(Underlined questions are not likely to appear in the exam. One may consider skipping them if one has limited time.)

Q1 Fill in the blanks with the correct numbers.

(1) Hexadecimal number $2024_{(16)}$ is equal to binary number $\boxed{}$.

(2) The number of integer solutions to the equation $y^2 - xy + x - y + 3 = 0$ is $\boxed{}$.

<u>(3)</u> If a, b are two decimal digits where $11! = 399a6b00$, then $(a, b) = \boxed{}$.

<u>(4)</u> If $a(n)$ denotes the number of even digits in a decimal number, the minimum natural number n such that $a(a(a(a(n)))) = 0$ is $n = \boxed{}$.

(5) The number of integer solutions to the equation $4x^2 - 4xy + 2y^2 - 4y + 4 - 25 = 0$ is $\boxed{}$.

(6) The maximum integer that can be represented by $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$, where a, b, c are positive integers, is $\boxed{}$.

Q2 There are two jugs with volumes a units and b units respectively. Consider measuring c units of water in total using the two jugs with the following actions:

1. Fill up a jug with water.
2. Empty a jug.
3. Pour water inside a jug to another jug until the jug is emptied or the other jug is filled up.

Fill in the blanks with the answers to the following question.

(1) In case $(a, b, c) = (3, 4, 1)$, find the minimum number of actions required.

(2) In case $(a, b, c) = (13, 17, 3)$, find the minimum number of actions required.

(3) In case $(a, b, c) = (4, 6, 1)$, is it possible to solve the puzzle? Fill in 1 for possible and 0 for impossible.

(1) $\boxed{}$

(2) $\boxed{}$

(3) $\boxed{}$

Q3 Let $n \geq 6$ be an integer that is divisible by 6. Let x, y, z be positive integers.

Fill in the blanks with the answers to the following questions.

(1) Find the number of pairs (x, y) satisfying $x + 3y = n$ in terms of n.

(2) Find the number of pairs (x, y, z) satisfying $x + y + 3z = n$ in terms of n.

(3) Find the number of pairs (x, y, z) satisfying $x + 2y + 3z = n$ in terms of n.

(1) $\boxed{}$

(2) $\boxed{}$

(3) $\boxed{}$

Brief Solutions and Comments

Q1(1) Ref: 2016 Math A Q1(7)

Question related to number system.

$$2024_{(16)} = 2 \cdot 16^3 + 2 \cdot 16 + 4 = 2^{13} + 2^9 + 2^2 = \boxed{10000000100100_{(2)}}.$$

This kind of question appears frequently in both explicit (base conversion) and implicit (decimal representation of decimal) forms. This question is a straight forward one with both concepts can be possibly involved.

Q1(2) Ref: Not specific

Question related to Diophantine equations.

We have $(x-y)(y-1)=3$.

Consider $3=3\cdot 1$, we have $(x-y,y-1)=(\pm 1,\pm 3),(\pm 3,\pm 1)$.

Note that for each set of equation, the solution (x,y) is unique and are both integer. Therefore, there are $\boxed{4}$ solutions.

This kind of Diophantine equations has not appear explicitly in past paper, but did implicitly. The methodology of solving this kind of equation is general, which is comparing the prime factors.

Q1(3) Ref: Not specific

Question related to divisibility test.

We first recognise that the last two digits 00 are came from $2 \cdot 5$ and 10. Therefore, b is the last digit of $3 \cdot 4 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 11$. Consider module 10, $b = 2 \cdot 2 \cdot 2 \cdot 1 = \boxed{8}$.
As 11! is divisible by 11, consider $10^n = (-1)^n \mod 11$, we have
$3 - 9 + 9 - a + 6 - 8 + 0 - 0 = 0 \mod 11$, i.e. $a = 1 \mod 11$. As $0 < a < 11$, we have $a = \boxed{1}$.

MEXT has asked a question related to divisibility test in Q2 of a Math A paper. However, it is unlikely to appear in Math B. If it does, use modules instead. There is no need to memorise those divisibility rules.

Q1(4) Ref: 2015 Math B Q1(4)

Question related to parity.

The minimum $a(a(a(n)))$ is 1.

The minimum $a(a(n))$ is 2.

The minimum $a(n)$ is 20 (0 is also even).

The minimum n is $\boxed{2 \times 10^{19}}$.

This kind of questions that composites multiple functions often require one the find the property of the composition function. Just bare in mind that the composited function usually share the same or a similar property with the original function.

Q1(5) Ref: 2009 Math A Q1(5)

Question related to Diophantine equations.

$$4x^2 - 4xy + 2y^2 - 4y + 4 - 25 = 0$$

$$(2x - y)^2 + (y - 2)^2 = 5^2$$

Recognise the Pythagoras pair (3,4,5), we have

$$(2x - y, y - 2) = (\pm 3, 4), (\pm 3, -4), (\pm 4, 3), (\pm 4, -3).$$

However, $2x - y$ and $y - 2$ are having opposite parity.

