$\boxed{№1}$ $\psi(x) = \dfrac{A}{x^2+a^2}$. $\psi^*(x) = \psi(x)$, т.к. $Im\,\psi(x)=0$. $\mathbb{R}^1$

$$\langle\psi(x)|\psi(x)\rangle = \int_{-\infty}^{+\infty}|\psi(x)|^2dx = 1 = \int_{-\infty}^{+\infty}\psi^*(x)\,\psi(x)\,dx = \int_{-\infty}^{+\infty}\frac{A^2dx}{(x^2+a^2)^2} =$$

$\int u\,dv = \int u\,dv - \int v\,du$

$$= \left| \cancel{u = \frac{1}{x} = x^{-1}, \; du = -1\cdot x^{-2}} \right| \left| \begin{matrix} u = x^{-1}, \; du = -1\cdot x^{-2} = \dfrac{-1}{x^2} \\ dv = \dfrac{x}{(x^2+a^2)^2}, \; v = \dfrac{-1}{2(x^2+a^2)^2} \end{matrix} \right| = -\underbrace{\frac{A^2}{2x(x^2+a^2)^2}\Bigg|_{-\infty}^{+\infty}}_{=0} - \;\textcircled{=}$$

$$-A^2\int_{-\infty}^{+\infty}\frac{dx}{2x^2(x^2+a^2)} = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{A^2dx}{x^2(x^2+a^2)} = \frac{A^2}{2a^2}\cdot\underbrace{\frac{1}{x}\Bigg|_{-\infty}^{+\infty}}_{0} + \frac{A^2}{2a^3}\cdot arctg\left(\frac{x}{a}\right)\Bigg|_{-\infty}^{+\infty} \;\textcircled{=}$$

$$\left[\frac{1}{x^2(x^2+a^2)} = \frac{P}{x^2} + \frac{Qx+M}{x^2+a^2} \Rightarrow P = \frac{1}{a^2}, \; M = -\frac{1}{a^2}, \; Q = 0\right]$$

$$\textcircled{=}\; A^2\cdot\frac{\pi}{2a^3} = 1 \Rightarrow \frac{2a^3}{\pi} = A^2, \; A = \sqrt{\frac{2a^3}{\pi}}$$

Ответ: $A = \sqrt{\dfrac{2a^3}{\pi}}$

$$\varphi(x) = \frac{B}{x+ib}; \quad x \in \mathbb{R}^1$$

$$\int_{-\infty}^{+\infty} \varphi(x)\,dx = 1 = \int_{-\infty}^{+\infty} \varphi^*(x)\varphi^*(x)\,dx = 1.$$

$$\varphi(x) = \frac{B(x-ib)}{(x+ib)(x-ib)} = \frac{B(x-ib)}{x^2+b^2}; \quad \varphi^*(x) = \frac{B(x+ib)}{x^2+b^2}$$

$$\langle \varphi(x)|\varphi(x)\rangle = \int_{-\infty}^{+\infty} \varphi(x)^* \cdot \varphi^*(x)\,dx = \int_{-\infty}^{+\infty} \frac{B(x+ib)}{(x^2+b^2)} \cdot \frac{B(x \mp ib)}{(x^2+b^2)}\,dx =$$

$$= \int_{-\infty}^{+\infty} \frac{B^2}{(x^2+b^2)}\,dx = B^2 \int_{-\infty}^{+\infty} \frac{dx}{x^2+b^2} = 1; \quad \int_{-\infty}^{+\infty} \frac{dx}{x^2+b^2} = \frac{1}{B^2}$$

$$\int_{-\infty}^{+\infty} \frac{dx}{x^2+b^2} = \frac{1}{b^2} \int_{-\infty}^{+\infty} \frac{dx}{\left(\frac{x}{b}\right)^2+1} = \frac{b}{b^2} \int_{-\infty}^{+\infty} \frac{d\left(\frac{x}{b}\right)}{\left(\frac{x}{b}\right)^2+1} = \frac{1}{b}\,\mathrm{arctg}\left(\frac{x}{b}\right)\Big|_{-\infty}^{+\infty} =$$

$$= \frac{1}{b}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \frac{\pi}{b} = \frac{1}{B^2} \Rightarrow B = \sqrt{\frac{b}{\pi}}$$

Ответ: $B = \sqrt{\frac{b}{\pi}}$

$\boxed{№3}$ $\langle \varphi | \psi \rangle = ?$ $\varphi(x) = \dfrac{B}{x+ib} = \dfrac{B(x-ib)}{(x+ib)(x-ib)}$; $\varphi^*(x) = \dfrac{B(x+ib)}{(x+ib)(x-ib)} =$

