Q1(1):

$$\sqrt{6+\sqrt{a}}+\sqrt{6-\sqrt{a}}=\sqrt{14}$$

$$(6+\sqrt{a})+(6-\sqrt{a})+2\sqrt{(6+\sqrt{a})(6-\sqrt{a})}=14$$

$$\sqrt{36-a}=1$$

$$36-a=1$$

$$a=\boxed{35}$$

Q1(2):

By using long division, $x^3=(x+1)(x^2-x+1)-1$. Hence, the reminder of the division of x^3 by x^2-x+1 is $\boxed{-1}$.

Note that $x^{2007}=x^{3*669}=(x^3)^{669}=((x+1)(x^2-x+1)-1)^{669}$. By using the binomial theorem, we can expand it as

$(x+1)^{669}(x^2-x+1)^{669}-669(x+1)^{668}(x^2-x+1)^{668}+...+(-1)^{669}$.

As all the first 668 terms contain x^2-x+1 as a factor, when it is divided by x^2-x+1, the remainder will be $(-1)^{669}=\boxed{-1}$.

Q1(3):

$$\log_2(x+1)\leq 3$$

$$x+1>0 \text{ and } x+1\leq 2^3$$

$$x>-1 \text{ and } x\leq 7$$

$$\boxed{-1 < x \leq 7}$$

Q1(4):

By completing the circle, rewrite $C : (x-1)^2 + y^2 = 5$. Therefore, we hsve the coordinates of center are $O(1,0)$ and the radius is $\sqrt{5}$.

As AP is tangent to C, we have $AP \perp AO$ and hence

$$AP^2 + AO^2 = PO^2$$

$$AP^2 + (\sqrt{5})^2 = (\sqrt{(4-1)^2 + 3^2})^2$$

$$AP = \sqrt{18-5} = \boxed{\sqrt{13}}$$

(Note: Although this question has itself introduced the coordinate system, analytical method is not recommended as the coordinates of A $(\frac{11-\sqrt{65}}{6}, \frac{5+\sqrt{65}}{6})$ are very ugly and the corresponding simultaneous equation is very difficult to solve.)

Q1(5):

As 16 is a factor to both x and y, we can rewrite $x = 16a$ and $y = 16b$, where $0 < a < b$ as $x < y$ and $x, y \neq 0$.

Then, the given condition $x + y = 96$ will become $a + b = 6$. All the integer solutions are $(a,b) = (1,5)$ and $(a,b) = (2,4)$.

However, when $(a,b) = (2,4)$, the greatest common divisor of x and y will become 32. Therefore, we have $(a,b) = (1,5)$ and hence $(x,y) = (\boxed{16}, \boxed{80})$.

Q2:

(1): By the cosine formula,

$$AC^2 = BA^2 + BC^2 - 2(BA)(BC)\cos\angle B$$

$$AC = \sqrt{5^2 + 4^2 - 2(5)(4)\cos 60^\circ} = \boxed{\sqrt{21}}$$

(2): By the sine formula,

$$\frac{AC}{\sin\angle B} = 2R$$

$$R = \frac{\sqrt{21}}{2\sin 60^\circ} = \boxed{\sqrt{7}}$$

(3): As the area of $\triangle ABC$ is fixed, the area of $ABCD$ attains to its maximum when the area of $\triangle ADC$ attains to its maximum. By that time, D lies above the centre with respect to AC (as for fixed base AC, the altitude is maximised by that time). By that time, we have $AD = AC$.

On the other hand, as $ABCD$ is a cyclic quadrilateral, $\angle D = 180^\circ - \angle B = 120^\circ$.

Then, by the cosine formula,

$$AC^2 = AD^2 + DC^2 - 2(AD)(DC)\cos\angle D$$

$$21 = 2AD^2 - 2AD^2\cos 120^\circ$$

$$AD = \sqrt{7}.$$

Therefore, the maximum area of $ABCD$=$\frac{1}{2}(BC)(BA)\sin\angle B$+$\frac{1}{2}(AD)(DC)\sin\angle D$

$= \frac{1}{2}(4)(5)\sin 60^\circ + \frac{1}{2}(\sqrt{7})(\sqrt{7})\sin 120^\circ$

$= 5\sqrt{3} + \frac{7}{4}\sqrt{3}$

$= \boxed{\dfrac{27}{4}\sqrt{3}}.$

Alternative to (3) See MEXT's official solution.

Q3:

(1): $F(x) = \int_1^x (3t^3 - x^2 t)dt$

$= \frac{3}{4}t^4 - \frac{x^2}{2}t^2|_1^x$

$= \frac{3}{4}x^4 - \frac{1}{2}x^4 - \frac{3}{4} + \frac{x^2}{2}$

$= \frac{1}{4}x^4 + \frac{1}{2}x^2 - \frac{3}{4}.$

Therefore, $F'(x) = \boxed{x^3 + x}.$

Alternative $F(x) = \int_1^x (3t^3 - x^2 t)dt = \int_1^x (3t^3)dt - x^2 \int_1^x tdt.$

By the fundament theorem of calculus and using the product rule of differentiation, $F'(x) = 3x^3 - (\frac{d}{dx}x^2)\int_1^x tdt - x^3$

$= 2x^3 - 2x[\frac{1}{2}t^2]_1^x$

$= 2x^3 - x^3 + x$

$= \boxed{x^3 + x}.$

(2): To find the extremum of $F(x)$, set $F'(x) = 0$, then we have $x = 0$.

$F''(x) = 3x^2 + 1.$

Conduct the second derivative test, as $F''(0) = 1 > 0$, $F(x)$ attains to its minimum when $x = 0$ and the corresponding value is $F(0) = \boxed{-\dfrac{3}{4}}.$

Alternative to (2) See MEXT's official solution, which used the first derivative test to test for the minimum.