# EC312 : Optimization Techniques

03/01/2023

- Cost Function / Objective function / Risk function
- Minimization or Maximization

1. Cost (Depends on no. of gates) • No. of gates ↑ — Area ↑, Power consumption ↑, Cost ↑
2. Delay
3. Speed

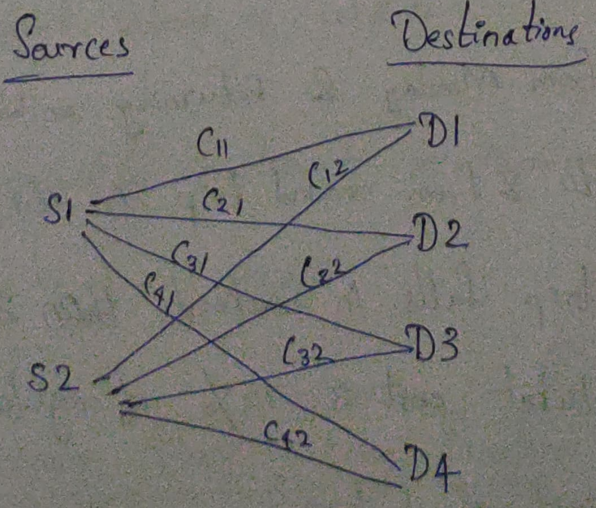
- for speed, Parallel processing — but it increases hardware

DSP system : Adders, Multipliers, Delay elements
Digital system : Logic gates
Analog System : R, C, Diodes, Transistors
- Minimizing cost, Maximizing Productivity

Building Blocks

04/01/2023

Ex : Coalmine — Transportation




Optimization

- To minimize Transportation cost

- Optimization always tries to optimize objective function.
(either maximization or minimization.

→ Optimizing or minimizing cost.

→ Optimizing or maximizing productivity.

- Consider constraints
  - Equality constraints
  - Non-equality constraints

- How to maximize or minimize objective function.

→ Objective function & Constraints are based on "Decision Variables"

Mathematical Modelling to Optimization Model:

Uses (1) Able to have alternatives.

Ex: VJA ⇄ BNG for 5 weeks Starting from Jan 2023 to Feb 3rd 2023

Constraints:

- Starting from Monday & Returning on Wednesday
- Regular to & from round trip is 10k/-
- If round trip falls in weekends then 20% discount
- One way ticket cost is 75% of regular fare

Q → Optimize cost of booking?

· Option 1: Mon to Wed – for 5 weeks (regular trip)

→ 50k

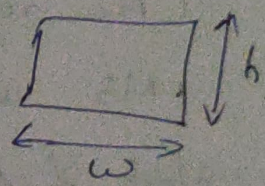
Option 2: 10 – one way tickets ⇒ 37.5k

Option 3: Jan $2^{nd}$ – Feb $1^{st}$ (round trip falls on weekends)
start end

⇒ $5 \times 10k \times 0.20$

Ex: Garden – Fencing · Length of fencing (fixed) 'L'

Constraint: $2(w+h) = L$



· To maximize / optimize the area

Objective function: $A = w \times h$

Decision Variables: $w$ & $h$

· Infinite no. of alternatives as we can choose different values of $w$ & $h$

→ Inequality constraint or Non-negative restriction

$w > 0, \ h > 0$

Linear Programming Problem →

# If objective function and constraints are linear in nature then that optimization is called as Linear Programming Problem (LPP)

Q. A pharmaceutical company produces 3 types of medicines M1, M2, M3. For manufacturing these medicines, 3 types of ingredients I1, I2, I3 are required.

1 M1 requires 3 units of I1, 2 of I2 and

1 M2 requires 2 of I1, 3 of I2, 4 of I3 and

1 M3 requires 4 of I1, 3 of I3. Company has stock of 40 of I1, 30 of I2, 45 of I3. Profit of M1, M2, M3 are Rs8, Rs13, Rs10. Formulate problem as an LPP so that company maximize the profit.

Sol.

| | M1 | M2 | M3 | Availability of Stock |
|---|---|---|---|---|
| I1 | 3 | 2 | 0 | 40 |
| I2 | 2 | 3 | 0 | 30 |
| I3 | 0 | 4 | 3 | 45 |

- Decision Variables : $x_1, x_2, x_3$

- No. of Mi manufactured } : $x_1, x_2, x_3$ ↑ ↑ ↑ M1 M2 M3

- Profit : $8x_1 + 13x_2 + 10x_3$

$$z = f(x)$$

- Objective function : (which needs to be Maximized)

<u>Optimize</u> / <u>Maximize</u> : $z = 8x_1 + 13x_2 + 10x_3$

- <u>Subject to</u> :

$$\left.\begin{array}{ll} (I_1 \to 40) & 3x_1 + 2x_2 + 4x_3 \leq 40 \\ (I_2 \to 30) & 2x_1 + 3x_2 \leq 30 \\ & 4x_2 + 3x_3 \leq 45 \\ & x_1, x_2, x_3 \geq 0 \end{array}\right\} \text{Constraints}$$

- Constraints and Objective function Linear in nature
$\Rightarrow$ LPP

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<u>2 Problem</u> : LPP to minimize total no. of waiters required.

- Hotel operates <u>24 hours</u>

| Period | Time | minimum no. of waiters required |
|---|---|---|
| $x_1$ 1 | 7am - 11am | 6 |
| $x_2$ 2 | 11am - 3pm | 12 |
| $x_3$ 3 | 3pm - 7pm | 8 |
| $x_4$ 4 | 7pm - 11pm | 16 |
| $x_5$ 5 | 11pm - 3am | 5 |
| $x_6$ 6 | 3am - 7am | 3 |

mi...

$z = x_1 + x_2 + x_3$

$z = x_1 + x_2$

$x_1 + x_2 \geq 6$

$x_2 + x_3 \geq 12$

$x_3 + x_4 \geq 8$

$x_4 + x_5 \geq 16$

$x_5 + x_6 \geq 5$

$x_6 + x_1 \geq 3$

. A waiter reports to the hotel manager at the beginning of the period and continues to work for 8 hours.

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· LPP - Linear Programming Problem

· Objective function

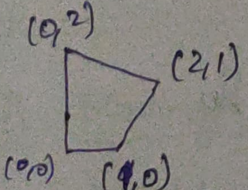
· Constraints → Inequality
→ Non-negative restrictions

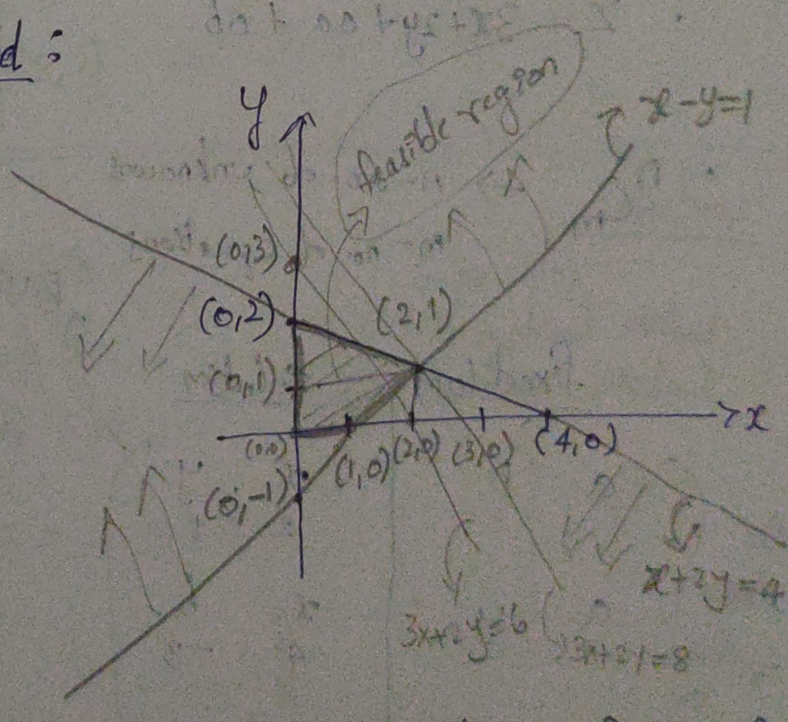
Ex: Objective function : Maximize
$Z = 3x + 2y$

Subject to : $x + 2y \le 4$, $x - y \le 1$ } Inequality constraints

$x \ge 0 ; y \ge 0$ } Non-negative restrictions.

Sol: Using Graphical Method :




Z value

$(0,0) \to 0$

$(1,0) \to 3$

$(2,1) \to 8$

$(0,2) \to 4$

· Maximization of objective function occur at $(2,1) \equiv (x,y)$

$\boxed{Z = 8}$

· Points on the line & in the region donot maximize (except the boundary points)

**Sol:** Using Algebraic Method : $Z = 3x + 2y$

$x + 2y \leq 4$

Introducing a new variable "a" : $x + 2y + a = 4$ ; $a \geq 0$

$x - y \leq 1$

Introducing a new variable "b" : $x - y + b = 1$ ; $b \geq 0$

- a, b are slack variables ; as they do not contribute anything to the objective function.

- $Z = 3x + 2y + 0a + 0b$

$\boxed{\text{fix non-basic variables and solve for Basic variables}}$

- ${}^{n}C_{m}$ → n - no. of unknowns, m - no. of equations

Non-basic variables | Basic Variables

$^{4}C_{2} \Rightarrow 6$

| fixed (0,0) | | Solution | | Z |
|---|---|---|---|---|
| a | b | x = 2 | y = 1 | 8 |
| a | y | x = 4 | b = -3 | - |
| a | x | b = 3 | y = 2 | 4 |
| b | y | a = 3 | x = 1 | 3 |
| b | x | a = 6 | y = -1 | - |
| x | y | a = 4 | b = 1 | 0 |

$a=0,\ x=0:$ $\qquad z = 3x+2y+0a+0b \qquad x+2y+a=4$

$x-y+b=1$

$\left.\begin{aligned} 2y &= 4 \\ b-y &= 1 \end{aligned}\right\} \quad \begin{aligned} y&=2 \\ b&=3 \end{aligned} \qquad z = 4$

$a=1,\ x=0$

$\left.\begin{aligned} 2y+1 &= 4 \\ b-y &= 1 \end{aligned}\right\} \quad \begin{aligned} y&=1.5 \\ b&=2.5 \end{aligned} \qquad z=3$

$a=0 \quad x=1$

$\left.\begin{aligned} 2y+1 &= 4 \\ b-y &= 1 \end{aligned}\right\} \quad \begin{aligned} y&=1.5 \\ b&=2.5 \end{aligned} \qquad z=3$

$\therefore (x,y) \equiv (2,1) \Rightarrow z=8.$

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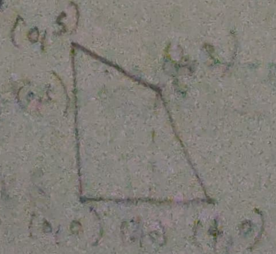
**Ex:** Maximize : $6x_1+5x_2$.

Solve using
(i) Graphical Method
(ii) Algebraic Method

Subject to : $x_1+x_2 \le 5$

$3x_1+2x_2 \le 12$

and $x_1 \ge 0\ ;\ x_2 \ge 0$




**Sol:** (i) $z = 6x_1+5x_2$

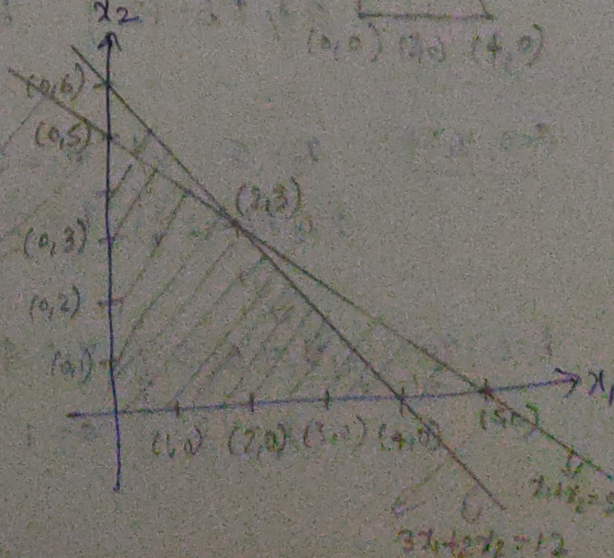
$(2,3) \Rightarrow z = 12+15 = 27$

$(0,5) \Rightarrow z = 25$

$(0,0) \Rightarrow z = 0$

$(4,0) \Rightarrow z = 24$

$\boxed{\therefore (x_1,x_2) = (2,3)}$



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(ii) Algebraic Method

$Z = 6x_1 + 5x_2 + 0a + 0b$

$x_1 + x_2 \le 5$

$3x_1 + 2x_2 \le 12$

$x_1 + x_2 + a = 5$ ; $a \ge 0$

$3x_1 + 2x_2 + b = 12$ ; $b \ge 0$

| fixed (0,0) | | Solution | | Z |
|---|---|---|---|---|
| a | b | $x_1$ | $x_2$ | |
| 0 | 0 | 2 | 3 | 27 |
| a | $x_2$ | $x_1$ | b | |
| 0 | 0 | 5 | −3 | — |
| a | $x_1$ | b | $x_2$ | |
| 0 | 0 | 2 | 5 | 25 |
| b | $x_2$ | a | $x_1$ | |
| 0 | 0 | 1 | 4 | 24 |
| b | $x_1$ | a | $x_2$ | |
| 0 | 0 | −1 | 6 | — |
| $x_1$ | $x_2$ | a | b | |
| 0 | 0 | 5 | 12 | 0 |

$\boxed{\therefore (x_1, x_2) = (2, 3)}$

$a=0, b=0$ : $x_1 + x_2 = 5$

$3x_1 + 2x_2 = 12$

$3x_1 + 2x_2 = 12$

$2x_1 + 2x_2 = 10$

$x_1 = 2$

$x_2 = 3$

$a=0, x_2=0$ : $x_1 + a = 5$ ; $3x_1 + b = 12$ ; $x_1 = 5$ ; $b = -3$

$a=0, x_1=0$ : $x_2 = 5$ ; $2x_2 + b = 12$ ; $x_2 = 5$ ; $b = 2$

$b=0, x_2=0$ : $x_1 + a = 5$ ; $3x_1 = 12$ ; $x_1 = 4$ ; $a = 1$

$b=0, x_1=0$ : $x_2 + a = 5$

$2x_2 = 12$

$\Rightarrow x_2 = 6$

$a = -1$

$x_1=0, x_2=0$ : $a = 5$

$b = 12$

1. As $n\uparrow$, no. of possible ways $\uparrow$

2. $z_1 > z_2$, no need calculate $z_2$ iteration

3. If $z_1$ is the max., then no need to move further.

<u>Iteration-1</u> : $Z = 6x_1 + 5x_2 + 0a + 0b$

$$\boxed{\begin{matrix} x_1 = 0 \\ x_2 = 0 \end{matrix}}$$

$a = 5 - x_1 - x_2$

$b = 12 - 3x_1 - 2x_2$

$Z = 6x_1 + 5x_2$

$(x_1, x_2) = (0, 0) \Rightarrow Z = 0$

$$\boxed{\begin{matrix} x_2 = 0 \\ b = 0 \end{matrix}}$$

<u>Iteration-2</u> : $x_1$ & $a$ - Basic variables

$a = 5 - x_1 \rightarrow x_1 = 5$ (max)

$b = 12 - 3x_1 \rightarrow x_1 = 4$ (max)

$$\boxed{b = 12 - 3x_1 - 2x_2}$$

$\hookrightarrow b = 0$

$3x_1 = 12 - 2x_2 - b$

$\Rightarrow x_1 = 4 - \frac{2}{3}x_2 - \frac{1}{3}b$

and $x_1 = 5 - x_2 - a$

(or)

$a = 5 - x_1 - x_2$

$\Rightarrow a = 5 - \left(4 - \frac{2}{3}x_2 - \frac{1}{3}b\right) - x_2$

$= 1 + \frac{2}{3}x_2 - x_2 + \frac{1}{3}b$

$= 1 - \frac{1}{3}x_2 + \frac{1}{3}b$

$= 1 - \frac{1}{3}x_2 + \frac{1}{3}(12 - 3x_1 - 2x_2)$

$$\boxed{a = 5 - x_2 - x_1}$$

$\hookrightarrow a = 1$

$\therefore Z = 6x_1 + 5x_2 = 6\left(4 - \frac{2}{3}x_2 - \frac{1}{3}b\right) + 5x_2$

$Z = 24 + x_2 - 2b$

$(x_2, b) = (0, 0) \Rightarrow Z = 24$

$$\boxed{\begin{matrix} x_1 = 4 - \frac{2}{3}x_2 - \frac{1}{3}b \\ a = 1 - \frac{1}{3}x_2 + \frac{b}{3} \end{matrix}}$$

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# Disadvantages of Algebraic Method:

1) As variables ↑, No. of possible combination/ways solutions also increases

2) Objective functions need to be calculated for all possible combinations.

3) We should be able to discard a combination whose solution is less than the existing optimized solution.

4) We should be able to terminate/stop the process, once we reach the optimized solution.

→ No. of iterations should be reduced.

→ Iterations which are not necessary, need to be discarded.

→ Process should be progressive towards optimized solution.

Iteration 3 : $\cancel{a}, \cancel{b} = 0$ and $x_1, a$ are slack variables

$x_1 + x_2 + a = 5$ and $3x_1 + 2x_2 + b = 12$

$x_1 = 5 - x_2 - a$

$2x_2 = 12 - 3x_1 - b$

$x_2 = 6 - \frac{3}{2}x_1 - \frac{b}{2}$

$x_1 = 5 - 6 + \frac{3}{2}x_1 + \frac{b}{2} - a$

$a = 5 - x_1 - x_2$

$-\frac{1}{2}x_1 = -1 + \frac{b}{2} - a$

$a = 6 - x_1$

$x_1 = 2 - b + 2a$

$\boxed{Z = 5x_1 + 5x_2}$

$z = 24 + x_2 - 2b$

Solving for $x_2$ : $x_2 = 5 - a - x_1$

$x_1 = 4 - \frac{2}{3}x_2 - \frac{1}{3}x_4 \rightarrow$ max of $x_2 = 6$

$x_3 = 1 - \frac{1}{3}x_2 + \frac{1}{3}x_4 \longrightarrow$ max of $x_2 = 3$

Non-basic variables : $(a, b) = (0,0)$

Basic Variables : $x_1, x_2$

$a = 5 - x_2 - x_1$

$a = 1 - \frac{x_2}{3} + \frac{b}{3}$

$\boxed{x_2 = 3 + b - 3a}$

$x_1 = 4 - \frac{2}{3}x_2 - \frac{1}{3}b$

$x_1 = 4 - \frac{2}{3}(3 + 6 - 3a) - \frac{1}{3}b$

$x_1 = 4 - 2 - b + 2a$

$\boxed{x_1 = 2 - b + 2a}$

$Z = 27 - 3a - b \rightarrow$ max $\boxed{Z = 27}$ — cannot be further proceeded

# ① Simplex Algorithm in Algebraic form

Maximize : $Z = 6x_1 + 5x_2 + 0x_3 + 0x_4$

Subject to: $x_1 + x_2 \leq 5$, $3x_1 + 2x_2 \leq 12$, $x_1, x_2 \geq 0$

$x_1 + x_2 + x_3 = 5$, $3x_1 + 2x_2 + x_4 = 12$, $x_3, x_4 \geq 0$

Iteration 1: $x_1, x_2 = 0, 0$ → Non basic variables & $x_3, x_4$ → Basic variables

$$x_3 = 5 - x_1 - x_2$$

$$x_4 = 12 - 3x_1 - 2x_2$$

$$\Rightarrow Z = 6x_1 + 5x_2 + 0x_3 + 0x_4$$

$(x_1, x_2) = (0, 0) \Rightarrow \boxed{Z = 0}$

- As in Z, $x_1$ coefficient > $x_2$ coefficient, make $x_1$ as Basic variable

→ make $x_2$ & $x_4$ as non basic variables

as $x_3 = 5 - x_1 - x_2 \rightarrow x_1(max) = 5$

$x_4 = 12 - 3x_1 - 2x_2 \rightarrow x_1(max) = 4 \rightarrow$ min

Iteration 2: $x_2, x_4 = 0, 0$ and $x_1, x_3$ – Basic variables

$$3x_1 + 2x_2 + x_4 = 12$$

$$x_1 = 4 - \frac{2}{3}x_2 - \frac{1}{3}x_4$$

$$x_3 = 5 - x_1 - x_2$$

$$x_3 = 5 - \left(4 - \frac{2}{3}x_2 - \frac{1}{3}x_4\right) - x_2$$

$$x_3 = 1 - \frac{1}{3}x_2 + \frac{1}{3}x_4$$

$$\Rightarrow Z = 6\left(4 - \frac{2}{3}x_2 - \frac{1}{3}x_4\right) + 5x_2$$

$$Z = 24 + x_2 - 2x_4 \qquad \boxed{Z = 24}$$

Iteration 3: $x_3, x_4 = 0, 0$ and $x_1, x_2$ - Basic variables

- Make $x_2$ as basic variable as in Z, coefficient of $x_2$ is greater than $x_4$.

