

Q1(1):

The boat has to sail with an angle  $\theta$  with AB so as to move in a straight path.

As the component of the velocity of the boat in the x-direction should cancel the speed of the river by that time, we have  $2V \sin \theta = V$ , i.e.  $\theta = 30^\circ$ . Therefore, the boat is moving along the straight line with a speed of  $2V \cos 30^\circ = \sqrt{3}V$ .

Hence, the time required =  $\boxed{\frac{L}{\sqrt{3}V}}$ .

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Q1(2):

For a revolving planet, its centripetal force of the circular motion is provided entirely by the gravitational attraction. Therefore, we have

$$m\left(\frac{2\pi}{T}\right)^2 a = \frac{GMm}{a^2}$$
$$T^2 = \boxed{\frac{4\pi^2}{GM}} a^3$$

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Q1(3):

Note that  $E$  is a component of the electric field due to the charge  $q'$ , where the another component cancelled the magnetic field due to the charge  $q$ .

Therefore, we have:

$$\begin{cases} k \frac{q'}{\sqrt{2}AC} \cos 45^\circ = E \\ k \frac{q'}{(\sqrt{2}AC)^2} \sin 45^\circ = -k \frac{q}{(AC)^2} \end{cases}$$

By the second equation, we have  $q = -\frac{1}{2\sqrt{2}}q'$ .

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Q1(4):

The change in internal energy of the gas is given by  $\Delta U = \frac{3}{2}p_0\Delta V = \frac{3}{4}p_0V$ .

The work done by the gas is given by  $W = p_0\Delta V = \frac{1}{2}p_0V$ .

By the first law of thermodynamics, we have  $Q = \Delta U + W = \frac{5}{4}p_0V$ .

**Alternative** The molar heat capacity of monoatomic gas under constant pressure is given by  $c_p = \frac{5}{2}R$ .

By  $pV = nRT$ , we have  $\Delta T = \frac{p_0}{R}\Delta V = \frac{p_0V}{2R}$ .

Therefore,  $Q = nc_v\Delta T = (\frac{5}{2}R)(\frac{p_0V}{2R}) = \frac{5}{4}p_0V$ .

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Q1(5):

Note that the optical path difference is equal to the optical path length that the refracted light travelled in the medium, i.e.  $2dn$ .

As the refracted light wave is reflected at the bottom, the phase difference is increased by  $\pi$ .

As constructive interference occurs, we have  $2dn = (m + \frac{1}{2})d$ .

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Q2:

(2): In the case,  $C_1$  and  $C_2$  are fully charged and has infinite resistance. Therefore, all the EMF of the battery is consumed by them.

As they are in series, the number of charges accumulated are the same. Let  $V_2$

and  $V_1$  be the voltage across  $C_2$  and  $C_1$  respectively. We have:

$$\begin{cases} V_1 + V_2 = V \\ Q = C_1 V_1 = C_2 V_2 \end{cases}$$

Therefore,  $V_2 = \frac{C_1}{C_2} V_1 = \frac{1}{2} V_1$  and hence  $V_1 = \frac{2}{3} V = \boxed{4} V$ .

(1): Continue (2),  $Q = C_1 V_1 = \boxed{8} \times 10^{-6} C$ .

(3): In the case, the charges in  $C_2$  are evenly distributed to  $C_2$  and  $C_3$ .

As the voltage across  $C_2$  and  $C_3$  is the same by that time, we have:

$$\begin{cases} Q_2 + Q_3 = 8 \times 10^{-6} \\ V = \frac{Q_2}{C_2} = \frac{Q_3}{C_3} \end{cases}$$

Therefore,  $Q_2 = \frac{16}{3} \times 10^{-6} C$  and  $Q_3 = \boxed{\frac{8}{3}} \times 10^{-6} C$ .

(4): By  $U = \frac{1}{2} QV = \frac{Q^2}{2C}$ , the total energy stored

$$= \frac{Q_2^2}{2C_2} + \frac{Q_3^2}{2C_3} = \boxed{\frac{16}{3}} \times 10^{-6} J.$$

(5): Initially, the total energy of the system  $= \frac{1}{2} C_2 V_2^2 = 8 \times 10^{-6} J$ .

The energy is lost in the form of Joule heat, which is valued

$$(8 - \frac{16}{3}) \times 10^{-6} = \boxed{\frac{8}{3}} \times 10^{-6} J.$$

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Q3:

Let  $N_A$  and  $N_B$  be the normal reactions acted on the plate, which is perpendicular to the corresponding inclined plane.

Let  $W$  be the weight.

By the balance of forces, we have:

$$\begin{cases} N_A \cos \alpha + N_B \cos \beta = W \\ N_A \sin \alpha = N_B \sin \beta \end{cases}$$

By solving, we have  $N_A = \frac{W \sin \beta}{\sin(\alpha + \beta)}$  and  $N_B = \frac{W \sin \alpha}{\sin(\alpha + \beta)}$ .

Let  $\theta$  be the angle between the left plane and the plate. Note that the angle between the right plane and the plate is hence  $\alpha + \beta - \theta$ . Also note the hidden condition  $0 < \theta < \alpha + \beta$ .

Take moment at the C.G. (which is in the middle of the plate), we have

$$N_A \cos \theta = N_B \cos(\alpha + \beta - \theta)$$

$$\sin \beta \cos \theta = \sin \alpha \cos(\alpha + \beta - \theta)$$

The balance is failed if there is no solution of  $\theta$  for  $0 < \theta < \alpha + \beta$ .

