Q1(1):

As $\omega^2+\omega+1=0$, we have $\omega^2=-\omega-1$, i.e. $\omega^{2+n}=-\omega^{1+n}-\omega^n$ for all $n\in\mathbb{N}$.

Then, $\omega^{2+n}=(\omega^n+\omega^{n-1})-\omega^n=\omega^{n-1}=\omega^{n-4}=...=\omega^{n-1-3k}$ for all $n,k\in\mathbb{N}$.

Therefore, $\omega^{10}+\omega^5+3$

$=\omega^{8-1-3\cdot 2}+\omega^{3-1-3\cdot 0}+3$

$=\omega+\omega^2+3$

$=\boxed{2}$ (as $\omega^2+\omega+1=0$).

Alternative There are many different approaches to this question if one cannot notice the relation $\omega^{2+n}=\omega^{n-1-3k}$ and I won't list out all of them here. Those approaches are using the relation $\omega^2+\omega+1=0$ to reduce the degree of the polynomial $\omega^{10}+\omega^5+3$. For example:

Use the relation $\omega^{10}=-\omega^9-\omega^8$ to reduce the polynomial to degree 9 and so on.

Use the relation $\omega^{10}=(-\omega-1)^5$, i.e. $\omega^{10}+\omega^5=-5\omega^4-10\omega^3-10\omega^2-5\omega-1$ by binomial expansion, to reduce the polynomial to degree 4 and so on.

Q1(2):

The binomial expansion of $(2x^4+\frac{1}{x^3})^7$ is $\sum\limits_{k=0}^{7}C_k^7(2x^4)^{7-k}(\frac{1}{x^3})^k$, i.e. $\sum\limits_{k=0}^{7}C_k^7 2^{7-k}x^{28-7k}$, where C_r^n is the binomial coefficient.

To find the constant term, we put a value of k to the general term such that the index of x of that term equals to 0.

Solving $28 - 7k = 0$, we have $k = 4$.

Therefore, put $k = 4$, the constant term is $C_4^7(2)^{7-4} = 35 \cdot 8 = \boxed{280}$.

Q1(3):

$-x < x^2 < 2x + 1$

$-x < x^2$ and $x^2 < 2x + 1$

$x(x+1) > 0$ and $(x - \frac{2+\sqrt{2^2-4(1)(-1)}}{2})(x - \frac{2-\sqrt{2^2-4(1)(-1)}}{2}) < 0$

$(x < -1$ or $x > 0)$ and $1 - \sqrt{2} < x < 1 + \sqrt{2}$

$\boxed{0 < x < 1 + \sqrt{2}}$

Q1(4):

Note that $|x(x-1)| = \begin{cases} x - x^2, \ x \in (0,1) \\ \\ x^2 - x, \ x \in (1,2) \end{cases}$.

Therefore, $\int_0^2 |x(x-1)| dx$

$= \int_0^1 |x(x-1)| dx + \int_1^2 |x(x-1)| dx$

$= \int_0^1 (x - x^2) dx + \int_1^2 (x^2 - x) dx$

$= [\frac{x^2}{2} - \frac{x^3}{3}]_0^1 + [\frac{x^3}{3} - \frac{x^2}{2}]_1^2$

$= \frac{1}{2} - \frac{1}{3} + \frac{8}{3} - 2 - \frac{1}{3} + \frac{1}{2}$

$= \boxed{1}$.

Q1(5):

If $\frac{1}{1-\sin\theta}+\frac{1}{1+\sin\theta}=6$, then

$$\frac{1+\sin\theta}{(1-\sin\theta)(1+\sin\theta)}+\frac{1-\sin\theta}{(1+\sin\theta)(1-\sin\theta)}=6$$
$$\frac{2}{1-\sin^2\theta}=6$$
$$\frac{1}{\cos^2\theta}=3$$

Moreover, divide both side of the identity $\sin^2\theta=1-\cos^2\theta$ by $\cos^2\theta$, we have

$\tan^2\theta=\frac{1}{\cos^2\theta}-1$.

As for $0<\theta<\frac{\pi}{2}$, $\tan\theta>0$, we have $\tan\theta=\sqrt{\frac{1}{\cos^2\theta}-1}=\sqrt{3-1}=\boxed{\sqrt{2}}$.

Q2:

Let the equation of C be $(x-a)^2+(y-b)^2=r^2$. As C touches the x-axis, the y-coordinate of the center of C is equal to the radius r. Therefore the equation of C becomes $(x-a)^2+(y-r)^2=r^2$.