- Make $x_3$ as non-basic variable as

$$x_1 = 4 - \frac{2}{3}x_2 - \frac{x_4}{3} \rightarrow x_2(max) = 6$$

$$x_3 = 1 - \frac{1}{3}x_2 + \frac{1}{3}x_4 \rightarrow x_2(max) = 3 \rightarrow min$$

$$\Rightarrow x_3 = 5 - x_2 - x_1$$

$$x_3 = 1 - \frac{x_2}{3} + \frac{x_4}{3}$$

$$\Rightarrow x_2 = 3 + x_4 - 3x_3$$

$$x_1 = 4 - \frac{2}{3}x_2 - \frac{1}{3}x_4$$

$$x_1 = 4 - \frac{2}{3}(3 + x_4 - 3x_3) - \frac{1}{3}x_4$$

$$\Rightarrow x_1 = 2 - x_4 + 2x_3$$

$$\Rightarrow Z = 6(2 - x_4 + 2x_3) + 5(3 + x_4 - 3x_3)$$

$$\Rightarrow Z = 27 - 3x_3 - x_4 \qquad \boxed{Z = 27}$$

$$x_3 \geq 0, x_4 \geq 0$$

$\boxed{\therefore \text{ Maximum : } Z = 27}$

# ② Simplex Algorithm – Tabular form Method

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① Initialization
② Iterations
③ Termination

**Maximize:** $z = 6x_1 + 5x_2 + 0x_3 + 0x_4$

**Subject to:** $x_1 + x_2 + x_3 = 5$

$3x_1 + 2x_2 + x_4 = 12$

$x_1, x_2, x_3, x_4 \geq 0$

| $C_j$ (Basic variables↓) | 6 $x_1$ | 5 $x_2$ | 0 $x_3$ | 0 $x_4$ | R.H.S | $\theta$ |
|---|---|---|---|---|---|---|
| 0 $x_3$ | 1 | 1 | 1 | 0 | 5 | $\frac{5}{1} = 5$ |
| 0 $x_4$ | 3 | 2 | 0 | 1 | 12 | $\frac{12}{3} = 4$ leave → (min) |
| $Z_j$ | $0\times1+0\times3 = 0$ | $0\times1+0\times2 = 0$ | $0\times1+0\times0 = 0$ | $0\times0+0\times1 = 0$ | | |
| $C_j - Z_j$ | 6 ↑enter (max) | 5 | 0 | 0 | $Z_j = 0$ | |
| 0 $x_3$ | 0 | $\frac{1}{3}$ | 1 | $-\frac{1}{3}$ | 1 ↳ $x_3 - x_4$ | $\frac{1}{1/3} = 3$ leave → (min) |
| 6 $x_1$ | 1 | $\frac{2}{3}$ | 0 | $\frac{1}{3}$ | 4 ↳ $\frac{x_4}{3}$ | $\frac{4}{2/3} = 6$ |
| $C_j - Z_j$ | 0 | 1 ↑enter (max) | 0 | $-2$ | $Z_j = 24$ | |
| 5 $x_2$ | 0 | 1 | 3 | $-1$ | $3 \; (x_3 \times 3)$ | $\Rightarrow x_3 \times 3$ |
| 6 $x_1$ | 1 | 0 | $-\frac{2}{3}\times 3 = -2$ | $\frac{1}{3} - \left(-1\times\frac{2}{3}\right) = 1$ | $4 - 3\times\frac{2}{3} = 2$ | $\Rightarrow x_1 - \frac{2}{3}x_2$ |
| $C_j - Z_j$ | 0 | 0 | $0-(15-12)$ $-3$ | $0-(-5+6)$ $-1$ | $5\times3+6\times2 = 27$ | Maximum $\therefore z = 27$ |

• Termination point: At which zeroes and –ve values are present

Ex: Maximi…

Subject to…

| $C_j$ | 6 $x_1$ … |
|---|---|
| 0 $x_3$ | 1 |
| 0 $x_4$ | 2 |
| 0 $x_5$ | 1 |
| $C_j - Z_j$ | 6 |
| 0 $x_3$ | $\frac{4}{5}$ |
| 0 $x_4$ | $2-\frac{3}{5}=$ … |
| 8 $x_2$ | $\frac{1}{5}$ |
| $C_j - Z_j$ | $6-\frac{8}{5}$ $=4.$… |
| 0 $x_3$ | 0 |
| 6 $x_1$ | 1 |
| 8 $x_2$ | 0 |
| $C_j - Z_j$ | 0 |

Ex: Maximize: $6x_1 + 8x_2$

Subject to: $x_1 + x_2 \le 10$

$2x_1 + 3x_2 \le 25$

$x_1 + 5x_2 \le 35$

$x_1, x_2 \ge 0$

$Z = 6x_1 + 8x_2 + 0x_3 + 0x_4 + 0x_5$

$x_1 + x_2 + x_3 = 10$

$2x_1 + 3x_2 + x_4 = 25$

$x_1 + 5x_2 + x_5 = 35$

$x_1, x_2, x_3, x_4, x_5 \ge 0$

St

| $C_j$ | | 6 | 8 | 0 | 0 | 0 | | |
|---|---|---|---|---|---|---|---|---|
| | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS | $\theta$ |
| 0 | $x_3$ | 1 | 1 | 1 | 0 | 0 | 10 | $\frac{10}{1}$ |
| 0 | $x_4$ | 2 | 3 | 0 | 1 | 0 | 25 | $\frac{25}{3} = 8.$ |
| 0 | $x_5$ | 1 | 5 | 0 | 0 | 1 | 35 | $\frac{35}{5} = 7$ leave |
| $C_j - Z_j$ | | 6 | 8 ↑ enter | 0 | 0 | 0 | 0 | |
| 0 | $x_3$ | $\frac{4}{5}$ | 0 | 1 | 0 | $-\frac{1}{5}$ | $10-7=3$ | $\frac{15}{4} = 3.$ |
| 0 | $x_4$ | $2-\frac{3}{5} = \frac{7}{5}$ | 0 | 0 | 1 | $-\frac{3}{5}$ | $25-21 = 4$ | $\frac{20}{7} = 2.$ leave |
| 8 | $x_2$ | $\frac{1}{5}$ | 1 | 0 | 0 | $\frac{1}{5}$ | 7 | 35 |
| $C_j - Z_j$ | | $6-\frac{8}{5} = 4.4$ ↑ enter | 0 | 0 | 0 | $-\frac{8}{5} = -1.6$ | 56 | |
| 0 | $x_3$ | 0 | 0 | 1 | $-\frac{4}{7}$ | $-\frac{1}{5}+\frac{12}{35} = \frac{5}{35} = \frac{1}{7}$ | $3-\frac{80}{35} = \frac{5}{7}$ | 5 leave |
| 6 | $x_1$ | 1 | 0 | 0 | $\frac{5}{7}$ | $-\frac{3}{7}$ | $\frac{20}{7}$ | $-\frac{20}{3}$ no leave (-ve) |
| 8 | $x_2$ | 0 | 1 | 0 | $-\frac{1}{7}$ | $\frac{1}{5}+\frac{3}{35} = \frac{10}{35} = \frac{2}{7}$ | $7-\frac{4}{7} = \frac{45}{7}$ | $\frac{45}{2} = 22.5$ |
| ~~$C_j - Z_j$~~ | | ~~0~~ | ~~0~~ | ~~0~~ | ~~$-\frac{22}{7}$~~ | ~~$\frac{2}{7}$~~ ↑ enter | ~~$\frac{480}{7}$~~ $= 68.57$ | |

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ization

tions

nation

0

$\theta$

$\frac{5}{1} = 5$

$\frac{12}{3} = 4$ leave (min)

$\frac{1}{1/3} = 3$ leave (min)

$\frac{4}{2/3} = 6$

$\Rightarrow x_3 \times 3$

$\Rightarrow x_1 - \frac{2}{3}x_2$

Maximum

$\boxed{\therefore Z = 27}$

lues are

| $C_j$ | 6 | 8 | 0 | 0 | 0 | | |
|---|---|---|---|---|---|---|---|
| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS | $\theta$ |
| 0 $x_3$ | $\frac{-7}{3}\times\frac{-1}{7}=\frac{1}{3}$ | 0 | 1 | $\frac{-4}{7}+\frac{5}{49}=\frac{-23}{49}$ | 0 | $\frac{5}{7}+\frac{20}{21}=\frac{35}{21}=\boxed{\frac{5}{3}}$ | $\frac{5}{3}\times\frac{3}{1}=5$ |
| 0 $x_5$ | $-\frac{7}{3}$ | 0 | 0 | $-\frac{5}{7}$ | 1 | $\frac{20}{7}\times\frac{-7}{3}=\frac{-20}{3}$ | $\frac{-20}{3}\times\frac{-3}{7}=\frac{20}{7}=2.2$ leave |
| 8 $x_2$ | $\frac{-7}{3}\times\frac{-2}{7}=\frac{+2}{3}$ | 1 | 0 | $\frac{-1}{7}+\frac{10}{49}=\frac{3}{49}$ | 0 | $\frac{45}{7}+\frac{40}{21}=\frac{175}{21}$ | $\frac{25}{3}\times\frac{3}{2}=\frac{25}{2}=12.$ |
| $C_j-Z_j$ | $6-\frac{16}{3}=2/3$ ↑ enter | 0 | 0 | $\frac{-24}{49}$ | 0 | $\frac{1400}{21}=\frac{200}{3}=66.67$ | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 $x_3$ | 0 | 0 | 1 | $\frac{-23}{49}-\frac{1}{3}\times\frac{15}{49}=\frac{-28}{49}$ | $\frac{-3}{7}\times\frac{-1}{3}=\frac{1}{7}$ | $\frac{5}{3}-\frac{20}{21}=\frac{15}{21}$ | $\frac{15}{21}\times\frac{7}{1}=5$ |
| 6 $x_1$ | 1 | 0 | 0 | $-\frac{5}{7}\times-\frac{3}{7}=\frac{15}{49}$ | $-\frac{3}{7}$ | $\frac{20}{7}$ | $\frac{20}{7}\times\frac{-7}{3}=\frac{-20}{3}$ → |
| 8 $x_2$ | 0 | 1 | 0 | $\frac{3}{49}-\frac{2}{3}\times\frac{15}{49}=\frac{-7}{49}=\frac{-1}{7}$ | $\frac{-3}{7}\times\frac{-2}{3}=\frac{2}{7}$ | $\frac{25}{3}-\frac{2}{3}\times\frac{20}{7}=\frac{135}{21}$ | $\frac{135}{21}\times\frac{7}{2}=\frac{135}{6}=22.5$ |
| $C_j-Z_j$ | 0 | 0 | 0 | $-\left(\frac{90}{49}-\frac{56}{49}\right)=-34/49$ | $-\left(\frac{-18}{7}+\frac{16}{7}\right)$ 2/7 ↑ | $\frac{120}{7}+\frac{1080}{21}=68.57$ | |

| $C_j$ | 6 | 8 | 0 | 0 | 0 | | |
|---|---|---|---|---|---|---|---|
| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS | $\theta$ |
| 0 $x_5$ | 0 | 0 | 7 | $-4$ | 1 | 5 | |
| 6 $x_1$ | 1 | 0 | $0+3$ | $\frac{5}{7}-\frac{12}{7}=-1$ | 0 | $\frac{20}{7}\times$ [illegible] $+\frac{5\times3}{7}=\frac{35}{7}=5$ | |
| 8 $x_2$ | 0 | 1 | $0-2$ | $\frac{-1}{7}+\frac{8}{7}=1$ | 0 | $\frac{45}{7}-\frac{10}{7}=5$ | |
| $C_j-Z_j$ | 0 | 0 | $-18+16=-4$ | $6-8=-2$ | 0 | $30+40=70$ | |

Termination step

$\therefore$ Maximum: $z = 70$

# Standard form of Linear Programming Problem (LPP)

27/01/2023

(1) (is always a minimization problem).
(2) (all constraints are linear equations)
(3) Decision variables are always non-negative.

## Scalar form

Minimize : $f(x_1, x_2, \dots, x_n) = c_1x_1 + c_2x_2 + \dots + c_nx_n$

Objective function

Subject to the constraints :

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

and $x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0$ (Non-negative restriction)

(decision variables)

## Vector / Matrix form

Minimize : $f(\vec{x}) = \vec{c}^T \cdot \vec{x}$

Subject to the constraints : $\vec{A}\vec{x} = \vec{B}$

and $\vec{x} \geq 0$ (non-negative restriction)

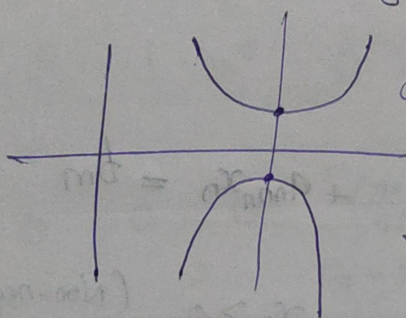
where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

$$\vec{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \bar{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

and $c_i, b_j, a_{ji}$ $(i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m)$

are known constants.

- How to solve maximization problem



$g(x)$ – minimum

|||

$f(x)$ – maximum

- We have '$m$' equations and '$n$' unknowns

Case1: if $m = n$ – Algebraic problem (1-solution)

Case2: if $m > n$ – Eqn.s ~~are~~ coincide

Case3: if $m < n$ – Infinite solution → The solution at which given objective function ~~is~~ maximum or minimum

Problem: Minimize: $Z = 3x_1 + 4x_2$ | Maximize: $z' = -3x_1 - 4x_2$

Subject to: $2x_1 + 3x_2 \geq 8$

$5x_1 + 2x_2 \geq 12$

$x_1, x_2 \geq 0$

Sol: Using Simplex Algorithm:

Maximize:

Introducing slack variables: $Z' = -3x_1 + 4x_2 - 0x_3 - 0x_4$

Subject to: $2x_1 + 3x_2 - x_3 = 8$ ; $x_3 \geq 0$

$5x_1 + 2x_3 - x_4 = 12$ ; $x_4 \geq 0$

| $c_j$ | -3 | -4 | 0 | 0 | | |
|---|---|---|---|---|---|---|
| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | RHS | $\theta$ |
| 0 $x_3$ | 2 | 3 | -1 | 0 | 8 | |
| 0 $x_4$ | 5 | 2 | 0 | -1 | 12 | |
| $c_j - z_j$ | -3 | -4 | 0 | 0 | | |

When nonbasic variables: $x_1, x_2 = 0, 0$

$x_3 = -8$ & $x_4 = -12$

Not a feasible solution

. It failed at the first step

i.e. Initialization step

. So introduce variables: Artificial (surplus) variables (which does not have physical significance)

- Slack variables : has physical significance (the leftout materials)
  Does not contribute to objective function

- Artifical variables : has no physical significance
  Does ~~not~~ contribute to objective function
  ( not ? )

Maximize: $Z' = -3x_1 - 4x_2 - 0x_3 - 0x_4 - Ma_1 - Ma_2$

Subject to: $2x_1 + 3x_2 - x_3 + a_1 = 8$

$5x_1 + 2x_2 - x_4 + a_2 = 12$

# $\boxed{\text{Big M Method}}$

| $C_j$ | | $-3$ | $-4$ | $0$ | $0$ | $-M$ | $-M$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $a_1$ | $a_2$ | RHS | $\theta$ |
| $-M$ | $a_1$ | 2 | 3 | $-1$ | 0 | 1 | 0 | 8 | 4 |
| $-M$ | $a_2$ | 5 | 2 | 0 | $-1$ | 0 | 1 | 12 | $\frac{12}{5} = 2.4$ → |
| $C_j - Z_j$ | | $-3+7M$ ↑ | $-4+5M$ | $-M$ | $-M$ | 0 | 0 | | |
| $-M$ | $a_1$ | 0 | $3 - \frac{4}{5} = \frac{11}{5}$ | $-1$ | $2/5$ | 1 | $-2/5$ | $8-4.8 = \frac{16}{5} = 3.2$ | $\frac{16}{11}$ → |
| $-3$ | $x_1$ | 1 | $2/5$ | 0 | $-1/5$ | 0 | $1/5$ | $2.4 = \frac{12}{5}$ | 6 |
| $C_j - Z_j$ | | 0 | $\frac{11M - 14}{5}$ ↑ | $-M$ | $\frac{2M-3}{5}$ | 0 | $\frac{2M+3}{5}$ | | |
| $-4$ | $x_2$ | 0 | 1 | $-5/11$ | $2/11$ | $5/11$ | $-2/11$ | $\frac{16}{11}$ | |
| $-3$ | $x_1$ | 1 | 0 | $2/11$ | $\frac{-15}{55} = \frac{-3}{11}$ | $\frac{-2}{11}$ | $\frac{15}{55} = \frac{3}{11}$ | $\frac{20}{11}$ | |
| $C_j - Z_j$ | | 0 | 0 | $-\frac{20}{11} + \frac{6}{11} = (-14/11)$ | $-\frac{9}{4} + \frac{8}{11} = (-1/11)$ | $(-M + \frac{20-6}{11})$ | $-M + \frac{1}{11}$ | $Z_j = \frac{-64-60}{11} = -124/11$ | |

$\boxed{\therefore \text{Maximum}: Z' = \frac{-124}{11}}$

$x_1 = 20/11$

$x_2 = 16/11$

$\boxed{\text{Minimum}: Z = \frac{124}{11}}$

31/01/2023

**Primal LPP**

**Two-Phase Method**

Minimize: $Z = 3x_1 + 4x_2$

Subject to: $2x_1 + 3x_2 \geq 8$

$5x_1 + 2x_2 \geq 12$, $(x_1, x_2 \geq 0)$

Sol: S.T: $2x_1 + 3x_2 - x_3 + a_1 = 8$

$5x_1 + 2x_2 - x_4 + a_2 = 12$

Maximize: $Z' = -3x_1 - 4x_2 - 0x_3 - 0x_4 - Ma_1 - Ma_2$

Dual LPP: $Z'' = 8y_1 + 12y_2$ ; Maximize ; S.T: $2y_1 + 5y_2 \leq 3$, $3y_1 + 2y_2 \leq 4$

Phase-01:

| | | 0 | 0 | 0 | 0 | -1 | -1 | | |
|---|---|---|---|---|---|---|---|---|---|
| | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $a_1$ | $a_2$ | RHS | $\theta$ |
| -1 | $a_1$ | 2 | 3 | -1 | 0 | 1 | 0 | 8 | $\frac{8}{2} = 4$ |
| -1 | $a_2$ | 5 | 2 | 0 | -1 | 0 | 1 | 12 | $\frac{12}{5} = 2.4 \rightarrow$ leave |
| $c_j - z_j$ | | 7 ↑ enter | 5 | -1 | -1 | 0 | 0 | | |
| -1 | $a_1$ | 0 | $3 - \frac{4}{5} = \frac{11}{5}$ | -1 | 2/5 | 1 | -2/5 | $8 - \frac{24}{5} = \frac{16}{5}$ | $\frac{16}{5} \times \frac{5}{11} = \frac{16}{11} = 1.4 \rightarrow$ |
| 0 | $x_1$ | 1 | 2/5 | 0 | -1/5 | 0 | 1/5 | $\frac{12}{5}$ | $\frac{12}{5} \times \frac{5}{2} = 6$ |
| $c_j - z_j$ | | 0 | $\frac{11}{5}$ ↑ | -1 | 2/5 | 0 | -7/5 | | |
| 0 | $x_2$ | 0 | 1 | -5/11 | 2/11 | 5/11 | -2/11 | $\frac{16}{5} \times \frac{5}{11} = \frac{16}{11}$ | |
| 0 | $x_1$ | 1 | 0 | $\frac{5}{11} \times \frac{2}{5} = \frac{2}{11}$ | $-\frac{1}{5} - \frac{4}{55} = \frac{-3}{11}$ | $-\frac{5}{11} \times \frac{2}{5} = \frac{-2}{11}$ | $\frac{1}{5} + \frac{4}{55} = \frac{15}{55} = \frac{3}{11}$ | $\frac{12}{5} - \frac{32}{55} = \frac{100}{55} = \frac{20}{11}$ | |
| $c_j - z_j$ | | 0 | 0 | 0 | 0 | -1 | -1 | | |

Termination step

Phase-02: $z' = -3x_1 - 4x_2 - 0x_3 - 0x_4$

Max:
$\therefore z' = \frac{-124}{11}$
$\Rightarrow z(\text{minimum}) = 124/11$

| | -3 | -4 | 0 | 0 | |
|---|---|---|---|---|---|
| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | RHS |
| $-4$ $x_2$ | 0 | 1 | $-5/11$ | $2/11$ | $16/11$ |
| $-3$ $x_1$ | 1 | 0 | $2/11$ | $-3/11$ | $20/11$ |
| $c_j - z_j$ | 0 | 0 | $\frac{-20}{11} + \frac{6}{11} = \frac{-14}{11}$ | $\frac{8}{11} - \frac{9}{4} = -\frac{1}{11}$ | $\frac{-64}{11} - \frac{60}{11} = \frac{-124}{11}$ |

Maximize: $z'' = 8y_1 + 12y_2$; S.T: $2y_1 + 5y_2 \le 3$ & $3y_1 + 2y_2 \le 4$

Dual LPP

| | 8 | 12 | 0 | 0 | | |
|---|---|---|---|---|---|---|
| | $y_1$ | $y_2$ | $S_1$ | $S_2$ | RHS | $\theta$ |
| 0 $S_1$ | 2 | 5 | 1 | 0 | 3 | $\frac{3}{2} = 1.5$ |
| 0 $S_2$ | 3 | 2 | 0 | 1 | 4 | $\frac{4}{3} = 1.33$ → leave |
| $c_j - z_j$ | 8 | 12 ↑ | 0 | 0 | | |
| 0 $S_1$ | $2 - \frac{15}{2} = \frac{-11}{2}$ | 0 | 1 | $-5/2$ | $3-10=-7$ | |
| 12 $y_2$ | $3/2$ | 1 | 0 | $1/2$ | 2 | |
| $c_j - z_j$ | $8-18=-10$ | 0 | 0 | 0 | 24 | |

Problem: Maximize: $Z = 4x + 5y$ — Use Big M method

Subject to: $2x + 3y \le 6$

$3x + y \ge 3$

$x, y \ge 0$

Sol: S.T.: $2x + 3y + S_1 = 6$ ; $S_1 \ge 0$

$3x + y - S_2 = 3$ , $S_2 \ge 0$

$3x + y - S_2 + a_1 = 3$ ; $a_1 \ge 0$

Maximize:

$Z = 4x + 5y + 0S_1 - 0S_2 - Ma_1$

| | | 4 | 5 | 0 | 0 | $-M$ | | |
|---|---|---|---|---|---|---|---|---|
| | | $x$ | $y$ | $S_1$ | $S_2$ | $a_1$ | RHS | $\theta$ |
| 0 | $S_1$ | 2 | 3 | 1 | 0 | 0 | 6 | $\frac{6}{2} = 3$ |
| $-M$ | $a_1$ | 3 | 1 | 0 | $-1$ | 1 | 3 | $\frac{3}{3} = 1 \rightarrow$ |
| $C_j - Z_j$ | | $4+3M$ | $5+M$ | 0 | $M$ | $-M-(-M) = 0$ | | |
| 0 | $S_1$ | 0 | $3 - \frac{2}{3} = \frac{7}{3}$ | 1 | $0 - 2 \times \frac{-1}{3} = \frac{2}{3}$ | $-\frac{2}{3}$ | $6 - 2 = 4$ | $4 \times \frac{3}{7} = \frac{12}{7} \rightarrow$ |
| 4 | $x$ | 1 | $\frac{1}{3}$ | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 1 | $1 \times \frac{3}{1} = 3$ |
| $C_j - Z_j$ | | 0 | $5 - \frac{4}{3} = \frac{11}{3}$ | 0 | $\frac{4}{3}$ | $-M - \frac{4}{3}$ | | |
| 5 | $y$ | 0 | 1 | $\frac{3}{7}$ | $\frac{2}{7}$ | $\frac{-2}{7}$ | $\frac{12}{7}$ | $6 \rightarrow$ |
| 4 | $x$ | 1 | 0 | $0 - \frac{1}{3} \times \frac{3}{7} = \frac{-1}{7}$ | $\frac{-1}{3} - \frac{2}{21} = \frac{-9}{21} = \frac{-3}{7}$ | $\frac{1}{3} + \frac{2}{21} = +\frac{3}{7}$ | $1 - \frac{4}{7} = \frac{3}{7}$ | $-1$ $\boxed{x - \frac{1}{3}y}$ $(\frac{-1}{3}) - \frac{1}{3}(\frac{2}{7})$ |
| $C_j - Z_j$ | | 0 | 0 | $\frac{-15}{7} + \frac{4}{7} = -\frac{11}{7}$ | $\frac{-10}{7} + \frac{12}{7} = +\frac{2}{7}$ | $-M + \frac{10}{7} - \frac{12}{7} = -M - \frac{2}{7}$ | $\frac{60}{7} + \frac{12}{7} = \frac{72}{7}$ | |
| 0 | $S_2$ | 0 | $\frac{7}{2}$ | $\frac{3}{2}$ | 1 | $-1$ | 6 | |
| 4 | $x$ | 1 | $\frac{3}{2}$ | $\frac{-1}{7} + \frac{9}{14} = \frac{7}{14} = +\frac{1}{2}$ | 0 | $\frac{3}{7} - \frac{3}{7} = 0$ | $\frac{3}{7} + \frac{18}{7} = 3$ | |

| | | 4 | 5 | 0 | 0 | −M | | |
|---|---|---|---|---|---|---|---|---|
| | | x | y | $S_1$ | $S_2$ | $a_1$ | RHS | θ |
| 0 | $S_2$ | 0 | 7/2 | 3/2 | 1 | −1 | 6 | $\frac{19}{7}$ |
| 4 | x | 1 | 3/2 | $\frac{-22}{3\times 49}$ | 0 | 0 | $\frac{20}{7}$ | $\frac{20}{7}\times\frac{7}{3}=30$ |
| | | 0 | $5-\frac{12}{2}=\frac{-3\times12}{7}=\frac{-23}{7}$ | $\frac{4\times 22}{3\times 49}$ | 0 | −M | | |
| 5 | y | | | | | | | |
| 4 | x | | | | | | | |

| | | 4 | 5 | 0 | 0 | −M | |
|---|---|---|---|---|---|---|---|
| | | x | y | $S_1$ | $S_2$ | $a_1$ | RHS |
| 0 | $S_2$ | 0 | 7/2 | 3/2 | 1 | −1 | 6 |
| 4 | x | 1 | 3/2 | +1/2 | 0 | 0 | 3 |
| | | 0 | $5-\frac{12}{2}=-1$ | −2 | 0 | −M | 12 |

$\boxed{\therefore \text{Maximum}: Z = 12}$

$(x, y) = (3, 0)$

19.(i) Maximi

S.T:

$6x_1 + 10x_2 = 30$

$25x_1 + 10x_2 = 5$

$19x_1 = 2$

$3x_1 + 5x_2 = 15 \to ($

$5x_1 + 2x_2 = 10 \to ($

Feasible regio

$(0,0) \to$

$(0,3) \to$

$(2,0) \to$

$(\frac{20}{19}, \frac{45}{19}) \to$

(ii) Min : Z

Extra $x_1 \leq 3$

$2x_1 + 2x_2 = 8$

$6x_1 + 2x_2 = 8$

$4x_1 = 0$

$(x_1, x_2) = (0, 4)$

$x_1 + x_2 = 4$ (4,0)(0,4)

