(MEXT has never included any "application of calculus" question explicitly in Q1 (for Math B). One may consider skipping them if one has limited time. Another plausible approach is to treat them as Q2 or Q3 questions.)

Q1 Fill in the blanks with the correct numbers.

(1) The area of the region bounded by the functions $y = x^2$ and $y = x$ is $\boxed{}$.

(2) The equation $x^3 - 2ax^2 + a^2x - 1 = 0$ has only one real root if and only if $a < \boxed{}$.

(3) The function $f(x) = |x^3 - 12x|$ has the maximum value $\boxed{}$ when $x = \boxed{}$ or $\boxed{}$.

(4) If the function $y = x^2$ $(0 \leq x \leq 1)$ is rotated along the y-axis, the volume of the solid obtained is $\boxed{}$ and the curved surface area of it is $\boxed{}$.

(5) If the graphs of $y = f(x)$ and $y = g(x)$ touch each other when $x = a$, then the reminder when $f(x) - g(x)$ is divided by $(x - a)^2$ is $\boxed{}$.

Q2 Let P, Q be two points on the coordinate plane such that $PQ = 1$. Let $A(a, b)$ be a point lies on PQ. Denote the origin as O. Fill in the blanks with the answers to the following questions.

(1) We confine the case to the first quadrant, let $\theta = \angle PQO$, express b in terms of a, θ.

(2) When θ varies, express the maximum value of b in terms of a.

(3) When P and Q varies, calculate the area of the locus of A.

(1) $b=$ $\boxed{}$

(2) maximum $b =$ $\boxed{}$

(3) $\boxed{}$

Q3 Consider the solid body B formed by rotating the sphere

$$(x-R)^2 + y^2 + z^2 = r^2 \quad (0 < r < R)$$

in the xyz-space along the z-axis. Fill in the blanks with the answers to the following questions.

(1) When the sphere is rotated by a very small angle θ, express the distance travelled by the center of the sphere in terms of r, R, θ.

(2) Express the volume of B in terms of r, R.

(3) In stead of a sphere, the solid body C is formed by rotating the shape $(\frac{x}{a}-R)^2+(\frac{y}{b})^2+(\frac{z}{c})^2 = r^2$. Express the volume of C in terms of r, R, a, b, c

(1) $\boxed{}$

(2) $\boxed{}$

(3) $\boxed{}$

Brief Solutions and Comments

Q1(1) Ref: Not specific

Question related to definite integral.

The two graphs intersect at $x = 0, 1$. Therefore, the area=$\int_0^1 (x - x^2)dx = \boxed{\dfrac{1}{6}}$.

A direct calculation of area using definite integral. Usually appears in the last subquestion of Q2. However, I moved it here as an independent question.

Q1(2) Ref: 2006 Math B Q1(2)

Question related to derivative test.

We consider the first derivative $3x^2 - 4ax + a^2 = (3x - a)(x - a)$. The extremun appears at $x = \frac{a}{3}, a$.

Then, the second derivative $6x - 4a$ suggests that the local maximum appears at $x = \frac{a}{3}$ if $a > 0$ and $x = a$ if $a < 0$.

If $a < 0$, $(a)^3 - 2a(a)^2 + a^2(a) - 1 = -1 < 0$ and therefore the equation has only one root.

If $a = 0$, obviously the equation has only one root.

If $a > 0$, the equation has only one root if and only if

$(\frac{a}{3})^3 - 2a(\frac{a}{3})^2 + a^2(\frac{a}{3}) - 1 < 0$, i.e. $\frac{4a^3}{27} - 1 < 0$, i.e. $0 < a < \frac{3\sqrt[3]{2}}{2}$.

Given the above, the equation has only one root if and only if $a < \boxed{\dfrac{3\sqrt[3]{2}}{2}}$.

A modified version of the model question that requires the use of differentiation. A much more complicated version has appear in 2015 Math B Q3.When such kind of question appears, sketch the graph of the given function using derivative test first if one is bad at imagination.

Q1(3) Ref: 2020 Math A Q1(8)

Question related to extremum.

Note that the function is symmetric along the y-axis. We can confine ourselves on $x > 0$.

Consider the first derivative of $x^3 - 12x$, $3x^2 - 12$, we have the extremum value when $x = 4$. By symmetry and the absolute value, it must be the maximum value. Hence the function has the maximum value $\boxed{16}$ when $x = \boxed{\pm 4}$.

A very standard question related to finding extremum using derivative. This type of questions are not difficult as there is a general methodology. However, always try to simplify the question using the hints given in the question (e.g. "$x =$ something or something" here) so as to save time.

Q1(4) Ref: 2020 Math B Q3

Question related to definte integral.

For $0 \leq x \leq 1$, $0 \leq y \leq 1$. The volume of the solid$=\int_0^1 \pi x^2 dy = \pi \int_0^1 y dy = \boxed{\frac{\pi}{2}}$.

