Singularity and limiting cases for oscillations in a viscous continuously stratified fluid problems

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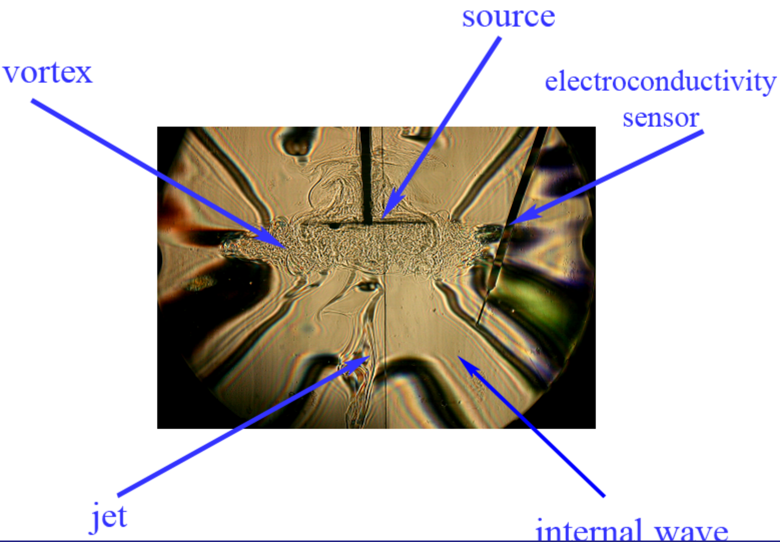
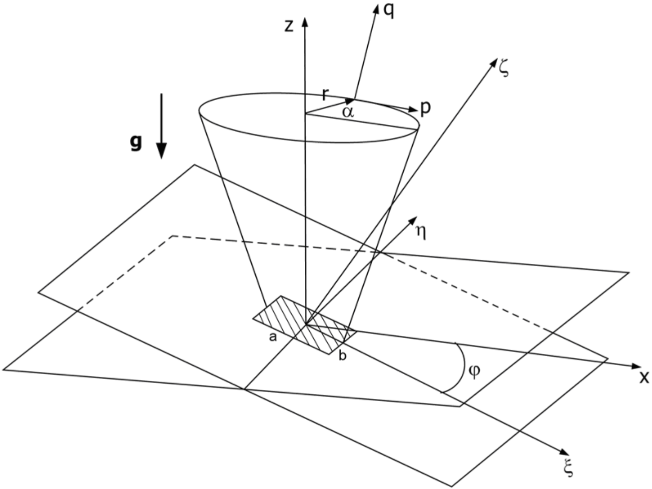
Seven International Scientific School for Young Scientists
"Waves and vortices in complex media".

Color shadow image of disk oscillation.



Formulation of the problem.



Governing equations and boundary conditions.

Governing equations

$$\frac{\partial \rho}{\partial t} + \mathbf{v}\nabla\rho = 0, \quad \mathrm{div}\ \mathbf{v} = 0$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}, \nabla)\,\mathbf{v} = \frac{1}{\rho}\nabla p + \nu\Delta v + \rho\mathbf{g}$$

$$\frac{\partial S}{\partial t} = \kappa_S \Delta S + \frac{v_z}{\Lambda}$$

Boundary conditions

$$\mathbf{v}\big|_\Gamma = \mathbf{u}_0 e^{-i\omega t}, \quad \kappa_S \left.\frac{\partial S}{\partial n}\right|_\Gamma = 0$$

$$v \to 0, \quad \rho \to \rho_0, \quad \partial P/\partial z \to \rho_0\left(z\right)g, r \to \infty$$

Toroidal-poloidar decomposition

$$\mathbf{v} = \nabla \times \mathbf{e}_z\Phi + \nabla \times \left(\nabla \times \mathbf{e}_z\Psi\right)$$

