EC252:

15/12/2021

- Random exp: some activity that results in an outcome.
- Event: Set of outcomes to which we assign probability.
  (A)
- Sample space: set of all possible outcomes.
  (Ω) : $\boxed{\text{Event} \subseteq \text{Sample space}}$

- Event space: collection of subsets of Sample space.
  ($\mathcal{F}$)

eg: Tossing a coin

$\Omega = \{H, T\}$

$\Omega = \mathbb{R} \rightarrow$ distance through which coin travelled.

$\Omega = \{1, 2, 3, \dots\} \rightarrow$ No. of flips.

$\mathcal{F} = \{\{H\}, \{T\}, \phi, \{H, T\}\} = 2^n \rightarrow$ power set of Ω.

$\rightarrow$ A set 's' is said to be countable if we can define a 1-1 mapping from $\mathbb{N}$ to set 's'.

eg: $S = \{1, 2, \dots\}$

$f : \mathbb{N} \rightarrow S \quad f(n) = n$

$S = \{0, 1, 2, \dots\}$

$f(n) = n-1, n \in \mathbb{N}$

($\mathbb{Q} = \frac{p}{q}, q \neq 0$ p, q are integers $\rightarrow$ Countable)

→ A collection of sets $\mathcal{F}$ is said to be σ-algebra if

(i) $\Omega \in \mathcal{F}$

(ii) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$

(iii) $A_1, A_2, A_3, \ldots$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

eg: $\Omega = \{1, 2, 3, 4, 5, 6\}$

$\mathcal{F} = \{\Omega, \phi, \{1, 3, 5\}, \{2, 4, 6\}\}$

- Probability:

A probability 'P' is a mapping from $\mathcal{F}$ to $[0,1]$

$$P : \mathcal{F} \to [0,1]$$

such that

(i) $P(\Omega) = 1$

(ii) If $A_1, A_2, \ldots \in \mathcal{F}$, then $A_i \cap A_j = \phi$

then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Properties:

(i) • $P(A^c) = 1 - P(A)$

Proof: $A \cap A^c = \phi$

$A_1 = A \quad A_2 = A^c$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P(A \cup A^c) = P(\Omega) = 1 = P(A) + P(A^c)$$

(ii) $A, B \in \mathcal{F}$ such that $A \subseteq B$ then $P(A) \leq P(B)$

Proof:

$B = A \cup (A^c \cap B)$

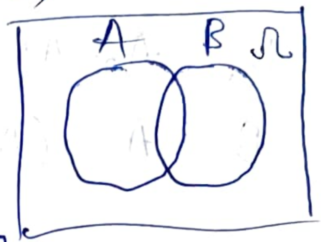
$B - A = A^c \cap B$

$$P(B) = P(A) + \underbrace{P(A^c \cap B)}_{\geq 0}$$

So $P(B) \geq P(A)$

(iii) $A, B \in \mathcal{F}$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Proof:

$$A \cup B = A \cup (A^c \cap B)$$

$$P(A \cup B) = P(A) + P(A^c \cap B)$$

$$B = (A \cap B) \cup (A^c \cap B)$$

$$P(B) = P(A \cap B) + P(A^c \cap B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

(iv) $P(\phi) = 0$

Proof:

$$A_i = \phi \;, \; i \geq 1$$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P(\phi) = \sum_{i=1}^{\infty} P(A_i)$$

$$P(\phi) = \sum_{i=1}^{\infty} P(\phi) \Rightarrow P(\phi) = 0$$

(v) If $A_1, A_2, \ldots A_n$ are disjoint events

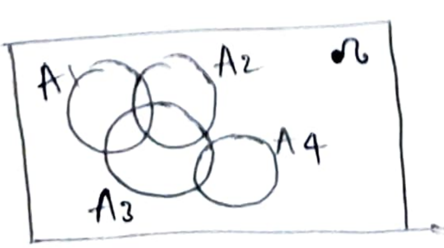
$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

(vi) Union bound:

let $A_1, A_2, \ldots A_n$ be events

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

Proof:



$B_1 = A_1$

$B_2 = A_2 - A_1$

$B_3 = A_3 - (A_1 \cup A_2)$

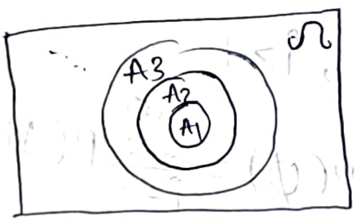
$B_i \subseteq A_i$

So, $P(B_i) \subsetneq P(A_i)$

$$P\left(\bigcup_{i=1}^{n} B_i\right) = P\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{i=1}^{n} P(B_i) \le \sum_{i=1}^{n} P(A_i)$$

→ Let $A_1, A_2, \dots A_n$ be sequence of events

(a) if $A_1 \subseteq A_2 \subseteq \dots \subseteq A_n$



$$P\left(\bigcup_{i=1}^{n} A_i\right) = \lim_{n \to \infty} P(A_n)$$

(b) $$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n \to \infty} P\left(\bigcup_{i=1}^{n} A_i\right)$$

(Continuity of probability)

$P(A \cup B \cap C)$

$P(C)$

$P($[illegible]

$P($

* Conditional probability :

If $P(A)$

$f_n(A)$; No. of times event A occurs.

$$P(A) = \frac{f_n(A)}{n}$$

$$P(A/B) = \frac{f_n(A\cap B)/n}{f_n(B)/n} = \frac{P(A\cap B)}{P(B)}$$

$\downarrow$

chance of A occuring when B occurs.

- $P(A\cup B/C) = P(A/C) + P(B/C) - P(A\cap B/C)$

Proof :

$$P(A\cup B/C) = \frac{P((A\cup B)\cap C)}{P(C)}$$

$$= \frac{P((A\cap C)\cup(B\cap C))}{P(C)}$$

$$= \frac{P(A\cap C)+P(B\cap C) - P(A\cap B\cap C)}{P(C)}$$

$$P(A\cup B/C) = \underline{P(A/C)+P(B/C) - P(A\cap B/C)}$$

$\rightarrow$ Let $A_1, A_2, A_3, \ldots A_n$ be disjoint events

$$P\left(\bigcup_{i=1}^{\infty} A_i/C\right) = \sum_{i=1}^{n} P(A_i/C)$$

Independence :

chance of occurence of one event doesn't alter the chance of occurence of another event.

$$P(A/B) = P(A)$$

$\rightarrow P(A\cap B) = P(A)P(B) \rightarrow$ independent events.

→ Events A_1, A_2, A_3 are statistically independent if

(i) $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$

(ii) $P(A_1 \cap A_2) = P(A_1)P(A_2)$

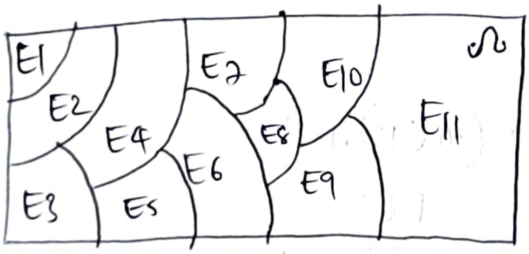
$P(A_1 \cap A_3) = P(A_1)P(A_3)$

$P(A_2 \cap A_3) = P(A_2)P(A_3)$

→ Let $\{E_i ; i = 1, 2, \ldots n\}$ be set of disjoint events

The set of events E_i is said to be partition of Ω

if $\bigcup_{i=1}^{n} E_i = \Omega$



→ Total probability

$$B_i = A \cap E_i$$

$$P(A) = P\left(\bigcup_{i=1}^{n} B_i\right) = \sum_{i=1}^{n} P(B_i)$$

$$= \sum_{i=1}^{n} P(A \cap E_i)$$

$$\boxed{P(A/E_i) = \frac{P(A \cap E_i)}{P(E_i)}}$$

$$\boxed{P(A) = \sum_{i=1}^{n} P(A/E_i) P(E_i)}$$

$$P(E_i/A) = \frac{P(A \cap E_i)}{P(A)}$$

$$\boxed{P(E_i/A) = \frac{P(A/E_i) P(E_i)}{\sum_{i=1}^{n} P(A/E_i) P(E_i)}}$$

eg: Two events A & B $P(A) = 3/4$ $P(B) = 1/3$
show that $\frac{1}{12} \leq P(A \cap B) \leq \frac{1}{3}$

Sol:- $\cancel{P(A)+P(B)} \; P(A \cup B) \leq 1$

$$P(A \cup B) = \frac{3}{4} + \frac{1}{3} - P(A \cap B)$$

$$= \frac{13}{12} - P(A \cap B) \leq 1$$

$$P(A \cap B) \geq \frac{1}{12}$$

$A \cap B \subseteq B$

$\downarrow$

$P(A \cap B) \leq P(B)$

$P(A \cap B) \leq 1/3$

$\downarrow$

$$\frac{1}{12} \leq P(A \cap B) \leq \frac{1}{3}$$

eg: Tossing 3 coins (balanced, identical). What is the probability at least one head?

$$P(\text{at least one } H) = 1 - \frac{1}{8} = 7/8$$

eg: Two identical coins (balanced)

$$P\left(\begin{matrix}\text{Both show}\\\text{head}\end{matrix} \middle| \begin{matrix}\text{coin 1 show}\\\text{head}\end{matrix}\right) = ? \quad \frac{1}{2}$$

$$P\left(\begin{matrix}\text{Both show}\\\text{head}\end{matrix} \middle| \begin{matrix}\text{atleast one}\\\text{head}\end{matrix}\right) = \frac{1}{3}$$

eg: Library with 1000 books

500 scientific

100 Engg

(i) Probability that all three are scientific $= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

(ii) Probability that all 3 are scientific in which only one is Engg.

$$= 3 \times 0.4 \times 0.4 \times 0.1$$

$$= \frac{3 \times 16}{1000}$$

eg:

	W	B
urn-1	2	3
urn-2	4	1
urn-3	3	4

choose an urn at random and pick a ball from the urn at random. We found ball to be white. What is the probability that it is picked from urn-I?

Sol:- $P(\text{Urn-I}/\text{White}) = \dfrac{P(\text{Urn I} \cap \text{White})}{\boxed{P(\text{White})}}$

$= \dfrac{2/5 \times \cancel{1/3}}{\cancel{1/3}\left(\frac{2}{5} + \frac{4}{5} + \frac{3}{7}\right)} = \dfrac{2/\cancel{5}}{\frac{57}{3\cancel{5}7}}$

$= \underline{14/57}$

eg: A, B throw alternatively a pair of dice.
A wins if he throws 6 before B throws 7.
B wins if he throws 7 before A throws 7
If A starts, s.t the chance of winning (for A) is —

Sol: $E_{A \text{ throws } 6} = \dfrac{5}{36}$ $\begin{pmatrix} 1 & 5 \\ 5 & 1 \\ 3 & 3 \\ 2 & 4 \\ 4 & 2 \end{pmatrix}$

$E_{B \text{ throws } 7} = \dfrac{6}{36}$ $\begin{pmatrix} 1 & 6 \\ 6 & 1 \\ 2 & 5 \\ 5 & 2 \\ 3 & 4 \\ 4 & 3 \end{pmatrix}$

$E_B^c = \dfrac{30}{36}$

$E_A^c = \dfrac{31}{36}$

$E_A, E_A^c E_B^c E_A, E_A^c E_B^c E_A^c E_B^c E_A, \dots$

$\dfrac{5}{36} + \dfrac{31}{36} \times \dfrac{30}{36} \times \dfrac{5}{36} + \dots$

$= \dfrac{30}{61}$ //

eg: Given, $P(A) > P(B)$

Show that $P(A/B) > P(B/A)$

Sol:

$$\frac{P(A)}{P(A\cap B)} > \frac{P(B)}{P(A\cap B)} \Rightarrow \frac{1}{P(B/A)} > \frac{1}{P(A/B)}$$

$$P(A/B) > P(B/A)$$

eg: Train X arrives at a station at random during $(0,T)$ and stops there for 'a' mins.

Train Y arrives independently during the same interval and stopping for 'b' mins.

(i) Find probability that X arrives before Y

(ii) Find probability that trains meet.

eg: An insurance company has equal no. of female and male drivers.

P(a male driver has an accident in a year) = α
independent of other year

P(a female driver " " " ") = β

(i) What is probability that selected driver will make a claim ? = $\alpha + \beta$

Sol: $P(\text{accident}) = P(\text{accident}/M)\,P(M) + P(\text{accident}/F)\,P(F)$

$$= \alpha \cdot \frac{1}{2} + \beta \cdot \frac{1}{2} = \frac{\alpha+\beta}{2}$$

(ii) P(selected driver make a claim in two consecutive years)

Sol: $P(B) = P(B/M)\,P(M) + P(\cancel{B}/\cancel{M})\,P(F)$

$$= \alpha^2 \cdot \frac{1}{2} + \beta^2 \cdot \frac{1}{2} = \frac{\alpha^2+\beta^2}{2}$$

(iii) Let A_1, A_2 be the events that a randomly choosen driver makes a claim in each of first and second year respectively.

Sol: $$P(A_2|A_1) = \frac{P(A_2 \cap A_1)}{P(A_1)} = \frac{\frac{\alpha^2+\beta^2}{\cancel{2}}}{\frac{\alpha+\beta}{\cancel{2}}} = \frac{\alpha^2+\beta^2}{\alpha+\beta}$$

$$P(A_2|A_1) - P(A_1) = \frac{\alpha^2+\beta^2}{\alpha+\beta} - \frac{\alpha+\beta}{2}$$

$$= \frac{(\alpha-\beta)^2}{2(\alpha+\beta)} \geq 0$$

$P(A_2|A_1) \geq P(A_1)$ ✓

(iv) $P(F/C) = \frac{P(F \cap C)}{P(C)} = \frac{P(C/F) \cdot P(F)}{P(C)}$

$$= \frac{\beta \cdot 1/2}{\frac{\alpha+\beta}{2}} = \frac{\beta}{\alpha+\beta}$$ //

eg: Test has 99% accuracy

Test has 5% false +ve reading.

$P(\text{a person has the disease}) = \frac{1}{10,000}$

One person tested positive (+ve) for the disease. What is the probability that person has disease?

Sol:- $P(D) = \frac{1}{10,000}$

$P(T \text{ is +ve}/D) = 0.99$

$P(T \text{ is +ve}/D^c) = 0.05$

$P(D/T \text{ is +ve}) = \frac{P(D \cap T \text{ is +ve})}{P(T \text{ is +ve})}$

$P(T \text{ is +ve}) = 0.99 \times \frac{1}{10000} + 0.05 \times \frac{9999}{10000}$

$$= \frac{500.94}{10000}$$

$\frac{P(T \text{ is +ve}/D)\, P(D)}{P(T \text{ is +ve})} = \frac{0.99 \times \frac{1}{\cancel{10000}}}{\frac{500.94}{\cancel{10000}}}$

$$= \frac{99}{50094} \simeq 0.002$$

$$\begin{array}{r} 4\;44 \\ 9999 \\ \times\; 5 \\ \hline 499.95 \\ 0.99 \\ \hline 500.94 \end{array}$$

eg: $A_1, A_2, \ldots A_n$ events

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j)$$

$$+ \sum_{i<j<k} P(A_i \cap A_j \cap A_k)$$

$$+ \cdots\cdots + (-1)^n P\left(\bigcap_{i=1}^{n} A_i\right)$$

Sol:- for $n=2 \rightarrow$ statement is true

True for l

for $l \rightarrow l+1$

$$P(A_1 \cup A_2 \cup A_3 \cdots \cup A_{l+1}) = P(B) + P(A_{l+1}) - P(B \cap A_{l+1})$$

Probability space:

$(\Omega, \mathcal{F}, P)$

Sample space (Ω), Event space ($\mathcal{F}$), probability mapping $P: \mathcal{F} \to [0,1]$

Set axioms are satisfied.

→ If Ω is finite or countable and $A \in \mathcal{F}$, $\omega \in \Omega$ (ω: atom)

$$P(\{\omega\}) = P_\omega$$

$$\text{then } P(A) = \sum_{\omega \in A} P_\omega \quad \left(A = \bigcup_{\omega \in A} \{\omega\}\right)$$

eg: $\Omega = \{1, 2, 3, 4, 5, 6\}$

Let $A = \{2, 4, 6\}$

$$P(A) = P_2 + P_4 + P_6$$

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$

$$\boxed{P(A) = 1/2}$$

→ Ω: uncountable

$\Omega = \mathbb{R}$

$A = (a, b)$; $a, b \in \mathbb{R}$, $a < b$

σ-algebra generated by open intervals (a,b)

(i) $\Omega \in \mathcal{F}$

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(iii) $A_1, A_2, \ldots A_n \in \mathcal{F}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

$\mathcal{F} = \{(a, b), (-\infty, a] \cup [b, \infty)$ include countable unions $\}$

↳ Borel σ-algebra of $\mathbb{R}$

can be generated by set of form $(-\infty, c]$, $c \in \mathbb{R}$

$$(a, b) = \{(-\infty, a] \cup [b, \infty)\}^c$$

$$\{b\} = \bigcap_{n=1}^{\infty} \left(b - \frac{1}{n}, b + \frac{1}{n}\right)$$

Random Variables:

A random variable X is a mapping $\Omega \to \mathbb{R}$ ($X : \Omega \to \mathbb{R}$) such that

$$\{\omega \in \Omega, X(\omega) \leq c, c \in \mathbb{R}\} \in \mathcal{F}$$

$$X^{-1}((-\infty, c]) \in \mathcal{F}$$

Based on range of X, we classify random variables into

(i) Discrete:

If the range of x is finite or countable then the random variable X is called Discrete Random Variable.

eg: $\Omega = \{1, 2, 3, 4, 5, 6\}$

$X(i) = i \quad , i = 1, 2, \cdots 6$

(ii) Continuous:

→ If the range of x is continuous (infinite) then x is called continuous Random Variable.

→ A random variable x is continuous if it's CDF $F_X(x)$ is continuous.

- Cumulative Distribution function (CDF):

CDF of X is defined as

$$F_X(x) = P(X \le x)$$
$$= P(\{\omega \in \Omega : X(\omega) \le x\}$$

→ If X is discrete

$$X \in \{x_1, x_2, \dots\}$$

$$P(X = x_i) = P(\{\omega \in \Omega ; X(\omega) = x_i\})$$
$$= p_i$$

$$\underbrace{P(X = i) = p_i}_{\text{Probability mass function (pmf)}}$$

$$P(X(-\infty, c]) = \sum_{x_i \le c} p_i$$

- Probability mass function (pmf):

$$P(X = x_i) = p_i \qquad X \in \{x_1, x_2, \dots\}$$

(i) $p_i \geq 0$ for all i

(ii) $\sum_i p_i = 1$

- Expectation or mean:

$$\mathbb{E}(X) = \sum_i x_i p_i$$

(i) Bernoulli distribution

$$X = \begin{cases} 0 & \text{wp } 1-p \\ 1 & \text{with probability } p \end{cases}$$

$$E[X] = 0(1-p) + 1(p)$$
$$= p$$

(ii) Binomial distribution

$$X \in \{0, 1, 2, 3, \ldots n\}$$

$$P(X=K) = \binom{n}{K} p^k (1-p)^{n-k}$$

$$E[X] = \sum_{K=0}^{n} k \binom{n}{K} p^k (1-p)^{n-k}$$

$$= \sum_{K=0}^{n} K \cdot \frac{n!}{K!(n-K)!} p^k (1-p)^{n-k}$$

$$= \sum_{K=0}^{n} \frac{n!}{(K-1)!(n-K)!} p^k (1-p)^{n-k}$$

$$= np \underbrace{\sum_{K=1}^{n} \frac{(n-1)!}{(K-1)!(n-K)!} p^{K-1} (1-p)^{n-k}}_{=1}$$

$$\therefore E[x] = np$$

(iii) Poisson distribution:

$$X = \{0, 1, 2, \ldots\}$$

λ is parameter of distribution

$$P[X=K] = \frac{e^{-\lambda} \lambda^k}{K!} \quad K = 0, 1, 2, \ldots$$

$$E[X] = \sum_{K=0}^{\infty} K \, P[X=K] = \sum_{K=1}^{\infty} K \cdot \frac{e^{-\lambda} \cdot \lambda^k}{K!}$$

$$= \lambda \cdot \sum_{K=1}^{\infty} \frac{e^{-\lambda} \cdot \lambda^{K-1}}{(K-1)!} = \lambda //$$

(iv) Geometric distribution:

$$X \in \{1, 2, 3, \dots\}$$

$$P(X=k) = p(1-p)^{k-1}$$

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k p(1-p)^{k-1}$$

$$= p \sum_{k=1}^{\infty} k(1-p)^{k-1}$$

$$= \frac{p}{(1-(1-p))^2} = \frac{1}{p}$$

$q = 1-p$

$$\sum_{k=1}^{\infty} k \cdot q^{k-1} = \sum_{k=0}^{\infty} \frac{d}{dq}[q^k]$$

$$= \frac{d}{dq}\left[\frac{1}{1-q}\right] = \frac{1}{(1-q)^2}$$

$$\sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} = \frac{1}{(1-(1-p))^2}$$

- Continuous Random Variables:

CDF
$$F_X(x) = P[X \le x]$$

$x_1 \le x_2$

$$\{\omega \in \Omega : X(\omega) \le x_1\} \subseteq \{\omega \in \Omega : X(\omega) \le x_2\}$$

→ Properties:

(i) $F_X(\infty) = 1$, $F_X(-\infty) = 0$

$$\{\omega : X(\omega) \le \infty\} = \Omega$$
$$\{\omega : X(\omega) \le -\infty\} = \phi$$

(ii) If $x_1 \le x_2$

$$F_X(x_1) \le F_X(x_2)$$

monotonically non-decreasing function.

