

Q1(1):

The prime factorisation of 2019 is $2019 = 3 \times 673^*$.

Therefore, the divisors of it are 1, 3, 673 and 2019.

There are totally $\boxed{4}$ positive divisors and the sum is $1 + 3 + 673 + 2019 = \boxed{2696}$.

(*: The tricky part of this question is that we don't know whether or not 673 is a prime. As $673 < 676 = 26^2$, if it is not a prime, it will be a multiple of a prime number smaller than 25. It is not difficult for us to exhaust all the cases: it is obvious that it is not a multiple of 2, 3, 5, 7, or 11 (see how to check divisibility if you are not sure). Besides, we check the divisibility of it by 13, 17, 19 and 23 by hand. We can see that 673 does not contain a prime factor. Hence, it itself is a prime.)

Q1(2):

Note that $\triangle OAB$ is a right-angled triangle with $\angle BAO = 90^\circ$. Consider the tangent ratio of it, we have $\tan \angle AOB = \frac{BA}{AO} = \frac{4}{3}$.

The angle between the angle bisector and the positive x-axis is equal to $\frac{1}{2}\angle AOB$ and hence the slope of it is given by $\tan \frac{1}{2}\angle AOB$, denote it as t .

Note that $\tan \angle AOB = \frac{2 \tan \frac{1}{2}\angle AOB}{1 - \tan^2 \frac{1}{2}\angle AOB} = \frac{2t}{1 - t^2}$.

Solving the equation:

$$\frac{2t}{1 - t^2} = \frac{4}{3}$$

$$2t^2 + 3t - 2 = 0$$

$$(2t - 1)(t + 2) = 0$$

$$t = \frac{1}{2} \text{ or } t = -2$$

Note that the angle bisector has a positive slope. Therefore, we have the equation of it, $y = \boxed{\frac{1}{2}}x$.

Alternative Let D be the intersection of the angle bisector and AB . Then, by the angle bisector theorem, we have $AD : DB = OA : OB = 3 : 5$. Therefore, the coordinates of D are $(3, \frac{3}{2})$.

The equation of the angle bisector is equal to the equation of OD , which is $y = \boxed{\frac{1}{2}}x$.

Q1(3):

The slope of the line through the two points $= \frac{9-1}{3+1} = 2$.

The slope of the tangent to the parabola at a point (t, t^2) is given by $y'|_{x=t} = 2t$.

Solving $2t = 2$ such that the tangent is parallel to the line, we have $t = 1$.

Therefore, the point of tangency is $(\boxed{1}, \boxed{1})$.

Then, using the point slope form of straight line, we have the equation $y - 1 =$

$2(x - 1)$, i.e. $y = \boxed{2}x + \boxed{-1}$.

Alternative As the slope of the tangent $= 2$, we may let the equation of tangent be $y = 2x + k$.

As it is tangent to the parabola, the equation $2x + k = x^2$ has only one solution.

Then, we have

$$\Delta = 4 + 4k = 0$$

$$k = -1$$

It gives the equation $y = \boxed{2}x + \boxed{-1}$.

Moreover, solving $2x - 1 = x^2$, we have $x = 1$. Hence, the point of tangency is $(\boxed{1}, \boxed{1})$.

Q1(4):

When $m = 0$, the line $y = m(x - 5) + 3$ touches the circle as $m = 0$ is the lower boundary such that the line intersects the circle. The equation becomes $y = 3$ by that time. As the centre of the circle is the origin, we have hence the radius, i.e. $r = \boxed{3}$.

To find another value of m such that the line tangents to the circle, i.e. the upper boundary of m , we consider the equation $x^2 + (m(x - 5) + 3)^2 = 9$.

When the line tangents to the circle, the equation has only one solution. Then, we have

$$\Delta = (2m(3 - 5m))^2 - 4(1 + m^2)((3 - 5m)^2 - 9) = 0$$

$$-8m^2 + 15x = 0$$

$$m = \frac{15}{8}$$

Therefore, the condition for the line to intersect the circle is $0 \leq m \leq \boxed{\frac{15}{8}}$.

Q1(5):

Note that $\sin x + \cos x = \sqrt{2}(\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x) = \sqrt{2} \sin(x + \frac{\pi}{4})^*$.

Therefore, for $|x| \leq \frac{\pi}{2}$, the maximum is $\boxed{\sqrt{2}}$ and the minimum is

$$\sqrt{2} \sin(-\frac{\pi}{2} + \frac{\pi}{4}) = \boxed{-1}.$$

(*: Search for how to express $a \sin \theta + b \cos \theta$ in a form of $R \sin(\alpha + \theta)$.)

Alternative Consider $(\sin x + \cos x)' = \cos x - \sin x$

To obtain an extremum, we set the derivative to be 0. For $|x| \leq \frac{\pi}{2}$, we have

$$x = \frac{\pi}{4}.$$

$$(\sin x + \cos x)'' = -\sin x - \cos x.$$

Conduct the second derivative test, when $x = \frac{\pi}{4}$, the second derivative is val-

ued $-\sqrt{2} < 0$. Therefore, a maximum of $\sin \frac{\pi}{4} + \cos \frac{\pi}{4} = \boxed{\sqrt{2}}$ is obtained.

Now, check the boundaries:

When $x = -\frac{\pi}{2}$, the expression is valued -1 .

When $x = \frac{\pi}{2}$, the expression is valued 1 .

Therefore, the minimum value is $\boxed{-1}$.

Q1(6):

6^{100} can be written as $a_0 \cdot 10^n + a_1 \cdot 10^{n-1} + \dots + a_n$, where $n, a_0, a_1, \dots, a_n \in \mathbb{N}$

and $0 < a_0, a_1, \dots, a_n \leq 9$.

Then, the number of digits will be $n + 1$.

