ME 231- Solid Mechanics

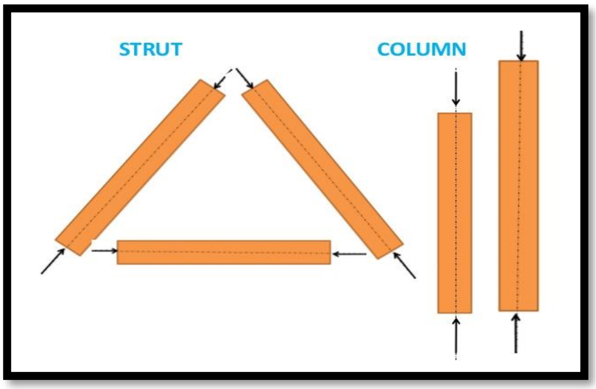
Term Paper- Analysis of Buckling of Columns

S.V.Koushik (11941070)

❖ Introduction:

Struts: Struts are the structural members that are subjected to an axial compressive load (i.e. the force which is compressive in nature and always acted along the axis of the member). Struts can be aligned either horizontally, vertically, or even in an inclined position.

Columns: Columns are also the structural members that are subjected to axial compressive load but they are always aligned vertically. By this, one could say that "All the columns are struts, but all struts are not columns".



But both the words are generally interchangeably used in reference to mechanical engineering.

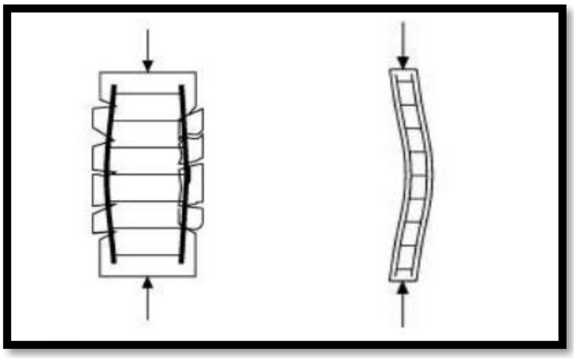
These are generally seen as vertical structural members in buildings.

The columns can be classified based on the type of its failure:

- **Short Columns:** These columns are generally failed due to the phenomenon called "Crushing".
- **Long Columns:** These columns are generally failed due to the phenomenon known as "Buckling or Crippling".
- **Intermediate or Medium Column:** These columns are failed due to the combined effect of both "Crushing" and "Buckling".

Now the following are some of the terminologies used in this analysis:

i. **Crushing: Crushing** is a type of failure where the applied stress exceeds the yield stress so as to cause plastic deformation.
ii. **Buckling: Buckling** is a form of deformation or deflection sideways when an axial compressive force is acted on it.



Crushing Buckling

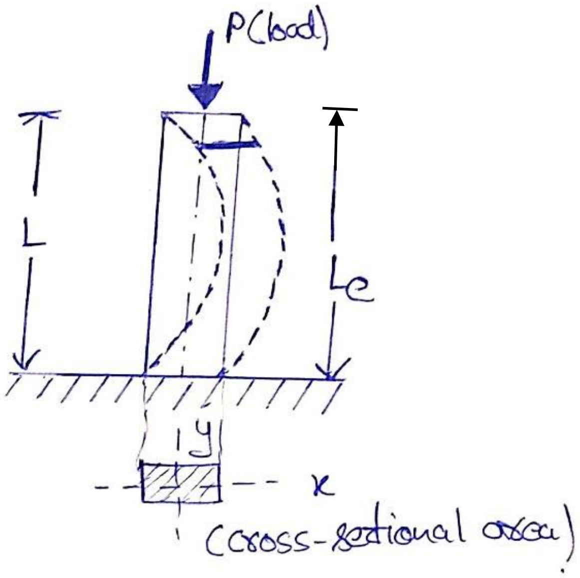
iii. **Critical Buckling Load: Critical buckling load** is the greatest load which does not cause the column to buckle i.e. deflection laterally. Above this load, the columns start to buckle and lose their stability and come to a state of unstable equilibrium.

