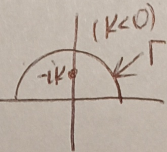
$\boxed{№1}$ $$\int_{-\infty}^{+\infty} \frac{e^{-ikx}dx}{(x+i)^3}$$

1) При $k<0$: $$\int_{-\infty}^{+\infty}\frac{e^{-ik\cdot x}dx}{(x+i)^3} = \left| \begin{array}{l} k\cdot x = t\\ dx=\frac{dt}{k}\\ x=t/k\end{array}\right| = \int_{-\infty}^{+\infty}\frac{e^{-it}dt}{k\left(\frac{t}{k}+i\right)^3} =$$

$$= \int_{-\infty}^{+\infty} \frac{e^{-it}dt}{\frac{k}{k^3}(t+ik)^3} = k^2\int_{-\infty}^{+\infty} \frac{e^{it}dt}{(t+ik)^3} = k^2\int\frac{e^{iz}dz}{(z+ik)^3} = k^2\cdot 2\pi i\cdot \operatorname{res} f(z)$$

$z=+\infty \not\in\Gamma$; $z=-ik \in \Gamma$



$$=\sum_i \operatorname{res}_i f(z) = -\frac{\cancel{i}\pi k^2}{e^k}\ (k>0)$$

2) При $k>0$ $$\int_{-\infty}^{+\infty}\frac{e^{-ikx}dx}{(x+i)^3} = k^2\int_\Gamma\frac{dz}{e^{iz}(z+ik)^3} = k^2\cdot 2\pi i\cdot \operatorname{res} f(z) \ominus$$

$-ik\notin\Gamma$, $\infty\notin\Gamma$

$\ominus$ 0

$$\operatorname*{res}_{z=-ik} f(z) = \frac{1}{2}\lim_{z\to -ik}\frac{d^2}{dz^2}\left((z+ik)^3\frac{e^{iz}}{(z+ik)^3}\right) = \frac{1}{2}\lim_{z\to -ik}\frac{d^2}{dz^2}e^{iz} =$$

$$=\frac{1}{2}\lim_{z\to -ik}\left[ie^{iz}\right]'_z = \frac{1}{2}\lim_{z\to -ik}\left[i^2 e^{iz}\right] = \frac{1}{2}(-1)e^{-iik} = \frac{-e^k}{2},\ k<0$$

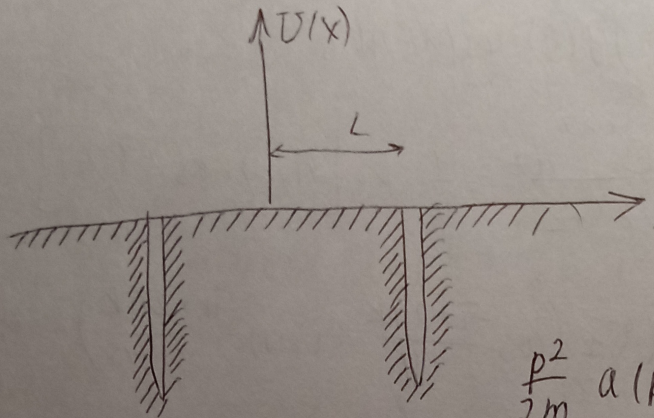
$\boxed{№8}$ $$I(p)=\int_0^{+\infty}e^{-pt}t^{z-1}dt = \left|\begin{array}{l} pt=u,\ t=\frac{u}{p}\\ dt=\frac{du}{p}\end{array}\right| = \int_0^{+\infty}e^{-u}\left(\frac{u}{p}\right)^{z-1}\frac{du}{p} =$$

$$=\frac{1}{p^z}\int_0^{+\infty}e^{-u}u^{z-1}du = \frac{1}{p^z}\Gamma(z)$$

$\boxed{№4}$ $U(x) = -\frac{\hbar^2 k}{m}\delta(x-L) - \frac{\hbar^2 k}{m}\delta(x+L)$

$$-\frac{\hbar^2}{2m}\psi''(x) + U(x)\psi(x) = E\psi$$

Решая в имп. предст., найти уровни энергии при $Lk \gg 1$



$$\frac{p^2}{2m}a(p) - \frac{\hbar^2 k}{m} i\hbar \frac{da(p)}{dp} = Ea(p)$$

$$\cancel{\frac{p^2}{2m}a(p) - \frac{\hbar^2 k}{m} \ldots = Ea(p)}$$

$$\frac{p^2}{2m}a(p) - \frac{\hbar^2 k}{m}\int\frac{dp'}{2\pi\hbar}\overbrace{\delta(x-L)}^{+\delta(x+L)}a(p') = \frac{-k\,a(p)}{2m}$$