As $\log_{10}(a_0 \cdot 10^n + a_1 \cdot 10^{n-1} + \dots + a_n) < \log_{10}(10^{n+1}) = n + 1$, we can find out

the value of $n + 1$ by considering the value of $\log_{10} 6^{100}$:

$$\begin{aligned}\log_{10} 6^{100} &= 100(\log_{10} 6) \\ &= 100(\log_{10}(2 \cdot 3)) \\ &= 100(\log_{10} 2 + \log_{10} 3) \\ &\approx 100(0.3010 + 0.4771) \\ &= 77.81\end{aligned}$$

Therefore, $n + 1 = 78$ and the number of digits is $\boxed{78}$.

On the other hand, we can now write 6^{100} as $a \cdot 10^{77}$, where $0 < a < 10$ is a rational number.

Then, $\log_{10} 6^{100} = \log_{10} a + 77$ and we have $\log_{10} a \approx 0.81$ by our above calculation.

As $\log_{10} 6.4 = \log_{10}(2^6 \cdot 10^{-1}) \approx 6 \cdot 0.3010 - 1 = 0.806^*$, we have $a \approx 6.4$ and we can judge that the integer part of a is 6. Note that the integer part of a is exactly the leading digit of 6^{100} . Therefore, the leading digit is $\boxed{6}$.

(*: Note that $\log_{10} 6 \approx 0.7781$, the question becomes “is the integer part of a 6 or 7”. It motivates us to construct an number between 6 and 7 using the multiplication of 2, 3, 5 and 10 (as we can only estimate the logarithm of those values) and see which logarithm is closer to 0.81. A easy-to-think and well-working number is thereby 6.4. In fact, $6.\dot{6}$, which is equal to $2 \cdot 3^{-1}$ is also a good choice.)

Q2:

(1): By (ii), we have $I(2, n) = I(1, n) + I(2, n - 1) = 1 + I(2, n - 1)$. Regard it as a recurrence of $I(2, n)$, with the initial value $I(2, 1) = 1$, we can solve $I(2, n) = \boxed{n}$.

Similarly, we have $I(3, n) = I(2, n) + I(3, n - 1) = n + I(3, n - 1)$. Solving the recurrence by telescoping:

$$\begin{aligned} \sum_{i=2}^n (I(3, i) - I(3, i - 1)) &= \sum_{i=2}^n i \\ I(3, n) - I(3, 1) &= \sum_{i=2}^n i \\ I(3, n) &= 1 + \sum_{i=2}^n i = \sum_{i=1}^n i = \boxed{\frac{n(n+1)}{2}} \end{aligned}$$

(2): Similar to the methodology in (1), we have $I(m, n) = 1 + \sum_{i=2}^n I(m - 1, i)$.

$$\begin{aligned} \text{Therefore, } I(5, 3) &= 1 + I(4, 2) + I(4, 3) \\ &= 1 + (1 + I(3, 2)) + (1 + I(3, 2) + I(3, 3)) \\ &= 3 + 2\left(\frac{2(3)}{2}\right) + \frac{3(4)}{2} \\ &= \boxed{15}. \end{aligned}$$

Alternative Use (ii) repeatedly, we have:

$$\begin{aligned} I(5, 3) &= I(5, 2) + I(4, 3) \\ &= I(5, 1) + I(4, 2) + I(3, 3) + I(4, 2) \\ &= 1 + 2(I(4, 1) + I(3, 2)) + 6 \\ &= 1 + 2(1 + 3) + 6 \\ &= \boxed{15}. \end{aligned}$$

Q3:

(1): $f'(x) = e^x$, $g'(x) = 1$ and $h'(x) = 1 + x$.

When $x < 0$, we have $e^x < 1$. Therefore, we have $f'(x) < g'(x)$.

On the other hand, as $f'(x) = e^x = 1 + x + \frac{x^2}{2} + \dots \geq 1 + x = h'(x)$, and the equality holds only if $x = 0$, we have $f'(x) > h'(x)$ for $x < 0$.

Given the above, we have $\boxed{h'(x)} < \boxed{f'(x)} < \boxed{g'(x)}$.

(2): Note that all the functions are convex downwards and valued 1 when $x = 0$.

Therefore, the larger the rate of change when $x < 0$, the smaller the value of the function when $x < 0$, which means the order of $f(x)$, $g(x)$ and $h(x)$ should be exactly the inverse of that in (1). Hence, we have $\boxed{g(x)} < \boxed{f(x)} < \boxed{h(x)}$.

Alternative $e^x = 1 + x + \frac{x^2}{2} + R(x)$, where $R(x)$, the remainder term, is negative but smaller in magnitude than the former term for $x < 0$. Therefore, we have $\boxed{g(x)} < \boxed{f(x)} < \boxed{h(x)}$.

$$(3): I_1 = \int_{-1}^0 |f(x) - g(x)| dx = \int_{-1}^0 (e^x - 1 - x) dx$$

$$= [e^x - x - \frac{1}{2}x^2]_{-1}^0$$

$$= 1 - e^{-1} - 1 + \frac{1}{2}$$

$$= \boxed{\frac{1}{2} - e^{-1}}.$$

$$I_2 = \int_{-1}^0 |f(x) - h(x)| dx = \int_{-1}^0 (h(x) - f(x)) dx \text{ (as } f(x) < h(x) \text{ for } x \in (-1, 0))$$

$$= \int_{-1}^0 (1 + x + \frac{x^2}{2} - e^x) dx$$

$$= [x + \frac{1}{2}x^2 + \frac{1}{6}x^3 - e^x]_{-1}^0$$

$$= -1 + 1 - \frac{1}{2} + \frac{1}{6} + e^{-1}$$

$$= \boxed{e^{-1} - \frac{1}{3}}.$$