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On Irregular Fuzzy Graphs

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Abstract

In this paper, neighbourly irregular fuzzy graphs, neighbourly total irregular fuzzy graphs, highly irregular fuzzy graphs and highly total irregular fuzzy graphs are introduced. A necessary and sufficient condition under which neighbourly irregular and highly irregular fuzzy graphs are equivalent is provided. Some results on neighbourly irregular fuzzy graphs are established.

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1. Introduction

Rosenfeld[7] considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs in 1975. Nagoor Gani and Radha[6] introduced regular fuzzy graphs, total degree and totally regular fuzzy graphs. Gnaana Bhragsam and Ayyaswamy[4] suggested a method to construct a neighbourly irregular graph of order n and also discussed some properties on neighbourly irregular graph. Yousef Alavi, etl.,[9] introduced k-path irregular graph and studied some properties on k-path irregular graphs. In this paper, neighbourly irregular fuzzy graphs, neighbourly total irregular fuzzy graphs, highly irregular fuzzy graphs and highly total irregular fuzzy graphs are introduced. A comparative study between neighbourly irregular and highly irregular fuzzy graphs is made. Also some results on neighbourly irregular fuzzy graphs are studied.

Throughout this paper only undirected fuzzy graphs are considered.

We review briefly some definitions which can be found in [1] – [9].

2. Preliminary

A fuzzy subset of a nonempty set S is a mapping σ: S→[0,1]. A fuzzy relation on S is a fuzzy subset of SxS. If μ and ν are fuzzy relations, then $\mu\circ\nu(u,w) = \mathrm{Sup}\{\mu(u,v)\Lambda \nu(v, w):v\in S\}$ and $\mu^k(u,v)= \mathrm{Sup}\{\mu(u ,u_1)\Lambda\nu(u_1 ,u_2)\Lambda\mu(u_2 ,u_3)\ \Lambda\ldots\Lambda\mu(u_{k-1} ,v):u_1 , u_2 ,\ldots. u_{k-1} \in S\}$, where 'Λ' stands for minimum. A fuzzy graph is a pair of functions $\sigma:V\to[0,1]$ and $\mu:V \times V\to[0,1]$, where for all $u,v\in V$ we have $\mu(u,v)\leq \sigma(u)\ \Lambda\ \sigma(v)$.

Definition 2.1: A graph G is called regular if every vertex is adjacent only to vertices having the same degree.

Definition 2.2: A graph G is called irregular, if there is a vertex which is adjacent only to vertices with distinct degrees.

Definition 2.3: A connected graph G is said to be highly irregular if every vertex of G is adjacent only to vertices with distinct degrees.

Definition 2.4: A connected graph G is said to be neighbourly irregular if no two adjacent vertices of G have the same degree.

Equivalently, a connected graph G is called neighbourly irregular if every two adjacent vertices of G have distinct degree.

Definition 2.5: A fuzzy graph $G = (\sigma,\mu)$ is a pair of functions $\sigma : V \to [\,0,1]$ and $\mu : V \times V \to [\,0,1]$, where for all $u,v \in V$, we have $\mu(u,v) \leq \sigma(u)\ \Lambda\ \sigma(v)$.

Definition 2.6: The fuzzy subgraph $H = (\tau,\rho)$ is called a fuzzy subgraph of $G = (\sigma,\mu)$ if $\tau(u) \leq \sigma(u)$ for all $u \in V$ and $\rho(u,v) \leq \mu(u,v)$ for all $u,v \in V$.

Definition 2.7: The underlying crisp graph of a fuzzy graph $G = (\sigma,\mu)$ is denoted by $G^*=(\sigma^*,\mu^*)$, where $\sigma^*=\{u\in V/ \sigma(u) > 0 \}$ and $\mu^* = \{ (u,v) \in VxV / \mu(u,v) > 0\}$.