$$-|E| = -\frac{k}{2m}$$

$\delta(x-L)$: $\delta_k = \int e^{-ikx}\delta(x)\,dx = 1$

$$\frac{p^2}{2m}a(p) - \frac{\hbar^2 k}{m}\overbrace{\int\frac{dp'}{2\pi\hbar}a(p')}^{C} = -\frac{k\,a(p)}{2m}$$

$$a(p) = \frac{\frac{\hbar^2 k}{m}C}{\frac{p^2}{2m}+\frac{k}{2m}} = \frac{2\hbar k L}{p^2+k} \;\hookrightarrow\; C = 2\hbar^2 k C\int_{-\infty}^{+\infty}\frac{dp'}{2\pi\hbar}\frac{1}{p'^2+\sqrt{k}^2} =$$

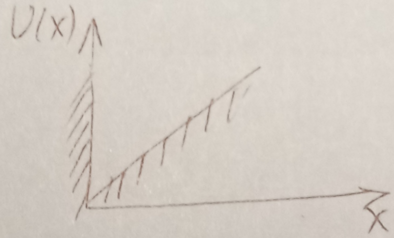
$$= \frac{1}{\sqrt k}\operatorname{arctg}\frac{p'}{k}\Big|_{-\infty}^{+\infty} = \frac{1}{\sqrt k}\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{\sqrt k}, \quad C = \frac{\cancel{2}\hbar^{\cancel{2}}\cdot \overset{\sqrt k}{\cancel{k}}\cdot C}{\cancel{2\pi}\cancel{\hbar}}\cdot\frac{\cancel{\pi}}{\cancel{\sqrt k}},$$

$$\hbar\sqrt k = 1 \Rightarrow \cancel{\ldots}\; k = \frac{1}{\hbar^2}, \Rightarrow E = -\frac{1}{2m\hbar^2},$$

$\delta(x+L)$: —〃—

№3

$$U(x) = \begin{cases} \alpha x, & x > 0 \\ \infty, & x = 0 \end{cases}$$



$\psi(0) = 0$ и

$$\hat{H}|\psi\rangle = E|\psi\rangle$$

$$\hat{H} = c|\vec{p}| + \alpha\hat{x} \qquad \hat{p} \to p, \quad \hat{x} \to i\hbar\frac{\partial}{\partial p}$$

$$c|\vec{p}|a(p) + \alpha i\hbar\frac{da(p)}{dp} = E a(p)$$

$$\psi(x) = \frac{1}{\sqrt{L}}\int e^{ipx/\hbar}\, a(p)\, \frac{dpL}{2\pi\hbar}$$

$x = 0$ $\psi(0)$ $\quad \psi(0) = \frac{1}{\sqrt{L}}\int\limits_{-\infty}^{+\infty} a(p)\, dp\, \frac{L}{2\pi\hbar} = 0$

$$\int\limits_{-\infty}^{+\infty} a(p)\, dp = 0$$

$$\frac{da}{dp} = \frac{1}{\alpha i\hbar}\left(E - \frac{p^2}{2m}\right)a(p)$$

$$\int\frac{da}{a} = \int -\frac{i}{\alpha\hbar}\left(E - \frac{p^2}{2m}\right)dp \qquad E = \frac{k^2}{2m}$$

$$\ln a = \frac{-i}{\alpha\hbar 2m}\left(p - \frac{p^3}{3}\right) + const$$

$$\lambda\left(t - \frac{t^3}{3}\right)$$

$$a(p) = Ce^{-i/(2m\alpha\hbar)(pk^2 - p^3/3)}$$

$$\int\limits_{-\infty}^{+\infty} a(p)\, dp = 0 = C\int e^{\frac{-i}{2m\alpha\hbar}\left(pk^2 - \frac{p^3}{3}\right)} dp = 0$$

$$\alpha i\hbar \frac{da(p)}{dp} = Ea(p) - c|p|a(p) = (E - c|p|)a(p)$$

$$\frac{da(p)}{a(p)} = (E - c|p|)dp \cdot \frac{1 i}{\alpha i^2 \hbar} = \frac{-i}{\alpha\hbar}$$

$$\ln a = \frac{-i}{\alpha\hbar}\left[Ep - \frac{cp^2}{2}\right] + C_1$$

$$a(p) = C_2 \exp\left\{\frac{-i}{\alpha\hbar}\left[Ep - \frac{cp^2}{2}\right]\right\}$$

$$\int_{-\infty}^{+\infty} a(p)dp = 0 : \int_{-\infty}^{+\infty} C_2 \exp\left\{\frac{-i}{\alpha\hbar}\left[Ep - \frac{cp^2}{2}\right]\right\}dp = 0$$

$p = kt$

$$E = kc : \quad k\int_{-\infty}^{+\infty} e^{\underbrace{-i\frac{k^2 c}{\alpha\hbar}}_{\lambda}\left(t - \frac{t^2}{2}\right)} dt$$

