

Modeling flow in a viscous continuously stratified fluid taking into account diffusivity effects

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Review

- ① J.V.S. Rayleigh
- ② J. Lighthill
- ③ Yu.D. Chashechkin
- ④ V.A. Gorodtsov
- ⑤ T.N. Stevenson
- ⑥ D.G. Hurley, G.J. Keady
- ⑦ B.R. Sutherland
- ⑧ Yu.V. Kistovich
- ⑨ A.V. Kistovich
- ⑩ B. Voisin

Color schlieren images oscillation of disk



Governing equations and boundary conditions

Governing equations

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \nabla \rho = 0, \quad \operatorname{div} \mathbf{v} = 0$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}, \nabla) \mathbf{v} = \frac{1}{\rho} \nabla p + \nu \Delta \mathbf{v} + \rho \mathbf{g}$$

$$\frac{\partial S}{\partial t} = \kappa_S \Delta S + \frac{v_z}{\Lambda}$$

Boundary conditions

$$\mathbf{v}|_{\Gamma} = \mathbf{u}_0 e^{-i\omega t}, \quad \kappa_S \left. \frac{\partial S}{\partial n} \right|_{\Gamma} = 0$$

$$v \rightarrow 0, \quad \rho \rightarrow \rho_0, \quad \partial P / \partial z \rightarrow \rho_0(z) g, \quad r \rightarrow \infty$$

Toroidal-poloidal decomposition

$$\mathbf{v} = \nabla \times \mathbf{e}_z \Phi + \nabla \times (\nabla \times \mathbf{e}_z \Psi)$$

System equations for functions Φ and Ψ

$$\left(\left(\frac{\partial}{\partial t} - D\Delta \right) \left(\frac{\partial}{\partial t} - \nu\Delta \right) \Delta + N^2 \Delta_{\perp} \right) \Phi = 0$$

$$\left(\frac{\partial}{\partial t} - \nu\Delta \right) \Psi = 0$$

$$\left(\left(\frac{\partial}{\partial t} - D\Delta \right) \left(\frac{\partial}{\partial t} - \nu\Delta \right) \Delta + N^2 \Delta_{\perp} \right) S = 0$$

where

$$\Delta_{\perp} = \partial_x^2 + \partial_y^2 \quad \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2 \quad N^2 = \sqrt{\frac{g}{\Lambda}}$$

Construction solution for Φ and Ψ in Fourier transform

$$\Phi = e^{-i\omega t} \sum_{j=1}^3 \int_{-\infty}^{+\infty} A_j(k_\xi, k_\eta) E_j dk_\xi dk_\eta$$

$$S = -\frac{\rho_0}{\Lambda} e^{-i\omega t} \sum_{j=1}^3 \int_{-\infty}^{+\infty} \frac{(k_\xi \cos \varphi - k_j \sin \varphi)^2 + k_\eta^2}{i\omega - \kappa_S k^2} A_j(k_\xi, k_\eta) E_j dk_\xi dk_\eta$$

$$\Psi = e^{-i\omega t} \int_{-\infty}^{+\infty} B(k_\xi, k_\eta) E_4 dk_\xi dk_\eta$$

where

$$E_j = \exp(ik_j\zeta + ik_\xi\xi + ik_\eta\eta), \quad k = \sqrt{k_j^2 + k_\xi^2 + k_\eta^2}$$

Viscous stratified fluid take into diffusion

Dispersion equation

$$\left(\nu \kappa_S \tilde{k}^6 - i\omega (\nu + \kappa_S) \tilde{k}^4 - \omega^2 \tilde{k}^2 + N^2 k_{\perp}^2 \right) \left(\tilde{k}^2 + \frac{\omega}{i\nu} \right) = 0$$

$$\tilde{k}^2 = 2k_{\zeta}^2 + k_{\perp}^2, \quad k_{\perp}^2 = k_{\xi}^2 + k_{\eta}^2$$

Viscous stratified fluid take into diffusion

Regular solution(waves)

$$k_1 = \frac{k_\xi \sin \varphi \cos \varphi \pm \kappa \cos \theta}{\mu_\theta} \pm \delta_N^2 (1 + \varepsilon) \frac{i \tan \theta \mu_\theta^4}{2\kappa \mu^4} + \dots$$

$$\mu = \sin^2 \varphi - \sin^2 \theta, \quad \mu_\theta = (k_\xi \sin \varphi \cos \varphi \pm \kappa \cos \varphi),$$

$$\varepsilon = Sc^{-1} = \frac{\kappa S}{\nu}, \quad \delta_N = \sqrt{\frac{\nu}{N}}$$