$6x_1 + 2x_2 = 8$ $(\frac{4}{3}, 0)(0,$

02/02/2023

1Q.(i) Maximize: $Z = 5x_1 + 3x_2$

S.T: $3x_1 + 5x_2 \le 15$ $\quad x_1, x_2 \ge 0$

$5x_1 + 2x_2 \le 10$

$6x_1 + 10x_2 = 30$

$25x_1 + 10x_2 = 50$

$19x_1 = 20$

$x_1 = \left(\frac{20}{19}\right)$

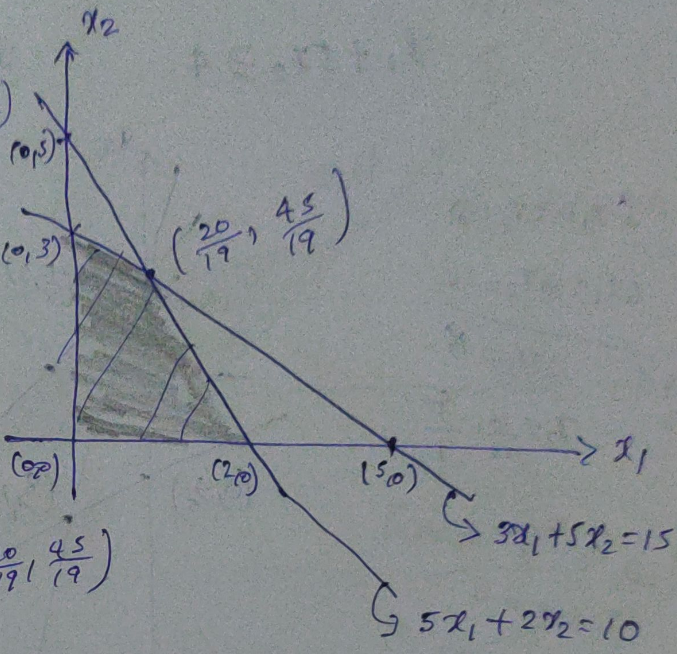
$5x_2 = 15 - 3\left(\frac{20}{19}\right)$

$5x_2 = \frac{225}{19}$

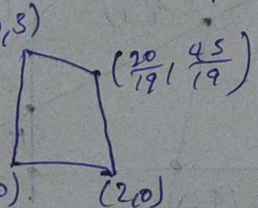
$x_2 = \left(\frac{45}{19}\right)$

$3x_1 + 5x_2 = 15 \rightarrow (5,0)\ (0,3)$

$5x_1 + 2x_2 = 10 \rightarrow (2,0)\ (0,5)$




Feasible region:




$\left(\frac{20}{19}, \frac{45}{19}\right)$

| | Z |
|---|---|
| $(0,0) \rightarrow$ | 0 |
| $(0,3) \rightarrow$ | 9 |
| $(2,0) \rightarrow$ | 10 ~~max~~ |
| $\left(\frac{20}{19}, \frac{45}{19}\right) \rightarrow$ | $\frac{100}{19} + \frac{135}{19} = \frac{235}{19} = 12.37 \rightarrow$ max |

$\boxed{\therefore \text{Maximum: } Z = 12.37}$

---

(ii) Min: $Z = 2x_1 + 3x_2$ $\quad$ S.T: $x_1 + x_2 \le 4$, $6x_1 + 2x_2 \ge 8$

Extra $\quad x_1 \le 3$, $x_2 \le 3$, $x_1, x_2 \ge 0$

$2x_1 + 2x_2 = 8$

$6x_1 + 2x_2 = 8$

$4x_1 = 0$

$(x_1, x_2) = (0,4)$

$x_1 + x_2 = 4$ $(4,0)(0,4)$ $(3,1)$ $(1,3)$

$6x_1 + 2x_2 = 8$ $\left(\frac{4}{3},0\right)(0,4)$ $\left(\frac{1}{3},3\right)$

$\left(\frac{1}{3},3\right) \rightarrow \frac{2}{3} + 9 = 9.67$

$(1,3) \rightarrow 10$

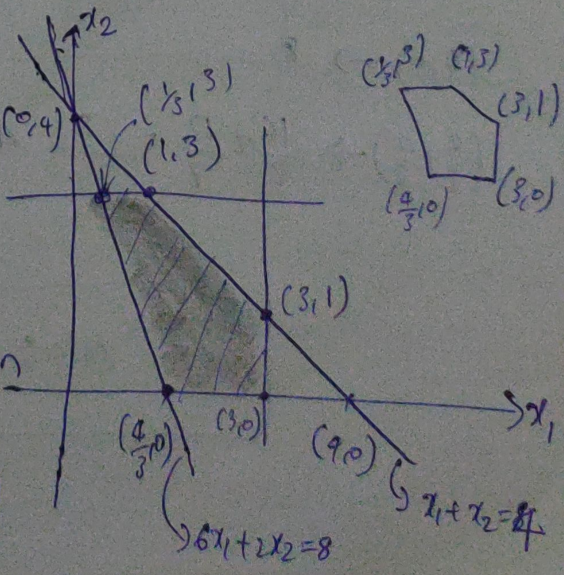
$(3,0) \rightarrow 6$

$(3,1) \rightarrow 9$

$\left(\frac{4}{3},0\right) \rightarrow \frac{8}{3} = 2.67$ $\hookrightarrow$ min

$\left(\frac{4}{3},0\right)$

$\boxed{\therefore \text{Min: } Z = 2.67}$



(ii) Min: $Z = 2x_1 + 3x_2$

S.T: $x_1 + x_2 \leq 4$ $\quad x_1 \leq 3$

$6x_1 + 2x_2 \geq 8$ $\quad x_2 \leq 3$

$x_1 + 5x_2 \geq 4$ $\quad x_1, x_2 \geq 0$

$6x_1 + 2x_2 = 8$

$6x_1 + 30x_2 = 24$

$28x_2 = 16$

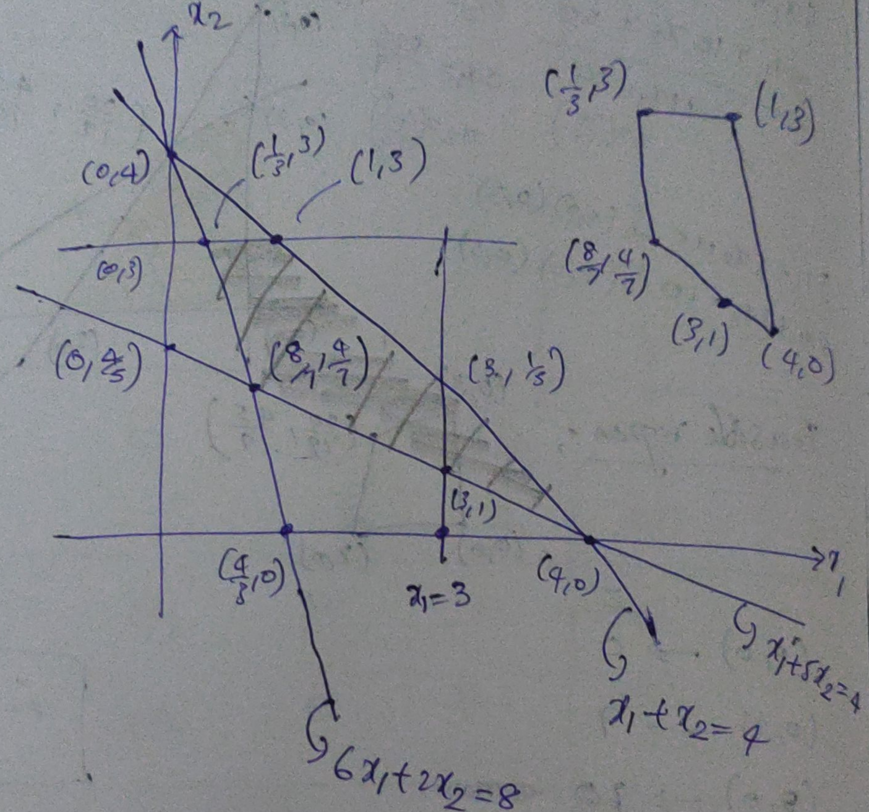
$x_2 = \frac{16}{28} = \frac{4}{7}$

$x_1 + 5x_2 = 4$

$x_1 + \frac{20}{7} = 4$

$x_1 = \frac{28-20}{7}$

$x_1 = 8/7$




$(\frac{1}{3},3) \longrightarrow 9.67$

$(1,3) \longrightarrow 10$

$(3,1) \longrightarrow 9$

$(4,0) \longrightarrow 8$

$(\frac{8}{7},\frac{4}{7}) \rightarrow \frac{16}{7} + \frac{12}{7} = \frac{28}{7} = 4 \rightarrow$ min

$\boxed{\therefore \text{Min: } Z = 4}$ at $(\frac{8}{7}, \frac{4}{7})$

(i) (ii) Max: $Z = 2x_1 + 2x_2$

Extra

S.T: $x_1 - x_2 \geq 1$ $\quad x_1, x_2 \geq 0$

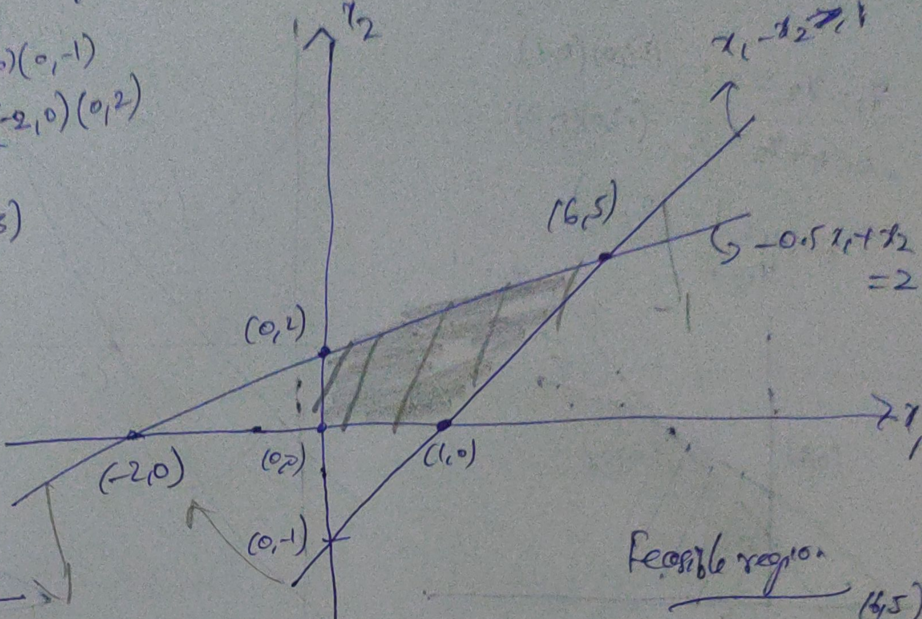
$-0.5x_1 + x_2 \leq 2$

$x_1 - x_2 = 1 \rightarrow (1,0)(0,-1)$

$-0.5x_1 + x_2 = 2 \rightarrow (-2,0)(0,2)$

$0.5x_1 = 3$

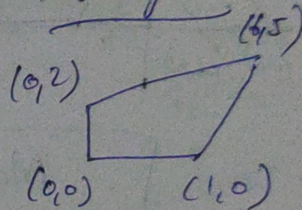
$x_1 = 6 \quad (6,5)$




| | Z |
|---|---|
| $(0,0) \rightarrow$ | 0 |
| $(0,2) \rightarrow$ | 4 |
| $(1,0) \rightarrow$ | 2 |
| $(6,5) \rightarrow$ | 22 $\rightarrow$ max |

$\therefore$ Max: $Z = 22$ at $(6,5)$

Feasible region



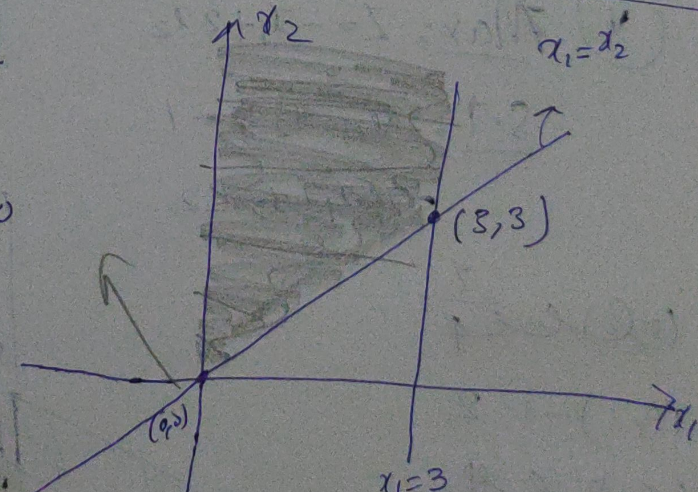

---

(iv) Max: $Z = -3x_1 + 2x_2$

S.T: $x_1 - x_2 \leq 0$

$x_1 \leq 3, \; x_1, x_2 \geq 0$

$x_1 < x_2$




| | Z |
|---|---|
| $(0,0) \rightarrow$ | 0 |
| $(3,3) \rightarrow$ | $-9+6 = -3$ |

(iii) Max: $Z = 2x_1 + 2x_2$

S.T: $x_1 - x_2 \geq -1$

$-0.5x_1 + x_2 \leq 2$

$x_1, x_2 \geq 0$

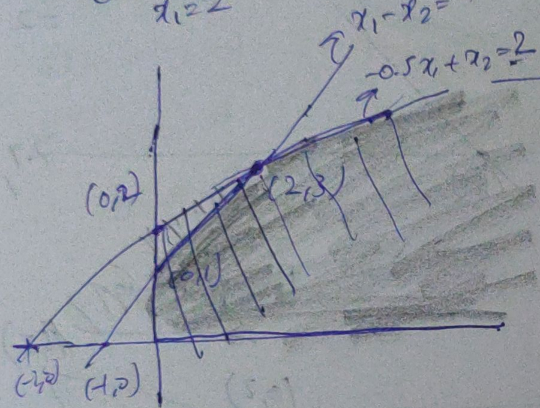
$(0,2) \rightarrow 4$
$(0,1) \rightarrow 2$
$(-2,0) \rightarrow -4$
$(-1,0) \rightarrow -2$
$(2,3) \rightarrow 4+6=10$ ↓ max

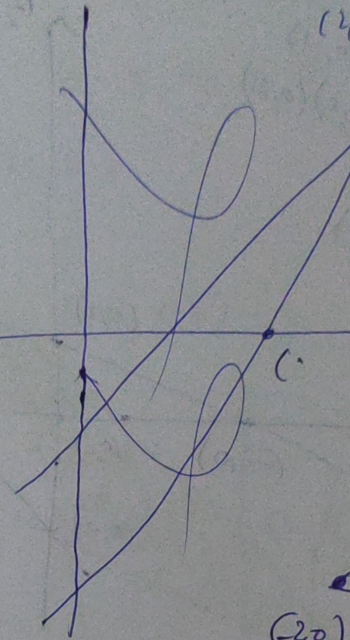
$x_1 - x_2 = -1$ $(-1,0)(0,1)$

$-0.5x_1 + x_2 = 2$ $(-2,0)(0,2)$

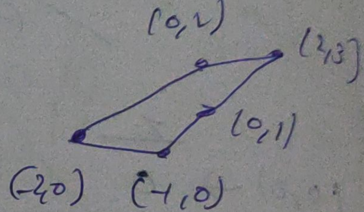
$0.5x_1 = 1$ $(2,3)$

$x_1 = 2$



feasible region




$\boxed{\therefore \text{Max} = 10}$ at (2,3)

---

(v) Max: $Z = -x_1 + 2x_2$ $x_1, x_2 \geq 0$.

S.T: $x_1 - x_2 \geq -1$

$-0.5x_1 + x_2 \leq 2$

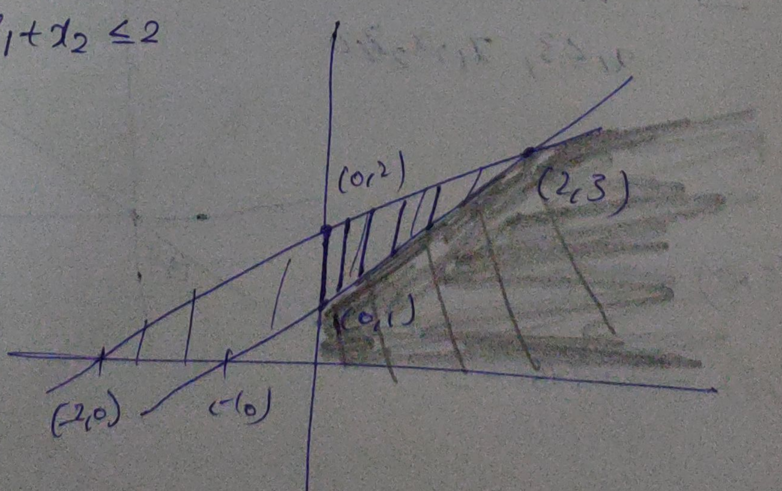
~~(0,2) → 1/4~~

$(0,1) \rightarrow 2$

~~(-2,0) → 4~~

~~(-1,0) → 2~~

$(2,3) \rightarrow -2+6=4$




(vi) Max: Z

S.T: $x_1$

$2x_1$

$2x_1 + x_2 = 2$

$x_1 + x_2 = 1$

. No f

---

(vii) Max:

S.T:

$x_1 - x_2 = 0$

$3x_1 - x_2 = -3$

$-2x_1 = +3$

$-\frac{9}{2} + \frac{3}{2} = \frac{-6}{2} = -3$

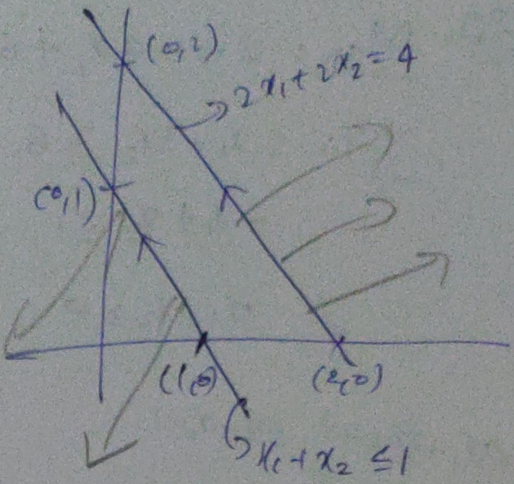
$\left(-\frac{3}{2}, -\frac{3}{2}\right)$

(vi) Max: $Z = 3x_1 - 2x_2$

S.T: $x_1 + x_2 \le 1$

$2x_1 + 2x_2 \ge 4$

$x_1, x_2 \ge 0$




$3x_1 + x_2 = 2$

$x_1 + x_2 = 1$

• No feasible solution

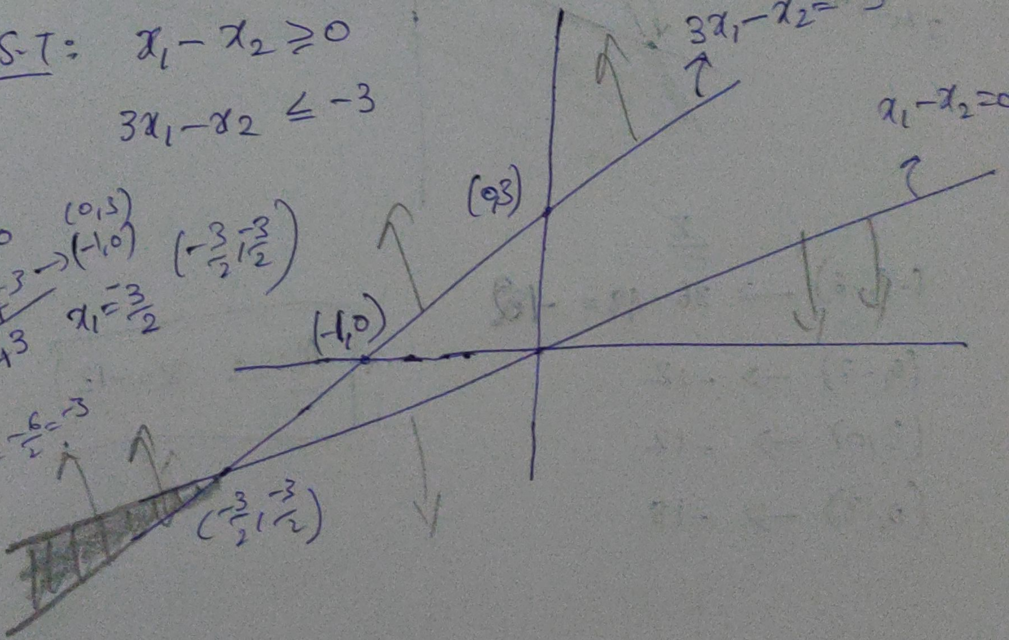
---

(vii) Max: $Z = x_1 + x_2$

S.T: $x_1 - x_2 \ge 0$

$3x_1 - x_2 \le -3$

$x_1, x_2 \ge 0$




(0,3)

$x_1 - x_2 = 0$ → (-1,0) $(-\frac{3}{2}, -\frac{3}{2})$

$3x_1 - x_2 = -3$

$-2x_1 = +3$ $\quad x_1 = \frac{3}{2}$

$-\frac{9}{2} + \frac{3}{2} = -\frac{6}{2} = -3$

$(-\frac{3}{2}, -\frac{3}{2}) \Rightarrow \frac{Z}{-3}$

20. Max: $z = -4x_1 + 6x_2$

S.T: $2x_1 - 3x_2 \geq 6$

$-x_1 + x_2 \leq 1$

$x_1, x_2 \geq 0$

$-3x_1 + 3x_2 = 3$ $(-x_1 + x_2 = 1)$

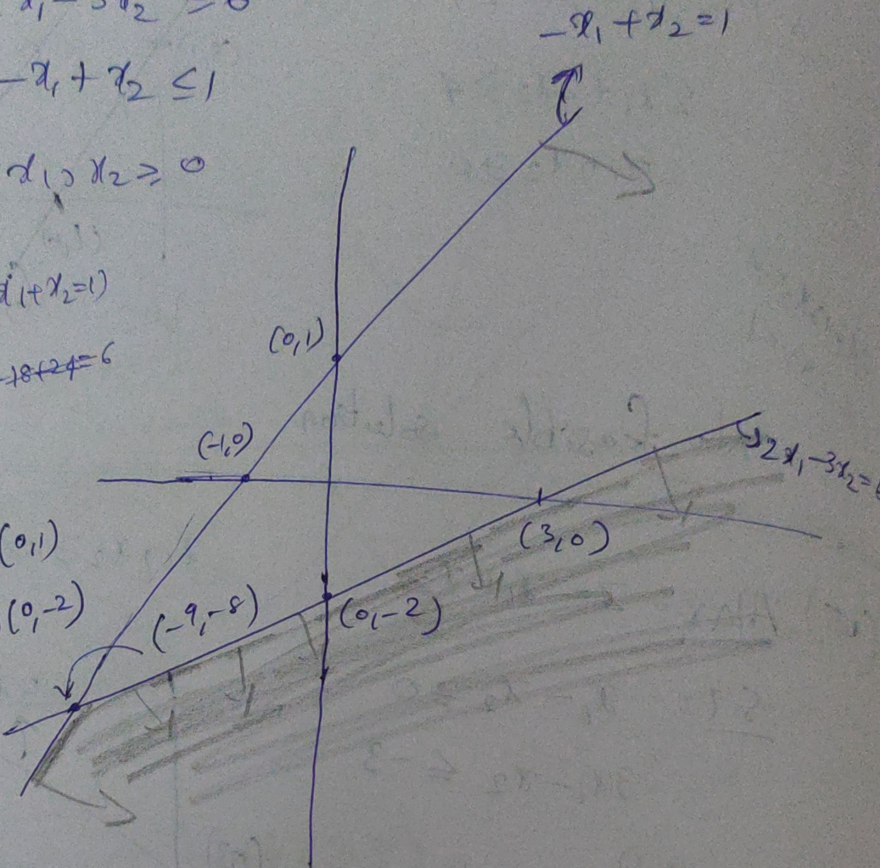
$2x_1 - 3x_2 = 6$ $\cancel{-18+24}=6$

$-x_1 = 9$

$(-9, -8)$

$-x_1 + x_2 = 1$ $(-1,0)(0,1)$

$2x_1 - 3x_2 = 6$ $(3,0)(0,-2)$




Z

$(-9,-8) \longrightarrow 36 - 48 = -12$

$(0,-2) \longrightarrow -12$

$(3,0) \longrightarrow -12$

$(0,-3) \longrightarrow -18$

$\boxed{\text{Max: } z = -12}$

07/02/2022

1. Solution is not altered even after any linear equations is multiplied with a scalar, from a given set of linear equations. $E_1, E_2 \Rightarrow 2E_1, 2E_2$

