned} & \lim_{x \rightarrow -\infty} \frac{2x+1}{\sqrt{x^2+1}} \\ &= \lim_{x \rightarrow -\infty} \frac{2+\frac{1}{x}}{-\sqrt{1+\frac{1}{x^2}}} \\ &= \boxed{-2} \end{aligned}$$

Note: As x takes negative value, it did not be put inside the square root directly.

Instead, its absolute value be put inside. Therefore, a negative sign is added.

Q1(11):

$$\begin{aligned} f(x) &= \ln \frac{\sqrt{x-1}}{x+1} = \frac{1}{2} \ln(x-1) - \ln(x+1) \\ f'(x) &= \boxed{\frac{1}{2(x-1)} - \frac{1}{x+1}} \end{aligned}$$

Q1(12):

Note that the function inside the integral is an odd function. Therefore, the integral is valued $\boxed{0}$.

Q2:

$$1): A^n = \begin{bmatrix} \left(\frac{1}{2}\right)^n & 0 \\ 0 & \left(\frac{1}{2}\right)^n \end{bmatrix}.$$

$$\begin{aligned} 2): S &= \sum_{k=1}^n A^k \\ &= \begin{bmatrix} \sum_{k=1}^n \left(\frac{1}{2}\right)^k & 0 \\ 0 & \sum_{k=1}^n \left(\frac{1}{2}\right)^k \end{bmatrix} \\ &= \begin{bmatrix} \frac{\left(\frac{1}{2}\right)^{n+1} - \frac{1}{2}}{\frac{1}{2} - 1} & 0 \\ 0 & \frac{\left(\frac{1}{2}\right)^{n+1} - \frac{1}{2}}{\frac{1}{2} - 1} \end{bmatrix} \\ &= \begin{bmatrix} 1 - \left(\frac{1}{2}\right)^n & 0 \\ 0 & 1 - \left(\frac{1}{2}\right)^n \end{bmatrix} \end{aligned}$$

$$3): \text{ As } \det(S) = \frac{1}{(1 - (\frac{1}{2})^n)^2}, S^{-1} = \begin{bmatrix} \frac{1}{1 - (\frac{1}{2})^n} & 0 \\ 0 & \frac{1}{1 - (\frac{1}{2})^n} \end{bmatrix}$$

Q3:

$$\begin{aligned}
1): & \int_0^{\frac{\pi}{2}} \sin^3 x dx \\
&= -\int_0^{\frac{\pi}{2}} (1 - \cos^2 x) d(\cos x) \\
&= -[\cos x - \frac{1}{3} \cos^3 x]_0^{\frac{\pi}{2}} \\
&= \boxed{\frac{2}{3}}
\end{aligned}$$

$$2): I_{2k+1} \cdot I_{2k} = \frac{\pi}{2} \cdot \boxed{\frac{1}{2k+1}} \text{ by the telescoping property.}$$

$$3): \text{ Similarly, } b_k = \boxed{\frac{1}{2k}}.$$

$$4): L = \lim_{k \rightarrow \infty} \frac{1}{k} \left(\frac{(2k)(2k-2)\dots 4 \cdot 2}{(2k-1)(2k-3)\dots 3 \cdot 1} \right)^2 = \lim_{k \rightarrow \infty} \frac{2k+1}{k} \left(\frac{\frac{\pi}{2} \cdot I_{2k+1}}{I_{2k}} \right).$$

$$\text{As } I_{2k+1} < I_{2k} < I_{2k-1}, \lim_{k \rightarrow \infty} \frac{2k+1}{k} \left(\frac{\frac{\pi}{2} \cdot I_{2k+1}}{I_{2k-1}} \right) < L < \lim_{k \rightarrow \infty} \frac{2k+1}{k} \left(\frac{\frac{\pi}{2} \cdot I_{2k+1}}{I_{2k+1}} \right).$$

$$\text{Moreover, } \lim_{k \rightarrow \infty} \frac{2k+1}{k} \left(\frac{\frac{\pi}{2} \cdot I_{2k+1}}{I_{2k-1}} \right) = \frac{\pi}{2} \cdot \lim_{k \rightarrow \infty} \frac{2k+1}{k} \left(\frac{2k}{2k+1} \right) = \pi.$$

$$\lim_{k \rightarrow \infty} \frac{2k+1}{k} \left(\frac{\frac{\pi}{2} \cdot I_{2k+1}}{I_{2k+1}} \right) = \frac{\pi}{2} \cdot \lim_{k \rightarrow \infty} \frac{2k+1}{k} = \pi.$$

$$\text{By squeeze theorem, } L = \boxed{\pi}.$$