$= \dfrac{B}{x-ib}$; $\psi(x) = \dfrac{A}{x^2+a^2}$; $\langle \varphi | \psi \rangle = \int\limits_{-\infty}^{+\infty} \varphi^*(x)\,\psi(x)\,dx =$

$$= \int\limits_{-\infty}^{+\infty} \frac{B}{x-ib}\cdot\frac{A}{x^2+a^2}\,dx = AB\int\limits_{-\infty}^{+\infty}\frac{dx}{(x-ib)(x^2+a^2)} \ominus \cancel{\frac{x^2+a^2}{a^2-b^2} + \left(-\frac{x}{a^2-b^2} + \frac{ib}{b^2-a^2}\right)(x-ib)}$$

$$\frac{1}{(x-ib)(x^2+a^2)} = \frac{C}{(x-ib)} + \frac{Dx+E}{(x^2+a^2)} = \frac{C(x^2+a^2)+(Dx+E)(x-ib)}{(x-ib)(x^2+a^2)}$$

$$\Rightarrow C = \frac{1}{a^2-b^2},\quad D = \frac{-1}{a^2-b^2},\quad E = \frac{ib}{b^2-a^2}$$

$$\ominus\ AB\left\{\frac{1}{a^2-b^2}\int\limits_{-\infty}^{+\infty}\frac{dx}{x-ib} - \frac{1}{a^2-b^2}\int\limits_{-\infty}^{+\infty}\cancel{\frac{x\,dx}{x^2+a^2}} + \frac{ib}{b^2-a^2}\int\limits_{-\infty}^{+\infty}\frac{dx}{x^2+a^2}\right\} \boxed{=}$$

интегр. => нечётн => = 0

$$\to AB\,\frac{ib}{b^2-a^2}\cdot\frac{1}{a}\,\mathrm{arctg}\left(\frac{x}{a}\right)\Bigg|_{-\infty}^{+\infty} = \frac{AB}{b^2-a^2}\cdot\frac{i\pi b}{a} = \left| \begin{array}{l} A = \dfrac{\sqrt{2a^3}}{\sqrt{\pi}} \\ B = \dfrac{\sqrt{b}}{\sqrt{\pi}} \end{array}\right| = \cancel{\frac{\sqrt{2}\,\sqrt{a}}{\sqrt{\pi a^3}\,\sqrt{\pi}\,(b^2-a^2)}\cdot\frac{i\pi b}{a}} =$$

$$\cancel{\frac{i b}{b^2-a^2}} = \frac{i\pi b\sqrt{2a^3}\,\sqrt{b}}{(b^2-a^2)\,a\,\sqrt{\pi}\,\sqrt{\pi}} = \frac{i\sqrt{2ab^3}}{(b^2-a^2)} = \frac{-i\sqrt{2ab^3}}{a^2-b^2}$$

$b = \sqrt{b^2}$, $a = \sqrt{a^2}$

$$\frac{i\sqrt{2ab^3}}{b^2-a^2}$$

$$\int\limits_{-\infty}^{+\infty}\frac{dx}{x-ib} = \int\limits_{-\infty}^{+\infty}\frac{x+ib}{x^2+b^2}\,dx = \int\limits_{-\infty}^{+\infty}\cancel{\frac{x\,dx}{x^2+b^2}} + ib\int\limits_{-\infty}^{+\infty}\frac{dx}{x^2+b^2} = ib\cdot\frac{1}{b}\,\mathrm{arctg}\left(\frac{x}{b}\right)\Bigg|_{-\infty}^{+\infty} =$$

0 симм. нечётн

$$= i\left[\mathrm{arctg}(+\infty) - \mathrm{arctg}(-\infty)\right] = i\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = i\pi$$

$$AB\,\frac{1}{a^2-b^2}\cdot i\pi = \frac{\sqrt{2a^3}}{\sqrt{\pi}}\cdot\frac{\sqrt{b}}{\sqrt{\pi}}\cdot\frac{i\pi}{(a^2-b^2)} = \frac{i\sqrt{2a^3b}}{(a^2-b^2)}$$

$$\boxed{=}\ \frac{i\sqrt{2a^3b}}{(a^2-b^2)} + \frac{i\sqrt{2ab^3}}{b^2-a^2} = \frac{i\sqrt{2ab}\cdot a - i\sqrt{2ab}\cdot b}{a^2-b^2} = \frac{i\sqrt{2ab}\,(a-b)}{(a-b)(a+b)} = \frac{i\sqrt{2ab}}{a+b}$$