As the point $(2,2)$ lies one C, we have $(2-a)^2+(2-r)^2=r^2$, i.e.

$r=\frac{8-4a+a^2}{4}$......(1).

Consider the slope of tangent to C at the point $(2,2)$. As the tangent also tangent to $y=\frac{1}{2}x^2$, the slope of it is equal to $\frac{dy}{dx}|_{x=2}=2$.

On the other hand, by doing implicit differentiation to the circle, we have

$2(x-a)+2(y-r)\frac{dy}{dx}=0$, i.e. $\frac{dy}{dx}=-\frac{x-a}{y-r}$.

Therefore, we have $\frac{dy}{dx}|_{(2,2)}=-\frac{2-a}{2-r}=2$, i.e. $r=\frac{6-a}{2}$......(2).

Combine equations (1) and (2), we have

$$\frac{6-a}{2}=\frac{8-4a+a^2}{4}$$

$$a^2-2a-4=0$$

$$a=1+\sqrt{5} \text{ or } a=1-\sqrt{5}(\text{rejected as the centre of } C \text{ lies in quadrant I})$$

Therefore, radius=$r=\frac{6-(1+\sqrt{5})}{2}=\boxed{\dfrac{5-\sqrt{5}}{2}}$.

Alternative See MEXT's official solution, which obtained the equation (2) using the slope of radius instead.

Alternative (without calculus) The equation (2) can also be obtained as the following:

The slope of radius at $(2,2)=\frac{r-2}{a-2}$. As the tangent is perpendicular to the radius, the slope of tangent at $(2,2)=-\frac{a-2}{r-2}$. Therefore, the equation of tangent is $y=-\frac{a-2}{r-2}(x-2)+2$ by using the point-slope from of straight line.

As it also tangent to the parabola, the equation

$$-\frac{a-2}{r-2}(x-2)+2=\frac{1}{2}x^2$$

$$\frac{1}{2}x^2+\frac{a-2}{r-2}x-2\frac{a-2}{r-2}-2=0$$

has only one solution.

Then,

$$\Delta=(\frac{a-2}{r-2})^2-4(\frac{1}{2})(-2\frac{a-2}{r-2}-2)=0$$

$$(\frac{a-2}{r-2}+2)^2=0$$

$$r = \frac{6-a}{2} ......(2)$$

Q3

(1): As $\triangle BAQ$ and $\triangle BCQ$ shares the common altitude with bases AQ and CQ respectively, the ratio $AQ : QC$ will be the ratio of their areas.

On the other hand, $\triangle BAQ : \triangle BCQ = (\frac{1}{2}(BA)(BQ)\sin\angle ABQ) : (\frac{1}{2}(BC)(BQ)\sin\angle CBQ)$ $= \frac{1}{2}(BA)(BP)\sin\angle ABP) : (\frac{1}{2}(BC)(BP)\sin\angle CBP) = \triangle BAP : \triangle BCP$ as $BQ : BQ = BP : BP = 1 : 1$.

Therefore, $AQ : QC = \triangle BAQ : \triangle BCQ = \triangle BAP : \triangle BCP = \boxed{2 : 3}$.

(2): As $AQ : QC = 2 : 3$, we have

$\vec{BQ} = \frac{1}{2+3}(2\vec{BC} + 3\vec{BA}) = \frac{1}{5}(2(\vec{c} - \vec{b}) + 3(\vec{a} - \vec{b})) = \frac{1}{5}(3\vec{a} - 5\vec{b} + 2\vec{c})$.

On the other hand, note that $\triangle ABC : \triangle APC = (2 + 3 + 5) : 5 = 2 : 1$. As they share the common base AC, the ratio of their areas is equal to the ratio of their altitudes, which is equal to the ratio $QB : QP$ by similarity.

Therefore, we have $BP : BQ = 1 : 2$ and hence $\vec{BP} = \frac{1}{2}\vec{BQ} = \frac{1}{10}(3\vec{a} - 5\vec{b} + 2\vec{c})$.

Hence, $\vec{OP} = \vec{OB} + \vec{BP} = \vec{b} + \frac{1}{10}(3\vec{a} - 5\vec{b} + 2\vec{c}) = \boxed{\frac{1}{10}(3\vec{a} + 5\vec{b} + 2\vec{c})}$.