Power and energy characteristics of the internal wave beams in a continuously stratified fluid.

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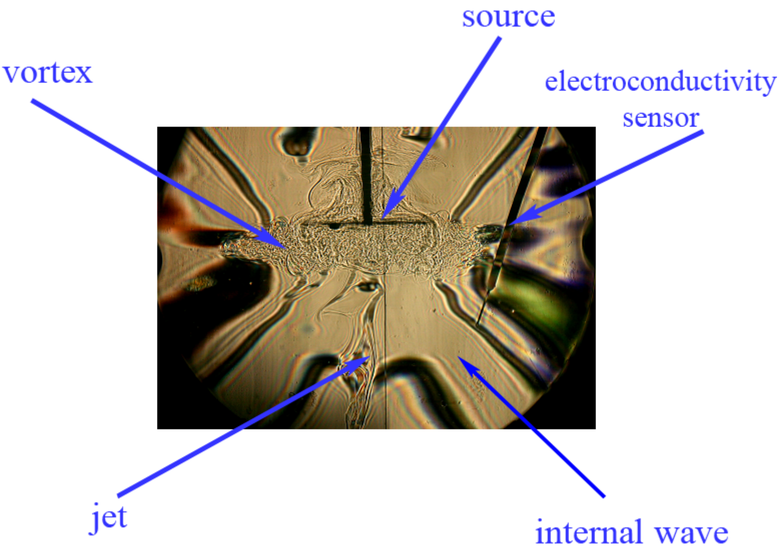
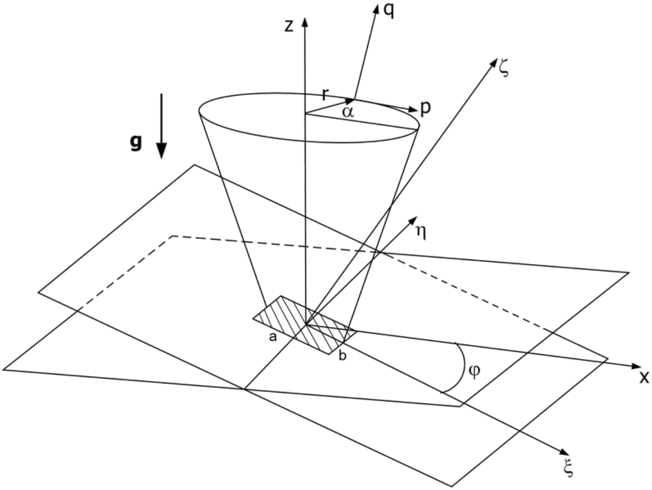
History of the problem.

1. J.V.S. Rayleigh
2. J. Lighthill
3. Yu.D. Chashechkin
4. V.A. Gorodtsov
5. T.N. Stevenson
6. D.G. Hurley, G.J. Keady
7. B.R. Sutherland
8. Yu.V. Kistovich
9. A.V. Kistovich
10. B. Voisin

Color shadow image of disk oscillation.



Formulation of the problem.



Governing equations and boundary conditions.

Governing equations

$$\frac{\partial\rho}{\partial t} + \mathbf{v}\nabla\rho = 0, \quad \mathrm{div}\ \mathbf{v} = 0$$

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}, \nabla)\,\mathbf{v} = \frac{1}{\rho}\nabla p + \nu\Delta v + \rho\mathbf{g}$$

$$\frac{\partial S}{\partial t} = \kappa_S \Delta S + \frac{v_z}{\Lambda}$$

Boundary conditions

$$\mathbf{v}|_{\Gamma} = \mathbf{u}_0 e^{-i\omega t}, \quad \kappa_S \left.\frac{\partial S}{\partial n}\right|_{\Gamma} = 0$$

$$v \to 0, \quad \rho \to \rho_0, \quad \partial P/\partial z \to \rho_0\,(z)\,g, r \to \infty$$

Toroidal-poloidar decomposition (TPD)

$$\mathbf{v} = \nabla \times \mathbf{e}_z\Phi + \nabla \times (\nabla \times \mathbf{e}_z\Psi)$$

System equation for scalar function Φ and Ψ

$$\left(\left(\frac{\partial}{\partial t} - D\Delta\right)\left(\frac{\partial}{\partial t} - \nu\Delta\right)\Delta + N^2\Delta_\perp\right)\ \Phi = 0$$

$$\left(\frac{\partial}{\partial t} - \nu\Delta\right)\ \Psi = 0$$

$$\left(\left(\frac{\partial}{\partial t} - D\Delta\right)\left(\frac{\partial}{\partial t} - \nu\Delta\right)\Delta + N^2\Delta_\perp\right)\ S = 0$$

where

$$\Delta_\perp = \partial_x^2 + \partial_y^2 \quad \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2 \quad N^2 = \sqrt{\frac{g}{\Lambda}}$$