(iii) The function $F_X(x)$ is continuous from the right

$$A_n = \left(-\infty, x + \frac{1}{n}\right]$$

$$A_1 \supseteq A_2 \supseteq A_3 \cdots$$

$$\bigcap_{n=1}^{\infty} A_n = (-\infty, x] = A$$

$$\lim_{n \to \infty} P(A_n) = P(A)$$

$$\lim_{n \to \infty} F_X\left(x + \frac{1}{n}\right) = F_X(x)$$

$$P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$$

$$X \in (x_1, x_2]$$

$$x_1 < X \leq x_2 = (-\infty, x_2] - (-\infty, x_1]$$

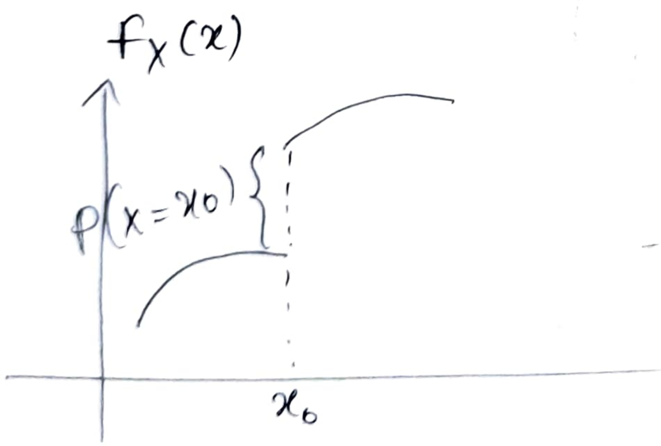
(iv) $P(X=x) = F_X(x) - F_X(x^-)$

$$F_X(x^-) = \lim_{\epsilon \to 0} F_X(x-\epsilon)$$

If r.v X is continuous, $F_X(x)$ is continuous

$$F_X(x^-) = F_X(x)$$

$\underline{P(X=x) = 0}$ $\qquad F_X(x) \in [0,1]$



$P(X=x_0)$ = The height of the jump.

CDF of Discrete R.V :

$X \in \{0,1\}$

$$X = \begin{cases} 0 & w.p \ 1-p \\ 1 & w.p \ p \end{cases}$$

→ Properties:

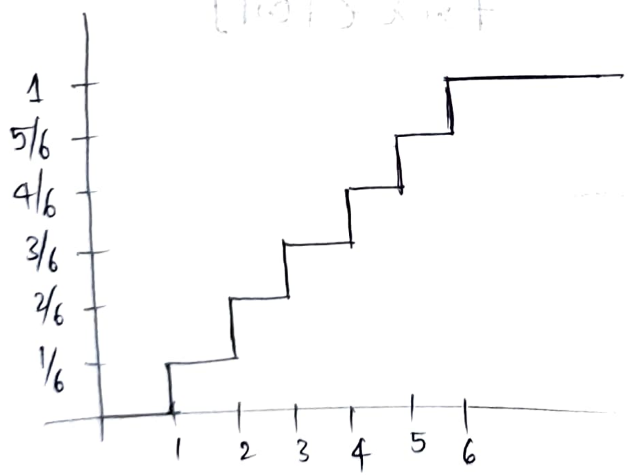
(i) $F_X(\infty) = 1$, $F_X(-\infty) = 0$

(ii) $F_X(x)$ is monotonically non-decreasing.

(iii) $F_X(x)$ is right continuous.

$X = \{1,2,3,4,5,6\}$

$P(X=i) = 1/6$



Probability density function (PDF):

The PDF of a random variable 'X' is a function $f_X(x)$ such that,

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$$

$$f_X(x) = \frac{d}{dx}(F_X(x))$$

$\Rightarrow$

→ **Properties:**

(i) $f_X(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f_X(x)\,dx = F_X(\infty) = 1$

(iii) $P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$

$$= \int_{-\infty}^{x_2} f_X(x)\,dx - \int_{-\infty}^{x_1} f_X(x)\,dx$$

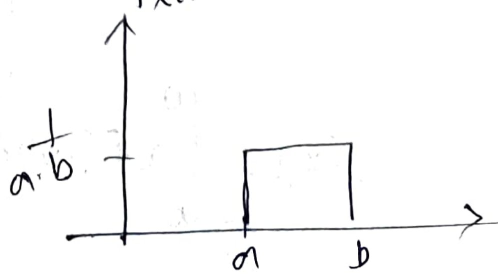
$$= \int_{x_1}^{x_2} f_X(x)\,dx$$

Expectation/mean of X:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

eg: (i) Uniform distribution $U[a,b]$ $f_X[x]$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$



$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

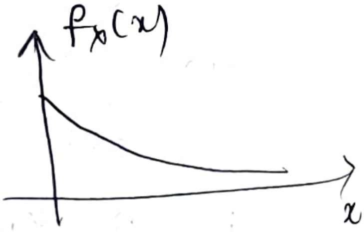
$$= \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$$

$$\mathbb{E}[X] = \frac{a+b}{2} \qquad \int_a^b \frac{x}{b-a}\,dx = \frac{b^2-a^2}{2(b-a)}$$

$$= \frac{b+a}{2}$$

(ii) Exponential distribution:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$



λ - parameter of distribution.

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{x=0} 0\,dx + \int_{x=0}^{\infty} \lambda e^{-\lambda x}\,dx$$

$$= \cancel{\int_x^{\infty}} \frac{\cancel{\lambda}\, e^{-\lambda x}}{-\cancel{\lambda}}\Bigg|_x^{\infty} = e^{-\lambda x}$$

$$\rightarrow (x=0)$$

$$= e^{-0} = 1$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \ge 0 \end{cases}$$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_0^{\infty} x \lambda e^{-\lambda x}\,dx = 1/\lambda$$

$$= \lambda\left[\frac{x \cdot e^{-\lambda x}}{-\lambda} + \int_0^{\infty} 1 \cdot \frac{e^{-\lambda x}}{+\lambda} \right]$$

$$\mathbb{E}[X] = \lambda\left[\frac{x\cdot e^{-\lambda x}}{-\lambda}\Big|_0^\infty - \frac{e^{-\lambda x}}{\lambda^2}\Big|_0^\infty\right] = \underline{\underline{1/\lambda}}$$

$$= \cancel{-e^{-\lambda x}\left[\frac{1}{\lambda}+x\right]}$$

(iii) Gaussian (or) Normal distribution:

(μ, σ^2)

$\mu \rightarrow$ mean.

$\sigma^2 \rightarrow$ Variance.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

→ If, $\mu = 0, \sigma^2 = 1 \rightarrow$ standard normal distribution

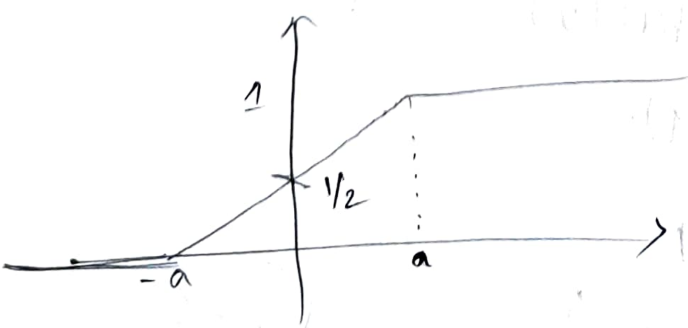
$$f_X(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}$$

Problems:

1. $$F_X(x) = \begin{cases} 0 & x < -a \\ \frac{1}{2}\left(\frac{x}{a}+1\right) & -a \le x \le a. \\ 1 & x > a \end{cases}$$

Verify that $F_X(x)$ is a valid CDF or not?

Sol: $F_X(\infty) = 1 \quad F_X(-\infty) =$



2. $f(x) = \begin{cases} x & 0 < x \leq 1 \\ 2-x & 1 < x \leq 2 \\ 0 & x > 2 \end{cases}$

find it's CDF ?

Sol: $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$

$F_X(x) = 0$ for $x \leq 0$

$x \in (0,1]$ $F_X(x) = \int_{-\infty}^{x} x\,dx = \frac{x^2}{2}$

$1 < x \leq 2$ $F_X(x) = \int_0^1 t\,dt + \int_1^x (2-t)\,dt$

$= \frac{1}{2} + \left(2t - \frac{t^2}{2}\right)_1^x$

$= \frac{1}{2} + \left(2x - \frac{x^2}{2} - \frac{3}{2}\right)$

$F_X(x) = 2x - \frac{x^2}{2} - 1$ //

$F_X(x) = 0$ for $x > 2$

3. A continuous r.v X has the pdf

$f_X(x) = K(1+x)$, $2 \leq x \leq 5$

find $P(X \leq 4)$.

Sol: $\int_2^5 K(1+x)\,dx = 1$

$K\left[x + \frac{x^2}{2}\right]_2^5 = 1$ $K = 2/27$

$K\left[5 + \frac{25}{2} - (2+2)\right] = 1$

$P(X \leq 4) = \int_{-\infty}^{4} f_X(x)\,dx = P(X < 4)$

$P(X = 4) = 0$ if X is a continuous random variable.

$P(X = a) = F_X(a) - F_X(a^-) = 0$ //

$\rightarrow P(X < 4) = \int_{-\infty}^{4} f_X(x)\,dx = \int_{2}^{4} \frac{2}{27}(1+x)\,dx$

$= \frac{2}{27}\left(x + \frac{x^2}{2}\right)_2^4$

$= \frac{2}{27}\left(4+8-(2+2)\right) = \frac{16}{27}$

$= 16/27$ //

4.

X	-2	-1	0	1	2	3
$P(x)$	0.1	k	0.2	$2k$	0.3	$8k$.

find $P(X < 2)$, $P(-2 < X < 2)$

Sol:-

$P(x) = 1 = 0.6 + 6k$

$0.4 = 6k \rightarrow \boxed{k = \frac{1}{15}}$

$P(X < 2) = 0.5$ //

$P(-2 < X < 2) = 0.4$ //

Mean $= 1.07 = \underline{x P(x)}$

5. A telephone call occurs at random in interval $[0, T]$.

$$P(t_1 \leq t \leq t_2) = \frac{t_2 - t_1}{T}$$

Define $X = t$ the time at which call occurs, find CDF of X.

Sol: $F_X(x) = P(X \leq x) = \frac{x}{T}, \quad x \leq T$

$F_X(x) = 1, \quad x > T$

6. $f(x) = C e^{-\gamma|x|}$, find c such that $f(x)$ is a valid pdf.

Sol: $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{0} C e^{-\gamma(-x)}\,dx + \int_{0}^{\infty} C e^{-\gamma x}\,dx$$

$$C\left(\frac{e^{\gamma x}}{\gamma}\right)_{-\infty}^{0} + C\left(\frac{e^{-\gamma x}}{-\gamma}\right)_{0}^{\infty}$$

$$C\left(\frac{1}{\gamma} - 0\right) + C\left(0 + \frac{1}{\gamma}\right) = \frac{2C}{\gamma} = 1$$

$$\boxed{C = \gamma/2}$$

7. $f(x) = \frac{D}{\tau^2 + x^2}$ find D s.t $f(x)$ is a valid pdf

Sol:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$D \times \frac{1}{\tau} \tan^{-1}\left(\frac{x}{6}\right)\Big|_{-\infty}^{+\infty} = 1$$

$$\frac{D}{6}\left[+\frac{\pi}{2} + \left(\pm\frac{\pi}{2}\right)\right] = 1$$

$$\boxed{D = \tau/\pi}$$

- **Variance:**

$$Var(X) = \mathbb{E}[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\,dx$$

$\mu = \mathbb{E}[X]$

$$= \sum_i (x_i - \mu)^2 P(x_i)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

μ, σ^2 are parameters
$\downarrow \quad \downarrow$
mean variance

$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

$\downarrow$

$$S = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$\frac{x-\mu}{\sigma} = t$$

$$\frac{1}{\sigma} dx = dt$$

$$S = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

$$S = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \qquad S = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

$$S^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}} dx\,dy$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$dx\,dy = |J|\,dr\,d\theta$$

$$|J| = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r$$

$$S^2 = \int_0^{2\pi}\int_0^{\infty} \frac{1}{2\pi} e^{-r^2/2} \, \underline{r\,dr\,d\theta}$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\infty} \frac{1}{2\pi} e^{-r^2/2} r\,dr\right)$$

$$= 2\pi \times \frac{1}{2\pi} \int_0^{\infty} e^{-r^2/2} r\,dr$$

$$S^2 = \int_0^{\infty} e^{-r^2/2} r\,dr = \int_0^{\infty} e^{-t} dt = e^{-t}\Big|_0^{\infty} = 0-(-1)$$

$$\left(\frac{r^2}{2} = t \Rightarrow r\,dr = dt\right) \qquad = 1$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \int_{-\infty}^{\infty} \frac{x-\mu}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx + \int_{-\infty}^{\infty} \frac{\mu}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$\Downarrow$$

$$= \sigma \int_{-\infty}^{\infty} \frac{t}{\sqrt{2\pi}} e^{-t^2/2} \quad + \quad \mu$$

$$\| $$

$$0$$

- Conditional distribution:

$A, B, \ P(B) \neq 0$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$A : \{\omega \in \Omega : X(\omega) \leq x\}$$

$$F_X(x/B) = P(X \leq x / B) = \frac{P(X \leq x \cap B)}{P(B)}$$

(i) $0 \leq F_X(x/B) \leq 1$

(ii) $F_X(x^+/B) = F_X(x/B)$

(iii) $F_X(x_1/B) \leq F_X(x_2/B)$, $x_1 < x_2$

$$f_X(x/B) = \frac{d}{dx}(F_X(x/B))$$

$$B = \{X \leq b\}$$

$$F_X(x/B) = \frac{P(X \leq x \cap X \leq b)}{P(X \leq b)}$$

$$F_X(x/B) = \begin{cases} \dfrac{F_X(x)}{F_X(b)} & x < b \\ 1 & x \geq b \end{cases}$$

$$f_X(x/B) = \begin{cases} \dfrac{f_X(x)}{\int_{-\infty}^{b} f_X(x)\,dx} & x < b \\ 0 & x \geq b \end{cases}$$

$$B = \{ b_1 < X \le b_2 \}$$

$$F_X(x|B) = \frac{P(\{X \le x\} \cap \{b_1 < X \le b_2\})}{P(B)}$$

$$= \begin{cases} \dfrac{F_X(x) - F_X(b_1)}{F_X(b_2) - F_X(b_1)} & b_1 < x < b_2 \\ 0 & x < b_1 \\ 1 & x \geq b_2 \end{cases}$$

$$f_X(x|B) = \begin{cases} \dfrac{f_X(x)}{F_X(b_2) - F_X(b_1)} & b_1 < x < b_2 \\ 0 & \text{otherwise.} \end{cases}$$

- **Memoryless property:**

$$P(x > m+t / x > t) = P(x > m)$$

(i) **Exponential distribution:**

$$f_x(x) = \lambda e^{-\lambda x}; \; x \geq 0$$

$$F_x(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$P(x > t) = 1 - P(x \leq t)$$
$$= e^{-\lambda t}$$

$$P\left(\frac{x > m+t}{x > t}\right) = \frac{P(x > m+t \cap x > t)}{P(x > t)}$$

$$= \frac{P(x > m+t)}{P(x > t)}$$

$$P(x > m+t / x > t) = \frac{e^{-\lambda(m+t)}}{e^{-\lambda t}} = e^{-\lambda m}$$

$$= P(x > m)$$

(ii) **Geometric distribution:**

$$P(x = i) = p(1-p)^{i-1} \quad i = 1, 2, \ldots$$

$$P(x > n) = \sum_{i=n+1}^{\infty} p(1-p)^{i-1}$$

$$= p(1-p)^{n} + p(1-p)^{n+1} + \ldots$$

$$= p(1-p)^{n}\left[1 + (1-p) + (1-p)^2 + \ldots\right]$$

$$= p(1-p)^{n} \times \frac{1}{1-(1-p)}$$

$$\therefore P(x > n) = \frac{\cancel{p}(1-p)^{n}}{\cancel{p}} = (1-p)^{n}$$

$$\therefore P(x > m+n) = (1-p)^{m+n}$$

$$P(x > m+n / x > n) = \frac{P(x > m+n)}{P(n)}$$

$$= \frac{(1-p)^{m+n}}{(1-p)^{n}}$$

$$= (1-p)^{m}$$

$$\boxed{\therefore P(x > m+n / x > n) = P(x > m)}$$

$f_X(x/A)$

let $A_1, A_2, \ldots, A_n$ be partitions of 'Ω'

$$\bigcup_{i=1}^{n} A_i = \Omega, \quad A_i \cap A_j = \phi \;, \; i \neq j$$

$$F_X(x) = P(X \le x)$$

$$= P(X \le x/A_1)\,P(A_1) + \cdots + (P(X \le x/A_n))\,P(A_n)$$

$$F_X(x) = F_X(x/A_1)\,P(A_1) + \cdots + F_X(x/A_n)\,P(A_n)$$

$$f_X(x) = f_X(x/A_1)\,P(A_1) + \cdots + f_X(x/A_n)\,P(A_n)$$

$$P(A/\{X = x\}) = \frac{P(A \cap \{X = x\})}{P(X = x)}$$

$$P(A/X = x) = \lim_{\Delta \to 0} P(A / x \le X \le x + \Delta)$$

$$= \lim_{\Delta \to 0} \frac{P(x \le X \le x + \Delta / A)\,P(A)}{P(x \le X \le x + \Delta)}$$

$$= \lim_{\Delta \to 0} \frac{(F_X(x+\Delta/A) - F_X(x/A))\,P(A)}{\left(\frac{F_X(x+\Delta) - F_X(x)}{\Delta}\right)}$$

$$P(A/X = x) = \frac{f_X(x/A)}{f_X(x)}\,P(A)$$

$$\int_{-\infty}^{\infty} f_X(x/A)\,dx = 1$$

$$\int_{-\infty}^{\infty} P(A/X = x)\,f_X(x)\,dx = \int_{-\infty}^{\infty} f_X(x/A)\,P(A)\,dx$$

$$= P(A)$$

$$\boxed{f_X(x/A) = \frac{P(A/X = x)\,f_X(x)}{\int_{-\infty}^{\infty} P(A/X = x)\,f_X(x)\,dx}} \longrightarrow \text{Baye's Rule}$$

EC252 : 19/01/2022

Asymptotic Approximations :

Eg: A fair coin tossed 1000 times. Find
(i) probability that heads will show 500 times
(ii) probability that heads will show 510 times

Sol: $P(\text{Head}) = P(\text{Tail}) = \frac{1}{2}$

$$\left.\begin{array}{l} \text{(i)}\quad {}^{1000}C_{500}\left(\tfrac{1}{2}\right)^{500}\left(\tfrac{1}{2}\right)^{500} \\ \text{(ii)}\quad {}^{1000}C_{510}\left(\tfrac{1}{2}\right)^{510}\left(\tfrac{1}{2}\right)^{490} \end{array}\right\} \text{Binomial Distribution.}$$

- **De Moivre-Laplace Theorem (Normal Approx) :**

If K is in the $\sqrt{npq}$ neighborhood of np

then, $$\binom{n}{k}p^{k}q^{n-k} \simeq \frac{1}{\sqrt{2\pi npq}}\, e^{-\frac{(k-np)^2}{2npq}}, \quad p+q=1$$

$np-\sqrt{npq}$ np $np+\sqrt{npq}$

$\Rightarrow$ For the above example problem,

(i) $$\binom{1000}{500}\left(\tfrac{1}{2}\right)^{500}\left(\tfrac{1}{2}\right)^{500} \simeq \frac{1}{\sqrt{2\pi\times1000\times\frac{1}{2}\times\frac{1}{2}}}\, e^{\cancel{-\frac{(500-500)}{2\times1000\times\frac{1}{2}\times\frac{1}{2}}}} \to 1$$

$$\simeq \frac{1}{\sqrt{\pi\times500}}$$

$\sqrt{npq} = \sqrt{1000\times\frac{1}{2}\times\frac{1}{2}} \simeq 15$

K lies in

$(485, 515)$

(ii) $\binom{1000}{510}\left(\frac{1}{2}\right)^{510}\left(\frac{1}{2}\right)^{490} \simeq \frac{1}{\sqrt{2\pi\times1000\times\frac{1}{2}\times\frac{1}{2}}} e^{-\frac{(510-500)^2}{2\times1000\times\frac{1}{2}\times\frac{1}{2}}}$

$$\simeq \frac{1}{\sqrt{\pi\times500}} e^{-\frac{100}{500}}$$

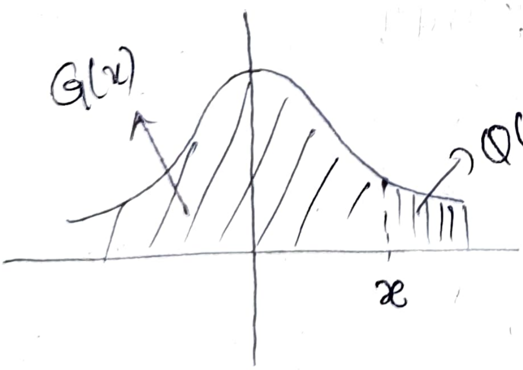
$$\simeq \frac{1}{\sqrt{\pi\times500}} e^{-0.2}$$

$$P(K_1 \le x \le K_2) = \sum_{k=K_1}^{K_2} \binom{n}{k} p^k q^{n-k}$$

$$\simeq \int_{K_1}^{K_2} \frac{1}{\sqrt{2\pi npq}} e^{-\frac{(x-np)^2}{2npq}} dx$$

$\Rightarrow$ Error function: $\text{erf}(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \underbrace{e^{-y^2/2}}_{} dy$

density of standard or normal function ($\mu=0$, $\sigma^2=1$)



$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-y^2/2} dy \qquad G(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2} dy$$

(i) $Q(x) + G(x) = 1$

(ii) $\text{erf}(x) = G(x) - 1/2$

(iii) $G(-x) = 1 - G(x)$

eg: A fair coin is tossed 10,000 times. What is the probability that the no. of heads is in b/w 4900 and 5100?

Sol:- $n = 10000 \quad k_1 = 4900 \quad k_2 = 5100$

$p = q = \frac{1}{2} \quad np = 5000 \quad \sqrt{npq} = \sqrt{2500} = 50$

$npq >> 1$

$x_1 = -2 \quad x_2 = 2$

$$P(k_1 \le X \le k_2) \simeq G(2) - G(-2)$$

$$= 2G(2) - 1$$

$$= 0.9545$$

(*) $x_1 = \dfrac{k_1 - np}{\sqrt{npq}} \quad x_2 = \dfrac{k_2 - np}{\sqrt{npq}}$

$$P(k_1 \le X \le k_2) \simeq G(x_2) - G(x_1)$$

EC252:

21/02/2022

Poisson Approximation:

eg: A system contains 1000 components. Each component fails independently of the others and the probability of it's failure in one month 10^{-3}. What is the probability that the system will function at the end of one month?

