2) Задание 1.
Пункт 1.

$$A = \begin{Vmatrix} \cos\theta & -i\sin\theta \\ i\sin\theta & -\cos\theta \end{Vmatrix}$$

① с.з.: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} \cos\theta - \lambda & -i\sin\theta \\ i\sin\theta & -\cos\theta - \lambda \end{vmatrix} = (\cos\theta - \lambda)(-\cos\theta - \lambda) - (-i\sin\theta)(i\sin\theta) =$$

$$= -(\cos^2\theta - \lambda^2) + i^2\sin^2\theta = -\cos^2\theta + \lambda^2 - \sin^2\theta = \lambda^2 - 1 = 0 \Rightarrow \left[\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= -1 \end{aligned}\right.$$ собственные значения оператора A

② с.в.

1) $\lambda_1 = 1$: $\begin{Vmatrix} \cos\theta - 1 & -i\sin\theta \\ i\sin\theta & -\cos\theta - 1 \end{Vmatrix} \cdot \begin{Vmatrix} \psi_1^1 \\ \psi_1^2 \end{Vmatrix} \cdot a = \vec{0}$, $\vec{\psi_1} = \begin{Vmatrix} \psi_1^1 \\ \psi_1^2 \end{Vmatrix}$

~~коэф. нормир.~~

$(\cos\theta - 1)\psi_1^1 - i\sin\theta\,\psi_1^2 = 0.$

Пусть $\psi_1^1 = 1 \Rightarrow \cos\theta - 1 = i\sin\theta\,\psi_1^2 \Rightarrow \psi_1^2 = \dfrac{\cos\theta - 1}{i\sin\theta} = i\dfrac{(1-\cos\theta)}{\sin\theta}$

$$\vec{\psi_1} = \begin{Vmatrix} 1 \\ i\dfrac{(1-\cos\theta)}{\sin\theta} \end{Vmatrix}$$

Отнормируем вектор:

$$|\vec{\psi_1}| = \sqrt{1^2 + \left[\frac{(1-\cos\theta)}{\sin\theta}\right]^2} = \sqrt{1 + \frac{1 - 2\cos\theta + \cos^2\theta}{\sin^2\theta}} =$$

$$= \sqrt{1 + \frac{1 - 2\cos\theta + 1 - \sin^2\theta}{\sin^2\theta}} = \sqrt{\cancel{1} + \frac{2}{\sin^2\theta} - \frac{2\cos\theta}{\sin^2\theta} - \cancel{\frac{\sin^2\theta}{\sin^2\theta}}} = \sqrt{\frac{2(1-\cos\theta)}{\sin^2\theta}}$$

Таким образом, отнормированный $\vec{\psi_1}$ имеет вид:

$$\vec{\psi_1} = \frac{\sin\theta}{\sqrt{2(1-\cos\theta)}} \cdot \begin{Vmatrix} 1 \\ i\dfrac{(1-\cos\theta)}{\sin\theta} \end{Vmatrix} = \begin{Vmatrix} \dfrac{\sin\theta}{\sqrt{2(1-\cos\theta)}} \\ i\dfrac{\sqrt{1-\cos\theta}}{\sqrt{2}} \end{Vmatrix}$$

2) $\lambda_2 = -1$

$$\begin{Vmatrix} \cos\theta + 1 & -i\sin\theta \\ i\sin\theta & -\cos\theta + 1 \end{Vmatrix} \cdot \begin{Vmatrix} \psi_2^1 \\ \psi_2^2 \end{Vmatrix} = \vec{0}$$

$$(\cos\theta + 1)\psi_2^1 - i\sin\theta \cdot \psi_2^2 = 0$$

Пусть $\psi_2^1 = 1 \Rightarrow \cos\theta + 1 = i\sin\theta \cdot \psi_2^2 \Rightarrow \psi_2^2 = \dfrac{\cos\theta + 1}{i\sin\theta}$

$$\vec{\psi_2} = \begin{Vmatrix} 1 \\ \dfrac{\cos\theta + 1}{i\sin\theta} \end{Vmatrix}; \quad |\vec{\psi_2}| = \sqrt{1 + \frac{(\cos\theta+1)^2}{\sin^2\theta}} = \sqrt{1 + \frac{\cos^2\theta}{\sin^2\theta} + \frac{2\cos\theta}{\sin^2\theta} + \frac{1}{\sin^2\theta}} =$$

$$= \sqrt{\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} + \frac{(2\cos\theta + 1)}{\sin^2\theta}} = \sqrt{\frac{2\cos\theta + 2}{\sin^2\theta}} = \sqrt{\frac{2(1+\cos\theta)}{\sin^2\theta}}$$

