(Underlined questions are unlikely to appear in the exam (Math B). Skip them if one has limited time.)

1. Fill in the blanks with the correct numbers.

(1) The total number of positive divisors of 2024 is $\boxed{}$, and the whole sum of those divisors is $\boxed{}$.

(2) For sets A, B, and C, we assume $A \cap B \subset C$. The number of elements in set A is 602, that is set C but not in either set A or set B is 222, that in set B but not in set C is 800, that in set C but not in set A is 622. Then, the number of elements included in set A, B, or C is $\boxed{}$.

(3) If p denotes the statment "$x < 2$" and q denotes the statement "$|x| < 2$", then the truth value (0 for false and 1 for true) of the statment "not p implies not q and not (p and q)" is $\boxed{}$.

(4) In the expansion of $(x+2y+z)^{10}$, the coefficient of $x^3y^2z^5$ is $\boxed{}$. If the multiplication between x, y, z are not commutative (i.e. $xy \neq yz$ and so on), the coefficient of $x^3y^2z^5$ is $\boxed{}$.

(5) There are three chests with a statement, only one of them is true. One and only one of the chests contains treasure.

Chest 1: This chest is empty.

Chest 2: This chest is empty.

Chest 3: The treaure is in Chest 2.

Then, the treasure is in Chest $\boxed{}$. Answer 0 if it cannot be deduced.

(6) We join every opposite mid-points on the sides of a cube and paths are created on its surface. The number of shortest routes to move from one vertex to its diagonally opposite vertex is $\boxed{}$.

Q2 There are three pegs and disks of different sizes. Initially, the disks are placed on the first peg in order of size, with the largest on the bottom. If there are n disks in total, let H_n be the number of moves required to move all the disks to another peg, where a larger disk can never be place on the top of a smaller disk.

(1) Find H_1, H_2, H_3, H_4 and write their values in this order.

(2) Express H_n in terms of H_{n-1}.

(3) Express H_n in terms of n.

(1) H_1, H_2, H_3, H_4 : $\boxed{}$ $\boxed{}$ $\boxed{}$ $\boxed{}$

(2) $H_n = \boxed{}$

(3) $H_n = \boxed{}$

Q3 Initially, there are a white balls and b black balls in a ball. In each action, a ball is picked randomly from the bag and k balls with the same colour of the picked ball is put into the bag.

(1) If $k = 1$, find the probability that a white ball is picked in the second action in terms of a, b.

(2) Express the probability that a white ball is picked in the second action in terms of a, b, k.

(3) Express the probability that a white ball is picked in the n-th action in terms of a, b, k, n.

(1) $\boxed{}$

(2) $\boxed{}$

(3) $\boxed{}$

Brief Solutions and Comments

Q1(1) Ref: 2019 Math B Q1(1), 2014 Q1(5)

Question related to counting.

As $2024 = 2^3 \cdot 11 \cdot 23$, there are $4 \cdot 2 \cdot 2 = \boxed{16}$ positive divisors and the sum is $(1+2+4+8)(1+11)(1+23) = \boxed{4320}$.

A very standard question. Just to make sure one is familiar with the prime factors of 2024.

Q1(2) Ref: 2022 Math A Q1(1)

Question related to sets.

Plot the Venn diagram (note that $A \cap B$ should be inside C). The number of element in $(C \cap B)\backslash A$ is $622 - 222 = 400$. Then, the total number of elements$=602 + 222 + 800 + 400 = \boxed{2024}$.

This kind of question is straightfowrd but sometime cumbersome. I recommend one draw a Venn diagram directly and count the number of element in each desired region rather than using set operation, which is very complicated.

Q1(3) Ref: 2016 Math B Q1(4), 2015 Math A Q1(4)

Question related to logic.

We first simplify the statement, "not q and not (p and q)" is equivalent to "not q and (not p or not q)", or more simply "not q". Then, the statement is equivalent to "q implies p".

Now, q is $-2 < x < 2$, which implies $x < 2$, i.e. p. Therefore, the statement has the truth value $\boxed{1}$.

A question mixed propositional equivalence (from Math A) and propositional logic (from Math B). I personally think this kind of question won't appear explicitly in Math B due to the limitation in format.

Q1(4) Ref: 2016 Math A Q1(1)

Question related to counting.

The coefficient of $x^3y^2z^5$ is equal to $2^2 = 4$ times that in the expansion of $(x+y+z)^{10}$. In the later case, we choose 3 x's, 2 y's and 5 z's in 10 distinctive $(x+y+z)$'s. The coefficient is equal to the number of choices, which is $\frac{10!}{3!2!5!} = 2520$. Therefore, the coefficient in the original expression is $4 \cdot 2520 = \boxed{10080}$. If the mutiplication is not commutative, the only way to obtain $x^3y^2z^5$ is the permutation $xxxyyzzzzz$, which has only one choice. Therefore, the coefficient is $\boxed{4}$.