iv. **Effective Length (L_e): Effective length** is defined as the distance between successive inflection points once the column has buckled. (Or) The actual length of the column that takes part in bending.

v. **Slenderness Ratio (λ):** It is defined as the ratio of effective length (L_e) of the column to that of the column's minimum radius of gyration (K_{min}).
This ratio generally differentiates the long columns from short and intermediate columns.

$$\boxed{\lambda = \frac{L_e}{K_{min}}}$$

where $\boxed{K_{min} = \sqrt{\frac{I_{min}}{A}}}$

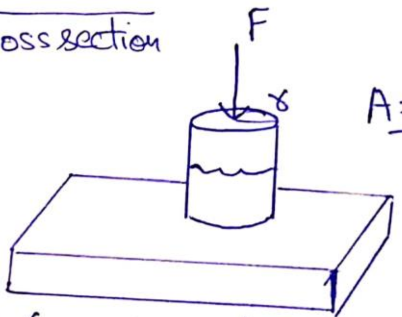


❖ Stability of Short Columns:

- The short columns are the columns that have its cross-sectional dimensions comparable to that of its length.
- Short columns are generally failed due to crushing i.e. the compression failure where the applied stress exceeds the yield stress leading to the permanent failure of the column.

$$\sigma_{Applied} = \frac{F_{applied}}{\text{Area of cross section}}$$

$$\sigma_{Applied} = \frac{F}{A}$$



$A = \pi r^2$

* let σ_y be the yield strength of the material (σ_y is material property)

* For making our column safe (columns to be stable)

$$\boxed{\sigma_{Applied} \leq \sigma_y}$$

$$\Rightarrow \boxed{\frac{F}{A} \leq \sigma_y}$$

The limit is σ_y (yield strength) for the ductile materials as they tend to show plastic deformation after the crossing the yield point. But, in the case of brittle materials, this limit is σ_{ut} (ultimate tensile strength) as they do not show the plastic deformation and fail after the ultimate tensile stress.

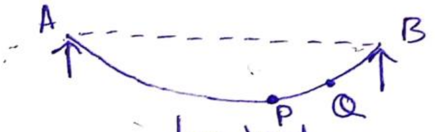
So, the critical load of short columns to be safe becomes $\boldsymbol{\sigma_y} \times \mathbf{A}$, where $\mathbf{A}$ is the area cross-section.

❖ Buckling of Long Columns:

Firstly, let us derive the deflection equation for beams which would be helpful in deriving Euler's equations of buckling in the case of slender or long columns.

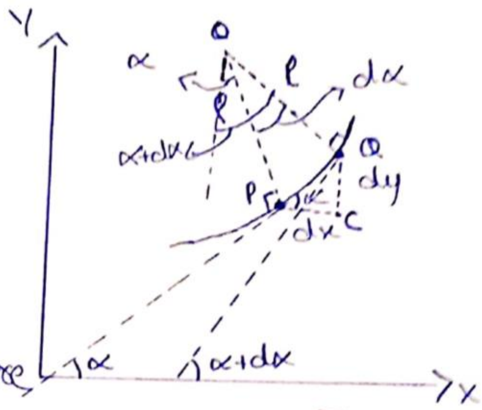
Deflection Equation in Beams:

Unloaded
(when no force is applied)



loaded
(when force is applied
and bending has started)

Considering points P and Q on the beam and representing them on co-ordinate axes such that P and Q are very close to each other



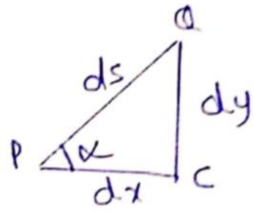
$\Rightarrow$ arc PQ $\simeq$ line PQ

and $OP = OQ = \rho$ = Radius of curvature

In Δ^{le} PQC, $QC = dy$ [change of length in y direction]

$PC = dx$ [change of length in x direction]