(1): When  $\alpha + \beta = \frac{\pi}{2}$ , we have

$$\sin\left(\frac{\pi}{2} - \alpha\right) \cos \theta = \sin \alpha \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\tan \alpha = \tan \theta$$

$$\theta = \alpha \in \left(0, \frac{\pi}{2}\right)$$

Therefore, the balance is always possible  $\boxed{(b)}$ .

(2): When  $\alpha = \beta$ , we have

$$\sin \alpha \cos \theta = \sin \alpha \cos(2\alpha - \theta)$$

$$\theta = \alpha \in (0, \pi)$$

Therefore, the balance is always possible  $\boxed{(c)}$ .

(3): When  $\alpha + \beta = \frac{\pi}{4}$ , we have

$$\sin(\frac{\pi}{4} - \alpha) \cos \theta = \sin \alpha \cos(\frac{\pi}{4} - \theta)$$

$$\cos \alpha \cos \theta - \sin \alpha \cos \theta = \sin \alpha \cos \theta + \sin \alpha \sin \theta$$

$$\tan \alpha = \frac{1}{2 + \tan \theta}$$

As  $0 < \theta < \frac{\pi}{4}$ , we have  $\frac{1}{2} < \frac{1}{2 + \tan \theta} < \frac{1}{3}$ . Therefore, for the equation to have solution, we have  $\frac{1}{2} < \tan \alpha < \frac{1}{3}$   $\boxed{(e)}$ .

(4): When  $\alpha = \frac{\pi}{4}$  and  $\beta = \frac{\pi}{3}$ , we have

$$\sin \frac{\pi}{3} \cos \theta = \sin \frac{\pi}{4} \cos(\frac{7\pi}{12} - \theta)$$

$$\frac{\sqrt{3}}{2} \cos \theta = \frac{\sqrt{2}}{2} (\frac{\sqrt{2} - \sqrt{6}}{4} \cos \theta + \frac{\sqrt{2} + \sqrt{6}}{4} \sin \theta)$$

$$\frac{\sqrt{3} - 1}{2} \cos \theta = \frac{\sqrt{3} + 1}{2} \sin \theta$$

$$\tan \theta = 2 - \sqrt{3}$$

Therefore, the case is possible  $\boxed{(b)}$ .

(5): The equation is equivalent to  $\frac{\sin \beta}{\sin \alpha} = \frac{\cos(\alpha+\beta-\theta)}{\cos \theta}$ .

Note that the R.H.S. is monotonic decrease for  $0 < \theta < \frac{3\pi}{10} < \frac{\pi}{2}$ .

When  $\theta = 0$ , R.H.S. =  $\frac{1}{\sqrt{2}}(\cos \frac{\pi}{20} + \sin \frac{\pi}{20})$  and L.H.S. =  $\frac{\sin \frac{\pi}{20}}{\frac{1}{\sqrt{2}}}$ .

As  $\tan \frac{\pi}{20} < \tan \frac{\pi}{4} = 1 < 2$ , we have  $R.H.S. > L.H.S.$

Therefore, the equation has no solutions for  $0 < \theta < \frac{3\pi}{10} < \frac{\pi}{2}$  and the case is impossible  $\boxed{(a)}$ .

Note: I think there should be a better way to solve this question but I can't figure it out.

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Q4:

(1): The molar fraction of  $O_2 = \frac{\frac{0.2}{2 \cdot 16}}{\frac{0.2}{2 \cdot 16} + \frac{0.8}{2 \cdot 14}} \times 100\% \approx 18\%$ .

By Avogadro's law, the volumetric fraction is equal to the molar fraction  $\approx \boxed{18\%}$ .

(2):

$$\frac{0.2m}{32} + \frac{0.8m}{28} = 1$$

$$m \approx 28.71798 \text{ g} \approx \boxed{0.0287 \text{ kg}}$$

(3): By  $pV = nRT$ , we have  $V = \frac{nRT}{p} = \frac{(1)(8.31)(273.15)}{0.1013 \times 10^6} \approx \boxed{0.0224 \text{ m}^3}$ .

(4): Combine the results of (2) and (3),  $\rho = \frac{m}{V} \approx \frac{0.0287}{0.0224} \approx \boxed{1.28 \text{ kg/m}^3}$ .

(5): As both oxygen and nitrogen are diatomic gas, the ratio =  $\boxed{\frac{7}{5}}$ .

Note: The  $\frac{c_p}{c_v}$  ratio is equal to  $\frac{5}{3}$ ,  $\frac{7}{5}$  and  $\frac{9}{7}$  respectively for monoatomic, diatomic and triatomic gases.

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Q5:

(1): Sound speed is proportional to the square root of temperature. Therefore,

$$\frac{T_{15}}{T_0} = \sqrt{\frac{288.15}{273.15}} \approx \boxed{1.027}.$$

(2): The sound speed at 273.15 K is 340 m/s. Therefore, the sound speed at 288.15 K is approximately 349.18 m/s.

Note that  $60 \text{ km/h} \approx 16.67 \text{ m/s}$ .

The general form of Doppler effect is given by  $f' = \frac{V - v_{\text{observer}}}{V - v_{\text{source}}} f$ .

Therefore, we have  $f' = \frac{349.18}{349.18 - 16.67} \cdot 440 \approx 462 \text{ Hz}$ .

Hence,  $\lambda = \frac{v}{f} \approx \frac{340}{462} \approx 0.73 \text{ m}$ .

(3): As calculated in (2),  $f' \approx \boxed{462 \text{ Hz}}$ .