2. Solution is not altered, even after any equation is replaced by its linear combination with the same set of linear equations. $E_1, E_2 \Rightarrow E_1+E_2, E_2$

Generalizing: (i) $m = n$ → No. of variables
→ No. of equations

$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \rightarrow E_1$

$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \rightarrow E_2$

$\vdots$

$a_{j1}x_1 + a_{j2}x_2 + \dots + a_{j,i-1}x_{i-1} + a_{ji}x_i + a_{j,i+1}x_{i+1} + \dots + a_{jn}x_n = b_j \rightarrow E_j$

$\vdots$

$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \rightarrow E_n$

→ Removing "$x_i$" from all the equations except from one equation, this operation is called as "Pivot Operation"

$$E_1' = E_1 - \frac{a_{1i}}{a_{ji}} E_j$$

$$E_2' = E_2 - \frac{a_{2i}}{a_{ji}} E_j$$

$$\vdots$$

$$E_n' = E_n - \frac{a_{ni}}{a_{ji}} E_j$$

$$\Rightarrow \left\{ \begin{array}{l} a_{11}' x_1 + a_{12}' x_2 + \dots + 0 x_i + \dots + a_{1n}' x_n = b_1' \\ a_{21}' x_1 + a_{22}' x_2 + \dots + 0 x_i + \dots + a_{2n}' x_n = b_2' \\ \vdots \\ a_{j1}' x_1 + a_{j2}' x_2 + \dots + a_{j\,i-1}' x_{i-1} + 1 \cdot x_i + a_{j\,i+1}' x_{i+1} + \dots \\ \qquad \dots + a_{jn}' x_n = b_j' \\ \vdots \\ a_{n1}' x_1 + a_{n2}' x_2 + \dots + 0 x_i + \dots + a_{nn}' x_n = b_n' \end{array} \right.$$

ly $\Rightarrow$

$$\Rightarrow \begin{cases} 1x_1 + 0x_2 + \cdots + 0x_i \cdots + 0x_n = b_1'' \\ 0x_1 + 1x_2 + \cdots + 0x_n = b_2'' \\ \vdots \\ 0x_1 + 0x_2 + \cdots + 1x_i + \cdots + 0x_n = b_i'' \\ \vdots \\ 0x_1 + 0x_2 + \cdots + 1x_n = b_n'' \end{cases}$$

Case 2: $n \geq m$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

Applying Pivot Operation:

$$\Rightarrow \begin{cases} 1x_1 + 0x_2 + \cdots + 0x_m + a^n_{1,m+1}x_{m+1} + \cdots + a^n_{1n}x_n = b_1'' \\ 0x_1 + 1x_2 + \cdots + 0x_m + a^n_{2,m+1}x_{m+1} + \cdots + a^n_{2n}x_n = b_2'' \\ \vdots \\ 0x_1 + 0x_2 + \cdots + 1x_m + a^n_{m,m+1}x_{m+1} + \cdots + a^n_{mn}x_n = b_m'' \end{cases}$$

$\longleftrightarrow$ Pivot variables $\longleftrightarrow$ Non-pivot or independent Variables $\downarrow$ Constant

$$\therefore x_i = \begin{cases} b_i'' , & i = 1, 2, \dots, m \\ 0 , & i = m+1, m+2, \dots, n \end{cases}$$

This solution called as

(Basic Solution) as the solution vector contains no more than 'm' nonzero terms..

→ Pivotal Variables / Basic Variables : $x_i$ ; $i = 1, 2, \dots, m$

→ Non-Basic Variables : $x_i$ ; $i = m+1, m+2, \dots, n$

---

**Problem:** Find all basic feasible solutions (BFSs) corresponding to the given system of linear equations.

$$2x_1 + 3x_2 - 2x_3 - 7x_4 = 1$$

$$x_1 + x_2 + x_3 + 3x_4 = 6$$

$$x_1 - x_2 + x_3 + 5x_4 = 4$$

**Sol:** Case 1:

Basic Variables : $x_1, x_2, x_3$

Non-Basic Variables : $x_4$

(1) $\Rightarrow -x_1 - 5x_3 - 16x_4 = -17 \Rightarrow 4x_1 + 4x_4 = 9$

(2) $\Rightarrow x_2 - x_4 = 1$

(3) $\Rightarrow x_1 + x_3 + 4x_4 = 5 \Rightarrow -4x_3 - 12x_4 = -12$

$x_3 + 3x_4 = 3$

$$\Rightarrow \begin{cases} x_1 + x_4 = 2 \\ x_2 - x_4 = 1 \\ x_3 + 3x_4 = 3 \end{cases} \qquad (x_1, x_2, x_3, x_4) = (2, 1, 3, 0)$$

# Basic solution assumes zero value for non basic variables.

# $f = C_1 x_1 + C_2 x_2 + \dots + C_n x_n$

$\Rightarrow f_0' = C_1 x_1 + C_2 x_2 + \dots \dots + C_n x_n - f$

$0 x_1 + 0 x_2 + \dots + 0 x_m - f + C_{m+1}'' x_{m+1} + \dots + C_{mn}'' x_n = -f_0''$

# Feasible solution is the solution which satisfies all given constraint

$\rightarrow$ BFS 1 $\Rightarrow x_4 = 0$ i.e. Assuming $x_4$ as Non basic variable

BFS 2 $\Rightarrow x_1 = 0$ i.e. " $x_1$ " "

BFS 3 $\Rightarrow x_2 = 0$ i.e. " $x_2$ " "

BFS 4 $\Rightarrow x_3 = 0$ i.e. " $x_3$ " "

- No. of BFSns : ${}^nC_m$

**Case 3 :** Basic Variable : $x_1, x_3, x_4$

Non-basic Variable : $x_2$

(1) $\Rightarrow 4x_1 + 5x_2 - x_4 = 13 \Rightarrow 4x_1 + 4x_2 = 12$

(2) $\Rightarrow x_2 - x_4 = 1 \Rightarrow x_2 - x_4 = 1$

(3) $\Rightarrow -5x_2 + 4x_3 + 17x_4 = 7 \Rightarrow 12x_2 + 4x_3 = 7 + 17 \times 1 = 24$

$$\begin{cases} x_1 + x_2 = 3 \\ x_2 - x_4 = 1 \\ 6x_2 + 2x_3 = 12 \end{cases}$$

| BFS | I | II | III | IV |
|---|---|---|---|---|
| $x_1$ | 2 | 0 | 3 | 1 |
| $x_2$ | 1 | 3 | 0 | 2 |
| $x_3$ | 3 | $-3$ | 6.0 | 0 |
| $x_4$ | 0 | 2 | $-1$ | 1 |

→ II & III solutions are not considered as $x_i \geq 0$, $i = 1,2,3,4$

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | RHS | |
|---|---|---|---|---|---|---|
| | 2 | 3 | $-2$ | $-7$ | 1 | $R_1$ |
| | 1 | 1 | 1 | 3 | 6 | $R_2$ |
| | 1 | $-1$ | 1 | 5 | 4 | $R_3$ |
| Pivot operation 1 | 1 | $3/2$ | $-1$ | $-7/2$ | $1/2$ | $R_1' \to R_1/2$ |
| | 0 | $1-\frac{3}{2}=\frac{-1}{2}$ | $1+1=2$ | $3+\frac{7}{2}=\frac{13}{2}$ | $6-\frac{1}{2}=\frac{11}{2}$ | $R_2' \to R_2 - R_1'$ |
| (Pivotal element is $x_1$) | 0 | $-1-\frac{3}{2}=\frac{-5}{2}$ | $1+1=2$ | $5+\frac{7}{2}=\frac{17}{2}$ | $4-\frac{1}{2}=\frac{7}{2}$ | $R_3' \to R_3 - R_1'$ |
| Pivot operation 2 | 1 | 0 | $-1+6=5$ | $\frac{-7}{2}=16$ | | $R_1'' \to R_1' - \frac{3}{2}R_2''$ |
| | 0 | 1 | $-4$ | $-13$ | $-11$ | $R_2'' \to R_2' \times -2$ |
| | | | | | | $R_3''$ |

Ex 3.3 Find all the basic solutions corresponding to the system of linear equations

$$2x_1 + 3x_2 - 2x_3 - 7x_4 = 1$$

$$x_1 + x_2 + x_3 + 3x_4 = 6$$

$$x_1 - x_2 + x_3 + 5x_4 = 4.$$

| | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | R.H.S | |
|---|---|---|---|---|---|---|---|
| | $x_1$ | 2 | 3 | $-2$ | $-7$ | 1 | $\rightarrow R1$ |
| | $x_2$ | 1 | 1 | 1 | 3 | 6 | $\rightarrow R2$ |
| | $x_3$ | 1 | $-1$ | 1 | 5 | 4 | $\rightarrow R3$ |
| | | 1 | $3/2$ | $-1$ | $-7/2$ | $1/2$ | $\rightarrow R_1/2 = R_1'$ |
| | | 0 | $-1/2$ | 2 | $13/2$ | $11/2$ | $\rightarrow R_2 - R_1/2 = R_2'$ |
| | | 0 | $-5/2$ | 2 | $17/2$ | $7/2$ | $\rightarrow R_3 - R_1/2 = R_3'$ |
| | | 1 | 0 | 5 | 16 | 17 | $\rightarrow R_1'' = R_1' - \frac{3}{2}R_2''$ |
| | | 0 | $+1$ | $-4$ | $-13$ | $-11$ | $\rightarrow R_2'' = R_2' * (-2)$ |
| | | 0 | 0 | $-8$ | $-24$ | $-24$ | $\rightarrow R_3'' = R_3' + \frac{5}{2}R_2''$ |
| | $x_1$ | 1 | 0 | 0 | 1 | 2 | $\rightarrow R_1''' = R_1'' - 5R_3'''$ |
| | $x_2$ | 0 | 1 | 0 | $-1$ | $+1$ | $\rightarrow R_2''' = R_2'' + 4R_3'''$ |
| If $x_4 = 0$ then the BFS is $[x_1 = 2; x_2 = 1; x_3 = 3]$ | $x_3$ | 0 | 0 | 1 | 3 | $+3$ | $\rightarrow R_3''' = R_3'/(-8)$ |
| | $x_1$ | 1 | 0 | $-1/3$ | 0 | 1 | $\rightarrow R_1^{IV} = R_1''' - R_3^{IV}$ |
| If $x_3 = 0$ then $[1, 2, 1]$ is the BFS | $x_2$ | 0 | 1 | $1/3$ | 0 | 2 | $\rightarrow R_2^{IV} = R_2''' + R_3^{IV}$ |
| | $x_4$ | 0 | 0 | $1/3$ | 1 | 1 | $\rightarrow R_3^{IV} = R_3'''/3$ |
| | $x_1$ | 1 | 1 | 0 | 0 | 3 | $\rightarrow R_1^{V} = R_1^{IV} + \frac{1}{3}R_2^{V}$ |
| if $x_2 = 0$ then $[3, 6, -1]$ is BS but not feasible. | $x_3$ | 0 | 3 | 1 | 0 | 6 | $\rightarrow R_2^{V} = 3R_2^{IV}$ |
| | $x_4$ | 0 | $-1$ | 0 | 1 | $-1$ | $\rightarrow R_3^{V} = R_3^{IV} - \frac{1}{3}R_2^{V}$ |

Problem: Maximize : $Z = 2x + 5y$

Use Simplex Algorithm Tabular Method

S.T : $2x + y \le 5$

$x + 2y \le 4$

$x \ge 0$

$y \ge 0$

Primal LPP means Original Problem

& Dual LPP

$\boxed{\text{Dual LPP:}}$ $\boxed{Z' = 5y_1 + 4y_2 \quad \text{S.T: } 2y_1 + y_2 \ge 2 \ \& \ y_1 + 2y_2 \ge 5}$

Sol $Z = 2x + 5y + 0S_1 + 0S_2$

$2x + y + S_1 = 5 \qquad S_1 \ge 0$

$x + 2y + S_2 = 4 \qquad S_2 \ge 0$

| | | 2 | 5 | 0 | 0 | | |
|---|---|---|---|---|---|---|---|
| | | $x$ | $y$ | $S_1$ | $S_2$ | RHS | $\theta$ |
| 0 | $S_1$ | 2 | 1 | 1 | 0 | 5 | 5 |
| 0 | $S_2$ | 1 | 2 | 0 | 1 | 4 | 2 → leave |
| | | | ↑ enter | | | | |
| | $C_j - Z_j$ | 2 | 5 | 0 | 0 | 0 | |
| 0 | $S_1$ | $3/2$ | 0 | 1 | $-1/2$ | 3 | |
| 5 | $y$ | $1/2$ | 1 | 0 | $1/2$ | 2 | |
| | $C_j - Z_j$ | $2 - \frac{5}{2} = -\frac{1}{2}$ | 0 | 0 | $-5/2$ | 10 | |

$\boxed{\therefore \text{ Maximum: } Z = 10}$ at $(0, 2)$ $(x, y)$

$\left\{ \begin{array}{l} x = 0 \\ y = 2 \\ S_1 = 3 \\ S_2 = 0 \end{array} \right.$

Problem : Maximize : $Z = 4x + 5y$    Use :- Two Phase Method

S.T : $2x + 3y \le 6$

$3x + y \ge 3$    $x, y \ge 0$

Sf  S.T : $2x + 3y + S_1 = 6$ ; $S_1 \ge 0$

$3x + y - S_2 + a_1 = 3$ ; $S_2, a_1 \ge 0$

Phase1

| $C_j$ | | 0 | 0 | 0 | 0 | -1 | | |
|---|---|---|---|---|---|---|---|---|
| | | $x$ | $y$ | $S_1$ | $S_2$ | $a_1$ | RHS | $\theta$ |
| 0 | $S_1$ | 2 | 3 | 1 | 0 | 0 | 6 | 3 |
| -1 | $a_1$ | 3 | 1 | 0 | -1 | 1 | 3 | 1 → leave |
| $C_j - Z_j$ : | | 3 ↑ | 1 | 0 | -1 | 0 | -3 | |
| 0 | $S_1$ | 0 | $3 - \frac{2}{3} = \frac{7}{3}$ | 1 | $\frac{2}{3}$ | $-\frac{2}{3}$ | 4 | |
| 0 | $x$ | 1 | $\frac{1}{3}$ | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 1 | |
| $C_j - Z_j$ : | | 0 | 0 | 0 | 0 | 0 | | |

Phase2

| | | 4 | 5 | 0 | 0 | | |
|---|---|---|---|---|---|---|---|
| | | $x$ | $y$ | $S_1$ | $S_2$ | RHS | $\theta$ |
| 0 | $S_1$ | 0 | $\frac{7}{3}$ | 1 | $\frac{2}{3}$ | 4 | $4 \times \frac{3}{7} = \frac{12}{3}$ → leave |
| 4 | $x_1$ | 1 | $\frac{1}{3}$ ↑ | 0 | $-\frac{1}{3}$ | 1 | 3 |
| $C_j - Z_j$ : | | 0 | $5 - \frac{4}{3} = \frac{11}{3}$ | 0 | $\frac{4}{3}$ | 4 | |
| 5 | $y$ | 0 | 1 | $\frac{3}{7}$ | $\frac{2}{7}$ | $\frac{12}{7}$ | 6 → leave |
| 4 | $x$ | 1 | 0 | $-\frac{1}{7}$ | $-\frac{1}{3} \cdot \frac{2}{21} = \frac{-3}{7}$ | $1 - \frac{4}{7} = \frac{3}{7}$ | -1 |
| $C_j - Z_j$ : | | 0 | 0 | $\frac{-15}{7} + \frac{4}{7} = -\frac{11}{7}$ | $\frac{-10}{7} + \frac{12}{7} = \frac{2}{7}$ $= \frac{2}{7}$ ↑ Enter | | |

| | | 4 | 5 | 0 | 0 | |
|---|---|---|---|---|---|---|
| | | $x$ | $y$ | $S_1$ | $S_2$ | RHS |
| 0 | $S_2$ | 0 | $7/2$ | $3/2$ | 1 | 6 |
| 4 | $X$ | 1 | $3/2$ | $\frac{-1}{7}+\frac{9}{14}=\frac{1}{2}$ | 0 | $\frac{3}{7}+\frac{18}{7}=3$ |
| | $C_j - Z_j$: | 0 | $\frac{5-12}{2}=-1$ | $-2$ | 0 | 12 |

$\boxed{\therefore \text{Maximum}: Z = 12}$ at $(x,y) = (3,0)$ $\left(\begin{array}{l} x=3 \\ y=0 \\ S_1=0 \\ S_2=6 \end{array}\right)$

09/02/2022

**Problem**: Maximize: $Z = 3x + 2y$

S.T : $2x + y \le 4$

$x + 2y \ge 9$, $x \ge 0$, $y \ge 0$.

Sol:

Maximize : $Z = 3x + 2y + 0S_1 - 0S_2 - \cancel{M} a_1$

S.T: $2x + y + S_1 = 4$ ; $S_1 \ge 0$

$x + 2y - S_2 + a_1 = 9$ ; $S_2 \ge 0$, $a_1 \ge 0$

(i) 2-Phase Method

(ii) Big-M Method

(iii) Graphical Method

Phase1:

| | | 0 | 0 | 0 | 0 | -1 | | |
|---|---|---|---|---|---|---|---|---|
| | | x | y | $S_1$ | $S_2$ | $a_1$ | RHS | $\theta$ |
| 0 | S1 | 2 | 1 | 1 | 0 | 0 | 4 | 4 → |
| -1 | $a_1$ | 1 | 2 | 0 | -1 | 1 | 9 | $\frac{9}{2}=4.5$ |
| $C_j-Z_j$ | | 1 | 2 ↑ | 0 | -1 | $-1+1=0$ | | |
| 0 | Y | 2 | 1 | 1 | 0 | 0 | 4 | 4 → |
| -1 | $a_1$ | -3 | 0 | -2 | -1 | 1 | $9-8=1$ | -0.5 |
| $C_j-Z_j$ | | -3 | 0 | -2 | -1 | $-1+1=0$ | | |

Termination step

$S_1$

$a_1$

$C_j-Z_j$

Phase-2:

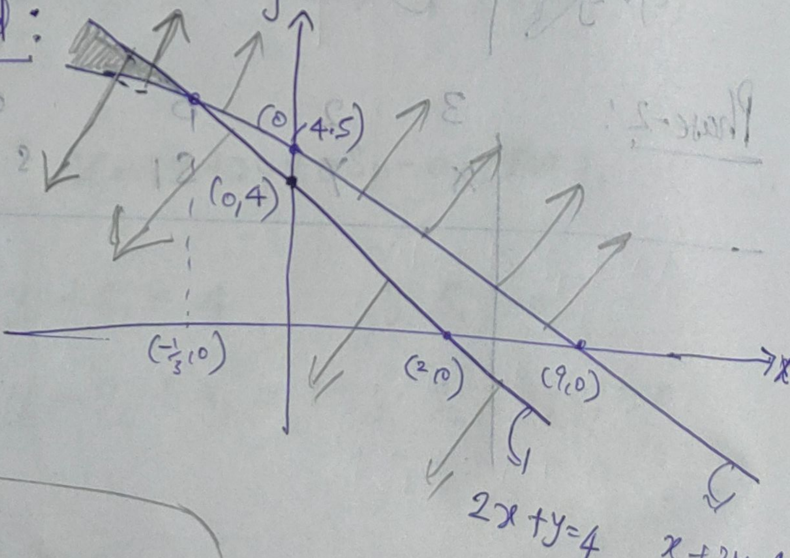
| | 3 | 2 | 0 | 0 |
|---|---|---|---|---|
| | x | Y | S1 | S2 |

## Big-M Method:

| | | 3 | 2 | 0 | 0 | -M | | |
|---|---|---|---|---|---|---|---|---|
| | | x | y | $S_1$ | $S_2$ | $a_1$ | RHS | θ |
| 0 | S1 | 2 | 1 | 1 | 0 | 0 | 4 | 4 → |
| -M | a1 | 1 | 2 | 0 | -1 | 1 | 9 | $\frac{9}{2}=4.5$ |
| $C_j - Z_j$ | | 3+M | 2+2M ↑ | 0 | -M | 0 | | |
| 2 | y | 2 | 1 | 1 | 0 | 0 | 4 | |
| -M | $a_1$ | -3 | 0 | -2 | -1 | 1 | 9-8=1 | |
| $C_j - Z_j$ | | 3-3M -4 = -1-3M | 0 | 2-2M | -M | 0 | | |

Termination step

## Graphical Method:




No feasible Solution

# Duality in LPP

- Every LPP is associated with another LPP called the Dual LPP.
- The original is called as the Primal LPP.

## In Dual Representation:

→ No. of decision variables = No. of constraints in Primal

→ Objective function becomes Maximum, if in Primal it is Minimum and viceversa.

→ No. of Constraints = No. of Decision variables in Primal.

→ The problem has (≥) type constraints, if in Primal it has (≤) type and vice versa.

→ RHS constants of the constraints are the objective coefficients in the Primal and viceversa function.

→ Coefficients of constraints of original/Primal problem which appear from left to right are placed from top to bottom in the constraints of the dual problem and vice versa.