Consider $PQ = ds$ ~~[chan~~



In Δ^{le} PQC, $\sin\alpha = \dfrac{dy}{ds}$ —— ①

$\cos\alpha = \dfrac{dx}{ds}$ —— ②

$\tan\alpha = \dfrac{dy}{dx}$ —— ③

Consider the arc PQO, and angle = $\dfrac{\text{Arc length}}{\text{radius}}$

$$\Rightarrow \quad d\alpha = \frac{ds}{\rho}$$

$$\Rightarrow \quad \rho = \frac{ds}{d\alpha}$$

This can be written as,

$$\rho = \frac{ds/dx}{d\alpha/dx}$$

$$\rho = \frac{1/\cos\alpha}{d\alpha/dx} \quad \text{[from eq (2)]}$$

$$\rho = \frac{\sec\alpha}{d\alpha/dx}$$

but from eq (3) $\Rightarrow$ $\tan\alpha = dy/dx$

$$\sec^2\alpha \frac{d\alpha}{dx} = \frac{d^2y}{dx^2}$$

$$\Rightarrow \quad \rho = \frac{\sec\alpha}{\left(\frac{d^2y}{dx^2}\right)\frac{1}{\sec^2\alpha}}$$

$$\rho = \frac{\sec^3\alpha}{\frac{d^2y}{dx^2}} \Rightarrow \frac{1}{\rho} = \frac{d^2y/dx^2}{\cancel{E}(1+\tan^2\alpha)^{3/2}}$$

As α is very small $\Rightarrow$ $\tan^2\alpha$ tends to zero

$$\Rightarrow \quad \frac{1}{\rho} = \frac{d^2y}{dx^2}$$

But, if we consider this ~~to~~ to be a pure ~~bendending~~ bending

we have, $\frac{1}{\rho} = \frac{M_b}{EI}$ where M_b is bending moment

I is moment of inertia about neutral axis

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{M_b}{EI}$$

$$\Rightarrow \quad \boxed{M_b = EI\frac{d^2y}{dx^2}}$$

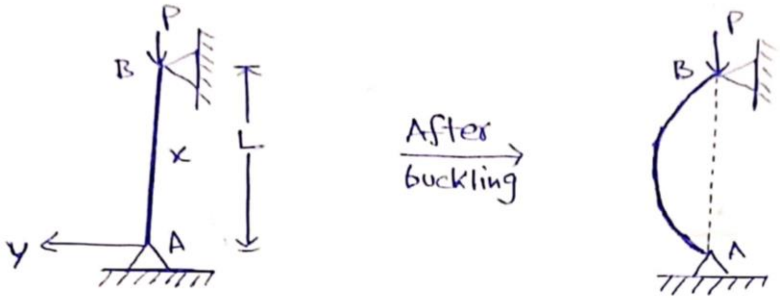
The formula for the Critical Buckling Load is given by **Euler,** so it is called the **Euler Buckling Load.**

The long columns can be divided on the basis of the end conditions:

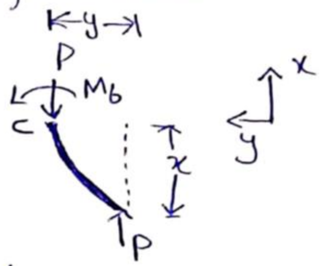
a) **Hinged at both the ends**
b) **Fixed at one end and Hinged at the other**
c) **Fixed at both the ends**
d) **Free at one end and Fixed at the other**

Since a column can be considered as a beam placed in a vertical position and subjected to an axial load, we could also use the above deflection equation in the case of the columns. Here, the axes are rotated and we consider the vertical axis as the x-axis and horizontal axes as the y-axis. The height of the column is measured in the x-direction and the deflection of the column corresponding to that height is measured in the y-direction.

a) Column with both ends Hinged:



By taking a section of column and analysing the stresses and bending moments.



