SMART
INDUSTRY
LABORATORY

Scheduling Algorithms (5)

- Production Planning Algorithm (2) -

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Simplex method

Basic Method to derive an optimal solution for Linear Programming problem,

finding only the extreme points increasing the value of the objective function.

Example of Production Planning

- Make two products called ProA and ProB
- Resources to make products, quantities of those for each product and the upper limits for those are given in the following table.

	ProA	ProB	Upper Limit
Material（kg）	4	11	440
Man Power（man-hour）	5	7	350
processing time（hour）	7	6	420

Profit for unit quantity of ProA is 40k¥.

Profit for unit quantity of ProB is 90k¥.

Decide the optimal quantities to get the largest total profit.

Transformation to Standard form

Objective	total benefit	: $\max z = 4x_1 + 9x_2$
subject to	material constraint	: $4x_1 + 11x_2 <= 440$
	man power constraint	: $5x_1 + 7x_2 <= 350$
	facility constraint	: $7x_1 + 6x_2 <= 420$
	non-negative constraint	: $x_1, x_2 >= 0$

$$\begin{array}{ll} \text{maximize} & z = 4x_1 + 9x_2 \\ \text{subject to} & 4x_1 + 11x_2 + s_1 = 440 \\ & 5x_1 + 7x_2 + s_2 = 350 \\ & 7x_1 + 6x_2 + s_3 = 420 \\ & x_1, x_2, s_1, s_2, s_3 >= 0 \end{array}$$

Standard Form

maximize $z = 4x_1 + 9x_2$

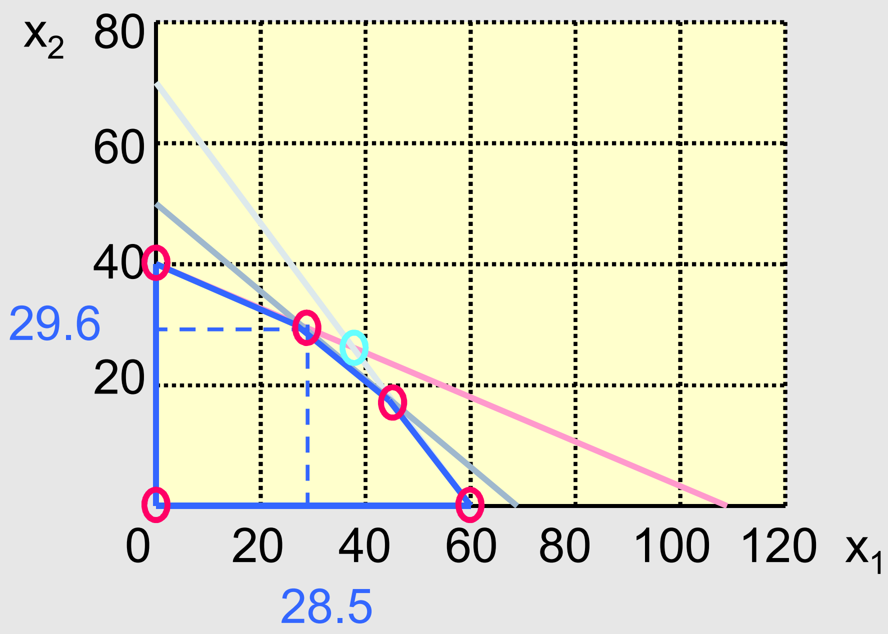
subject to

$$
\begin{aligned}
4x_1 + 11x_2 + s_1 &= 440 \\
5x_1 + 7x_2 + s_2 &= 350 \\
7x_1 + 6x_2 + s_3 &= 420
\end{aligned}
$$

$x_1, x_2, s_1, s_2, s_3 >= 0$

Select x_1, x_2 as independent variables,
substitute 0 for these variables.
Values of s_1, s_2, s_3 are decided.

Extreme Points in the all of the intersection point of constraint equations



x_1	x_2	s_1	s_2	s_3
0	0	440	350	420
0	40	0	70	180
0	50	-110	0	120
0	70	-330	-140	0
110	0	0	-200	-350
70	0	160	0	-70
60	0	200	50	0
28.5	29.6	0	0	42.6
37.4	26.4	0	-21.8	0
44.2	18.4	60.8	0	0

Simplex method

Step1: Make the following table (Simplex Table),
select the basic variables.

c_j	Basic Variable	4	9	0	0	0	0	
		x_1	x_2	s_1	s_2	s_3	b_i	b_i/a_{ij}
0	s_1	4	11	1	0	0	440	
0	s_2	5	7	0	1	0	350	
0	s_3	7	6	0	0	1	420	
	z_j							
	$c_j - z_j$							