Sol: K = no. of components failed in one month

$$P(K=0) = \binom{1000}{0} p^0 (1-p)^{1000} \qquad \lambda = np = 1$$

$$= (0.999)^{1000} \qquad P(k=0) \simeq e^{-1} \cdot \frac{1^0}{0!}$$

$$= e^{-1}$$

$$= \underline{0.367}$$

- Poisson theorem:

$n \to \infty$, $p \to 0$ such that $np \to \lambda$

then $$\binom{n}{k} p^k (1-p)^{n-k} \xrightarrow[\substack{n\to\infty \\ p\to 0}]{} e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$k = 0, 1, 2, \ldots$

Proof:

$$= \frac{n!}{(n-k)!\,(k!)} p^k (1-p)^{n-k}$$

$$= \frac{n(n-1)\cdots(n-(k-1))\,\cancel{(n-k)!}}{\cancel{(n-k)!}\; k!} p^k (1-p)^{n-k}$$

$$= \frac{n(n-1)\cdots(n-k+1)}{n^k} \cdot \frac{(np)^k}{k!} \left(1 - \frac{np}{n}\right)^{n-k}$$

$$= \frac{\overbrace{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)}^{\to 1}}{\left(1-\frac{\lambda}{n}\right)^k} \cdot \frac{\lambda^k}{k!} \left(1-\frac{\lambda}{n}\right)^n$$

$$= \frac{\lambda^k}{k!} \cdot e^{-\lambda}$$

eg: An order of 3000 parts is received. The probability that a part is defective equals 10^{-3}. What is the probability that more than 5 parts are defective.

Sol:-

$$P(K>5) = 1 - P(K \le 5)$$

$$= 1 - \sum_{K=0}^{5} \binom{3000}{K} (p)^K (1-p)^{3000-K}$$

$$P(K>5) = 1 - \sum_{K=0}^{5} \binom{3000}{K} (10^{-3})^K (0.999)^{3000-K}$$

$$(\lambda = np = 3000 \times 10^{-3} = 3)$$

$$P(K>5) = 1 - \sum_{K=0}^{5} \frac{3^K}{K!} e^{-3}$$

$$= 1 - e^{-3}\left[\frac{3^0}{0!} + \frac{3^1}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \frac{3^5}{5!}\right]$$

$$= 1 - e^{-3}\left[1 + 3 + 4.5 + 4.5 + 3.375 + 2.025\right]$$

$$= 1 - e^{-3}\left[18.4\right]$$

$$= 1 - 0.916$$

$$\therefore P(K>5) = \underline{0.084}$$

eg: An insurance company has issued policies to 1,00,000 people for a premium of \$500/person. In the event of causality, the probability of which is assumed to be 0.001, the company pays \$200,000/causality.

(a) What is the probability that company will suffer a loss?

$$K > \frac{\cancel{500}^{250} \times 1,\cancel{00,000}}{\cancel{2,00,000}} \Rightarrow K > 250$$

$\downarrow$

Company suffers loss.

$$P(K>250) = \sum_{K=250}^{1,00,000} \binom{100000}{k} p^k (1-p)^{n-k}$$

$$\left(\begin{aligned} \lambda &= np = 10^5 \times 10^{-3} \\ \lambda &= 10^2 \end{aligned} \right)$$

$$P(K>250) = \sum_{K=250}^{1,00,000} \frac{e^{-\lambda} \cdot \lambda^k}{K!}$$

$n = 250$

$\lambda = 100$

$$\Delta = \frac{\lambda}{n} e^{1-\frac{\lambda}{n}}$$

$$= \frac{100}{250} e^{1-\frac{100}{250}}$$

$$(\Delta)^{250} \simeq 0$$

$$\Delta = 0.4 e^{0.6} = 0.7288$$

Because the upper bound and lower bound of $P(K \geq n)$ are close to zero

$$P(K \geq 250) \simeq 0$$

$$\frac{e^{-\lambda}\cdot\lambda^n}{n!} < \sum_{k=n}^{\infty} \frac{e^{-\lambda}\cdot\lambda^k}{k!}$$

$$= e^{-\lambda}\cdot\left[\frac{\lambda^n}{n!} + \frac{\lambda^{n+1}}{(n+1)!} + \cdots\right]$$

$$= \frac{e^{-\lambda}\cdot\lambda^n}{n!}\left[1 + \frac{\lambda}{n+1} + \frac{\lambda^2}{(n+1)\underbrace{(n+2)}_{(n+1)}} + \cdots\right]$$

$$\leq \frac{e^{-\lambda}\cdot\lambda^n}{n!}\left[1 + \frac{\lambda}{n+1} + \frac{\lambda^2}{(n+1)^2} + \frac{\lambda^3}{(n+1)^3} + \cdots\right]$$

$$= \frac{e^{-\lambda}\cdot\lambda^n}{n!}\left[\frac{1}{1-\frac{\lambda}{n+1}}\right]$$

$$\boxed{\frac{e^{-\lambda}\cdot\lambda^n}{n!} < P(K \geq n) < \frac{e^{-\lambda}\cdot\lambda^n}{n!}\left[\frac{1}{1-\frac{\lambda}{n+1}}\right]}$$

- Stirling's formula:

$$n! = \sqrt{2\pi n}\cdot n^n\cdot e^{-n}$$

Substituting $n!$ value in above equation,

$$\frac{e^{-\lambda}\cdot\lambda^n}{\sqrt{2\pi n}\cdot n^n\cdot e^{-n}} < P(K \geq n) < \frac{e^{-\lambda}\cdot\lambda^n}{\sqrt{2\pi n}\cdot n^n\cdot e^{-n}}\left(\frac{1}{1-\frac{\lambda}{n+1}}\right)$$

$$\left(\frac{\lambda}{n}\right)^n \cdot \frac{e^{n-\lambda}}{\sqrt{2\pi n}} < P(k \geq n) < \frac{e^{-\lambda} \cdot \lambda^n}{\sqrt{2\pi n} \cdot n^n \cdot e^{-n}} \left(\frac{1}{1-\frac{\lambda}{n+1}}\right)$$

$$\left(\frac{\lambda}{n}\right)^n \cdot \frac{\left(e^{\left(1-\frac{\lambda}{n}\right)}\right)^n}{\sqrt{2\pi n}} < P(k \geq n) < \left(\frac{\lambda}{n}\right)^n \cdot \frac{\left(e^{1-\frac{\lambda}{n}}\right)^n}{\sqrt{2\pi n}} \left(\frac{1}{1-\frac{\lambda}{n+1}}\right)$$

$\rightarrow \Delta = \frac{\lambda}{n} e^{1-\frac{\lambda}{n}}$ with this def of Δ, we can write the above inequality as,

$$\boxed{\frac{\Delta^n}{\sqrt{2\pi n}} < P(k \geq n) < \frac{\Delta^n}{\sqrt{2\pi n}} \left(\frac{1}{1-\frac{\lambda}{n+1}}\right)}$$

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Functions of One Random Variable:

Let X be a continuous random variable with CDF $F_X(x)$.

$g(x)$ be real valued function $g: \mathbb{R} \to \mathbb{R}$

$X: \Omega \to \mathbb{R}$ range of X. $(R(X)) \subseteq \mathbb{R}$

Let $Y = g(X)$

$$Y: \Omega \to \mathbb{R} \mid Y(\omega) = g(X(\omega))$$

Y is a random variable if
$$\{\omega \in \Omega : Y(\omega) \le y\} \in \mathcal{F} \,;\, \forall y \in \mathbb{R}$$

CDF of Y, $F_Y(y) = P(Y \le y) = P(g(X) \le y)$

If $g(\cdot)$ is a Borel function (measurable function) then Y is a random variable - continuous function

→ We need to find $F_Y(y)$ interms of F_X, $g(\cdot)$

$$F_Y(y) = P(Y \le y) = P(g(X) \le y)$$

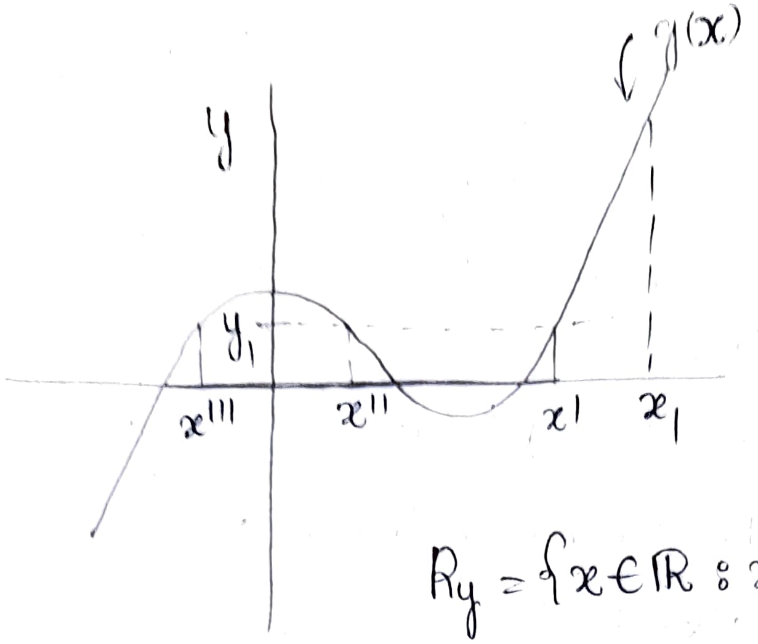
$$\{\omega \in \Omega : g(X(\omega)) \le y\}$$

$$g(x) \le y$$

$$R_y = \{x \in \mathbb{R} : g(x) \le y\}$$

$$F_Y(y) = P_X(R_y) = \underbrace{P\{\omega \in \Omega : X(\omega) \in R_y\}}_{\text{using CDF of } X.}$$

eg:



$$R_y = \{x \in \mathbb{R} : x \le x_1\}$$

$$R_{y_1} = \{x'' \le x \le x'\} \cup \{x \le x'''\}$$

$$F_Y(y) = P(R_{y_1}) = P(x'' \le X \le x') + P(X \le x''')$$

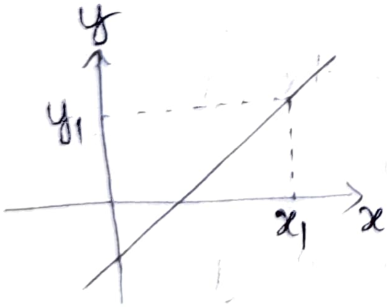
$$= F_X(x') - F_X(x'') + F_X(x''')$$

eg: (i) $a > 0$

$$P(aX + b \le y) = P(aX \le y - b)$$

$$= P\left(X \le \frac{y-b}{a}\right)$$

$$= F_X\left(\frac{y-b}{a}\right)$$



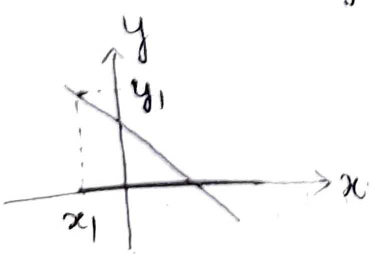
(ii) $a < 0$

$$P(Y \le y_1) = P(aX + b \le y_1)$$

$$= P(aX \le y_1 - b)$$

$$= P\left(X \ge \frac{y_1 - b}{a}\right)$$

$$= 1 - P\left(X < \frac{y_1 - b}{a}\right)$$



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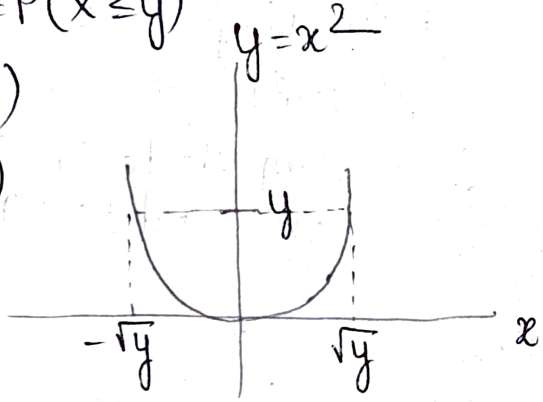
$$f_Y(y) = \begin{cases} F_X\left(\frac{y-b}{a}\right), & a>0 \\ 1 - F_X\left(\frac{y-b}{a}\right), & a<0 \end{cases}$$

$$\boxed{f_X(x) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)} \longrightarrow \text{PDF expression.}$$

eg: $g(x) = x^2$, $Y = X^2$

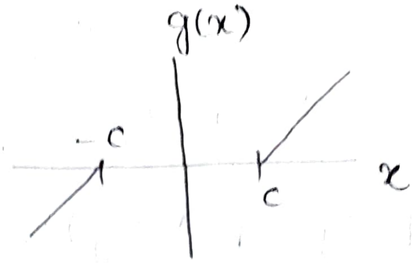
$$F_Y(y) = P(Y \le y) = P(X^2 \le y)$$
$$= P(-\sqrt{y} \le X \le \sqrt{y})$$
$$F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$



$$F_Y(y) = \begin{cases} F_X(\sqrt{y}) - F_X(-\sqrt{y}), & y \geq 0 \\ 0, & y<0 \end{cases}$$

$$f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}), \quad y \geq 0$$

eg:
$$g(x) = \begin{cases} x - c, & x > c \\ 0, & -c < x < c \\ x + c, & x < -c \end{cases}$$



$Y = g(x)$

$f_Y(y) = P(Y \leq y)$

$y = 0, \quad f_Y(y) = P(-c \leq X \leq c) \cup P(X + c \leq y)$
$= f_X(c) - f_X(-c) + f_X(y - c)$

$y > 0, \quad f_Y(y) = P(X - c \leq y)$
$= P(X < y + c)$
$= f_X(y + c)$

$y < 0, \quad f_Y(y) = P(X + c \leq y)$
$= P(X \leq y - c)$
$= f_X(y - c)$

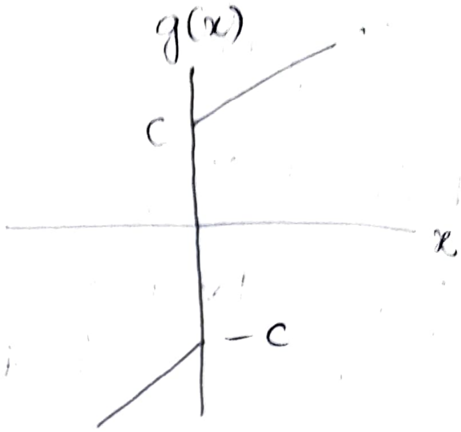
$$f_Y(y) = \begin{cases} F_X(y + c), & y > 0 \\ f_X(c) - f_X(-c) + f_X(y - c), & y = 0 \\ f_X(y - c), & y < 0 \end{cases}$$

eg:

$$g(x) = \begin{cases} x+c & , x \geq 0 \\ x-c & , x < 0 \end{cases}$$

$Y = g(x)$

$F_Y(y) = P(Y \leq y)$



$y \in (-c, c)$, $F_Y(y) = P(Y \leq y)$

$= P(X - c \leq y)$

$g(x) \leq y$, $x \leq 0$

$P(Y \leq y) = P(X \leq 0) = \underline{F_X(0)}$

$y \geq c$, $g(x) \leq y$

$F_Y(y) = P(Y \leq y) = P(X + c \leq y)$

$= P(X \leq y - c)$

$= \underline{F_X(y-c)}$

$y \leq -c$, $g(x) \leq y$

$F_Y(y) = P(Y \leq y) = P(X - c \leq y)$

$= P(X \leq y + c)$

$= \underline{F_X(y+c)}$

Fundamental Theorem :- $Y = g(x)$

To find $f_Y(y)$ for a specific y, solve the equation $y = g(x)$,

Let $x_1, x_2, x_3, \dots x_n$ be the real roots of $y = g(x)$

Then, PDF is given by,

$$\underset{(PDF)}{f_Y(y)} = \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} + \dots + \frac{f_X(x_n)}{|g'(x_n)|},$$

$g'(x)$ is the derivative of $g(x)$

eg: $Y = X^2$

$x_1 = \sqrt{y}$ $\quad g'(x) = 2x$

$x_2 = -\sqrt{y}$

$$f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|}$$

$$= \frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}}$$

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left(f_X(\sqrt{y}) + f_X(-\sqrt{y})\right)$$

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eg: $Y = \frac{1}{X}$, $g(x) = \frac{1}{x}$ $g'(x) = -\frac{1}{x^2}$

$$y = g(x) \Rightarrow x_1 = \frac{1}{y}$$

$$f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|}$$

$$f_Y(y) = \frac{f_X(1/y)}{\left(\left|-\frac{1}{x_1^2}\right|_{x_1 = 1/y}\right)}$$

$$\boxed{f_Y(y) = \frac{1}{y^2} f_X(1/y)}$$

$$f_X(x) = \frac{\alpha/\pi}{x^2+\alpha^2}$$; cauchy density with parameter 'α'.

$(Y = 1/X)$ $$f_Y(y) = \frac{\alpha/\pi}{1+\alpha^2 y^2} = \frac{1/\alpha\pi}{y^2 + 1/\alpha^2}$$; Cauchy density with parameter '$1/\alpha$'.

$$f_Y(y) = \frac{1}{y^2}\left[\frac{\alpha/\pi}{\frac{1}{y^2}+\alpha^2}\right]$$

$$= \frac{1}{\cancel{y^2}}\left[\frac{\alpha/\pi \times \cancel{y^2}}{1+\alpha^2 y^2}\right]$$

$$f_Y(y) = \frac{\alpha/\pi}{1+\alpha^2 y^2}$$

eg: $Y = \sqrt{x}$, $g(x) = \sqrt{x}$ $\quad g'(x) = \frac{1}{2\sqrt{x}}$

$$y = g(x) \Rightarrow x_1 = y^2$$

$y > 0$, $f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|}$

$$= \frac{f_X(y^2)}{\left|\frac{1}{2y}\right|} \qquad U(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$f_Y(y) = 2y f_X(y^2) U(x)$$

- Chi-square density: n-degrees of freedom,

$$f_X(x) = \frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} x^{\frac{n}{2}-1} e^{-\frac{x}{2}} U(x)$$

$$\boxed{\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt}$$

$Y = \sqrt{X}$

$$f_Y(y) = 2y f_X(y^2) U(y)$$

$$= 2y \cdot \frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} y^{n-2} e^{-y^2} U(y)$$

$$f_Y(y) = 2 \cdot \frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} y^{n-1} e^{-y^2} U(y)$$

↳ chi-density with n-degrees of freedom

$n = 3$;

$$f_Y(y) = \sqrt{2/\pi}\, y^2 e^{-y^2/2} \rightarrow \text{Maxwell density}$$

$n = 2$;

$$f_Y(y) = y e^{-y^2/2} U(y) \rightarrow \text{Rayleigh density}$$

eg: $Y = a\sin(X+\theta)$, $a>0$

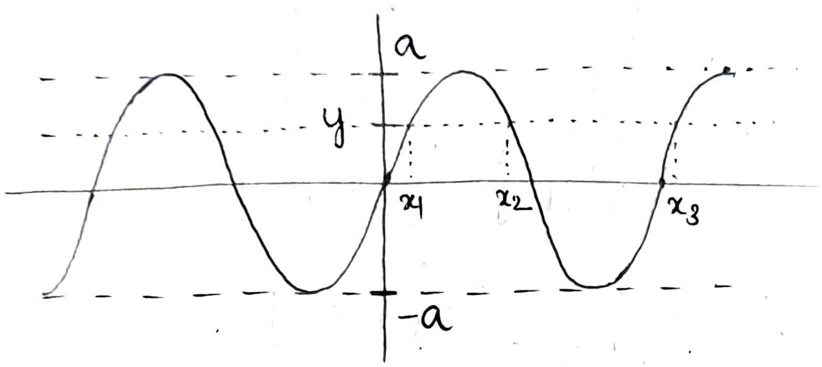
$g(x) = a\sin(x+\theta)$

$|y|>a$, $f_Y(y) = 0$

$|y| \le a \rightarrow x_1 = \sin^{-1}\left(\frac{y}{a}\right) - \theta$

$x_2 = \pi - x_1$.