# → If the $i^{th}$ primal variable is unrestricted in sign, the the $j^{th}$ dual constraint is = type & viceversa.

**Problem: Primal:**

Max: $Z = 6x_1 + 5x_2$

S.T: $x_1 + x_2 \le 5$

$3x_1 + 2x_2 \le 12$

$x_1, x_2 \ge 0$

**Dual Representation:**

Min: $W = 5y_1 + 12y_2$

S.T: $y_1 + 3y_2 \ge 6$

$y_1 + 2y_2 \ge 5$

$y_1, y_2 \ge 0$

**Problem Primal:**

Max: $Z = -5x_1 + 2x_2$

S.T: $-x_1 + x_2 \le -2$

$2x_1 + 3x_2 \le 5$

$x_1, x_2 \ge 0$

Sol: Max: $Z = -5x_1 + 2x_2$

S.T: $x_1 - x_2 \ge 2$

$2x_1 + 3x_2 \le 5$

$x_1, x_2 \ge 0$

**Dual:** (Refer Table)

Min: $W = 2y_1 + 5y_2$

S.T: $y_1 + 2y_2 \ge -5$

$-y_1 + 3y_2 \ge 2$

$y_1 \le 0, \; y_2 \ge 0$

**Problem: Primal:**

Max: $Z = 3x_1 + 2x_2$

S.T: $3x_1 + 4x_2 \le 12$

$3x_1 + 2x_2 \le 16$

$x_1 + x_2 \le 3$

$x_1, x_2 \ge 0$

**Dual:**

Min: $W = 12y_1 + 16y_2 + 3y_3$

$3y_1 + 3y_2 + 0y_3 \ge 3$

$4y_1 + 2y_2 + y_3 \ge 2$

$y_1, y_2, y_3 \ge 0$

Problem Primal:

$$\text{Max}: Z = -5x_1 + 2x_2$$

$$\text{S.T}: -x_1 + x_2 \le -2$$

$$2x_1 + 3x_2 \le 5$$

$$x_1, x_2 \ge 0$$

Sol: $\text{Max}: Z = -5x_1 + 2x_2$

$\text{S.T}: x_1 - x_2 \ge 2$

$x_1, x_2 \ge 0$ $\quad 2x_1 + 3x_2 \le 5$

Dual: (Refer Table)

$$\text{Min}: W = 2y_1 + 5y_2$$

$$\text{S.T}: y_1 + 2y_2 \ge -5$$

$$-y_1 + 3y_2 \ge 2$$

$$y_1 \le 0, \; y_2 \ge 0$$

---

al:

$$W = 12y_1 + 16y_2 + 3y_3$$

$$3y_1 + 3y_2 + 0y_3 \ge 3$$

$$4y_1 + 2y_2 + y_3 \ge 2$$

$$y_1, y_2, y_3 \ge 0$$

---

Problem: Primal

$$\text{Max}: Z = x_1 + x_2$$

$$\text{S.T}: 2x_1 + x_2 = 5$$

$$3x_1 - x_2 = 6$$

$x_1, x_2$ (unrestricted) (urs)

Dual

$$\text{Min}: W = 5y_1 + 6y_2$$

$$\text{S.T}: 2y_1 + 3y_2 = 1$$

$$y_1 - y_2 = 1$$

$y_1, y_2$ (urs)

---

Problem: Primal

$$\text{Max}: Z = x_1 - 2x_2 + 3x_3$$

$$\text{S.T}: -2x_1 + x_2 + 3x_3 = 2$$

$$2x_1 + 3x_2 + 4x_3 = 1$$

$$x_1, x_2, x_3 \ge 0$$

Dual

$$\text{Min}: W = 2y_1 + y_2$$

$$-2y_1 + 2y_2 \ge 1$$

$$y_1 + 3y_2 \ge -2$$

$$3y_1 + 4y_3 \ge 3$$

$y_1, y_2$ (urs)

---

| | Primal | Dual | |
|---|---|---|---|
| Constraints | >= | <= | Variables |
| | <= | >= | |
| | = | unrestricted | |
| Variables | >= | >= | Constraints |
| | <= | <= | |
| | unrestricted | = | |
| Objective function | Max | Min | objective functio |

Maximize: $Z = 4x_1 + 3x_2$ 10/02/2023

S.T: $2x_1 + 3x_2 \le 8$

$3x_1 + 2x_2 \le 12$

$x_1, x_2 \ge 0$

$Z = 4x_1 + 3x_2 + 0S_1 + 0S_2$

$2x_1 + 3x_2 + S_1 = 8$ ; $S_1 \ge 0$, $x_1 \ge 0$, $x_2 \ge 0$

$3x_1 + 2x_2 + S_2 = 12$ ; $S_2 \ge 0$, $x_2 \ge 0$, $x_2 \ge 0$

| | 4 $x_1$ | 3 $x_2$ | 0 $S_1$ | 0 $S_2$ | RHS | $\theta$ |
|---|---|---|---|---|---|---|
| 0 $S_1$ | 2 | 3 | 1 | 0 | 8 | $\frac{8}{2} = 4$ → |
| 0 $S_2$ | 3 ↑ | 2 | 0 | 1 | 12 | $\frac{12}{3} = 4$ |
| $C_j - Z_j$ | 4 | 3 | 0 | 0 | | |
| 4 $x_1$ | 1 | $3/2$ | $1/2$ | 0 | 4 | |
| 0 $S_2$ | 0 | $2 - \frac{9}{2} = -\frac{5}{2}$ | $-3/2$ | 1 | $12 - 12 = 0$ | |
| $C_j - Z_j$ | 0 | $3 - \frac{12}{2} = -3$ | $-2$ | 0 | 16 | |

Solution: $x_1 = 4$, $x_2 = 0$, $S_1 = 0$, $S_2 = 0$

Case(i) $Z = 16$

Solution: $x_1 = 4$, $x_2 = 0$, $S_1 =$ [illegible], $S_2 = 0$

Case(ii)

| | 4 | 3 | 0 | 0 | | |
|---|---|---|---|---|---|---|
| | $x_1$ | $x_2$ | $S_1$ | $S_2$ | RHS | $\theta$ |
| 0 $S_1$ | 2 | 3 | 1 | 0 | 8 | 4 |
| 0 $S_2$ | 3 | 2 | 0 | 1 | 12 | 4 $\rightarrow$ |
| $C_j - Z_j$ | 4 $\uparrow$ | 3 | 0 | 0 | | |
| 0 $S_1$ | 0 | $3-\frac{4}{3}=\frac{5}{3}$ | 1 | $-2/3$ | $8-4\times2=0$ | 0 leave $\rightarrow$ |
| 4 $x_1$ | 1 | $2/3$ $\uparrow$ | 0 | $1/3$ | $\frac{12}{3}=4$ | 6 |
| $C_j - Z_j$ | 0 | $3-\frac{8}{3}=\frac{1}{3}$ | 0 | $\frac{-4}{3}$ | 16 | |
| ~~0 $S_1$~~ | ~~$-\frac{3}{2}\times\frac{5}{3}=-\frac{5}{2}$~~ | ~~0~~ | ~~1~~ | ~~$-\frac{2}{3}-\frac{5}{3}\times\frac{1}{2}=-3/2$~~ | ~~$0-\frac{6\times5}{3}=-10$~~ | |
| ~~3 $x_2$~~ | ~~$3/2$~~ | ~~1~~ | ~~0~~ | ~~$1/2$~~ | ~~$4\times\frac{3}{2}=6$~~ | |
| ~~$C_j - Z_j$~~ | ~~$4-\frac{9}{2}=-\frac{1}{2}$~~ | ~~0~~ | ~~0~~ | ~~$-3/2$~~ | ~~18~~ | |
| 3 $x_2$ | 0 | 1 | $3/5$ | $-2/5$ | 0 | |
| 4 $x_1$ | 1 | 0 | $-2/5$ | $3/5$ | 4 | |
| $C_j - Z_j$ | 0 | 0 | $-1/5$ | $-6/5$ | 16 | |

# Degeneracy

Infimum

Book : by SS Rao (only Theory)

- 1.4, 1.5, 1.6
- 2 (rough idea)
- # 3 ~~4~~
- 4.3

? Why in Simplex algorithm ~~Δ~~ we consider ... and min. value in θ

Minor-1

- Image processing
- Signal processing
- VLSI
- Digital
- Bio - Signal Proc

Course Project

Abstract - Minor-1

problem solution - Minor-2

? OT problem related to EC in real time

Contact : faculty

# Unconstrained Optimization :

Minimize : $f(x)$

$x \in$ Field

$F \subset R^n$

(feasible region) (n-dimensional vector space)

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad f : R^n \longrightarrow R$$

(1-dimensional space)

- Global minimum, Strict global min., Local min, Strict local min.

→ A $x \in F$ then $f(x)$ is said to be Global minimum if

$f(x) \leq f(y)$ ; $\forall y \in F$

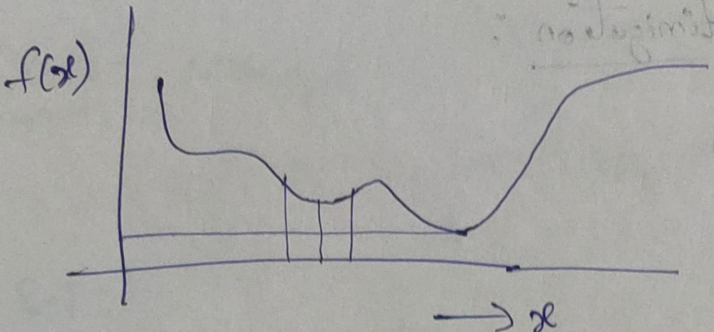
→ A $x \in F$, then $f(x)$ is said to be Strictly global minimum

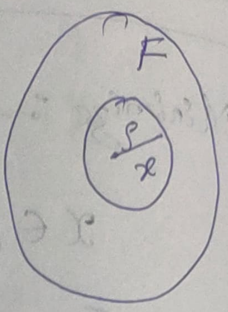
if $f(x) < f(y)$ ; $\forall y \in F$ and $y \neq x$

→ A $x \in F$, then $f(x)$ is said to be Local Minimum

$\rho > 0$, $f(x) \leq f(y)$, $\forall y \in F$ and $\|y - x\|_2 < \rho$

- Local minimum defined for a given interval.



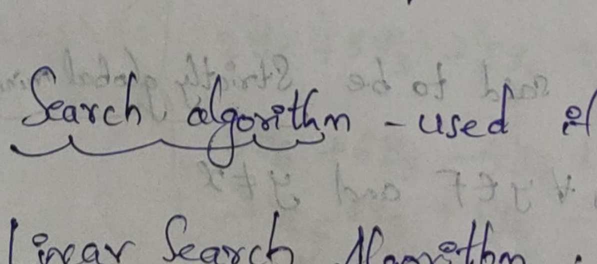




→ A $x \in F$, $f(x)$ is said to be strictly local minimum if $\rho > 0$, $f(x) < f(y)$ ; $\forall y \in F$, $y \neq x$ and $\|y-x\| < \rho$

## Unimodal function :

(it keeps on) (increasing) (or) (keeps on decreasing)

- Either only increase or only decrease.




— Unimodal functions

- Search algorithm - used if $f(x)$ is a unimodal function.

## Linear Search Algorithm :

$x$ $f(x)$ (minimize)

① Initially, Guess $x_1$

② Step size $S$

$x_2 = x_1 + S$ ; $f(x_1)$ $f(x_2)$

Case1 if $f(x_1) < f(x_2)$ $\Rightarrow$ ~~min~~ available from $(-\infty, x_1)$

$\hookrightarrow$ search region is $(-\infty, x_1)$

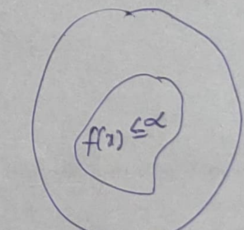
Case2 if $f(x_1) > f(x_2)$ $\hookrightarrow$ $x_3 = x_1 - s$,

$\hookrightarrow$ $x_3 = x_2 + s$

$x_3 = x_1 + 2s$

- The above method for 1-dimension
- For 'n'-dimension, as it is difficult to calculate objective function for each & every problem, we go for progressive step-size.
- Compact set - means closed region

★ Level set :- $L(\alpha) = \{x : f(x) \leq \alpha\}$

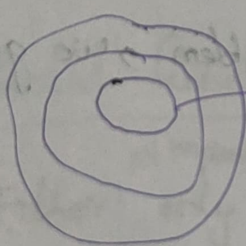



- Derivative of a function gives ~~as~~ the rate at which function is growing.
- More slope $\rightarrow$ high rate
- Gradient : $\vec{\nabla} f = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$
- Divergence : $\vec{\nabla} \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$ ; $\vec{F} = F_x \hat{a}_x + F_y \hat{a}_y + F_z \hat{a}_z$
- Curl : $\vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$

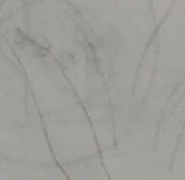
↳ it is a local property.

→ <u>Gradient</u> gives us the direction of rate at which function is growing "rapidly" and its magnitude gives the rate value, it's always ⊥r to the surface i.e. tangent

- Gradient calculated for scalar fields.
- Gradient gives a vector as o/p.
- Contour - Each contour represents different elevations




$(f = x e^{-x^2-y^2})$




$\vec{\nabla} f \cdot \vec{d}$ ← Directional derivative

$$\boxed{\vec{\nabla} f^T \cdot \vec{d}}$$

($\vec{\nabla} f$ - column vector)

- Rate at which 'f' growing in specified ($\vec{d}$) direction

9/03/2023

Ex: $f(x,y) = xy^2 + x^3y$

Find (i) $\vec{\nabla} f$

Sol: $\vec{\nabla}.f = i\frac{\partial}{\partial x}f + j\frac{\partial f}{\partial y}$ ~~$+ k\frac{\partial f}{\partial z}$~~

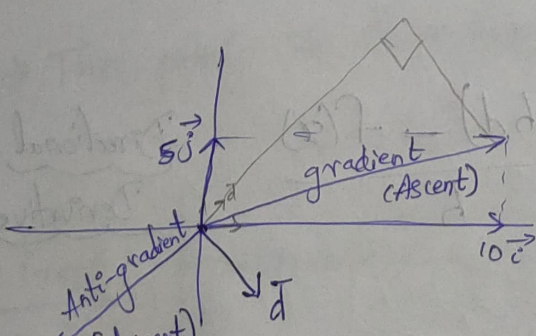
$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}\right)(xy^2 + x^3y)$

$= \vec{i}(y^2 + 3x^2y) + \vec{j}(2xy + x^3)$

At $(1,2)$ $\vec{\nabla}f(x,y) = ?$

$\vec{\nabla}f(1,2) = (4+6)\vec{i} + (4+1)\vec{j}$

$= 10\vec{i} + 5\vec{j}$




$\vec{d} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$

- Gradient gives direction of point at which its growing rapidly. (with max. rate)
- Anti-gradient - The direction at which point descending rapidly. (max. rate).

Directional Derivative: $\vec{\nabla}f.\vec{d}$

$(10\vec{i} + 5\vec{j})\left(\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}\right) = \frac{10}{\sqrt{2}} + \frac{5}{\sqrt{2}} = \frac{15}{\sqrt{2}}$

* rate at which $(1,2)$ growing in $\vec{d}$

. Usually, vectors representation as column vectors

$$\vec{\nabla} f = \begin{bmatrix} \frac{\partial}{\partial x} f \\ \\ \frac{\partial}{\partial y} f \end{bmatrix} = \begin{bmatrix} y^2 + 3x^2 y \\ \\ 2xy + x^3 \end{bmatrix} \qquad \vec{d} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\vec{\nabla} f_{(1,2)}^{T} \cdot \vec{d} = \begin{bmatrix} 10 & 5 \end{bmatrix}_{1\times 2} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \\ \frac{1}{\sqrt{2}} \end{bmatrix}_{2\times 1} = \frac{15}{\sqrt{2}}$$

- $$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ Definition of Derivative

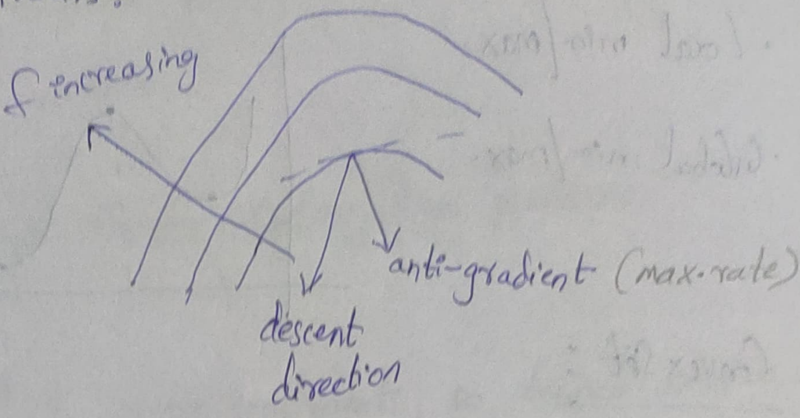
$$\vec{\nabla} f' \cdot \vec{d} = \lim_{h \to 0} \frac{f(x + hd) - f(x)}{h}$$ Directional Derivative

→ Let $f : \mathbb{R}^n \to \mathbb{R}$. A vector $d \in \mathbb{R}^n$ is said to be a descent direction for $f$ in $x$, if $\exists\ \delta > 0$ such that

$f(x_0 + \lambda d) < f(x_0)$ ; ($\lambda$ - constant, $\vec{d} \in \mathbb{R}^n$)

- If $\nabla f$ exists & is continuous, if $\vec{\nabla} f(x_0)' \vec{d} < 0$ then direction $d$ is a descent direction for $f$ at $x_0$.

Equi ampl. countours:




→ The gradient $\nabla f(x_0)$ is a direction orthogonal ($\perp$) to the level surface and it is a direction of ascent and hence the anti-gradient $-\nabla f(x_0)$ is a descent direction.

$$\cos\theta = \frac{a \cdot b}{\|a\|\,\|b\|}$$

→ The point $x_0$ is a local minimum of $f$, iff

$$\vec{\nabla} f(x_0) = 0$$

(when $\nabla f$ exists)

Proof: (by contradiction) $\vec{\nabla} f(x_0) \neq 0$

(Anti-gradient) $\vec{d} = -\vec{\nabla} f(x_0)$

. if $x_0$ is not a local minimum point then

$$f(x_0 + \lambda \vec{d}) < f(x_0)$$

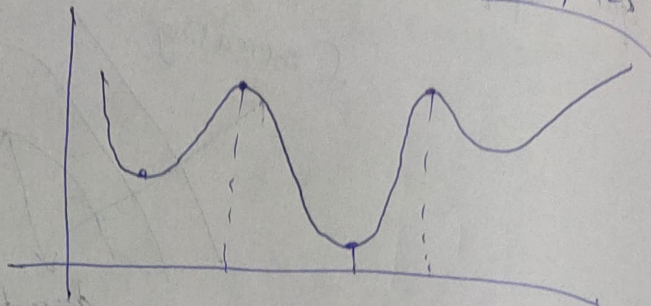
$$f(x_0 - \lambda \nabla f(x_0)) < f(x_0)$$

. This contradicts the hypothesis and so $x_0$ is a Local Minimum

Hence Proved

03/03/2023

- Local min/max.
- Global min/max.



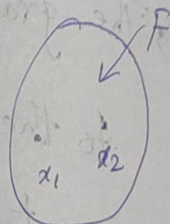
## Convex Set:

: A set is said to be Convex.

F:

$x_1, x_2$ if $\lambda_1 x_1 + \lambda_2 x_2 \in F$

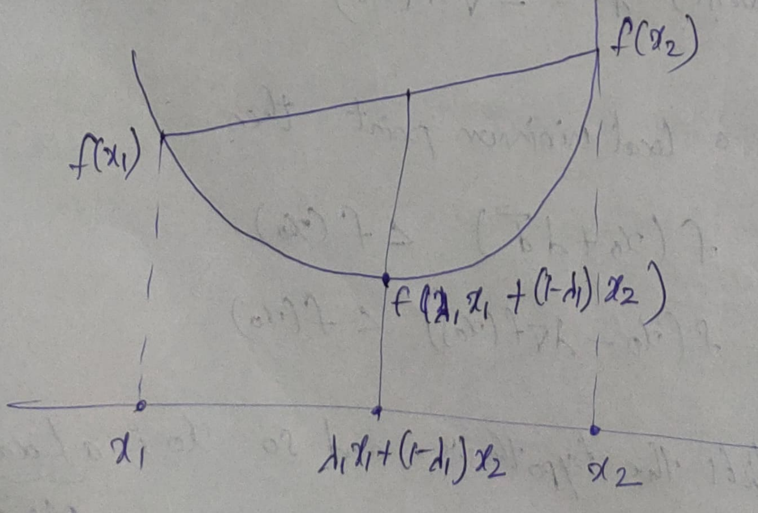
then $\lambda_1 + \lambda_2 = 1$




$$y = \lambda_1 x_1 + (1-\lambda_1) x_2 \quad ; \quad 0 \le \lambda_1 \le 1$$

- f: defined on Convex set is Convex function.

Condition: $f(\lambda_1 x_1 + (1-\lambda_1) x_2) \le \lambda_1 f(x_1) + (1-\lambda_1) f(x_2)$




Convex function

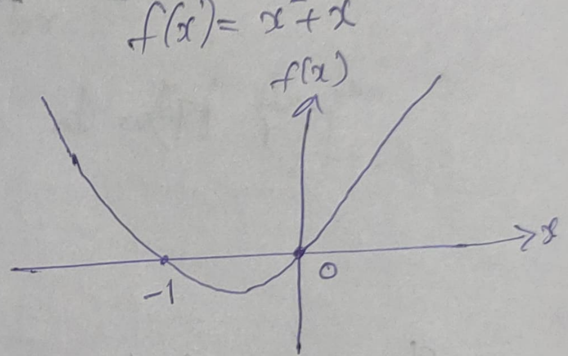
→ If not Convex, it is concave.

- All LPP deals with convex function i.e. Convex Optimization problem.

- LPPs are always Convex Optimization problems.

Ex: $f(x) = x^2 + x$ no constraints, so defined over $(-\infty, \infty)$

Find minimum value?



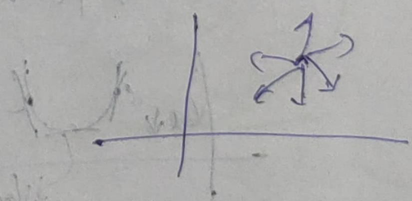

Sol:

At $x=3$: $f(3) = 9+3 = 12$

Methods: Gradient descent ✓
Steepest descent ✓
Steepest Ascent
Gradient Ascent

Gradient Descent Algorithm:

$f(x)$, $-\vec{\nabla} f$, $\frac{-\vec{\nabla} f}{|\vec{\nabla} f|}$



Ex: $f(x,y) = 4x^2 - 4xy + 2y^2$ min value?

