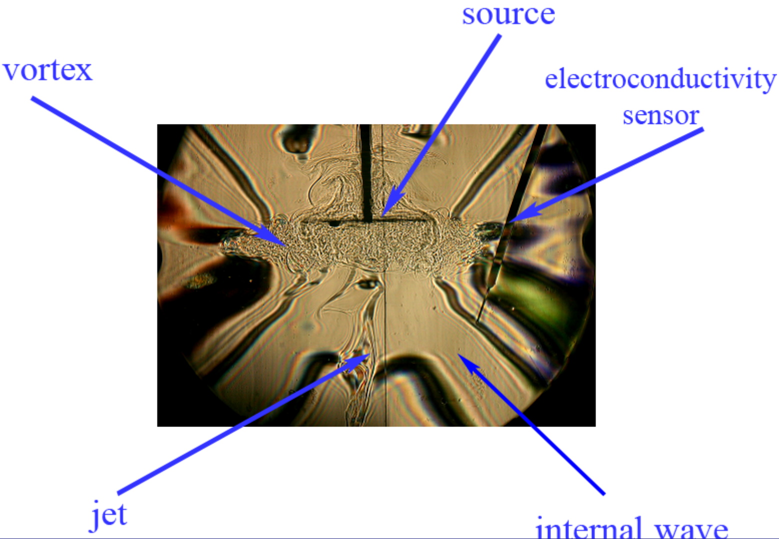
Singularity and limiting cases for oscillations in a viscous continuously stratified fluid problems

Alexey Vasiliev

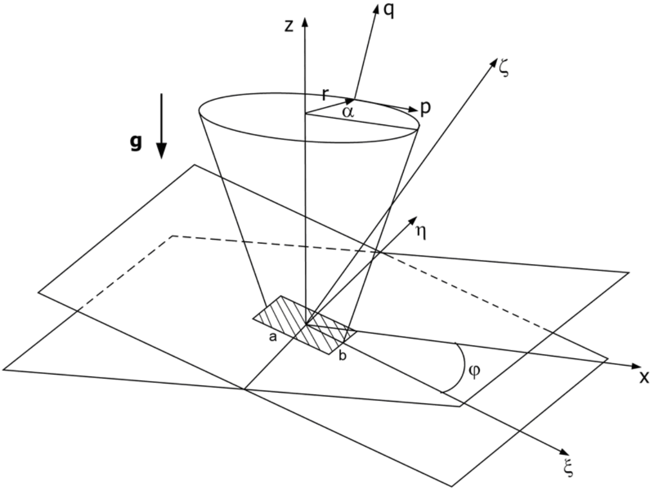
Laboratory of Fluid Mechanics
Institute for Problems in Mechanics of the RAS
Moscow, Russia

Seven International Scientific School for Young Scientists
"Waves and vortices in complex media".

Color shadow image of disk oscillation.



Formulation of the problem.



Governing equations and boundary conditions.

Governing equations

$$\frac{\partial \rho}{\partial t} + \mathbf{v}\nabla\rho = 0, \quad \mathrm{div}\ \mathbf{v} = 0$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}, \nabla)\,\mathbf{v} = \frac{1}{\rho}\nabla p + \nu\Delta v + \rho\mathbf{g}$$

$$\frac{\partial S}{\partial t} = \kappa_S \Delta S + \frac{v_z}{\Lambda}$$

Boundary conditions

$$\mathbf{v}\big|_\Gamma = \mathbf{u}_0 e^{-i\omega t}, \quad \kappa_S \left.\frac{\partial S}{\partial n}\right|_\Gamma = 0$$

$$v \to 0, \quad \rho \to \rho_0, \quad \partial P/\partial z \to \rho_0(z)\, g, r \to \infty$$

Toroidal-poloidar decomposition

$$\mathbf{v} = \nabla \times \mathbf{e}_z\Phi + \nabla \times (\nabla \times \mathbf{e}_z\Psi)$$

