Course Title (CS-3.14) Homework Set #17

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Exercise 1

Give an appropriate positive constant c such that $f(n) \leq c \cdot g(n)$ for all $n > 1$.

1. $f(n) = n^2 + n + 5$, $g(n) = 3n^2$
2. $f(n) = n\sqrt{n} + n^2$, $g(n) = n^2$
3. $f(n) = n^2 - n$, $g(n) = n^2/2$

Solution

Part One

$$\begin{aligned} n^2 + n + 5 = & \\ & \leq n^2 + n^2 + n^2 \\ & = 3n^2 \\ & \leq c \cdot 3n^2 \end{aligned}$$

choose $c = 1$.

Part Two

$$\begin{aligned} n\sqrt{n} + n^2 = & \\ & = n^{3/2} + n^2 \\ & \leq n^{4/2} + n^2 \\ & = 2n^2 \\ & \leq c \cdot n^2 \end{aligned}$$

choose $c = 2$.

Part Three

$$\begin{aligned} n^2 - n = & \\ & \leq n^2 \\ & \leq c \cdot \frac{n^2}{2} \end{aligned}$$

again choose $c = 2$.

Exercise 2

Let $\Sigma = \{0, 1\}$. Construct a DFA A that recognizes all binary numbers with even number of ones.

Solution

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ be the desired automaton, and let $Q = \{q_0, q_1\}$ where state q_0 corresponds to a binary number with even number of ones and q_1 the vase of odd number of ones. We obviously have $\Sigma = \{0, 1\}$ and $F = \{q_0\}$.

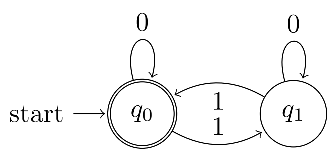

Figure 1: The DFA $\mathcal{A}$

If A is in state q_0, inputting 1 transists to state q_1, while by having even number of ones (state q_0), inputting 1 does not result in a change of state etc, as in Figure 1. Hence the transition matrix of δ is

δ	0	1
q_0	q_0	q_1
q_1	q_1	q_0

Therefore a word $w \in \{0, 1\}^\star$ is recognizable by $\mathcal{A}$ if-f $\delta^\star(q_0, w) = q_0$, which is exactly when w consists of an even number of ones.

Exercise 3

Write a pseudocode for the **Insertion-Sort** algorithm.

Solution

```
function INSERTION-SORT(Array A)
    i = 1
    while i < length(A) do
        j = i
        while j > 0 and A[j − 1] > A[j] do
            swap(A[j], A[j − 1])
            j ← j − 1
        end while
        i ← i + 1
    end while
end function
```

Algorithm 1: Insertion-Sort

Exercise 4

Given a first order linear regression $Y_i = \beta_0 + \beta_1 x_i + e_i$ with $i = 1, \ldots, n$, $\mathbb{E}[e_i] = 0$, with $\mathrm{Var}[e_i] = \sigma_e^2$ and $\mathrm{Cov}[e_i, e_j] = 0, \forall i \neq j$, where ϵ_i is an error term. Obtain the least squares estimators (LSE) $\hat{\beta}_0$, $\hat{\beta}_1$.

Proof.

We should minimize the Residual Sum of Squares (RSS):

$$RSS = \sum_{i=1}^{n} \left(Y_i - \hat{Y}_i\right)^2 = \sum_{i=1}^{n} \left(Y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i\right)^2$$

taking the partial derivatives with respect to $\hat{\beta}_0$ and $\hat{\beta}_1$, we get:

$$\frac{\partial}{\partial \hat{\beta}_0}(RSS) = -2 \sum_{i=1}^{n} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)$$

$$\frac{\partial}{\partial \hat{\beta}_1}(RSS) = -2 \sum_{i=1}^{n} x_i (Y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)$$

which means

$$\sum_{i=1}^{n} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0 \tag{1}$$

$$\sum_{i=1}^{n} Y_i x_i - \hat{\beta}_0 x_i - \hat{\beta}_1 x_i^2) = 0 \tag{2}$$

Expanding the sums and solving for $\hat{\beta}_0$ and $\hat{\beta}_1$ we obtain

$$\hat{\beta}_0 = \overline{Y} - \hat{\beta}_1 \overline{x}$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} x_i Y_i - \frac{\sum_{i=1}^{n} x_i \sum_{i=1}^{n} Y_i}{n}}{\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}} \tag{3}$$

□

Exercise 5

Prove *Liouville's Theorem*: Every bounded entire function in the complex plane $\mathbb{C}$ is constant (every holomorphic function f for which there exists $M > 0$ such that $|f(z)| \leq M \ \forall z \in \mathbb{C}$ is constant).

