Preface

Since 2024, Japan government introduced a new syllabus "Mathematics C" that collected a list of advanced topics that are usually studied in college in other countries. Some of the topics are originally included and some are newly added. All topics collected in this document have never appeared in past papers (up to 2020) but they can often be some useful tools for one to solve some tricky questions. Judging from the fact that MEXT has started including a wider syllabus which covers more unusual topics in Math A, in addition to the change in syllabus in Japan, there is a chance for these topics to appear as independent questions after 2024. As they are difficult to be set as Q1 questions, I will keep only 5 subquestions in Q1, like most of the other documents. However, in Q2 and Q3, I will present one application (in Q2) and one theoretical question (in Q3). One can choose whether to attempt or to skip base on one's own judgement. Here I list out a brief index of topics that have been classified into this category. One can go to Japan MEXT's official site to see a full guidance.

Vector: Vector product added. Vector calculus excluded.

Curves: Parametric curves, Polar curves, Conic sections. Calculus included.

Complex Number: Euler's formula excluded (but I recommend to have a look). Argand plane included. Calculus excluded.

Matrix: Determinant - Applications exluded. Characteristic polynomial included (but it is werid to teach characteristic polynomial and vector product without teaching determinant).

Graph Theory: Eulerian path and adjacency matrix. Directed graph excluded. Applications excluded.

Statistics*: Applications in proportional representation. Interpret of distribution curve. regression.

* I personally don't think they will ever be included in an exam. Not only the MEXT's one but even Japan's universities' entrance exams, as long as calculator is not allowed.

Q1 Fill in the blanks with the correct numbers.

(1) The unit normal vector to the plane $x + 2y + 3z + 4 = 0$ are ($\boxed{}$, $\boxed{}$, $\boxed{}$) and ($\boxed{}$, $\boxed{}$, $\boxed{}$).

(2) The complex number $\frac{1+\sqrt{-3}}{\sqrt{2}+\sqrt{-2}}$ has an argument $\boxed{}$.

(3) If $A = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 0 & 4 \\ 5 & 6 & 0 \end{pmatrix}$, E be the 3×3 identity matrix, then

$A^3 - 37A - 56E = \boxed{}$.

(4) The foci of hyperbola are points F, F' lie on the axis of the hyperbola such that for every points P on the hyperbola, $FP - F'P =$ (Constant). The foci of the hyperbola $9x^2 - 16y^2 = 144$ are ($\boxed{}$, $\boxed{}$) and ($\boxed{}$, $\boxed{}$).

(5) If z is a complex number satisfying $|z - (3 + 4i)| = 16$, then the maximum value of $|z|$ is $\boxed{}$ and the minimum value of $|z|$ is $\boxed{}$.

Vector

Q2 Consider a regular tetrahedral with side length 1. Let $0 < \lambda < 1$ be a real number. Denote a point P such that $AP : PB = \lambda : (1-\lambda)$ and a point Q such that $BQ : QC = \lambda : (1-\lambda)$. Moreover, denote $\vec{a} = \vec{OA}$, $\vec{b} = \vec{OB}$, and $\vec{c} = \vec{OC}$. Fill in the blanks with the answers to the following questions.

(1) Express $\vec{OP}$ and $\vec{OQ}$ in terms of $\vec{a}, \vec{b}, \vec{c}, \lambda$.

(2) Express the range of $\cos \angle POQ$ in terms of λ.

(1) $\vec{OP} = \boxed{}$ $\quad \vec{OQ} = \boxed{}$.

(2) $\boxed{}$

Q3 There are two orthogonal unit vectors $\vec{a}, \vec{b}$ in $\mathbb{R}^3$. Let $\vec{c}$ be a vector such that $\vec{c} \times \vec{b} = \vec{a} - \vec{c}$. Fill in the blanks with the answers to the following questions.

(1) Express $(\vec{c} \times \vec{b}) \times \vec{b}$ in terms of $\vec{c}$.

(2) Find $|\vec{c}|$.

(3) Express $\vec{c}$ in terms of $\vec{a}, \vec{a} \times \vec{b}$.

(1) $(\vec{c} \times \vec{b}) \times \vec{b} = \boxed{}$

(2) $|\vec{c}| = \boxed{}$

(3) $\vec{c} = \boxed{}$

Curves

Q2 Consider the circle $x^2 + y^2 = r^2$. A bendable string of length πr is fixed on $A(0, r)$. Fill in the blanks with the answers to the following questions.

