

Q1:

Put  $(x, y) = (-2, 41)$  and  $(x, y) = (5, 20)$  into the functions respectively, we have equations:

$$4A + 2B + C = 41 \dots (1)$$

$$25A - 5B + C = 20 \dots (2).$$

On the other hand, by completing the square, we have  $y = A(x - \frac{B}{2A})^2 - \frac{B^2}{4A} + C$ .

As the function is minimized at  $x = 2$ , we have  $\frac{B}{2A} = 2$ , i.e.  $4A - B = 0 \dots (3)$ .

$$(2)-(1): 21A - 7B = -21, \text{ i.e. } 3A - B = -3 \dots (4).$$

$$(3)-(4): A = \boxed{3}.$$

$$\text{Substitute } A = 3 \text{ into (3), } B = 4A = \boxed{12}.$$

$$\text{Substitute } (A, B) = (3, 12) \text{ into (1), } C = 41 - 4A - 2B = 41 - 12 - 24 = \boxed{5}.$$

Moreover, when  $x = 2$ , we obtain the minimum value of the function, which is

$$y = -\frac{B^2}{4A} + C = -\frac{12^2}{12} + 5 = -12 + 5 = \boxed{-7}.$$

(Note: The minimum value can also be calculated by putting  $x = 2$  into the function.)

**Alternative (with calculus)** The equation (3) can also be obtained as the following:

$$y' = 2Ax - B.$$

When  $y$  attains to its extremum,  $y' = 0$ . Hence, by putting  $x = 2$ , we have

$$4A - B = 0 \dots (3).$$

Q2:

As  $x$  satisfies  $x^2 + 2x - 2 = 0$ , we have  $x^3 = -2x^2 + 2x$ . Therefore,  $P$  can be rewritten as  $P = (-2x^2 + 2x) + x^2 + ax + 1 = -x^2 + (a + 2)x + 1$ .

Moreover, as  $x^2 = -2x + 2$ ,  $P = -(-2x + 2) + (a + 2)x + 1 = (a + 4)x - 1$ .

As  $P$  is independent on the value of  $x$ , we have  $a + 4 = 0$ , i.e.  $a = \boxed{-4}$ .

In this case, the value of  $P$  is  $\boxed{-1}$ .

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Q3:

(i):

$$x^2 - 3x - 10 < 0$$

$$\iff (x - 5)(x + 2) < 0$$

$$\iff \boxed{-2 < x < 5}$$

(ii):

$$|x - 2| < a$$

$$\iff -a < x - 2 < a$$

$$\iff 2 - a < x < a + 2$$

In case  $x^2 - 3x - 10 < 0 \implies |x - 2| < a$ , we have

$$2 - a \leq -2 \text{ and } 5 \leq a + 2$$

$$\iff 4 \leq a \text{ and } 3 \leq a$$

$$\iff \boxed{a \geq 4}$$

(iii) In case  $|x - 2| < a \implies x^2 - 3x - 10 < 0$ , we have

$$-2 \leq 2 - a \text{ and } a + 2 \leq 5$$

$$\iff a \leq 4 \text{ and } a \leq 3$$

$$\iff a \leq 3$$

On the other hand, for the inequality  $|x - 2| < a$  holds with a solution, we have the hidden condition  $a > 0$ .

Combine the above, we have  $\boxed{0 < a \leq 3}$ .

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Q4:

(1):  $X = x \leq 4$  if and only if  $x \in \{1, 2, 3, 4\}$ , where the set has 4 elements.

On the other hand, the universal set,  $\{1, 2, 3, 4, 5, 6\}$ , has 6 elements.

Therefore,  $P(X = x \leq 4) = \frac{4}{6} = \frac{2}{3}$ .

Then,  $P(B) = (P(X = x \leq 4))^3 = \left(\frac{2}{3}\right)^3 = \boxed{\frac{8}{27}}$ .

Similarly,  $P(C) = (P(X = x \leq 3))^3 = \left(\frac{3}{6}\right)^3 = \boxed{\frac{1}{8}}$ .

(2):  $P(B) = P(A \cup C) \iff P(B) = P(A) + P(C)$  as  $A$  and  $C$  are mutually exclusive.

Therefore,  $P(A) = P(B) - P(C) = \frac{8}{27} - \frac{1}{8} = \boxed{\frac{37}{216}}$ .