Proof. Let f be an entire function. Then it can be represented by its Taylor series around zero:

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

by Cauchy's integral formula:

$$a_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{2\pi i} \oint_{C_r} \frac{f(\zeta)}{\zeta^{k+1}} \, d\zeta$$

where C_r is the positively oriented circle around 0 of radius $r > 0$. Suppose f is bounded: i.e. there exists a positive constant M such that $|f(z)| \leq M \ \forall z$. Then

$$\begin{aligned}
|a_k| &\leq \frac{1}{2\pi} \oint_{C_r} \frac{|f(\zeta)|}{|\zeta|^{k+1}} \, |d\zeta| \\
&\leq \frac{1}{2\pi} \oint_{C_r} \frac{M}{r^{k+1}} \, |d\zeta| \\
&= \frac{M}{2\pi r^{k+1}} \oint_{C_r} |d\zeta| \\
&= \frac{M}{2\pi r^{k+1}} 2\pi r \\
&= \frac{M}{r^k} \\
&\to 0
\end{aligned}$$

as r goes to infinity (since f is analytic on the entire compelx plane). Hence $a_k = 0 \ \forall k \geq 1$. Thus $f(z) = a_0$. □

Exercise 6

Prove the fundamental theorem of algebra using Liouville's Theorem.

Exercise 7

Show that any prime $p > 3$ is next to a multiple of 6.

Proof. We equivalently show that every prime $p > 3$ is either $p \equiv 1 \bmod 6$ or $p \equiv -1 \bmod 6 \equiv 5 \bmod 6$, that is, of the form $p = 6n + 1$ or $p = 6n + 5$. By Euclidean division, every integer is of the form $n = 6q + r$ where q is non-negative integer and $r = 0, 1, \ldots, 5,\ r \leq q$. If p is of of the form $6q$ or $6q + 2$ or $6q + 4$, then p is even, therefore not prime. If p is of the form $6q + 3$, then q is divisible by 3 and greater than 3, and therefore not prime. The above leaves as the only candidates for primality greater than 3 integers of the form $p = 6q + 1$ and $p = 6q + 5 = 6(q + 1) - 1$. In fact, by Dirichlet's Theorem an arithmetic progression $an + b,\ n = 1, 2, \ldots$ generates infinitely many primes if-f $(a, b) = 1$[1].

□

Exercise 8

In a bin there are 6 black socks and 4 blue socks. If you select two socks at random, find the probability that you select two socks of the same color.

Solution

Picking two socks of the same color means picking either two black or two blue socks. The probability of picking the first black sock is 6/10 and 5/9 for picking the second black sock due to sampling without replacement. So the probability of picking a black pair is $\frac{6}{10} * \frac{5}{9} = \frac{1}{3}$. Similarly, the probability of picking a blue pair is $\frac{2}{15}$. Total probability of picking a pair of black or red socks is $p = \frac{1}{3} + \frac{2}{15} = \frac{7}{15}$. Another solution using binomial coefficients instead of counting yields

$$\frac{\binom{6}{2}}{\binom{10}{2}} + \frac{\binom{4}{2}}{\binom{10}{2}} = \frac{21}{45} = \frac{3}{15}$$

Exercise 224

Evaluate $\displaystyle\sum_{k=1}^{5} k^2$ and $\displaystyle\sum_{k=1}^{5} (k-1)^2$.

Exercise 225

Find the derivative of $f(x) = x^4 + 3x^2 - 2$.

Exercise 9

Evaluate the integrals $\displaystyle\int_0^1 (1 - x^2)\mathrm{d}x$ and $\displaystyle\int_1^\infty \frac{1}{x^2}\mathrm{d}x$.

[1] https://mathworld.wolfram.com/DirichletsTheorem.html

Exercise 10

Find the derivative of $\alpha^x,\ a \in \mathbb{N}$.

Solution

Rewrite $\alpha^x \equiv e^{\ln \alpha^x}$, hence $e^{\ln \alpha^x} = e^{x \ln \alpha}$. Differentiating with respect to x, we obtain

$$\frac{d}{dx} e^{x \ln \alpha} = \ln(\alpha) e^{x \ln \alpha} = \ln(\alpha)\alpha^x$$

Exercise 243

Prove Goldbach's conjecture, that every even integer greater than 2 is the sum of two prime numbers.

Exercise 250

Prove the Riemann Hypothesis.