System equation for scalar function Φ and Ψ

$$\left(\left(\frac{\partial}{\partial t} - D\Delta\right)\left(\frac{\partial}{\partial t} - \nu\Delta\right)\Delta + N^2\Delta_\perp\right)\ \Phi = 0$$

$$\left(\frac{\partial}{\partial t} - \nu\Delta\right)\ \Psi = 0$$

$$\left(\left(\frac{\partial}{\partial t} - D\Delta\right)\left(\frac{\partial}{\partial t} - \nu\Delta\right)\Delta + N^2\Delta_\perp\right)\ S = 0$$

where

$$\Delta_\perp = \partial_x^2 + \partial_y^2 \quad \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2 \quad N^2 = \sqrt{\frac{g}{\Lambda}}$$

The viscous stratified fluid taking into account effects of diffusion

Dispersion equation

$$\left(\nu\kappa_S\tilde{k}^6 - i\omega\left(\nu+\kappa_S\right)\tilde{k}^4 - \omega^2\tilde{k}^2 + N^2k_\perp^2\right)\left(\tilde{k}^2 + \frac{\omega}{i\nu}\right) = 0$$

$$\tilde{k}^2 = 2k_\zeta^2 + k_\perp^2, \quad k_\perp^2 = k_\xi^2 + k_\eta^2$$

Regular solution

$$k_1 = \frac{k_\xi \sin\varphi\cos\varphi \pm \kappa\cos\theta}{\mu_\theta} \pm \delta_N^2\left(1+\varepsilon\right)\frac{i\ \tan\theta\mu_\theta^4}{2\kappa\mu^4} + \ldots$$

$$\mu = \sin^2\varphi - \sin^2\theta, \quad \mu_\theta = \left(k_\xi\sin\varphi\cos\varphi \pm \kappa\cos\varphi\right),$$

$$\varepsilon = Sc^{-1} = \frac{\kappa_S}{\nu}, \quad \delta_N = \sqrt{\frac{\nu}{N}}$$

The viscous stratified fluid taking into account effects of diffusion

Singular solutions

$$k_{2,\,3} \approx \sqrt{\frac{i\omega\left(\varepsilon + 1 \pm \lambda_{\nu\kappa}\right)}{\varepsilon}}, \quad \lambda_{\nu\kappa} = \frac{2}{\sin\theta}\sqrt{(1+\varepsilon)^2 - \frac{4\varepsilon\mu}{\sin^2\theta}}$$

$$k_4 = \sqrt{\frac{2i}{\delta_\nu^2} - k^2}, \quad \delta_\nu = \delta_N\sqrt{\frac{2}{\sin\theta}}$$

The viscous stratified fluid taking into account effects of diffusion

$$v_\xi \approx -\frac{1-i}{2}\delta_\varphi G_1 + \frac{i\delta_N^2}{\sqrt{|\mu|}}\tan^2\varphi\ G_2 - G_3,$$

$$Q_n = u_0 \int\limits_{-\infty}^{+\infty} \frac{g_n}{k_\eta^2\cos\varphi + k_\xi\beta}\, dk_\xi dk_\eta, \quad \beta = k_\xi\cos\varphi - k_1\sin\varphi,$$

$$g_0 = \gamma k_\xi k_\eta e_1, \quad g_1 = k_\xi e_1\left(k_\eta^2\sin\varphi - k_1\beta\right),$$

$$g_2 = k_\eta Q\exp\left(\frac{i-1}{\delta_\nu\xi}\right)\left[k_\eta^2\cos\varphi + k_\xi\left(k_1-\gamma\right)\right], \quad Q = e^{i\left(k_\xi\xi + k_\eta\eta\right)},$$

$$g_3 = k_\eta Q\exp\left(-\frac{i+1}{\delta_\nu\xi}\right)\left(k_\eta^2\sin\varphi + k_\xi\beta\sin\varphi\right), \quad \gamma = k_\xi\sin\varphi + k_1\cos\varphi$$