$\lambda >> 1$

$S(t) = -t + \frac{t^2}{2}$, $S'(t) = -1 + t = 0$ $\quad S(1) = -\frac{1}{2}$

$S''(t) = 1$ $\quad \varphi = \frac{-\lambda}{2} + \frac{\pi}{4}$ $\quad t = 1$

$$\int_{-\infty}^{+\infty} a(p)dp = \sqrt{\frac{2\pi}{\lambda \cdot 1}}\, e^{-\frac{1}{2}i\lambda + i\frac{\pi}{4}} = 0$$

$\cos\varphi + i\sin\varphi = 0$

$\cos$: $\quad \frac{\pi}{4} - \frac{\lambda}{2} = \frac{-\pi}{2} + \pi n$, $n = 0, 1, 2, \ldots$

$$\frac{\lambda}{2} = \frac{3\pi}{4} - \pi n, \quad \lambda = \frac{3\pi}{2} - 2\pi n = \frac{k^2 c}{\alpha\hbar}$$

$$k^2 = \frac{\alpha\hbar}{c}\left(\frac{3\pi}{2} - 2\pi n\right), \quad k = \sqrt{\frac{\alpha\hbar}{c}\left(\frac{3\pi}{2} - 2\pi n\right)}$$

$$E_n = c\sqrt{\frac{\alpha\hbar}{c}\left(\frac{3\pi}{2} - 2\pi n\right)}, \quad n = \ldots$$

$\sin$:

$$k^2 = \frac{\alpha\hbar}{c}\left(\frac{\pi}{2} - 4\pi n\right)$$

$$E_n = c\sqrt{\frac{\alpha\hbar}{c}\left(\frac{\pi}{2} - 4\pi n\right)}$$

№2 Разложить в интеграл Фурье $u(x) = \frac{1}{x^4+1}$

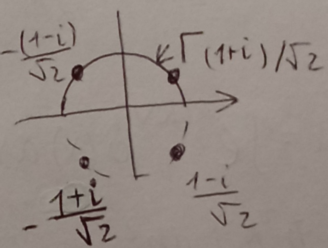
$$u(x) = \int_0^{+\infty} \{ a(\lambda)\cos(\lambda x) + b(\lambda)\sin(\lambda x)$$

Для чётной функции $a(\lambda) = \frac{2}{\pi}\int_0^{+\infty} u(\xi)\cdot\cos(\lambda\xi)\,d\xi =$

$$= \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\cos(\lambda\xi)\,d\xi}{\xi^4+1} = \frac{1}{\pi}\cdot Re\int_{-\infty}^{+\infty}\frac{e^{i\lambda z}\,dz}{z^4+1} = \frac{1}{\pi}\cdot Re\Big[2\pi i\sum_i res\, f(z)\Big]$$

$$z^4+1=0 \Rightarrow \left[\begin{array}{l} z = -\frac{1+i}{\sqrt2} \\ z = \frac{1+i}{\sqrt2} \in \Gamma \\ z = \frac{1-i}{\sqrt2} \\ z = -\frac{1-i}{\sqrt2} \in \Gamma \end{array}\right.$$

$$1\sqrt[4]{1}\left(\cos\frac{-\pi+2\pi k}{4} + i\sin\frac{-\pi+2\pi k}{4}\right)\ k=0,1,2,\dots$$



$$\boxed{\underset{z=a}{res} = \frac{\varphi(a)}{\psi'(a)}}$$ $\varphi(z), \psi(z)$ – ан. в т. а, $\varphi(a), \psi(a), \psi'(a) \neq 0$, a – простой полюс

$$res_1 f(z) = \frac{e^{i\lambda\left(\frac{1+i}{\sqrt2}\right)}}{4\left(\frac{1+i}{\sqrt2}\right)^3} =$$

$$= \frac{e^{i\lambda\left(\frac{1+i}{\sqrt2}\right)}}{4e^{i\frac{\pi}{4}\cdot3}} = \frac14 e^{i\frac{\lambda}{\sqrt2} - \frac{\lambda}{\sqrt2} - 3\frac{i\pi}{4}} = \frac14 e^{-\frac{\lambda}{\sqrt2}}\cdot e^{i\left(\frac{\lambda}{\sqrt2} - \frac{3\pi}{4}\right)}$$

$$-\frac{i\lambda}{\sqrt2} + i^2\frac{\lambda}{\sqrt2}$$

$$res_2 f(z) = \frac{e^{-i\lambda\frac{1-i}{\sqrt2}}}{4e^{i\frac{3\pi}{4}\cdot3}} = \frac14 e^{-i\frac{\lambda}{\sqrt2} - \frac{\lambda}{\sqrt2} - 9i\frac{\pi}{4}} \ominus$$