System equation for scalar function Φ and Ψ

$$\left(\left(\frac{\partial}{\partial t} - D\Delta\right)\left(\frac{\partial}{\partial t} - \nu\Delta\right)\Delta + N^2\Delta_\perp\right)\ \Phi = 0$$

$$\left(\frac{\partial}{\partial t} - \nu\Delta\right)\ \Psi = 0$$

$$\left(\left(\frac{\partial}{\partial t} - D\Delta\right)\left(\frac{\partial}{\partial t} - \nu\Delta\right)\Delta + N^2\Delta_\perp\right)\ S = 0$$

where

$$\Delta_\perp = \partial_x^2 + \partial_y^2 \quad \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2 \quad N^2 = \sqrt{\frac{g}{\Lambda}}$$

The viscous stratified fluid taking into account effects of diffusion

Dispersion equation

$$\left(\nu\kappa_S\tilde{k}^6 - i\omega\left(\nu+\kappa_S\right)\tilde{k}^4 - \omega^2\tilde{k}^2 + N^2 k_\perp^2\right)\left(\tilde{k}^2 + \frac{\omega}{i\nu}\right) = 0$$

$$\tilde{k}^2 = 2k_\zeta^2 + k_\perp^2, \quad k_\perp^2 = k_\xi^2 + k_\eta^2$$

Regular solution

$$k_1 = \frac{k_\xi \sin\varphi\cos\varphi \pm \kappa\cos\theta}{\mu_\theta} \pm \delta_N^2\left(1+\varepsilon\right)\frac{i\ \tan\theta\mu_\theta^4}{2\kappa\mu^4} + \ldots$$

$$\mu = \sin^2\varphi - \sin^2\theta, \quad \mu_\theta = \left(k_\xi\sin\varphi\cos\varphi \pm \kappa\cos\varphi\right),$$

$$\varepsilon = Sc^{-1} = \frac{\kappa_S}{\nu}, \quad \delta_N = \sqrt{\frac{\nu}{N}}$$

The viscous stratified fluid taking into account effects of diffusion

Singular solutions

$$k_{2,\,3} \approx \sqrt{\frac{i\omega\left(\varepsilon + 1 \pm \lambda_{\nu\kappa}\right)}{\varepsilon}}, \quad \lambda_{\nu\kappa} = \frac{2}{\sin\theta}\sqrt{\left(1+\varepsilon\right)^2 - \frac{4\varepsilon\mu}{\sin^2\theta}}$$

$$k_4 = \sqrt{\frac{2i}{\delta_\nu^2} - k^2}, \quad \delta_\nu = \delta_N\sqrt{\frac{2}{\sin\theta}}$$

The viscous stratified fluid taking into account effects of diffusion

$$v_\xi \approx -\frac{1-i}{2}\delta_\varphi G_1 + \frac{i\delta_N^2}{\sqrt{|\mu|}}\tan^2\varphi\; G_2 - G_3$$

$$v_\eta \approx (1-i)\,\delta_\varphi G_0 + \frac{1-i}{2}\delta_\nu \tan\varphi\, G_2 - (1-i)\,\delta_\varphi \frac{\cos\varphi}{\sin^2\varphi}G_3$$

$$v_\zeta \approx \frac{i-1}{2}\delta_\varphi \int\limits_{-\infty}^{+\infty} u_0 k_\xi e_1\, dk_\xi dk_\eta + \frac{i\delta_N^2}{2\sqrt{|\mu|}}\tan\varphi\, G_2 - \frac{1+i}{\delta_\varphi \sin\varphi}G_3$$

The viscous stratified fluid. Friction source. Rectangle. 3D. $\varphi = 0$.

$$v_\xi = \int\limits_{-\infty}^{+\infty} k_\eta^2 L_3 \, dk_\xi dk_\eta + \int\limits_{-\infty}^{+\infty} k_\xi^2 L_1 \, dk_\xi dk_\eta,$$

$$v_\eta = \int\limits_{-\infty}^{+\infty} L_3 \, dk_\xi dk_\eta + \int\limits_{-\infty}^{+\infty} k_\xi k_\eta L_1 \, dk_\xi dk_\eta, \quad v_\zeta = \int\limits_{-\infty}^{+\infty} k_\eta k_\perp^2 L_2 \, dk_\xi dk_\eta$$

$$L_m = u_0 Q \frac{k_1^{2-m} \exp\left(ik_1\zeta\right) + k_2^{2-m} \exp\left(ik_2\zeta\right)}{\left(k_\eta^2 - k_\xi^2\right)\left(k_2 - k_1\right)}, \quad m = 1, 2,$$

$$L_3 = \frac{u_0 Q}{k_\eta^2 - k_\xi^2} \exp\left(-\frac{1-i}{\delta_\nu}\zeta\right)$$

