SMART
INDUSTRY
LABORATORY

Scheduling Algorithms (6)

- Production Planning Algorithm (3) -

Graduate School of Information, Production and Systems

Shigeru FUJIMURA

Simplex Method (Penalty Method)

＜Example＞

Suppose that food F1 per unit consists of one unit of nutrient A and three units of nutrient B, whose price is 900Yen per unit. And also suppose that food F2 per unit consists of four units of nutrient A, and one unit of nutrient B, whose price is 1000Yen per unit.

And suppose that food F3 per unit consists of one unit of nutrient A and B, whose price per unit is 200Yen. In this situation, derive the minimum price of foods that 4 units of nutrient A and 9 units of nutrient B can be taken.

Nutrient \ Food	F_1	F_2	F_3	Amount
A	1	4	1	4
B	3	1	1	9
Price per unit	9	10	2	

Simplex Method (Penalty Method)

Minimize

$z = 9y_1 + 10y_2 + 2y_3$

subject to

$y_1 + 4y_2 + y_3 >= 4$

$3y_1 + y_2 + y_3 >= 9$

$y_1, y_2, y_3 >= 0$

Nutrient \ Food	F_1	F_2	F_3	Amount
A	1	4	1	4
B	3	1	1	9
Price per unit	9	10	2	

surplus Variable

Minimize

$z = 9y_1 + 10y_2 + 2y_3$

subject to

$y_1 + 4y_2 + y_3 - t_1 = 4$

$3y_1 + y_2 + y_3 - t_2 = 9$

$y_1, y_2, y_3, t_1, t_2 >= 0$

"No initial feasible solution "

Artificial Variable

Minimize

$z = 9y_1 + 10y_2 + 2y_3 + Ma_1 + Ma_2$

subject to

$y_1 + 4y_2 + y_3 - t_1 + a_1 = 4$

$3y_1 + y_2 + y_3 - t_2 + a_2 = 9$

$y_1, y_2, y_3, t_1, t_2, a_1, a_2 >= 0$

"Initial feasible solution "

$(y_1, y_2, y_3, t_1, t_2, a_1, a_2) = (0, 0, 0, 0, 0, 4, 9)$

Simplex Method (Penalty Method)

A method setting the initial feasible solution
by adding “Artificial Variables” into constraint equation.

Impose the enough large amount of penalty on the Artificial Variable in the objective function.

- To maximize the objective function, set the enough small amount of number “–M” for the coefficients for the artificial variables .

- To minimize the objective function, set the enough large amount of number “M” for the coefficients for the artificial variables.

Simplex Method (Penalty Method)

Minimize

$$z = 9y_1 + 10y_2 + 2y_3 + Ma_1 + Ma_2$$

subject to

$$y_1 + 4y_2 + y_3 - t_1 \qquad + a_1 \qquad = 4$$
$$3y_1 + y_2 + y_3 \qquad - t_2 \qquad + a_2 = 9$$
$$y_1, y_2, y_3, t_1, t_2, a_1, a_2 >= 0$$

c_j	Basic Variable	9	10	2	0	0	M	M		
		y_1	y_2	y_3	t_1	t_2	a_1	a_2	b_i	b_i/a_{ij}
M	a_1	1	4	1	-1	0	1	0	4	1
M	a_2	3	1	1	0	-1	0	1	9	9
	z_j	4M	5M	2M	-M	-M				
	$c_j - z_j$	9-4M	10-5M	2-2M	M	M				

Simplex Method (Penalty Method)

c_j	Basic Variable	9	10	2	0	0	M	M		
		y_1	y_2	y_3	t_1	t_2	a_1	a_2	b_i	b_i/a_{ij}
M	a_1	1	4	1	-1	0	1	0	4	1
M	a_2	3	1	1	0	-1	0	1	9	9
	z_j	4M	5M	2M	-M	-M				
	$c_j - z_j$	9-4M	10-5M	2-2M	M	M				

c_j	Basic Variable	9	10	2	0	0	M	M		
		y_1	y_2	y_3	t_1	t_2	a_1	a_2	b_i	b_i/a_{ij}
10	y_2	1/4	1	1/4	-1/4	0	1/4	0	1	4
M	a_2	11/4	0	3/4	1/4	-1	-1/4	1	8	32/11
	z_j	10/4+11M/4		10/4 +3M/4	-10/4+M/4	-M	10/4-M/4			
	$c_j - z_j$	26/4-11M/4		-2/4 -3M/4	10/4-M/4	M	-10/4+5M/4			