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Q5:

Square the first equality, we have

$$(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx) = 3^2 = 9.$$

By the second equality, we have  $x^2 + y^2 + z^2 = 9$ .

Substitue it into the equality we got, we have  $xy + yz + zx = \boxed{0}$ .

$$\begin{aligned} \text{Next, } (x^2 + y^2 + z^2)^2 &= x^4 + y^4 + z^4 + \boxed{2}((xy)^2 + (yz)^2 + (zx)^2) \\ &= x^4 + y^4 + z^4 + 2((xy + yz + zx)^2 - 2(xy^2z + yz^2x + zx^2y)) \\ &= x^4 + y^4 + z^4 + 2((0)^2 - 2xyz(x + y + z)) \\ &= x^4 + y^4 + z^4 + 2(-2(-2)(3)) \\ &= x^4 + y^4 + z^4 + 24 = (9)^2 = 81. \end{aligned}$$

Therefore,  $x^4 + y^4 + z^4 = 81 - 24 = \boxed{57}$ .

Q6:

(1): We have

$$\triangle ADF = \frac{1}{2}(AD)(AF) \sin \angle A = \frac{1}{2}(AD)(DF) \sin 60^\circ = \frac{\sqrt{3}}{4} AD \cdot DF.$$

On the other hand, by the circle power theorem, we have  $AD \cdot DF = AG \cdot AE$ .

$$\text{Therefore, } \frac{\triangle ADF}{AG \cdot AE} = \frac{\frac{\sqrt{3}}{4} AG \cdot AE}{AG \cdot AE} = \boxed{\frac{\sqrt{3}}{4}}.$$

(2): As both  $BE, BD$  and  $CE, CF$  are tangent to the circle, we have

$$BE = BD = 4 \text{ and } CE = CF = 2.$$

$$\text{Then, } BC = BE + EC = 4 + 2 = \boxed{6}.$$

On the other hand,  $AF = AD = x$ . By the cosine formula, we have

$$BC^2 = AB^2 + AC^2 - 2(AB)(AC) \cos \angle A$$

$$6^2 = (x + 4)^2 + (x + 2)^2 - 2(x + 4)(x + 2) \cos 60^\circ$$

$$36 = x^2 + 8x + 16 + x^2 + 4x + 4 - x^2 - 6x - 8$$

$$x^2 + \boxed{6}x - \boxed{24} = 0$$

$$x = \frac{-6 \pm \sqrt{6^2 - 4(1)(-24)}}{2} = -3 \pm \sqrt{33}$$

As  $AD > 0$ , we have  $AD = \boxed{-3 + \sqrt{33}}$ .

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Q7:

If the x-coordinate of  $P$  is  $\alpha$ , then the x-coordinates of  $Q$  and  $T$  will also be  $\alpha$ . Moreover, by symmetry along the y-axis (as the axis of symmetry of the parabolas is  $x = 0$ ), the x-coordinate of  $R$  will be  $-\alpha$ .

As  $Q, R$  lie on a straight line parallel to the x-axis, we have

$$QR = \alpha - (-\alpha) = \boxed{2}\alpha.$$

Moreover, substitue  $x = \alpha$  into the equations of the two parabolas respectively, the y-coordinates of  $Q$  and  $T$  are  $-\alpha^2 + 4$  and  $\frac{1}{2}\alpha^2 - 2$  respectively.

As  $Q, P, T$  lie on a straight line parallel to the y-axis, we have

$$PQ = (-\alpha^2 + 4) - 0 = \boxed{4} - \alpha^2 \text{ and } PT = 0 - (\frac{1}{2}\alpha^2 - 2) = \boxed{2} - \frac{1}{2}\alpha^2.$$

$$\text{Then, we have } l = 2(QR + RT) = 2(2\alpha + (4 - \alpha^2 + 2 - \frac{1}{2}\alpha^2)) = \boxed{12} + \boxed{4}\alpha - \boxed{3}\alpha^2.$$

By completing the square, we have  $l = -3(\alpha - \frac{2}{3})^2 + \frac{40}{3}$ .

As  $-2(\alpha - \frac{2}{3})^2 \leq 0$  and the equality holds when  $\alpha = \frac{2}{3}$ , we have  $l \leq \frac{40}{3}$ .

When  $\alpha = \boxed{\frac{2}{3}}$ ,  $l$  is maximized and the value of it is  $\boxed{\frac{40}{3}}$ .