Definition 2.8: Let $G = (\sigma,\mu)$ be a fuzzy graph. The degree of a vertex u is

$$d_G(u) = d(u) =\sum_{u\neq v}\mu(u,v) = \sum_{uv\in E}\mu(u,v)$$

Definition 2.9: Let $G = (\sigma,\mu)$ be a fuzzy graph on G^*: (V,E). If $d_G(v) = k$ for all $v\in V$, i.e., if each vertex has the same degree k, then G is said to be a regular fuzzy graph of degree k or a k-regular fuzzy graph.

Definition 2.10: Let $G = (\sigma,\mu)$ be a fuzzy graph on G^*. The total degree of a vertex $u\in V$ is defined by $td_G(u)=\sum_{u\neq v}\mu(u,v)+\sigma(u)=\sum_{uv\in E}\mu(u,v)+\sigma(u)=d_G(u) + \sigma(u)$.

If each vertex of G has the same total degree k, then G is said to be a totally

regular fuzzy graph of total degree k or a k-totally regular fuzzy graph.
Definition 2.11: A path ρ in a fuzzy graph is a sequence of distinct vertices u_0, u_1, $u_2 \ldots\ldots, u_n$ such that $\mu(u_{i-1},u_i) > 0$, $1 \le i \le n$. The path ρ is called a cycle if $u_0 = u_n$ and $n \ge 3$.
Definition 2.12: The complement of a fuzzy graph $G = (\sigma,\mu)$ is a fuzzy graph $G^c = (\sigma^c,\mu^c)$, where $\sigma^c = \sigma$ and $\mu^c(u,v) = \sigma(u) \Lambda \sigma(v) - \mu(u,v)$ for all u,v in V.
Definition 2.13: A fuzzy graph $G = (\sigma,\mu)$ is a complete fuzzy graph if $\mu(u,v) = \sigma(u) \Lambda \sigma(v)$ for all $u,v \in \sigma^*$.
Definition 2.14: Let $G = (\sigma,\mu)$ be a fuzzy graph such that $G^* = (V,E)$ is a cycle. Then G is a fuzzy cycle if and only if there does not exist a unique edge (x,y) such that $\mu(x,y) = \Lambda\{\mu(u,v)/(u,v) > 0\}$.

3. Neighbourly irregular and Highly irregular fuzzy graphs

Definition 3.1: Let $G = (\sigma,\mu)$ be a fuzzy graph. Then G is irregular, if there is a vertex which is adjacent to vertices with distinct degrees.
Example 3.2: Define $G = (\sigma,\mu)$ by $\sigma(u) = 0.4$, $\sigma(v) = 0.6$, $\sigma(w) = 0.4$, $\sigma(x) = 0.2$, $\sigma(y) = 0.5$ and $\mu(u,v) = 0.2$, $\mu(v,w) = 0.4$, $\mu(w,x) = 0.3$, $\mu(x,y) = 0.2$, $\mu(u,y) = 0.3$.
Definition 3.3: Let $G = (\sigma,\mu)$ be a connected fuzzy graph. G is said to be a neighbourly irregular fuzzy graph if every two adjacent vertices of G have distinct degree.
Example 3.4: Define $G = (\sigma,\mu)$ by $\sigma(u) = 0.8$, $\sigma(v) = 0.9$, $\sigma(w) = 0.7$, $\sigma(x) = 0.6$, and $\mu(u,v) = 0.8$, $\mu(v,w) = 0.4$, $\mu(w,x) = 0.6$, $\mu(x,u) = 0.6$.
Definition 3.5: Let $G = (\sigma,\mu)$ be a fuzzy graph. Then G is totally irregular, if there is a vertex which is adjacent to vertices with distinct total degrees.
Definition 3.6: If every two adjacent vertices of a fuzzy graph $G = (\sigma,\mu)$ have distinct total degree, then G is said to be a neighbourly total irregular fuzzy graph.
Definition 3.7: Let $G = (\sigma,\mu)$ be a connected fuzzy graph. G is said to be a highly irregular fuzzy graph if every vertex of G is adjacent to vertices with distinct degrees.
Example 3.8: Define $G = (\sigma,\mu)$ by $\sigma(u) = 0.4$, $\sigma(v) = 0.3$, $\sigma(w) = 0.5$, $\sigma(x) = 0.7$, and $\mu(u,v) = 0.3$, $\mu(v,w) = 0.2$, $\mu(w,x) = 0.3$, $\mu(x,u) = 0.4$.
Proposition 3.9:
A highly irregular fuzzy graph need not be a neighbourly irregular fuzzy graph.
Example 3.10: Define $G = (\sigma,\mu)$ by $\sigma(u) = 0.7$, $\sigma(v) = 0.8$, $\sigma(x) = 0.5$, $\sigma(y) = 0.9$, $\sigma(z) = 0.6$ and $\mu(u,v) = 0.4$, $\mu(v,x) = 0.4$, $\mu(x,y) = 0.5$, $\mu(u,y) = 0.4$, $\mu(y,z) = 0.6$.
To every vertex $v \in V$, the adjacent vertices have distinct degrees. Hence G is highly irregular. But $d(u) = d(v) = 0.8$. So G is not neighbourly irregular.
Proposition 3.11:
A neighbourly irregular fuzzy graph need not be a highly irregular fuzzy graph.
Example 3.12: Define $G = (\sigma,\mu)$ by $\sigma(u) = 0.6$, $\sigma(v) = 0.4$, $\sigma(w) = 0.5$, $\sigma(y) = 0.5$, $\sigma(x) = 0.7$ and $\mu(u,v)=0.6$, $\mu(v,w) = 0.3$, $\mu(w,y) = 0.2$, $\mu(u,y) = 0.4$, $\mu(w,x) = 0.5$, $\mu(x,y) = 0.3$.
No two adjacent vertices has the same degree. So G is a neighbourly irregular