A question analogue to those counting questions in Math A. However, MEXT has never included a counting question in Math B explicitly (that in 2016 is of different type). If it does, it should be like 2015 Math B Q1(5), which has hidden the requirement of counting behind a seemingly irrelavent question.

Q1(5) Ref: 2016 Math B Q1(4)

Question related to logic.

If the statement on Chest 1 is true, the treasure is either in Chest 2 or Chest 3. As the statement on Chest 2 is false, the treasure is in Chest 2. However, then the statement on Chest 3 is also true, which contradicts the fact that only one statement is true.

If the statement on Chest 2 is true, the treasure is in Chest 1, where the statements on Chest 1 and Chest 2 are false.

If the statement on Chest 3 is true, the statement on Chest 1 is also true.

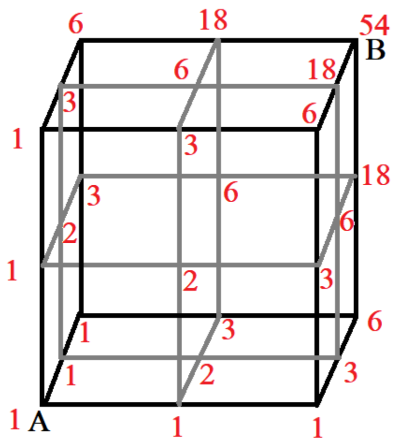
Given the above, the treasure is in Chest $\boxed{1}$.

This question is analogue to the model question, which is also a logic puzzle. However, I personally think MEXT won't include such kind of question frequently.

Q1(6) Ref: 2016 Math B Q1(5).

Question related to counting.

Count directly:



Similar to the model question, counting directly won't cost too much times but using counting techniques may complicate the question. Better to double check the answer with a different approach in the exam if one has enough time.

Q2 Ref: 2015 Math B Q2, 2019 Math B Q2

Question related to counting and recurrence.

(1) Obviously, $H_1 = \boxed{1}$.

For $n = 2$, we first move the smaller disk to a peg, then move the larger disk to another peg, and finally move the smaller disk to the top of the larger disk. Therefore, $H_2 = \boxed{3}$.

For $n = 3$, we first move the smallest disk to a peg, then move the middle disk to another peg and move the smallest disk to the top of it. Then, move the largest disk to a new peg and move the smallest disk to the original peg. Finally, put the middle and smallest disk to the top of the largest disk in order. Therefore, $H_3 = \boxed{7}$.

For $n = 4$, we can move the top 3 disks to a new peg with $H_3 = 7$ moves (as the remaining disk on the original peg is the largest, it can be regarded as an empty peg). After that, we move the largest disk to the empty peg with 1 move. Finally, we can move the remaining 3 disks to the top of it with another $H_3 = 7$ moves. Therefore, $H_4 = \boxed{15}$.

(2) With the same fashion in (1), we have $H_n = \boxed{2H_{n-1} + 1}$.

(3) Suppose $H_n + k = 2(H_{n-1} + k)$, we have $k = 1$. Therefore, $H_n + 1 = 2^n$ and hence $H_n = \boxed{2^n - 1}$.

If a counting question appears in Q2, it is likely to be asking one to find the general pattern through the specific cases. This question is analogue to the model question, but the only different is the “general pattern” is a recurrence ralation instead of a closed form expression. Solving recurrance is also a common technique for “‘discrete mathematics” questions, but MEXT seldom ask it explicitly. Therefore, I present them implicitly in this paper only.

Q3 Ref: Not specific

Question related to equation with absolute value involved.

(1) If a white ball is picked in the first action: $\frac{a}{a+b} \cdot \frac{a+1}{a+b+1} = \frac{a(a+1)}{(a+b)(a+b+1)}$.

If a black ball is picked in the first action: $\frac{b}{a+b} \cdot \frac{a}{a+b+1} = \frac{ab}{(a+b)(a+b+1)}$.

The total probability=$\frac{a(a+1)}{(a+b)(a+b+1)} + \frac{ab}{(a+b)(a+b+1)} = \boxed{\dfrac{a}{a+b}}$.

(2) Similarly, the probability=$\frac{a(a+k)}{(a+b)(a+b+k)} + \frac{ab}{(a+b)(a+b+k)} = \boxed{\dfrac{a}{a+b}}$.

(3) Let p_n be the probability, we have $p_1 = p_2 = \frac{a}{a+b}$.

Let x, y be the number of white ball and black ball before the $(n-1)$-th action, we have

$$p_n = \frac{x(x+k)}{(x+y)(x+y+k)} + \frac{xy}{(x+y)(x+y+k)} = \frac{x}{x+y} = p_{n-1}$$

Therefore, $p_n = p_1 = \boxed{\dfrac{a}{a+b}}$, which is independent on n.

The probability question appeared in Q2 (2020) was about exhaustion. However, Q3 are usually more tricky than Q2 and seemingly tedious techniques like exhaustion won't appear in Q3. Therefore, I set this question though it is unlikely to appear in the exam.