Applying moment equilibrium at C,

$$M_b + Py = 0$$

$$M_b = -Py$$

By using deflection equation, $EI\frac{d^2y}{dx^2} = M_b$

$$\Rightarrow EI\frac{d^2y}{dx^2} = -Py$$

$$\Rightarrow EI\frac{d^2y}{dx^2} + Py = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + \frac{Py}{EI} = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + A^2y = 0 \quad \left[\text{Consider } A^2 = \frac{P}{EI}\right]$$

Put $y = e^{mx} \Rightarrow m^2e^{mx} + A^2y = 0$

$$y(A^2 + m^2) = 0$$

$$\Rightarrow A^2 + m^2 = 0$$

$$\Rightarrow m = \pm iA \Rightarrow y = e^{\pm iAx}$$

$$\Rightarrow y = C_1 \sin Ax + C_2 \cos Ax$$

Applying boundary conditions $\Rightarrow$ At $x = 0$, $y = 0$

$$\Rightarrow \boxed{0 = C_2}$$

At $x = L$, $y = 0$

$$\Rightarrow 0 = C_1 \sin AL$$

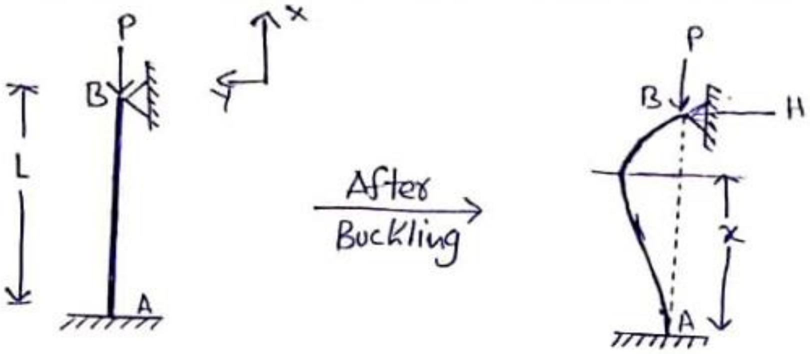
$$\Rightarrow AL = n\pi \quad [n = 1, 2, 3, \cdots]$$

$$\Rightarrow A^2 = \frac{n^2\pi^2}{L^2} \Rightarrow \frac{P}{EI} = \frac{n^2\pi^2}{L^2}$$

for $n = 1$, $P_{cr} = \frac{n^2\pi^2EI}{L^2} \Rightarrow \boxed{P_{cr} = \frac{\pi^2EI}{L^2}}$

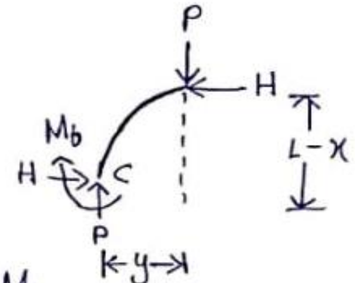
Here, I is the moment of Inertia about the axis which gives minimum value as the column buckles in that way (about that axis)

b) Column with one end Fixed and other is Hinged:



Considering a section of buckled column.

Applying moment equilibrium at C



$$\Rightarrow \quad -M_b + (-P(y)) + H(L-x) = 0$$

$$\Rightarrow \quad M_b = H(L-x) - Py.$$

By using deflection equation, $EI\frac{d^2y}{dx^2} = M_b$

$$\Rightarrow EI\frac{d^2y}{dx^2} = H(L-x) - Py.$$

$$\Rightarrow \frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{H}{EI}(L-x)$$

$$\Rightarrow \frac{d^2y}{dx^2} + A^2y = A^2\frac{H}{P}(L-x) \quad \left[\text{Consider } A^2 = \frac{P}{EI}\right]$$

By solving the differential equation similar to the previous case

$$\Rightarrow y = C_1 \sin Ax + C_2 \cos Ax + \frac{H}{P}(L-x)$$

Applying boundary conditions,

At $x=0,\ y=0 \Rightarrow 0 = C_2 + \frac{HL}{P} \Rightarrow C_2 = -\frac{HL}{P}$

At $x=L,\ y=0 \Rightarrow 0 = C_1 \sin AL - \frac{HL}{P}\cos AL$ — ①

At $x=0,\ \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = C_1 A\cos Ax - C_2 A\sin Ax - \frac{H}{P}$

(fixed end)

$$\Rightarrow 0 = C_1 A - \frac{H}{P}$$

$$\Rightarrow C_1 = \frac{H}{PA}$$

$$\text{eq ①} \Rightarrow \tan AL = \frac{\cancel{H}}{\cancel{P}A} \Big/ \frac{\cancel{H}L}{\cancel{P}} = AL$$

