

Q1(1):

By testing the potential rational roots given by the rational root theorem  $\pm 1$ ,  
we have  $x = -1$  is a root.

Then, we can do the factorisation by the long division:

$$x^3 + 4x^2 + 4x + 1 = 0$$

$$(x + 1)(x^2 + 3x + 1) = 0$$

$$x = \boxed{-1, \frac{-3 \pm \sqrt{5}}{2}}$$

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Q1(2):

$$\cos 2x + \cos x = 0$$

$$2 \cos^2 x + \cos x - 1 = 0$$

$$(2 \cos x - 1)(\cos x + 1) = 0$$

$$\cos x = \frac{1}{2}, -1$$

$$x = \boxed{\frac{\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}}$$

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Q1(3):

$$3^{x+1} + \frac{1}{3^x} < 4$$

$$3 \cdot 3^{2x} - 4 \cdot 3^x + 1 < 0$$

$$(3 \cdot 3^x - 1)(3^x - 1) < 0$$

$$\frac{1}{3} < 3^x < 1$$

$$\boxed{-1 < x < 0}$$

Q1(4):

$$\log_2 \sqrt{2x-1} < \log_4 x$$

$$\log_4 (2x-1) < \log_4 x$$

$$2x-1 < x$$

$$x < 1$$

Moreover, for  $\sqrt{2x-1}$  to be defined, we have  $x > \frac{1}{2}$ .

Therefore, the solution is  $\boxed{\frac{1}{2} < x < 1}$ .

Q1(5):

$$\begin{aligned} & \sum_{n=0}^{120} \frac{1}{\sqrt{n+1} + \sqrt{n}} \\ &= \sum_{n=0}^{120} (\sqrt{n+1} - \sqrt{n}) \\ &= \sqrt{121} - 0 \\ &= \boxed{11} \end{aligned}$$

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Q1(6):

Number of  $n$  such that  $n$  is divisible by 2 and divisible by 5 are 100 and 40 respectively.

Moreover, number of  $n$  such that  $n$  is divisible by both 2 and 5 (i.e. divisible by 10) is 20.

Therefore, number of  $n$  such that  $n$  is divisible by 2 or 5 is  $100+40-20=120$ .

Number of  $n$  such that  $n$  is not divisible by 2 nor 5  $=200-120=\boxed{80}$ .

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Q1(7):

We have  $\vec{c} \cdot \vec{a} = 0$  and  $\vec{c} \cdot \vec{b} = 0$ , i.e.

$$\begin{cases} 3s + 2t = 0 \\ s - t = -5 \end{cases}$$

By solving, we have  $s = \boxed{-2}$  and  $t = \boxed{3}$ .

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Q1(8):

$$\frac{dy}{dx} = \frac{1}{2}e^{\frac{x}{2}}.$$

Let the point of tangency be  $(k, e^{\frac{k}{2}})$ , then the equation of it is  $y - e^{\frac{k}{2}} = \frac{1}{2}e^{\frac{k}{2}}(x - k)$ .

As it passes through  $(0,0)$ , we have  $-e^{\frac{k}{2}} = \frac{1}{2}e^{\frac{k}{2}}(-k)$ , i.e.  $k = 2$ .

Therefore, the equation of it is  $y = \frac{1}{2}e(x - 2) + e = \boxed{\frac{e}{2}x}$ .

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Q1(9):

Note that  $a_n$  is a geometric sequence with the first term 5 and the common ratio  $\frac{3}{4}$ .

Therefore, the required sum  $= \frac{5}{1 - \frac{3}{4}} = \boxed{20}$ .

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Q1(10):

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{1 - \sqrt{1 - \sin x}}{x} \\ &= \left( \lim_{x \rightarrow 0} \frac{\sin x}{x} \right) \left( \lim_{x \rightarrow 0} \frac{1}{1 + \sqrt{1 - \sin x}} \right) \\ &= \boxed{\frac{1}{2}} \end{aligned}$$

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Q1(11):

$$f'(x) = 1 - \frac{2}{2x+1}$$

$$f'(x) < 0$$

$$1 - \frac{2}{2x+1} < 0$$

$$2x + 1 < 2$$

$$x < \frac{1}{2}$$

Moreover, for  $\ln(2x+1)$  to be defined, we have  $x > -\frac{1}{2}$ .

Therefore, the solution is  $\boxed{-\frac{1}{2} < x < \frac{1}{2}}$ .

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Q1(12):

$$\begin{aligned} & \int_0^\pi x \sin x dx \\ &= \int_0^\pi x d(-\cos x) \\ &= -x \cos x \Big|_0^\pi + \int_0^\pi \cos x dx \\ &= \pi + [\sin x]_0^\pi \\ &= \boxed{\pi} \end{aligned}$$

**Alternative**

$$\begin{aligned} \int_0^\pi x \sin x dx &= \int_0^\pi (\pi - x) \sin(\pi - x) dx = \pi \int_0^\pi \sin x dx - \int_0^\pi x \sin x dx \\ \int_0^\pi x \sin x dx &= \frac{1}{2} [-\pi \cos x]_0^\pi = \boxed{\pi} \end{aligned}$$

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Q2:

$$1): A^2 = a^2 \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

$$A^3 = a^3 \begin{bmatrix} -2 & -2 \\ 2 & -2 \end{bmatrix}$$

$$A^4 = a^4 \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}$$

$$\text{Therefore, } a^4 = \frac{1}{4}, \text{ i.e. } a = \boxed{\frac{1}{\sqrt{2}}}.$$

2): Note that  $\begin{bmatrix} 0 & 1 \\ x & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$

We are to find the minimum  $n$  such that  $A^n$  is in the form of  $\begin{bmatrix} 0 & 1 \\ x & 0 \end{bmatrix}.$

As  $A^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  and  $A^4 = -I$ , i.e.  $A^{4n+k} = (-1)^n A^k$ , the minimum  $n$  will be  $4 \cdot 1 + 2 = \boxed{6}.$

3):  $A^{2014} = A^{503 \cdot 4 + 2} = (-1)^{503} A^2 = \boxed{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}$

Q3:

1):  $f'(x) = (1-x)^n - nx(1-x)^{n-1}.$

Solving  $f'(x) = 0$ , we have  $x = \boxed{\frac{1}{n+1}}.$

2): For  $f'(x) = 0$ , we have  $x = \frac{1}{n+1}$  and  $x = 1.$

As we have:

$$f(0) = 0$$

$$f\left(\frac{1}{n+1}\right) = \frac{n^n}{(n+1)^{n+1}}$$

$$f(1) = 0$$

We have  $a_n = \frac{n^n}{(n+1)^{n+1}}.$

Therefore,  $\lim_{n \rightarrow \infty} (n+1)a_n = \lim_{n \rightarrow \infty} \frac{n^n}{(n+1)^n} = \lim_{n \rightarrow \infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \boxed{\frac{1}{e}}.$

$$\begin{aligned}
3): & \int_0^1 x(1-x)^n dx \\
&= -\frac{1}{n+1} \int_0^1 x d((1-x)^{n+1}) \\
&= -\frac{1}{n+1} x(1-x)^{n+1} \Big|_0^1 + \frac{1}{n+1} \int_0^1 (1-x)^{n+1} dx \\
&= -\frac{1}{(n+1)(n+2)} [(1-x)^{n+2}]_0^1 \\
&= \boxed{\frac{1}{(n+1)(n+2)}}
\end{aligned}$$