$x_3 = 2\pi + x_1$



$x_1, x_2, \ldots\ldots$ are roots of $y = g(x)$

$g'(x) = a\cos(x+\theta) = \sqrt{a^2-y^2}$

$x_n = \sin^{-1}\left(\frac{y}{a}\right) - \theta$

$\sin(x_n+\theta) = y/a$

$\sin^2(x_n+\theta) + \cos^2(x_n+\theta) = 1$

$\frac{y^2}{a^2} + \cos^2(x_n+\theta) = 1$

$\cos(x_n+\theta) = \sqrt{1-\frac{y^2}{a^2}}$

$a\cos(x_n+\theta) = \sqrt{a^2-y^2}$

$$\boxed{f_Y(y) = \sum_{n=-\infty}^{\infty} \frac{f_X(x_n)}{\sqrt{a^2-y^2}} \; ; |y|<a}$$

$$f_X(x) = \begin{cases} \dfrac{2x}{\pi^2}, & 0 < x < \pi \\ 0 & \text{else} \end{cases}$$

$$f_Y(y) = \sum_{n=-\infty}^{\infty} \frac{f_X(x_n)}{\sqrt{a^2-y^2}}$$

$$= \frac{1}{\sqrt{1-y^2}} f_X(x_1) + \frac{1}{\sqrt{1-y^2}} f_X(x_2)$$

$$= \frac{2x_1}{\pi^2\sqrt{1-y^2}} + \frac{2x_2}{\pi^2\sqrt{1-y^2}}$$

$$= \frac{2}{\pi^2\sqrt{1-y^2}} [x_1 + x_2]$$

$$= \frac{2}{\pi^2\sqrt{1-y^2}} [\cancel{x_1} + \pi - \cancel{x_1}]$$

$$f_Y(y) = \frac{2}{\pi\sqrt{1-y^2}}$$

$$f_Y(y) = \begin{cases} \dfrac{2}{\pi\sqrt{1-y^2}}, & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

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$Y = g(X)$, X is a r.v with CDF $F_X(x)$

We can choose g appropriately so that Y is a uniform random variable for any r.v 'X'.

choose $g(x) = F_X(x)$

then $Y = F_X(X)$

$$F_Y(y) = P(Y \le y) = P(F_X(X) \le y)$$
$$= P(F_X(X) \le y)$$
$$= P(X \le F_X^{-1}(y))$$
$$= F_X(F_X^{-1}(y))$$
$$= y$$

$X \sim U[0,1]$ $\quad Y = g(X)$

By appropriately choosing g, we can obtain the r.v 'Y' with a given $F_Y(y)$

$$g(x) = F_Y^{-1}(x)$$

$$\tilde{F}_Y(y) = P(Y \le y) = P(g(X) \le y)$$
$$= P(F_Y^{-1}(X) \le y)$$
$$= P(X \le \underbrace{F_Y(y)}_{[0,1]})$$
$$= F_Y(y)$$

$U \sim \text{Unif}[0,1]$

$$f_U(u) = 1 , \quad 0 \le u \le 1$$

$$F_U(u) = \begin{cases} u & 0 \le u \le 1 \\ 1 & \text{if } u > 1 \\ 0 & u < 0 \end{cases}$$

$X \sim f_X(x)$ $\quad$ $Y \sim f_Y(y)$

$g = ?$ so that $Y = g(X)$

$$X \downarrow U \downarrow Y$$

$U = g_1(X)$ $\quad$ $g_1(x) = f_X(x)$

$Y = g_2(U)$ $\quad$ $g_2(x) = f_Y^{-1}(x)$

$$Y = g_2(g_1(X)) = (g_2 \circ g_1)(X)$$

$$= \underline{f_Y^{-1}(f_X(x))}$$

- Mean and Variance:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx \longrightarrow \text{continuous}$$

$$E[X] = \sum_x x P(X = x) \longrightarrow \text{discrete}$$

$Y = g(X)$

$$\boxed{E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx} \longrightarrow \text{Mean/Expectation}$$

$$\boxed{E[g(X)] = \sum_x g(x) P(X = x)}$$

Variance $\sigma_X^2 = var(X) = \mathbb{E}[(X-\mu)^2]$

for continuous r.v

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\,dx$$

($\mu \to$ mean)

for discrete r.v $\sigma_X^2 = \sum_{x} (x-\mu)^2 P(X=x)$

$Y = aX + b$

$\mathbb{E}[Y] = a\,\mathbb{E}[X] + b$

<u>example</u>: $f_X(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$

$Var(X) = ?$

$$Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

$$= 2\int_0^{\infty} \frac{(x-\mu)^2}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

$x - \mu = t$

$$= 2\int_0^{\infty} \frac{t^2}{\sqrt{2\pi\sigma^2}} e^{-t^2/2\sigma^2}\,dt$$

eg:

$$X = \begin{cases} 0 & \text{with pr } 1-p \\ 1 & \text{with pr } p \end{cases}$$

$$\mathbb{E}[X] = 0(1-p) + 1(p)$$
$$= p$$

$$\text{Var}(x) = \sum_{x} (x-p)^2 \, p(X=x)$$

$$= p^2 \, p(X=0) + (1-p)^2 \, p(X=1)$$
$$= p^2 (1-p) + (1-p)^2 (p)$$
$$= p(1-p)(\cancel{p} + 1 - \cancel{p})$$

$$\underline{\text{Var}(x) = p(1-p)}$$

- **Moments:**

nth moment of a r.v $X = \mathbb{E}[X^n]$

$\mathbb{E}[X^2]$: second moment

→ Central moments

the nth central moment $\mu_n = \mathbb{E}[(X-\mu)^n]$

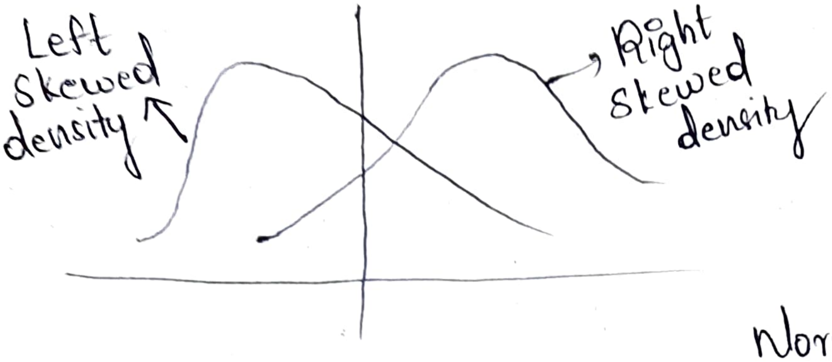
$\mu_1 = 0$ $\quad \mathbb{E}[X-\mu] = E[X] - E[\mu] = \mu - \mu = 0$

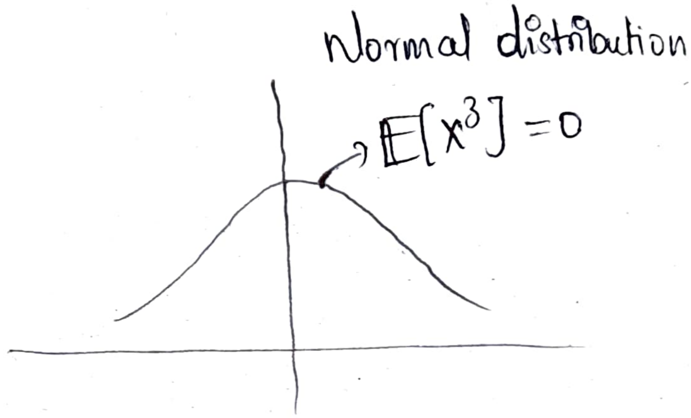
$\mu_2 = \text{Var}(x)$ $\quad \mathbb{E}[(X-\mu)^2] = \text{Var}(x)$

$$\text{Var}(X) = \mathbb{E}[(X-\mu)^2] = \mathbb{E}[x^2 - 2x\mu + \mu^2]$$
$$= \mathbb{E}[X^2] - 2\mu \underbrace{\mathbb{E}[X]}_{\mu} + \mu^2$$
$$= \mathbb{E}[X^2] - 2\mu^2 + \mu^2$$

$$\boxed{\mathbb{E}[X^2] = \text{Var}(x) + \mu^2}$$

$\mathbb{E}[X^3]$: skewness





eg: $X \sim N(0,1)$ $\quad Y = aX + b$

find mean and variance of Y.

$$\mathbb{E}[Y] = \mathbb{E}[aX+b] = a\,\overset{0}{\cancel{\mathbb{E}[X]}} + b = \underline{\underline{b}}.$$

$$\begin{aligned}
Var(Y) &= \mathbb{E}[(Y-b)^2] \\
&= \mathbb{E}[(aX+\cancel{b}-\cancel{b})^2] \\
&= \mathbb{E}[a^2X^2] \\
&= a^2\,\mathbb{E}[X^2] \\
&= a^2 \times 1
\end{aligned}$$

$$\underline{\underline{Var(Y) = a^2}}$$

• Characteristic function of R.V. X :

characteristic fn of X

$$\phi_X(\omega) = \mathbb{E}[e^{j\omega X}] = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$$

characteristic fn $\phi_X(\omega) \xleftrightarrow{FT}$ pdf $f_X(x)$

Moment generating fn $\Psi_X(s) = \mathbb{E}[e^{sX}]$

$$= \int_{-\infty}^{\infty} e^{sx} f_X(x)\, dx$$

(Laplace - Transform)

$$\frac{d^n}{ds^n}(\Psi_X(s)) = \frac{d^n}{ds^n}\mathbb{E}[e^{sX}]$$

$$= \frac{d^{n-1}}{ds^{n-1}}\left[\mathbb{E}[X e^{sX}]\right]$$

$$\mathbb{E}[X^n] = \left.\frac{d^n}{ds^n}(\Psi_X(s))\right|_{s=0}$$

eg: $Y = aX + b$

Given characteristic fn $\phi_X(\omega)$ of X, find $\phi_Y(\omega)$

Sol: $\phi_Y(\omega) = \mathbb{E}[e^{j\omega Y}] = \mathbb{E}[e^{j\omega(aX+b)}]$

$$= \mathbb{E}[e^{j\omega aX} \cdot e^{j\omega b}]$$

$$= e^{j\omega b}\,\mathbb{E}[e^{j\omega aX}]$$

$$\therefore \phi_Y(\omega) = e^{j\omega b}\,\phi_X(\omega a)$$

eg: Compute the moments of $X \sim N(0, \sigma^2)$ from definition

$$E[X^n] = \int_{-\infty}^{\infty} x^n \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} dx$$

if n is odd : $E[X^n] = 0$

n is even :

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} dx = 1$$

$$\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} dx = \sqrt{2\pi\sigma^2}$$

Let $\alpha = \frac{1}{2\sigma^2}$

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$$

differentiate 'k' times,

$$\int_{-\infty}^{\infty} x^{2k} e^{-\alpha x^2} dx = \sqrt{\pi} \frac{d^k}{d\alpha^k}\left(\frac{1}{\sqrt{\alpha}}\right)$$

$$\int_{-\infty}^{\infty} \cancel{(-1)^k} x^{2k} e^{-\alpha x^2} dx = \sqrt{\pi} \frac{d^k}{d\alpha^k}(\alpha^{-1/2})$$

$$= \frac{\sqrt{\pi}\, \cancel{(-1)^k} (+3)(+5)\ldots(2k-1)}{2^k} \alpha^{-\left(\frac{2k+1}{2}\right)}$$

$$= \frac{\sqrt{\pi}\,(1 \times 3 \times 5 \times \ldots (2k-1))}{2^k} \cdot \frac{1}{\sqrt{\alpha^{2k+1}}}$$

$$\int_{-\infty}^{\infty} x^{2k} e^{-\alpha x^2} dx = \sqrt{\pi} \, \frac{(1\times 3\times 5 \cdots \times (2k-1))}{2^k} \cdot \frac{1}{\sqrt{\alpha^{2k+1}}}$$

$$\left((\alpha)^{2k+1} = \frac{1}{(2\sigma^2)^{2k+1}} \right)$$

$$= \sqrt{\pi} \, \frac{(1\times 3\times 5 \cdots \times (2k-1))}{2^k} \cdot \sqrt{2}^{2k+1} \cdot \sigma^{2k+1}$$

$$= \sqrt{\pi} \times \frac{(1\times 3\times 5 \cdots \times (2k-1))}{\cancel{2^k}} \cdot \cancel{2^k} \cdot \sqrt{2} \, \sigma^{2k} \cdot \sqrt{\sigma^2}$$

$$= \sqrt{2\pi\sigma^2} \, (1\times 3\times 5 \cdots \times (2k-1)) \times \sigma^{2k}$$

$n = 2k$

$$E[x^n] = (1\times 3\times 5 \times \cdots \times (n-1)) \, \sigma^n$$

$$E[x^n] = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (1\times 3\times 5 \cdots \times (n-1)) \, \sigma^n & n \text{ is even} \end{cases}$$

eg: S.T $\phi_x(\omega) = \exp\{j\mu\omega - \frac{1}{2}\sigma^2\omega^2\}$.

$X \sim N(\mu, \sigma^2)$

Let $Y = \frac{X-\mu}{\sigma}$, $Y \sim N(0,1)$

Sol:

$$\phi_Y(\omega) = \mathbb{E}[e^{j\omega Y}]$$

$$\Psi_Y(s) = \int_{-\infty}^{\infty} e^{sy} \cdot \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy$$

$$sy - \frac{y^2}{2} = \frac{2sy - y^2}{2} = \frac{2sy - y^2 + s^2 + s^2}{2}$$

$$= -\frac{(y-s)^2 - s^2}{2}$$

$$\Psi_Y(s) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(y-s)^2}{2}} \cdot e^{\frac{s^2}{2}}}{\sqrt{2\pi}}\, dy$$

$$= e^{s^2/2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-s)^2}{2}}\, dy}_{N(s,1)}$$

$$\Psi_Y(s) = e^{s^2/2}$$

$$\phi_Y(\omega) = e^{(j\omega)^2/2} = e^{-\omega^2/2}$$

$$X = \sigma Y + \mu$$

$$\phi_X(\omega) = \mathbb{E}[e^{j\omega X}]$$

$$= \mathbb{E}[e^{j\omega(\sigma Y + \mu)}]$$

$$= e^{j\omega\mu}\, \mathbb{E}[e^{j\omega\sigma Y}] = e^{j\omega\mu}\phi_Y(\omega\sigma)$$

$$= e^{j\omega\mu} \cdot e^{-\sigma^2\omega^2/2}$$

EC252:

04/02/2022

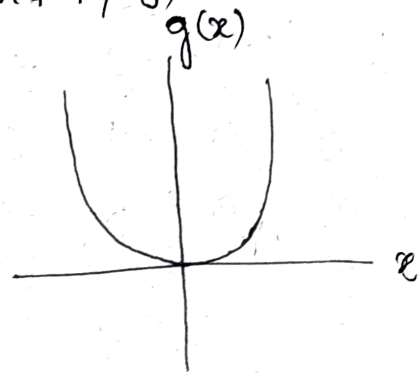
Problems:

1. Let X be a random variable between $(-2c, 2c)$

$Y = g(x) \quad g(x) = x^2$

Find and sketch, $f_Y(y)$ and $F_Y(y)$

$$f_X(x) = \begin{cases} \frac{1}{4c}, & -2c < x < 2c \\ 0, & \text{else} \end{cases}$$



roots of $g(x) = y$

$$x_1 = \sqrt{y}, \quad x_2 = -\sqrt{y}$$

$$f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} \quad \text{for } 0 < y < 4c^2$$

$$f_Y(y) = \begin{cases} \frac{\frac{1}{4c}}{2\sqrt{y}} + \frac{\frac{1}{4c}}{2\sqrt{y}} & 0 < y < 4c^2 \\ 0 & y \geq 4c^2 \end{cases}$$

$$f_Y(y) = \frac{1}{4c\sqrt{y}}, \quad 0 < y < 4c^2$$

$$F_Y(y) = \begin{cases} \frac{\sqrt{y}}{2c}, & 0 < y < 4c^2 \\ 1\cancel{0}, & y \geq 4c^2 \end{cases}$$

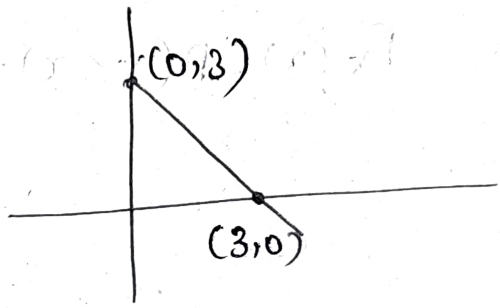
2. $f_X(x) = 2e^{-2x}U(x)$

$$U(x) = \begin{cases} 1 & , \text{ if } x \geq 0 \\ 0 & , \text{ else} \end{cases}$$

$Y = -4X + 3$ find $F_Y(y)$ and $f_Y(y)$

Sol: $X \geq 0$

$f_Y(y) = 0 \quad y > 3$



$g(x) = y$

$$x_1 = \frac{y-3}{-4}$$

$$f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} = \frac{2e^{-\cancel{2}\left(\frac{y-3}{\cancel{+4}\,2}\right)}}{4}$$

$$f_Y(y) = \begin{cases} \frac{1}{2}e^{\left(\frac{y-3}{2}\right)} & , y \leq 3 \\ 0 & , y > 3 \end{cases}$$

$$F_Y(y) = \begin{cases} e^{\left(\frac{y-3}{2}\right)} & , y \leq 3 \\ 1 & , y > 3 \end{cases}$$

3. U is a uniform random variable between $(0,1)$

$$X = (U-0.5)^{+} = \max\{0, U-0.5\}$$

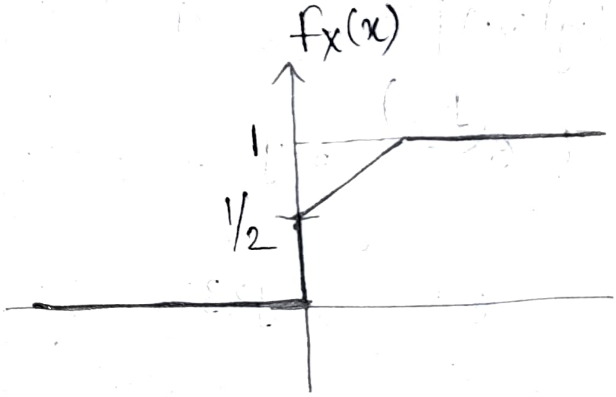
$$= \begin{cases} U-0.5, & U \geq 0.5 \\ 0, & U < 0.5 \end{cases}$$

(i) find and sketch CDF, $F_X(x)$

$$F_X(x) = P(X \leq x) = \begin{cases} 0 & \text{if } x < 0 \\ 1/2 & x = 0 \\ P(U-0.5 \leq x) & x > 0, \; x \leq 0.5 \\ 1 & x > 0.5 \end{cases}$$

$0 < x < 0.5$

$$P(U-0.5 \leq x) = P(U \leq x+0.5) = 0.5 + x$$



(ii) find characteristic fn $\phi_X(\omega)$

$$\phi_X(\omega) = \mathbb{E}[e^{j\omega x}]$$

$$f_X(x) = \begin{cases} \frac{1}{2}\delta(0) + 1 & \text{if } x \in (0, 0.5) \\ 0 & \text{else} \end{cases}$$

4. X is a random variable with pdf,

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(i) $P(X \geq 0.4 / X \leq 0.8)$

(ii) $Y = -\ln(x)$ find Pdf, CDF of Y.

Sol: (i)

$$P(x \geq 0.4 / x \leq 0.8) = \frac{P(0.4 \leq x \leq 0.8)}{P(x \leq 0.8)}$$

$$= \frac{\int_{0.4}^{0.8} 2x\,dx}{\int_{0}^{0.8} f_X(x)\,dx}$$

$$= \frac{x^2 \Big|_{0.4}^{0.8}}{x^2 \Big|_{0}^{0.8}} = \frac{0.64 - 0.16}{0.64}$$

$$P(x \geq 0.4 / x \leq 0.8) = \frac{0.48}{0.64} = \frac{3}{4} = 0.75$$

(ii) $Y = -\ln(x)$ $\quad x_1 = e^{-y}$

$$f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|}, \quad y \geq 0$$

$$= \frac{2e^{-y}}{1/e^{-y}} = 2e^{-2y}, \quad y \geq 0$$

EC252: 09/02/2022.

Multiple Random Variables:

Assume that you have invested 'x' in MF. The returns is 'y'.

SBI MF
- Cipla (the return in investment is a random variable)
- Reliance r.V_2
- Airtel r.V_3
- Colgate r.V_4
- Ranbaxy ⋮ r.V_n

The random variables may have dependency.

- **Joint Discrete Random variables:**

Let X, Y be discrete r.V such that

$$P_X(x) = P(X = x)$$

Joint PMF of X and Y is defined as

$$P_{X,Y}(x,y) = P(X=x, Y=y)$$
$$= P((X=x) \text{ and } (Y=y))$$

$X \in \mathcal{X} = \{x_1, x_2, \dots\}$

$Y \in \mathcal{Y} = \{y_1, y_2, \dots\}$

(i) $P(\Omega) = 1$

$$\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} P_{XY}(x,y) = 1$$

$$P_{XY}(x,y) \geq 0$$

(ii) $A = \{ (x_i, y_j) ; \begin{matrix} i = 1, \dots k \\ j = 1, \dots l \end{matrix} \}$

$$P((X,Y) \in A) = \sum_{x_i, y_j} P_{XY}(x_i, y_j)$$

- Marginal PMFs:

$$P_{XY}(x, y) = P(X = x, Y = y)$$

$$P(X = x) = P(X = x, Y = y_1) + P(X = x, Y = y_2) + \dots$$

(law of total probability)

Marginal PMF of r.v 'X'

$$= \sum_{y_j \in Y} P_{XY}(x, y_j)$$

$$P(Y = y) = \sum_{x_i \in X} P_{XY}(x_i, y)$$

Marginal PMF of r.v. 'Y'.

eg:

	$Y=0$	$Y=1$	$Y=2$
$X=0$	$1/6$	$1/4$	$1/8$
$X=1$	$1/8$	$1/6$	$1/6$

$$P(x,y) = \begin{bmatrix} 1/6 & 1/4 & 1/8 \\ 1/8 & 1/6 & 1/6 \end{bmatrix}$$

$$\Rightarrow \sum_{(x,y)} P_{XY}(x,y) = 1$$

$$\sum_{(x,y)} P_{XY}(x,y) = 3\times\frac{1}{6} + 2\times\frac{1}{8} + \frac{1}{4}$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{4}$$

$$= \underline{\underline{1}}$$

Marginal PMF's:

$$P(X=0) = P(X=0, Y=0) + P(X=0, Y=1) + P(X=0, Y=2)$$

$$= \frac{1}{6} + \frac{1}{4} + \frac{1}{8} = \underline{\underline{\frac{13}{24}}}$$

$$P(X=1) = \frac{1}{8} + \frac{1}{6} + \frac{1}{6} = \frac{1}{8} + \frac{1}{3} = \underline{\underline{\frac{11}{24}}}$$

$$P(Y=0) = P(X=0, Y=0) + P(X=1, Y=0)$$

$$= \frac{1}{6} + \frac{1}{8} = \underline{\underline{\frac{7}{24}}}$$

$$P(Y=1) = \frac{1}{4} + \frac{1}{6} = \underline{\underline{\frac{10}{24}}} \qquad P(Y=2) = \frac{1}{8} + \frac{1}{6} = \underline{\underline{\frac{7}{24}}}$$

$$P(X=0, Y\leq 1) = P(X=0, Y=0) + P(X=0, Y=1)$$
$$= \frac{1}{6} + \frac{1}{4} = \underline{\underline{\frac{10}{24}}}$$

- **Conditional PMF:**

from above example,

$$P(Y=1/X=0) = \frac{P(X=0, Y=1)}{P(X=0)} = \frac{1/4}{13/24} = \frac{6}{13}$$

$$\boxed{P(Y\in B/X\in A) = \frac{P(X\in A, Y\in B)}{P(X\in A)}}$$

$\Rightarrow$ We say that the r.v 'X' and 'Y' are independent if $\boxed{P_{XY}(x,y) = P_X(X=x)\cdot P_Y(Y=y)}$ $\forall\ x,y \in x_{xy}$

If X, Y are independent

$$\underline{P(Y\in B|X\in A) = P(Y\in B)}$$

from above example,

$$P(X=0, Y=0) = 1/6$$

$$P(X=0)P(Y=0) = \frac{13}{24}\times\frac{7}{24} = \frac{91}{576} \neq \frac{1}{6};$$

