Monte Carlo Simulations for Financial Models

Dawid Kubkowski

January 26, 2025

Contents

1 Introduction

This report explores Monte Carlo methods applied to financial models. The focus is on simulating Brownian motion, Geometric Brownian motion and their applications to pricing European and Asian options.

1.1 Brownian Motion

Roughly speaking, a stochastic process $\mathbf{B} = (B(t))_{t \le T}$ is a **Brownian motion** if $B(t_0) = 0$ at $t_0 = 0$, and for any $0 \le t_1 < \ldots < t_n \le T$, the vector $(B(t_1), \ldots, B(t_n))$ is a zero-mean multivariate normal random variable $\mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$ with covariance matrix

$$\mathbf{\Sigma}(i, j) = \mathrm{Cov}\left(B(t_i), B(t_j)\right) = \min(t_i, t_j), \quad i, j = 1, \ldots, n.$$

In this project, we consider $T = 1$ and equally spaced time points $(t_1, t_2, \ldots, t_n) = \left(\frac{1}{n}, \frac{2}{n}, \ldots, 1\right)$.

1.2 Geometric Brownian Motion

The evolution of stocks (assets) is often modeled as geometric Brownian motion–GBM(μ, σ)–which is defined by

$$S(t) = S(0) \exp\left(\left(r - \frac{\sigma^2}{2}\right) t + \sigma B(t)\right), \qquad 0 \le t \le T,$$

where $B(t)$ $(0 \le t \le T)$ is Brownian motion. In computing option prices, often the interest rate r and volatility σ are known; we then make computations for GBM(r, σ). Denote $\mu^* = r - \sigma^2/2$. Then we have

$$S(t) = S(0) \exp\left(\mu^* t + \sigma B(t)\right), \qquad 0 \le t \le T.$$

1.3 European and Asian Call Options

We are interested in estimating the following (called an *option*, with discounted payoff at time 1) with price given by the formula

$$I = e^{-r} E(A_n - K)_+,$$

where

$$A_n = \frac{1}{n} \sum_{i=1}^{n} S(i/n)$$

and $S(t)$ is given in the previous section.

In the case $n = 1$, this is called a **European call option**; otherwise, it is called an **Asian call option**.

1.4 Black-Scholes Formula

In the case $n = 1$ (i.e., European call option), the exact value of $E(A_1 - K)_+ = E(S(1) - K)_+$ is provided by the Black-Scholes formula (where Φ is the c.d.f. of $\mathcal{N}(0, 1)$):

$$E(S(1) - K)_+ = S(0)\Phi(d_1) - K e^{-r} \Phi(d_2),$$

where

$$d_1 = \frac{1}{\sigma}\left[\log\left(\frac{S(0)}{K}\right) + r + \frac{\sigma^2}{2}\right],$$

and

$$d_2 = d_1 - \sigma.$$

2 Antithetic Estimator in the Project

In this project, the **antithetic estimator** is used to reduce the variance of the Monte Carlo simulation when estimating the price of a European call option ($n = 1$). The antithetic variates method leverages the symmetry of the standard normal distribution to create pairs of negatively correlated random variables, which helps to reduce the overall variance of the estimator.

2.1 How the Antithetic Estimator is Applied

1. **Simulating Brownian Motion**:

- For each simulation path, we generate standard normal random variables Z_{2i-1} to simulate the Brownian motion $B(t)$.
- Instead of generating an independent Z_{2i} for the next simulation, we use $Z_{2i} = -Z_{2i-1}$. This ensures that the pairs (Z_{2i-1}, Z_{2i}) are negatively correlated.

2. **Generating Stock Prices**:

- Using the Geometric Brownian Motion (GBM) model, the stock price at time $t = 1$ is simulated as:

$$S(1) = S(0) \exp\left(\mu^* \cdot 1 + \sigma B(1)\right),$$

where $B(1)$ is the value of Brownian motion at time $t = 1$, and $\mu^* = r - \frac{\sigma^2}{2}$.

