(Unerdlined questions are unlikely to appear in the exam. One may consider skipping them if one has limited time.)

Q1 Fill in the blanks with the correct numbers.

(1) $\lim\limits_{x \to 1} \frac{\sin(x^2-1)}{x-1} = \boxed{}$.

<u>(2)</u> $\sum\limits_{k=0}^{10} \binom{10}{k} k = \boxed{}$, where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

<u>(3)</u> $\lim\limits_{n \to \infty} \frac{1}{n} \sqrt[n]{n(n+1)...(2n-1)(2n)} = \boxed{}$.

(4) $\int_0^{2024} \{x\} dx = \boxed{}$, where $\{x\}$ denotes the fraction part of x.

(5) $\int_0^2 x^2(x-2)^7 dx = \boxed{}$.

Q2 Consider the function of a pair of natural numbers (m, n) that is defined by an integral

$$I(m, n) = \int_0^1 x^{n-1}(1-x)^{m-1} dx$$

Fill in the blanks with the answers to the following questions.

(1) Express $I(m, 1)$ in terms of m,

(2) Express $I(m, n)$ in terms of m, n, and $I(m+1, n-1)$.

(3) Express $I(m, n)$ in terms of m, n.

(1) $I(m, 1) = \boxed{}$

(2) $I(m, n) = \boxed{}$

(3) $I(m, n) = \boxed{}$

Q3 For real number x, functions $f(x), g(x)$ are differentiable with respect to x and satisfy the following conditions:

$$\frac{d}{dx}(f(x)+g(x)) = e^x, \quad \frac{d}{dx}(f(x)^2+g(x)^2) = e^{2x} - e^{-2x},$$

where $f(0) = 1$ and $g(0) = 0$. Fill in the blanks with the answers to the following questions.

(1) Express $f(x) + g(x)$ in terms of x.

(2) Express $f(x)g(x)$ in terms of x.

(3) Express $f(x)$ and $g(x)$ in terms of x.

(1) $f(x) + g(x) = \boxed{}$

(2) $f(x)g(x) = \boxed{}$

(3) $f(x) = \boxed{}$ $\quad g(x) = \boxed{}$

Brief Solutions and Comments

Q1(1) Ref: Not specific

Question related to limit.

$$\lim_{x\to1}\frac{\sin(x^2-1)}{x-1} = \lim_{x\to1}\frac{(x+1)\sin(x^2-1)}{x^2-1} = (1+1)\cdot 1 = \boxed{2}$$

A very standard type of question that often appers in college of technology's papers. If it appears in Math B, it should be more or less this type.

Q1(2) Ref: 2018 Math A Q1(10)

Question related to differentiation.

As $(x+1)^{10} = \sum_{k=0}^{10} \binom{10}{k} x^k$, we have $10(x+1)^9 = \sum_{k=0}^{10} \binom{10}{k} k x^{k-1}$.

Therefore, $\sum_{k=0}^{10} \binom{10}{k} k = 10(1+1)^9 = \boxed{5120}$.

I don't know how else can MEXT set a differentiation question though this kind of question is unlikely to appear in the exam.

Q1(3) Ref: Not specific

Question related to limit.

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\sqrt[n]{n(n+1)...(2n-1)(2n)} &= \lim_{n\to\infty}\sqrt[n]{1(1+\frac{1}{n})...(1+\frac{n-1}{n})(1+\frac{n}{n})} \\
&= e^{\lim\limits_{n\to\infty}\frac{1}{n}(\ln 1+\ln(1+\frac{1}{n})+...+(1+\frac{n-1}{n})+(1+\frac{n}{n})} \\
&= e^{\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}(\ln(1+\frac{k(1-0)}{n})\frac{1-0}{n})} \\
&= e^{\int_0^1 \ln(1+x)dx} \\
&= e^{x\ln(1+x)-(x-\ln(1+x))|_0^1} \\
&= e^{2\ln 2-1} \\
&= \boxed{\frac{4}{e}}
\end{aligned}$$

A very standard question type in Japan of evaluation a limit of infinite sum using the Riemann sum. However, it has never appeared in MEXT.