Solution for Φ and Ψ

$$\Phi = e^{-i\omega t} \sum_{j=1}^{3} \int_{-\infty}^{+\infty} A_j\left(k_\xi, k_\eta\right) E_j \; dk_\xi dk_\eta,$$

$$S = -\frac{\rho_0}{\Lambda} e^{-i\omega t} \sum_{j=1}^{3} \int_{-\infty}^{+\infty} \frac{\left(k_\xi \cos\varphi - k_j \sin\varphi\right)^2 + {k_\eta}^2}{i\omega - \kappa_S \, k^2} A_j\left(k_\xi, k_\eta\right) E_j dk_\xi dk_\eta,$$

$$\Psi = e^{-i\omega t} \int_{-\infty}^{+\infty} B\left(k_\xi, k_\eta\right) \; E_4 dk_\xi dk_\eta$$

where

$$E_j = \exp\left(i k_j \zeta + i k_\xi \xi + i k_\eta \eta\right), \quad k = \sqrt{k_j^2 + k_\xi^2 + k_\eta^2}$$

The viscous stratified fluid taking into account effects of diffusion.

Dispersion equation

$$\left(\nu\kappa_S\tilde{k}^6 - i\omega\left(\nu+\kappa_S\right)\tilde{k}^4 - \omega^2\tilde{k}^2 + N^2k_\perp^2\right)\left(\tilde{k}^2 + \frac{\omega}{i\nu}\right) = 0$$

$$\tilde{k}^2 = 2k_\zeta^2 + k_\perp^2, \quad k_\perp^2 = k_\xi^2 + k_\eta^2$$

The viscous stratified fluid taking into account effects of diffusion.

Regular solution

$$k_1 = \frac{k_\xi \sin\varphi\cos\varphi \pm \kappa\cos\theta}{\mu_\theta} \pm \delta_N^2 (1+\varepsilon) \frac{i\ \tan\theta\mu_\theta^4}{2\kappa\mu^4} + \ldots$$

$$\mu = \sin^2\varphi - \sin^2\theta, \quad \mu_\theta = (k_\xi \sin\varphi\cos\varphi \pm \kappa\cos\varphi),$$

$$\varepsilon = Sc^{-1} = \frac{\kappa_S}{\nu}, \quad \delta_N = \sqrt{\frac{\nu}{N}}$$

Singular solutions

$$k_{2,\,3} \approx \sqrt{\frac{i\omega(\varepsilon + 1 \pm \lambda_{\nu\kappa})}{\varepsilon}}, \quad \lambda_{\nu\kappa} = \frac{2}{\sin\theta}\sqrt{(1+\varepsilon)^2 - \frac{4\varepsilon\mu}{\sin^2\theta}}$$

$$k_4 = \sqrt{\frac{2i}{\delta_\nu^2} - k^2}, \quad \delta_\nu = \delta_N\sqrt{\frac{2}{\sin\theta}}, \quad \delta_\varphi = \delta_N\sqrt{\frac{2\sin\theta}{|\mu|}}, \quad \delta_\kappa = \delta_N\sqrt{\frac{2\varepsilon}{\sin\theta}}$$

The viscous stratified fluid taking into account effects of diffusion. Vertical component of velocity.

$$v_\zeta \approx \int\limits_{-\infty}^{+\infty} A_1 \left(k_\eta^2 \sin\varphi - k_\xi \beta_1\right) e^{ik_1\zeta + ik_\xi\xi + ik_\eta\eta} dk_\xi dk_\eta -$$

$$-ie^{\frac{i-1}{\delta_\nu}\zeta} \sin\varphi \int\limits_{-\infty}^{+\infty} Be^{ik_\xi\xi + ik_\eta\eta} \, dk_\xi dk_\eta -$$

$$-\frac{i+1}{\delta_\varphi} e^{\frac{i-1}{\delta_\varphi}\zeta} \cos\varphi \int\limits_{-\infty}^{+\infty} A_2 k_\xi e^{ik_\xi\xi + ik_\eta\eta} dk_\xi dk_\eta -$$

$$-\frac{1+i}{\delta_\kappa} \sqrt{\frac{\sin\theta}{2}} e^{-\frac{\sqrt{\sin\theta}}{\delta_\kappa\sqrt{2}}\zeta + \frac{i\zeta}{\delta_\kappa\sqrt{2}}} \cos\varphi \int\limits_{-\infty}^{+\infty} A_3 k_\xi e^{ik_\xi\xi + ik_\eta\eta} dk_\xi dk_\eta$$