Отнормированный $\vec{\psi_2} = \dfrac{\sin\theta}{\sqrt{2(1+\cos\theta)}} \cdot \begin{Vmatrix} 1 \\ \dfrac{\cos\theta + 1}{i\sin\theta} \end{Vmatrix} = \begin{Vmatrix} \dfrac{\sin\theta}{\sqrt{2(1+\cos\theta)}} \\ \dfrac{\sqrt{\cos\theta + 1}}{i\sqrt{2}} \end{Vmatrix}$

Проверка ортогональности собственных векторов

$\langle \vec{\psi_1} | \vec{\psi_2} \rangle = 0$? Коэфф. перед вектором не влияет, поэтому

$$\langle \vec{\psi_1} | \vec{\psi_2} \rangle = \int \vec{\psi_1}^*(\theta)\vec{\psi_2}(\theta)\,dt$$

$$\vec{\psi_1}^* \cdot \vec{\psi_2} = \begin{Vmatrix} 1 & -i\dfrac{(1-\cos\theta)}{\sin\theta} \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ \dfrac{\cos\theta + 1}{i\sin\theta} \end{Vmatrix} = 1 - i\frac{(1-\cos\theta)}{\sin\theta} \cdot \frac{(\cos\theta+1)}{i\sin\theta} =$$

$= 1 - \dfrac{1-\cos^2\theta}{\sin^2\theta} = 1 - \dfrac{\sin^2\theta}{\sin^2\theta} = 0 \Rightarrow$ векторы ортогональны.

Задание 1 пункт 2. $e^{i\alpha A}$ – ? $\alpha \in \mathbb{R}$

$$e^{\hat{S}} = I + \frac{\hat{S}}{1!} + \frac{\hat{S}^2}{2!} + \dots, \quad A = \begin{Vmatrix} \cos\theta & -i\sin\theta \\ i\sin\theta & -\cos\theta \end{Vmatrix}$$

$$A^2 = \begin{Vmatrix} \cos\theta & -i\sin\theta \\ i\sin\theta & -\cos\theta \end{Vmatrix} \cdot \begin{Vmatrix} \cos\theta & -i\sin\theta \\ i\sin\theta & -\cos\theta \end{Vmatrix} =$$

$$= \begin{Vmatrix} \cos^2\theta - i^2\sin^2\theta & \overbrace{-i\sin\theta\cdot\cos\theta + i\sin\theta\cdot\cos\theta}^{=0} \\ \underbrace{i\sin\theta\cdot\cos\theta - i\sin\theta\cdot\cos\theta}_{0} & -i^2\sin^2\theta + \cos^2\theta \end{Vmatrix} = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix};$$

$$i\alpha A = \begin{Vmatrix} i\alpha & 0 \\ 0 & i\alpha \end{Vmatrix}$$

$$A^3 = A \cdot A^2 = A \cdot I = A$$

Таким образом, видим, что $A^{2k} = I$, $A^{2k+1} = A$

$$e^{i\alpha A} = I + i\alpha\frac{\hat{A}}{1!} + (i\alpha)^2\frac{\hat{A}^2}{2!} + (i\alpha)^3\frac{\hat{A}^3}{3!} + \dots + (i\alpha)^n\frac{\hat{A}^n}{n!} + \dots =$$

$$= I\left(1 + (i\alpha)^2\cdot\frac{1}{2!} + (i\alpha)^4\cdot\frac{1}{4!} + \dots\right) + \hat{A}\left(\frac{i\alpha}{1!} + \frac{(i\alpha)^3}{3!} + \frac{(i\alpha)^5}{5!} + \dots\right) =$$

$$= I\left(1 - \alpha^2\frac{1}{2!} + \alpha^4\cdot\frac{1}{4!} - \alpha^6\cdot\frac{1}{6!} + \dots\right) + i\hat{A}\left(\frac{\alpha}{1!} - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \dots\right) =$$

$$= I\cdot\cos\alpha + i\hat{A}\cdot\sin\alpha$$

Задание 2. $A = \frac{1}{\sqrt{2}} \begin{Vmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{Vmatrix}$

