Q1 Fill in the blanks with the correct numbers.

(1) $\sin^2 1° + \sin^2 2° + ... + \sin^2 90° = \boxed{}$.

(2) If $\log_2 x = \log_4(3x+8) + \frac{1}{2}$, then $x = \boxed{}$.

(3) The maximum value of the function $3\cos^2 x + 4\sin x\cos x + 5$ is $\boxed{}$,

and the minimum value of that is $\boxed{}$.

(4) If $(\frac{1-\cos 3\theta}{\sin 3\theta})(\frac{1-\cos\theta}{\sin\theta}) = 1$ with $\frac{\pi}{6} < \theta < \frac{\pi}{3}$, it follows that $\theta = \boxed{}$.

(5) If the value of $\log_2(\log_3(\log_2 x))$ is an integer, then the minimum value of x is $\boxed{}$.

(Warning: MEXT has not included any questions in Q2 and Q3 with only "special functions" involved. Skip the following two questions if one has limited time.)

Q2 Consider the function $y = 4^x - 2^{x+1} + 9$. Fill in the blanks with the answers to the following questions.

(1) Let X denote 2^x. Express y in terms of X.

(2) Calculate minimum of y, and the values of x at which y attains them.

(1) $y = \boxed{}$

(2) The minimum is $\boxed{}$ at $x = \boxed{}$.

Q3 We suppose $\cos n\theta$ can be expressed as a polynomial of $\cos\theta$, which is denoted as $T_n(\cos\theta)$. Fill in the blanks with the answers to the following questions.

(1) Express $T_2(x)$ and $T_3(x)$ in terms of x.

(2) Express $T_n(x)$ in terms of $x, T_{n-1}(x)$, and $T_{n-2}(x)$.

(3) Express the coefficient of x^n in $T_n(x)$ in terms of n.

(1) $T_2(x) = \boxed{}$ $T_3(x) = \boxed{}$

(2) $T_n(x) = \boxed{}$

(3) $\boxed{}$

Brief Solutions and Comments

Q1(1) Ref: 2020 Math B Q1(1)

Question related to trigonometric functions.

$$\sum_{k=1}^{90} \sin^2 k^\circ = \sum_{k=1}^{44^\circ} \sin^2 k^\circ + \sum_{k=1}^{44} \sin^2 (90-k)^\circ + \tfrac{1}{2} + 1 = \sum_{k=1}^{44} 1 + \tfrac{3}{2} = \boxed{\frac{91}{2}}.$$

Direct calculation using the property of special function. A very straight forward question type.

Q1(2) Ref: Not specific

Question related to logarithmic equation.

$$\log_2 x = \log_4(3x+8) + \frac{1}{2}$$

$$\log_4 x^2 = \log_4(6x+16)$$

$$x^2 - 6x - 16 = 0$$

$$x = \boxed{8} \quad \text{(-2 is rejected as } x > 0\text{)}$$

This type of questions require one to use the property of logarithm on solving equation (or inequality). Although they are often straight forward, one should keep an eye on the domain where the logarithm is defined to avoid giving wrong answer (or missed answer).

Q1(3) Ref: Not specific

Question related to trigonometric function.

As $3\cos^2 x + 4\sin x\cos x + 5 = (\sin x + 2\cos x)^2 + 4 = 5\sin^2(x+\alpha) + 4$, where $\tan\alpha = 2$, we have the maximum is $5 + 4 = \boxed{9}$ and the minimum is $\boxed{4}$.

Finding the maximum and minium of trigonometric functions often appears in Q1. This kind of questions are usually solved by combining the sum of sine and cosine into a signle sine function and use the fact that $|\sin\theta| \leq 1$ (if the domain is not confined). It is enough if one can handle this general methodology.

Q1(4) Ref: Not specific

Question related to trigonometric equation.

Note that $\frac{1-\cos\theta}{\sin\theta} = \frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$, we have $\tan\frac{3\theta}{2} = \tan(\frac{\pi}{2} - \frac{\theta}{2})$.
For $\frac{\pi}{6} < \theta < \frac{\pi}{3}$, we have $\theta = \boxed{\frac{\pi}{4}}$.