Ответ

$\boxed{N^{\circ}4}$

$$\int_{-\infty}^{+\infty} f(x)\delta(x-x_0)\,dx = f(x_0) \quad *$$

Задание: Доказать $\delta(f(x)) = \sum_{x_i} \frac{1}{|f'(x_i)|}\delta(x-x_i)$

Домножим на произв. $g(x)$ и возьмём интеграл

$$\int_{-\infty}^{+\infty} g(x)\delta(f(x))dx = \sum_{x_i} \frac{\int_{-\infty}^{+\infty} g(x)\cdot\delta(x-x_i)\,dx}{|f'(x_i)|} \overset{\text{по св-ву }*}{=\!=} \sum_{x_i} \frac{g(x_i)}{|f'(x_i)|}$$

Это утверждение и будем доказывать

Δ Разложим $f(x)$ в ряд Тейлора в окрестности x_i:

$$f(x) = \underbrace{f(x_i)}_{0 \text{ из усл.}} + f'(x_i)(x-x_i) + O(x^2) = f'(x_i)(x-x_i) + O(x^2)$$

$$\int_{-\infty}^{+\infty} g(x)\delta(f(x))dx = \int_{-\infty}^{+\infty} g(x)\delta\left[f'(x_i)(x-x_i)\right]dx = \sum_i \int_{x_i-\varepsilon}^{x_i+\varepsilon} g(x)\delta\left[f'(x_i)(x-x_i)\right]dx =$$

$$= \left| \begin{matrix} y = x-x_i, & x = y+x_i \\ dy = dx & \end{matrix} \right| = \sum_i \int_{-\varepsilon}^{\varepsilon} g(y+x_i)\delta\left[f'(x_i)\cdot y\right]dy \ (=)$$

по св-ву δ-функции, $\delta(\alpha x) = \frac{1}{|\alpha|}\delta(x)$, откуда

$$(=)\ \sum_i \int_{-\varepsilon}^{\varepsilon} g(y+x_i)\cdot\frac{1}{|f'(x_i)|}\cdot\delta(y)\,dy = \sum_i \frac{g(x_i)}{|f'(x_i)|}$$

Или же $\sum_i \int_{x_i-\varepsilon}^{x_i+\varepsilon} g(x)\delta(f'(x_i)(x-x_i))dx = \sum_i \int_{x_i-\varepsilon}^{x_i+\varepsilon} \frac{g(x)}{|f'(x_i)|}\delta(x-x_i)\,dx$ $\square$

$\boxed{№5}$

$$\Psi(x) = \sqrt{\frac{2}{a}} \sin\frac{\pi x}{a}, \quad x \in [0,a]$$

Базисные вектора $f_1(x) = \alpha_1 e^{\frac{i\pi x}{a}}$; $f_2(x) = \alpha_2 e^{\frac{-i\pi x}{a}}$

1. $\langle f_1 | f_2 \rangle = 0$.

$$\Delta \ \langle f_1 | f_2 \rangle = \int_0^a f_1^*(x) f_2(x)\,dx = \int_0^a \alpha_1 e^{\frac{-i\pi x}{a}} \cdot \alpha_2 e^{\frac{-i\pi x}{a}}\,dx = \alpha_1\alpha_2 \int_0^a e^{\frac{-2i\pi x}{a}}\,dx \ominus$$

$$\left(f_1^*(x) = \alpha_1 e^{\frac{-i\pi x}{a}}\right)$$

$$\ominus\ \alpha_1\alpha_2 \frac{-a}{2i\pi} e^{\frac{-2i\pi x}{a}}\Big|_0^a = \alpha_1\alpha_2 \frac{-a}{2i\pi}\left[e^{-2i\pi} - e^0\right] =$$

$$= \alpha_1\alpha_2 \frac{-a}{2i\pi}\Big[\underbrace{\cos(2\pi)}_{1} + i\cancelto{0}{\sin(2\pi)} - 1\Big] = 0. \ \square$$

2. α_1, α_2 – из усл. нормировки.

Условие нормировки: $\langle f_1 | f_1 \rangle = 1$, $\langle f_2 | f_2 \rangle = 1$

$$\int_0^a \underbrace{\alpha_1 e^{\frac{-i\pi x}{a}}}_{f_1^*} \cdot \underbrace{\alpha_1 e^{\frac{i\pi x}{a}}}_{f_1}\,dx = \alpha_1^2 \int_0^a e^0\,dx = \alpha_1^2 \int_0^a dx = a\alpha_1^2 = 1 \Rightarrow \alpha_1 = \sqrt{\frac{1}{a}}$$

$$\int_0^a \underbrace{\alpha_2 e^{\frac{i\pi x}{a}}}_{f_2^*} \cdot \underbrace{\alpha_2 e^{\frac{-i\pi x}{a}}}_{f_2}\,dx = \alpha_2^2 \int_0^a dx = 1 \Rightarrow \alpha_2 = \frac{1}{\sqrt{a}} = \alpha_1$$

Ответ: $\alpha_1 = \alpha_2 = \frac{1}{\sqrt{a}}$.

