# Project Euler Problem 190

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**Problem 190:** Let $S_m = (x_1, x_2, \ldots, x_m)$ be the $m$-tuple of positive real numbers with $x_1 + x_2 + \ldots + x_m = m$ for which $P_m = x_1 \cdot x_2^2 \cdot \ldots \cdot x_m^m = m$ is maximised. For example, it can be verified that $[P_{10}] = 4112$ ( [ ] is the integer part function). Find $\sum [P_m]$ for $2 \le m \le 15$.

*Proof.* We are looking to maximize the multivariate function

$$f_m(x_1, x_2, \cdots, x_m) = x_1 x_2^2 \cdot \ldots \cdot x_m^m$$

subject to the constraint function

$$\begin{aligned} x_1 + x_2 + \ldots + x_m = m &\Rightarrow \\ x_1 + x_2 + \ldots x_m - m = 0 &\Rightarrow \\ g(x_1, x_2, \ldots, x_m) = 0 & \end{aligned}$$

We use the method of *Lagrange Multipliers* for extrema under constraints:

**Proposition 0.1.** *Let $U$ be an open subset of $\mathbb{R}^{\ltimes}$ and $f, g : U \to \mathbb{R}^{\ltimes}$ be $C^1$ functions. If $x_0$ is a local extrema of $f$ subject to $g(x) = K$ with $\nabla g(x_0) \not\equiv 0$ then there exists $\lambda_0 \not\equiv 0$ (called a Lagrange multiplier) such that: $\nabla f(x_0) = \lambda_0 \nabla g(x_0)$.*

The Method of Lagrange Multipliers allows us to first solve the following system of equations $\nabla f(x) = \lambda \nabla g(x)$, $g(x) - K = 0$, then plug in all solutions into $f$ to identify the minimum and maximum values.

$$\nabla f(x_1, \ldots, x_m) = (x_2^2 x_3^3 \ldots x_m^m, 2x_1 x_2 x_3^3 \ldots x_m^m, \ldots, m x_1 x_2^2 \ldots x_m^{m-1}))$$

and

$$\nabla g(x_1, \ldots, x_m) = (1, 1, \ldots, 1)$$

$$\begin{aligned}
\lambda_0 = m x_1 x_2^2 \ldots x_m^{m-1} &\Rightarrow \\
\lambda_0 = \frac{m x_1 x_2^2 \ldots x_m^m}{x_m} &\Rightarrow \\
\lambda_0 = \frac{m f(x_1, \ldots, x_m)}{x_m} &\Rightarrow \\
x_m = \frac{m f(x_1, \ldots, x_m)}{\lambda_0}
\end{aligned}$$

So $x_k = \dfrac{k f(x_1, \ldots, x_m)}{\lambda_0} = k x_1$ for $1 \leq k \leq m$. Hence

$$\begin{aligned}
g(x_1, \ldots, x_m) = 0 \Rightarrow \\
\sum_{i=1}^{m} x_i - m = 0 \Rightarrow \\
x_1 + 2x_1 + \ldots m x_1 - m = 0 \Rightarrow \\
x_1 \frac{m(m+1)}{2} - m = 0 \Rightarrow \\
x_1 = \frac{2}{m+1}
\end{aligned}$$

So for any $j = 1, \ldots, m : x_j = \dfrac{2j}{m+1}$. Taking the product and the floor function for $P_m$ we now have that

$$[P_m] = \left[ \prod_{i=1,\ldots,m} \left( \frac{2k}{m+1} \right)^k \right]$$

Summing through $m = 2, \ldots, 15$ we obtain the solution.

$\square$