1. Найдем с.з. A. $\det(A - \lambda E) = 0$: $\begin{vmatrix} -\lambda & -i/\sqrt{2} & 0 \\ i/\sqrt{2} & -\lambda & -i/\sqrt{2} \\ 0 & i/\sqrt{2} & -\lambda \end{vmatrix} =$

$$= -\lambda\left[(-\lambda)\cdot(-\lambda) - \left(\frac{-i}{\sqrt{2}}\cdot\frac{i}{\sqrt{2}}\right)\right] + \frac{i}{\sqrt{2}}\left[\left(-\lambda\cdot\frac{i}{\sqrt{2}} - 0\right)\right] + 0 =$$

$$= -\lambda\left[\lambda^2 + \frac{i^2}{2}\right] + \frac{i}{\sqrt{2}}\cdot\frac{-i\lambda}{\sqrt{2}} = -\lambda\left(\lambda^2 - \frac{1}{2}\right] + \frac{\lambda}{2} = -\lambda^3 + \frac{\lambda}{2} + \frac{\lambda}{2} = 0;$$

$$-\lambda^3 + \lambda = 0, \quad \lambda(1-\lambda^2) = 0 \Rightarrow \left[\begin{array}{l} \lambda = 0 \\ \lambda = 1 \\ \lambda = -1 \end{array}\right.$$

2. Найдем соотв. с.в.

1) λ_0, $\vec{\psi_0}$ $\quad \begin{Vmatrix} 0 & -i/\sqrt{2} & 0 \\ i/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & i/\sqrt{2} & 0 \end{Vmatrix} \cdot \begin{Vmatrix} \psi_0^1 \\ \psi_0^2 \\ \psi_0^3 \end{Vmatrix} = \vec{0}$

$0\cdot\psi_0^1 - \frac{i}{\sqrt{2}}\cdot\psi_0^2 + 0\cdot\psi_0^3 = 0 \Rightarrow \psi_0^2 = 0$

$\frac{i}{\sqrt{2}}\psi_0^1 + 0\cdot\psi_0^2 - \frac{i}{\sqrt{2}}\psi_0^3 = 0 \Rightarrow \psi_0^1 = \psi_0^3$

$\Rightarrow \vec{\psi_0} = \begin{Vmatrix} 1 \\ 0 \\ 1 \end{Vmatrix} \cdot \frac{1}{\sqrt{2}}$ – отнормированный

2) λ_1, $\vec{\psi_1}$: $\quad \begin{Vmatrix} -1 & -i/\sqrt{2} & 0 \\ i/\sqrt{2} & -1 & -i/\sqrt{2} \\ 0 & i/\sqrt{2} & -1 \end{Vmatrix} \cdot \begin{Vmatrix} \psi_1^1 \\ \psi_1^2 \\ \psi_1^3 \end{Vmatrix} = \vec{0}$

$-\psi_1^1 - \frac{i}{\sqrt{2}}\psi_1^2 = 0 \Rightarrow \psi_1^1 = \frac{-i}{\sqrt{2}}\psi_1^2 \qquad \psi_1^3 = \frac{\psi_1^2 i}{\sqrt{2}},$

$\frac{i}{\sqrt{2}}\psi_1^1 - \psi_1^2 - \frac{i}{\sqrt{2}}\psi_1^3 = 0$

$\psi_1^2 = 1, \quad \psi_1^1 = \frac{-i}{\sqrt{2}}, \psi_1^3 = i/\sqrt{2},$

$|\vec{\psi_1}| = \sqrt{1 + \frac{1}{2} + \frac{1}{2}} = \sqrt{2}.$

$\vec{\psi_1} = \frac{1}{\sqrt{2}} \begin{Vmatrix} -i/\sqrt{2} \\ 1 \\ i/\sqrt{2} \end{Vmatrix}$ – отнормированный