5) Разл. $|\psi\rangle$ по базису $|f_{1,2}\rangle$ (найти c_1, c_2: $|\psi\rangle = c_1|f_1\rangle + c_2|f_2\rangle$

$$f_1(x) = \alpha_1 e^{i\pi x/a}, \quad f_2(x) = \alpha_2 e^{-i\pi x/a}$$

$$\psi(x) = \sqrt{\frac{2}{a}} \sin\frac{\pi x}{a}, \quad f_1^*(x) = \alpha_1 e^{-i\pi x/a}, \quad f_2^*(x) = \alpha_2 e^{i\pi x/a}$$

1) $$\langle f_1|\psi(x)\rangle = \int_0^a f_1^*(x)\cdot\psi(x)\,dx = \int_0^a \alpha_1 e^{\frac{-i\pi x}{a}}\cdot\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)dx =$$

$$= \alpha_1\sqrt{\frac{2}{a}}\int_0^a e^{\frac{-i\pi x}{a}}\cdot\sin\left(\frac{\pi x}{a}\right)dx = \alpha_1\sqrt{\frac{2}{a}}\int_0^a\left(\cos\left(\frac{-\pi x}{a}\right) + i\sin\left[\frac{-\pi x}{a}\right]\right)\cdot\sin\left(\frac{\pi x}{a}\right)dx =$$

$$= \alpha_1\sqrt{\frac{2}{a}}\int_0^a\cos\left(\frac{\pi x}{a}\right)\cdot\sin\left(\frac{\pi x}{a}\right)dx - i\alpha_1\sqrt{\frac{2}{a}}\int_0^a \sin^2\left[\frac{\pi x}{a}\right] dx =$$

$$= \alpha_1\sqrt{\frac{2}{a}}\int_0^a\sin\left(\frac{\pi x}{a}\right)\cdot d\sin\left[\frac{\pi x}{a}\right]\cdot\frac{a}{\pi} - i\alpha_1\sqrt{\frac{2}{a}}\int \ldots \frac{a}{\pi} =$$

$$= \alpha_1\sqrt{\frac{2}{a}}\cdot\frac{a}{\pi}\,\cancel{\sin^2\left(\frac{\pi x}{a}\right)\cdot\frac{1}{2}\Big|_0^a}^{\,0} - i\alpha_1\sqrt{\frac{2}{a}}\,\underbrace{\frac{a}{\pi}\int_0^\pi \sin^2(u)\,du}_{=\frac{a}{2}} =$$

$$\int\sin^2 u\,du = \int\frac{1}{2}du - \int\frac{\cos 2u}{2}du = \frac{u}{2}\Big| - \cancel{\frac{\sin 2u}{4}}\Big|^{0}$$

$$= -i\alpha_1\sqrt{\frac{2}{a}}\cdot\frac{a}{2} = -i\alpha_1\frac{\sqrt{2}\,a}{\sqrt{a}\,2} = -i\alpha_1\frac{\sqrt{a}}{\sqrt{2}} = \frac{\alpha_1\sqrt{a}}{\sqrt{2}\,i} = \left|\alpha_1 = \frac{1}{\sqrt{a}}\right| = \frac{1}{\sqrt{2}\,i} = \frac{-i}{\sqrt{2}}$$

$$\frac{1}{i} = \frac{-i\cdot i}{i} = -i$$

2) $$\langle f_2(x)|\psi(x)\rangle = \int_0^a f_2^*(x)\,\psi(x)\,dx = \int_0^a \underbrace{\alpha_2}_{\frac{1}{\sqrt{a}}} e^{i\pi x/a}\cdot\sqrt{\frac{2}{a}}\sin\frac{\pi x}{a}\,dx =$$

аналогично решению выше

$$= \frac{\sqrt{2}}{a}\int_0^a\left[\cos\left(\frac{\pi x}{a}\right) + i\sin\left(\frac{\pi x}{a}\right)\right]\cdot\sin\left(\frac{\pi x}{a}\right)dx = 0 + \frac{i}{\sqrt{2}}$$

Ответ: $$|\psi\rangle = \underbrace{\frac{-i}{\sqrt{2}}}_{c_1}|f_1\rangle + \underbrace{\frac{i}{\sqrt{2}}}_{c_2}|f_2\rangle$$