Singular solution

$$k_{2,3} \approx \sqrt{\frac{i\omega(\varepsilon + 1 \pm \lambda_{\nu\kappa})}{\varepsilon}}, \quad \lambda_{\nu\kappa} = \frac{2}{\sin \theta} \sqrt{(1 + \varepsilon)^2 - \frac{4\varepsilon\mu}{\sin^2 \theta}}$$

$$k_4 = \sqrt{\frac{2i}{\delta_\nu^2} - k^2}, \quad \delta_\nu = \delta_N \sqrt{\frac{2}{\sin \theta}}, \quad \delta_\varphi = \delta_N \sqrt{\frac{2 \sin \theta}{|\mu|}}, \quad \delta_\kappa = \delta_N \sqrt{\frac{2\varepsilon}{\sin \theta}}$$

Viscous stratified fluid take into diffusion. Vertical component of the velocity

$$v_{\zeta} \approx \int_{-\infty}^{+\infty} A_1 (k_{\eta}^2 \sin \varphi - k_{\xi} \beta_1) E_1 dk_{\xi} dk_{\eta} -$$

$$-ie^{\frac{i-1}{\delta\nu}\zeta} \sin \varphi \int_{-\infty}^{+\infty} B E_{\xi\eta} dk_{\xi} dk_{\eta} - \frac{i+1}{\delta_{\varphi}} e^{\frac{i-1}{\delta_{\varphi}}\zeta} \cos \varphi \int_{-\infty}^{+\infty} A_2 k_{\xi} E_{\xi\eta} dk_{\xi} dk_{\eta} -$$

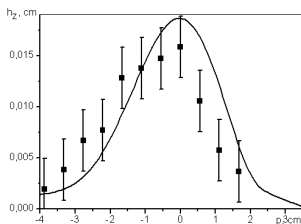
$$- \frac{1+i}{\delta_{\kappa}} \sqrt{\frac{\sin \theta}{2}} e^{-\frac{\sqrt{\sin \theta}}{\delta_{\kappa}\sqrt{2}}\zeta + \frac{i\zeta}{\delta_{\kappa}\sqrt{2}}} \cos \varphi \int_{-\infty}^{+\infty} A_3 k_{\xi} E_{\xi\eta} dk_{\xi} dk_{\eta}$$

where

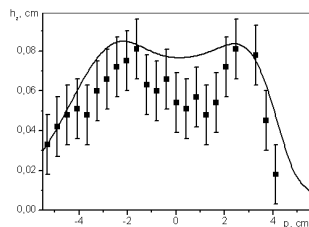
$$E_{\xi\eta} = \exp(ik_{\xi}\xi + ik_{\eta}\eta)$$

Comparison theoretical analyze and measurement. Source is disk

$$u_0 = 0.25 \text{ cm s}^{-1}$$



$$R = 1.75 \text{ cm}, N = 1.0 \text{ s}^{-1}, \omega = 0.57 \text{ s}^{-1}$$



$$R = 4.0 \text{ cm}, N = 1.26 \text{ s}^{-1}, \omega = 1.11 \text{ s}^{-1}$$

Why OpenFOAM?

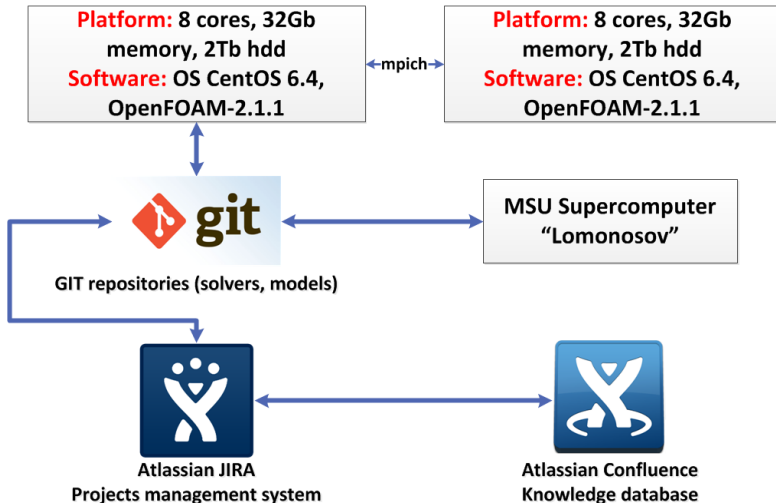
Pluses:

- 1 OpenFOAM free and open source, under the GNU general public licence (GPL).
- 2 Support of community
<http://www.cfd-online.com/Forums/openfoam/>,
<http://openfoamwiki.net/>
- 3 Support open-source Linux platform (openSUSE, Ubuntu, Fedora and etc)

Minuses:

- 1 OpenFOAM has no GUI to create grids, but can use other applications such as: GMSH (<http://geuz.org/gmsh/>), Salome (<http://www.salome-platform.org/>) or commercial mesh generators such as: Icem CFD (www.ansys.com), Gambit (www.ansys.com), pro*star Star-CD (www.cd-adapco.com)