Simplex Method (Penalty Method)

c_j	Basic Variable	9	10	2	0	0	M	M		
		y_1	y_2	y_3	t_1	t_2	a_1	a_2	b_i	b_i/a_{ij}
10	y_2	1/4	1	1/4	-1/4	0	1/4	0	1	4
M	a_2	11/4	0	3/4	1/4	-1	-1/4	1	8	32/11
	z_j	10/4+11M/4		10/4 +3M/4	-10/4+M/4	-M	10/4-M/4			
	$c_j - z_j$	26/4-11M/4		-2/4 -3M/4	10/4-M/4	M	-10/4+5M/4			

c_j	Basic Variable	9	10	2	0	0	M	M		
		y_1	y_2	y_3	t_1	t_2	a_1	a_2	b_i	b_i/a_{ij}
10	y_2	0	1	2/11	-3/11	1/11	3/11	-1/11	3/11	3/2
9	y_1	1	0	3/11	1/11	-4/11	-1/11	4/11	32/11	32/3
	z_j			47/11	-21/11	-26/11	21/11	26/11		
	$c_j - z_j$			-25/11	21/11	26/11	M-21/11	M-26/11		

Simplex Method (Penalty Method)

c_j	Basic Variable	9	10	2	0	0	M	M		
		y_1	y_2	y_3	t_1	t_2	a_1	a_2	b_i	b_i/a_{ij}
10	y_2	0	1	2/11	-3/11	1/11	3/11	-1/11	3/11	3/2
9	y_1	1	0	3/11	1/11	-4/11	-1/11	4/11	32/11	32/3
	z_j			47/11	-21/11	-26/11	21/11	26/11		
	$c_j - z_j$			-25/11	21/11	26/11	M-21/11	M-26/11		

c_j	Basic Variable	9	10	2	0	0	M	M		
		y_1	y_2	y_3	t_1	t_2	a_1	a_2	b_i	b_i/a_{ij}
2	y_3	0	11/2	1	-3/2	1/2	3/2	-1/2	3/2	-1
9	y_1	1	-3/2	0	1/2	-1/2	-1/2	1/2	5/2	5
	z_j		-5/2		3/2	-7/2	-3/2	7/2		
	$c_j - z_j$		25/2		-3/2	7/2	M+3/2	M-7/2		

Simplex Method (Penalty Method)

c_j	Basic Variable	9	10	2	0	0	M	M		
		y_1	y_2	y_3	t_1	t_2	a_1	a_2	b_i	b_i/a_{ij}
2	y_3	0	11/2	1	-3/2	1/2	3/2	-1/2	3/2	-1
9	y_1	1	-3/2	0	1/2	-1/2	-1/2	1/2	5/2	5
	z_j	9	-5/2	2	3/2	-7/2	-3/2	7/2	51/2	
	$c_j - z_j$	0	25/2	0	-3/2	7/2	M+3/2	M-7/2		

c_j	Basic Variable	9	10	2	0	0	M	M		
		y_1	y_2	y_3	t_1	t_2	a_1	a_2	b_i	b_i/a_{ij}
2	y_3	3	1	1	0	-1	0	1	9	
0	t_1	2	-3	0	1	-1	-1	1	5	
	z_j	6	2			-2	0	2	18	
	$c_j - z_j$	3	8			2	M	M-2		

Simplex Method (Penalty Method)

Minimize

$z = 9y_1 + 10y_2 + 2y_3$

subject to

$y_1 + 4y_2 + y_3 >= 4$

$3y_1 + y_2 + y_3 >= 9$

$y_1, y_2, y_3 >= 0$

Nutrient \ Food	F_1	F_2	F_3	Amount
A	1	4	1	4
B	3	1	1	9
Price per unit	9	10	2	

surplus Variable

Minimize

$z = 9y_1 + 10y_2 + 2y_3$

subject to

$y_1 + 4y_2 + y_3 - t_1 = 4$

$3y_1 + y_2 + y_3 - t_2 = 9$

$y_1, y_2, y_3, t_1, t_2 >= 0$

"No initial feasible solution "

