1. Fill in the blanks with the correct numbers.

(1) If $\frac{x}{x-2} > 2$, then $\boxed{\quad} < x < \boxed{\quad}$.

(2) For a constant k, the equation $|x^2 - 3x + 2| = kx$ has the maximum number of solutions when $\boxed{\quad} < k < \boxed{\quad}$, where the number of solutions is $\boxed{\quad}$.

(3) The range of the function $y = \frac{x^2+x+1}{x+1}$ $(x \neq -1)$ is $y \leq \boxed{\quad}$ or $y \geq \boxed{\quad}$.

(4) For the equation $x^2 - px = 1 - p$, the sum of the squares of the roots is minimised when $p = \boxed{\quad}$ and the maximum value is $\boxed{\quad}$.

(5) If the equation $x^3 + px^2 - x + 2 = 0$ has one and only one rational root, then the constant $p = \boxed{\quad}$.

(6) The expression $x^2 + 13y^2 - 4xy - 6y + 7$ takes its maximum value $\boxed{\quad}$ when $x = \boxed{\quad}$ and $y = \boxed{\quad}$.

(Warning: MEXT did not included any "equations" questions in Q2 or Q3 recently. Skip the following two questions if one has limited time.)

Q2 Consider the equation

$$x^2 + 7x - 5 = 5\sqrt{x^3 - 1}$$

Fill in the blanks with the answers to the following questions.

(1) Put $y = \sqrt{\frac{x^2+x+1}{x-1}}$, solve y.

(2) Find the product of roots of the equation.

(1) $\boxed{}$

(2) $\boxed{}$

Q3 Consider the number of solutions of the equation

$$|x^2 - 2| = |2x^2 + ax - 1|$$

when a varies. Fill in the blanks with the answers to the following questions.

(1) Express ax in terms of x.

(2) Find the range of a such that the equation has maximum numbers of solutions.

(3) Find the range of a such that the equation has minimum numbers of solutions.

(1) $ax = \boxed{}$

(2) $\boxed{}$

(3) $\boxed{}$

Brief Solutions and Comments

Q1(1) Ref: 2017 Math B Q1(3)

Question related to inequality.

$$\frac{x}{2-x} > 2$$
$$\frac{3x-4}{2-x} > 0$$
$$\boxed{\frac{4}{3} < x < 2}$$

A seeminglys easy but tricky question. If the question does not require two answers, one might easily get the wrong answer $x < 2$ by multiplying both sides by $2 - x$. Check carefully whether or not the steps one has written down are valid (does the $\iff$ relation holds?) even if the question is in a very simple form.

Q1(2) Ref: 2019 Math A Q1(2), 2013 Math B Q3(1)

Question related to equation with absolute value.

Sketch a graph and observe that the maximum number of solutions is obtained when $k > \boxed{0}$.

Moreover, $y = kx$ is tangent to $-(x^2 - 3x + 2)$ when $\Delta = (3-k)^2 - 8 = 0$, i.e. $k = 3 - 2\sqrt{2}$, we have $k < \boxed{3 - 2\sqrt{2}}$.

By that time, the number of solutions is $\boxed{4}$.

A very standard question type that appeared in MEXT several times. For all equations with absolute valule involved, I highly recommend one combine the question with geometry, or the calculation, which requires one to do case by case, will be very cumbersome.

Q1(3) Ref: Not specific

Question related to discriminant.

Regard the function as an equation of x, $x^2 + (1-y)x + (1-y) = 0$.

We have $\Delta = (1-y)^2 - 4(1-y) = (y-1)(y+3) \geq 0$, i.e. $y \leq \boxed{-3}$ or $y \geq \boxed{1}$.

Using the discriminant to determine the range of a function has appeared in the exam several times implicitly. Here I set it as a question explicitly so as to arouse one's familiarity with this methodology.

Q1(4) Ref: Not specific

Question related to Vieta's theorem.

Let α, β be the roots, we have $\alpha + \beta = p$ and $\alpha\beta = p - 1$. Therefore, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = p^2 - 2p + 2 = (p - 1)^2 + 1$.