$$\ominus\ \frac14 e^{-\frac{\lambda}{\sqrt2}}\cdot e^{i\left(\frac{-\lambda}{\sqrt2} - \frac{9\pi}{4}\right)} = \frac14 e^{\frac{-\lambda}{\sqrt2}}\Big[\cos\overbrace{\left(\frac{-\lambda}{\sqrt2} - \frac{9\pi}{4}\right)}^{\varphi_2} + i\sin\varphi_2\Big]$$

$$res_1 f(z) = \frac14 e^{-\frac{\lambda}{\sqrt2}}\cdot\Big[\cos\overbrace{\left(\frac{\lambda}{\sqrt2} - \frac{3\pi}{4}\right)}^{\varphi_1} + i\sin(\varphi_1)\Big]$$

1) $-\dfrac{\exp\left[\frac{1}{\sqrt{2}}(1+i)\lambda i\right]}{\left(\frac{\sqrt{2}}{2}\right)^3(1+i-1+i)(1+i+1+i)(1+i+1-i)} = -\dfrac{\exp\left(\frac{1}{\sqrt{2}}(1+i)\lambda i\right)}{2\sqrt{2}\,(i-1)}$

2) $-\dfrac{\exp\left[\frac{1}{\sqrt{2}}\lambda i(i-1)\right]}{\left(\frac{\sqrt{2}}{2}\right)^3(i-1-1+i)(i-1+1+i)(i-1-1-i)} = -\dfrac{\exp\left(\frac{1}{\sqrt{2}}\lambda i(i-1)\right)}{2\sqrt{2}\,(i+1)}$

1)+2) $= -\dfrac{\exp\left\{\frac{\lambda i}{\sqrt{2}}(1+i)\right\}(1+i)+\exp\left\{\frac{\lambda i}{\sqrt{2}}(i-1)\right\}(i-1)}{4\sqrt{2}} =$

$= \dfrac{-1}{4\sqrt{2}}\exp\left\{-\dfrac{\lambda}{\sqrt{2}}\right\}\left[(1+i)\exp\left\{\dfrac{\lambda i}{\sqrt{2}}\right\}+(i-1)\exp\left\{\dfrac{-\lambda i}{\sqrt{2}}\right\}\right]$

$\mathrm{Re}\left[2\pi i \cdot a\right] =$

$\dfrac{\pi}{\sqrt{2}}\left[\sin\dfrac{\lambda}{\sqrt{2}}+\cos\dfrac{\lambda}{\sqrt{2}}\right]\exp\left\{\dfrac{-\lambda}{\sqrt{2}}\right\}$

Ответ: $u(x)=\int_0^{+\infty}\dfrac{1}{\sqrt{2}}\left\{\sin\dfrac{\lambda}{\sqrt{2}}+\cos\dfrac{\lambda}{\sqrt{2}}\right\}\exp\left\{\dfrac{-\lambda}{\sqrt{2}}\right\}\cos\lambda x\,dx$

$\boxed{N^o 7}$ $I(\lambda) = \int\limits_{-\infty}^{+\infty} \frac{e^{i\lambda\left(x+\frac{x^4}{4}\right)}}{x^2+1} dx$

Метод стационарной фазы

$f(x), S(x) \in C^2[a,b]$, $S(x)$ на (a,b) имеет ед. экстремум в $x_0 \in (a,b)$, $S''(x_0) \neq 0$, $f(x_0) \neq 0$)

$\hookrightarrow$ при $\lambda \to \infty$: $\int\limits_a^b f(x)\, e^{i\lambda S(x)} dx \sim f(x_0)\sqrt{\frac{2\pi}{\lambda |S''(x_0)|}} \exp\left\{i\left(\lambda S(x_0) + \frac{\pi}{4} \operatorname{sign} S''(x_0)\right)\right\}$

$S(x) = x + \frac{x^4}{4}$, $S'(x) = 1 + x^3$, $S''_{xx}(x) = 3x^2$, $x_0 = -1$ – экстремум

$S''_{xx}\Big|_{x_0} = 3$, $\operatorname{sign} S''(x_0) = 1$, $S(x_0) = -1 + \frac{1}{4} = -\frac{3}{4}$

$f(x_0) = \frac{1}{x_0^2+1} = \Big|_{x_0=-1} = 1/2$

$I(\lambda) = \frac{1}{2}\sqrt{\frac{2\pi}{\lambda \cdot 3}} \exp\left\{i\left(\lambda \cdot \frac{-3}{4} + \frac{\pi}{4} \cdot 1\right)\right\} = \sqrt{\frac{\pi}{6\lambda}}\, e^{i\left[\frac{\pi}{4} - \frac{3\lambda}{4}\right]},$