Calculation of periodic flows in a continuously stratified fluid

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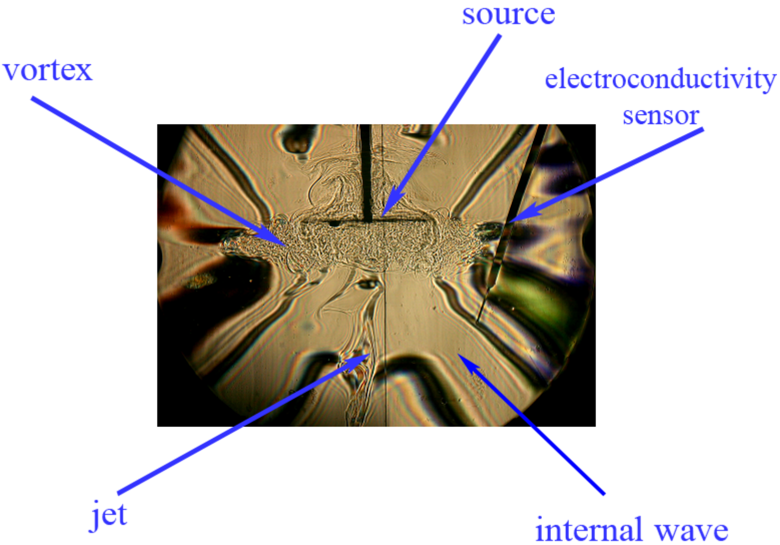
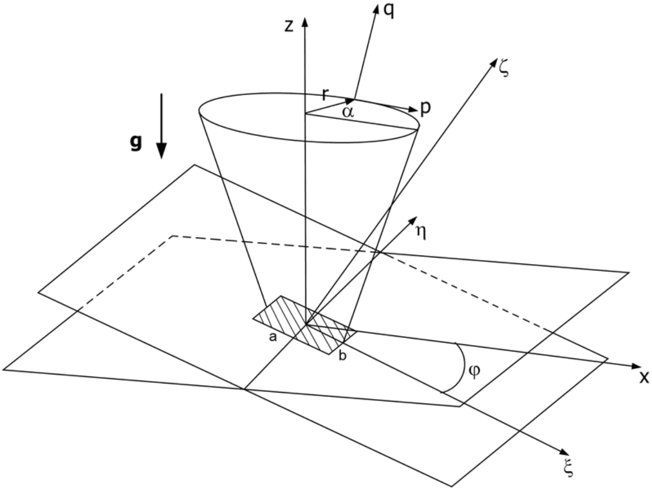
History of the problem.

1. J.V.S. Rayleigh
2. J. Lighthill
3. Yu.D. Chashechkin
4. V.A. Gorodtsov
5. T.N. Stevenson
6. D.G. Hurley, G.J. Keady
7. B.R. Sutherland
8. Yu.V. Kistovich
9. A.V. Kistovich
10. B. Voisin

Color shadow image of disk oscillation.



Formulation of the problem.



Governing equations

$$\frac{\partial \rho}{\partial t} + \mathbf{v}\nabla\rho = 0, \quad \mathrm{div}\ \mathbf{v} = 0$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}, \nabla)\,\mathbf{v} = \frac{1}{\rho}\nabla p + \nu\Delta v + \rho\mathbf{g}$$

$$\frac{\partial S}{\partial t} = \kappa_S \Delta S + \frac{v_z}{\Lambda}$$

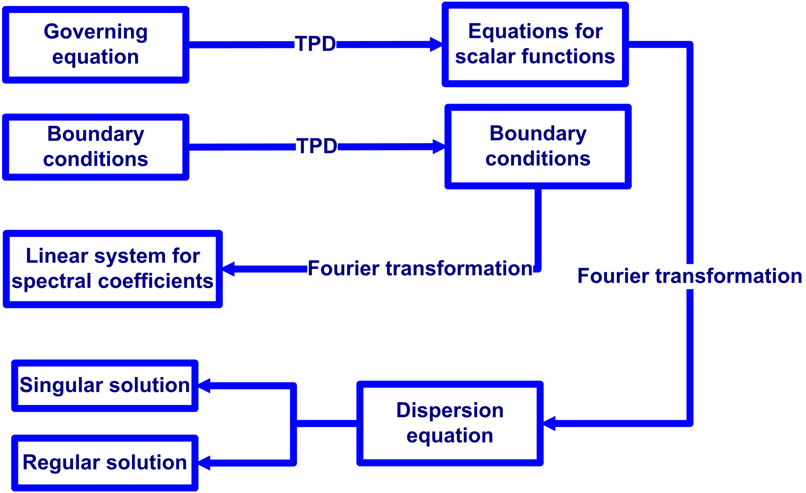
Boundary conditions

$$\mathbf{v}\big|_{\Gamma} = \mathbf{u}_0 e^{-i\omega t}, \quad \kappa_S \left.\frac{\partial S}{\partial n}\right|_{\Gamma} = 0$$

$$v \to 0, \quad \rho \to \rho_0, \quad \partial P/\partial z \to \rho_0\,(z)\,g, r \to \infty$$