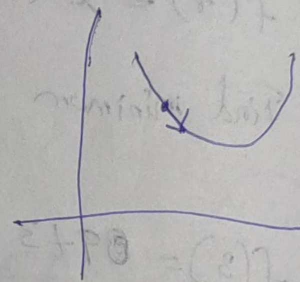
Sol $\vec{\nabla} f = \vec{i}(8x - 4y) + \vec{j}(-4x + 4y)$

$$= \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 8x - 4y \\ -4x + 4y \end{bmatrix}$$

Sol: Assume initial guess $(2,3)$

$$\vec{\nabla} f = \begin{bmatrix} 8x - 4y \\ -4x + 4y \end{bmatrix} \qquad \vec{\nabla} f(2,3) = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

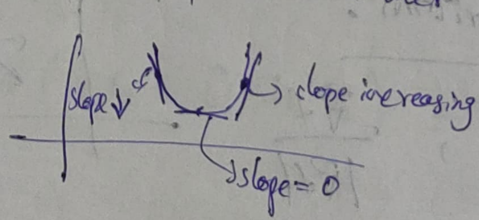
$$\frac{-\vec{\nabla} f(2,3)}{\|\vec{\nabla} f\|} = \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$



$$f(\bar{x} - t\bar{d})$$

07/03/2023

- $f'(x) > 0$ - function is increasing → it has min. point & so $f''(x) > 0$
- $f'(x) < 0$ - " " is decreasing → it has max. point & so $f''(x) < 0$
- $f'(x) = 0$ → point is either min or max, ∵ $f'(x)$ → gives the slope




- FONC - First order Necessary condition
- SONC - Second Order Necessary condition
- SOSC - Second order Sufficient Condition

Ex: Minimize $f(x,y) = 4x^2 - 4xy + 2y^2$, $f : R^2 \to R^1$

Sol: Steepest Descent method:


$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 8x - 4y \\ 4y - 4x \end{bmatrix}$$

- Gradient always gives a vector
- Gradient gives the direction at which function is growing rapidly.
- Gradient is a local property.
- Step value changes after every iteration.
- Initial guess $(2,3) \equiv (x_0, y_0)$

Unconstrained Non-LPP Optimized problems

$$\nabla f(x_0, y_0) = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

- $f(\bar{x} + \alpha \bar{d}) \leq f(\bar{x})$ ; $\alpha \geq 0$
- $\bar{d} = -\vec{\nabla} f$

$$\alpha^* = \arg\min_{\alpha} f(\bar{x} - \alpha \vec{\nabla} f)$$

$= \phi'(\alpha)$

$$\boxed{\vec{\nabla} f\left((x,y) - \alpha \vec{\nabla} f(x_0,y_0)\right) \cdot \vec{\nabla} f(x_0,y_0) = 0}$$

Sol: $\cancel{f\{[2\ 3]^T - \alpha[4\ \ 4]^T\} = f([2-4\alpha,\ 3-4\alpha]^T)}$

$\cancel{\text{let } \phi(\alpha) = f([2-4\alpha,\ 3-4\alpha]^T)}$

$\cancel{\nabla f(2-4\alpha, 3-4\alpha)}$

$$\cancel{\nabla \phi(\alpha) = \begin{bmatrix} -16\alpha + 4\alpha \\ -4\alpha + 8\alpha \end{bmatrix} = \begin{bmatrix} -12\alpha \\ 4\alpha \end{bmatrix}}$$

$\rightarrow f((x_0, y_0) - \alpha \vec{\nabla} f(x_0, y_0))$

$$= f([2\ \ 3]^T - \alpha[4\ \ 4]^T)$$

$$= f([2-4\alpha\ \ \ 3-4\alpha]^T)$$

$$= f(2-4\alpha,\ 3-4\alpha)$$

let $\phi(\alpha) = f(2-4\alpha,\ 3-4\alpha)$

$$\nabla \phi(\alpha) = \begin{bmatrix} 8(2-4\alpha) - 4(3-4\alpha) \\ 4(3-4\alpha) - 4(2-4\alpha) \end{bmatrix} = \begin{bmatrix} 16 - 32\alpha - 12 + 16\alpha \\ 12 - 16\alpha - 8 + 16\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 4 - 16\alpha \\ 4 \end{bmatrix}$$

W.K.T

$\Rightarrow \vec{\nabla} f(\vec{x_0} - \alpha \vec{\nabla} f(x_0, y_0)) \cdot \vec{\nabla} f(x_0, y_0) = 0$

$\Rightarrow (4 - 16\alpha)(4) + (4)(4) = 0$

$\Rightarrow 4 - 16\alpha + 4 = 0$

$\Rightarrow 16\alpha = 8$

$\Rightarrow \alpha = 1/2$

Now new guess:

$x_1 = \vec{x_0} + \alpha \vec{d} = \vec{x_0} - \alpha \vec{\nabla} f(2,3)$

$\vec{x_0} = (2,3)$

$= \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\cdot\ \vec{x_1} = (0, 1)$

$\vec{\nabla} f = \begin{bmatrix} 8x - 4y \\ 4y - 4x \end{bmatrix}$ $\quad \vec{\nabla} f(0,1) = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$

$\rightarrow f((0,1) - \alpha(-4,4)) = f([4\alpha \quad 1-4\alpha]^T)$

$\vec{\nabla} f([4\alpha \quad 1-4\alpha]^T) = \begin{bmatrix} 8(4\alpha) - 4(1-4\alpha) \\ 4(1-4\alpha) - 4(4\alpha) \end{bmatrix} = \begin{bmatrix} 32\alpha - 4 + 16\alpha \\ 4 - 16\alpha - 16\alpha \end{bmatrix}$

$= \begin{bmatrix} 48\alpha - 4 \\ 4 - 32\alpha \end{bmatrix}$

$\because \vec{\nabla} f([4\alpha \quad 1-4\alpha]^T) \cdot \vec{\nabla} f(0,1) = 0$

$$\Rightarrow (48\alpha-4)(4)+(4-32\alpha)(4)=0$$

$$\Rightarrow 4-48\alpha+4-32\alpha=0 \quad \Rightarrow 1-12\alpha+1-8\alpha=0$$

$$\Rightarrow 2-20\alpha=0$$

$$\Rightarrow \alpha=1/10$$

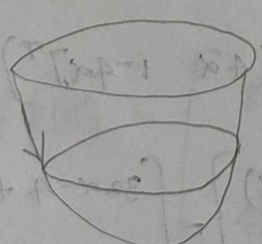
$$\Rightarrow 80\alpha=8$$

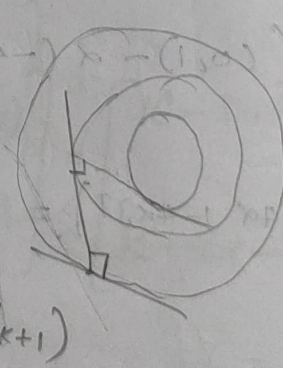
$$\Rightarrow \alpha=1/10$$

---

| | | Descent direction | Stepvalue | New estimate |
|---|---|---|---|---|
| I1 | $\begin{bmatrix}2\\3\end{bmatrix}$ | $-\begin{bmatrix}4\\4\end{bmatrix}$ | $\alpha=1/2$ | $\begin{bmatrix}0\\1\end{bmatrix}$ |
| I2 | $\begin{bmatrix}0\\1\end{bmatrix}$ | $-\begin{bmatrix}-4\\4\end{bmatrix}$ | $\alpha=1/10$ | $\begin{bmatrix}2/5\\3/5\end{bmatrix}$ |
| I3 | $\begin{bmatrix}2/5\\3/5\end{bmatrix}$ | | | $\begin{bmatrix}0\\2/10\end{bmatrix}$ |

$$\vec{x_2}=\begin{bmatrix}0\\1\end{bmatrix}-\frac{1}{10}\begin{bmatrix}-4\\4\end{bmatrix}=\begin{bmatrix}2/5\\3/5\end{bmatrix}$$

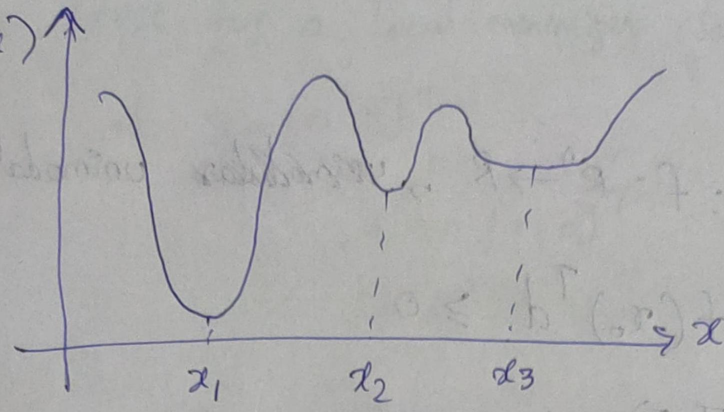




$$\nabla f(x_k) \perp \nabla f(x_{k+1})$$

→ The value of $\vec{x}$, at which the function takes minimum value, that $\vec{x}$ is defined as Minimizer.

\#




- $x_1$ – Global minimizer
- $x_2$ – strict Local minimizer
- $x_3$ – Local minimizer

- $f(\vec{x}) = f(x_1, x_2, \ldots, x_n)$

Gradient:

$$\vec{\nabla} f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

Hessian

$$\vec{\nabla}^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \frac{\partial^2 f}{\partial x_2 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2} \\ \vdots & & & \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

# FONC :

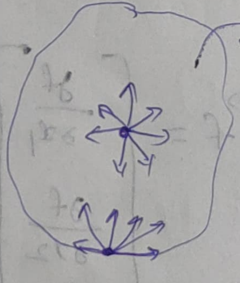
- A function : $f : R^n \to R$ , unimodal

FONC: $\vec{\nabla} f(x_0)^T d \geq 0$

(directional derivative) $d^T \vec{\nabla} f(x_0) \geq 0$

then $x_0$ is local minimizer. ($x_0$ is a boundary point)

- $f \in C^1 \to$ function is first order differentiable and continuous
- $f \in C^2 \to$ function is $2^{nd}$ order differentiable & continuous.
- $\Omega / F \subset R^n$


feasible region

For Interior points case :

$$\left.\begin{array}{l} \nabla f(x_0)^T d \geq 0 \\ \nabla f(x_0)^T d \leq 0 \end{array}\right\}$$

$\Rightarrow \vec{\nabla} f(x_0) = 0$ ; $x_0$ - is an interior point

---

Example 6.3 (in KPChong book) : minimize : $x_1^2 + 0.5 x_2^2 + 3x_2 + 4.5$

Subject to : $x_1, x_2 \geq 0$ (feasible region)

(a) Is the FONC for a local minimizer satisfied at $x = [1 \ 3]^T$

(b) Is FONC for a local minimizer satisfied

at $x = [0 \;\; 3]^T$

(c) " " at $x = [1, 0]^T$

(d) " " at $x = [0 \;\; 0]^T$

Sol: $f(x) = x_1^2 + 0.5x_2^2 + 3x_2 + 4.5$ ; $x_1, x_2 \geq 0$

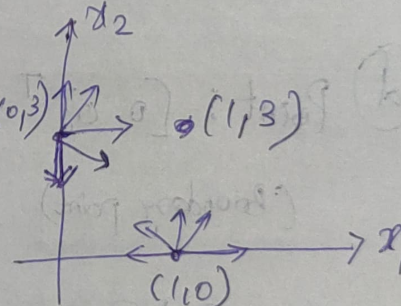
(a)

Point: (1,3) (Interior point) $\vec{\nabla} f = \begin{bmatrix} 2x_1 \\ x_2 + 3 \end{bmatrix}$ $\vec{\nabla} f(1,3) = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(b)

Point: (0,3) (Boundary point) $\vec{\nabla} f = \begin{bmatrix} 2x_1 \\ x_2 + 3 \end{bmatrix}$ $\vec{\nabla} f(0,3) = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$

$d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$




from graph: $d_1 \geq 0$
$d_2$ ∀ rs

Necessary Condition $\vec{\nabla} f(0,3) \cdot d \geq 0 \Rightarrow 6d_2$ is not always greater than zero

- FONC is not satisfied for $[0 \;\; 3]^T$

- FONC is not satisfied for $[1 \;\; 3]^T$

(c) Point : $[1 \ 0]^T$ (Boundary point)

$$\vec{\nabla} f = \begin{bmatrix} 2x_1 \\ x_2 + 3 \end{bmatrix} \qquad \vec{\nabla} f(1,0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

From graph : $d_1$ urs → limited to $-1$

$d_2 \geq 0$

<u>FONC to be satisfied</u> : $\vec{\nabla} f(1,0)^T d = 2d_1 + 3d_2$ not always greater than zero

∴ FONC is not satisfied at $[1 \ 0]^T$

(d) Point : $[0 \ 0]^T$ (Boundary point)

$$\vec{\nabla} f = \begin{bmatrix} 2x_1 \\ x_2 + 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

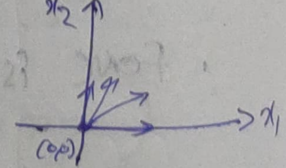
$$d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

From graph : $d_1 \geq 0$

$d_2 \geq 0$

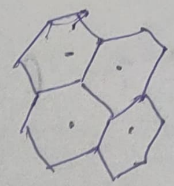
<u>FONC to be satisfied</u> : $\vec{\nabla} f(0,0)^T d = 3d_2 \geq 0$ (True)

∴ FONC is satisfied at $[0 \ 0]^T$

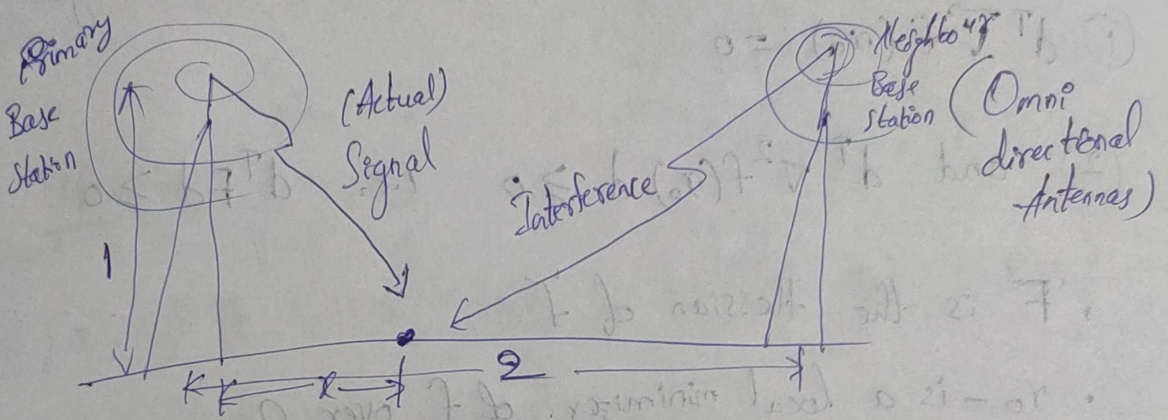


Q. Two antennas

Example 6.4



- Cell site (Hexagonal shape)




**Maximize:** $\dfrac{\text{Signal Power}}{\text{Interference signal Power}}$    find $x = ?$

- Signal Power $\propto \dfrac{1}{(\sqrt{1+x^2})^2}$    $\left(\because \text{Power} \propto \dfrac{1}{d^2}\right)$

- Interference signal Power $\propto \dfrac{1}{(\sqrt{1+(2-x)^2})^2}$

- Therefore,

**Maximize:** $f(x) = \dfrac{1+(2-x)^2}{1+x^2} = \dfrac{5-4x+x^2}{1+x^2}$

**Using FONC:** $f'(x) = 0 \Rightarrow f'(x) = \dfrac{4(x^2-2x-1)}{(1+x^2)^2} = 0$

($x_0 \to$ interior point / Boundary point - both same)

$\to x_0 = 1 \pm \sqrt{2}$

- Here, function is one dimentional

$\boxed{\therefore x_0 = 1 - \sqrt{2}}$

(As $1+\sqrt{2}$ is beyond 2)

SONC : $f : \mathbb{R}^n \to \mathbb{R}$, $f \in C^2$, $\Omega \subset \mathbb{R}^n$

feasible region

① $d^T \vec{\nabla} f(x_0) = 0$

② and $d^T \nabla^2 f(x_0)\, d \geq 0$ i.e, $d^T F d \geq 0$

- 'F' is the Hessian of 'f'
- $x_0$ – is a local minimizer of f over $\Omega$

SOSC : $f \in C^2$

① $\nabla f(x_0) = 0$

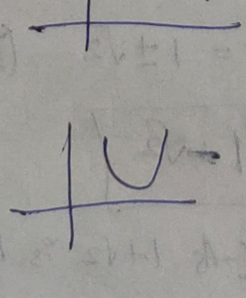
② $F(x_0) > 0$

---

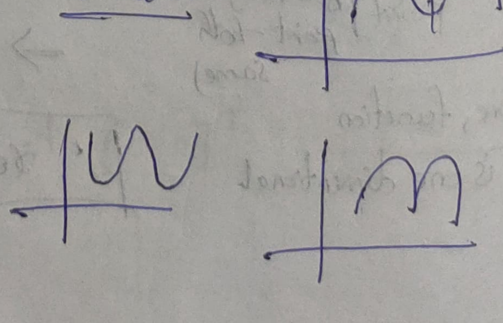
Line Search algorithms : searching in a single line. (Unimodal) 09/03/2023

1. Golden-section search
2. Fibonacci method
3. Newtons method
4. Secant method
5. Bisection method

Unimodal :



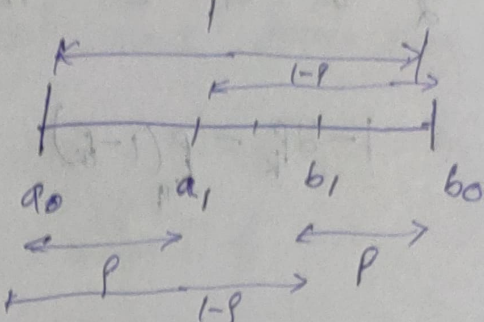
Bimodal :



07/08/2023

# 1. Golden - Section Search :

$f(x) \in [a_0, b_0]$




$[a_0, b_0]$

Iteration 1 : $[a_1, b_1]$

$[a_1, b_1] \ni a_1 - a_0 = b_0 - b_1 = \rho(b_0 - a_0)$ ; $\rho < \frac{1}{2}$

ρ – shorter length
(1-ρ) – Longer length

Case 1: $f(a_1) < f(b_1)$

Case 2: $f(a_1) > f(b_1)$

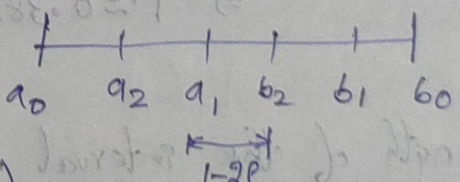
· For a minimizing problem :

In Case 1 i.e. $f(a_1) < f(b_1)$ then minimizer : $x_0 \in [a_0, b_1]$

In Case 2 i.e. $f(a_1) > f(b_1)$ then minimizer : $x_0 \in [a_1, b_0]$

Iteration 2 : $[a_2, b_2]$

Case 1 : $[a_0, b_1]$




$a_2 - a_0 = b_1 - b_2 = \rho_{k+1}(b_1 - a_0)$

· To reduce computational calculations, make points $b_2$ and $a_1$ as same, i.e. $b_2 = a_1$

$[a_2, b_2] = [a_2, a_1]$

$$a_2 - a_0 = b_1 - b_2 = \rho_k (b_1 - a_0)$$

$$1 - 2\rho_k = \rho_{k+1}(1 - \rho_k) \quad ; \rho_k - \rho \text{ value of } k^{th} \text{ iteration}$$

# In Golden- ~~section~~ search, $\rho_k = \rho_{k+1}$ i.e. $\rho$ value is same for all iterations

# In fibonacci method, different values of $\rho$ is considered for different iterations

<u>In Golden- section search</u>

$$\Rightarrow 1 - 2\rho = \rho(1-\rho)$$

$$\Rightarrow 1 - 2\rho = \rho - \rho^2$$

$$\Rightarrow \rho^2 - 3\rho + 1 = 0$$

$$\Rightarrow \rho = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3-\sqrt{5}}{2}$$

$$\Rightarrow \rho \simeq 0.382$$

# Length of the interval in $n^{th}$ iteration is $(1-\rho)^n$

$$\cdot\ 1 - \rho = 1 - \left(\frac{3-\sqrt{5}}{2}\right) = \frac{\sqrt{5}-1}{2} \qquad \cdot\ \frac{\rho}{1-\rho} = \frac{3-\sqrt{5}}{\sqrt{5}-1} = \frac{\sqrt{5}-1}{2} = 1-\rho$$

$$\therefore \frac{\rho}{1-\rho} = \frac{1-\rho}{1}$$ ; the ratio of shorter segment to longer equals to ratio of longer to sum of two.

. This rule was referred to by ancient Greek geometers as the <u>Golden section</u>

Example 7:1

$$f(x) = x^4 - 14x^3 + 60x^2 - 70x$$ defined over $[0, 2]$

Value of $x$ that minimizes $f(x)$? (use Golden-section Search method)

- Locate $x$ value to within a range of 0.3

## 2. Fibonacci Method :

$$\rho_{k+1} = \frac{1-2\rho_k}{1-\rho_k}$$

$$\therefore \boxed{\rho_{k+1} = 1 - \frac{\rho_k}{1-\rho_k}}$$

- Length of the intervals : $(1-\rho_1)\ (1-\rho_2)\ \dots\dots\ (1-\rho_N)$

$\rho_{-1} = 0,\ \rho_0 = 1,\ \rho_1 = 1,\ \rho_2 = 2,\ \rho_3 = 3,\ \rho_4 = 5$
8, 13, 21, 34, ...

# The rate at which the interval length narrows down is much faster in Fibonacci Method than in Golden-section search algorithm.

$$\rho_1 = 1 - \frac{F_N}{F_{N+1}}, \dots\dots, \rho_N = 1 - \frac{F_1}{F_2}$$

## 3. Bisection Method :

$[a_0, b_0]$

Iteration 1 :

- $x_0 = \frac{a_0+b_0}{2}$, $f'(x_0) > 0 \Rightarrow \left[a_0, \frac{a_0+b_0}{2}\right]$

$f'(x_0) < 0 \Rightarrow \left[\frac{a_0+b_0}{2}, b_0\right]$

# Rate at which interval gets narrower & narrower is $(1/2^N)$ in Bisection Method i.e. $(0.5)^N$

## 4. Newtons Method :

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \cdots$$

$$\rightarrow x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$$

$$q(x) = f(x^k) + f'(x^k)((x-x^k)) + \frac{1}{2} f''(x^k)(x-x^k)^2$$

$$q'(x) = 0 \Rightarrow f'(x^k) + f''(x^k)(x-x^k) = 0$$

Setting $x = x^{k+1}$ $\Rightarrow x^{k+1} = x^k - \frac{f'(x^k)}{f''(x^k)}$

## 4. Newtons Method:

$$\left\{ f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \cdots \right\}$$

$$\rightarrow x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$$

Taylorseries:

$$q(x) = f(x^k) + f'(x^k)((x - x^k)) + \frac{1}{2} f''(x^k)(x - x^k)^2$$

$f(x_k)$ is a constant

$$q'(x) = 0 \Rightarrow f'(x^k) + f''(x^k)(x - x^k) = 0$$

Setting $x = x^{k+1}$ $\Rightarrow x^{k+1} = x^k - \frac{f'(x^k)}{f''(x^k)}$

10/08/2023

## Line-Search Algorithms

1. Golden-section Search
2. Fibonacci method

No use of $f'(x)$ & $f''(x)$

3. Bisection — $f'(x)$ is used

4. Newtons method
5. Secant method

Both $f'(x)$ & $f''(x)$ are used

## 4. Newton's Method:

Let $q(x) = f(x_k) + (x - x_k) f'(x_k) + \frac{(x - x_k)^2}{2!} f''(x_k) + \cdots$

FONC: $q'(x) = 0 \Rightarrow q'(x) = f'(x_k) + \frac{2}{2!}(x - x_k) f''(x_k) = 0$

$$\Rightarrow f'(x_k) + (x - x_k) f''(x_k) = 0$$

$$\Rightarrow (x - x_k) f''(x_k) = -f'(x_k)$$

$$\Rightarrow x - x_k = \frac{-f'(x_k)}{f''(x_k)}$$

$$\Rightarrow x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$$

$$\boxed{\therefore x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}}$$

$f'(x_k) > 0$, $f'(x_{k+1}) < 0$

5. Secant Method:

- If $f''(x_k)$ is not available

$$f''(x_k) = \frac{f'(x_k) - f'(x_{k-1})}{x_k - x_{k-1}}$$

$$x_{k+1} = x_k - \frac{(x_k - x_{k-1}) f'(x_k)}{f'(x_k) - f'(x_{k-1})}$$

$$\boxed{x_{k+1} = \frac{x_{k-1} f'(x_k) - x_k f'(x_{k-1})}{f'(x_k) - f'(x_{k-1})}}$$

Example 7.4: $f(x) = \frac{1}{2}x^2 - \sin x$, find the minimizer $x_*$?

Sol: $f'(x) = x - \cos x$

$f''(x) = 1 + \sin x$

Let $x_0 = 0.5$

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$$

$$x_1 = 0.5 - \frac{[0.5 - \cos(0.5)]}{[1 + \sin(0.5)]}$$

$$= 0.7552$$

$$x_2 = x_1 - \frac{f'(x_1)}{f''(x_1)} = 0.7552 - \frac{[0.7552 - \cos(0.7552)]}{[1 + \sin(0.7552)]}$$

$$= 0.7391$$

$$x_3 = x_2 - \frac{f'(x_2)}{f''(x_2)} = 0.7390$$

$$x_4 = x_3 - \frac{f'(x_3)}{f''(x_3)} = 0.7390$$

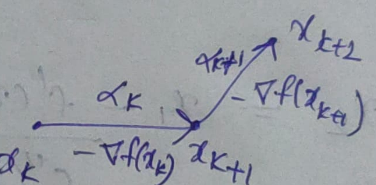
$f(x)$ $\qquad\qquad$ $f'(x)$

$f(x_1, x_2, \ldots, x_n) = f(\vec{x})$ $\qquad\qquad$ $\nabla f(\vec{x})$ — gradient

$f(\vec{x} + \alpha \vec{d})$ $\qquad\qquad$ $\nabla f(x + \alpha d)^T d$ — Directional Derivative

# If $\{x_k\}_{k=0}^{\infty}$ is a steepest descent sequence for a given function $f: \mathbb{R}^n \to \mathbb{R}$, then for each $k$ the vector $x_{k+1} - x_k$ is orthogonal to the vector $x_{k+2} - x_{k+1}$.