- For each pair (Z_{2i-1}, Z_{2i}), we compute two stock prices:

$$S_{2i-1}(1) = S(0) \exp\left(\mu^* \cdot 1 + \sigma Z_{2i-1}\right),$$

$$S_{2i}(1) = S(0) \exp\left(\mu^* \cdot 1 + \sigma Z_{2i}\right).$$

3. **Calculating Payoffs**:

- For each pair of stock prices, we compute the discounted payoffs:

$$Y_{2i-1} = e^{-r} \max(S_{2i-1}(1) - K, 0),$$

$$Y_{2i} = e^{-r} \max(S_{2i}(1) - K, 0).$$

- These payoffs are negatively correlated due to the use of antithetic variates.

4. **Averaging the Payoffs**:

- The antithetic estimator is computed as the average of all payoffs:

$$\hat{Y}_R^{\text{anti}} = \frac{1}{R} \sum_{j=1}^{R} Y_j^{\text{anti}},$$

where Y_j^{anti} are the payoffs generated using the antithetic variates method.

2.2 Why the Antithetic Estimator is Effective

- **Variance Reduction**:

- The antithetic variates method exploits the negative correlation between Y_{2i-1} and Y_{2i}. Since the payoffs are negatively correlated, the variance of the estimator $\hat{Y}_R^{\text{anti}}$ is reduced compared to the crude Monte Carlo estimator.
- Mathematically, the variance of the antithetic estimator is:

$$\text{Var}(\hat{Y}_R^{\text{anti}}) = \text{Var}(\hat{Y}_R^{\text{CMC}})(1 + \text{Corr}(Y_1, Y_2)),$$

where $\text{Corr}(Y_1, Y_2) < 0$. This ensures that $\text{Var}(\hat{Y}_R^{\text{anti}}) < \text{Var}(\hat{Y}_R^{\text{CMC}})$.

Later on we will use coefficient $= (1 + \text{Corr}(Y_1, Y_2))$.

3 Control Variate Estimator in the Project

The **control variate estimator** is another variance reduction technique used in Monte Carlo simulations. It leverages the correlation between the random variable of interest Y and a control variate X, whose expected value is known. By using the control variate, we can adjust the crude Monte Carlo estimator to reduce its variance.

3.1 How the Control Variate Estimator is Applied

1. **Choosing the Control Variate**:

   - In this project, for the case $n = 1$ (European call option), we use $X = B(1)$, the value of Brownian motion at time $T = 1$, as the control variate.
   - The expected value of X is known: $\mathbb{E}[X] = 0$, since $B(1) \sim \mathcal{N}(0, 1)$.

2. **Simulating Pairs** (Y, X):

   - For each simulation, we generate $B(1)$ and compute the stock price $S(1)$ using the Geometric Brownian Motion (GBM) model:
   $$S(1) = S(0) \exp\left(\mu^* \cdot 1 + \sigma B(1)\right),$$
   where $\mu^* = r - \frac{\sigma^2}{2}$.
   - The discounted payoff Y is calculated as:
   $$Y = e^{-r} \max(S(1) - K, 0).$$
   - The pair (Y, X) is recorded for each simulation.

3. **Computing the Control Variate Estimator**:

   - The control variate estimator is defined as:
   $$\hat{Y}_R^{\text{CV}} = \frac{1}{R} \sum_{j=1}^{R} \left(Y_j - \beta(X_j - \mathbb{E}[X])\right),$$
   where β is a coefficient chosen to minimize the variance of the estimator. The optimal β is given by:
   $$\beta = \frac{\text{Cov}(Y, X)}{\text{Var}X}.$$
   - Since $\mathbb{E}[X] = 0$, the estimator simplifies to:
   $$\hat{Y}_R^{\text{CV}} = \frac{1}{R} \sum_{j=1}^{R} \left(Y_j - \beta X_j\right).$$

3.2 Theoretical Variance of the Control Variate Estimator

The variance of the control variate estimator is given by:

$$\begin{aligned}
\text{Var}(\hat{Y}_R^{\text{CV}}) &= \frac{1}{R}\left(\text{Var}Y + \frac{(\text{Cov}(Y,X))^2}{(\text{Var}X)^2} \cdot \text{Var}X - \frac{2(\text{Cov}(Y,X))^2}{\text{Var}X}\right) \\
&= \frac{\text{Var}Y}{R}\left(1 - \frac{(\text{Cov}(Y,X))^2}{\text{Var}X\,\text{Var}Y}\right) = \frac{\text{Var}Y}{R}(1-\rho^2) \\
&= \text{Var}\hat{Y}_R^{\text{CMC}}(1-\rho^2),
\end{aligned}$$

where $\rho = \text{Corr}(Y, X)$ is the correlation between Y and X. The variance is thus reduced by a factor of $1 - \rho^2$, which we will later on use as coefficient.