The viscous stratified fluid taking into account effects of diffusion

$$v_\eta \approx (1-i)\,\delta_\varphi G_0 + \frac{1-i}{2}\delta_\nu \tan\varphi\, G_2 - (1-i)\,\delta_\varphi \frac{\cos\varphi}{\sin^2\varphi} G_3$$

$$v_\zeta \approx \frac{i-1}{2}\delta_\varphi \int\limits_{-\infty}^{+\infty} u_0 k_\xi e_1\, dk_\xi dk_\eta + \frac{i\delta_N^2}{2\sqrt{|\mu|}} \tan\varphi\, G_2 - \frac{1+i}{\delta_\varphi \sin\varphi} G_3$$

The viscous stratified fluid. Friction source. Rectangle. 3D. $\varphi = 0$.

$$v_\xi = \int\limits_{-\infty}^{+\infty} k_\eta^2 L_3 \, dk_\xi dk_\eta + \int\limits_{-\infty}^{+\infty} k_\xi^2 L_1 \, dk_\xi dk_\eta,$$

$$v_\eta = \int\limits_{-\infty}^{+\infty} L_3 \, dk_\xi dk_\eta + \int\limits_{-\infty}^{+\infty} k_\xi k_\eta L_1 \, dk_\xi dk_\eta, \quad v_\zeta = \int\limits_{-\infty}^{+\infty} k_\eta k_\perp^2 \, L_2 \, dk_\xi dk_\eta$$

$$L_m = u_0 Q \frac{k_1^{2-m} \exp\left(ik_1\zeta\right) + k_2^{2-m} \exp\left(ik_2\zeta\right)}{\left(k_\eta^2 - k_\xi^2\right)\left(k_2 - k_1\right)}, \quad m = 1, 2,$$

$$L_3 = \frac{u_0 Q}{k_\eta^2 - k_\xi^2} \exp\left(-\frac{1-i}{\delta_\nu}\zeta\right)$$

The viscous stratified fluid. Friction source. Plate. 2D.

$$v_\xi = \frac{u_0}{\pi^2} \int\limits_{-\infty}^{+\infty} \frac{1}{k_\xi \left(k_1 - k_2\right)} \sin \frac{k_\xi a}{2} \left(k_1 e^{ik_1 \zeta} - k_2 e^{ik_2 \zeta}\right) e^{ik_\xi \xi} dk_\xi,$$

$$v_\eta = 0,$$

$$v_\zeta = \frac{u_0}{\pi^2} \int\limits_{-\infty}^{+\infty} \frac{1}{k_2 - k_1} \sin \frac{k_\xi a}{2} \left(e^{ik_1 \zeta} + e^{ik_2 \zeta}\right) e^{ik_\xi \xi} dk_\xi,$$

$$k_1 = k_\xi \cot\left(\varphi + \theta\right) \pm \frac{i k_\xi^3 \delta_N^2}{2 \cos\theta \sin^4\left(\varphi - \theta\right)}, \quad k_2 = \frac{1+i}{\delta_\varphi}$$

Homogeneous fluid $N \to 0$

Solutions of the dispersion equation

$$k_1 = k_\perp \quad k_2 = \frac{1+i}{\delta_\varphi} \quad k_3 = \frac{1+i}{\delta_\kappa} \quad k_4 = k_2$$

$$M_j = a_j \exp\left(-\sigma_\kappa \xi + i\frac{\zeta}{\sqrt{2}\delta_\kappa}\right) \sin^2\varphi \int\limits_{-\infty}^{+\infty} A_3 Q b_j dk_\xi dk_\eta$$

$$\sigma_\kappa = \frac{\delta_N}{\delta_\nu \delta_\kappa}$$

Homogeneous fluid $N \rightarrow$

Velocity components

$$v_\xi = \int\limits_{-\infty}^{+\infty} A_1 e_1 \left(k_\eta^2 \sin\varphi + k_1\beta_1\right) dk_\xi dk_\eta$$

$$+i \exp\left(-\frac{1-i}{\delta_\nu}\zeta\right) \int\limits_{-\infty}^{+\infty} \left(\frac{2}{\delta_\nu^2} + B\cos\varphi\right) Q dk_\xi dk_\eta + M_1$$

$$v_\eta = \int\limits_{-\infty}^{+\infty} A_1 \gamma_1 e_1 \kappa_1 dk_\xi dk_\eta$$