We have $2x = \pm(\text{odd/even}) + (\text{even/odd} + 2)$, where the right hand side is always odd. Therefore, the equation has $\boxed{0}$ integer solution.

Common ways to solve quadratic Diophantine equations are using Pythagoras pairs $x^2 + y^2 = z^2$ or using the fact that $A_1^2 + A_2^2 + ... + A_n^2 = 0$ if and only if $A_1 = A_2 = ... = A_n = 0$. Both ways has appeared in MEXT but this kind of questions are not common recently.

Q1(6) Ref: 2017 Math B Q1(6)

Question related to Diophantine equations.

Note that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \leq \frac{1}{1}+\frac{1}{2}+\frac{1}{3} < 2$. If 1 can be represented by the expression, 1 is the required maximum integer. However, $1=\frac{1}{2}+\frac{1}{3}+\frac{1}{6}$. Hence, $\boxed{1}$ is the required maximum integer.

This kind of Diophantine equations (Egytian fraction) has a system of theory and algorithm to solve. However, they are very time-costing. In the exam, one should not treat it too precisly. Simply exhausting the cases and find the answer will do. For this question, one can even write down 1 without think as one cannot fill in "does not exist" as the answer, which ensured the existence of pair (a, b, c) that can express 1 using the expression.

Q2 Ref: Not specific

Question related to Diophantine equations.

(1) We first fill up the 3 units jug, and then pour the 3 units of water into the 5 units jug. Then, fill up the 3 units jug and pour 2 units of water into the 5 units jug with 1 unit of water remained. Now, empty the 5 units jug and 1 unit of water is measured. The minumum number of actions is $\boxed{5}$.

(2) Let x, y be the times that the $13, 17$ units jugs have been filled up respectively. We want to solve $13x+17y = 3$, i.e. $13(x+y)+4y = 3$. A trivial solution is $(x + y, y) = (3, -9)$, i.e. $(x, y) = (12, -9)$. Therefore, the general solution is $(x, y) = (17k + 12, -13k - 9)$. It is obvious that the number of actions is minimised when $|x| + |y|$ is minimised. By that time, $k = -1$ and $(x, y) = (-5, 4)$. Now we interpretate this solution with the valid actions:

$(0, 0) \to (0, 17) \to (13, 4) \to (0, 4) \to (4, 0) \to (4, 17) \to (13, 8) \to (0, 8) \to (8, 0) \to (8, 17) \to (13, 12) \to (0, 12) \to (12, 0) \to (12, 17) \to (13, 16) \to (0, 16) \to (13, 3) \to (0, 3)$

Where there are $\boxed{17}$ steps in total.

(3) Similar to (2), we want to solve $4x + 6y = 1$. However, the left hand side is divisible by 2 whereas the right hand side is not. Therefore, there is no solution $\boxed{0}$.

Number theory is included in Japan's "Mathematics A" syllabus. The questions that MEXT set for this syllabus were all very interesting. Hope that one can find them interesting if one encounters such kind of questions in the exam.

Q3 Ref: 2012 Math A Q3

Question related to Diophantine equations.

(1) As $x = n - 3y$, we have $1 \leq y \leq \frac{n}{3} - 1$. Therefore, there are $\boxed{\dfrac{n-3}{3}}$ such pairs.

(2) We have $x + y = n - 3z$, where $1 \leq z \leq \frac{n}{3} - 1$. For $x = (n - 3z) - y$, $1 \leq y \leq n - 3z - 1$ and therefore there are $n - 3z - 1$ pairs of (x, y). Hence, the number of pairs $(x, y, z) = \sum\limits_{z=1}^{\frac{n}{3}-1} (n - 3z - 1) = \boxed{\dfrac{(n-2)(n-3)}{6}}$.

(3) We have $x + 2y = n - 3z$, where $1 \leq z \leq \frac{n}{3} - 1$. For $x = (n - 3z) - 2y$, $1 \leq y \leq \frac{n-3z}{2} - 1$ when z is even and $1 \leq y \leq \frac{n-3z-1}{2}$ when z is odd. Therefore, there are $\frac{n-3z}{2} - 1$ pairs and $\frac{n-3z-1}{2}$ pairs of (x, y) when z is even and odd respectively. As $\frac{n}{3} - 1$ is odd, write $\frac{n}{3} - 1 = 2k + 1$, the total number of pairs (x, y, z) is $\sum\limits_{m=1}^{k} (\frac{n-3(2m)}{2} - 1) + \sum\limits_{m=0}^{k} \frac{n-3(2m+1)-1}{2} = \boxed{\dfrac{n^2 - 6n + 12}{12}}$.

As there was a "number theory" question in Q3, it is better for me to adapt that question directly. However, I couldn't come up with other similar situation. So, I just modified the model question a bit. This kind of questions are not too difficult, but one should be extremely careful, especially when dealing with summation.