(1) Consider only the part $x \geq 0$. It is clear that the string sketch a quarter circle until it becomes tangent to the circle. Let O be the origin. Let B be the point of tangency and let $\theta = \angle AOB$. Let $C(a, b)$ be the other end of the string. Express a and b in terms of r, θ.

(2) Now, when the other end of the string is free to move along the whole coordinate plane as far as possible, express the area of the region sketched by the string in terms of r.

(3) In case the circle is replaced by the sphere $x^2 + y^2 + z^2 = r^2$, under the same setting, express the volume of the region sketched by the string in terms of r.

(1) $a = \boxed{}$ $b = \boxed{}$

(2) $\boxed{}$

(3) $\boxed{}$

Q3 For the ellipse $(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$, the foci are points $F(c, 0)$ and $F'(-c, 0)$ $(c > 0)$ such that $XF + XF'$ is constant for any points X on the ellipse. Fill in the blanks with the answers to the following questions

(1) Express c in terms of a, b.

(2) Consider only the $x \geq 0$ part of the ellipse, define the eccentricity $e = \frac{c}{a}$. There is a line parallel to the y-axis called the directrix such that for any points X on the ellipse, the shortest distance from X to the directrix is $\frac{1}{e}FX$. Find the equation of the directrix in terms of a, b.

(3) Write the $x \geq 0$ part of the ellipse in polar form with origin F, express r in terms of a, e, θ.

(1) $f =$ $\boxed{}$

(2) $\boxed{}$

(3) $r =$ $\boxed{}$

Complex Number

Q2 For a complex number a satisfying $|a| \leq 2$ and a complex number b satisfying $|b-(8+6i)| = 3$. Let $c = \frac{a+b}{2}$. Fill in the blanks with the answers to the following questions.

(1) When b is fixed as $u + vi$, where u, v are real numbers, write the equation of locus of c using xy-coordinates in terms of u, v.

(2) When b varies, find the area of the locus of c.

(1) $\boxed{}$

(2) $\boxed{}$

Q3 Let $P(z)$ be a point distinct from $O(0)$. Let $Q(w)$ be a point on OP such that $OP \cdot OQ = 1$. Fill in the blanks with the answers to the following questions.

(1) Express z in terms of w.

(2) If P varies on $|z - \alpha| = r$, where $|\alpha| \neq r$, express the area bounded by the locus of Q in terms of $r, |a|$.

(1) $z = \boxed{\qquad\qquad}$

(2) $\boxed{\qquad\qquad}$

Matrix

Q2 Let $A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ and E be the 2×2 identity matrix. Fill in the blanks with the answers to the following questions.

(1) Express A^{-1} in terms of A.

(2) If k is a natural number, express A^{2k-1} and A^{2k} in terms of k, A, E.

(3) The point $P_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ has underwent the linear transformation A for $2k$ times to obtain points $P_0, P_1, P_2, ..., P_{2k}$. Express the area of the polygon $P_0P_2...P_{2k}P_{2k-1}P_{2k-3}...P_1$ in terms of k.

(1) $A^{-1} = \boxed{}$

(2) $A^{2k-1} = \boxed{}$ $A^{2k} = \boxed{}$

(3) $\boxed{}$

Q3 Consider the recurrence $a_{n+2} = r_1 a_{n+1} + r_2 a_n$, where $a_1 = a_2 = 1$ and r_1, r_2 are positive real constants. Fill in the blanks with the answers to the following questions.

(1) In case $r_1 = r_2 = 1$, the recurrence defines the Fibonacci number F_n. Find a matrix A such that $\begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix} = A \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$ for all natural number n.

(2) By assuming that there exists a number λ such that $A\mathbf{x} = \lambda\mathbf{x}$ for a non-zero vector $\mathbf{x}$, express F_n in terms of n.

(3) Suppose the recurrence has a general term $a_n = c_1 x_1^n + c_2 x_2^n$, express c_1 and c_2 in terms of x_1, x_2.

(4) Find an equation with coefficients expressed in terms of r_1, r_2 such that x_1 and x_2 are solutions of the equation.

(1) $A = \boxed{}$

(2) $F_n = \boxed{}$

(3) $c_1 = \boxed{}$ $c_2 = \boxed{}$

(4) $\boxed{}$

Graph Theory

Q2 Consider a graph with the set of vertices $V = \{v_1, v_2, v_3, v_4\}$ which has the adjacency matrix

$$A = \begin{pmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix}.$$

Fill in the blanks with the answers to the following questions.