Artificial Variable

Minimize

$z = 9y_1 + 10y_2 + 2y_3 + Ma_1 + Ma_2$

subject to

$y_1 + 4y_2 + y_3 - t_1 + a_1 = 4$

$3y_1 + y_2 + y_3 - t_2 + a_2 = 9$

$y_1, y_2, y_3, t_1, t_2, a_1, a_2 >= 0$

"Optimal solution"

$(y_1, y_2, y_3, t_1, t_2, a_1, a_2) = (0, 0, 9, 5, 0, 0, 0)$

Duality Principle

Primal Problem <P>	Dual Problem <D>

Primal Problem <P>

$$\text{maximize } z = \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to } \sum_{j=1}^{n} a_{1j} x_j \le b_1$$

....

$$\sum_{j=1}^{n} a_{mj} x_j \le b_m$$

$$x_1 \ge 0, \ \dots, \ x_n \ge 0$$

Maximize ---------------------------

$z = 4x_1 + 9x_2$

subject to ---------------------------

$x_1 + 3y_2 <= 9$

$4x_1 + x_2 <= 10$

$x_1 + x_2 <= 2$

$x_1, x_2 >= 0$

Dual Problem <D>

$$\text{minimize } z = \sum_{i=1}^{m} b_i y_i$$

$$\text{subject to } \sum_{i=1}^{m} a_{i1} y_i \ge c_1$$

....

$$\sum_{i=1}^{m} a_{in} y_i \ge c_n$$

$$y_1 \ge 0, \ \dots, \ y_m \ge 0$$

Minimize ------------------------------

$z = 9y_1 + 10y_2 + 2y_3$

subject to ------------------------------

$y_1 + 4y_2 + y_3 >= 4$

$3y_1 + y_2 + y_3 >= 9$

$y_1, y_2, y_3 >= 0$

- In case that there is an optimal solution of either primal problem or dual problem, the other problem also has the solution and the maximum value of the primal problem is the same with the minimum value of the dual problem.
- When the objective function of either of these problems is not bounded, the other problem has no feasible solution.

Primal Problem

c_j	basic Variable	4	9	0	0	0	0	
		x_1	x_2	s_1	s_2	s_3	b_i	b_i/a_{ij}
0	s_1	1	3	1	0	0	9	3
0	s_2	4	1	0	1	0	10	10
0	s_3	1	1	0	0	1	2	2
	z_j	0	0	0	0	0	0	
	$c_j - z_j$	4	9	0	0	0		

c_j	basic Variable	4	9	0	0	0	0	
		x_1	x_2	s_1	s_2	s_3	b_i	b_i/a_{ij}
0	s_1	-2	0	1	0	-3	3	
0	s_2	3	0	0	1	-1	8	
9	x_2	1	1	0	0	1	2	
	z_j	9	9	0	0	9	18	
	$c_j - z_j$	-5	0	0	0	-9		

Primal Problem

c_j	Basic Variable	4	9	0	0	0	0	
		x_1	x_2	s_1	s_2	s_3	b_i	b_i/a_{ij}
0	s_1	-2	0	1	0	-3	3	
0	s_2	3	0	0	1	-1	8	
9	x_2	1	1	0	0	1	2	
	z_j	9	9	0	0	9	18	
	$c_j - z_j$	-5	0	0	0	-9		

$$c_0 = \begin{bmatrix} 0 & 0 & 9 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} = c_0 B = \begin{bmatrix} 0 & 0 & 9 \end{bmatrix}$$

- Set vector (row vector) of c_j to $\mathbf{c}_0$, which is the coefficient for basic variable in the final step of the Simplex method on the primal problem
- Set matrix of a_{ij} to **B** which is the coefficient for each initial basic variable of each constraint equation in the final step of the Simplex method on the primal problem.
- The optimal solution of dual problem can be taken as $\mathbf{c}_0\mathbf{B}$.

Application 1: Production Planning Problem

A problem to determine the production balance for Month/Week/Day time Range.

＜Example＞

A factory takes orders for the next 3 months and the required product numbers of items are 550, 800 and 200 for each month. The delivery of product should be finished in due time.