The curved surface area$=\int_0^1 2\pi x\sqrt{1^2 + (\frac{dy}{dx})^2} dx = 2\pi \int_0^1 x\sqrt{1+4x^2} dx = \boxed{\frac{(5\sqrt{5}-1)\pi}{6}}$.

Another common application of definite integral is to find the volume of a shape obtained by rotation. Although finding volume using the integral of cross section area is more common in MEXT, using rotation did appear once in the model question.

Q1(5) Ref: Not specific

Question related to tangent.

The equation of tagent is $l(x) = f'(a)(x-a) + f(a) = g'(a)(x-a) + g(a)$.

Consider the degree of reminder, we may write

$$f(x) = q_1(x)(x-a)^2 + q_2(x-a) + r.$$

However, by the reminder theorem, $r = f(a)$. Hence

$$\frac{f(x) - f(a)}{x-a} = q_1(x)(x-a) + q_2.$$

Now, take the limit of both sides $x \to a$, we have $f'(a) = q_2$. Therefore,

$$f(x) = q(x)(x-a)^2 + f'(a)(x-a) + f(a) = q(x)(x-a)^2 + l(x)$$

and similarly

$$g(x) = Q(x)(x-a)^2 + l(x).$$

Therefore, $f(x) - g(x)$ is divisible by $(x-a)^2$ and the reminder is $\boxed{0}$.

Unlike Math A, MEXT has never asked the equation of tangent explicitly in Math B, even in Q2 or Q3. Therefore, I complicated the question a bit here so that the concept of equation of tangent is used implicitly. This question is motivated by 2010 Math B Q6. Althought the model question is very unusua, it is amazing to see that so many terms of a polynomial can be replace using the first and section derivative.

Q2 Ref: 2014 Math B Q2

Question related to modelling.

(1) As $PQ = 1$ $p = \sin\theta$ and $q = \cos\theta$, therefore the equation of PQ is $y = \sin\theta - x\tan\theta$. Hence, $b = \boxed{\sin\theta - a\tan\theta}$.

(2) $\frac{db}{d\theta} = \cos\theta - \frac{a}{\cos^2\theta}$. Solving $\frac{db}{d\theta} = 0$, we have $a = \cos^3\theta$.

As $\cos\theta$ is decreasing, θ decreases as a increases.

When $a < \cos^3\theta$, $\frac{db}{d\theta} > 0$.

When $a > \cos^3\theta$, $\frac{db}{d\theta} < 0$.

Hence, by the first derivative test on $\underline{\theta}$, b attains its maximum when $a = \cos^3\theta$.

By that time, $b = (1 - a^{\frac{2}{3}})^1 2 - a \cdot \frac{(1-a^{\frac{2}{3}})^{\frac{1}{2}}}{a^{\frac{1}{3}}} = \boxed{(1 - a^{\frac{2}{3}})^{\frac{3}{2}}}$.

(3) By (2), the locus is described as $y \leq (1 - x^{\frac{2}{3}})^{\frac{3}{2}}$ in the first quadrant, where the area is that bounded by $y = (1 - x^{\frac{2}{3}})^{\frac{3}{2}}$ and the x, y axes.

The area in the first quadrant is

$$\begin{aligned}
\int_0^1 (1 - x^{\frac{2}{3}})^{\frac{3}{2}} dx = 3\int_0^1 u^4\sqrt{1-u^2}du \\
&= 3\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^2\theta d\theta \\
&= \frac{3}{16}\int_0^{\frac{\pi}{2}} (\frac{1+\cos 6\theta}{2} - \cos 4\theta - \cos 2\theta + \frac{1+\cos 2\theta}{2})d\theta \\
&= \frac{3\pi}{32}
\end{aligned}$$

By symmetry, the total area in four quadrants is $\boxed{\frac{3\pi}{8}}$.

If a calculus question appears in Q2, it is likely to be modelling. The most difficult part of such kind of questions will be the first subquestion as tools in other topics are often involved and one has to understand the situation. After one has modelled the situation successfully, simply follow the instructions in the remaining parts and the whole question can be solved easily.

Q3 Ref: 2017 Math B Q3, 2016 Math B Q2, 2018 Math A Q3

Question related to functional equation.

(1) Note that the path of the centre is an arc, where its length is $\boxed{R\theta}$ by the definition of θ.

(2) The cross section area of B is πr^2.

Therefore, the volume is $\int_0^{2\pi} \pi r^2 d(R\theta) = \boxed{2\pi^2 r^2 R}$

(3) The new shape is obtained by enlarge the sphere along the x,y,z axes by a, b, c times respectively. Therefore, the volume of it is $\boxed{2\pi^2 r^2 Rabc}$.

This kind of question that requires candidates to find a volume of a solid by integrating the cross section area is very common in Math B. This question concentrated a lot of concepts that has appeared in past papers (see those references). One should definitely follow up this topic if one encountered difficulty solving this question.