A standard question of trigonometric equations. This kind of question appeared in MEXT did not involve too much complicated calculations, but rather more observations. Think carefully before start working on it. Also be careful of the domain provided.

Q1(5) Ref: 2015 Math B Q1(4)

Question related to logarithm.

To obtain the smallest value of x, we take $\log_2(\log_3(\log_2 x)) = 1$, i.e. $x = \boxed{512}$.

An in fact very easy question seemingly to be difficult (which are the last two subquestions in Q1 often be like). Stay calm and one can come up with the essence of the question.

Q2 Ref: 2015 Math B Q3

Question related to exponential.

(1) $y = (2^x)^2 - 2(2^x) + 9 = \boxed{X^2 + 2X + 9}$.

(2) $y = (X-1)^2 + 8 \geq \boxed{8}$, where the equality holds when $X = 1$, i.e. $x = \boxed{0}$.

This kind of question is too simple for a Q2 question. MEXT is unlike to ask it independently but more likely to mix it with other topics like quadratic equations or differentiation (e.g. the model question). One thing to remind when using substitution is to check carefully the domain of the substituted function.

Q3 Ref: Not specific

Question related to trigonometric function.

(1) $\cos 2\theta = 2\cos^2\theta - 1$. Therefore, $T_2(x) = \boxed{2x^2 - 1}$.

$\cos 3\theta = \cos 2\theta \cos\theta - \sin 2\theta \sin\theta = 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta = 4\cos^3\theta - 3\cos\theta$

Therefore, $T_3(x) = \boxed{4x^3 - 3x}$.

(2) As $T_3(x) = 2xT_2(x) - T_1(x)$, we guess $T_n(x) = \boxed{2xT_{n-1}(x) - T_{n-2}(x)}$. A proof is given in **Appendix**.

(3) It is obvious that $\deg(T_n) = n$. Initially ($T_1(x)$), the leading coefficient is 1. In each iteration, the leading coefficient is multiplied by 2. Therefore, the coefficient of x^n is $\boxed{2^{n-1}}$.

Unlike logarithmetic or exponential function, trigonometric function has the potential to be set as a Q2 or Q3 question. However, it is unlike to happen and in fact every such kind of question can be solved by those identities one has learnt (memorised) effortlessly. Moreover, if one knows the Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$, those identities can even not be learnt. Therefore, one has no need to worry about it.

Appendix

Proof using trigonometric identity:

Proof. We want to prove

$$\cos n\theta = 2\cos\theta\cos(n-1)\theta - \cos(n-2)\theta,$$

i.e.

$$\cos n\theta + \cos(n-2)\theta = 2\cos\theta\cos(n-1)\theta.$$

In fact, we have

$$\cos n\theta = \cos\theta\cos(n-1)\theta - \sin\theta\sin(n-1)\theta$$

and

$$\cos(n-2)\theta = \cos((n-1)\theta - \theta) = \cos\theta\cos(n-1)\theta + \sin\theta\sin(n-1)\theta.$$

Adding the two equations together gives us the desired equation. $\square$

Proof using Euler's identity:

Proof. By Euler's identity, we have $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$.

We want to prove

$$\frac{e^{in\theta}+e^{-in\theta}}{2} = 2(\frac{e^{i\theta}+e^{-i\theta}}{2})(\frac{e^{i(n-1)\theta}+e^{-i(n-1)\theta}}{2}) - \frac{e^{i(n-2)\theta}+e^{-i(n-2)\theta}}{2}.$$

In fact, we have

$$\begin{aligned}
&2(\frac{e^{i\theta}+e^{-i\theta}}{2})(\frac{e^{i(n-1)\theta}+e^{-i(n-1)\theta}}{2}) - \frac{e^{i(n-2)\theta}+e^{-i(n-2)\theta}}{2} \\
&= \frac{e^{in\theta}+e^{-in\theta}}{2} + \frac{e^{i(n-2)\theta}+e^{-i(n-2)\theta}}{2} - \frac{e^{i(n-2)\theta}+e^{-i(n-2)\theta}}{2} \\
&= \frac{e^{in\theta}+e^{-in\theta}}{2},
\end{aligned}$$

which completes the proof immediately. □