3) $\lambda_{-1}, \overrightarrow{\psi_{-1}}$

$$\begin{Vmatrix} 1 & -i/\sqrt{2} & 0 \\ i/\sqrt{2} & 1 & -i/\sqrt{2} \\ 0 & i/\sqrt{2} & 1 \end{Vmatrix} \cdot \begin{Vmatrix} \psi_{-1}^1 \\ \psi_{-1}^2 \\ \psi_{-1}^3 \end{Vmatrix} = \vec{0}$$

$$\psi_{-1}^1 = \psi_{-1}^2 \cdot \frac{i}{\sqrt{2}}, \quad \psi_{-1}^3 = \frac{-\psi_{-1}^2 \cdot i}{\sqrt{2}}, \quad |\overrightarrow{\psi_{-1}}| = \sqrt{2} \text{ (ненорм)}$$

$$\overrightarrow{\psi_{-1}} = \frac{1}{\sqrt{2}} \begin{Vmatrix} i/\sqrt{2} \\ 1 \\ -i/\sqrt{2} \end{Vmatrix} \text{ — отнормированный вектор.}$$

Проверим ортогональность

$$\langle \overrightarrow{\psi_0} | \overrightarrow{\psi_1} \rangle = \begin{Vmatrix} 1 \\ 0 \\ 1 \end{Vmatrix}^T \cdot \begin{Vmatrix} -i/\sqrt{2} \\ 1 \\ i/\sqrt{2} \end{Vmatrix} = \begin{Vmatrix} 1 & 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} -i/\sqrt{2} \\ 1 \\ i/\sqrt{2} \end{Vmatrix} = \frac{-i}{\sqrt{2}} + \frac{i}{\sqrt{2}} = 0.$$

$$\langle \overrightarrow{\psi_0} | \overrightarrow{\psi_{-1}} \rangle = \begin{Vmatrix} 1 & 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} i/\sqrt{2} \\ 1 \\ -i/\sqrt{2} \end{Vmatrix} = \frac{i}{\sqrt{2}} - \frac{i}{\sqrt{2}} = 0$$

$$\langle \overrightarrow{\psi_1} | \overrightarrow{\psi_{-1}} \rangle = \underbrace{\begin{Vmatrix} \frac{+i}{\sqrt{2}} & 1 & \frac{-i}{\sqrt{2}} \end{Vmatrix}}_{\overrightarrow{\psi_1}^*} \cdot \begin{Vmatrix} i/\sqrt{2} \\ 1 \\ -i/\sqrt{2} \end{Vmatrix} = \frac{+i^2}{2} + 1 + \frac{i^2}{2} =$$

$$= -\frac{1}{2} + 1 - \frac{1}{2} = 0.$$

Попарные скалярные произведения $= 0 \Rightarrow$ в-ра ортогональны.

Задание 2 пункт 2.

Найти $e^{i\theta A}$, $A = \begin{Vmatrix} 0 & -i/\sqrt{2} & 0 \\ i/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & i/\sqrt{2} & 0 \end{Vmatrix}$

$$A^2 = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{Vmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{Vmatrix} \cdot \begin{Vmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{Vmatrix} = \frac{1}{2} \begin{Vmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{Vmatrix}$$

$$A^3 = A \cdot A^2 = \frac{1}{2\sqrt{2}} \begin{Vmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{Vmatrix} = \frac{1}{2\sqrt{2}} \begin{Vmatrix} 0 & -2i & 0 \\ 2i & 0 & -2i \\ 0 & 2i & 0 \end{Vmatrix} =$$

$$= +\frac{1}{\sqrt{2}} \begin{Vmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{Vmatrix} = A$$

$$e^{i\theta A} = I + \frac{i\theta A}{1!} + \frac{i^2\theta^2 A^2}{2!} + \frac{i^3\theta^3 A^3}{3!} + \ldots$$

$$e^{i\theta A} = I + \frac{i\theta A}{1!} + \frac{i^2\theta^2}{2!}A^2 + \frac{i^3\theta^3 A}{3!} + \frac{i^4\theta^4 A^2}{4!} + \frac{i^5\theta^5 A}{5!} + \ldots =$$

$$= I + \frac{i\theta A}{1!} - \frac{\theta^2}{2!}A^2 - \frac{i\theta^3 A}{3!} + \frac{\theta^4 A^2}{4!} + \frac{i\theta^5 A}{5!} + \ldots =$$

$$= I + iA\left[\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots\right] + A^2\left[-\frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right] =$$

$$= I + iA\left[\frac{\theta}{1!} - \frac{\theta^3}{3!} + \ldots\right] + A^2\left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right] - A^2 =$$

$$= I + iA\sin\theta + A^2\cos\theta - A^2 = \boxed{I + iA\sin\theta + A^2(\cos\theta - 1)}$$