Q1(4) Ref: 2009 Math B Q1(4)

Question related to integration.

Note that $\{x\} = x - [x]$, where $[x]$ denotes the integer part of x.

Therefore, $\int_0^{2024} \{x\} dx = \int_0^{2024} x dx - \int_0^{2024} [x] dx$.

Where $\int_0^{2024} x dx = \frac{2024^2}{2}$, and $\int_0^{2024} [x] dx = \sum_{k=0}^{2023} \int_k^{k+1} k dx = \sum_{k=0}^{2023} k = \frac{2023 \cdot 2024}{2}$.

Therefore, the original integral$= \frac{2024(2024-2023)}{2} = \boxed{1012}$.

MEXT did not include integral that can be directly calculated in Math B. Instead, MEXT tends to include integral that requires candidates to think about the property of function. One example is the absolute value (see the model question). Here I illustrated another impressive example.

Q1(5) Ref: Not specific

Question related to integration.

$$\begin{aligned}
\int_0^2 x^2(x-2)^7 dx &= \frac{1}{8}x^2(x-2)^8|_0^2 - \frac{1}{4}\int_0^2 x(x-2)^8 dx \\
&= -\frac{1}{36}x(x-2)^9|_0^2 + \frac{1}{36}\int_0^2 (x-2)^9 dx \\
&= \frac{1}{360}(x-2)^{10}|_0^2 \\
&= \boxed{\frac{128}{45}}
\end{aligned}$$

Although MEXT has never asked explicitly, when solving some problems (especially those involve the calculation of area using integration), some integrals that are easy to find the primitive function but difficult to calculate the numerical value (which is the required answer) appears. Here I set that kind of integrals as an independent question for one's reference.

Q2 Ref: 2019 Math B Q2

Question related to integration.

(1) $I(m, 1) = \int_0^1 (1-x)^{m-1} dx = \boxed{\dfrac{1}{m}}$.

(2) $I(m, n) = \int_0^1 x^{n-1}(1-x)^{m-1} dx$

$= -\frac{1}{m} x^{n-1}(1-x)^m|_0^1 + \frac{n-1}{m} \int_0^1 x^{n-2}(1-x)^m dx = \boxed{\dfrac{n-1}{m} I(m+1, n-1)}$.

(3) By (2), we have

$I(m, n) = \frac{(n-1)(n-2)\ldots 1}{m(m+1)\ldots(m+n-2)} I(m+n-1, 1) = \frac{(m-1)!(n-1)!}{(n+m-2)!} I(m+n-1, 1)$.

As $I(m+n-1, 1) = \frac{1}{m+n-1}$, we have $I(m, n) = \boxed{\dfrac{(m-1)!(n-1)!}{(m+n-1)!}}$.

The model question is a question related to recurrence. However, I am pretty sure it used an integral as its backdrop. Therefore, I present here a similar integral version of the question with the Beta function.

Q3 Ref: 2017 Math A Q3

Question related to functional equation.

(1) $f(x) + g(x) = e^x + C$. As $f(0) + g(0) = 1$, $C = 0$.

Therefore, $f(x) + g(x) = \boxed{e^x}$.

(2) $f(x)^2 + g(x)^2 = \frac{e^{2x}+e^{-2x}}{2}$.

Therefore, $f(x)g(x) = \frac{1}{2}((f(x) + g(x))^2 - (f(x)^2 + g(x)^2)) = \boxed{\dfrac{e^{2x} - e^{-2x}}{4}}$.

(3) $f(x)$ and $g(x)$ are roots of the equation $y^2 - e^x y + (\frac{e^{2x}-e^{-2x}}{2}) = 0$.

Solving, we have $y = \frac{e^x \pm e^{-2x}}{2}$.

Comparing the initial values, we have $f(x) = \boxed{\dfrac{e^x + e^{-x}}{2}}$ and $g(x) = \boxed{\dfrac{e^x - e^{-x}}{2}}$.

This question is a direct complicated version of the model version. This kind of question appears frequently in Math A. Generally they are not too difficult but one should think carefully and make sure one has got use of all the information provided.