Hardware, software and workflow process for analyze and solving



Structure of the solver. IGWFoam.C

Navier - Stokes equations

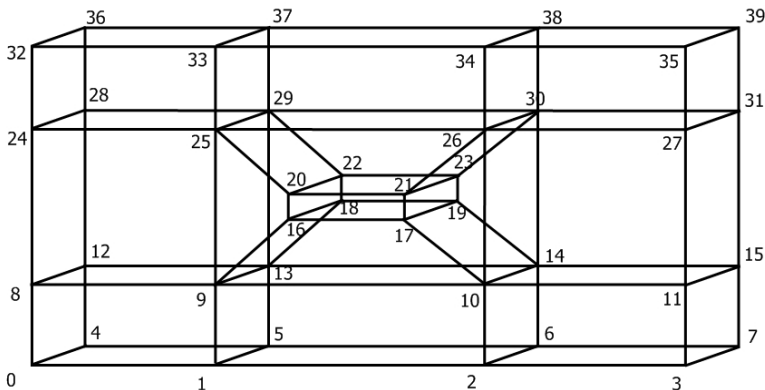
```
fvVectorMatrix UEqn (
    fvm::ddt(U) + fvm::div(phi, U)
    - fvm::laplacian(nu, U) - S*g
);
solve(UEqn == -fvc::grad(p)/dens0);
```

Equations for salinity S and density $dens$

```
fvScalarMatrix SEqn (
    fvm::ddt(S) + fvm::div(phi, S)
    - fvm::laplacian(DS, S)
    - U.component(vector::Z)/Lambda
);
SEqn.solve();
dens = dens0*(1.0-Z/Lambda+S);
```

Create O-grid model

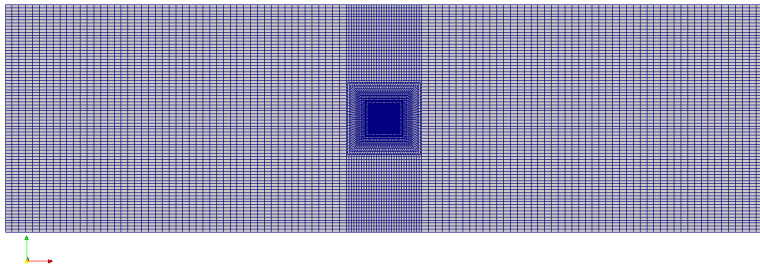
To construct the mesh used a standard utility of OpenFOAM
blockMesh or **pyFoam** (Python for OpenFOAM)



Create mesh

Create mesh using blockMesh utility:

```
[user@server]$ blockMesh -case name-of-model
```

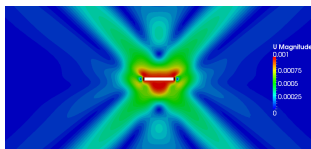


Boundary conditions:

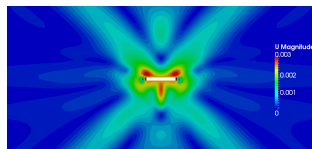
Top, Botton, Left, Right: freeSteram (U, p), zeroGradient (S)

Plate: oscillating (U), fixedGradient (S)

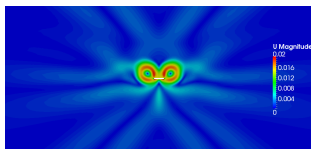
Different velocities $L_x = 1$ cm, $N = 0.9$ s⁻¹, $\omega = 0.54$ s⁻¹
 Module of velocity. Source - horizontal plate. Type - piston



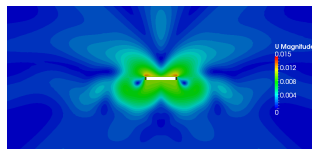
$$u_0 = 0.001 \text{ m s}^{-1}$$



$$u_0 = 0.0025 \text{ m s}^{-1}$$

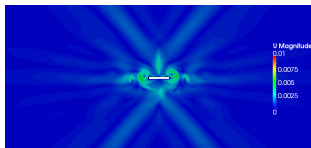


$$u_0 = 0.025 \text{ m s}^{-1}$$

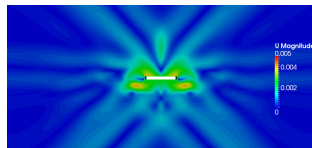


$$u_0 = 0.01 \text{ m s}^{-1}$$

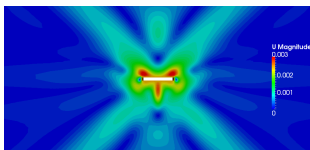
Different viscosity $L_x = 1$ cm, $N = 0.9$ s⁻¹, $\omega = 0.54$ s⁻¹
 Module of velocity. Source - horizontal plate. Type - piston