fuzzy graph. But to the vertex u, the adjacent vertices v and y has the same degree (= 0.9). So G is not highly irregular.

Theorem 3.13:

Let G=(σ,μ) be a fuzzy graph. Then G is highly irregular fuzzy graph and neighbourly irregular fuzzy graph if and only if the degrees of all vertices of G are distinct.

Proof:

Let G be a fuzzy graph with n vertices $u_1, u_2, u_3\ldots\ldots,u_n$.

Assume that G is highly irregular and neighbourly irregular fuzzy graphs.

Let the adjacent vertices of u_1 be $u_2, u_3\ldots\ldots,u_n$ with degrees $k_2, k_3\ldots\ldots k_n$ respectively. As G is highly irregular, $k_2 \neq k_3 \neq \ldots\ldots\ldots \neq k_n$.

$d(u_1)$ cannot be either of $k_2, k_3\ldots\ldots k_n$ as G is neighbourly irregular.

Therefore, the degrees of all vertices of G are distinct.

Conversely assume that the degrees of all vertices of G are distinct.

This means that every two adjacent vertices have distinct degrees and to every vertex the adjacent vertices have distinct degrees.

Hence G is neighbourly irregular and highly irregular fuzzy graphs.

Theorem 3.14:

A fuzzy graph G = (σ,μ) where G^* is a cycle with vertices 3 is neighbourly irregular and highly irregular if and only if the weights of the edges between every pair of vertices are all distinct.

Proof:

For, if the weights of any two edges are the same, it violates the definition of neighbourly irregular and highly irregular fuzzy graphs.

The converse part follows from the definition of neighbourly irregular and highly irregular fuzzy graphs.

4. Properties of Neighbourly Irregular fuzzy graphs

Proposition 4.1:

If a fuzzy graph G = (σ,μ) is neighbourly irregular, then G^c is not be neighbourly irregular.

Proof:

For, the non – adjacent vertices in G having the same degree, are then adjacent vertices and also of the same degree in G^c. Therefore, G^c is not a neighbourly irregular fuzzy graph.

Proposition 4.2:

The converse of the above result is not true.(i.e) If a fuzzy graph G = (σ,μ) is not neighbourly irregular, then its complement is not neighbourly irregular.