(1) How many paths with length 2 are there in total?

(2) How many trianglar subgraphs are there in total?

(1) $\boxed{}$

(2) $\boxed{}$

Q3 Consider colouring the vertices of a graph G with k distinct colours such that no adjacent vertices have the same colour. Denote the minimum k such that the colouring is possible for G as $\mathcal{X}(G)$. Fill in the blanks with the answers to the following questions.

(1) Denote a complete graph, that is, a graph with any pairs of two distinct vertices are connected, with n vertices, as K_n. Express $\mathcal{X}(K_n)$ in terms of n.

(2) Let G be a simple graph, that is a graph that is undirected and does not have any loops or multiple edges, with at most n edges incident to a vertex. Express the maximum $\mathcal{X}(G)$ in terms of n.

(1) $\mathcal{X}(K_n) = \boxed{}$

(2) $\boxed{}$

Brief Solutions

Q1(1) The unit normal vector of the given plane is equal to that of the plane

$x + 2y + 3z = 0$.

However, the vector equal is $(1, 2, 3) \cdot (x, y, z) = 0$. Therefore, the plane is is

perpendicular to the vector $(1, 2, 3)$.

Now, the unit vector of it is $\boxed{(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}})}$. However, then, $\boxed{(-\frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}})}$

is also a unit normal vector of the plane.

Q1(2) Rewrite the complex number in polar form $\frac{\frac{1}{2}e^{i\frac{\pi}{3}}}{\frac{1}{2}e^{i\frac{\pi}{4}}} = e^{i\frac{\pi}{12}}$, the argument is $\boxed{\dfrac{\pi}{12}}$.

Without Euler's formula: Rewrite the complex number in polar form

$(\frac{1}{2}(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}))(\frac{1}{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}))^{-1} = (\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4})$ by De Moivre's theorem. Moreover, it is equal to

$(\cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}) + i(\sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}) = \cos\frac{\pi}{12} + i\sin\frac{\pi}{12}$, which has an argument of $\boxed{\dfrac{\pi}{12}}$.

Q1(3) The characteristic polynomial of A is $-\lambda^3 + 37\lambda + 56$. By Cayley-Hamilton theorem, $A^3 - 37A - 56E = \mathbf{0}$.

Q1(4) Note that the foci have to be symmetric along the y-axis to follow the definition. Let $F = (c, 0)$ and $F' = (-c, 0)$, for every point $P(x, y)$, we have $FP - F'P = \sqrt{(c-x)^2 + y^2} - \sqrt{(c+x)^2 + y^2}$.

Specifically, consider the distance between the two x-intercepts, $4 + 4 = 8$, we have

$$\sqrt{(c-x)^2 + y^2} - \sqrt{(c+x)^2 + y^2} = 8$$

$$(c-x)^2 + y^2 = 64 + (c+x)^2 + y^2 + 16\sqrt{(c+x)^2 + y^2}$$

$$(c^2 - 16)x^2 - 16y^2 = 16(c^2 - 16)$$

Comparing the coefficients, $c = \pm 5$. Therefore, the foci are $\boxed{(\pm 5, 0)}$.

Q1(5) Note that the locus of z is a circle centred at $(3, 4)$ with a radius 4.

The distance between $(3, 4)$ and the origin is 5.

Therefore, the minimum $|z| = 5 - 4 = \boxed{1}$ and the maximum $|z| = 5 + 4 = \boxed{9}$.

Vector

Q2 (1) $\vec{OP} = \boxed{\lambda\vec{b} + (1-\lambda)\vec{a}}$, $\vec{OQ} = \boxed{\lambda\vec{c} + (1-\lambda)\vec{b}}$.

(2) Let $\cos\angle POQ = \frac{\vec{OP}\cdot\vec{OQ}}{|\vec{OP}||\vec{OQ}|}$.

As every side of a regular tetrahedral is an equilateral triangle, i.e. the angle between any two edged is $60°$, the dot product between any two vectors $\vec{a}, \vec{b}, \vec{c}$ is $1 \cdot 1 \cdot \cos 60° = \frac{1}{2}$.

We have $|\vec{OP}|^2 = \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda)\vec{a}\cdot\vec{b} = \lambda^2 - \lambda + 1$.