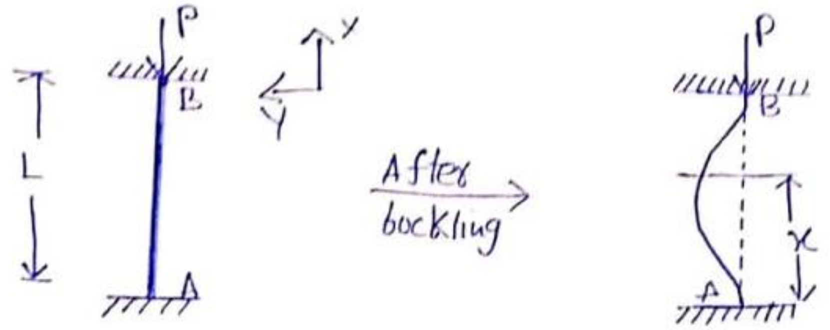
$$\Rightarrow AL = 4.49 \quad (\text{smallest value})$$

$$\Rightarrow \sqrt{\frac{P_{cr}}{EI}}L = 4.49$$

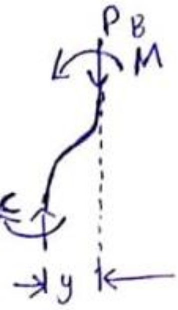
$$\Rightarrow \boxed{P_{cr} \simeq \frac{2\pi^2 EI}{L^2}}$$

$[\ P_{cr} = (20.16EI)/L^2\ ]$

c) Column with both ends Fixed:



Considering a section of buckled column



Applying moment equilibrium at C

$\Rightarrow \quad -M_b + M - Py = 0$ [Here M is the moment due to fixed end]

$$M_b = M - Py$$

By using deflection equation, $EI \frac{d^2y}{dx^2} = M_b$

$$\Rightarrow \quad EI \frac{d^2y}{dx^2} = M - Py$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{M}{EI} - \frac{P}{EI} y$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = A^2 \frac{M}{P} - A^2 y \qquad \left[\text{consider } A^2 = \frac{P}{EI}\right]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + A^2 y = A^2 \frac{M}{P}$$

By solving the differential equation similar to the previous case

$$\Rightarrow \quad y = C_1 \sin Ax + C_2 \cos Ax + \frac{M}{P}$$

By applying boundary conditions,

At $x = 0, y = 0 \Rightarrow 0 = C_2 + \frac{M}{P} \Rightarrow \underline{C_2 = -\frac{M}{P}}$

At $x = 0, \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = C_1 A \cos Ax + (-C_2 A \sin Ax)$

$$\Rightarrow \quad 0 = C_1 A \Rightarrow \underline{C_1 = 0}$$

At $x = L, y = 0 \Rightarrow 0 = C_2 \cos AL + \frac{M}{P}$

$$0 = \frac{M}{P}(1 - \cos AL)$$

$$\Rightarrow \quad \underline{\cos AL = 1}$$

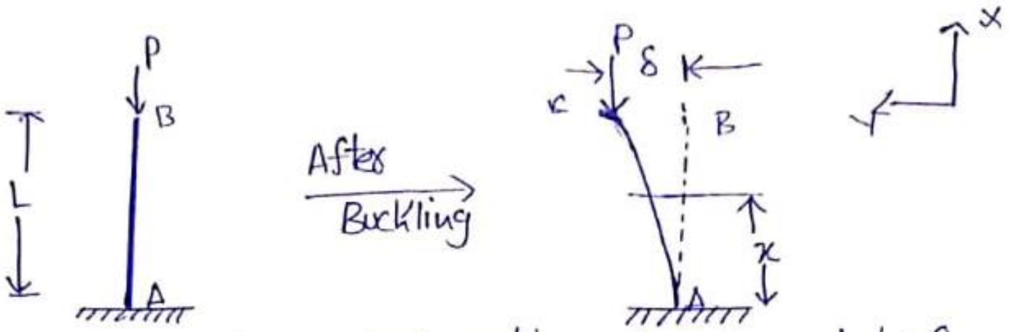
$$\Rightarrow \quad \underline{AL = 2n\pi}$$