Therefore, the required value is maximised when $p = \boxed{1}$ as $\boxed{1}$.

Very standard question with Vieta's theorem, which has appeared several times in the papers for college of technology and implicitly in the past papers. If it appears explicitly in Math B, it is likely to be this kind of question.

Q1(5) Ref: Not specific

Question related to rational root theorem.

By the rational root theorem, possible rational roots are ± 1 and ± 2.

If 1 is a root, $p = -2$. By long division, $x^3 - 2x^2 - x + 2 = (x+1)(x-1)(x-2)$.

Therefore, $p \neq -2$ and ± 1, 2 cannot be the only rational root.

Now the only possibility is -2, where $p = \boxed{1}$. There is no need to check other roots as the rational root theorem ensured there are no other rational roots.

MEXT has never put an "equation" question in the last two subquestions of Q1. However, judging from the past paper, the last two subquestions are likely to be some questions that will be extremely cumbersome if one haven't treated well. In here, one may conduct the long division four times if one didn't make use of the rational root theorem properly.

Q1(6) Ref: Probably a question from past paper, but I forgot which one exactly.

Question related to completing the square.

By completing the square,

$$x^2 + 13y^2 - 4xy - 6y + 7 = (x - 2y)^2 + 9y^2 - 6y + 7 = (x - 2y)^2 + (3y - 1)^2 + 6$$

As $(x - 2y)^2 + (3y - 1)^2 \geq 0$, the expression take its maximum value $\boxed{6}$ when $y = \boxed{\frac{1}{3}}$ and $x = 2y = \boxed{\frac{2}{3}}$.

A standard completing square question. However, the expression which appeared as bivariable might confuss one at first glance. The best strategy for this kind of question is to treat it variable by variable and step by step.

Q2 Ref: Not specific

Question related to quadratic equation.

(1)

$$x^2 + 7x - 5 = 5\sqrt{x^3 - 1}$$

$$(x^2 + x + 1) + 6(x - 1) = 5\sqrt{(x - 1)(x^2 + x + 1)}$$

$$y^2 + 6 = 5y$$

$$y = \boxed{2 \text{ or } 3}$$

(2) If $y = 2$, $x^2 - 3x + 5 = 0$, where $\Delta < 0$ implies the equation has no solution.

Therefore, $y = 3$, i.e. $x^2 - 8x + 10 = 0$.

By Vieta's theorem, the product of roots is $\boxed{10}$.

I personally do not think that MEXT will include an "equations" question in Q2 due to a lot of limitations. Just for fun.

Q3 Ref: 2013 Math B Q3

Question related to equation with absolute value involved.

(1) We have $x^2 - 2 = 2x^2 + ax - 1$ or $x^2 - 2 = -(2x^2 + ax - 1)$, i.e. $ax = \boxed{-x^2 - 1 \text{ or } -3x^2 + 3}$.

(2) Similar to Q1(2), we sketch a graph and observe that the maximum number of solutions is obtained when the line barely passes through the point of tangent to $-x^2 - 1$ and haven't passed throught the intersection point of the parabolas, or when the line has passes throught the point of intersection of the two parabolas.

For the line tangent to $y = -x^2 - 1$, we have $\Delta = a^2 - 4 = 0$, i.e. $a = \pm 2$.

For the two parabola intersects, we have $-x^2 - 1 = -3x^2 + 3$, i.e. $x = \pm 2$, i.e. $a = \pm\frac{3\sqrt{2}}{2}$.

Hence, for the former case, we have $\boxed{2 < |a| < \frac{3\sqrt{2}}{2}}$ and for the latter case, we have $\boxed{|a| < \frac{3\sqrt{2}}{2}}$.

(Accept any other equivalent expression.)

(3) Similarly, the minimum number of solutions is obtained when the line haven't touched the point of tangent to $-x^2 - 1$ yet, i.e. $\boxed{|a| < 2}$.

This question is a complicated version of 2013 Math B Q3. I personally think the model question is already the limit for MEXT. However, just in case a question as complex as this one appeared, stay calm and follow the standard way (break down into cases and sketch a graph) will turn everything okay.