So, 'X' and 'Y' are not independent.

eg:

	$Y=0$	$Y=1$
$X=0$	$1/4$	$1/4$
$X=1$	$1/4$	$1/4$

$P(X=0) = \frac{1}{4}+\frac{1}{4} = 1/2$

$P(X=1) = \frac{1}{4}+\frac{1}{4} = 1/2$

$P(Y=0) = \frac{1}{4}+\frac{1}{4} = 1/2$

$P(Y=1) = \frac{1}{4}+\frac{1}{4} = 1/2$

$P(X=0, Y=0) = \frac{1}{4}$

$P(X=0)\cdot P(Y=0) = \frac{1}{2}\times\frac{1}{2} = \frac{1}{4} = P(X=0, Y=0)$

So, 'x' and 'y' are independent

eg:

	$Y=0$	$Y=1$
$X=0$	$1/4$	$1/4$
$X=1$	0	$1/2$

$P(X=0) = 1/2$ $\quad$ $P(Y=0) = 1/4$

$P(X=1) = 1/2$ $\quad$ $P(Y=1) = 3/4$

$P(X=0, Y=0) = 1/4$

$P(X=0)\cdot P(Y=0) = \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8} \neq \frac{1}{4}$

So, 'x' and 'y' are not independent.

eg:

X \ Y	y_1	y_2	y_3
x_1	1/12	1/6	0
x_2	0	1/9	1/5
x_3	1/18	1/4	2/15

$$P[x_1] = \frac{1}{12} + \frac{1}{6} = \frac{3}{12} = \underline{\underline{\frac{1}{4}}}$$

$$P[x_2] = \frac{1}{9} + \frac{1}{5} = \underline{\underline{\frac{14}{45}}}$$

$$P[x_3] = \frac{1}{18} + \frac{1}{4} + \frac{2}{15} = \frac{10+45+24}{180} = \underline{\underline{\frac{79}{180}}}$$

$$P[y_1] = \frac{1}{12} + \frac{1}{18} = \frac{3+2}{36} = \underline{\underline{\frac{5}{36}}}$$

$$P[y_2] = \frac{1}{6} + \frac{1}{9} + \frac{1}{4} = \frac{6+4+9}{36} = \underline{\underline{\frac{19}{36}}}$$

$$P[y_3] = 0 + \frac{1}{5} + \frac{2}{15} = \underline{\underline{\frac{1}{3}}}$$

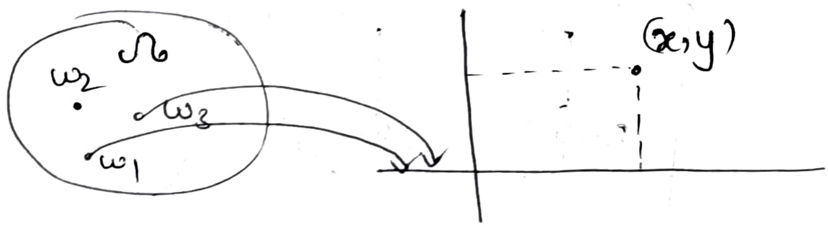
$$P[Y = y_3 / X = x_2] = \frac{P[X = x_2, Y = y_3]}{P[X = x_2]} = \frac{1/\cancel{5}}{\frac{14}{\cancel{45}\,9}} = \underline{\underline{\frac{9}{14}}}$$

$$P[X = x_2 / Y = y_1] = \frac{P[X = x_2, Y = y_1]}{P[Y = y_1]} = \underline{\underline{0}}$$

EC252:

10/02/2022

Let X and Y be continuous random variables



$$F_X(x) = P(\{\omega \in \Omega : X(\omega) \leq x\})$$

Joint CDF,

$$F_{XY}(x, y) = P(\{\omega \in \Omega : X(\omega) \leq x, Y(\omega) \leq y\})$$

$$= P(X \leq x, Y \leq y)$$

Properties:

(i) $0 \leq F_{XY}(x, y) \leq 1$

$\rightarrow F_{XY}(\infty, \infty) = 1$

$\rightarrow F_{XY}(-\infty, -\infty) = 0$

(ii) $F_{XY}(-\infty, y) = P(X \leq -\infty, Y \leq y)$

$= 0$ //

(iii) $F_{XY}(x, -\infty) = P(X \leq x, Y \leq -\infty)$

$= 0$ //

(iv) $F_{XY}(\infty, y) = P(X \leq \infty, Y \leq y)$

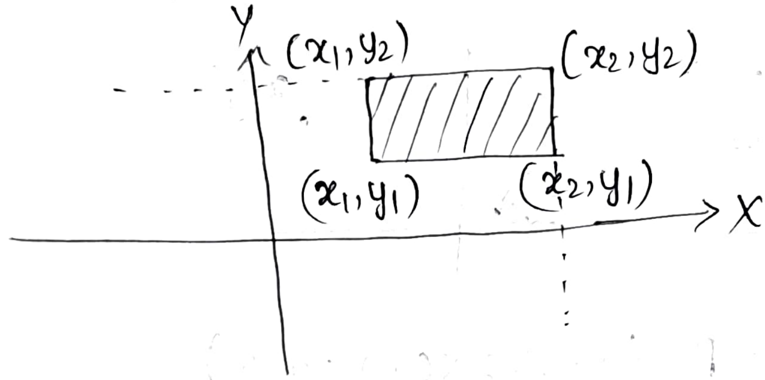
$= P(Y \leq y) = \underline{F_Y(y)}$

(v) $F_{XY}(x, \infty) = P(X \leq x, Y \leq \infty)$

$= P(X \leq x) = \underline{F_X(x)}$

$F_X(x)$ and $F_Y(y)$ are marginal CDFs

(vi) $P(x_1 < X \le x_2,\ y_1 < Y \le y_2)$



$$P((x_1 < X \le x_2), (y_1 < Y \le y_2)) = F_{XY}(x_2, y_2) - F_{XY}(x_2, y_1) - F_{XY}(x_1, y_2) + F_{XY}(x_1, y_1)$$

(vii) $F_{XY}(x,y)$ is non-decreasing interms of x & y.

(viii) $F_{XY}(x,y)$ is right-continuous

→ The joint pdf X, Y denoted by $f_{XY}(x,y)$ is a function such that,

$$F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(\theta_x, \theta_y)\, d\theta_y\, d\theta_x$$

$$\boxed{f_{XY}(x,y) = \frac{\partial^2}{\partial x\, \partial y} F_{XY}(x,y)}$$

(i) $f_{XY}(x,y) \geq 0$

(ii) $\int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(x,y) = 1$

(iii) $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx$$

(iv) $P(x_1 < X \leq x_2, \; y_1 < Y \leq y_2)$

$$= \int_{x_1}^{x_2} \int_{y_1}^{y_2} f_{XY}(x,y)\, dy\, dx$$

$\hookrightarrow$ area of f_{XY} under the rectangle.

- Conditional distribution:

$$f_{X|Y}(x|M) = \frac{P(X \leq x \cap M)}{P(M)}, \quad M = \{Y \leq y\}$$

$$= \frac{P(X \leq x, Y \leq y)}{P(Y \leq y)}$$

$$f_{X|Y}(x \mid y - \Delta y \leq y \leq y + \Delta y)$$

$$= \frac{P(X \leq x \cap y - \Delta y \leq y \leq y + \Delta y)}{\cancel{P}\; F_X(y+\Delta y) - F_X(y - \Delta y)}$$

$$\boxed{f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}}$$

$\rightarrow$ The random variables X and Y are said to be independent if $f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$

$$\therefore f_{X|Y}(x|y) = f_X(x)$$

Join distribution and its properties:

$$F_{X,Y}(x,y) = P(X \le x, Y \le y)$$

properties:

1. $F_{X,Y}(-\infty,-\infty) = 0$, $F_{X,Y}(-\infty,y) = 0$, $F_{XY}(x,-\infty) =$
2. $F_{XY}(\infty,\infty) = 1$
3. $0 \le F_{XY}(x,y) \le 1$
4. $F_{XY}(x,y)$ is nondecreasing function of both x and y
5. $P(x_1 < X \le x_2, y_1 < Y \le y_2)$

$$= F_{XY}(x_2,y_2) + F_{X,Y}(x_1,y_1) - F_{XY}(x_1,y_2) - F_{XY}(x_2,y_1)$$

6. $F_{XY}(x,\infty) = F_X(x)$, $F_{XY}(\infty,y) = F_Y(y)$

Joint density fn:

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$$

properties of pdf.

(1) $f_{X,Y}(x,y) \ge 0$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(\theta_1,\theta_2)\,d\theta_2\,d\theta_1$$

4. $$F_X(x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{x,y}(\theta_1,\theta_2)\,d\theta_2\,d\theta_1$$

$$F_Y(y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f_{x,y}(\theta_1,\theta_2)\,d\theta_1\,d\theta_2$$

5. $$P(x_1 < X \leq x_2, y_1 < Y \leq y_2) = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{XY}(x,y)\,dx\,dy$$

6. $$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy \rightarrow \text{marginal pdf of } x$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx \rightarrow$$

Conditional distributions and density

Point Conditioning:

$$B = \{y - \Delta y \leq Y \leq y + \Delta y\}$$

$$F_X(x / y - \Delta y < Y \leq y + \Delta y) = \frac{\int_{y-\Delta y}^{y+\Delta y}\int_{-\infty}^{x} f_{X,Y}(\theta_1,\theta_2)\,d\theta_1\,d\theta_2}{\int_{y-\Delta y}^{y+\Delta y} f_Y(y)\,dy}$$

If Δy is very small, we can write the above equation as

$$F_X(x / y - \Delta y < Y \leq y + \Delta y) = \frac{\int_{-\infty}^{x} f}{f_Y(y)\, 2\Delta y}$$

Thus in the limit $\Delta y \to 0$

$$F_X(x/Y = y) = \frac{\int_{-\infty}^{x} f_{X,Y}(\theta, y)\, d\theta}{f_Y(y)}$$ for every y such that $f_Y(y) \neq 0$

Differenting we get

$$f_X(x/Y=y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Similalary)

$$f_X(y/X=x) = \frac{f_{X,Y}(x,y)}{f_X(x)}.$$

Interval Condititioning:

$$B = \{ y_a < Y \leq y_b \}$$

$$F_X\left[x / y_a < Y \le y_b\right) = \frac{F_{XY}(x, y_b) - F_{XY}(x, y_a)}{F_Y(y_b) - F_Y(y_a)}$$

$$= \frac{\int_{y_a}^{y_b} \int_{-\infty}^{x} f_{X,Y}(\alpha, y)\, d\alpha\, dy}{\int_{y_a}^{y_b} \int_{-\infty}^{\infty}}$$

X and Y are statistically independent if

$$F_{X,Y}(x, y) = F_X(x) F_Y(y)$$

$$f_{XY}(x, y) = f_X(x) f_Y(y)$$

Problem-1:

$$F_{X,Y}(x, y) = u(x)u(y)\left[1 - e^{-ax} - e^{-ay} + e^{-a(x+y)}\right]$$

where $u(\cdot)$ is the unit step ft^n, $a > 0$.

- find marginal CDFs.
- find joint pdf and marginal pdf's.

Marginal CDF's

$$F_{XY}(x, \infty) = F_X(x)$$

$$F_X(x) = F_{XY}(x, \infty)$$

$$= u(x)\left[1 - e^{-ax}\right]$$

$$F_Y(y) = F_{XY}(\infty, y)$$

$$= u(y)\left[1 - e^{-ay}\right]$$

$$f_{XY}(x,y) = \frac{d^2 F_{XY}(x,y)}{dx\,dy} \longrightarrow \text{Joint pdf}$$

$$= \frac{d}{dy}\left[ae^{-ax} - ae^{-a(x+y)}\right]$$

$$= a^2 e^{-a(x+y)}, \quad x>0, y>0$$

$$f_X(x) = \frac{d}{dx} F_X(x) \text{ (or) } = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dy$$

$$= ae^{-ax}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dx = ae^{-ay}$$

Q] Find the constant 'b' such that the following are valid joint density functions.

(a) $$f_{XY}(x,y) = \begin{cases} bx(1-y), & 0<x<0.5 \\ & 0<y<1 \\ 0 & \text{else.} \end{cases}$$

$$\int_{y=0}^{1}\int_{x=0}^{0.5} bx(1-y)\,dx\,dy = 1$$

$$b\int_0^1 \left.\frac{x^2}{2} - \frac{x^2}{2}y\,\right|_0^{1/2} dy = 1$$

$$\left(\tfrac{1}{2}\right)^2 \times \frac{1}{2}\left[y - \frac{y^2}{2}\right]_0^1 = \frac{1}{b}$$

$$\frac{1}{8}\left[\frac{1}{2}\right] = \frac{1}{b} \Rightarrow \boxed{b = 16}$$

Formula: $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$

Joint pdf with $b = 16$

Marginal $f_{XY}(x,y) = 16x(1-y)$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_0^1 16(x - xy)\,dy$$

$$16x\left[\tfrac{1}{2}\right] = 8x$$

(b) $f_{XY}(x,y) = \begin{cases} b(x^2+4y^2) & , 0 \le |x| < 1, \\ & \quad 0 \le y < 2 \\ 0 & \text{else} \end{cases}$

$$\int_0^2 \int_{-1}^{1} b(x^2+4y^2)\,dx\,dy = 1$$

$$2\int_0^2 \int_0^1 (x^2+4y^2)\,dx\,dy = 1/b$$

$$2\int_0^2 \frac{x^3}{3} + 4y^2 x \Big|_0^1 \,dy = 1/b$$

$$\int_0^2 (1/3 + 4y^2)\,dy = \frac{1}{2b}$$

$$\frac{1}{3}y + \frac{4}{3}y^3 \Big|_0^2 = \frac{1}{2b}$$

$$\frac{2}{3} + \frac{32}{3} = \frac{1}{2b} \Rightarrow \frac{34}{3} = \frac{1}{2b}$$

$$\boxed{b = \frac{3}{68}}$$

Marginaly

$f_Y(y)$

$$= \int_{-1}^{1} \frac{3}{68}(x^2+4y^2)\,dx$$

$$= \frac{3(2)}{68}\left[\frac{x^3}{3} + 4xy^2\right]_0^1 = \frac{3}{34}\left[\frac{1}{3} + 4y^2\right.$$

$$f_X(x) = \int_0^2 \frac{3}{68}\left[x^2 y + \frac{4}{3}y^3\right]dy$$

$$= \frac{3}{68}\left[x^2 \frac{y^2}{2} + \frac{4}{3}\frac{y^4}{4}\right]_0^2$$

→ Independent

EC252:

Problems:

(1) $$f_{X,Y}(x,y) = \begin{cases} \frac{1}{8}(6-x-y), & 0<x<2, 2<y<4 \\ 0, & \text{otherwise} \end{cases}$$

find (a) $P(X<1, Y<3) = ?$

$$\int_0^1 \int_2^3 \frac{1}{8}(6-x-y)\,dy\,dx$$

$$\int_0^1 \frac{1}{8}\left(6y-xy-\frac{y^2}{2}\right)\Big|_2^3 dx$$

$$\int_0^1 \frac{1}{8}\left(6-x-\frac{5}{2}\right)dx$$

$$\frac{1}{8}\int_0^1 \frac{7}{2}-x\,dx = \frac{1}{8}\left[\frac{7}{2}x-\frac{x^2}{2}\right]_0^1 = \frac{1}{8}\left[\frac{7}{2}-\frac{1}{2}\right]$$

$$= \frac{1}{8}\times\cancel{\frac{6}{2}}^{3} = 3/8$$

(b) $$P(X<1 / Y<3) = \frac{P(X<1, Y<3)}{P(Y<3)} = \frac{3/8}{\int_0^2\int_2^3 \frac{1}{8}(6-x-y)\,dy\,dx}$$

$$\frac{1}{8}\int_0^2 \left(\frac{7}{2}-x\right)dx = \frac{1}{8}\left[\frac{7}{2}x-\frac{x^2}{2}\right]_0^2 = \frac{1}{8}\left[7-2\right] = 5/8$$

$$\therefore P(X<1 / Y<3) = \frac{3/\cancel{8}}{5/\cancel{8}} = 3/5$$

② The joint pdf of random variables X and Y is

$$f_{XY}(x,y) = A e^{-(x+y)}, \quad x \geq 0,\ y \geq 0$$

ⓐ find 'A' such that $f_{XY}(x,y)$ is a valid pdf

ⓑ find $f_X(x)$, $f_Y(y)$.

Sol: $$\int_0^\infty \int_0^\infty f_{XY}(x,y)\,dy\,dx = 1$$

$$\int_0^\infty \int_0^\infty A e^{-(x+y)}\,dy\,dx = 1$$

$$\int_0^\infty A e^{-y}(-e^{-x})\Big|_0^\infty dy = 1$$

$$\int_0^\infty A e^{-y}(+1)\,dy = \cancel{0}\ 1$$

$$\int_0^\infty A e^{-y}\,dy = 1$$

$$A(-e^{-y})\Big|_0^\infty = 1$$

$$A[0+1] = 1$$

$$\therefore \boxed{A=1}$$

$$f_X(x) = \int_0^\infty e^{-(x+y)}\,dy = e^{-x}(-e^{-y})\Big|_0^\infty = e^{-x}(0+1) = e^{-x}$$

$$f_Y(y) = \int_0^\infty e^{-(x+y)}\,dx = e^{-y}(-e^{-x})\Big|_0^\infty = e^{-y}(0+1) = e^{-y}$$

Both marginal pdf's are independent.

③

$$f_{XY}(x,y) = \begin{cases} C(1+xy) & , 0 \le x \le 6, 0 \le y \le 5 \\ 0 & , \text{otherwise} \end{cases}$$

ⓐ find 'c'

ⓑ $f_{X|Y}(x|Y=3)$

Sol:

$$\int_0^6 \int_0^5 C(1+xy)\,dy\,dx = 1$$

$$\int_0^6 C\left(y + \frac{xy^2}{2}\right)\Big|_0^5 dx = 1$$

$$\int_0^6 C(5 + x(25/2))\,dx = 1$$

$$C\left(5x + \frac{25}{2}\frac{x^2}{2}\right)\Big|_0^6 = 1$$

$$C\left(30 + \frac{25}{\cancel{2}}(\cancel{18}\,9)\right) = 1$$

$$\underline{\underline{C = 1/255}}$$

$$f_{X|Y}(X|Y=3) = \frac{f_{XY}(x,3)}{f_Y(3)} = \frac{\frac{\cancel{1}}{\cancel{255}}(1+3x)}{\int_0^6 \frac{\cancel{1}}{\cancel{255}}(1+xy)\,dx\Big|_{y=3}}$$

$$= \frac{1+3x}{x + y\frac{x^2}{2}\Big|_0^6\Big|_{y=3}} = \frac{1+3x}{6+54} = \underline{\underline{\frac{1+3x}{60}}}$$

$$= \frac{1+3x}{6+18\cancel{y}\Big|_{y=3}}$$

(4) $f_{XY}(x,y) = Ae^{-|x|-2|y|}$

check whether X and Y are independent or not.

Sol: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Ae^{-|x|-2|y|}\,dx\,dy = 1$

$$A\int_{-\infty}^{\infty} e^{-2|y|}\,dy \int_{-\infty}^{\infty} e^{-|x|}\,dx = 1$$

$$\int_{-\infty}^{\infty} e^{-|x|} = \int_{-\infty}^{0} e^{x}\,dx + \int_{0}^{\infty} e^{-x}\,dx$$

$$= e^{x}\Big|_{-\infty}^{0} - e^{-x}\Big|_{0}^{\infty}$$

$$= 1-(-1) = 2$$

$$\int_{-\infty}^{\infty} e^{-2|y|}\,dy = \int_{-\infty}^{0} e^{2y}\,dy + \int_{0}^{\infty} e^{-2y}\,dy$$

$$= \frac{1}{2}\left[e^{2y}\Big|_{-\infty}^{0}\right] - \frac{1}{2}\left[e^{-2y}\Big|_{0}^{\infty}\right]$$

$$= \frac{1}{2}[1] - \frac{1}{2}(-1) = 1$$

$$A(1)(2) = 1$$

$$\therefore \boxed{A = 1/2}$$

$$f_X(x) = \int_{-\infty}^{\infty} \frac{1}{2} e^{-|x|-2|y|}\, dy$$

$$= \frac{1}{2} e^{-|x|} \int_{-\infty}^{\infty} e^{-2|y|}\, dy$$

$$= \frac{1}{2} e^{-|x|} \quad //$$

$$f_Y(y) = \int_{-\infty}^{\infty} \frac{1}{2} e^{-|x|-2|y|}\, dx$$

$$= \frac{1}{2} e^{-2|y|} \int_{-\infty}^{\infty} e^{-|x|}\, dx$$

$$= e^{-2|y|} \quad //$$

$$f_{XY}(x,y) = f_X(x) \cdot f_X(y)$$

$\therefore$ X and Y are independent

(5) $f_{XY}(x,y) = c[1-x-y] \quad 0 \le x \le 1-y \;\&\; 0 \le y \le 1$

check whether X & Y are independent.