Select s_1, s_2, s_3 as the basic variables.
Solution: $(z, x_1, x_2, s_1, s_2, s_3) = (0, 0, 0, 440, 350, 420)$
all variables are non-negative $\Rightarrow$ extreme point in the feasible region.
$(x_1, x_2) = (0, 0)$, $z = 0$

Simplex method

Step2: Evaluate whether the solution is an optimal solution or not.

c_j	Basic Variable	4	9	0	0	0	0	
		x_1	x_2	s_1	s_2	s_3	b_i	b_i/a_{ij}
0	s_1	4	11	1	0	0	440	
0	s_2	5	7	0	1	0	350	
0	s_3	7	6	0	0	1	420	
	z_j	0	0					
	$c_j - z_j$	4	9					

When the value of a non-basic variable is changed to positive number from 0,
if it makes the value of the objective function increase,
this solution is not the optimal solution.
If there is not such variable, this solution is the optimal solution.
In this example, the coefficients of x_1 and x_2 are positive,
so if the variables are changed to positive, z will increase.
Then it is not the optimal solution.

Simplex method

Step3: Change the combination of the basic variables and non-basic variables.
(Find the other extreme point)
Step3-1: Select the variable to change from non-basic variable to basic variable.
(Select the variable that coefficient for z is biggest positive.
Because it will increase the objective function value most.)

c_j	Basic Variable	4	9	0	0	0	0	
		x_1	x_2	s_1	s_2	s_3	b_i	b_i/a_{ij}
0	s_1	4	11	1	0	0	440	
0	s_2	5	7	0	1	0	350	
0	s_3	7	6	0	0	1	420	
	z_j	0	0					
	$c_j - z_j$	4	9					

Simplex method

Step3-2: Select the variable to change from basic variable to non-basic variable.

Calculate the Increase limit according to the following equation.
Select the basic variable that the increase limit is the non-negative smallest one

Increase limit = bi / aij

bi: Right side fixed value bi of the focused constraint equation

aij: coefficient value of the non-basic variable selected in Step3-1 and the focused constraint equation

c_j	Basic Variable	4	9	0	0	0	0	
		x_1	x_2	s_1	s_2	s_3	b_i	b_i/a_{ij}
0	s_1	4	11	1	0	0	440	40
0	s_2	5	7	0	1	0	350	50
0	s_3	7	6	0	0	1	420	70
	z_j	0	0					
	$c_j - z_j$	4	9					

Simplex method

Step3-3: Remake Simplex table for new basic variable

pivot: the intersection of new basic variable (column) and new non-basic variable (row)

① To set 1 to the pivot, Divide all values of the row by the pivot value.

② Set 0 to cells of the other rows of the pivot column

(ex: for s_2 row, firstly all values of the pivot row times 7,

then subtract these values from the values of s_2 row)

c_j	Basic Variable	4	9	0	0	0	0	
		x_1	x_2	s_1	s_2	s_3	bi	bi/aij
0	s_1	4	11	1	0	0	440	
0	s_2	5	7	0	1	0	350	
0	s_3	7	6	0	0	1	420	

c_j	Basic Variable	4	9	0	0	0	0	
		x_1	x_2	s_1	s_2	s_3	bi	bi/aij
9	x_2	4/11	1	1/11	0	0	40	
0	s_2	27/11	0	-7/11	1	0	70	
0	s_3	53/11	0	-6/11	0	1	180	

Simplex method

Repeat Step2～3 until finding the optimal solution

c_j	Basic Variable	4	9	0	0	0	0	
		x_1	x_2	s_1	s_2	s_3	bi	bi/aij
9	x_2	4/11	1	1/11	0	0	40	
0	s_2	27/11	0	-7/11	1	0	70	
0	s_3	53/11	0	-6/11	0	1	180	
	z_j	36/11		9/11				
	$c_j - z_j$	8/11		-9/11				

c_j	Basic Variable	4	9	0	0	0	0	
		x_1	x_2	s_1	s_2	s_3	bi	bi/aij
9	x_2	4/11	1	1/11	0	0	40	110
0	s_2	27/11	0	-7/11	1	0	70	28.5
0	s_3	53/11	0	-6/11	0	1	180	37.4
	z_j	36/11		9/11				
	$c_j - z_j$	8/11		-9/11				

$$z_j = \Sigma c_i * a_{ij}$$

Meaning of Zj

c_j	Basic Variable	4	9	0	0	0	0	
		x_1	x_2	s_1	s_2	s_3	bi	bi/aij
9	x_2	4/11	1	1/11	0	0	40	
0	s_2	27/11	0	-7/11	1	0	70	
0	s_3	53/11	0	-6/11	0	1	180	
	z_j	36/11		9/11				
	$c_j - z_j$	8/11		-9/11				

$z_j = \Sigma c_i * a_{ij}$

Const 1: $4/11*x_1 + 1*x_2 + 1/11*s_1 + 0*s_2 + 0*s_3 = 40$

if x_1 increases 1, then x_2 has to decrease 4/11 and z will decrease 9 * 4 /11.