3.3 Why the Control Variate Estimator is Effective

- **Variance Reduction**:

- The control variate estimator reduces the variance of the crude Monte Carlo estimator by exploiting the correlation between Y and X. The higher the correlation ρ, the greater the variance reduction.
- The variance reduction factor $1 - \rho^2$ shows that if Y and X are highly correlated ($\rho \approx 1$), the variance of the estimator can be significantly reduced.

- **Unbiased Estimation**:

- The control variate estimator remains unbiased because the adjustment term $\beta(X_j - \mathbb{E}[X])$ has an expected value of zero.

3.4 Summary

In this project, the control variate estimator is used to improve the efficiency of Monte Carlo simulations for pricing European call options. By using a control variate $X = B(1)$ with known expected value, the method reduces the variance of the estimator while maintaining unbiasedness. The effectiveness of the control variate estimator depends on the correlation ρ between Y and X. For significant variance reduction, Y and X should be highly correlated, and the pair (Y, X) should be easy to simulate.

4 Simulation

In this section, we focus on implementing Monte Carlo simulations to estimate the price of European and Asian call options. We will use the following fixed parameters throughout the simulation:

$$r = 0.05, \quad \sigma = 0.25, \quad \mu^* = r - \frac{\sigma^2}{2} = -0.0125, \quad S(0) = 100, \quad K = 100.$$

The goal is to estimate the option price I, given by:

$$I = e^{-r} E(A_n - K)_+,$$

where A_n is the average stock price over time points as defined previously. We explore the following estimation techniques:

4.1 Crude Monte Carlo Estimator

The crude Monte Carlo estimator is a straightforward method that relies on generating R independent simulations of the underlying stock price $S(t)$ over time, calculating the discounted payoff for each simulation, and then averaging the results. This approach provides a baseline for comparing the accuracy and efficiency of other methods.

4.2 Stratified Estimator

The stratified sampling method divides the simulation range into strata, ensuring better coverage of the sample space. We implement both proportional and optimal allocation schemes for stratification to improve the estimation of I.

4.3 Antithetic Estimator (for $n = 1$)

To reduce variance in the simulation, we use an antithetic estimator. Specifically, for each standard normal random variable Z_{2i-1}, we pair it with $Z_{2i} = -Z_{2i-1}$. By simulating pairs of dependent variables in this way, the estimator exploits symmetry to achieve variance reduction while preserving unbiased estimation.

4.4 Control Variate Estimator (for $n = 1$)

For the case $n = 1$, we introduce a control variate $X = B(1)$, the value of Brownian motion at time $T = 1$. Since the expected value of X is known, we use this information to adjust the crude Monte Carlo estimator, reducing the variance of the estimation.

4.5 Comparison and Analysis

In the analysis, for $n = 1$ we will cover the estimation results of the option price, comparing the Monte Carlo estimates with the exact value provided by the Black-Scholes formula. Additionally, we will examine the bias and variance of the different estimators to assess their accuracy and reliability.

For $n \geq 2$, the focus will be on comparing the estimations obtained using various methods, particularly evaluating and analyzing the variance of the estimators.

5 Results

5.1 European call option, $n = 1$

5.1.1 Stratified estimation

$R = 10^3$, Proportional Allocation Results				
	CMC	Stratified estimator		
		number of strata m		
		5	10	20
$\hat{Y}$	12.4045	12.7940	11.8229	11.7487
$\mathrm{Var}(\hat{Y}_R)$	0.3563	0.2909	0.2519	0.2445
bias	0.0685	0.4580	0.5131	0.5873

(a) Proportional Allocation, $R = 10^3$

$R = 10^3$, Optimal Allocation Results				
	CMC	Stratified estimator		
		number of strata m		
		5	10	20
$\hat{Y}$	12.4045	11.6816	11.9978	12.2448
$\mathrm{Var}(\hat{Y}_R)$	0.3563	0.1690	0.1555	0.1499
bias	0.0685	0.6544	0.3382	0.0912

