1. Fill in the blanks with the correct numbers.

(1) If $x > 0$, $y > 0$, and $x+y = 3$, then $\boxed{} \geq xy^2 > 0$ with equality if and only if $x = \boxed{}$ and $y = \boxed{}$.

(2) If z is a root of the equation $z^2+z+1 = 0$, then $z^6-2z^3+1 = \boxed{}$.

(3) If the functions $f(x) = \frac{x+2}{3x+4}$, $g(x) = \frac{1-px^2}{3x^2-1}$ satisfy the relation $f(g(x)) = 1 - 2x^2$ $(x \neq -\frac{4}{3}, \pm\sqrt{\frac{1}{3}})$, then the constant $p = \boxed{}$.

(4) $\sum\limits_{k=0}^{10} (-1)^k \binom{10}{k} = \boxed{}$, where $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.

(5) If a, b, c are positive, $a + b + c = 1$ and $(1 - a)(1 - b)(1 - c) = 8abc$, then $(a, b, c) = (\boxed{}, \boxed{}, \boxed{})$.

(6) Let x is a positive integer and $1 < x < 100$. If $x^{2024} - 1$ is divisible by $(x - 1)^2$, the maximum value of x is $\boxed{}$.

(Warning: MEXT has never included any "formula and polynomial" questions in Q2 or Q3 (for Math B). Skip the following two questions if one has limited time.)

2. Consider $x = \sqrt{a^2 + a + 1} - \sqrt{a^2 - a + 1}$. Fill in the blanks with the answers to the following questions.

(1) Express a^2 in terms of x.

(2) Find the range of x.

(1) $a^2 = \boxed{}$

(2) $\boxed{}$

3. Consider a function

$$y = \frac{2}{\frac{4(x^2+4x+53)}{x+2} + \frac{9(x+2)}{x^2+4x+53}}$$

defined in $x > -2$. Fill in the blanks with the answers to the following questions.

(1) Let $t = (x+2) + \frac{49}{x+2}$, express y in terms of t.

(2) Find the range of t.

(3) Find the maximum of y and the corresponding value of x.

(1) $y = \boxed{}$

(2) $\boxed{}$

(3) maximum $y = \boxed{}$ when $x = \boxed{}$

Brief Solutions and Comments

Q1(1) Ref: 2018 Math B Q1(1).

Question related to AM-GM inequality.

As $(\frac{x+\frac{y}{2}+\frac{y}{2}}{3})^3 \geq x(\frac{y}{2})(\frac{y}{2}) = \frac{1}{4}xy^2$, we have $xy^2 \leq 4(\frac{3}{3})^3 = \boxed{4}$.

The equality holds if and only if $x = \frac{y}{2} = \frac{y}{2}$, i.e. $x = \boxed{1}$ and $y = \boxed{2}$.

Another approach to this kind of questions is using differentiation to find the extremum, which is easier to come up with but often more cumbersome in calculation.

Although the trivial fact that $xy^2 > 0$ is provided here, it might not necessarily be provided in the exam (Ref: 2015 Math B Q1(1)).

Q1(2) Ref: Not specific.

Question related to the root of unity.

Note that $z^3-1=(z-1)(z^2+z+1)=0$. Therefore, $z^6-2z^3+1=(z^3-1)^2=\boxed{0}$.

MEXT has used several ways to express the root of unity (we denote the n-th root of unity as ζ_n), like $\zeta_3=\omega=\frac{1+\sqrt{3}i}{2}$ (Ref: 2015 Math A Q1(7)), $\zeta_6=\cos\frac{1}{3}\pi+\sqrt{-1}\sin\frac{1}{3}\pi$ (Ref: 2018 Math B Q1(5)), or providing the original equation (Ref: several questions for college of technology). The general methodology to solve this kind of questions is factorisation. Direct calculation also work, but it is not recommended as it is highly cumbersome.

Q1(3) Ref: 2014 Math B Q1(3), 2020 Math B Q1(2).

Question related to inverse function.

Solving $x = \frac{f^{-1}(x)+2}{3f^{-1}(x)+4}$, we have $f^{-1}(x) = \frac{4x-2}{1-3x}$.
Then, $g(x) = f^{-1}(1-2x^2) = \frac{4-8x^2-2}{1-3+6x^2} = \frac{1-\boxed{4}x^2}{3x^2-1}$.

This kind of question can be solved by direct calculation easily (but very cumbersome). Using the inverse function will be a tricky way to simplify the question in a very large extent (one may assume all the functions provided are invertible if the question is exactly this type).

Q1(4) Ref: 2017 Math B Q1(5).

Question related to binomial expansion.

$(1-1)^{10} = \sum_{k=0}^{10} \binom{10}{k} = \boxed{0}$.