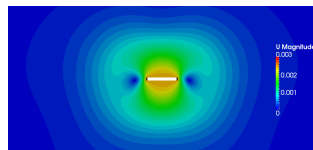
$$\nu = 10^{-8} \text{ m}^2 \text{ s}^{-1}$$



$$\nu = 10^{-7} \text{ m}^2 \text{ s}^{-1}$$

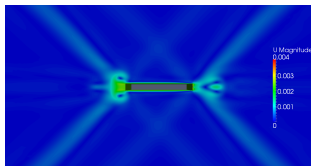


$$\nu = 10^{-6} \text{ m}^2 \text{ s}^{-1}$$

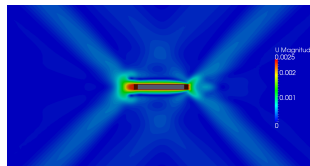


$$\nu = 10^{-5} \text{ m}^2 \text{ s}^{-1}$$

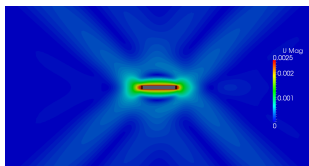
Different viscosity $L_x = 1$ cm, $N = 0.9$ s⁻¹, $\omega = 0.54$ s⁻¹
 Module of velocity. Source - horizontal plate. Type - friction



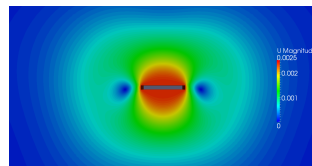
$$\nu = 10^{-8} \text{ m}^2 \text{ s}^{-1}$$



$$\nu = 10^{-7} \text{ m}^2 \text{ s}^{-1}$$

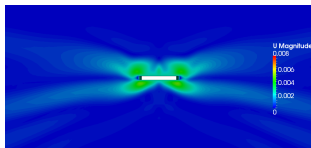


$$\nu = 10^{-6} \text{ m}^2 \text{ s}^{-1}$$

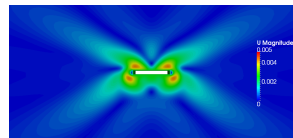


$$\nu = 10^{-5} \text{ m}^2 \text{ s}^{-1}$$

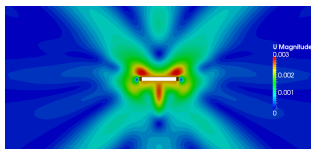
Different stratification $L_x = 1$ cm, $\nu = 10^{-2}$ cm² s⁻¹, $\omega = 0.54$ s⁻¹
 Module of velocity. Source - horizontal plate. Type - piston



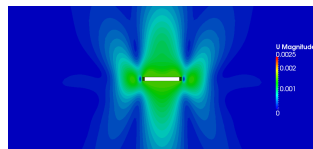
$$N = 2.8 \text{ s}^{-1}$$



$$N = 1.46 \text{ s}^{-1}$$

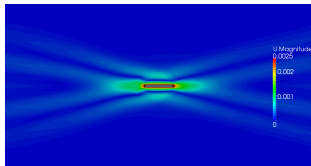


$$N = 2.8 \text{ s}^{-1}$$

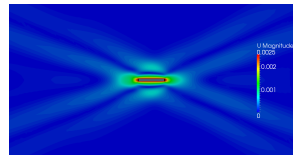


$$N = 0.58 \text{ s}^{-1}$$

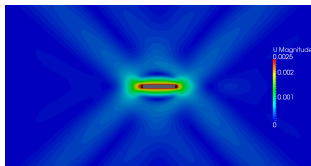
Different stratification $L_x = 1 \text{ cm}$, $\nu = 10^{-2} \text{ cm}^2 \text{ s}^{-1}$, $\omega = 0.54 \text{ s}^{-1}$
 Module of velocity. Source - horizontal plate. Type - piston



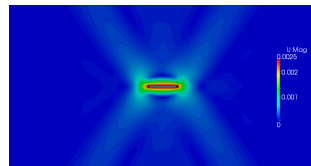
$$N = 2.8 \text{ s}^{-1}$$



$$N = 1.46 \text{ s}^{-1}$$



$$N = 2.8 \text{ s}^{-1}$$



$$N = 0.58 \text{ s}^{-1}$$

Conclusion

- 1 In the general case, in a viscous stratified fluid there are two types of solutions: regular (waves) and three type singular solutions. Two of them don't have analogues in homogeneous fluid Their properties are defined viscosity, stratification, diffusion and the geometry of the problem;
- 2 For a complete description of the flow of fluid you must consider all parameters (viscosity, stratification diffusion);
- 3 Create solver for calculation of the internal gravity waves in a continuesly stratified fluid;
- 4 Calculations case horizontal plate.