Similarly, $|\vec{OQ}|^2 = \lambda^2 - \lambda + 1$.

Therefore, $|\vec{OP}||\vec{OQ}| = \lambda^2 - \lambda + 1$.

On the other hand,

$\vec{OP}\cdot\vec{OQ} = \frac{1}{2}(\lambda^2 + (1-\lambda)^2 + \lambda(1-\lambda) + 2\lambda(1-\lambda)) = -\frac{1}{2}\lambda^2 + \frac{1}{2}\lambda + \frac{1}{2}$.

We have $\cos\angle POQ = \frac{-\lambda^2+\lambda+1}{2(\lambda^2-\lambda+1)} = \frac{1}{\lambda^2-\lambda+1} - \frac{1}{2} = \frac{1}{(\lambda-\frac{1}{2})^2+\frac{3}{4}} - \frac{1}{2}$.

For $0 < \lambda < 1$, we have $\boxed{\dfrac{1}{2} < \cos\angle POQ \leq \dfrac{5}{6}}$.

Q3 (1) As $(\vec{c} \times \vec{b}) \perp \vec{b}$, $(\vec{c} \times \vec{b}) \cdot \vec{b} = 0$. Similarly $\vec{a} \cdot \vec{b} = 0$.

We have $(\vec{c} \times \vec{b}) \cdot \vec{b} = (\vec{a} - \vec{c}) \cdot \vec{b}$, i.e. $\vec{c} \cdot \vec{b} = 0$, i.e. $\vec{b} \perp \vec{c}$.

Now, by the definition of vector product, $(\vec{c} \times \vec{b}) \times \vec{b} = \boxed{-\vec{c}}$.

(2) We have $|\vec{c} \times \vec{b}|^2 = |\vec{a} - \vec{c}|^2$.

However, as $\vec{b} \perp \vec{c}$, $|\vec{c} \times \vec{b}| = |\vec{c}|$.

On the other hand, $|\vec{a} - \vec{c}|^2 = 1 - 2\vec{a} \cdot \vec{c} + |\vec{c}|^2$.

Therefore, we have $\vec{a} \cdot \vec{c} = \frac{1}{2}$.

Now, we have $(\vec{c} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} - |\vec{c}|^2$, which gives us $|\vec{c}| = \boxed{\dfrac{\sqrt{2}}{2}}$.

(3) As $(\vec{c} \times \vec{b}) \times \vec{b} = -\vec{c}$, we have $(\vec{a} - \vec{c}) \times \vec{b} = -\vec{c}$, which gives $\vec{a} \times \vec{b} - \vec{c} \times \vec{b} = -\vec{c}$.

Again, we have $(\vec{a} \times \vec{b}) - (\vec{a} - \vec{c}) = -\vec{c}$, which gives $\vec{c} = \boxed{\dfrac{1}{2}\vec{a} - \dfrac{1}{2}\vec{a} \times \vec{b}}$.

Curves

Q2 (1) Note that $B = (r\sin\theta, r\cos\theta)$ and $\widehat{\mathrm{AB}} = r\theta$.

Then, by some basic geometry, we have $\vec{BC} = (r(\pi-\theta)\cos\theta, -r(\pi-\theta)\sin\theta)$.

Therefore, $C = (r(\sin\theta + (\pi-\theta)\cos\theta), r(\cos\theta - (\pi-\theta)\sin\theta))$.

i.e. $a = \boxed{r(\sin\theta + (\pi-\theta)\cos\theta)}$ and $b = \boxed{r(\cos\theta - (\pi-\theta)\sin\theta)}$.

(2) The area of the quarter circle is $\frac{1}{4}\pi(\pi r)^2 = \frac{1}{2}\pi^3 r^2$.

For the parametric curve $\begin{cases} x = r(\sin\theta + (\pi-\theta)\cos\theta) \\ y = r(\cos\theta - (\pi-\theta)\sin\theta) \end{cases}$, where $0 \le \theta \le \pi$, the area is given by $-\int_0^\pi (r(\cos\theta - (\pi-\theta)\sin\theta))d(r(\sin\theta + (\pi-\theta)\cos\theta))$, which is equal to $\frac{1}{6}\pi^3 r^2$. Therefore, the total area for $x \ge 0$ is $\frac{2}{3}\pi^3 r^2$.

By symmetry, the total area is $\boxed{\dfrac{4}{3}\pi^3 r^2}$.