for min load i.e critical load $n = 1$

$$\sqrt{\frac{P_{cr}}{EI}}\, L = 2\pi$$

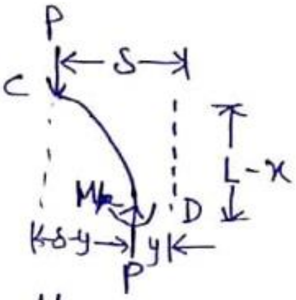
$$\boxed{P_{cr} = \frac{4\pi^2 EI}{L^2}}$$

d) Column with one end Fixed and the other end free:



We consider the maximum deformation of Column to be δ.

Consider a section of buckled column,



Applying moment equilibrium at D,

$$\Rightarrow \quad -M_b + P(\delta - y) = 0$$

$$\Rightarrow \quad M_b = P(\delta - y)$$

By using deflection equation, $EI\dfrac{d^2y}{dx^2} = M_b$

$$\Rightarrow \quad EI\frac{d^2y}{dx^2} = P(\delta - y)$$

$$\frac{d^2y}{dx^2} = A^2(\delta - y) \quad \left[\text{Considering } A^2 = \sqrt{\frac{P}{EI}}\right]$$

$$\frac{d^2y}{dx^2} + A^2y = A^2\delta.$$

By solving the differential equation similar to the previous case

$$\Rightarrow \quad y = C_1 \sin Ax + C_2 \cos Ax + \delta$$

Applying boundary conditions,

At $x = 0,\ y = 0 \Rightarrow 0 = C_2 + \delta \Rightarrow C_2 = -\delta$

At $x = 0,\ \dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = C_1 A \cos Ax - C_2 A \sin Ax$

$$\Rightarrow \quad 0 = C_1 - 0 \Rightarrow C_1 = 0$$

At $x = L,\ y = \delta \Rightarrow \cancel{\delta} = C_2 \cos AL + \cancel{\delta}$

$$\Rightarrow \quad \delta \cos AL = 0$$

$$\Rightarrow \quad AL = \frac{n\pi}{2}$$

for critical load, $n = 1$

$$\Rightarrow \quad \sqrt{\frac{P_{cr}}{EI}} = \frac{\pi}{2}$$

$$\Rightarrow \quad \boxed{P_{cr} = \frac{\pi^2 EI}{4L^2}}$$

In this way, Euler's Buckling load can be given $\boxed{\mathbf{P_E = P_{cr} = \pi E^2 I_{min}/L^2_e}}$

Where $L_e = L$ for columns with both ends hinged

$L_e = L/\sqrt{2}$ for columns with one end hinged and another end fixed

$L_e = L/2$ for columns with both ends fixed

$L_e = 2L$ for columns with one end fixed and another end free

But this is only applied for the long columns since we derived only when there is buckling.

❖ Rankine's Formula for Columns:

For any kind of columns maybe short, maybe long, maybe intermediate Rankine's Formula is used.

The empirical relation is given by:

$$\mathbf{1/P_R = 1/P_C + 1/P_E}$$

Where $\mathbf{P_R}$ is **Rankine's buckling load**

$\mathbf{P_C}$ is **Compressive load ($\mathbf{P_C = \sigma A}$)**

$\mathbf{P_E}$ is **Euler's Buckling load ($\mathbf{P_E = P_{cr} = \pi E^2 I_{min}/L^2_e}$)**

Substituting the values, we finally get,

$$\boxed{\mathbf{P_R = \sigma A/ (1+ ((\sigma/\pi^2 E) (L_e/K_{min})^2))}}$$

This can be applied to any type of column.

❖ References Used:

- **Continuum Mechanics Website (*www.continuummechanics.org*)**
- **Wikipedia (*www.wikipedia.org*)**
- **Official Website of NPTEL (*nptel.ac.in*)**
- **Engineering Tool Box Website (*www.engineeringtoolbox.com*)**