Разложить $|\psi\rangle = \begin{Vmatrix} 1 \\ 0 \\ 0 \end{Vmatrix}$ в базисе с.в. A

$$\vec{\psi}_0 = \frac{1}{\sqrt{2}} \begin{Vmatrix} 1 \\ 0 \\ 1 \end{Vmatrix}; \quad \vec{\psi}_1 = \frac{1}{\sqrt{2}} \begin{Vmatrix} -i/\sqrt{2} \\ 1 \\ i/\sqrt{2} \end{Vmatrix}; \quad \vec{\psi}_{-1} = \frac{1}{\sqrt{2}} \begin{Vmatrix} i/\sqrt{2} \\ 1 \\ -i/\sqrt{2} \end{Vmatrix}$$

$$|\vec{\psi}\rangle = a|\vec{\psi}_0\rangle + b|\vec{\psi}_1\rangle + c|\vec{\psi}_{-1}\rangle$$

$$\begin{cases} 1 = a \cdot \frac{1}{\sqrt{2}} - b\frac{i}{2} + c\frac{i}{2} \\ 0 = a \cdot 0 + b \cdot \frac{1}{\sqrt{2}} + c \cdot \frac{1}{\sqrt{2}} \Rightarrow b = -c \\ 0 = a \cdot \frac{1}{\sqrt{2}} + b\frac{i}{2} - c\frac{i}{2} \end{cases}$$

$$\begin{cases} 0 = \frac{a}{\sqrt{2}} - c\frac{i}{2} - c\frac{i}{2} \Rightarrow ic = \frac{a}{\sqrt{2}} \Rightarrow -c = \frac{ai}{\sqrt{2}}, \ c = \frac{-ai}{\sqrt{2}} \\ 1 = \frac{a}{\sqrt{2}} + c\frac{i}{2} + c\frac{i}{2} \Rightarrow 1 = \frac{a}{\sqrt{2}} + ic = \frac{2a}{\sqrt{2}} \Rightarrow a = \frac{\sqrt{2}}{2} \end{cases}$$

$$c = \frac{-ai}{\sqrt{2}} = \frac{-\sqrt{2}}{\sqrt{2}}\frac{i}{2} = -\frac{i}{2}, \quad b = -c = \frac{i}{2}$$

Ответ: $$\begin{Vmatrix} 1 \\ 0 \\ 0 \end{Vmatrix} = \frac{\sqrt{2}}{2} \cdot \frac{1}{\sqrt{2}} \begin{Vmatrix} 1 \\ 0 \\ 1 \end{Vmatrix} + \frac{i}{2} \cdot \frac{1}{\sqrt{2}} \begin{Vmatrix} -i/\sqrt{2} \\ 1 \\ i/\sqrt{2} \end{Vmatrix} - \frac{i}{2} \cdot \frac{1}{\sqrt{2}} \begin{Vmatrix} i/\sqrt{2} \\ 1 \\ -i/\sqrt{2} \end{Vmatrix}$$

Задание 3. $\hat{T}_a\psi(x) = \psi(x+a)$

1. Является ли $\hat{T}_a$ эрмитовым?

$$\int_{-\infty}^{+\infty}\psi^*(x)\,\hat{T}_a\varphi(x)\,dx = \int_{-\infty}^{+\infty}\psi^*(x)\,\varphi(x+a)\,dx = \left|\begin{matrix} x+a=t,\ dx=dt \\ x=t-a \end{matrix}\right| =$$

$$= \int_{-\infty}^{+\infty}\psi^*(t-a)\,\varphi(t)\,dt \equiv \int_{-\infty}^{+\infty}\psi^*(x-a)\,\varphi(x)\,dx = $$

$$\ne \int_{-\infty}^{+\infty}\psi^*(x+a)\,\varphi(x)\,dx \ !$$

$$= \int_{-\infty}^{+\infty}\left(\hat{T}_a^{+}\psi(x)\right)^*\varphi(x)\,dx$$

$$\hat{T}_a^{+}\psi(x) = \psi(x-a) \Rightarrow \hat{T}_a^{+} = \hat{T}_{-a},$$ оператор не эрмитов, но унитарный.