(3) Consider rotating along the z-axis, the volume$=\int_0^\pi \frac{4}{3}\pi^3 r^2 d(r\theta) = \boxed{\dfrac{4}{3}\pi^4 r^3}$.

Q3 (1) With a calculation similar to Q1(4), we have $c = \boxed{\sqrt{a^2 - b^2}}$.

(2) We consider the case F, X, P are collinear. Then, $X = (a, 0)$ and the point on the directrix $P = (p, 0)$.

We have $FX = a - c$ and $\frac{1}{e}FX = \frac{a}{c}(a - c)$.

Then, $p = a + \frac{a}{c}(a - c) = \frac{a^2}{c} = \frac{a^2}{\sqrt{a^2-b^2}}$ and hence the equation is $\boxed{x = \dfrac{a^2}{\sqrt{a^2 - b^2}}}$.

(3) As $FX = e(\frac{a}{e} - x)$, where x is the x-coordinate of X, we have

$$r = e(\frac{a}{e} - (r\cos\theta + ae))$$

$$r = \boxed{\frac{a(1 - e^2)}{1 + e\cos\theta}}$$

Complex Number

Q2 (1) As $c - \frac{1}{2}b = \frac{1}{2}a$, $|c - \frac{1}{2}b| \leq 1$.

Therefore, c is a filled circle centred at $\frac{1}{2}b$ with radius 1, where the equation is $\boxed{(x - \dfrac{u}{2})^2 + (y - \dfrac{v}{2})^2 \leq 1}$.

(2) When b varies, $\frac{b}{2}$ varies as $|\frac{b}{2} - (4 + 3i)| = \frac{3}{2}$, which is a circle centred at $(3, 4)$ with radius $\frac{3}{2}$.

Then, the area of the locus$=\pi((\frac{3}{2} + 1)^2 - (\frac{3}{2} - 1)^2) = \boxed{6\pi}$.

Q3 (1) As Q lies on OP, we may write $z = \lambda w$.

As $OP \cdot OQ = 1$, we have $\lambda |w|^2 = 1$, i.e. $\lambda = \frac{1}{|w|^2}$.

Now, $z = \frac{w}{w\bar{w}} = \boxed{\dfrac{1}{\bar{w}}}$.

(2) We have $|\frac{1}{\bar{w}} - \alpha| = r$, which can be rewritten as

$$(|\alpha|^2 - r^2)w\bar{w} - \bar{\alpha}w - \alpha\bar{w} + 1 = 0.$$

As $|\alpha| \neq r$, we have $w\bar{w} - \frac{\bar{\alpha}}{|\alpha|^2 - r^2}w - \frac{\alpha}{|\alpha|^2 - r^2}\bar{w} + \frac{\alpha\bar{\alpha}}{(|\alpha|^2 - r^2)^2} = (\frac{r}{|\alpha|^2 - r^2})^2$.

Simplifying, $|w - \frac{\alpha}{|\alpha|^2 - r^2}| = \frac{r}{|\alpha|^2 - r^2}$.

Therefore, Q forms a circle with area $\boxed{\dfrac{\pi r^2}{(|\alpha|^2 - r^2)^2}}$.

Matrix

Q2 (1) By Cayley-Hamilton theorem, $\frac{1}{2}A^2 = E$, i.e. $A^{-1} = \boxed{\dfrac{1}{2}A}$.

(2) As $A^2 = 2E$, $A^{2k} = \boxed{2^k E}$ and $A^{2k-1} = 2^k A^{-1} = \boxed{2^{k-1}A}$.

(3) By (2), we have $P_{2n} = \begin{pmatrix} 0 \\ \\ 2^n \end{pmatrix}$ and $P_{2n-1} = \begin{pmatrix} 2^{n-1} \\ \\ 0 \end{pmatrix}$.
Therefore, the shape is simply a quadrilateral $P_0 P_{2k} P_{2k-1} P_1$.

The area of the shape$=\frac{1}{2}(2^{k-1})(2^k) - \frac{1}{2}(1)(1) = \boxed{\dfrac{2^{2k-1}-1}{2}}$.

Q3 (1) Obviously $A = \boxed{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}$.

(2) λ satisfies the characteristic equation $\lambda^2 - \lambda - 1 = 0$, i.e. $\lambda = \frac{1 \pm \sqrt{5}}{2}$.