Sol: $\int_0^1 \int_0^{1-y} c[1-x-y]\,dx\,dy = 1$

$\int_0^1 c\left[x - \frac{x^2}{2} - xy\right]_0^{1-y} dy = 1$

$c\int_0^1 \left(1-y - \frac{(1-y)^2}{2} - (1-y)y\right) dy = 1$

$c\int_0^1 \frac{(1-y)^2}{2}\, dy = 1$

$\frac{c}{2}\int_0^1 1+y^2-2y\, dy = 1$

$\frac{c}{2}\left[y + \frac{y^3}{3} - y^2\right]_0^1 = 1$

$\frac{c}{2}\left[\cancel{1} + \frac{1}{3} - \cancel{1}\right] = 1 \Rightarrow \boxed{c=6}$

$f_X(x) = \int_0^1 6(1-x-y)\,dy = 6\left[y - xy - \frac{y^2}{2}\right]\Big|_0^1$

$= 6\left[1 - x - \frac{1}{2}\right]$

$= 3 - 6x$ // $(0 \le x \le 1-y)$

$f_Y(y) = \int_0^{1-y} 6(1-x-y)\,dx$

$= 6\left(x - \frac{x^2}{2} - xy\right)\Big|_0^{1-y} = 6\left(1-y - \frac{(1-y)^2}{2} - (1-y)y\right)$

$= \cancel{6}^{3}\left(\frac{(1-y)^2}{\cancel{2}}\right) = 3(1-y)^2$

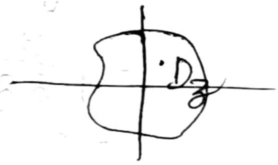
$\boxed{0 \le y \le 1}$

02/08/2022

Function of two Random Variables:

Let X, Y be random variables with joint pdf $f_{XY}(x,y)$ then,

$$Z = g(X,Y)$$

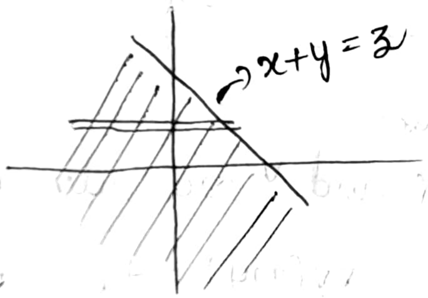


$$F_Z(z) = P(Z \le z)$$

$$= P(g(X,Y) \le z) = \iint_{D_z} f_{XY}(x,y)\,dx\,dy$$

Case-(i):

eg: $Z = X + Y$ $\quad F_Z(z) = ?$



$$F_z(z) = P(Z \le z)$$

$$= P(X + Y \le z)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_X(x,y)\,dx\,dy$$

$$f_z(z) = \frac{\partial}{\partial z} F_Z(z)$$

- **Leibnitz Rule:**

$$F(z) = \int_{a(z)}^{b(z)} f(x,z)\,dx$$

$$\frac{dF(z)}{dz} = \cancel{\frac{d}{dy}} \frac{d}{dz}(b(z))\, f(b(z),z) - \frac{d}{dz}(a(y))\, f(a(z),z) + \int_{a(z)}^{b(z)} \frac{\partial}{\partial z} f(x,z)\,dx$$

In the previous example,

$$f_Z(z) = \frac{d}{dz}\left(F_Z(z)\right)$$

$$= \int_{-\infty}^{\infty} \frac{\partial}{\partial z}\left(\int_{-\infty}^{z-y} f_{XY}(x,y)\right)dy$$

$$= \int_{-\infty}^{\infty} 1 \cdot f_{XY}(z-y, y)\, dy$$

$$\therefore f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(z-y, y)\, dy$$

- Special case:

If X and Y are independent, $f_Z(z)$ is convolution

$$f_{XY}(x,y) = f_X(x) f_Y(y)$$

$$\therefore f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\, dy$$

(or)

$$\therefore f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx$$

eg: X, Y are IID with uniform distribution on (0,1)

Find $f_Z(z)$ of $Z = X + Y$

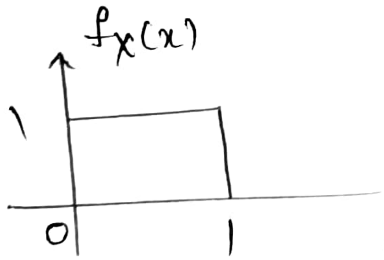
Sol:

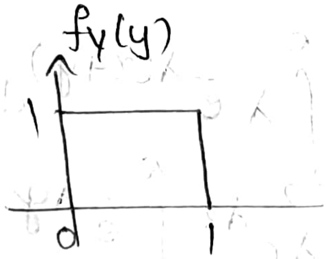
$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} 1 & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Z(z) = \int_{-\infty}^{\infty} \cancel{f_{XY}(x,y)}\, f_X(z-y) f_Y(y)\, dy$$

$$f_Z(z) = \begin{cases} z & 0 < z \le 1 \\ 2-z & 1 < z < 2 \end{cases}$$

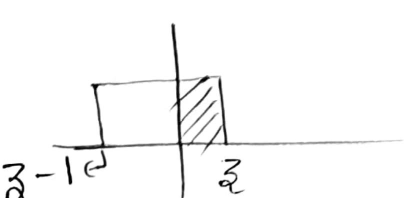




$$f_Z(z) = f_X(x) \circledast f_Y(y)$$

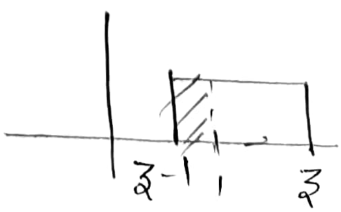
$$= \int_0^1 f_X(z-y) \cdot f_Y(y)\, dy$$

$0 < z \le 1$



$$f_Z(z) = z = \int_0^1 dz$$

$1 < z < 2$



$$f_Z(z) = 2 - z$$

eg: X and Y are two independent random variables with pdf

$$f_X(x) = \lambda e^{-\lambda x} u(x)$$
$$f_Y(y) = \lambda e^{-\lambda y} u(y)$$

} independent and identically distributed

$\downarrow$

$X \neq Y$

But $f_X(x) = f_Y(y)$.

Find $f_Z(z)$ for $Z = X + Y$

Sol:
$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y)\, f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \lambda e^{-\lambda(z-y)} U(z-y)\, \lambda \cdot e^{-\lambda y}\, U(y)\, dy$$

$$= \int_{0}^{y} \lambda^2 e^{-\lambda z}\, dy$$

$$= \lambda^2 e^{-\lambda z} (y)_0^y$$

$$\therefore f_Z(z) = \lambda^2 y e^{-\lambda z}$$

- $Z = X - Y$, $f_{XY}(x,y) \longrightarrow$ Case-(ii)

$$F_Z(z) = P(X - Y \le z)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z+y} f_{XY}(x,y)\, dx\, dy$$

$$f_Z(z) = \frac{d}{dz}\left(F_Z(z)\right)$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(z+y, y)\, dy$$

If X and Y are independent,

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z+y) \cdot f_Y(y)\, dy$$

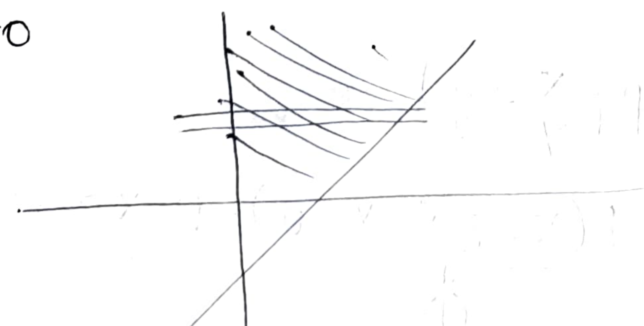
$$= f_X(-z) \circledast f_Y(y)$$

$\rightarrow X \geq 0,\ Y \geq 0$

$f_X(x) = 0,\ x < 0 \qquad f_Y(y) = 0,\ y < 0$

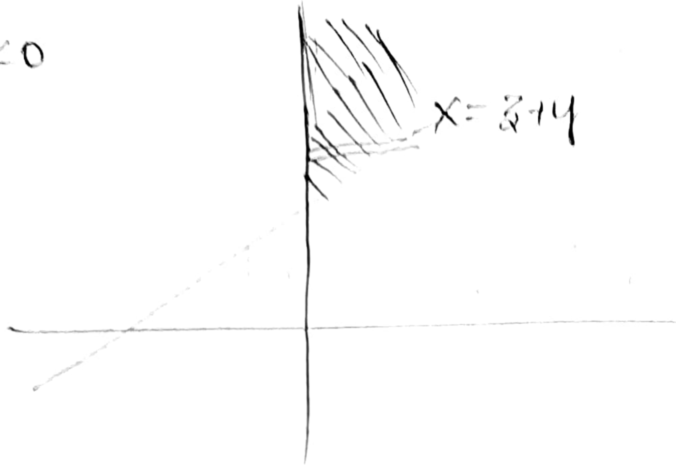
Z can be '-ve' or '+ve'

if $z \geq 0$



$$F_Z(z) = \int_{y=0}^{y=\infty} \int_{x=0}^{z+y} f_{XY}(x,y)\, dx\, dy$$

if $z<0$



$$f_Z(z) = \int_{y=-z}^{\infty} \int_{0}^{z+y} f_{XY}(x,y)\,dx\,dy$$

$$f_Z(z) = \begin{cases} \int_{0}^{\infty} f_{XY}(z+y,y)\,dy & \text{if } z \geq 0 \\ \int_{-z}^{\infty} f_{XY}(z+y,y)\,dy & \text{if } z<0 \end{cases}$$

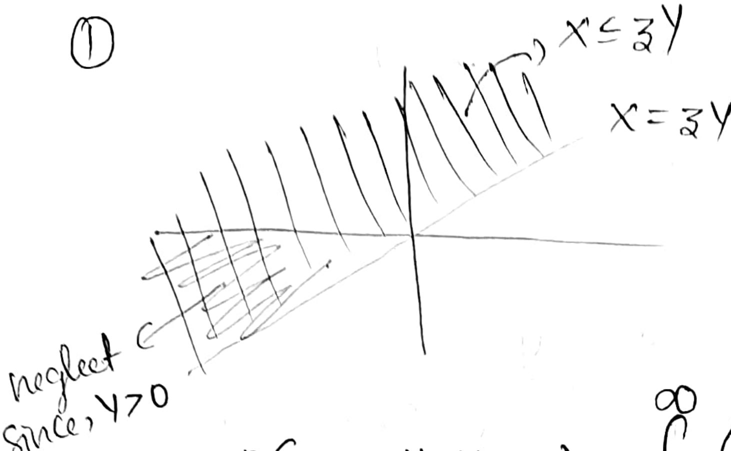
<u>case-(iii):</u>

$Z = \frac{X}{Y}$ $\quad f_{XY}(x,y)$

$$f_Z(z) = P\left(\frac{X}{Y} \leq y\right)$$

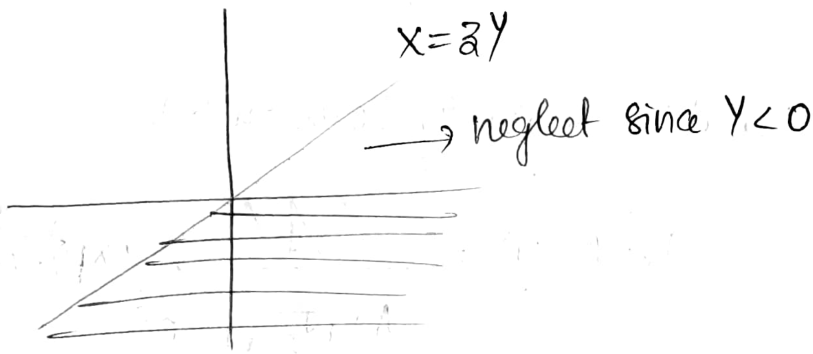
$$= \underbrace{P(X \leq zY,\ Y \geq 0)}_{1} + \underbrace{P(X \geq zY,\ Y<0)}_{2}$$

(1)



$$P(X \leq zY, Y \geq 0) = \int_{0}^{\infty}\int_{-\infty}^{zy} f_{XY}(x,y)\,dx\,dy$$

(2)



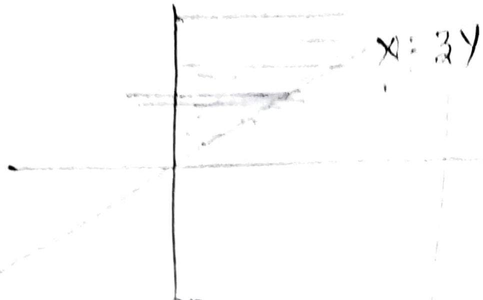
$$P(X \geq zY, Y<0) = \int_{-\infty}^{0}\int_{zy}^{\infty} f_{XY}(x,y)\,dx\,dy$$

$$f_Z(z) = \int_{0}^{\infty} y\, f_{XY}(zy,y)\,dy + \int_{-\infty}^{0} -y\, f_{XY}(x,y)\,dy$$

$$= \int_{-\infty}^{\infty} |y|\, f_{XY}(zy,y)\,dy$$

Specialcase: $f_X(x) = 0, \quad x<0$

$f_Y(y) = 0, \quad y<0$



$$f_z(z) = \int_{y=0}^{\infty} \int_{x=0}^{zy} f_{xy}(x,y)\,dx\,dy$$

$$f_z(z) = \int_{0}^{\infty} f_{xy}(zy, y)\,dy$$

<u>eg:</u> X and Y are jointly Gaussian R.V's

$$\cancel{f_{xy}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-e^2}} \exp\left\{\frac{-1}{2(1-e^2)}\right.}$$

$$f_{xy}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-e^2}} \exp\left\{\frac{-1}{2(1-e^2)}\left[\frac{x^2}{\sigma_1^2} - \frac{2exy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2}\right]\right\}$$

$Z = \frac{X}{Y}$. Show that Z has cauchy distribution centered at $\frac{e\sigma_1}{\sigma_2}$

<u>Sol:-</u>

$$f_Z(z) = \int_{-\infty}^{\infty} |y|\, f_{xy}(yz, y)\,dy$$

$$= 2\int_{0}^{\infty} \frac{y}{2\pi\sigma_1\sigma_2\sqrt{1-e^2}} \exp\left\{\frac{-1}{2(1-e^2)}\left[\frac{y^2z^2}{\sigma_1^2} - \frac{2ey^2z}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2}\right]\right\}$$

$$f_Z(z) = \frac{2}{2\pi\sigma_1\sigma_2\sqrt{1-e^2}} \int_{0}^{\infty} y\, e^{-y^2/2\sigma_0}\,dy$$

where, $\sigma_0 = \dfrac{1-e^2}{\left[\dfrac{z^2}{\sigma_1^2} - \dfrac{2ez}{\sigma_1\sigma_2} + \dfrac{1}{\sigma_2^2}\right]}$

$$\frac{y^2}{2\sigma_0} = t$$

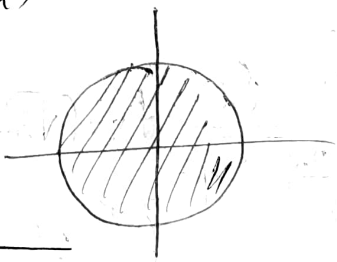
$$y\,dy = \sigma_0\,dt$$

$$f_Z(z) = \frac{2\sigma_0}{2\pi\sigma_1\sigma_2\sqrt{1-e^2}}\, \frac{e^{-t}}{-1}\Big|_0^{\infty}$$

$$\boxed{f_Z(z) = \frac{\sigma_0}{\pi\sigma_1\sigma_2\sqrt{1-e^2}}}$$

Case-(iv): $Z = X^2+Y^2$ with $f_{XY}(x,y)$

$z>0$ $F_Z(z) = P(X^2+Y^2 \le z)$



$$\therefore F_Z(z) = \int_{-\sqrt{z}}^{+\sqrt{z}} \int_{-\sqrt{z-y^2}}^{+\sqrt{z-y^2}} f_{XY}(x,y)\,dx\,dy$$

$$f_Z(z) = \frac{d}{dz}\left(F_Z(z)\right)$$

$$\therefore f_Z(z) = \int_{-\sqrt{z}}^{+\sqrt{z}} f_{XY}\left(\sqrt{z-y^2}, y\right) dy$$

$$\therefore f_Z(z) = \int_{-\sqrt{z}}^{+\sqrt{z}} \left\{ \frac{1}{2\sqrt{z-y^2}} \left\{ f_{XY}(\sqrt{z-y^2}, y) + f_{XY}(-\sqrt{z-y^2}, y) \right\} dy \right\}$$

Case-(V): $Z = \sqrt{X^2+Y^2}$ with pdf $f_{XY}(x,y)$

$$F_Z(z) = P(Z \le z)$$

$$= P(\sqrt{X^2+Y^2} \le z)$$

$$= P(X^2+Y^2 \le z^2)$$

$$\therefore F_Z(z) = \int_{y=-z}^{+z} \int_{-\sqrt{z^2-y^2}}^{+\sqrt{z^2-y^2}} f_{XY}(x,y)\, dx\, dy$$

$$f_Z(z) = \frac{d}{dz}(F_Z(z))$$

$$f_Z(z) = \int_{y=-z}^{+z} \frac{\cancel{2}z}{\cancel{2}\sqrt{z^2-y^2}} f_{XY}(\sqrt{z^2-y^2}, y)\, dy + \int_{y=-z}^{+z} \frac{\cancel{2}z}{\cancel{2}\sqrt{z^2-y^2}} f_{XY}(-\sqrt{z^2-y^2}, y)\, dy$$

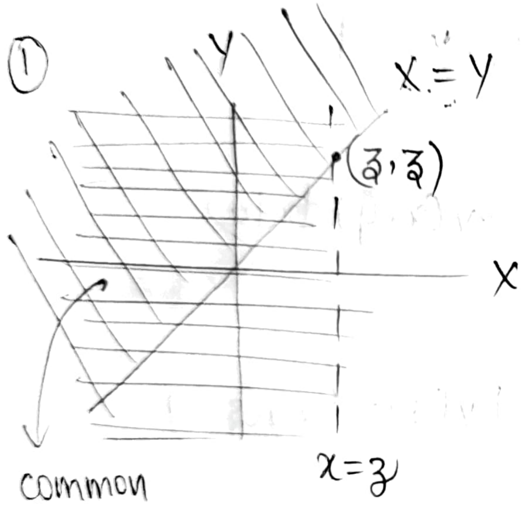
$$\boxed{f_Z(z) = \int_{-z}^{+z} \frac{z}{\sqrt{z^2-y^2}} \left[f_{XY}(\sqrt{z^2-y^2}, y) + f_{XY}(-\sqrt{z^2-y^2}, y) \right] dy}$$

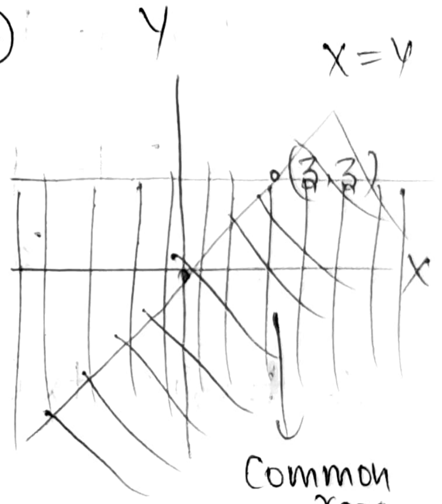
Case-(vi): $Z = \mathrm{Max}(X, Y)$

$$Z = \begin{cases} X, & X > Y \\ Y, & X < Y \end{cases}$$

$$F_Z(z) = P(Z \le z) = P\big((X \le z, X \ge Y) \cup (Y \le z, Y > X)\big)$$

$$= \underbrace{P[X \le z, X \ge Y]}_{①} + \underbrace{P[Y \le z, Y > X]}_{②}$$





$$F_Z(z) = P(X \le z, Y \le z)$$

$$= \int_{-\infty}^{z} \int_{-\infty}^{z} f_{XY}(x, y)\, dx\, dy$$

$$\therefore \underline{F_Z(z) = F_{XY}(z, z)}$$

→ If X and Y are independent,

$$F_Z(z) = F_X(z) \cdot F_Y(z)$$

$$\boxed{f_Z(z) = f_X(z) \cdot F_Y(z) + F_X(z)\, f_Y(z)}$$

Case - vii: $Z = \text{Min}(x, y)$

$$Z = \begin{cases} X, & X \le Y \\ Y, & X > Y \end{cases}$$

$$F_Z(z) = P(Z \le z)$$

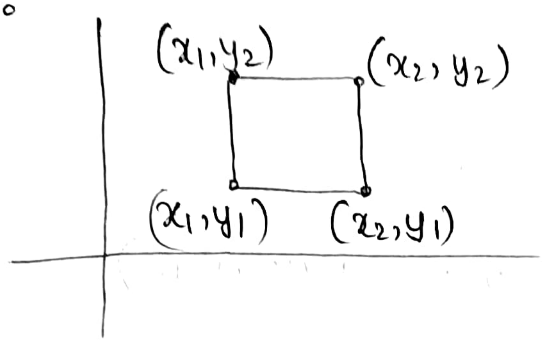
$$= \underbrace{P[X \le z, X \le Y]}_{1} + \underbrace{P[Y \le z, X > Y]}_{2}$$

$$F_Z(z) = 1 - P[X > z, Y > z]$$

$$= 1 - \int_z^{\infty} \int_z^{\infty} f_{XY}(x, y)\, dx\, dy$$

$$\therefore F_Z(z) = F_X(z) + F_Y(z) - F_{XY}(z, z)$$

NOTE:



$$F_{XY}(x_1 < X \le x_2, y_1 < Y \le y_2)$$
$$= F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1)$$
$$= 1 - F_X(z) - F_Y(z) + F_{XY}(z, z)$$
$$[x_1 = z, x_2 = \infty, y_1 = z, y_2 = \infty]$$

eg: X and Y are two iid exponential random variables with parameter λ.
find $Z = \min[X, Y]$
find pdf of Z.

Sol: $f_X(x) = \lambda e^{-\lambda x} U(x)$
$f_Y(y) = \lambda e^{-\lambda y} U(y)$

$$f_Z(z) = f_X(z) + f_Y(z) - \frac{d}{dz}(F_X(z) \cdot F_Y(z))$$
$$= f_X(z) + f_Y(z) - F_X(z) f_Y(z) - f_X(z) F_Y(z)$$
$$= f_X(z)[1 - F_Y(z)] + f_Y(z)[1 - F_X(z)]$$

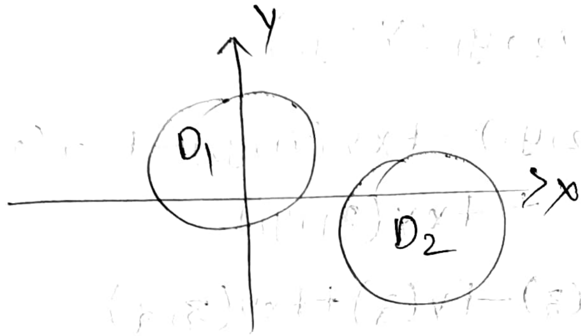
CDF of exponential r.v is $F_X(x) = 1 - e^{-\lambda x}, x > 0$

$$f_Z(z) = [\lambda e^{-\lambda z}][e^{-\lambda z}] + [\lambda e^{-\lambda z}][e^{-\lambda z}]$$
$$= 2\lambda e^{-2\lambda z}, \quad z > 0$$

$\hookrightarrow$ exponential with parameter '2λ'.

