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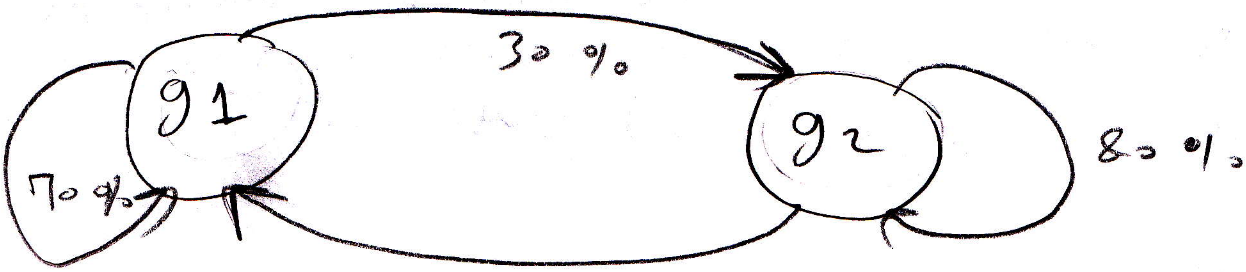
Homework 2

"Mohammad Rami Koujan"



where: g1 represents the group of people who purchased the newspaper one day.

g2 represents the group of people who have not purchased the newspaper yet.

So, the Markov matrix for the previous Markov process is:

$$M = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$$

Let $V_0 = \begin{bmatrix} 750 \\ 250 \end{bmatrix}$ be the number of people in each group at Day 0.

$$\Rightarrow V_1 = \begin{bmatrix} 0.7(750) + 0.2(250) \\ 0.3(750) + 0.8(250) \end{bmatrix} = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} V_0$$

$$\Rightarrow V_{n+1} = M V_n = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} V_n = M^{n+1} V_0$$

1) Since $V_n = M^n V_0$, and m_{ij} is the probability that a

Citizen in group j one day will be in group i the next day
The Probability that a person who purchased a paper today will purchased on Day 2 is calculated from matrix M^2

$$M^2 = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}\begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} = \begin{bmatrix} 0.55 & 0.3 \\ 0.45 & 0.7 \end{bmatrix}$$

$\Rightarrow m_{11} = 0.55$

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- for Day 3: $$M^3 = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}\begin{bmatrix} 0.55 & 0.3 \\ 0.45 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.475 & 0.35 \\ 0.525 & 0.65 \end{bmatrix}$$

$\Rightarrow m_{11}$ for Day 3 $= 0.475$

- for Day n we should calculate M^n which is done by factorization: $M = S \Lambda S^{-1}$

$\Rightarrow$ eigenvalues & eigenvectors should be computed:

$$Mx = \lambda x \Rightarrow (M - I\lambda)x = 0, \quad \det\left(\begin{bmatrix} 0.7-\lambda & 0.2 \\ 0.3 & 0.8-\lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow (0.7-\lambda)(0.8-\lambda) - (0.3)(0.2) = 0$$

$$0.56 - 0.7\lambda - 0.8\lambda + \lambda^2 - 0.06 = 0$$

$$\lambda^2 - 1.5\lambda + 0.5 = 0 \quad \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 0.5 \end{cases}$$

$\lambda_1 \neq \lambda_2 \Rightarrow$ we can diagonalize M

- x_1? in the $N(M - I\lambda_1)$

$$\Rightarrow \begin{bmatrix} -0.3 & 0.2 \\ 0.3 & -0.2 \end{bmatrix} x_1 = 0 \Rightarrow x_1 = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

- x_2 is in the $N(M - \lambda_2 I)$:

$\Rightarrow \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.3 \end{bmatrix} X_2 = 0 \Rightarrow X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ 3

Checking: $\begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

& $\begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = 0.5 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

- for x_1: $\begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}\begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} = (1)\, x_1$

$S = \begin{bmatrix} \frac{2}{3} & -1 \\ 1 & 1 \end{bmatrix}$, $S^{-1} = \frac{3}{5}\begin{bmatrix} 1 & 1 \\ -1 & \frac{2}{3} \end{bmatrix}$, $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$