Calculation of forces acting on the radiating surface

$$\tilde{f}_i = P\delta_{ik} n_k + \mu \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) n_k$$

$$P = -\rho_0 \sum_{j=1}^{3} \int\limits_{-\infty}^{+\infty} A_j \left[\omega \left(k_\xi \cos\varphi + k_j \sin\varphi \right) + \nu k^2 \right] E_j \; dk_\xi dk_\eta$$

The results of calculations of the pressure for different types of the sources. Friction plate. 3D

$$P_1^w \approx \frac{\rho_0 \omega \; u_0 ab \delta_N}{2\; \pi^{3/2} \sqrt{|\mu|}} sin\theta \left[cos\varphi sin\theta sin \left(\frac{\pi}{4} - \alpha \right) - cos\theta sin\varphi \right] \; G \left(\frac{1}{2}, \; p, \; q \right.$$

$$P_1^b \approx -i\rho_0 \omega u_0 \delta_\varphi exp \left(\frac{i-1}{\delta_\varphi} + \frac{i\pi}{4} \right)$$

$$G\left(n, \; p, q\right) = \frac{1}{\sqrt{psin\theta + qcos\theta}} \int\limits_0^{+\infty} dk_p \; k_p^n exp \left(ik_p p - \frac{k_p^3 \delta_N^2 q}{2cos\theta} \right).$$

$$|P_1^w|_{max} = \frac{\rho_0 \omega \; u_0 absin\theta \; sin\left(\theta - \varphi\right)}{6\sqrt{2} \; q \; \pi \sqrt{|\mu|}}$$

The results of calculations of the pressure for different types of the sources. Piston plate. 3D

$$P_2^w \approx -\frac{\rho_0\omega\ u_0 ab}{\pi} e^{-\frac{i\pi}{4}} \sqrt{\frac{sin\theta}{\pi}}\ G\left(-\frac{1}{2},\ p,\ q\right)$$

$$P_2^b \approx \frac{i+1}{\sqrt{|\mu|}} i\rho_0\omega u_0\delta_N \frac{\sqrt{sin\theta}ctg\varphi}{\pi^2} W_2$$

$$W_2 = \int\limits_{-\infty}^{+\infty} \frac{k_\eta^2 cos\varphi + k_1^{(0)}\sigma}{k_\xi k_\eta} W_\delta\left(k_\xi,\ k_\eta\right)\ dk_\xi dk_\eta$$

$$|P_2^w|_{max} = \frac{\rho_0\omega\ u_0 ab}{3\ \pi\sqrt{\pi}} \left(\frac{2tan^2\theta\ sin\theta}{q^4\delta_N^2}\right)^{1/6} \Gamma\left(\frac{1}{6}\right.$$

The results of calculations of the pressure for different types of the sources. Friction disk. 3D

$$P_3^w \approx \frac{\rho_0\omega\ u_0 R^2 \delta_N}{2\sqrt{\pi\,|\mu|}} sin\theta\ \left[cos\varphi sin\theta sin\left(\frac{\pi}{4}-\alpha\right)\ -cos\theta sin\varphi\right]\ G\left(\frac{1}{2},\ p,\right.$$

$$P_3^b \approx \rho_0\omega u_0 R\ \frac{1-i}{\sqrt{|\mu|}}\sqrt{sin\theta}\ exp\left(\frac{i-1}{\delta_\varphi}\right)$$

$$|P_3^w|_{max} = \frac{\rho_0\omega\ u_0 R^2 sin\theta\ sin\left(\theta-\varphi\right)}{6\sqrt{2}\ q\sqrt{|\mu|}}$$

The results of calculations of the pressure for different types of the sources. Piston disk. 3D

$$P_4^w \approx -\frac{\rho_0\omega\ u_0 R^2}{\pi} e^{-i\pi/4}\sqrt{sin\theta}\, G\left(-\frac{1}{2},\ p,\ q\right)$$

$$P_4^b \approx \frac{i+1}{\sqrt{|\mu|}} i\rho_0\omega u_0\delta_N \frac{\sqrt{sin\theta}ctg\varphi}{\pi^2} W_2$$

$$|P_4^w|_{max} = \frac{\rho_0\omega\ u_0 R^2}{3\sqrt{\pi}}\left(\frac{2tan^2\theta\ sin\theta}{q^4\delta_N^2}\right)^{1/6}\Gamma\left(\frac{1}{6}\right)$$