Toroidal-poloidar decomposition (TPD)

$$\mathbf{v} = \nabla \times \mathbf{e}_z\Phi + \nabla \times (\nabla \times \mathbf{e}_z\Psi)$$


Governing equation
TPD
Equations for scalar functions
Boundary conditions
TPD
Boundary conditions
Linear system for spectral coefficients
Fourier transformation
Fourier transformation
Singular solution
Regular solution
Dispersion equation

Regular solution

$$k_1 = \frac{k_\xi \sin\varphi\cos\varphi \pm \kappa\cos\theta}{\mu_\theta} \pm \delta_N^2 \left(1+\varepsilon\right) \frac{i\ \tan\theta \mu_\theta^4}{2\kappa\mu^4} + \ldots$$

$$\mu = \sin^2\varphi - \sin^2\theta, \quad \mu_\theta = \left(k_\xi \sin\varphi\cos\varphi \pm \kappa\cos\varphi\right),$$

$$\varepsilon = Sc^{-1} = \frac{\kappa_S}{\nu}, \quad \delta_N = \sqrt{\frac{\nu}{N}}$$

Singular solutions

$$k_{2,\,3} \approx \sqrt{\frac{i\omega\left(\varepsilon + 1 \pm \lambda_{\nu\kappa}\right)}{\varepsilon}}, \quad \lambda_{\nu\kappa} = \frac{2}{\sin\theta}\sqrt{\left(1+\varepsilon\right)^2 - \frac{4\varepsilon\mu}{\sin^2\theta}}$$

$$k_4 = \sqrt{\frac{2i}{\delta_\nu^2} - k^2}, \quad \delta_\nu = \delta_N\sqrt{\frac{2}{\sin\theta}}, \quad \delta_\varphi = \delta_N\sqrt{\frac{2\sin\theta}{|\mu|}}, \quad \delta_\kappa = \delta_N\sqrt{\frac{2\varepsilon}{\sin\theta}}$$

The viscous stratified fluid taking into account effects of diffusion. Vertical component of velocity.

$$v_\zeta \approx \int\limits_{-\infty}^{+\infty} A_1 \left(k_\eta^2 \sin\varphi - k_\xi \beta_1\right) e^{ik_1\zeta + ik_\xi \xi + ik_\eta \eta} dk_\xi dk_\eta -$$

$$-ie^{\frac{i-1}{\delta_\nu}\zeta} \sin\varphi \int\limits_{-\infty}^{+\infty} Be^{ik_\xi \xi + ik_\eta \eta}\, dk_\xi dk_\eta -$$

$$-\frac{i+1}{\delta_\varphi} e^{\frac{i-1}{\delta_\varphi}\zeta} \cos\varphi \int\limits_{-\infty}^{+\infty} A_2 k_\xi e^{ik_\xi \xi + ik_\eta \eta} dk_\xi dk_\eta -$$

$$-\frac{1+i}{\delta_\kappa}\sqrt{\frac{\sin\theta}{2}}\, e^{-\frac{\sqrt{\sin\theta}}{\delta_\kappa\sqrt{2}}\zeta + \frac{i\zeta}{\delta_\kappa\sqrt{2}}} \cos\varphi \int\limits_{-\infty}^{+\infty} A_3 k_\xi e^{ik_\xi \xi + ik_\eta \eta} dk_\xi dk_\eta$$

The viscous stratified fluid. Friction source. Rectangle. 3D. $\varphi = 0$.

$$v_\xi = \int\limits_{-\infty}^{+\infty} k_\eta^2 L_3 \, dk_\xi dk_\eta + \int\limits_{-\infty}^{+\infty} k_\xi^2 L_1 \, dk_\xi dk_\eta,$$

$$v_\eta = \int\limits_{-\infty}^{+\infty} L_3 \, dk_\xi dk_\eta + \int\limits_{-\infty}^{+\infty} k_\xi k_\eta L_1 \, dk_\xi dk_\eta, \quad v_\zeta = \int\limits_{-\infty}^{+\infty} k_\eta k_\perp^2 L_2 \, dk_\xi dk_\eta$$

$$L_m = u_0 Q \frac{k_1^{2-m} \exp\left(ik_1\zeta\right) + k_2^{2-m} \exp\left(ik_2\zeta\right)}{\left(k_\eta^2 - k_\xi^2\right)\left(k_2 - k_1\right)}, \quad m = 1, 2,$$

$$L_3 = \frac{u_0 Q}{k_\eta^2 - k_\xi^2} \exp\left(-\frac{1-i}{\delta_\nu}\zeta\right)$$

$$v_\xi = \frac{u_0}{\pi^2} \int\limits_{-\infty}^{+\infty} \frac{1}{k_\xi \left(k_1 - k_2\right)} \sin \frac{k_\xi a}{2} \left(k_1 e^{ik_1\zeta} - k_2 e^{ik_2\zeta}\right) e^{ik_\xi \xi} dk_\xi,$$