Solving the vectors $\mathbf{x}$, $\mathbf{x} = \begin{pmatrix} \frac{1 \pm \sqrt{5}}{2} \\ 1 \end{pmatrix}$.

Therefore, $A = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}^{-1}$.

$A^n = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (\frac{1+\sqrt{5}}{2})^n & 0 \\ 0 & (\frac{1-\sqrt{5}}{2})^n \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}^{-1}$.

Let me skip the clumsy calculation.

On the other hand, let $B_{n+1} = \begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix} = AB_n = A^2 B_{n-1} = ... = A^n B_1$.

Expanding all the matrix multiplication, we have $F_n = \boxed{\frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n)}$.

(3) Consider $a_1 = c_1 x_1 + c_2 x_2 = 1$ and $a_2 = c_1 x_1^2 + c_2 x_2^2 = 1$, we have $c_1 = \boxed{\frac{1 - x_2}{x_1(x_1 - x_2)}}$ and $c_2 = \boxed{\frac{1 - x_1}{x_2(x_2 - x_1)}}$

(4) As $\begin{pmatrix} a_{n+2} \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} r_1 & r_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix}$, with a similar argument in (2), we have the characteristic equation $\boxed{x^2 - r_1 x - r_2 = 0}$.

Graph Theory

Q2 (1) Consider this: The sum of the number of $v_1 - v_k$ edges times the number of $v_k - v_2$ edges is equal to the number of $v_1 - v_k - v_2$ paths, which has a length 2, which can be found by the matrix multiplication

$$\begin{pmatrix} 2 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix}.$$

To simplify the repeatition of this process, we simply compute

$$A^2 = \begin{pmatrix} 5 & 1 & 4 & 2 \\ 1 & 6 & 2 & 4 \\ 4 & 2 & 5 & 1 \\ 2 & 4 & 1 & 6 \end{pmatrix},$$

where a_{ij} is exactly the number of $v_j - v_i$ paths with length 2 as deduced.

The total number of paths with length 2 is the sum of all elements, which is $\boxed{50}$.

(2) All subgraphs that is a cycle with length 3 is triangular. Therefore, we want to find the number of $v_k - v_k$ paths with length 3.

Similar to (1), we compute

$$A^3 = \begin{pmatrix} 4 & 16 & 5 & 14 \\ 16 & 8 & 14 & 11 \\ 5 & 14 & 4 & 16 \\ 14 & 11 & 16 & 8 \end{pmatrix},$$

the number of desired paths is $4 + 8 + 4 + 8 = 24$.

However, consider the path $v_1 - v_2 - v_3 - v_1$, where the corresponding subgraph has vertices $V = \{v_1, v_2, v_3\}$ and edges $E = \{v_1 - v_2, v_2 - v_3, v_3 - v_1\}$. Then, the paths

$v_1 - v_2 - v_3 - v_1$,

$v_1 - v_3 - v_2 - v_1$,

$v_2 - v_1 - v_3 - v_2$,

$v_2 - v_3 - v_1 - v_3$,

$v_3 - v_1 - v_2 - v_3$,

$v_3 - v_2 - v_1 - v_3$

are actually correspond to the same subgraph.

Therefore, the total number of triangular subgraphs is $\frac{24}{6} = \boxed{4}$.

Q3(1) Obviously $\mathcal{X}(K_1) = 1$, $\mathcal{X}(K_2) = 2$. We assume $\mathcal{X}(K_n) = n$.

In fact, if $\mathcal{X}(K_n) = n$, when a new vertex is added and connected to every other vertices, a new colour is required to colour the new vertex, which proved $\mathcal{X}(K_{n+1}) = n + 1$.

Therefore, by induction, we have $\mathcal{X}(K_n) = \boxed{n}$.

(2) When $n = 1$, each vertex is connected to one other vertex only. Therefore $\mathcal{X}(G) = 2$.

When $n = 2$, it is clear that the extreme case is attended when the graph is triangular. In that case, $\mathcal{X}(G) = 3$.

We guess $\mathcal{X}(G) = n + 1$.

In fact, when $n = m + 1$, we can delete the vertex with $m + 1$ edges, and the degree of the graph becomes at most m. Then, if $\mathcal{X}(G) = m + 1$ when $n = m$, it follows obviously that $\mathcal{X}(G) = m + 2$ when $n = m + 1$ as we need to use a different colour to colour the deleted vertex.

Therefore, by induction, we have $\mathcal{X}(K_n) = \boxed{n + 1}$.