2. Найти $\hat{T}_a^{+} = \hat{T}_{-a}$

3. $\hat{T}_a^{+}\hat{T}_a\,\psi(x) = \hat{T}_a^{+}\psi(x+a) = \hat{T}_{-a}\psi(x+a) = \psi(x)$,

$\cancel{\hat{T}_a^{+}\hat{T}_a}$ $\hat{T}_a^{+}\hat{T}_a = id = I$

4. Ядро оператора $T(x,x')$.

$\cancel{T|\psi(x)\rangle = \int K(x,x')\varphi(x)\,d\mu(x)}$

$$\hat{T}_a|\psi(x)\rangle = \int \hat{T}_a(x,x')\,\psi(x')\,dx' = \psi(x+a)$$

$\cancel{T_a(x,x') = \delta(x'-(x+a))\frac{d}{dx'}}$?

$$\underbrace{\hat{T}_a(x,x') = \delta(x-x'+a),}_{\text{ядро}}\ \frac{d}{dx}\ ?$$

Задание 4. $\hat{S} = \alpha\left(x^2\frac{d}{dx} - \frac{d}{dx}x^2\right)$

1) $\hat{S}^+$

$$\boxed{\int\varphi^*(x)\,\hat{S}^+\psi(x)\,dx = \int\psi(x)\,\hat{S}^*\varphi^*(x)\,dx}$$

$$\boxed{\langle\hat{S}\psi|\varphi\rangle = \langle\psi|\hat{S}^+\varphi\rangle}$$

$$\langle\varphi|\hat{S}\psi\rangle = \langle\hat{S}^+\varphi|\psi\rangle$$

$$\langle\varphi|\hat{S}\psi\rangle = \int\varphi^*(x)\,\alpha\left(x^2\frac{d}{dx} - \frac{d}{dx}x^2\right)\psi(x)\,dx =$$

$$= \alpha\int\varphi^*(x)\left(x^2\frac{d[\psi(x)]}{dx} - \frac{d[x^2\psi(x)]}{dx}\right)dx = \qquad x^2\frac{d[\psi(x)]}{dx} + \psi(x)\cdot 2x\,dx$$

$$= \alpha\int\varphi^*(x)\,x^2\frac{d[\psi(x)]}{dx}dx - \alpha\int\varphi^*(x)\frac{d[x^2\psi(x)]}{dx}dx =$$

$$= \underbrace{\alpha\int\varphi^*(x)\,x^2\frac{d[\psi(x)]}{dx}dx - \alpha\int\varphi^*(x)\,x^2\frac{d[\psi(x)]}{dx}dx}_{\text{сокращаются взаимно}} \ominus$$

$$\ominus\,\alpha\int\varphi^*(x)\cdot\psi(x)\cdot 2x\,dx = -\alpha\int\varphi^*(x)\cdot 2x\,\psi(x)\,dx =$$

$$= \int\left[\alpha^*\cdot 2x^*\cdot\varphi(x)\right]^*\psi(x)\,dx$$

$$\hat{S}^+|\varphi(x)\rangle = -\alpha\cdot 2x\cdot\varphi(x) \Rightarrow \hat{S}^+ = -2\alpha x,$$ ~~верно $\forall\alpha$~~

Эрмитов оператор (нестрого — самосопряжённый), если $\hat{S}^* = \hat{S}$, т.е. $-\alpha x = -\alpha^* x^*$, т.е. $\alpha = \frac{\alpha^* x^*}{x}$

если $x \in \mathbb{R}$, то $\alpha = \alpha^*$ и $\alpha \in \mathbb{R}$

т.е. $\exists\,\alpha: \hat{S}^+ = \hat{S}$

Найти ядро оператора $S(x,x')$

$$\hat{S}\,\psi(x) = \underbrace{-2\alpha x\,\psi(x)}_{\psi(x')} = \int \hat{S}(x,x')\,\psi(x')\,dx' =$$

$$= \underline{\int (-2\alpha x)\,\delta(-x+x')\,\psi(x')\,dx'}$$

По сути, для <u>не</u> равенства нулю x должно быть равно x'

интегральное ядро

По определению, $\hat{A}[\psi(x)] = \int K(x,x')\,\varphi(x')\,dx'$

Тогда ядро: $(-2\alpha x)\,\delta(-x+x')\,\frac{d}{dx'}$?