Example 4.3: Define G = (σ,μ) by σ(u) = 0.6 , σ(v) = 0.5, σ(w) = 0.6, σ(x) = 0.5, σ(y) = 0.7 and μ(u,v) = 0.5, μ(u,w) = 0.1, μ(u,x) = 0.5, μ(u,y) = 0.6, μ(v,w) = 0.4, μ(w,x) = 0.4, μ(x,y) = 0.3.Clearly G and G^c are not neighbourly irregular since d(v) = d(w) = d(x) = d(y) = 0.9 in G and d(v) = d(x) = 1.1 in G^c.

Proposition 4.4:

A complete fuzzy graph need not be neighbourly irregular.

Example 4.5: Define $G = (\sigma,\mu)$ by $\sigma(u) = 0.5$, $\sigma(v) = 0.4$, $\sigma(x) = 0.7$, $\sigma(w) = 0.5$ and $\mu(u,v) = 0.4$, $\mu(v,x) = 0.4$, $\mu(x,w) = 0.5$, $\mu(w,u) = 0.5$, $\mu(u,x) = 0.5$,$\mu(v,w) = 0.4$

G is a complete fuzzy graph but not neighbourly irregular.

Proposition 4.6:

A neighbourly irregular fuzzy graph need not be a neighbourly total irregular fuzzy graph.

Example 4.7: Define $G = (\sigma,\mu)$ by $\sigma(u) = 0.5$, $\sigma(v) = 0.6$, $\sigma(x) = 0.9$, $\sigma(w) = 0.4$, $\sigma(y) = 0.2$ and $\mu(u,v) = 0.5$, $\mu(v,w) = 0.4$, $\mu(x,w) = 0.3$, $\mu(x,y) = 0.1$, $\mu(u,y) = 0.2$, $\mu(u,w) = 0.4$, $\mu(v,x) = 0.6$.

No two adjacent vertices of G has the same degree. Therefore G is a neighbourly irregular fuzzy graph. Since $td(u)=td(w)=1.6$, G is not a neighbourly total irregular fuzzy graph.

Proposition 4.8:

A neighbourly total irregular fuzzy graph need not be a neighbourly irregular fuzzy graph.

Example 4.9: Define $G = (\sigma,\mu)$ by $\sigma(u) = 0.5$, $\sigma(v) = 0.6$, $\sigma(x) = 0.6$, $\sigma(w) = 0.4$, $\sigma(y) = 0.2$ and $\mu(u,v) = 0.3$, $\mu(v,w) = 0.4$, $\mu(x,w) = 0.4$, $\mu(x,y) = 0.2$ $\mu(u,y) = 0.2$, $\mu(u,w) = 0.4$, $\mu(v,x) = 0.5$.

Every two adjacent vertices of G has distinct total degrees. So G is a neighbourly total irregular fuzzy graph.As the adjacent vertices v and w has the same degree, $d(v) = d(w) = 1.2$, G is not a neighbourly irregular fuzzy graph.

Theorem 4.10:

Let $G = (\sigma,\mu)$ be a fuzzy graph. If G is neighbourly irregular and σ is a constant function, then G is a neighbourly total irregular fuzzy graph.

Proof:

Assume that $G = (\sigma,\mu)$ is a neighbourly irregular fuzzy graph.

(i.e) The degrees of every two adjacent vertices are distinct.

Consider two adjacent vertices u_1 and u_2 with distinct degrees k_1 and k_2 respectively.

(i.e) $d(u_1) = k_1$ and $d(u_2) = k_2$ where $k_1 \neq k_2$

Also assume that $\sigma(u_1) = \sigma(u_2) = c$, a constant and $c \in [0,1]$

Therefore $td(u_1) = d(u_1) + \sigma(u_1) = k_1 + c$

$td(u_2) = d(u_2) + \sigma(u_2) = k_2 + c$

To prove: $td(u_1) \neq td(u_2)$

Suppose $td(u_1) = td(u_2)$

$k_1 + c = k_2 + c$

$k_1 - k_2 = c - c = 0$

$k_1 = k_2$, a contradiction to $k_1 \neq k_2$

Therefore $td(u_1) \neq td(u_2)$

(i.e) For any two adjacent vertices u_1 and u_2 with distinct degrees, its total degrees are also distinct, provided σ is a constant function.