(b) Optimal Allocation, $R = 10^3$

$R = 10^6$, Proportional Allocation Results				
	CMC	Stratified estimator		
		number of strata m		
		5	10	20
$\hat{Y}$	12.3157	12.3421	12.3373	12.3460
$\mathrm{Var}(\hat{Y}_R)$	$3.41e-04$	$2.63e-04$	$2.55e-04$	$2.51e-04$
bias	0.0203	0.0061	0.0013	0.0100

(c) Proportional Allocation, $R = 10^6$

$R = 10^6$, Optimal Allocation Results				
	CMC	Stratified estimator		
		number of strata m		
		5	10	20
$\hat{Y}$	12.3157	12.3313	12.3339	12.3287
$\mathrm{Var}(\hat{Y}_R)$	$3.41e-04$	$1.68e-04$	$1.57e-04$	$1.53e-04$
bias	0.0203	0.0047	0.0021	0.0073

(d) Optimal Allocation, $R = 10^6$

Figure 1: Comparison of Proportional and Optimal Allocation results for different values of R.

The results demonstrate the impact of increasing the number of replications R and the efficiency of variance reduction techniques, such as proportional and optimal stratified allocation, when compared to the Crude Monte Carlo (CMC) estimator. A few key observations can be made:

1. **Effect of Increasing Replications (R):**
   As the number of replications increases from $R = 10^3$ to $R = 10^6$, both the bias and variance of the estimators decrease significantly. This highlights the convergence properties of Monte Carlo methods, where increasing the sample size leads to more precise estimates. For example, the variance of the CMC estimator decreases from 0.3563 for $R = 10^3$ to 3.41×10^{-4} for $R = 10^6$.

2. **Variance Reduction Techniques:**
   Both proportional and optimal stratified estimators exhibit lower variances compared to the CMC estimator, validating the effectiveness of stratification. This is consistent across all configurations of the number of strata (m). For instance, under optimal allocation with $R = 10^6$, the variance reduces to 1.53×10^{-4} with $m = 20$, compared to 3.41×10^{-4} for the CMC estimator.

3. **Bias of Stratified Estimators:**
   While stratified estimators generally exhibit lower variance, their bias tends to vary based on the allocation method and the number of strata. For $R = 10^3$ the bias is larger than the CMC estimator, but for $R = 10^6$ is lower. These are results of 1 simulation with R replications so it is hard to generalize but it seems that stratified estimation provide more accurate results.

4. **Comparison Between Proportional and Optimal Allocation:**
Optimal allocation consistently outperforms proportional allocation in terms of variance reduction. For example, with $R = 10^6$ and $m = 20$, the variance under optimal allocation is 1.53×10^{-4}, compared to 2.51×10^{-4} under proportional allocation. This aligns with theoretical expectations, as optimal allocation is designed to minimize variance by allocating more samples to strata with higher variability.

5. **Number of Strata (m):**
Increasing the number of strata generally reduces variance for both allocation methods. However, this effect diminishes as m increases, and the improvement is less pronounced for higher replication counts (R).

In summary, the results underscore the benefits of variance reduction techniques, especially stratification, in achieving more precise estimates. Larger replication counts and optimal allocation strategies consistently yield better performance, demonstrating their importance in practical Monte Carlo simulations.

5.1.2 Antithetic and Control variate estimators

$R = 10^3$			
	CMC	Antithetic	Control Variate
$\hat{Y}$	12.4045	11.8738	12.1795
$\mathrm{Var}(\hat{Y}_R)$	0.3563	0.1641	0.0975
Coefficient	1.0000	0.5377	0.2661
bias	0.0685	0.4622	0.1564

(a) $R = 10^3$

$R = 10^6$			
	CMC	Antithetic	Control Variate
$\hat{Y}$	12.3157	12.3510	12.3475
$\mathrm{Var}(\hat{Y}_R)$	$3.41e-04$	$1.91e-04$	$9.7e-05$
Coefficient	1.0000	0.5560	0.2821
bias	0.0203	0.0150	0.0115

(b) $R = 10^6$

Figure 2: Comparison of results for Antithetic and Control variate estimators with different values of R.