$\Rightarrow M = \begin{bmatrix} \frac{2}{3} & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & \frac{2}{3} \end{bmatrix}\left(\frac{3}{5}\right)$

$M^n = S \Lambda^n S^{-1} = \begin{bmatrix} \frac{2}{3} & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0.5^n \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & \frac{2}{3} \end{bmatrix}\left(\frac{3}{5}\right)$

$\Rightarrow M^n = \begin{bmatrix} \frac{2}{3} & -0.5^n \\ 1 & 0.5^n \end{bmatrix}\begin{bmatrix} \frac{3}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix} = \begin{bmatrix} \frac{2}{5} + 0.5^n\left(\frac{3}{5}\right) & \frac{2}{5} - 0.5^n\left(\frac{2}{5}\right) \\ \frac{3}{3} - 0.5^n\left(\frac{3}{5}\right) & \frac{3}{5} + 0.5^n\left(\frac{2}{5}\right) \end{bmatrix}$

$\Rightarrow M_{11}$ on Day n will be: $M_{11} = \frac{2}{5} + 0.5^n\left(\frac{3}{5}\right)$, $M_{11} \underset{n\to\infty}{=} \frac{2}{5}$

(for $n=2$, $M_{11} = \frac{2}{5} + 0.5^2\left(\frac{3}{5}\right) = 0.55$

and for $n=3$, $M_{11} = \frac{2}{5} + 0.5^3\left(\frac{3}{5}\right) = 0.475$)

2) The sales figures on Day 2: $V_2 = M^2 V_0$

$V_2 = \begin{bmatrix} 0.55 & 0.3 \\ 0.45 & 0.7 \end{bmatrix}\begin{bmatrix} 750 \\ 250 \end{bmatrix} = \boxed{\begin{matrix} 487.5 \\ 512.5 \end{matrix}}$, around 487 will buy it, around 513 will not

- on Day 3: $V_3 = M^3 V_0 = \begin{bmatrix} 0.475 & 0.35 \\ 0.525 & 0.65 \end{bmatrix}\begin{bmatrix} 750 \\ 250 \end{bmatrix} = \boxed{\begin{matrix} 443.75 \\ 556.25 \end{matrix}}$

So, around 444 people will purchase it on Day 3, and about 556 will not

- on Day n: $V_n = M^n V_0$

$$\Rightarrow V_n = \begin{bmatrix} \frac{2}{5}+0.5^n\left(\frac{3}{5}\right) & \frac{2}{5}-0.5^n\left(\frac{2}{5}\right) \\ \frac{3}{5}-0.5^n\left(\frac{3}{5}\right) & \frac{3}{5}+0.5^n\left(\frac{2}{5}\right) \end{bmatrix} \begin{bmatrix} 750 \\ 250 \end{bmatrix}$$

$$\Rightarrow V_n = \begin{bmatrix} \left(\frac{2}{5}+0.5^n\left(\frac{3}{2}\right)\right)750+\left(\frac{2}{5}-0.5^n\left(\frac{2}{5}\right)\right)250 \\ \left(\frac{3}{5}-0.5^n\left(\frac{3}{5}\right)\right)750+\left(\frac{3}{5}+0.5^n\left(\frac{2}{5}\right)\right)250 \end{bmatrix}$$

3) Since M is a Markov matrix $\Rightarrow$ the columns of M will approach the steady state eigenvector $\left(\begin{bmatrix} +\frac{2}{3} \\ 1 \end{bmatrix}\right.$ which is the eigenvector corresponding to eigenvalue $\lambda_1 = 1)$

$$\lim_{n\to\infty} M^n = \begin{bmatrix} \frac{2}{3} & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{3}{5} \end{bmatrix}$$

each of these columns equal to: $\frac{1}{\sum x_{1i}} X_1 = \frac{1}{\frac{5}{3}} \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} =$

$$\frac{3}{5}\begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ \frac{3}{5} \end{bmatrix}$$

$\Rightarrow$ the sales figures are likely to be stable

$$\boxed{V_n = M^n V_0}$$

$$\lim_{n\to\infty} V_n = \begin{bmatrix} \frac{2}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 750 \\ 250 \end{bmatrix} = \begin{bmatrix} 400 \\ 600 \end{bmatrix}$$