The results of calculations of the pressure for different types of the sources. Composite. 3D

$$P_5^w \approx -\frac{\rho_0\omega\ u_0 ab}{2\pi^{3/2}} e^{i\pi/4} sin^{3/2}\theta\ cos\left(\frac{\pi}{4}-\alpha\right)\ G\left(\frac{1}{2},\ p,\ q\right)$$

$$P_5^b \approx -i\rho_0\omega u_0\delta_N \frac{\sqrt{sin\theta} ctg\varphi}{\pi^2} exp\left(\frac{i-1}{\delta_\varphi}\right) W_4$$

$$W_4 = \int\limits_{-\infty}^{+\infty} \frac{k_\eta^2 sin\varphi + k_1^{(0)}\sigma}{k_\xi k_\eta}\ V_\delta\ dk_\xi dk_\eta$$

$$V_\delta\left(k_\xi,\ k_\eta\right) = \frac{sin\frac{k_\xi a}{2} sin\frac{k_\eta b}{2}}{k_\eta^2 cos\varphi - k_\xi\sigma} exp\left\{ik_\xi\xi + ik_\eta\eta + \frac{i-1}{4}\delta_\nu\zeta\left(k_\xi^2 + k_\eta^2\right)\right\}$$

$$|P_5^w|_{max} = \frac{\sqrt{2}\ \rho_0\omega\ u_0 ab^2 sin^{3/2}\theta}{12\ q\delta_N^2}$$

The components of the viscous stress tensor and virtical component of force in the wave beams

Components of the viscous stress tensor

$$\sigma_{xx} = -p_i^w - 2i\rho_0\nu A_i^w cos^2\alpha sin2\theta cos\alpha Q_i\left(n,\ p,\ q\right)$$

$$\sigma_{yy} = -p_i^w + 4i\rho_0\nu A_i^w cos\alpha sin\alpha sin\theta cos^2\theta Q_i\left(n,\ p,\ q\right)$$

$$\sigma_{zz} = -p_i^w - 4i\rho_0\nu A_i^w cos\theta cos^2\theta Q_i\left(n,\ p,\ q\right)$$

$$Q_i\left(n,\ p,\ q\right) = f\left(\alpha,\ \theta,\ \varphi\right)\ G\left(n,\ p, q\right),$$

Vertical component of the force

$$f = cos\varphi sin\theta - sin\left(\frac{\pi}{4} - \alpha\right)\ sin\varphi cos\theta$$

$$F_z \approx i\rho_0\omega sin\theta sin^2\varphi sin\alpha f\left(\alpha,\ \theta,\ \varphi\right) A_m^w \int\limits_{-a/2}^{a/2} dp \int\limits_{-b/2}^{b/2} dq\, G\left(n+1,\ p, q\right)$$

Forces on the axis $h\left(0,q\right)$. Far field. Plate

Type of sources	A_j^w	f	n
Friction plate	$\frac{i}{4}\frac{u_0\, ab\delta_N}{\sqrt{\pi^3\lvert\mu\rvert}}$	f	$3/2$
Piston plate	$\frac{1-i}{(2\pi)^{3/2}}\frac{u_0 ab}{\sqrt{sin\theta}}$	1	$1/2$
Composite	$\frac{1-i}{(2\pi)^{3/2}}\frac{u_0\pi R^2}{\sqrt{2sin\theta}}$	$\cos\left(\frac{\pi}{4}-\alpha\right)$	$1/2$

Forces on the axis $h\,(0,q)$. Far field. Disk

Type of sources	A_j^w	f	n
Friction disk	$\frac{1-i}{(2\pi)^{3/2}}\frac{u_0\ ab^2\sqrt{sin\theta}}{2}$	f	$3/2$
Piston disk	$\frac{i}{4}\quad\frac{u_0\pi R^2}{\sqrt{\pi^3\|\mu\|}}\delta_N$	1	$3/2$

Conclusions:

1. In general case there are the two type of flow: regular solution (internal waves) and three types of singular components of flow (boundary layers). Two of them have no analogue in a homogeneous fluid, their thickness is defined by dissipative factors;
2. Near the source viscosity and diffusion is basic factors;
3. The obtained results show that it is necessary to consider influence of dissipative factors (viscosity, stratification, diffusion).