Const 2: $27/11*x_1 + 0*x_2 - 7/11*s_1 + 1*s_2 + 0*s_3 = 70$

if x_1 increases 1, then s_2 has to decrease 27/11 and z will not change.

Const 3: $53/11*x_1 + 0*x_2 + 1/11*s_1 + 0*s_2 + 1*s_3 = 40$

if x_1 increases 1, then s_3 has to decrease 53/11 and z will not change.

Simplex method

Repeat Step2～3 until to find the optimal solution

c_j	Basic Variable	4	9	0	0	0	0	
		x_1	x_2	s_1	s_2	s_3	bi	bi/aij
9	x_2	4/11	1	1/11	0	0	40	110
0	s_2	27/11	0	-7/11	1	0	70	28.5
0	s_3	53/11	0	-6/11	0	1	180	37.4
	z_j	36/11		9/11				
	$c_j - z_j$	8/11		-9/11				

c_j	Basic Variable	4	9	0	0	0	0	
		x_1	x_2	s_1	s_2	s_3	bi	bi/aij
9	x_2	4/11	1	1/11	0	0	40	
0	s_2	1	0	-7/27	11/27	0	770/27	
0	s_3	53/11	0	-6/11	0	1	180	
	z_j							
	$c_j - z_j$							

Simplex method

c_j	Basic Variable	4	9	0	0	0	0	
		x_1	x_2	s_1	s_2	s_3	bi	bi/aij
9	x_2	4/11	1	1/11	0	0	40	
0	s_2	1	0	-7/27	11/27	0	770/27	
0	s_3	53/11	0	-6/11	0	1	180	
	z_j							
	$c_j - z_j$							

c_j	Basic Variable	4	9	0	0	0	0	
		x_1	x_2	s_1	s_2	s_3	bi	bi/aij
9	x_2	0	1	55/297	-44/297	0	800/27 =29.6	
4	x_1	1	0	-7/27	11/27	0	770/27 =28.5	
0	s_3	0	0	209/297	-53/27	1	1150/27 =42.6	
	z_j	4	9	17/27 =0.630	8/27 =0.296		380.8	
	$c_j - z_j$			-0.630	-0.296			

The optimal solution （$z, x_1, x_2, s_1, s_2, s_3$) = (380.8, 28.5, 29.6, 0, 0, 42.6)
The extreme point for the optimal solution (x_1,x_2) = (28.5, 29.6)
The optimal value z = 380.8

Analysis of Simplex table

c_j	Basic Variable	4	9	0	0	0	0	
		x_1	x_2	s_1	s_2	s_3	bi	bi/aij
9	x_2	0	1	55/297	-44/297	0	800/27 =29.6	
4	x_1	1	0	-7/27	11/27	0	770/27 =28.5	
0	s_3	0	0	209/297	-53/27	1	1150/27 =42.6	
	z_j	4	9	17/27 =0.630	8/27 =0.296		380.8	
	$c_j - z_j$			-0.630	-0.296			

There is a margin in 42.6 hours in facility using time.

$s_1 = 0, s_2 = 0$

$209/297 * s_1 - 53/27 * s_2 + s_3 = 42.6$

$s_3 = 42.6$

facility constraint (upper limit for facility is 420 hours) :

$7x_1 + 6x_2 + s_3 = 420$

Analysis of Simplex table

c_j	Basic Variable	4	9	0	0	0	0	
		x_1	x_2	s_1	s_2	s_3	bi	bi/aij
9	x_2	0	1	55/297	-44/297	0	800/27 =29.6	
4	x_1	1	0	-7/27	11/27	0	770/27 =28.5	
0	s_3	0	0	209/297	-53/27	1	1150/27 =42.6	
	z_j	4	9	17/27 =0.630	8/27 =0.296		380.8	
	$c_j - z_j$			-0.630	-0.296			

This solution means all of the given material and man-power are used.

From the following equation,

$$z = 380.8 - 0.630s_1 - 0.296s_2$$

- If we can use more 1 kg for material,
  the optimal value will increase by 0.630 × 10k¥.
- If we can use 1 man-hour for man power,
  the optimal vale will increase by 0.296 × 10k¥.

these values are called shadow price or marginal price

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Thank you