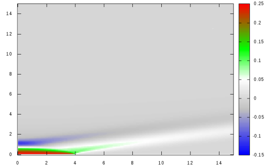
$$v_\eta = 0,$$

$$v_\zeta = \frac{u_0}{\pi^2} \int\limits_{-\infty}^{+\infty} \frac{1}{k_2 - k_1} \sin \frac{k_\xi a}{2} \left(e^{ik_1\zeta} + e^{ik_2\zeta}\right) e^{ik_\xi \xi} dk_\xi,$$

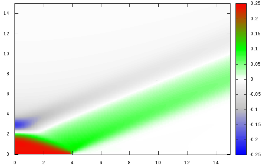
$$k_1 = k_\xi \cot\left(\varphi + \theta\right) \pm \frac{i k_\xi^3 \delta_N^2}{2\cos\theta \sin^4\left(\varphi - \theta\right)}, \quad k_2 = \frac{1+i}{\delta_\varphi}$$

Type of sources	Plane(2D)	Plane(3D)	Disk(3D)
Friction	$\frac{u_0}{N}\frac{a}{{\delta_N}^{1/3}}\frac{1}{q^{2/3}}$	$\frac{u_0}{N}\frac{S}{\delta_N^{2/3}q^{4/3}}$	$\frac{u_0}{\pi N}\frac{S}{{\delta_N}^{2/3}q^{4/3}}$
Piston	$\frac{u_0}{N}\frac{a}{{\delta_N}^{2/3}}\frac{1}{q^{1/3}}$	$\frac{u_0}{N}\frac{S}{\delta_N q}$	$\frac{u_0}{\pi N}\frac{S}{\delta_N q}$
Composite	$\frac{u_0}{N}\frac{a}{\delta_N^{4/3}}\frac{a}{q^{2/3}}$	$\frac{u_0}{N}\frac{S}{\delta_N^{5/3}}\frac{b}{q^{4/3}}$	

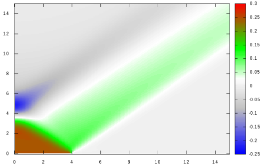
Comparison of different stratification $R = 4$ cm, $\nu = 10^{-2}$ cm/s, $\omega = 0.7\ s^{-1}$ Vertical component of velocity. Source - horizontal disk.



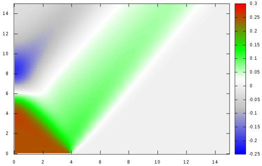
$N = 4.11\ s^{-1}$



$N = 1.4\ s^{-1}$

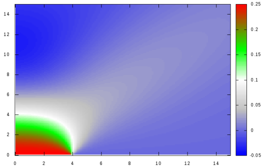


$N = 0.99\ s^{-1}$

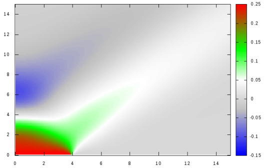


$N = 0.81\ s^{-1}$

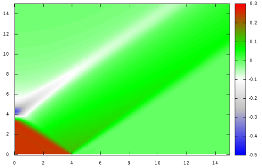
Comparison of different viscosity $R = 4$ cm, $N = 0.9\ s^{-1}$, $\omega = 0.63\ s^{-1}$ Vertical component of velocity. Source - horizontal disk.



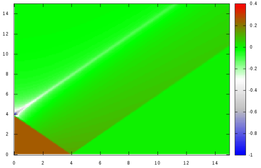
$\nu = 1\ cm^2/s$



$\nu = 0.1\ cm^2/s$

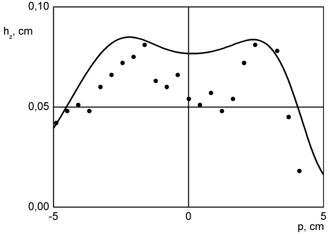


$\nu = 10^{-3}\ cm^2/s$

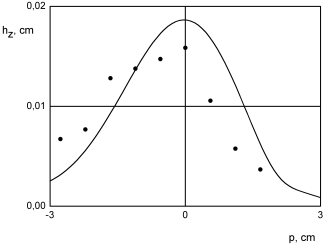


$\nu = 10^{-5}\ cm^2/s$

Comparison of calculations and measurements. Disk 3D.



$u_0 = 0.25\ cm/s;\ N = 1.0\ s^{-1};$
$\omega = 0.57\ s^{-1};\ R = 4.0\ cm.$



$u_0 = 0.25\ cm/s;\ N = 1.26\ s^{-1};$
$\omega = 1.11\ s^{-1};\ R = 1.8\ cm.$

1. In general case there are the two type of flow: regular solution (internal waves) and three types of singular components of flow (boundary layers). Two of them have no analogue in a homogeneous fluid, their thickness is defined by dissipative factors;
2. Near the source viscosity and diffusion is basic factors;
3. The obtained results show that it is necessary to consider influence of dissipative factors (viscosity, stratification, diffusion).