It takes 23,000Yen per item in the daytime work and it is possible to make 680 items per one month. And also it is possible to increase the productivity up to another 50, but it costs 28,000Yen in overtime work.

The additional cost to stock products is 2000Yen per one month. How should we make a production schedule for 3 months satisfying the demand of 3 months with minimum cost?

Application 1: Production Planning Problem

x_i: Production Quantities in month i
y_i: Production Quantities of overtime in month i
s_i: Quantities of inventory at the end of the month i
s_0 (initial inventory) = 0。

Minimize $z = 23{,}000 \times (x_1 + x_2 + x_3)$
$+ 28{,}000 \times (y_1 + y_2 + y_3)$
$+ 2{,}000 \times (s_1 + s_2)$

Subject to
$x_1 + y_1 >= 550$
$x_2 + y_2 + s_1 >= 800$
$x_3 + y_3 + s_2 >= 200$
$x_i <= 680, \; y_i <= 50$
$x_1 + y_1 - s_1 = 550$
$x_2 + y_2 + s_1 - s_2 = 800$
$x_i, y_i >= 0, \; s_i >= 0$
$i = 1, 2, 3$

Common Characteristic of Production Planning Problems

- Divide the whole product time period into time bucket, and deal with the *total* production quantity in each bucket.
- Final goal is to minimize the cost.
- Production capacity, inventory capacity, demand and so on are given as constraints
- Production quantities, required material quantities, used facilities and so on are prepared as variables for constraints and objective function.

Practical Difficulties of Production Planning Problems

- In case of producing large number of product or coordinating at the various process stages of production, production planning problem becomes a little bit complicated.
- If set-up time is important to decide the capacity of the process, it is necessary to make a precise capacity schedule.
- In case of determination of the production balance according to the demand, the reliability of demand forecasting data should be more important.

Application 2: Mixing Optimization Problem

A problem to determine the mixing ratio of components with different qualities to make a product with the required quality

＜Example＞
At the oil refinery, there are 3 kinds of component oil 1, 2, 3. By mixing these components, 3 kinds of gasoline A, B, C are produced. Detailed qualities of property are described as following. This quality is linear for volume. Give production quantity for each gasoline and mixing ratio for each component satisfying the maximum profit.

Comp.	available Volume (BL/day)	Octane num
1	3,800	107
2	2,652	93
3	4,081	87

Gasoline	Octane num (more than)	Profit/BL
A	87	4.5
B	89	4.9
C	98	6.0

Application 2: Mixing Optimization Problem

x_{ij}: Quantity of component i used for gasoline j

j = A, B, C $\quad$ i = 1, 2, 3

Maximize
$$z = 4.5 \times (x_{1A} + x_{2A} + x_{3A}) + 4.9 \times (x_{1B} + x_{2B} + x_{3B}) + 6.0 \times (x_{1C} + x_{2C} + x_{3C})$$

Subject to

$$x_{1A} + x_{1B} + x_{1C} =< 3800$$
$$x_{2A} + x_{2B} + x_{2C} =< 2652$$
$$x_{3A} + x_{3B} + x_{3C} =< 4081$$
$$(107x_{1A} + 93\ x_{2A} + 87x_{3A}) / (x_{1A} + x_{2A} + x_{3A}) >= 87$$
$$(107x_{1B} + 93\ x_{2B} + 87x_{3B}) / (x_{1B} + x_{2B} + x_{3B}) >= 89$$
$$(107x_{1C} + 93\ x_{2C} + 87x_{3C}) / (x_{1C} + x_{2C} + x_{3C}) >= 98$$
$$x_{ij} >= 0, \quad i = 1, 2, 3, \quad j = \text{A, B, C}$$

Common Characteristic of Mixing Optimization Problems

- It deals with the production process mixing raw materials.
- In mixing process, the quality of product is calculated by the linear combination of mixing ratios and qualities of components.
- Available component quantity, required production quantity and property characteristic are given.

Practical Difficulties of Mixing Optimization Problems

- In case of adapting to time-series change or taking into account the component purchase, it will be more complicated.
- If the quality function is non-linear, the function should be translated to linear function.

etc.

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Thank you