The above argument is true for every pair of adjacent vertices in G.

Theorem 4.11:

Let $G = (\sigma,\mu)$ be a fuzzy graph. If G is a neighbourly total irregular and σ is a constant function, then G is a neighbourly irregular fuzzy graph.

Proof:
Assume that $G = (\sigma,\mu)$ is a neighbourly total irregular fuzzy graph.
(i.e) The total degree of every two adjacent vertices are distinct.
Consider two adjacent vertices u_1 and u_2 with degrees k_1 and k_2.
(i.e) $d(u_1) = k_1$ and $d(u_2) = k_2$
Also assume that $\sigma(u_1) = \sigma(u_2) = c$, a constant where $c\in[0,1]$ and $td(u_1) \neq td(u_2)$
To prove: $d(u_1) \neq d(u_2)$
As $td(u_1) \neq td(u_2)$
$k_1 + c \neq k_2 + c$
$k_1 \neq k_2$
(i.e) The degrees of adjacent vertices of G are distinct.
This is true for every pair of adjacent vertices in G.

Proposition 4.12:
Let $G = (\sigma,\mu)$ be a fuzzy graph. If G is both neighbourly irregular and neighbourly total irregular fuzzy graph, then σ need not be a constant function.

Example 4.13: Define $G = (\sigma,\mu)$ by $\sigma(u) = 0.3$, $\sigma(v) = 0.5$, $\sigma(x) = 0.2$, $\sigma(w) = 0.3$ and $\mu(u,v) = 0.3$, $\mu(v,w) = 0.3$, $\mu(x,w) = 0.2$, $\mu(u,x) = 0.2$
G is neighbourly irregular and neighbourly total irregular, but σ is not a constant function.

Proposition 4.14:
If a fuzzy graph $G = (\sigma,\mu)$ is neighbourly irregular, the fuzzy subgraph $H = (\tau,\rho)$ of G need not be neighbourly irregular.

Example 4.15: For the fuzzy graph $G = (\sigma,\mu)$ given by $\sigma(u) = 0.7$, $\sigma(v) = 0.6$, $\sigma(w) = 0.4$, $\sigma(x) = 0.5$ and $\mu(u,v) = 0.5$, $\mu(u,w) = 0.4$, $\mu(w,x) = 0.4$, $\mu(v,x) = 0.3$, define fuzzy subgraphs $H_1 = (\tau,\rho)$ by $\tau(u)=0.6$, $\tau(v)=0.4$, $\tau(w)=0.4$, $\tau(x)=0.3$ and $\rho(u,v) = 0.4$, $\rho(u,w) = 0.4$, $\rho(w,x) = 0.3$, $\rho(v,x) = 0.3$ and $H_2 = (\tau,\rho)$ by $\tau(u) = 0.7$, $\tau(v)=0.5$, $\tau(w)=0.4$, $\tau(x)=0.3$ and $\rho(u,v)=0.4$, $\rho(u,w)=0.3$, $\rho(w,x)=0.4$, $\rho(v,x)=0.3$.
Clearly H_1 is a neighbourly irregular fuzzy graph but H_2 is not.

Proposition 4.16:
Let $G = (\sigma,\mu)$ be either a neighbourly irregular or a neighbourly total irregular fuzzy graph, where G^* is a cycle, then G need not be a fuzzy cycle.

Example 4.17: Define $G = (\sigma,\mu)$ by $\sigma(u) = 0.8$, $\sigma(v) = 0.9$, $\sigma(w) = 0.8$, $\sigma(x) = 0.5$ and $\mu(u,v) = 0.3$, $\mu(u,x) = 0.6$, $\mu(v,w) = 0.4$, $\mu(w,x) = 0.5$
Clearly G is neighbourly irregular and neighbourly total irregular fuzzy graph, but G is not a fuzzy cycle since there exists a unique edge (u,v) such that
$\mu(u,v) = 0.3 = \Lambda\{\mu(x,y) / (x,y) > 0\}$.

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