The viscous stratified fluid. Friction source. Plate. 2D.

$$v_\xi = \frac{u_0}{\pi^2} \int\limits_{-\infty}^{+\infty} \frac{1}{k_\xi \left(k_1 - k_2\right)} \sin \frac{k_\xi a}{2} \left(k_1 e^{ik_1\zeta} - k_2 e^{ik_2\zeta}\right) e^{ik_\xi \xi} dk_\xi,$$

$$v_\eta = 0,$$

$$v_\zeta = \frac{u_0}{\pi^2} \int\limits_{-\infty}^{+\infty} \frac{1}{k_2 - k_1} \sin \frac{k_\xi a}{2} \left(e^{ik_1\zeta} + e^{ik_2\zeta}\right) e^{ik_\xi \xi} dk_\xi,$$

$$k_1 = k_\xi \cot\left(\varphi + \theta\right) \pm \frac{ik_\xi^3 \delta_N^2}{2\cos\theta \sin^4\left(\varphi - \theta\right)}, \quad k_2 = \frac{1+i}{\delta_\varphi}$$

Homogeneous fluid $N->0$

Solutions of the dispersion equation

$$k_1 = k_\perp \quad k_2 = \frac{1+i}{\delta_\varphi} \quad k_3 = \frac{1+i}{\delta_\kappa} \quad k_4 = k_2$$

Homogeneous fluid $N->0$

$$v_\xi = \int\limits_{-\infty}^{+\infty} A_1 e_1 \left(k_\eta^2 \sin\varphi + k_1\beta_1\right) dk_\xi dk_\eta$$
$$+i \exp\left(-\frac{1-i}{\delta_\nu}\zeta\right) \int\limits_{-\infty}^{+\infty} \left(\frac{2}{\delta_\nu^2} + B\cos\varphi\right) Q dk_\xi dk_\eta + M_1$$

$$v_\eta = \int\limits_{-\infty}^{+\infty} A_1 \gamma_1 e_1 \kappa_1 dk_\xi dk_\eta$$
$$-\frac{1-i}{\delta_\nu} \exp\left(\frac{1-i}{\delta_\nu}\zeta\right) \int\limits_{-\infty}^{+\infty} (A_2\kappa_\eta - B\sin\varphi) Q dk_\xi dk_\eta - M_2$$

$$v_\zeta = \int\limits_{-\infty}^{+\infty} A_1 e_1 \left(k_\eta^2 \sin\varphi - k_\eta\beta\right) dk_\xi dk_\eta$$
$$+i \exp\left(-\frac{1-i}{\delta_\nu}\zeta\right) \int\limits_{-\infty}^{+\infty} \left(\frac{1-i}{\delta_\varphi} A_2 k_\xi - B\sin\varphi\right) Q dk_\xi dk_\eta - M_3$$

Displacement on the axis $h\left(0,q\right)$. Far field.

Type of sources	Plane(2D)	Plane(3D)	Disk(3D)
Friction	$\frac{u_0}{N}\frac{a}{{\delta_N}^{1/3}}\frac{1}{q^{2/3}}$	$\frac{u_0}{N}\frac{S}{\delta_N^{2/3}q^{4/3}}$	$\frac{u_0}{\pi N}\frac{S}{{\delta_N}^{2/3}q^{4/3}}$
Piston	$\frac{u_0}{N}\frac{a}{{\delta_N}^{2/3}}\frac{1}{q^{1/3}}$	$\frac{u_0}{N}\frac{S}{\delta_N q}$	$\frac{u_0}{\pi N}\frac{S}{\delta_N q}$
Composite	$\frac{u_0}{N}\frac{a}{\delta_N^{4/3}}\frac{a}{q^{2/3}}$	$\frac{u_0}{N}\frac{S}{\delta_N^{5/3}}\frac{b}{q^{4/3}}$	

Conclusions:

1. In general case there are the two type of flow: regular solution (internal waves) and three types of singular components of flow (boundary layers). Two of them have no analogue in a homogeneous fluid, their thickness is defined by dissipative factors;
2. Near the source viscosity and diffusion is basic factors;
3. The obtained results show that it is necessary to consider influence of dissipative factors (viscosity, stratification, diffusion).