The results provide a comparison between the Crude Monte Carlo (CMC) estimator, the Antithetic estimator, and the Control Variate estimator for different values of R. Key observations are summarized below:

1. **Antithetic Estimator:**
   - The Antithetic estimator demonstrates a significant reduction in variance compared to the CMC estimator. This is evident from the "Coefficient" row, which shows the ratio of the variance of the Antithetic estimator to the CMC variance.
   - For $R = 10^3$, the variance is reduced by approximately 46% (Coefficient ≈ 0.5377), with the variance of the Antithetic estimator being 0.1641 compared to 0.3563 for CMC.
   - For $R = 10^6$, the variance reduction remains consistent (Coefficient ≈ 0.5560).

   While the variance reduction is significant, the Antithetic estimator introduces a slightly higher bias compared to CMC, particularly noticeable for $R = 10^3$ (bias = 0.4622).

2. **Control Variate Estimator:**
   - The Control Variate estimator significantly reduces the variance compared to both CMC and Antithetic estimators. For $R = 10^3$, the variance is reduced by approximately 73% relative to CMC (Coefficient $\approx$ 0.2661, with a variance of 0.0975).
   - For $R = 10^6$, the reduction remains substantial, with a coefficient of ≈ 0.2821 and variance 9.7×10^{-5}, showcasing the effectiveness of this approach for large R.
   - Additionally, the bias of the Control Variate estimator is lower than Antithetic for both values of R, indicating its robustness.

Conclusion: The Antithetic estimator reduces variance effectively but is outperformed by the Control Variate estimator, which achieves a far greater variance reduction (up to 73%) while maintaining a lower bias. As R increases, the performance of all estimators improves, but the relative advantage of Control Variates is particularly notable.

5.2 Asian call option, $n > 1$

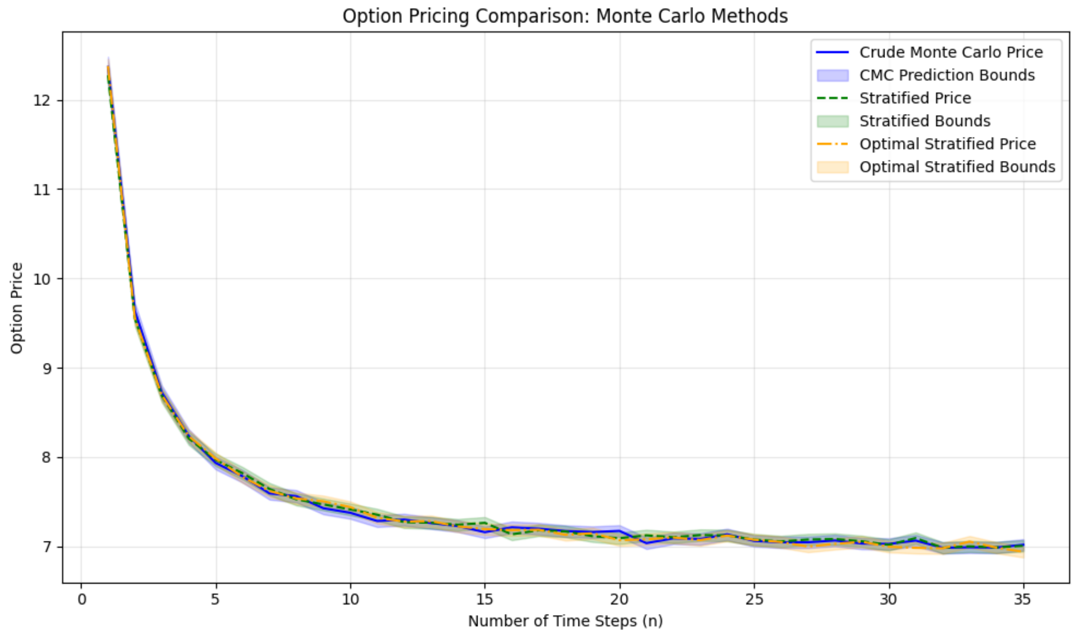

Figure 3: Comparison of Monte Carlo methods for option pricing.

The plot demonstrates that the option price converges to approximately 7 as the number of time steps n increases. Based on the analysis for $n = 1$, stratified estimators are expected to exhibit lower variance compared to other methods.

6 Figures

Table 1: Figures and Functions

Figure	Function
Figure1	monte_carlo_pricing(S0, K, r, sigma, R, n)
Figure2	antithetic_control(S0, K, r, sigma, R, n)
Figure3	simulate_and_plot_option_prices(S0, K, r, sigma, R=$10^5, m = 10, max\_n = 35$)