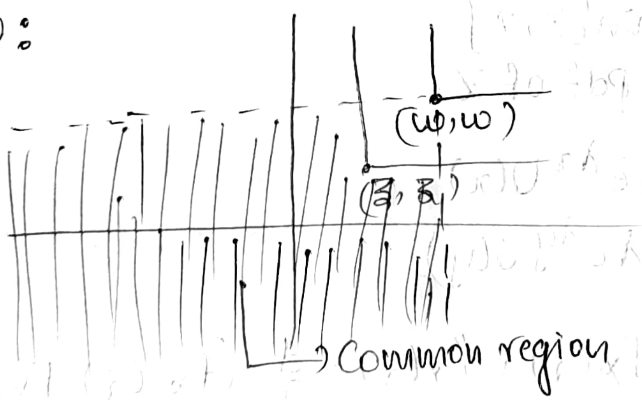
$$Z = g_1(X, Y)$$
$$W = g_2(X, Y)$$

$$F_{ZW}(z, w) = P(Z \leq z, W \leq w)$$
$$= P(g_1(x,y) \leq z, g_2(x,y) \leq w)$$



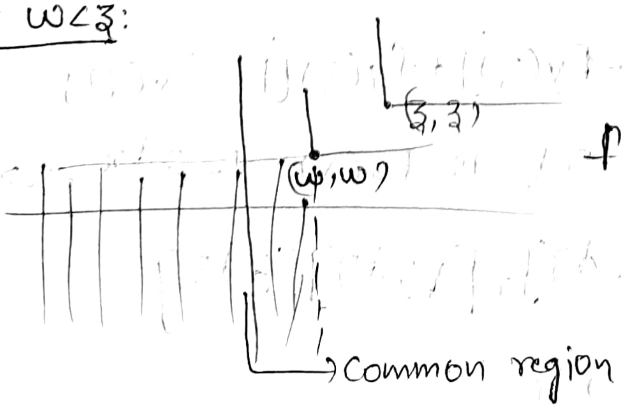
eg: $Z = \min(X, Y)$
$W = \max(X, Y)$
$$F_{ZW}(z, w) = P(Z \leq z, W \leq w)$$

if $w \geq z$:



$$F_{ZW}(z, w) = F_{XY}(z, w) + F_{XY}(w, z) - F_{XY}(z, z)$$

if $w < z$:



$$F_{ZW}(z, w) = F_{XY}(w, w)$$

$$\therefore f_{ZW}(z,w) = \begin{cases} f_{XY}(w,w) & , z > w \\ f_{XY}(z,w) + f_{XY}(w,z) - f_{XY}(z,z) & , z \le w \end{cases}$$

Result:

$$\left.\begin{array}{l} Z = g_1(X,Y) \\ W = g_2(X,Y) \end{array}\right\} \Rightarrow \begin{array}{l} X = h_1(z,w) \\ Y = h_2(z,w) \end{array}$$

$$J_i(z,w) = \begin{bmatrix} \frac{\partial h_1}{\partial z} & \frac{\partial h_1}{\partial w} \\ \frac{\partial h_2}{\partial z} & \frac{\partial h_2}{\partial w} \end{bmatrix}_{x=x_i,\, y=y_i}$$

$i = 1, 2, \cdots n$

$$f_{ZW}(z,w) = \sum_i |J(z,w)|_i \, f_{XY}(x_i, y_i)$$

$$= \sum \frac{1}{|J(x,y)|} f_{XY}(x_i, y_i)$$

$$J(x,y) = \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{bmatrix}$$

$$|J(z,w)| = \frac{1}{|J(x,y)|}$$

---

eg: X and Y are iid Gaussian R.V's with $\mu = 0, \underset{\text{Variance}}{\sigma^2}$

$$R = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

find $f_{R\theta}(r, \theta)$.

Sol:-

$$X = R\cos\theta$$

$$Y = R\sin\theta$$

$$|J(r, \theta)| = \begin{bmatrix} \cos\theta & -R\sin\theta \\ \sin\theta & R\cos\theta \end{bmatrix} = R$$

$$|J(x, y)| = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix}$$

$$= \frac{x^2}{(x^2+y^2)^{3/2}} + \frac{y^2}{(x^2+y^2)^{3/2}}$$

$$= \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{R}$$

$$\therefore |J(r, \theta)| = \frac{1}{|J(x, y)|}$$

$$f_{R\theta}(r, \theta) = r \cdot f_{xy}(r\cos\theta, r\sin\theta)$$

$$= r \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^2 \cdot e^{-\frac{r^2}{2\sigma^2}}$$

$$\therefore \boxed{f_{R\theta}(r, \theta) = \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2}}$$

eg: X and Y are iid exponential R.V's with parameter λ.

$Z = X+Y$

$W = X-Y$ Find, $f_{ZW}(z,w)$

Sol:

$X = \frac{Z+W}{2}$

$Y = \frac{Z-W}{2}$

$$|J(z,w)| = \left|\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\right|$$

$$= +1/2$$

$$f_{ZW}(z,w) = \frac{1}{2} f_{XY}\left(\frac{z+w}{2}, \frac{z-w}{2}\right)$$

$$= \frac{1}{2} \lambda^2 e^{-\lambda\left(\frac{z+w}{2}\right)} \cdot e^{-\lambda\left(\frac{z-w}{2}\right)}$$

$$= \frac{1}{2} \lambda^2 e^{-\frac{\lambda}{2}(z+\cancel{w}+z-\cancel{w})}$$

$$\therefore f_{ZW}(z,w) = \frac{1}{2}\lambda^2 e^{-\lambda z} \quad 0<|w|<z<\infty$$

Result: $Z = aX + bY$

$W = cX + dY$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} Z \\ W \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

$\downarrow$

linear transformation

$$|J(x, y)| = ad - bc$$

$$|J(z, w)| = |AD - BC|$$

$$X = AZ + BW$$

$$Y = CZ + DW$$

$$\boxed{f_{ZW}(z, w) = \frac{1}{|ad - bc|} f_{XY}(Az + Bw, Cz + Dw)}$$

$$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y)\, dx\, dy$$

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

$$\mathbb{E}[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\, dx\, dy \neq \mathbb{E}[X] \cdot \mathbb{E}[Y]$$

(in general)

if X and Y are independent,

$$\mathbb{E}[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, f_X(x) \cdot f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x f_X(x)\, dx \int_{-\infty}^{\infty} y \cdot f_Y(y)\, dy$$

$$\therefore \underline{\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]}$$

- Covariance (C_{XY}):

$$C_{XY} = \mathbb{E}[(X-\mu_x)(Y-\mu_y)]$$

$$= \mathbb{E}[XY - \mu_x Y - \mu_y X + \mu_x \mu_y]$$

$$= \mathbb{E}[XY] - \mu_x \mathbb{E}[Y] - \mu_y \mathbb{E}[X] + \mu_x \mu_y$$

$$= \mathbb{E}[XY] - \mu_x \mu_y - \cancel{\mu_y \mu_x} + \cancel{\mu_x \mu_y}$$

$$\therefore C_{XY} = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

→ R.V's X and Y are said to be Uncorrelated if $C_{XY} = 0 \Rightarrow \mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$

→ If X and Y are independent then they are uncorrelated but the converse is not true.

→ Correlation coefficient : (ρ_{XY})

$$\rho_{XY} = \frac{C_{XY}}{\sigma_X \sigma_Y}$$

Standard deviation of X, $\sigma_X = \sqrt{Var(X)} = \sqrt{\mathbb{E}[(X-\mu)^2]}$

$$|\rho_{XY}| \le 1$$

Proof: $\mathbb{E}[a(X-\mu_x) + (Y-\mu_y)]^2 \geq 0$

$$\mathbb{E}[a^2(X-\mu_x)^2 + 2a(X-\mu_x)(Y-\mu_y) + (Y-\mu_y)^2]$$

$$= a^2 \sigma_X^2 + 2a C_{XY} + \sigma_Y^2$$

$$\Delta \Rightarrow 4C_{XY}^2 - 4\sigma_X^2 \sigma_Y^2 \le 0$$

$$C_{XY}^2 \le \sigma_X^2 \sigma_Y^2$$

$$|C_{XY}| \le \sigma_X \sigma_Y \Rightarrow |\rho_{XY}| \le 1$$

→ R.V's X and Y are said to be orthogonal, if $E[XY] = 0$

↓

X and Y are uncorrelated and $\mu_X = 0$

eg: X and Y are two R.V's that are uniformly distributed between (0,1). Suppose X and Y are independent

$Z = X+Y$; $W = X-Y$

(i) Are Z and W independent?

(ii) Find Covariance of Z and W (Cov(Z,W))

<u>Sol:</u> $f_{ZW}(z,w) = f_Z(z) \cdot f_W(w)$

X $\quad f_X(x) = \begin{cases} 1 & 1 \geq x \geq 0 \\ 0 & \cancel{x<0} \text{ otherwise} \end{cases}$

$f_Y(y) = \begin{cases} 1 & 1 \geq x \geq 0 \\ 0 & \cancel{x<0} \text{ otherwise} \end{cases}$

$Z = X+Y$ $\quad X = \frac{Z+W}{2}$, $Y = \frac{Z-W}{2}$

$W = X-Y$

$$f_{ZW}(z,w) = |\sum_i |J(z,w)| \, f_{XY}(x_i, y_i)$$

$$J(z,w) = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -1/2$$

$$f_{ZW}(z,w) = \frac{1}{2} f_{XY}\left(\frac{z+w}{2}, \frac{z-w}{2}\right)$$

$$= \begin{cases} 1/2 , & 0 < z < 2, \; -1 < w < 1 \\ & |w| < z, \; z+w < 2 \\ & z-w < 2 \\ 0 & , \text{otherwise} \end{cases}$$

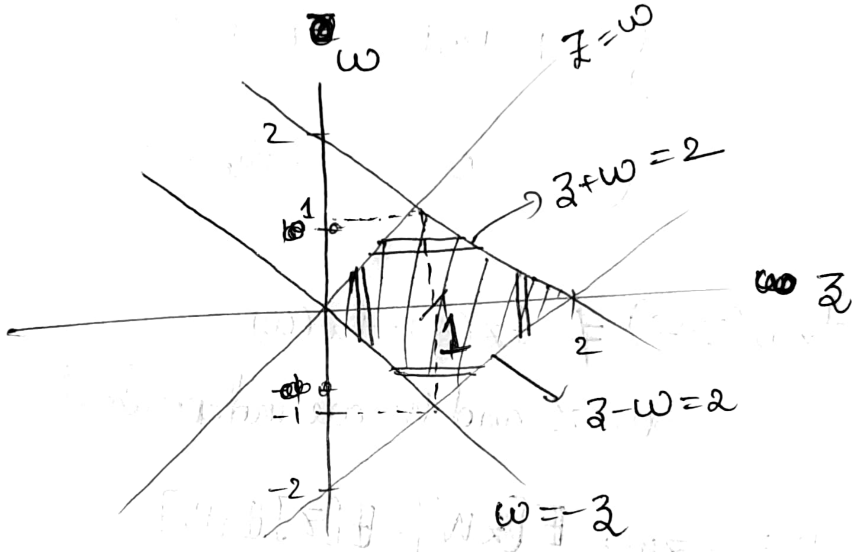
$Z = X + Y$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) \cdot f_Y(y)\, dy$$

$$f_Z(z) = \begin{cases} z, & 0 < z \leq 1 \\ 2 - z, & 1 < z < 2 \end{cases}$$

$\cancel{Z}\, W = X - Y$

$$\cancel{f_Z} \quad f_W(w) = \int_{-\infty}^{\infty} f_X(z+y) \cdot f_Y(y)\, dy$$



$$f_Z(z) = \int_{-\infty}^{\infty} f_{ZW}(z, w)\, dw$$

$$= \begin{cases} \int_{-z}^{z} \frac{1}{2}\, dw & 0 < z < 1 \\ \int_{z-2}^{2-z} \frac{1}{2}\, dw & 1 < z < 2 \end{cases}$$

$$= \begin{cases} z & 0 < z < 1 \\ \frac{1}{2}(4 - 2z) & 1 < z < 2 \end{cases} \Rightarrow \begin{cases} z & 0 < z < 1 \\ 2 - z & 1 < z < 2 \end{cases}$$

$$f_W(w) = \int_{-\infty}^{\infty} f_{ZW}(z,w)\,dz$$

$$= \begin{cases} \displaystyle\int_{|w|}^{2-|w|} \frac{1}{2}\,dz & |w|<1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1-|w| & |w|<1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{ZW}(z,w) \neq f_Z(z)\cdot f_W(w)$$

So, Z and W are independent.

$$C(z,w) = C_{ZW} = \mathbb{E}[ZW] - \mathbb{E}[Z]\mathbb{E}[W]$$

$$\mathbb{E}[zw] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} zw\, f_{zw}(z,w)\,dz\,dw$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{zw}(z,w)\,dw\,dz$$

$$\mathbb{E}[ZW] = \mathbb{E}[(X+Y)(X-Y)]$$

$$= \mathbb{E}[X^2-Y^2] = \mathbb{E}[X^2] - \mathbb{E}[Y^2]$$

$$= \int_{-\infty}^{\infty} x^2 f_X(x)\,dx - \int_{-\infty}^{\infty} y^2 f_Y(y)\,dy$$

$$= \int_0^1 x^2\,dx - \int_0^1 y^2\,dy$$

$$\therefore \mathbb{E}[ZW] = 1/3 - 1/3 = 0$$

$\mathbb{E}[z] \Rightarrow \mathbb{E}[z] = \mathbb{E}[x+y]$

$= \mathbb{E}[x] + \mathbb{E}[y] = 1$

$\mathbb{E}[W] \Rightarrow \mathbb{E}[W] = \mathbb{E}[x-y]$

$= \mathbb{E}[x] - \mathbb{E}[y]$

$= 0 //$

$\therefore C(z, w) = \mathbb{E}[zw] - \mathbb{E}[z]\mathbb{E}[w]$

$= 0 - 1(0)$

$= 0 //$

eg: X, Y and Z are R.V's with mean '0' and

$Var(X) = Var(Y) = Var(Z) = 20$

$Cov(X, Y) = cov(X, Z) = 10$

$Cov(Y, Z) = 5$

(i) find $Cov(3X+Y, 3X+Z)$

Sol: $Cov(x, y) = \mathbb{E}[xy] - \mathbb{E}[x] \cdot \mathbb{E}[y]$

$\mathbb{E}[XY] = 10$

$cov(x, z) = 10 \Rightarrow \mathbb{E}[XZ] = 10$

$cov(Y, Z) = 5 \Rightarrow \mathbb{E}[YZ] = 5$

(i) $\mathbb{E}$ ~~cov~~ $((3x+y - \mathbb{E}[3x+y])(3x+z - \mathbb{E}[3x+z]))$

$= \mathbb{E}[(3x+y)(3x+z)]$

$= \mathbb{E}[9x^2 + 3xz + 3xy + yz]$

$= 9\mathbb{E}[x^2] + 3\mathbb{E}[xz] + 3\mathbb{E}[xy] + \mathbb{E}[yz]$

$= 9 \times 20 + 3 \times 10 + 10 \times 3 + 5 = 245 //$

- Joint characteristic function:

$$\phi_X(\omega) = \mathbb{E}[e^{j\omega X}]$$

$$= \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$$

$$= \mathcal{F}[f_X(x)] \rightarrow \text{fourier transform.}$$

$\therefore \phi_{X,Y}(\omega_1, \omega_2) = \mathbb{E}[e^{j(\omega_1 X + \omega_2 Y)}]$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j(\omega_1 x + \omega_2 y)} f_{XY}(x,y)\, dx\, dy$$

$$\left(\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{XY}(x,y)\, dx\, dy \right)$$

→ Conditional expectation:

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

$$\mathbb{E}[g(X)|M] = \int_{-\infty}^{\infty} g(x)\, f_{X|M}(x|m)\, dx$$

$$\mathbb{E}[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx$$

$$\mathbb{E}[Y|X=x] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)\, dy$$

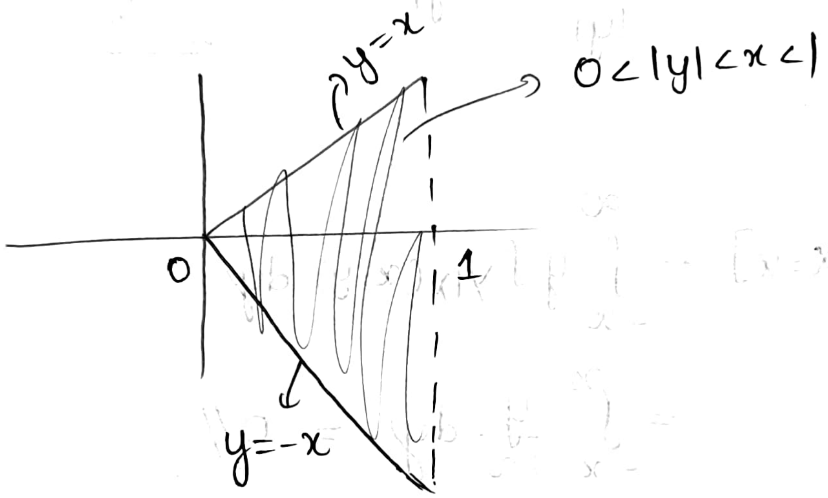
→ Conditional variance:

$$\sigma^2_{X|Y} = \mathbb{E}_{X|Y}\left[(X-\mu_{X|Y})^2\right]$$

eg: $f_{XY}(x,y) = \begin{cases} 1, & 0<|y|<x<1 \\ 0, & \text{else} \end{cases}$

find $\mathbb{E}[X|Y=y]$ and $\mathbb{E}[Y|X=x]$

Sol:



$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_{-x}^{x} 1\,dy = 2x, \quad 0<x<1$$

$$f_Y(y) = \int_{y}^{1} 1\,dx, \quad y>0$$

$$= \int_{-y}^{1} 1\,dx, \quad y<0$$

$$= 1-|y|, \quad |y|<1$$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{1}{1-|y|}, \quad 0<|y|<x<1$$

$$f_{Y|X}(x,y) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{1}{2x}, \quad 0<|y|<x<1$$

$$\mathbb{E}[X|Y=y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x,y)\, dx$$

$$= \int_{|y|}^{1} \frac{x}{1-|y|}\, dx = \underline{\underline{\frac{1+|y|}{2}}}, \quad |y|<1$$

$$\mathbb{E}[Y|X=x] = \int_{-\infty}^{\infty} y\, f_{Y|X}(x,y)\, dy$$

$$= \int_{-x}^{x} \frac{y}{2x}\, dy = 0 //$$

$\mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[Y]$ | $\mathbb{E}[\mathbb{E}[g(X,Y)|X]] = \mathbb{E}[g(X,Y)]$

Proof: $g(X) = \mathbb{E}[Y|X]$

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} f_X(x)\left(\int_{-\infty}^{\infty} y \cdot f_{Y|X}(x|y)\, dy\right) dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y \cdot f_{XY}(x,y)\, dx\, dy$$

$$= \underline{\mathbb{E}[Y]}$$

$$\left(\mathbb{E}[g(X,Y)] = \iint g(x,y) \cdot f_{XY}(x,y)\, dx\, dy \right.$$

→ X and Y are R.V's with joint pdf $f_{XY}(x,y)$

$$Z = aX + bY \implies \mu_Z = a\mu_X + b\mu_Y$$

$$Var(Z) = \mathbb{E}[(Z-\mu_Z)^2]$$

$$= \mathbb{E}[(a(X-\mu_X) + b(Y-\mu_Y))^2]$$

$$= \mathbb{E}[a^2(X-\mu_X)^2 + b^2(Y-\mu_Y)^2 + 2ab(X-\mu_X)(Y-\mu_Y)]$$

$$\therefore Var(Z) = a^2 Var(X) + b^2 Var(Y) + 2ab\, Cov(X,Y)$$

- **Central limit theorem (CLT):**

Let $X_1, X_2, X_3 \ldots$ be iid RV's with mean μ and variance σ^2

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i - \mu}{\sigma} = N(0,1)$$

- **Covariance matrix:**

$$V = \begin{bmatrix} X \\ Y \end{bmatrix}, \quad \mu = \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}$$

$$\text{Covariance matrix} = E[\underbrace{(V-\mu)}_{2\times1}\underbrace{(V-\mu)^T}_{1\times2}]$$

$$= \mathbb{E}\left[\begin{bmatrix} X-\mu_X \\ Y-\mu_Y \end{bmatrix} \begin{bmatrix} X-\mu_X & Y-\mu_Y \end{bmatrix}\right]$$

$$\Sigma = \begin{bmatrix} \sigma_X^2 & C_{XY} \\ C_{XY} & \sigma_Y^2 \end{bmatrix}$$

and Y are jointly guassian if

$$f_{XY}(x,y) = f_V(v)$$

$$= \frac{1}{2\pi|\Sigma|} \exp\left\{-\frac{(v-u)^T \Sigma^{-1} (v-u)}{2}\right\}$$

$$\bar{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} \qquad f_{\bar{X}}(x_1, x_2, \ldots, x_n)$$

$$= P(X_1 \le x, \ldots, X_n \le x_n)$$

$$\therefore f_{\bar{X}}(x_1, \ldots, x_n) = \frac{\partial^n f}{\partial x_n \cdots \partial x_1}$$

$$\therefore f_{X_1}(x_1) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{\bar{X}}(x_1, \ldots, x_n)\, dx_2 \cdots dx_n$$

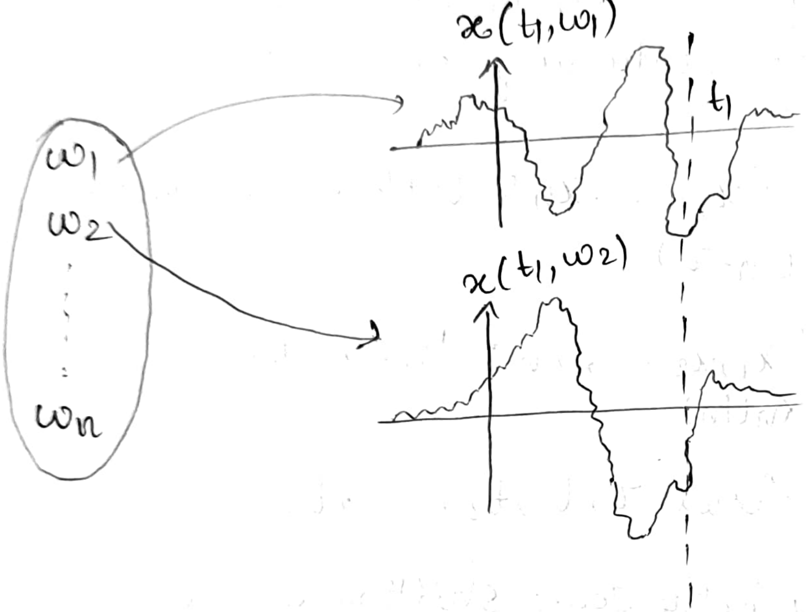
Random Processes :

30/03/22

A random variable 'X' is a mapping ($X : \Omega \to \mathbb{R}$) such that

$$\{\omega \in \Omega,\ X(\omega) \leq c\} \in \mathcal{F} \quad \text{where } c \in \mathbb{R}$$

$$X^{-1}((-\infty, c])$$



→ We define a random process $X(t;\omega)$ or $X(t)$ as an ensemble of time functions such that together with a probability rule that assigns probability to any meaningful event associated with an observation of one of the sample functions of the random process.

eg: $X(t) = A\cos(2\pi f_c t + \Theta)$

A, f_c are constants

$\Theta \sim U[-\pi, \pi]$

$$X(t, \omega) = A\cos(2\pi f_c t + \Theta(\omega))$$

for $n \geq 1$ (nth order distribution function)

$$f_{X(t_1)X(t_2)\cdots X(t_n)}(x_1, x_2, \ldots x_n ; t_1, t_2, \ldots t_n)$$

$$= P(X(t_1) \leq x_1, \ldots \ldots, X(t_n) \leq x_n)$$

→ A random process $X(t)$ is said to be strict-sense stationary or strictly stationary if for each n,

$$f_{X(t_1+\tau)\cdots X(t_n+\tau)}(x_1, \ldots x_n ; t_1+\tau, t_2+\tau, \ldots t_n+\tau)$$

$$= f_{X(t_1)\cdots X_n(t_n)}(x_1, x_2, \ldots, x_n ; t_1, t_2, \ldots t_n)$$

forall $\tau, t_1, t_2, \ldots, t_n$

(stationary in the sense shifting constant 'τ')

(i) for $n=1$

$$f_{X(t_1+\tau)}(x; t_1+\tau) = f_{X(t_1)}(x_1, t_1)$$

$$= f_X(x) \text{ if independent of time.}$$

(ii) for $n=2$

$$f_{X(t_1+\tau)X(t_2+\tau)}(x_1, x_2 ; t_1+\tau, t_2+\tau)$$

$$= f_{X(t_1)X(t_2)}(x_1, x_2 ; t_1, t_2)$$

$$= f_{X(t_1)X(t_2)}(x_1, x_2 ; t_2-t_1)$$

→ Mean

$$\mu_{X(t)}(t_1) = \int_{-\infty}^{\infty} x \cdot f_X(x, t_1)\, dx$$

if $X(t)$ is strictly stationary then,

$$\mu_X(t) = \int_{-\infty}^{\infty} x f_X(x)\, dx$$ independent of time.