Задание 3. $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$B \cdot B^{-1} = \frac{1}{ad-bc} \begin{Vmatrix} a & b \\ c & d \end{Vmatrix} \cdot \begin{Vmatrix} d & -b \\ -c & a \end{Vmatrix} = \begin{Vmatrix} ad-bc & -ab+ba \\ cd-dc & ac-bc \end{Vmatrix} \cdot \frac{1}{ad-bc} = I$$

$$O(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$O^{+} = (O^{T})^{*} = O^{T} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$O(\theta) \cdot O^{-1}(\theta) = I \ : \ \begin{pmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

~~Итак~~ Отсюда (сразу) видно, что $a = \cos\theta$, $c = -\sin\theta$ ($т.к.\ \cos^2\theta + \sin^2\theta = 1$), $b = +\sin\theta$, $d = \cos\theta$

т.е. $O^{-1} = \begin{pmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

Задание 6. $\langle \varphi | \hat{A}\psi \rangle = \langle \hat{A}^{+}\varphi | \psi \rangle$

Доказать: $(\hat{A}\hat{B})^{+} = \hat{B}^{+}\hat{A}^{+}$

Δ 1) $\langle \varphi | \hat{A} \hat{B}\psi \rangle = \langle \underbrace{\hat{A}^{+}\varphi}_{\varphi'} | \hat{B}\psi \rangle = \langle \hat{B}^{+}\hat{A}^{+}\varphi | \psi \rangle$

ψ' - фиксируем φ' - фиксируем

2) $\langle \varphi | \underbrace{\hat{A}\hat{B}}_{\hat{G}} \psi \rangle = \langle \underbrace{(\hat{A}\hat{B})^{+}}_{G^{+}} \varphi | \psi \rangle$ $\Rightarrow (\hat{A}\hat{B})^{+} = \hat{B}^{+}\hat{A}^{+}$. $\square$

Задание 7. $\hat{A}$ - пол. опр., если $\forall |\psi\rangle \mapsto \langle \psi | \hat{A}\psi \rangle \geq 0$.

Доказать: $\forall \hat{C}$ / лин. оп

$\hat{C}^{+}\hat{C}$ - пол. опр.

Δ $(\hat{C}\hat{C}^{+})^{+} = (\hat{C}^{+})^{+}\hat{C}^{+} = \hat{C}\hat{C}^{+}$

Если $\varphi \in V$ - с.в. $\hat{C}\hat{C}^{+}$, $\lambda \neq 0$, $\lambda \in \mathbb{R}$ - с.з., соотв v,

то $\lambda = \dfrac{\langle \lambda\varphi | \varphi \rangle}{\langle \varphi | \varphi \rangle} = \dfrac{\langle \hat{C}\hat{C}^{+}\varphi | \varphi \rangle}{\langle \varphi | \varphi \rangle} = \dfrac{\langle \hat{C}^{+}\varphi | \hat{C}^{+}\varphi \rangle}{\langle \varphi | \varphi \rangle} = \dfrac{\| \hat{C}^{+}\varphi \|^2}{\| \varphi \|^2} > 0$ $\square$

Задание 8. $A(x,x') = \dfrac{1}{(x-i)(x'+i)}$ $\quad \hat{A}$ на $\mathbb{R}^1$

1. с.з ; 2) с.в. в координатном описании.

$$\hat{A}\psi(x) = \int_{-\infty}^{+\infty} \frac{1}{(x-i)(x'+i)} \psi(x')\, dx' = \lambda \psi(x)$$

$$\frac{1}{x-i} \int_{-\infty}^{+\infty} \frac{1 \cdot (x'-i)}{(x'-i)(x'+i)} \psi(x')\, dx' = \lambda\ \psi(x)$$

$$\psi(x) = \frac{1}{x-i}$$, поскольку все ост. зав. от x'

Тогда $\psi(x') = \dfrac{1}{x'-i}$, Получаем:

$$\int_{-\infty}^{+\infty} \frac{\cancel{(x'-i)}}{(x'-i)(x'+i)} \cdot \frac{1}{\cancel{(x'-i)}} dx' = \lambda \Rightarrow \int_{-\infty}^{+\infty} \frac{dx'}{x'^2+1} = \operatorname{arctg} x \Big|_{-\infty}^{+\infty} = \pi$$