Exercise 2: Matrix and Gradient

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

とする。

1. $\boldsymbol{x}^T\boldsymbol{y} = \boldsymbol{y}^T\boldsymbol{x} = x_1y_1 + \ldots + x_ny_n$ を示せ。
2. $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}$ を成分 $(x_i, a_{i,j})$ で表せ。
3. $\frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{a}^T\boldsymbol{x} = \boldsymbol{a}$ を示せ。
4. $\frac{\partial}{\partial \boldsymbol{x}}(\boldsymbol{a}^T\boldsymbol{x} + b)^2$ を求めよ。

1. $x^T y = (x_1, x_2, \cdots, x_n)\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$

$y^T x = (y_1, y_2, \cdots, y_n)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$

thus, $x^T y = y^T x = x_1y_1 + x_2y_2 + \cdots + x_ny_n$

2. $x^T A x = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i a_{ij} x_j$

3. $a^T x = \sum_{i=1}^{n} a_i x$

thus $\frac{\partial}{\partial x}(a^T x) = \sum_{i=1}^{n} a_i = a$

4. $\frac{\partial}{\partial \boldsymbol{x}}(\boldsymbol{a}^T\boldsymbol{x} + b)^2$ を求めよ。

let $u = a^T x + b$

then $\frac{\partial}{\partial x}(a^T x + b)^2 = \frac{\partial}{\partial x}(u^2) = 2u \cdot \frac{\partial}{\partial x}(u)$

according to 3, $\frac{\partial}{\partial x}(a^T x) = a$

thus $\frac{\partial}{\partial x}(a^T x + b)^2 = 2(a^T x + b) \cdot a$