→ Auto-correlation

$$R_{XX}(t_1, t_2) = \mathbb{E}\left[X(t_1)\, X(t_2)\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2; t_1, t_2)\, dx_1\, dx_2$$

$$R_{XX}(t_1, t_2) = R_X(t_2 - t_1)$$

→ A random process $X(t)$ is a wide-sense stationary or widely stationary if

(i) $\mu_{X(t)}$ is independent of time.

(ii) $R_{XX}(t_1, t_2)$ depends only on the $t_2 - t_1$.

→ $R_X(\tau) = \mathbb{E}\left[X(t+\tau)\, X(t)\right]$

(i) Mean square value of $X(t) \Rightarrow \mathbb{E}[X^2(t)] = R_X(0)$

(ii) $R_X(\tau) = R_X(-\tau)$

$$R_X(\tau) = \mathbb{E}\left[X(t+\tau)\, X(t)\right]$$

$$R_X(-\tau) = \mathbb{E}\left[X(t-\tau)\, X(t)\right]$$

$$= \mathbb{E}\left[X(t)\, X(t-\tau)\right]$$

$$= R_X(\tau)$$

(iii) $R_X(\tau) \leq R_X(0)$

$$E\left[(X(t+\tau)+X(t))^2\right] \geq 0$$

$$E\left[X^2(t+\tau) + 2X(t+\tau)X(t) + X^2(t)\right] \geq 0$$

$$2R_X(0) + 2R_X(\tau) \geq 0$$

$$\underline{R_X(0) \geq -R_X(\tau)}$$

eg: $U[a,b]$

$$f_X(x) = \frac{1}{b-a}, \quad a \leq x \leq b$$

eg: $X(t) = A\cos(2\pi f_c t + \Theta)$

A, f_c are constants

$\Theta \sim U[-\pi, \pi]$

$$f_\Theta(\theta) = \frac{1}{2\pi}, \quad -\pi \leq \theta \leq \pi$$

(i) $\mu_X = E[X(t)]$

$$= \int_{-\pi}^{\pi} A\cos(2\pi f_c t + \theta) \cdot \frac{1}{2\pi}\, d\theta$$

$\therefore \mu_X = 0 //$

(ii) $E[X(t+\tau)X(t)]$

$$= E\left[A^2\cos(2\pi f_c t + 2\pi f_c \tau + \Theta)\cos(2\pi f_c t + \Theta)\right]$$

$$= \frac{A^2}{2\pi}\int_{-\pi}^{\pi} \cos(2\pi f_c t + 2\pi f_c \tau + \theta)\cos(2\pi f_c t + \theta)\, d\theta$$

$$= \frac{A^2}{2\pi} \times \frac{1}{2}\int_{-\pi}^{\pi} \cos(4\pi f_c t + 2\pi f_c \tau + 2\theta) + \cos(2\pi f_c \tau)\, d\theta$$

$$\mathbb{E}[X(t+\tau)X(t)] = \frac{A^2}{2\pi} \times \frac{1}{2} \left[\int_{-\pi}^{\pi} \cancel{\cos(4\pi f_c t + 2\pi f_c \tau + 2\theta)\, d\theta}^{\,0} + \int_{-\pi}^{\pi} \cos(2\pi f_c \tau)\, d\theta \right.$$

$$= \frac{A^2}{\cancel{2\pi}} \times \frac{1}{2} \times \cos(2\pi f_c \tau)(\cancel{2\pi})$$

$$= \underline{\frac{A^2}{2}\cos(2\pi f_c \tau)}$$

• **Auto Covariance:**

$$C_{XX}(t_1, t_2) = \mathbb{E}\left[(X(t_1) - \mu_X(t_1))(X(t_2) - \mu_X(t_2))\right]$$

• **Joint Random processes:**

→ $X(t)$, $Y(t)$

$n \geq 0, m \geq 0 \quad n+m \geq 1$

$$f_{X(t_1)\ldots X(t_n), Y(u_1)\ldots Y(u_m)}(x_1, \ldots, x_n, y_1, \ldots y_m ; t_1, \ldots t_n, u_1, \ldots u_m)$$

$$= P(X(t_1) \leq x_1 \ldots X(t_n) \leq x_n, Y(u_1) \leq y_1 \ldots Y(u_m) \leq y_m)$$

→ Cross-correlation

$$R_{XY}(t, u) = \mathbb{E}[X(t)Y(u)]$$

$$R_{YX}(t, u) = \mathbb{E}[Y(t)X(u)] \neq R_{XY}(t, u)$$

→ if $X(t)$ & $Y(u)$ jointly stationary then,

$$R_{XY}(\tau) = \mathbb{E}[X(t+\tau)Y(t)]$$

→ Correlation matrix

$$R(t, u) = \begin{bmatrix} R_{XX}(t, u) & R_{XY}(t, u) \\ R_{YX}(t, u) & R_{YY}(t, u) \end{bmatrix}$$

→ If $X(t)$ and $Y(t)$ are jointly stationary then,

$$R(\tau) = \begin{bmatrix} R_{XX}(\tau) & R_{XY}(\tau) \\ R_{YX}(\tau) & R_{YY}(\tau) \end{bmatrix}$$

→ $R_{XY}(-\tau) = \mathbb{E}[X(t-\tau)Y(t)]$

$$= \mathbb{E}[Y(t)X(t-\tau)] = R_{YX}(\tau)$$

eg: $X_1(t) = X(t)\cos(2\pi f_c t + \theta)$ $X(t)$ is a stationary Random process.

$X_2(t) = X(t)\sin(2\pi f_c t + \theta)$

$\theta \sim U[0, 2\pi]$ is independent of $X(t)$

find $R_{12}(\tau, t) = \mathbb{E}[X_1(t+\tau)X_2(t)]$

Sol:- $= \mathbb{E}[X_1(t+\tau)X_2(t)]$

$$= \mathbb{E}[X(t+\tau)\cos(2\pi f_c t + \theta + 2\pi f_c \tau) X(t)\sin(2\pi f_c t + \theta)]$$

$$= \mathbb{E}\left[\frac{X(t+\tau)X(t)}{2} 2\sin(2\pi f_c t + \theta)\cos(2\pi f_c t + \theta + 2\pi f_c \tau)\right]$$

$$= \mathbb{E}\left[\frac{X(t+\tau)X(t)}{2} \times \left(\sin(4\pi f_c t + 2\theta + 2\pi f_c \tau) - \sin(2\pi f_c \tau)\right)\right]$$

$$= \frac{1}{2}\mathbb{E}[X(t+\tau)X(t)\sin(4\pi f_c t + 2\theta + 2\pi f_c \tau)] \rightarrow 0$$

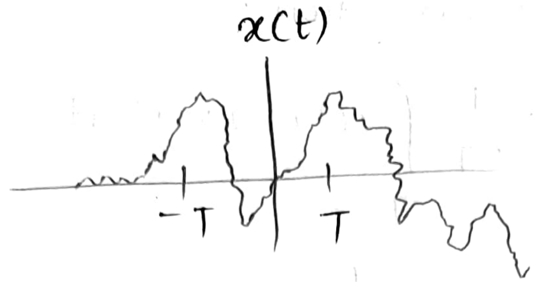
$$- \frac{1}{2}\mathbb{E}[X(t+\tau)X(t)\sin(2\pi f_c \tau)]$$

∘∘ If X and Y are independent, then

$$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)]$$

$$= -\frac{R_X(\tau)}{2}\sin(2\pi f_c \tau)$$

- Ergodicity:



Time-average of sample $x(t)$ is;

$$\mu_x(T) = \frac{1}{2T}\int_{-T}^{T} x(t)\,dt$$

$$\downarrow$$

$$\mathbb{E}[\mu_x(T)] = \mathbb{E}\left[\frac{1}{2T}\int_{-T}^{T} X(t)\,dt\right]$$

$$= \frac{1}{2T}\int_{-T}^{T} \underbrace{\mu_x(t)}_{\text{statistical average.}}\,dt$$

If $X(t)$ is stationary,

$$\mathbb{E}[\mu_x(T)] = \mu_x$$

$$Var(x) = \mathbb{E}[(X-\mu_x)^2] \geq 0$$

$Var(x) = 0 \Rightarrow X = \mu_x$ with probability 1.

$\downarrow$

At this instant, random variable becomes const.

→ A random process $X(t)$ is said to be ergodic in mean (or) mean-ergodic if

(i) $\lim_{T\to\infty} \mu_x(T) = \mu_x$

(ii) $\lim_{T\to\infty} Var(\mu_x(T)) = 0$

• <u>Time-average autocorrelation</u>:

$$R_X(\tau, T) = \frac{1}{2T}\int_{-T}^{T} x(t+\tau)\,x(t)\,dt$$

$$E[R_X(\tau,T)] = E\left[\frac{1}{2T}\int_{-T}^{T} X(t+\tau)\,X(t)\,dt\right]$$

$$= \frac{1}{2T}\int_{-T}^{T} R_X(\tau, t)\,dt$$

If $X(t)$ is stationary,

$$E[R_X(\tau,T)] = R_X(\tau)$$

→ A Random process is said to be ergodic in autocorrelation if,

(i) $\lim_{T\to\infty} R_X(\tau, T) = R_X(\tau)$

(ii) $\lim_{T\to\infty} Var(R_X(\tau,T)) = 0$

$$x(t) \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ h(t)\end{matrix}} \longrightarrow y(t)$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$$\mathbb{E}[y(t)] = \mathbb{E}\left[\int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau\right]$$

$$\mathbb{E}[Y(t)] = \mathbb{E}\left[\int_{-\infty}^{\infty} h(\tau)\,X(t-\tau)d\tau\right]$$

$$\mu_Y(t) = \int_{-\infty}^{\infty} h(\tau)\,\mathbb{E}[X(t-\tau)]\,d\tau$$

$$\mu_Y(t) = \int_{-\infty}^{\infty} h(\tau)\,\mu_X(t-\tau)\,d\tau$$

If $X(t)$ is stationary,

$$\mu_Y(t) = \mu_X \int_{-\infty}^{\infty} h(\tau)d\tau$$

$$\therefore \underline{\mu_Y(t) = \mu_X\,H(0).}$$

$$\left(\begin{aligned} H(f) &= \mathcal{F}\{h(t)\} \\ &= \int_{-\infty}^{\infty} h(t)\,e^{-j2\pi ft}\,dt \end{aligned}\right)$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$R_Y(t_1,t_2) = \mathbb{E}\left[Y(t_1)\cdot Y(t_2)\right]$$

$$Y(t_1) = \int_{-\infty}^{\infty} h(\tau)\cdot X(t_1-\tau)\,d\tau$$

$$Y(t_2) = \int_{-\infty}^{\infty} h(\tau)\, X(t_2-\tau)\,d\tau$$

$$R_Y(t_1,t_2) = \mathbb{E}\left[\int_{-\infty}^{\infty} h(\tau)\cdot X(t_1-\tau)\,d\tau \int_{-\infty}^{\infty} h(\tau)\, X(t_2-\tau)\,d\tau\right]$$

$$= \mathbb{E}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)\,h(\tau_2)\,X(t_1-\tau_1)\,X(t_2-\tau_2)\,d\tau_1\,d\tau_2\right]$$

→ for stable LTI system,

$$R_Y(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)\,h(\tau_2)\,R_X(t_1-\tau_1,\, t_2-\tau_2)\,d\tau_1\,d\tau_2$$

→ If $X(t)$ is stationary,

WSS

(i) $\mathbb{E}[X(t)]$ is independent of time.

(ii) $R_X(t_1,t_2)$ depends only on t_2-t_1.

→ $X(t)$ is nth-order strict-sense stationary if

$$f_{X(t_1)\,X(t_2)\cdots X(t_N)}(x_1,\dots,x_N;\, t_1,t_2,\dots t_N)$$

$$= f_{X(t_1+\tau)\cdots X(t_N+\tau)}(x_1\dots x_N;\, t_1,\dots t_N)$$

$$R_Y(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau+\tau_2-\tau_1)\,d\tau_1\,d\tau_2$$

where ~~[illegible]~~ $\tau = t_1 - t_2$

$\Downarrow$

$R_Y(0)$

Mean square,

$$\mathbb{E}[Y^2(t)] = R_Y(0)$$

$$H(f) = \int_{-\infty}^{\infty} h(\tau)e^{-j2\pi f t}\,d\tau$$

$$h(\tau_1) = \int_{-\infty}^{\infty} H(f)\,e^{j2\pi f\tau_1}\,df$$

$$\mathbb{E}[Y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} H(f)e^{j2\pi f\tau_1}\,df\right)h\tau_2\,R_X(\tau_2-\tau_1)\,d\tau_1\,d\tau_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(f)\,df\;h(\tau_2)R_X(\tau_2-\tau_1)e^{j2\pi f\tau_1}\,d\tau_2\,d\tau_1$$

$\alpha = \tau_2 - \tau_1$

$$= \int_{-\infty}^{\infty} H(f)\int_{-\infty}^{\infty} h(\tau_2)e^{j2\pi f\tau_2}\Bigg(\underbrace{\int_{-\infty}^{\infty} R_X(\alpha)e^{-j2\pi f\alpha}\,d\alpha}_{\text{fourier Transform of autocorrelation } R_X(\tau)}\Bigg)d\tau_2\,df$$

$$= \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)\,df$$

$\uparrow$

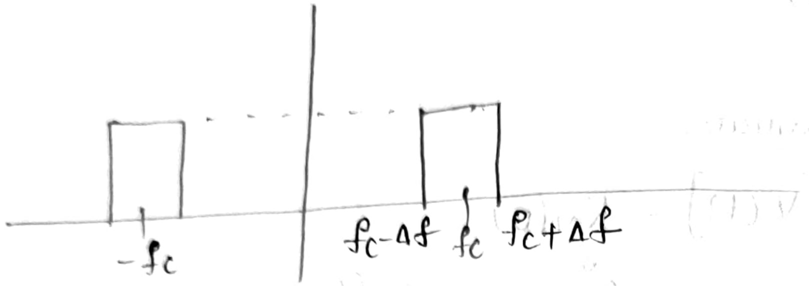
Power spectral density (PSD)

(or)

Power density spectrum.

$$E[Y^2(t)] = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)\,df$$

$$\simeq 2\Delta f\, S_X(f)$$



$\therefore$ dimension of $S_X(f)$ is Watt/Hz.

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)\, e^{j2\pi f\tau}\, df$$

(i) $S_X(0) = \int_{-\infty}^{\infty} R_X(\tau)\,d\tau$

$$E[X^2(t)] = R_X(0) = \int_{-\infty}^{\infty} S_X(f)\,df$$

(ii) $S_X(f) \geq 0$

(iii) $S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j2\pi f\tau}\, d\tau$

$$S_X(-f) = \int_{-\infty}^{\infty} R_X(\tau)\, e^{j2\pi f\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} R_X(-\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$u = -\tau$

$$= \int_{-\infty}^{\infty} R_X(u)\, e^{-j2\pi f u}\, du$$

$$= S_X(f)$$

(iv) $P_X(f) = \dfrac{S_X(f)}{\int_{-\infty}^{\infty} S_X(f)}$ Satisfies properties of a pdf.

eg: $X(t) = A\cos(2\pi f_c t + \Theta)$ $\Theta \sim U[-\pi, \pi]$

find PSD of $X(t)$

Sol: $\mathbb{E}[X^2(t)] = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)\, df$

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$$R_X(\tau) = \mathbb{E}[X(t+\tau)X(t)]$$

$$= \mathbb{E}\left[A^2\cos(2\pi f_c t + 2\pi f_c \tau + \theta)\cos(2\pi f_c t + \theta)\right]$$

$$= \frac{A^2}{2\pi}\int_{-\pi}^{\pi} \cos(2\pi f_c t + 2\pi f_c \tau + \theta)\cos(2\pi f_c t + \theta)\, d\theta$$

$$= \frac{A^2}{2\pi} \times \frac{1}{2}\int_{-\pi}^{\pi} \underbrace{\cos(4\pi f_c t + 2\pi f_c \tau + 2\theta)}_{0} + \cos(2\pi f_c \tau)\, d\theta$$

$$= \frac{A^2}{\cancel{2\pi}} \times \frac{1}{2} \times \cos(2\pi f_c \tau) \times \cancel{2\pi}$$

$$R_X(\tau) = \frac{A^2\cos(2\pi f_c \tau)}{2}$$

$$S_X(f) = \int_{-\infty}^{\infty} \frac{A^2}{2}\cos(2\pi f_c \tau)\, e^{-j2\pi f\tau}\, d\tau$$

$$= \frac{A^2}{2}\int_{-\infty}^{\infty} \frac{e^{j2\pi f_c \tau} + e^{-j2\pi f_c \tau}}{2} \cdot e^{-j2\pi f\tau}\, d\tau$$

$$= \frac{A^2}{2} \times \frac{1}{2}\int_{-\infty}^{\infty} \left(e^{-j2\pi(f-f_c)\tau} + e^{-j2\pi(f+f_c)\tau}\right) d\tau$$

$$\therefore S_X(f) = \frac{A^2}{4}\left[\delta(f-f_c) + \delta(f+f_c)\right]$$

→ $X(t)$ is a Random process with

$$R_X(\tau) = \begin{cases} A^2\left(1-\frac{|\tau|}{T}\right), & |\tau| < T. \\ 0, & \text{otherwise.} \end{cases}$$

find $S_X(f)$.

Sol:

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$$= \int_{-T}^{T} A^2\left(1-\frac{|\tau|}{T}\right) e^{-j2\pi f\tau}\, d\tau$$

$$= \int_{-T}^{0} A^2 (2)\, e^{-j2\pi f\tau}\, d\tau + \int_{0}^{\tau} \cancel{0\, d\tau}^{\,0}$$

$$= 2A^2 \int_{-T}^{0} e^{-j2\pi f\tau}\, d\tau$$

$$= \frac{2A^2}{-j2\pi f} \times e^{-j2\pi f\tau}\Big|_{-T}^{0}$$

$$= \frac{jA^2}{\pi f}\left[1 + e^{j2\pi fT}\right]$$

$$S_X(f) = A^2 T \,\text{sinc}^2(fT)$$

$\rightarrow$ $$S_Y(f) = \int_{-\infty}^{\infty} R_Y(\tau) e^{-j2\pi f \tau} d\tau$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) R_X(\tau - \tau_1 + \tau_2) d\tau_1 d\tau_2 \right) e^{-j2\pi f \tau} d\tau$$

$$u = \tau - \tau_1 + \tau_2 , \quad \tau = u + \tau_1 - \tau_2$$

$$= \left(\int_{-\infty}^{\infty} h(\tau_1) e^{-j2\pi f \tau_1} d\tau_1 \right) \left(\int_{-\infty}^{\infty} R_X(u) e^{-j2\pi f u} du \right) \int_{-\infty}^{\infty} h(\tau_2) e^{j2\pi f \tau_2} d\tau_2$$

$\rightarrow$ $X(t), Y(t)$ are joint stationary R.P's

$$R_{XY}(\tau) = \mathbb{E}[X(t+\tau) Y(t)]$$

cross-spectral density,

$$S_{XY}(f) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j2\pi f \tau} d\tau$$

$$S_{YX}(f) = \int_{-\infty}^{\infty} R_{YX}(\tau) e^{-j2\pi f \tau} d\tau$$

$$\boxed{S_{XY}(f) = S_{YX}^{*}(f) = S_{YX}(-f)}$$