A very simple but tricky question. Just relate the question to the binomial expansion if it is about a sum of binomial coefficients.

Q1(5) Ref: 2019 Math A Q1(7).

Question related to AM-GM inequality.

We have

$$(1-a)(1-b)(1-c) = 1-(a+b+c)+(ab+bc+ac)-abc = abc(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})-abc.$$

On the other hand, by the AM-GM inequality, we have

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 3\sqrt[3]{\frac{1}{abc}}$$

and

$$\sqrt[3]{abc} \leq \frac{a+b+c}{3} = \frac{1}{3}.$$

Combine the two equations together, we have $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 9$.

It follows that $(1-a)(1-b)(1-c) \geq 8abc$ and the equality holds if and only if $a = b = c = \boxed{\dfrac{1}{3}}$.

Although it cannot be judged directly that the question is related to the AM-GM inquality, it can be seen if one attempt it step by step. This kind of question (difficult to start but in fact very standard) often appear in the last one or two questions of Q1. Stay calm and one will ace.

Q1(6) Ref: 2020 Math B Q(4)

Question related to factor theorem and reminder theorem.

We have $x^{2024} - 1 = (x-1)\sum_{k=0}^{2023} x^k$. Therefore, x^{2024} is divisible by $(x-1)^2$ if and only if $\sum_{k=0}^{2023} x^k$ is divisible by $x-1$.

By the factor theorem, $\sum_{k=0}^{2023} x^k = (x-1)q(x) + r(x)$, we have $r(1) = 2024$, i.e. the reminder when $r(x)$ is divided by $x-1$ is 2024.

For $r(x)$ divisible by $x-1$, 2024 is divisible by $x-1$. As $2024 = 22 \cdot 92$, maximum $x-1$ is 92, i.e. maximum x is $\boxed{93}$.

Unlike Math A, factor theorem and reminder theorem do not appear as direct questions in Math B. However, attempt this kind of question step by step using the two theorems repeatedly and one can find the break through. The model question involved only a cubic polynomial and some people may attempt it by long division directly. However, that approach is implausible as that will cost a lot of time. Regarding it, I set the order of the polynomial 2024 here.

Q2 Ref: Not specific

Question related to roots.

(1)

$$x + \sqrt{a^2 - a + 1} = \sqrt{a^2 + a + 1}$$

$$x^2 + 2x\sqrt{a^2 - a + 1} + (a^2 - a + 1) = a^2 + a + 1$$

$$x^2 - 2a = -2x\sqrt{a^2 - a + 1}$$

$$x^4 - 4ax^2 + 4a^2 = 4x^2(a^2 - a + 1)$$

$$4(x^2 - 1)a^2 = x^2(x^2 - 4)$$

$$a^2 = \boxed{\frac{x^2(x^2 - 4)}{4(x^2 - 1)}}$$

(2) As $a^2 \geq 0$, we have $\frac{x^2-4}{x^2-1} \geq 0$.

However, since $\sqrt{a^2 + |a| + 1} > |a|$, we have

$$|x| = \left|\frac{2a}{\sqrt{a^2 + a + 1} + \sqrt{a^2 - a + 1}}\right| < \left|\frac{2a}{a}\right| = 2.$$

Therefore, $x^2 - 4 < 0$, which forces $x^2 - 1 < 0$, i.e. $\boxed{-1 < x < 1}$.

(Note: $x < -2$ or $x > 2$ are not accepted.)

Very unlikely to appear in the exam (even for Math A). Just for fun.

Q3 Ref: 2014 Math A Q3

Question related to AM-GM inequality.

(1) Note that $\frac{x^2+4x+53}{x+2} = (x+2) + \frac{49}{x+2}$, we have $y = \boxed{\dfrac{2}{4t+\frac{9}{t}}}$.

(2) By the AM-GM inequality, $t \geq 2\sqrt{(x+2)(\frac{49}{x+2})} = 14$, i.e. $\boxed{t \geq 14}$.

(3) By the AM-GM inequality, $\frac{2}{4t+\frac{9}{t}} \leq \sqrt{(\frac{1}{4t})(\frac{t}{9})} = \frac{1}{6}$. However, the equality holds if and only if $4t = \frac{9}{t}$, i.e. $t = \frac{3}{2}$, contradicting $t \geq 14$. Therefore, the maximum will be obtained when t is minimum, i.e. $t = 14$, $y = \boxed{\dfrac{28}{793}}$. And by (2), $x + 2 = \frac{49}{x+2}$, i.e. $x = \boxed{5}$.

A similar question did appear in Math A (the model question), but not Math B. I personally think such kind of question is unlikely to appear as a Q3 question in Math B limited by the format (and in fact it is way too simple).