$$-\frac{1-i}{\delta_\nu} \exp\left(\frac{1-i}{\delta_\nu}\zeta\right) \int\limits_{-\infty}^{+\infty} \left(A_2\kappa_\eta - B\sin\varphi\right) Q dk_\xi dk_\eta - M_2$$

$$v_\zeta = \int\limits_{-\infty}^{+\infty} A_1 e_1 \left(k_\eta^2 \sin\varphi - k_\eta\beta\right) dk_\xi dk_\eta$$

$$+i \exp\left(-\frac{1-i}{\delta_\nu}\zeta\right) \int\limits_{-\infty}^{+\infty} \left(\frac{1-i}{\delta_\varphi} A_2 k_\xi - B\sin\varphi\right) Q dk_\xi dk_\eta - M_3$$

Homogeneous fluid. Horizontal plane $N \to 0$, $\varphi = 0$

$$v_\xi = \int\limits_{-\infty}^{+\infty} \left(k_\eta^2 L_3 dk_\xi dk_\eta + k_\eta^2 L_1\right) dk_\xi dk_\eta$$

$$v_\eta = \int\limits_{-\infty}^{+\infty} \left(L_3 + k_\xi k_\eta L_1\right) dk_\xi dk_\eta$$

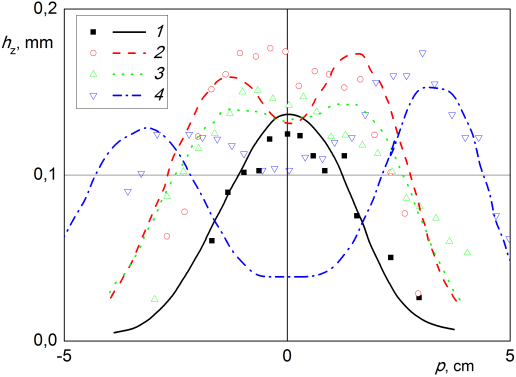
$$v_\zeta = \int\limits_{-\infty}^{+\infty} k_\eta k_\perp^2 L_2 dk_\xi dk_\eta$$

$$L_m = u_0 Q \frac{k_1^{2-m} e^{ik_1\zeta} + k_1^{2-m} e^{ik_2\zeta}}{(k_\eta^2 - k_\xi^2)(k_2 - k_1)}, \quad L_3 = \frac{u_0 Q}{k_\eta^2 - k_\xi^2} \exp\left(-\frac{1-i}{\delta_\nu}\zeta\right)$$

Displacement on the axis $h\left(0,q\right)$. Far field.

Type of sources	Plane(2D)	Plane(3D)	Disk(3D)
Friction	$\frac{u_0}{N}\frac{a}{{\delta_N}^{1/3}}\frac{1}{q^{2/3}}$	$\frac{u_0}{N}\frac{S}{\delta_N^{2/3}q^{4/3}}$	$\frac{u_0}{\pi N}\frac{S}{{\delta_N}^{2/3}q^{4/3}}$
Piston	$\frac{u_0}{N}\frac{a}{{\delta_N}^{2/3}}\frac{1}{q^{1/3}}$	$\frac{u_0}{N}\frac{S}{\delta_N q}$	$\frac{u_0}{\pi N}\frac{S}{\delta_N q}$
Composite	$\frac{u_0}{N}\frac{a}{\delta_N^{4/3}}\frac{a}{q^{2/3}}$	$\frac{u_0}{N}\frac{S}{\delta_N^{5/3}}\frac{b}{q^{4/3}}$	

Comparison of the laboratory experiment (point) and calculations (lines) of the vertical displacement.



Conclusions:

1. In general case there are the two type of flow: regular solution (internal waves) and three types of singular components of flow (boundary layers). Two of them have no analogue in a homogeneous fluid, their thickness is defined by dissipative factors;
2. Near the source viscosity and diffusion is basic factors;
3. The obtained results show that it is necessary to consider influence of dissipative factors (viscosity, stratification, diffusion).