Coin Change Problem

- Finding the number of ways of making changes for a particular amount of cents, n, using a given set of denominations C={c₁...c_d} (e.g, the US coin system: {1, 5, 10, 25, 50, 100})
 - An example: $n = 4,C = \{1,2,3\}$, solutions: $\{1,1,1,1\}$, $\{1,1,2\},\{2,2\},\{1,3\}$.
- Minimizing the number of coins returned for a particular quantity of change (available coins {1, 5, 10, 25})
 - 30 Cents (solution: 25 + 2, two coins)
 - 67 Cents?
- 17 cents given denominations = {1, 2, 3, 4}?

Sudoku Puzzle

		4	
	4		3
3		2	
	2		1

5			1	7	3		2
3	2			6			
	1		8			9	
	7			9	6		
6							8
		9	6			7	
	6			3		4	
			9			2	3
7		3	4	8			9

Find the Fewest Coins: Casher's algorithm

- Given 30 cents, and coins {1, 5, 10, 25}
- Here is what a casher will do: always go with coins of highest value first
 - Choose the coin with highest value 25
 - 1 quarter
 - Now we have 5 cents left
 - 1 nickel

The solution is: 2 (one quarter + one nickel)

Greedy Algorithm Does not Always Give Optimal Solution to Coin Change Problem

- Coins = $\{1, 3, 4, 5\}$
- 7 cents = ?
- Greedy solution:
 - 3 coins: one 5 + two 1
- Optimal solution:
 - 2 coins: one 3 + one 4

Find the Fewest Coins: Divide and Conquer

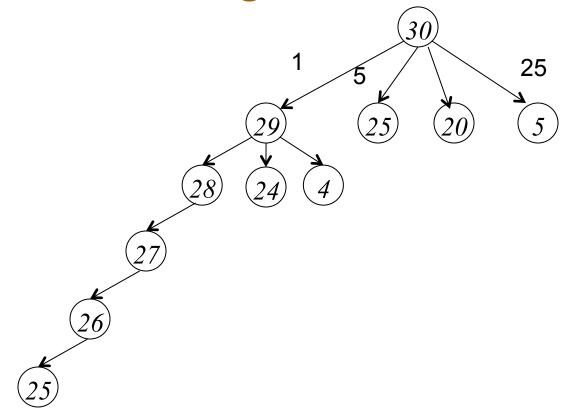
- 30 cents, given coins {1, 5, 10, 25, 50}, we need to calculate MinChange(30)
- Choose the smallest of the following:
 - 1 + MinChange(29) #give a penny
 - 1 + MinChange(25) #give a nickel
 - 1 + MinChange(10) #give a dime
 - 1 + MinChange(5) #give a quarter
- Do not know MinChange(29), MinChange(25), MinChange(10), MinChange(5)?

Coin Change Problem: A Recursive Algorithm

MinChange(<u>M</u>)

- 2. if M = 0
- 3. return 0
- 4. $v \leftarrow \infty$
- 5. for c in denominations $\leq M$
- 6. $v \leftarrow \min \{ MinChange(M-c) + 1, v \}$
- 7. return *v*

Recursive Algorithm Is Not Efficient



 It recalculates the optimal coin combination for a given amount of money repeatedly

How to avoid computing the same function multiple times We're re-computing values in our algorithm

- more than once
- Save results of each computation for 0 to M
- This way, we can do a reference call to find an already computed value, instead of recomputing each time
- Running time M*d, where M is the value of money and **d** is the number of denominations

Coin Change Problem: Save the Intermediate Results

```
MinChange(M)
     if minChange[M] not empty
        return minChange[M]
    if M=0
        return 0
6. for c in denominations \leq M
         v \leftarrow \min \{ MinChange(M-c) + 1, v \}
     minChange[M] = v
8.
     return v
```

The Change Problem: Dynamic Programming

```
    MinChangeDP(M)
    minCoin[0] ← 0
    for m ← 1 to M
    minCoin[m] ← infinity
    for c in denominations ≤ M
    if minCoin[m-c] + 1 < minCoin[m]</li>
    minCoin[m] ← minCoin[m-c] + 1
    return minCoin[M]
```

Greedy algorithm outputs optimal solutions for coin values 10, 5, 1

Proof:

Let N be the amount to be paid. Let the optimal solution be P=A*10 + B*5 + C. Clearly $B \le 1$ (otherwise we can decrease B by 2 and increase A by 1, improving the solution). Similarly, $C \le 4$.

Let the solution given by GreedyCoinChange be P=a*10 + b*5 + c. Clearly $b \le 1$ (otherwise the algorithm would output 10 instead of 5). Similarly $c \le 4$.

From $0 \le C \le 4$ and P = (2A + B)*5 + C we have $C = P \mod 5$.

Similarly c=P mod 5, and hence c=C. Let Q=(P-C)/5.

From $0 \le B \le 1$ and Q = 2A + B we have $B = Q \mod 2$.

Similarly b=Q mod 2, and hence b=B.

Thus a=A, b=B, c=C, i.e., the solution given by GreedyCoinChange is optimal.