<u>Proof</u>: $\quad x_{k+1} - x_k = ?$




$\Rightarrow x_{k+1} - x_k = -\alpha_k \nabla f(x_k)$ $\quad$ ; $\alpha_k$ – magnitude (step-size value)

$-\nabla f(x_k) \to$ descent direction

$\Rightarrow x_{k+2} - x_{k+1} = -\alpha_{k+1} \nabla f(x_{k+1})$

<u>RTP</u>: $(x_{k+1} - x_k)$ orthogonal to $(x_{k+2} - x_{k+1})$

$\Rightarrow \langle \nabla f(x_k), \nabla f(x_{k+1}) \rangle = 0$ (RTP condition)

$\phi_k(\alpha) = f(x_k - \alpha \nabla f(x_k))$

Generalized Newton's Method

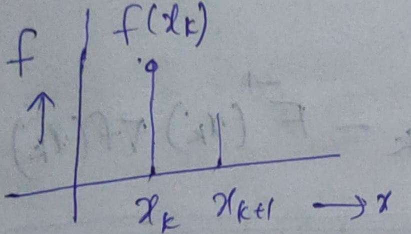
pg - 180/642 — 162

14/08/2023

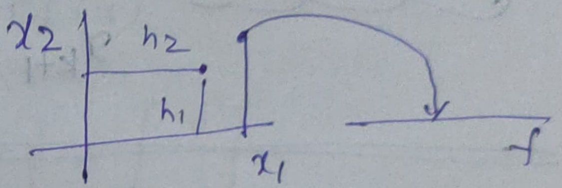
Taylor series of a function:

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \cdots$$

$$f(x_{k+1}) = q(x) \approx f(x_k) + (x - x_k) f'(x_k) + \frac{(x-x_k)^2}{2!} f''(x_k) + \cdots$$




$L = x_k + h$




N-dimensional case:

$$f(x_{k+1}) = q(x) \approx f(x_k) + (x - x_k)^T \nabla f(x_k) + \frac{1}{2!} (x - x_k)^T F(x - x_k)$$

$$\therefore f(x+h) \approx f(x) + h^T \nabla f(x) + \frac{1}{2!} h^T F(h)$$

$F = \nabla^2 f \rightarrow$ Hessian

$\rightarrow f(x_k) = f(x_1, x_2, \ldots, x_n)$

$\rightarrow x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$ (from $q'(x) = 0$)

$\boxed{q'(x) = f'(x_k) + (x - x_k) f''(x_k)}$

Newton's Method : $x = x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$

Steepest Descent Method : $x_{k+1} = x_k - \alpha \nabla f(x_k)$

In N-Dimension case :

Newtons Method :

$$x_{k+1} = x_k - F^{-1}(x_k) \nabla f(x_k)$$

---

Ex : $f(x,y) = 4x^2 - 4xy + 2y^2$

$x^0 = [2, 3]$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 8x - 4y \\ 4y - 4x \end{bmatrix} \qquad \nabla f(x^0) = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$\nabla^2 f = F = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ -4 & 4 \end{bmatrix}$$

$$F^{-1} = \frac{1}{(32-16)} \begin{bmatrix} 4 & 4 \\ 4 & 8 \end{bmatrix}^T = \frac{1}{16} \begin{bmatrix} 4 & 4 \\ 4 & 8 \end{bmatrix}$$

$$F^{-1}\nabla f(x^0) = \frac{1}{16}\begin{bmatrix} 4 & 4 \\ 4 & 8 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 4 \\ 4 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 8 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

- $$x_{k+1} = x_k - F^{-1}\nabla f(x)$$

$$= \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- $$f(x^0) = 4(4) - 4(2)(3) + 2(9)$$

$$= 16 - 24 + 18$$

$$= 10$$

- $$f(x^1) = 4(0) - 4(0) + 2(0)$$

$$= 0$$

---

# 1. Solve for $d^{(k)}$ in

$$F(x^k)\, d^k = -g^k$$

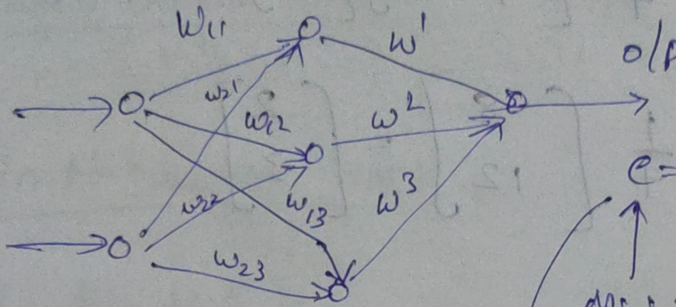
$$\Rightarrow \boxed{d^k = -F^{-1}(x_k)\, g_k}$$

$$\boxed{\begin{matrix} F = \nabla^2 f \\ g = \nabla f \end{matrix}}$$

2. Set: $x_{k+1} = x_k + d_k$

# OT in ML / AI

MLP – Multi-layer perceptron




$e = \hat{f}(x) - f(x)$

Minimizing the error function – interms of weights

- Dual n/w application, Machine learning problem
- 5-weight problem

---

$$f(x) = E[x] = \sum_{i=1}^{m} w_i [e_i(x)]^p$$ – (Weighted summation)

$$w_i [e_i(x)]^p \longrightarrow [e_i(x)]^p$$

10th chapter

$\longrightarrow$ $E[x] = \sum_i e_i^p$ – $\underbrace{\text{least } p^{th} \text{ optimization technique}}$

$\longrightarrow$ if, $p=2$: $E = \sum_i e_i^2$ – $\underbrace{\text{least square optimization technique}}$

$$E(x_1, x_2) = \sum_i^m e_i^2 (x_1, x_2)$$ (wrt, $p=2$)

(2-dimensional)

FONC : $x^*$, $\nabla f(x^*) = 0$

$\hookrightarrow \nabla E(x) = 0$

$\boxed{\text{for } p=2}$

$$\nabla E(x) = \begin{bmatrix} \frac{\partial}{\partial x_1} E \\ \frac{\partial}{\partial x_2} E \\ \vdots \\ \frac{\partial}{\partial x_n} E \end{bmatrix} = \begin{bmatrix} 2 \sum_i \left(e_i \cdot \frac{\partial e_i}{\partial x_1}\right) \\ 2 \sum_i \left(e_i \cdot \frac{\partial e_i}{\partial x_2}\right) \\ \vdots \\ 2 \sum_i \left(e_i \cdot \frac{\partial e_i}{\partial x_n}\right) \end{bmatrix}$$

$$\Rightarrow \nabla E(x) = 2 \begin{bmatrix} \sum_i \left(e_i \cdot \frac{\partial e_i}{\partial x_1}\right) \\ \sum_i \left(e_i \cdot \frac{\partial e_i}{\partial x_2}\right) \\ \vdots \\ \sum_i \left(e_i \cdot \frac{\partial e_i}{\partial x_n}\right) \end{bmatrix}$$

$\boxed{\text{for } p}$

$$\nabla E(x) = p \begin{bmatrix} \sum_i e_i^{p-1} \cdot \frac{\partial e_i}{\partial x_1} \\ \sum_i e_i^{p-1} \cdot \frac{\partial e_i}{\partial x_2} \\ \vdots \\ \sum_i e_i^{p-1} \cdot \frac{\partial e_i}{\partial x_n} \end{bmatrix}$$

$$J(x) = \begin{bmatrix} \frac{\partial e_1}{\partial x_1} & \frac{\partial e_1}{\partial x_2} & \cdots & \frac{\partial e_1}{\partial x_n} \\ \frac{\partial e_2}{\partial x_1} & \frac{\partial e_2}{\partial x_2} & \cdots & \frac{\partial e_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial e_m}{\partial x_1} & \frac{\partial e_m}{\partial x_2} & \cdots & \frac{\partial e_m}{\partial x_n} \end{bmatrix}_{m \times n}$$

$$= \left[ \frac{\partial e_i}{\partial x_k} \right]$$

Jacobian Matrix

$$e = \begin{bmatrix} e_1(x) \\ e_2(x) \\ \vdots \\ e_m(x) \end{bmatrix}$$

$\boxed{\nabla E(x) = 2 J_{(x)}^T e_{(x)}}$ for $p = 2$

for Least $p^{th}$ method:

$$\boxed{\nabla E(x) = p\, J^T(x)\, e^{p-1}(x)}$$

FONC: $\nabla E(x^1) \leq \nabla E(x^0)$

$$x^1 = x^0 + \delta$$

$$\nabla E(x+\delta x) = p\, J^T(x+\delta x)\, e^{p-1}(x+\delta x)$$

$$\nabla E(x+\delta x) = 2\, J^T(x+\delta x)\, e(x+\delta x$$

The initial parameter vector $x$ will usually not be at the minimum thus $\nabla E(x)$ will not be zero.

However, $\nabla E(x) = p J^T(x) e^{p-1}(x)$ can be used to obtain a change $\delta x$ such that the magnitude of gradient is reduced.

The increment $\delta x$ can be found by using above expression to express the gradient at $x + \delta x$

as $$\nabla E(x+\delta x) = 2 J^T(x+\delta x)\, e(x+\delta x)$$

$$\nabla E(x+\delta x) = p J^T(x+\delta x)\, e^{p-1}(x+\delta x)$$

<u>Approximation</u> :-

$$\nabla E(x+\delta x) \approx 2 J^T(x)\, e(x+\delta x)$$

$$\nabla E(x+\delta x) \approx p J^T(x)\, e^{p-1}(x+\delta x)$$

- As Jacobian does not vary very much, it will be true as the minimum is approached.

$$e(x+\delta x) \approx e(x) + \nabla e^T(x)\, \delta(x) + \frac{1}{2!}\, \delta^T(x)\, \nabla^2 e(x)\, \delta(x)$$

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## Least $p^{th}$ optimization

$$E(x) = \sum_{i=1}^{m} [e_i(x)]^p \qquad \bar{x} = [x_1, x_2, \dots, x_n]^T_{1\times n}$$

$$e = [e_1, e_2, \dots, e_m]^T_{1\times m}$$

$$J = \begin{bmatrix} \frac{\partial e_1}{\partial x_1} & \frac{\partial e_1}{\partial x_2} & \cdots & \frac{\partial e_1}{\partial x_n} \\ \frac{\partial e_2}{\partial x_1} & \frac{\partial e_2}{\partial x_2} & \cdots & \frac{\partial e_2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial e_m}{\partial x_1} & \frac{\partial e_m}{\partial x_2} & \cdots & \frac{\partial e_m}{\partial x_n} \end{bmatrix}_{m\times n}$$

$$\boxed{\nabla E(x) = \underset{(n\times 1)}{\nabla E(x)} = P\cdot \underset{(n\times m)}{J^T}\, \underset{(m\times 1)}{e^{p-1}}}$$

## Least squares optimization

$$E(x) = \sum_{i=1}^{m} e_i^2(x)$$

$$\nabla E = \begin{bmatrix} \frac{\partial E}{\partial x_1} \\ \frac{\partial E}{\partial x_2} \\ \vdots \\ \frac{\partial E}{\partial x_n} \end{bmatrix} = 2 \begin{bmatrix} \sum_i e_i \frac{\partial e_i}{\partial x_1} \\ \sum_i e_i \frac{\partial e_i}{\partial x_2} \\ \vdots \\ \sum_i e_i \frac{\partial e_i}{\partial x_n} \end{bmatrix}_{n\times 1}$$

$$\boxed{\nabla E = 2J^T e}$$

$$\Downarrow$$

$$\nabla E(x+\delta x) = 2J^T(x+\delta x)\, e(x+\delta x)$$

$$\Rightarrow \nabla E(x+\delta x) \simeq 2J^T(x)\, e(x+\delta x)$$

FONC: $\nabla E(x^*) = 0$

Initial guess: $x^0$, $\nabla E(x^0) \neq 0$

$$x_1 = x^0 + \delta x$$

$$\nabla E(x+\delta x) \longrightarrow 0$$

$$\nabla E(x+\delta x) = P\, J^T(x+\delta x)\, e^{(P-1)}_{(x+\delta x)}$$

$\Rightarrow \nabla E(x+\delta x) \approx p\, J^T(x)\, e^{p-1}(x+\delta x)$

For least squares method:

\# $e(x+\delta x) \approx e(x) + \delta x\, e'(x+\cancel{\ })$

$\Rightarrow e(x+\delta x) \approx e(x) + \nabla e^T(x+\cancel{\ }) \cdot \delta x$

$$\delta x = \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \vdots \\ \delta x_n \end{bmatrix}_{n\times 1}$$

$$\nabla e^T(x+\delta x) = \begin{bmatrix} \frac{\partial e}{\partial x_1} & \frac{\partial e}{\partial x_2} & \cdots & \frac{\partial e}{\delta x_n} \end{bmatrix}_{m\times n}$$

$= J(x)$

$$\frac{\partial e}{\partial x_1} = \begin{bmatrix} \frac{\partial e_1}{\partial x_1} \\ \frac{\partial e_2}{\partial x_2} \\ \vdots \\ \frac{\partial e_m}{\partial x_n} \end{bmatrix}_{m\times 1}$$

$\Rightarrow e(x+\delta x) \approx e(x) + J\, \delta(x)$

$\approx e(x) + J(x+\delta x)\, \delta(x)$

$\approx e(x) + J(x)\, \delta(x)$

$\longrightarrow$ 1

FONC:

$2 J^T(x)\, e(x+\delta x) = 0$ from eq 1

$\Rightarrow 2 J^T(x) \left[ e(x) + J(x)\, \delta(x) \right] = 0$

$\Rightarrow J^T e(x) + J^T J\, \delta(x) = 0$

$\Rightarrow J^T J\, \delta(x) = -J^T e(x)$

$\Rightarrow J^T(\mathring{x})\, J(\mathring{x})\, \delta(x) = -J^T(\mathring{x})\, e(\mathring{x})$

$\therefore$ For least squares method

$$J^T(\overset{\circ}{x})\, J(\overset{\circ}{x})\, \delta(x) = -J^T(\overset{\circ}{x})\, e(\overset{\circ}{x})$$

$$x^{(1)} = x^{(0)} + \delta(x)$$

For least $p^{th}$ method

$$\nabla E(x+\delta x) \approx p\, J^T(x)\, e^{p-1}(x+\delta x) = 0$$

$$e^{p-1}(x+\delta x) \approx e^{p-1}(x) + \delta(x)(p-1)\, e^{p-2}(x) \frac{\partial e}{\partial x}$$

$$\approx e^{p-1}(x) + (p-1)\, e^{p-2}(x)\, \nabla e(x)\, \delta(x)$$

$$\approx e^{p-1}(x) + (p-1)\, D J \delta(x)$$

$$D = \text{diag.}\left[ e_1^{p-2}(x) \quad e_2^{p-2}(x) \cdots e_m^{p-2}(x) \right]$$

Therefore:

$$\nabla E(x+\delta x) = p J^T(x)\, e^{p-1}(x+\delta x) = 0$$

$$\Rightarrow p J^T(x) \left[ e^{p-1}(x) + (p-1) D J \delta(x) \right] = 0$$

$$\Rightarrow J^T D J \delta(x) = \frac{-1}{p-1} J^T e^{p-1}$$

∴ For least $p^{th}$ method

$$J^T D J \, \delta(x) = -\frac{1}{p-1} J^T e^{p-1}$$

Problem: $E(x) = (x_1^2 + x_2^2 + x_1 x_2 - 4)^2 + 4x_1^2 x_2^2 + 9x_1^2 x_2^4$

Sol: Initial guess: $(x_1, x_2) = (1, 1)$

$$\overset{\circ}{x} = [1 \quad 1]^T$$

$e_1 = x_1^2 + x_2^2 + x_1 x_2 - 4$

$e_2 = 2x_1 x_2$

$e_3 = 3x_1 x_2^2$

$$E(x) = \sum_{i=1}^{3} e_i^2$$

Least Squares Method

$$J = \begin{bmatrix} \frac{\partial e_1}{\partial x_1} & \frac{\partial e_1}{\partial x_2} \\ \frac{\partial e_2}{\partial x_1} & \frac{\partial e_2}{\partial x_2} \\ \frac{\partial e_3}{\partial x_1} & \frac{\partial e_3}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 & 2x_2 + x_1 \\ 2x_2 & 2x_1 \\ 3x_2^2 & 6x_1 x_2 \end{bmatrix}$$

$$e = \begin{bmatrix} x_1^2 + x_2^2 + x_1 x_2 - 4 \\ 2x_1 x_2 \\ 3x_1 x_2^2 \end{bmatrix} \Rightarrow e(\overset{\circ}{x}) = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

$$J(\overset{\circ}{x}) = \begin{bmatrix} 3 & 3 \\ 2 & 2 \\ 3 & 6 \end{bmatrix}$$

WKT.

$$J^T(\overset{\circ}{x})\, J(\overset{\circ}{x})\, \delta(x) = -J^T(\overset{\circ}{x})\, e(\overset{\circ}{x})$$

$$\begin{bmatrix} 3 & 2 & 3 \\ 3 & 2 & 6 \end{bmatrix}_{2\times 3} \begin{bmatrix} 3 & 3 \\ 2 & 2 \\ 3 & 6 \end{bmatrix}_{3\times 2} \delta(x) = -\begin{bmatrix} 3 & 2 & 3 \\ 3 & 2 & 6 \end{bmatrix}_{2\times 3} \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1}$$

$$\begin{bmatrix} 9+4+9 & 9+4+18 \\ 9+4+18 & 9+4+36 \end{bmatrix} \delta(x) = -\begin{bmatrix} -3+4+9 \\ -3+4+18 \end{bmatrix}$$

$$\begin{array}{r} 13 \\ 18 \\ \hline 31 \end{array} \quad \begin{array}{r} 18 \\ 4 \\ \hline 22 \end{array}$$

$$\begin{bmatrix} 22 & 31 \\ 31 & 49 \end{bmatrix} \delta(x) = -\begin{bmatrix} 10 \\ 19 \end{bmatrix}$$

$$\Rightarrow \delta(x) = -\begin{bmatrix} 22 & 31 \\ 31 & 49 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 19 \end{bmatrix}$$

$$= -\frac{1}{(22\times 49 - 31^2)} \begin{bmatrix} 49 & -31 \\ -31 & 22 \end{bmatrix}^T \begin{bmatrix} 10 \\ 19 \end{bmatrix}$$

$$\Rightarrow \delta(x) = -\frac{1}{117}\begin{bmatrix} 49 & -31 \\ -31 & 22 \end{bmatrix}\begin{bmatrix} 10 \\ 19 \end{bmatrix}$$

$$= -\frac{1}{117}\begin{bmatrix} 490 - 31\times 19 \\ -310 + 22\times 19 \end{bmatrix}$$

$$= \frac{-1}{117}\begin{bmatrix} -99 \\ 108 \end{bmatrix} = \begin{bmatrix} 0.8462 \\ -0.9231 \end{bmatrix}$$

$$\therefore \delta(x) = \begin{bmatrix} 0.8462 \\ -0.9231 \end{bmatrix}$$

$$x^{(1)} = x^{(0)} + \delta(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0.8462 \\ -0.9231 \end{bmatrix}$$

$$= \begin{bmatrix} 1.8462 \\ 0.0769 \end{bmatrix}$$

$$e = \begin{bmatrix} x_1^2 + x_2^2 + x_1x_2 - 4 \\ 2x_1x_2 \\ 3x_1x_2^2 \end{bmatrix} \Rightarrow e(x^1) = \begin{bmatrix} -0.4434 \\ 0.28394 \\ 0.03275 \end{bmatrix}$$

$$\boxed{\begin{aligned} E(\overset{\circ}{x}) &= 14 \\ E(x^{(1)}) &= 0.2784 \\ &\approx 0.279 \end{aligned}}$$

17/03/2023

$$E(x) = e_1^4 + e_2^4 + e_3^4$$

$$e_1 = x_1^2 + x_2^2 + x_1 x_2 - 4$$

$$e_2 = 2x_1 x_2$$

$$e_3 = 3x_1 x_2^2$$

$$J^T D J\, \delta(x) = \frac{-1}{P-1} J^T e^{P-1}(x) \quad ; D = diag(e_1^{P-2}(x), e_2^{P-2}(x), \ldots e_m^{P-2}(x))$$

$$J^T J\, \delta(x) = - J^T e(x)$$

Sol:

$$e^{P-1}(x) = \begin{bmatrix} e_1^{P-1}(x) \\ e_2^{P-1}(x) \\ e_3^{P-1}(x) \end{bmatrix} \qquad e(x) = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

P=4:

$$e^3(x) = \begin{bmatrix} -1 \\ 8 \\ 27 \end{bmatrix}$$

$$D = \begin{bmatrix} e_1^2 & 0 & 0 \\ 0 & e_2^2 & 0 \\ 0 & 0 & e_3^2 \end{bmatrix}_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}_{3\times3}$$

$$J = \begin{bmatrix} \frac{\partial e_1}{\partial x_1} & \frac{\partial e_1}{\partial x_2} \\ \frac{\partial e_2}{\partial x_1} & \frac{\partial e_2}{\partial x_2} \\ \frac{\partial e_3}{\partial x_1} & \frac{\partial e_3}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 2 & 2 \\ 3 & 6 \end{bmatrix}_{3\times2}$$

$$\therefore\ J^T D J\, \delta(x) = \frac{-1}{p-1} J^T e^{p-1}(x)$$

$\underline{\underline{P=4}}$

$$J^T D J\, \delta(x) = \frac{-1}{(4-1)} J^T e^{3}(x)$$

$$\begin{bmatrix} 3 & 2 & 3 \\ 3 & 2 & 6 \end{bmatrix}_{2\times3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}_{3\times3} \begin{bmatrix} 3 & 3 \\ 2 & 2 \\ 3 & 6 \end{bmatrix}_{3\times2} \delta(x)$$

$$= \frac{-1}{3} \begin{bmatrix} 3 & 2 & 3 \\ 3 & 2 & 6 \end{bmatrix}_{2\times3} \begin{bmatrix} -1 \\ 8 \\ 27 \end{bmatrix}_{3\times1}$$

$$\begin{bmatrix} 3 & 2 & 3 \\ 3 & 2 & 6 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 8 & 8 \\ 27 & 54 \end{bmatrix} \delta(x) = \frac{-1}{3} \begin{bmatrix} -3+16+81 \\ -3+16+162 \end{bmatrix}$$

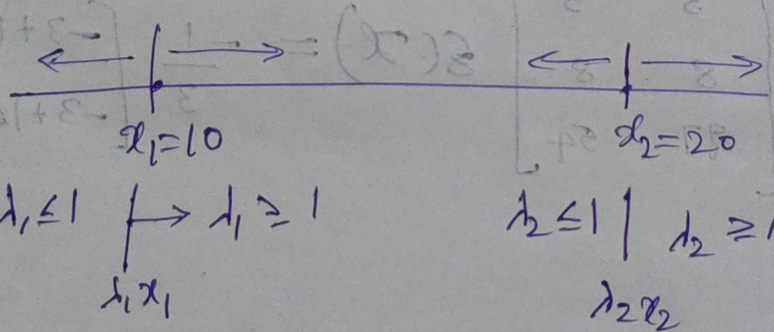
- MAE: Mean absolute error

$$E = |e_1| + |e_2| + |e_3|$$

☆ A general property of least-$p^{th}$ optimization technique is that the larger the power $p$, the slower the rate of convergence.

---

<u>Convex Set</u>: $S \in R^n$, $x_1$ & $x_2$, $y = \lambda x_1 + (1-\lambda) x_2$, if $y \in S$ then 'S' is a convex set.

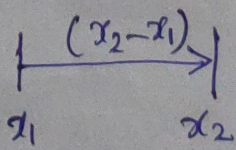


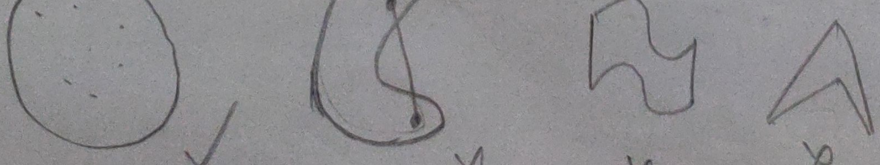

$$y = \lambda_1 x_1 + \lambda_2 x_2 \quad ; \quad \lambda_1 + \lambda_2 = 1$$

$$= x_1 + \lambda_2 (x_2 - x_1) \qquad \lambda_i \geq 0$$

$$= (x_1 - x_2)\lambda_1 + x_2$$

$y = x_1 + \theta (x_2 - x_1)$ ; here $(x_2 - x_1)$ is a vector

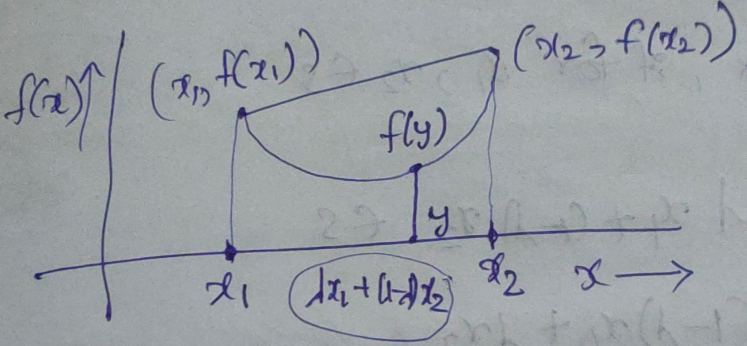


## Convex function:

. A function 'f' is convex on a convex set S if it satisfies $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$

$\# \; f(y) = f(\lambda x_1 + (1-\lambda) x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$




$y = \lambda_p x_1 + (1-\lambda_p) x_2$

Concave : $f(\alpha x + (1-\alpha) y) \geq \alpha f(x) + (1-\alpha) f(y)$

Condition for Convexity :

→ A function is convex if and only if $f''(x) \geq 0 \;\; \forall x \in S$

→ $\forall d \in S, \;\; d^T \nabla^2 f \, d \geq 0 \Rightarrow d^T F d \geq 0$

F - Hessian matrix : $F = \nabla^2 f$

21/03/2023

First order condition for Convexity

Convex function: $f(x) \quad f: R^n \to R'$

domain f is convex

(i) $x \in S$, 's' is convex

S is convex, if for $x_1, x_2 \in S$

# First order condition for convexity

27/03/2023

**Convex function:** $f(x)$ $f: R^n \to R^1$

Book
Griva..., Nash-S. So
3 Chapter
linear & nonlinear optimization

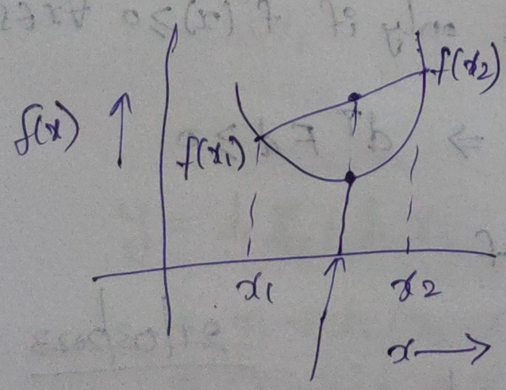
domain $f$ is convex

(i) $\exists\ x \in S$, $S$ is convex

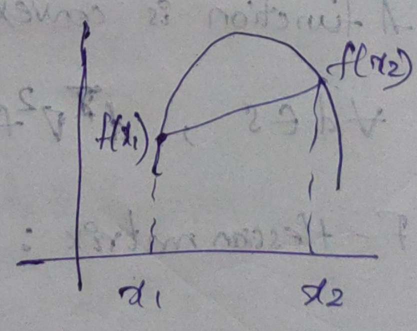
$S$ is convex, if for $x_1, x_2 \in S$

$$\exists\ y = \lambda x_1 + (1-\lambda) x_2 \in S$$
$$= (1-\lambda) x_1 + \lambda x_2 \checkmark$$

(ii) $f(\lambda x_1 + (1-\lambda) x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2) \checkmark$




Convex

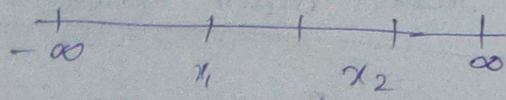



Concave

**Strictly Convex:** $f(\lambda x_1 + (1-\lambda) x_2) < \lambda f(x_1) + (1-\lambda) f(x_2)$

**Strictly Concave:** $\lambda f(x_1) + (1-\lambda) f(x_2) < f(\lambda x_1 + (1-\lambda) x_2)$

• Real line is a convex set :




## First Order Condition :

• As, it is difficult to check for all points, we have come up with First order & second order conditions.

$$f(y) \geq \underbrace{f(x) + \nabla f(x)^T \overbrace{(y-x)}^{\text{represents a vector like } \vec{d}}}_{\text{directional derivative}}$$

**Proof :** If $f$ is convex then

$$f(\lambda x_2 + (1-\lambda)x_1) \leq \lambda f(x_2) + (1-\lambda) f(x_1)$$

$$f(\lambda x_2 + x_1 - \lambda x_1) \leq \lambda f(x_2) + f(x_1) - \lambda f(x_1)$$

$$f(x_1 + \lambda(x_2 - x_1)) \leq f(x_1) + \lambda\left[f(x_2) - f(x_1)\right]$$

$$\frac{f(x_1 + \lambda(x_2 - x_1)) - f(x_1)}{\lambda} \leq f(x_2) - f(x_1)$$

$$\lim_{\lambda \to 0} (x_2 - x_1) \frac{f(x_1 + \lambda(x_2 - x_1)) - f(x_1)}{\lambda(x_2 - x_1)} \leq f(x_2) - f(x_1)$$

$$\Rightarrow (x_2 - x_1)^T \nabla f(x_1) = \nabla f(x_1)^T (x_2 - x_1) \leq f(x_2) - f(x_1)$$

$$\Rightarrow f(x_2) \geq f(x_1) + \nabla f(x_1)^T (x_2 - x_1)$$

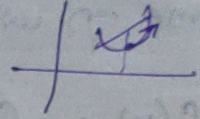
$\boxed{\text{Hence, Proved}}$

## Second order Condition for Convexity : $f \in C^2$

$$d^T \nabla^2 f(x)\, d \geq 0 \quad \equiv f''(x) \geq 0 \text{ in 1D}$$

### First order condition

$$f(y) \geq f(x) + \nabla f^T(x)(y-x)$$



$\downarrow$

function growing in the given specified direction

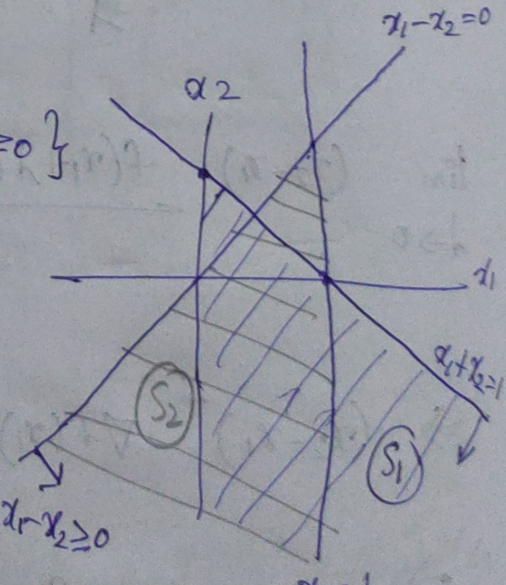
---

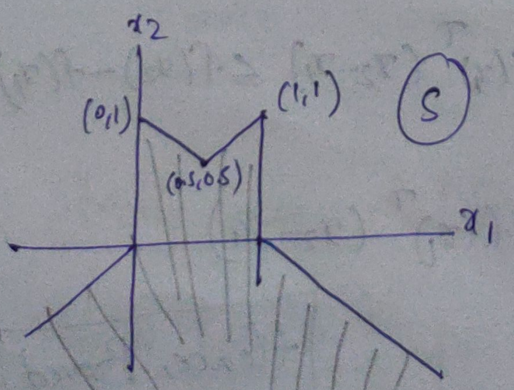
Example: 3.2 Let $S_1 = \{x : x_1 + x_2 \leq 1, x_1 \geq 0\}$ and $S_2 = \{x : x_1 - x_2 \geq 0, x_1 \leq 1\}$ and let $S = S_1 \cup S_2$. Prove that $S_1$ & $S_2$ are both convex sets but $S$ is not a convex set.

→ This shows that union of convex sets is not necessarily convex.

Sol:- $S_1 = \{x : x_1 + x_2 \leq 1, x_1 \geq 0\}$

$S_2 = \{x : x_1 - x_2 \geq 0, x_1 \leq 1\}$

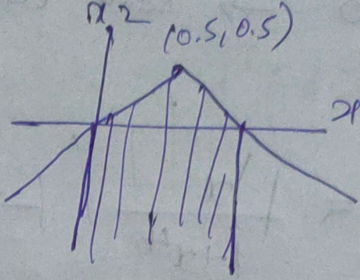


→ $S_1$ & $S_2$ are convex sets

→ But their union $S$ i.e. $S_1 \cup S_2$ is not a convex set.

$S^{(1)} = S \cap S_2$




Convex set ✓

→ Their intersection is a convex set.

---

Ex: 3.19

Check whether convex, concave or both even strictly or not

(v) $f(x) = 4 - 5x + 3x^2$

$f'(x) = -5 + 6x$

$f''(x) = 6 > 0$ → Strictly convex set

(vi) $f(x) = 2x^4 + 3x^3 + 4x^2$

$f'(x) = 8x^3 + 9x^2 + 8x$

$f''(x) = 24x^2 + 18x + 8 > 0$ → strictly convex set

(iv) $f(x) = \sqrt{1+x^2}$

$f'(x) = \dfrac{x}{\sqrt{1+x^2}}$

$$f''(x) = \frac{(\sqrt{1+x^2})(1) - x\left(\frac{x}{\sqrt{1+x^2}}\right)}{(1+x^2)} = \frac{1+x^2-x^2}{(1+x^2)^{3/2}}$$

$$= \frac{1}{(1+x^2)^{3/2}} \geq 0$$ → ~~Strictly convex~~

→ Convex set

(iii) $f(x) = 7x - 15$

$f'(x) = 7$

$f''(x) = 0 \rightarrow$ Both convex & concave set

(vii) $f(x) = \dfrac{x}{1+x^4}$ $\rightarrow$ neither convex nor concave set

$$f'(x) = \frac{(1+x^4) - x(4x^3)}{(1+x^4)^2} = \frac{1-3x^4}{(1+x^4)^2}$$

$$f''(x) = \frac{(1+x^4)^2(-12x^3) - (1-3x^4)\,2(1+x^4)(4x^3)}{(1+x^4)^4}$$

$$\cancel{= \frac{-2}{(1+x^4)^3}}$$

$$= \frac{(x^8+2x^4+1)(-12x^3) - 8x^3(1-3x^4)(1+x^4)}{(1+x^4)^4}$$

$$= \frac{4x^3}{(1+x^4)^4}\left[-3x^8 - 6x^4 - 3 - 8(1+x^4-3x^4-3x^8)\right]$$

$$= \frac{4x^3}{(1+x^4)^4}\left[21x^8 + 18x^4 - 8x^4 - 11\right]$$

$$= \frac{4x^3}{(1+x^4)^4}\left[21x^8 + 10x^4 - 11\right]$$ – can take both +ve & -ve values

$\rightarrow$ neither convex nor concave set.

Positive Semidefinite matrix

$F = \nabla^2 f$

$\rightarrow$ $d^T F d \geq 0$ (convex set)

· Positive definite matrix : $d^T F d > 0$ (strictly convex set)

---

Ex 3.20 : Determine $f(x_1, x_2) = 2x_1^2 - 3x_1x_2 + 5x_2^2 - 2x_1 + 6x_2$

is convex, concave, both or neither for $x \in R^2$.

Sol: $d^T F . d$ ?

$F = \nabla^2 f$ ; $\nabla f = \begin{bmatrix} 4x_1 - 3x_2 - 2 \\ 10x_2 - 3x_1 + 6 \end{bmatrix}$ $\nabla^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial f}{\partial x_1 \partial x_2} \\ \frac{\partial f}{\partial x_2 \partial x_1} & \frac{\partial f}{\partial x_2^2} \end{bmatrix}$

$F = \begin{bmatrix} 4 & -3 \\ -3 & 10 \end{bmatrix}$ let $d = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$[y_1 \ y_2] \underbrace{\begin{bmatrix} 4 & -3 \\ -3 & 10 \end{bmatrix}}_{2\times2} \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{2\times1} = [y_1 \ y_2] \begin{bmatrix} 4y_1 - 3y_2 \\ -3y_1 + 10y_2 \end{bmatrix}$$

$$= 4y_1^2 - 3y_1y_2 - 3y_1y_2 + 10y_2^2$$

$$= 4y_1^2 - 6y_1y_2 + 10y_2^2$$

· Calculating eigen values of $F$ $\longrightarrow$

$$F = \begin{bmatrix} 4 & -3 \\ -3 & 10 \end{bmatrix}$$

$$|F - \lambda I| = \begin{vmatrix} 4-\lambda & -3 \\ -3 & 10-\lambda \end{vmatrix}$$

$$= (4-\lambda)(10-\lambda) - (-3)(-3)$$

$$= \lambda^2 - 14\lambda + 40 - 9$$

$$= \lambda^2 - 14\lambda + 31$$

$$\lambda = \frac{14 \pm \sqrt{14^2 - 4(31)}}{2}$$

$$\Rightarrow \underbrace{\lambda = \frac{14 \pm 8.48}{2}}$$

$$\lambda = \frac{14 \pm \sqrt{72}}{2} \rightarrow \text{+ve values}$$

- Eigen values are +ve (positive definite) $\Rightarrow$ $\underbrace{\text{Strictly convex set}}$

---

## Optimality conditions for convex optimization problems

LPP : Simplex, Big M, 2-phase, Dual LPP

Unconstrained Non-LPP : Line Search $\Leftarrow$ 3

- Steepest Descent Method
- Least $p^{th}$ opt.
- Least Square opt.

$\downarrow$ all the constraints considered as ~~inactive~~

# Constrained Optimization problems: (non-LPP) Richard W.

→ Active constraints vs Inactive constraints

- Initial guess always should be in feasible region.
- When a constraint is active, often it can be used to remove one of the parameters from the error function.

**Example 11.1:** Minimize

$$f(x) = x_1^2 + 2x_2^2 + 3x_3^2 + 4x_4^2$$

Constraints: (a) $x_1 \geq 0.4$ (b) $x_2 + x_3 \geq 0.5$

(c) $x_3 + x_4 \geq 0.6$

- 8 cases
- This procedure varies the parameters, which are related by the active constraints, until a minimum is reached.
- The active constraints could be used to simplify error function, which could then be minimized by an unconstrained optimization program.

Cases: (1) none might be active (2) a (3) b (or)
(4) c might be active (5) a & b (6) a & c
(7) b & c (8) a, b & c might be active.

- But as no. of possibilities increase, it's so large that it is impractical to analyze all of them.

## Transformations:

- ~~Making~~ Converting Constrained problem to Unconstrained problem

Ex: $x_1 \geq 5 \Rightarrow x_1 = 5 + z_1^2$ ; $z_1$ can take any value

Restrictions (replace)

$y_1 \leq 1 \Rightarrow y_1 = 1 - z_2^2$

$-3 \leq x \leq 5 \Rightarrow x = (-3) + (5-(-3))(\sin z)^2$

$x_{iL} \leq x \leq x_{iU} \Rightarrow x = x_{iL} + (x_{iU} - x_{iL}) \sin^2 z$

## Penality functions:

Restrictions

$y_1 \le 1 \Rightarrow y_1 = 1 - z_2^2$

$-3 \le x \le 5 \Rightarrow x = (-3) + (5-(-3))(\sin z)^2$

$x_{iL} \le x \le x_{iU} \Rightarrow x = x_{iL} + (x_{iU} - x_{iL}) \sin^2 z$

## Penality functions:

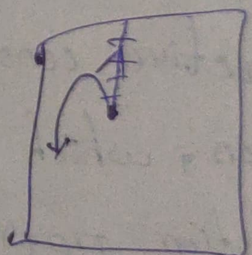
23/03/2023

For constraints like: $x_1 + x_2 \le 5$, $x_1 - 2x_2 \ge 10$

$$E^*(x) = E(x) + \text{penality function}$$

$$5 - x_1 - x_2 \ge 0, \quad x_1 - 2x_2 - 10 \ge 0$$

Penality function $= \sum_{i=1}^{q} \frac{1}{C_i(x)}$



$$= \gamma \sum_{i=1}^{q} \frac{1}{C_i(x)}$$

- As $x$ approaches a boundary of the feasible region the penality function will become very large - which is exactly what is desired.

Problem: given Area of sheet $A_0$ (rectangular one), objective is to make cylinder ~~with~~ to maximize volume.

Sol

Volume of cylinder $= \pi r^2 h$

Surface area of cylinder $= 2\pi r^2 + 2\pi r h$

Maximize: $V(r,h) = \pi r^2 h$

$A_0 = 2\pi r^2 + 2\pi r h$

$= 2\pi r (r+h)$

$\Rightarrow h = \dfrac{A_0 - 2\pi r^2}{2\pi r}$

$\Rightarrow V(r,h) = \pi r^2 h = \pi r^2 \left( \dfrac{A_0 - 2\pi r^2}{2\pi r} \right)$

$= \dfrac{A_0 r}{2} - \pi r^3$

$\Rightarrow V(r) = \dfrac{A_0 r}{2} - \pi r^3$

Based on FONC: $\dfrac{dV}{dr} = 0$

$\Rightarrow \dfrac{A_0}{2} - 3\pi r^2 = 0$

$\Rightarrow r^2 = \dfrac{A_0}{6\pi}$

$\Rightarrow r = + \sqrt{\dfrac{A_0}{6\pi}}$

$$r = \sqrt{\frac{A_0}{6\pi}} \Rightarrow h = \frac{A_0 - 2\pi r^2}{2\pi r}$$

$$\Rightarrow h = \frac{1}{2\pi}\sqrt{\frac{6\pi}{A_0}}\left[A_0 - 2\pi\left(\frac{A_0}{6\pi}\right)\right]$$

$$= \sqrt{\frac{3}{2\pi A_0}}\left[A_0 - \frac{A_0}{3}\right]$$

$$= \frac{2A_0}{3}\sqrt{\frac{3}{2\pi A_0}}$$

$$\Rightarrow h = \sqrt{\frac{2A_0}{3\pi}}$$

$$\boxed{\therefore\ r = \sqrt{\frac{A_0}{6\pi}}\ ,\ h = \sqrt{\frac{2A_0}{3\pi}}}$$

---

Book

**Lagrange's Multiplier Method :** Stephen Boyd - Convex Opt. Cambridge Uni.

(used when presence of equality constraints) — Lagrange's multiplier

Ex: $J(r, h, \lambda) = V(r, h) + \lambda\left(A(r, h) - A_0\right)$

→ Here, '$\lambda$' units in 'meters'; as dimensions should be matched (always)

→ $\frac{\partial J}{\partial r} = 0$ , $\frac{\partial J}{\partial h} = 0$ , $\frac{\partial J}{\partial \lambda} = 0$

$$J(r, h, \lambda) = \pi r^2 h + \lambda\left[2\pi r^2 + 2\pi r h - A_0\right]$$

$$\frac{\partial J}{\partial r} = 2\pi r h + \lambda\left[4\pi r + 2\pi h\right] \rightarrow (1)$$

$$\frac{\partial J}{\partial h} = \pi r^2 + 2\pi r \lambda \rightarrow (2)$$

$$\frac{\partial J}{\partial \lambda} = 2\pi r^2 + 2\pi r h - A_0 \rightarrow (3)$$

$$(1) \Rightarrow 2\pi r h + 4\pi r \lambda + 2\pi h \lambda = 0$$

$$\Rightarrow rh + 2r\lambda + h\lambda = 0$$

$$(2) \Rightarrow \pi r^2 + 2\pi r \lambda = 0$$

$$\Rightarrow r^2 + 2r\lambda = 0$$

$$(3) \Rightarrow 2\pi r^2 + 2\pi r h - A_0 = 0$$

$$\Rightarrow 2\pi r(r+h) = A_0$$

$$\Rightarrow h = \frac{A_0 - 2\pi r^2}{2\pi r}$$

$$\Rightarrow \lambda = -\frac{r}{2}$$

$$\Rightarrow r\left(\frac{A_0 - 2\pi r^2}{2\pi r}\right) + 2r\left(-\frac{r}{2}\right) + \left(\frac{A_0 - 2\pi r^2}{2\pi r}\right)\left(-\frac{r}{2}\right) = 0$$

$$\Rightarrow \frac{A_0 - 2\pi r^2}{2\pi} - r^2 - \frac{(A_0 - 2\pi r^2)}{4\pi} = 0$$

$$\Rightarrow 2A_0 - 4\pi r^2 - 4\pi r^2 - A_0 + 2\pi r^2 = 0$$

$$\Rightarrow A_0 - 6\pi r^2 = 0$$

$$\Rightarrow r^2 = \frac{A_0}{6\pi}$$

$$\Rightarrow r = +\sqrt{\frac{A_0}{6\pi}}$$

$$\boxed{\therefore r = \sqrt{\frac{A_0}{6\pi}}}$$

$$h = \sqrt{\frac{2A_0}{3\pi}}$$

Example: Minimize : $f_0(x)$

S.T : $f_i(x) \le 0$ ; $i = 1, \dots, m$

$h_i(x) = 0$ ; $i = 1, \dots, p$

257/730

KKT Condition for non-convex problems

Sol:

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

→ Vectors $\lambda$ & $\nu$ are called dual variables or Lagrange multipliers

06/04/2023

LPP

max/min $f(x)$

Constraints $h_i(x) = 0$ ; $i = 1, 2, \dots l$

$g_j(x) \le / \ge b_j$ ; $j = 1, 2, \dots m$

NLPP

Unconstrained

$f(x)$

Constrained → (i) Equality

(ii) Inequality

$L(x, \lambda) = obj + \lambda(\text{equality constraint})$

Lagrange's Multiplier Method

FONC: $\nabla L = 0$ $\qquad L = f + \lambda h$

$\nabla f \pm \lambda \nabla h = 0$

$\lambda = \frac{\nabla f}{\nabla h}$ ⇒ rate of change of objective function w.r.t constraints

- $\lambda \rightarrow$ Sensitivity parameter

Shadow price - gives the indication of obj function / profit w.r.t constraints / raw materials

- $\lambda$ is unrestricted in sign when there are equality constraints.

$L(x, \lambda_1, \lambda_2) = obj + \lambda_1 \left(\begin{matrix}\text{equality}\\ \text{constraint 1}\end{matrix}\right) + \lambda_2 \left(\begin{matrix}\text{equality}\\ \text{constraint 2}\end{matrix}\right)$

- Stationary values / points - which first derivation goes to zero are point at

- By Hessian matrix, it is decided whether function goes to max. or min at that stationary point

$x^T \nabla^2 f x > 0 \rightarrow$ Positive definite (Eigen values)

$x^T \nabla^2 f x \geq 0 \rightarrow$ +ve semidefinite

$x^T \nabla^2 f x < 0 \rightarrow$ Negative definite

$x^T \nabla^2 f x \leq 0 \rightarrow$ -ve semidefinite

- Principle Diagnol elements
- <u>Principal Minors</u>.

$D_1 = |a_{11}|$ ($\leq a_{11} \Rightarrow$ matrix should have principle diagnol elements less than)

$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ ($\leq a_{22} \Rightarrow a_{11}, a_{22}$ ✓)

$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{33} & a_{33} \end{vmatrix}$ ($\leq a_{33} \Rightarrow a_{11}, a_{22}, a_{33}$ ✓)

... $D_N$

- $D_1 > 0$, $D_2 > 0$, $D_3 > 0$ ... $D_N > 0 \Rightarrow$ +ve definite

<u>Ex:</u> $\begin{bmatrix} 4 & -3 \\ -3 & 10 \end{bmatrix}$ $D_1 = 4 > 0$
$D_2 = 40 - 9 = 31 > 0$

↳ Positive definite

- $D_1 \geq 0$, $D_2 \geq 0$, $D_3 \geq 0$, ... $D_N \geq 0 \Rightarrow$ +ve semidefinite

- $D_1 < 0$, $D_2 > 0$, $D_3 < 0$, ... , $D_N < 0 \Rightarrow$ -ve definite
(Concave) (alternative signs)

- $D_1 \leq 0$, $D_2 \geq 0$, $D_3 \leq 0$, ... ; $D_N \leq 0 \Rightarrow$ -ve semidefinite

---

<u>KKT conditions</u>: (FONC)

<u>Optimize</u>: $f(x)$ <u>S.T</u>: $h_i(x)$, $g_j(x) \leq / \geq b_j$

→ Used when there are both equality & inequality constraints present.

$g_j(x) \leq b_j$

<u>New inequality constraint</u>: $g_j(x) + \overset{\text{slack variable}}{S_j^2} = b_j$

$$g_j(x) + S_j^2 - b_j = 0$$

$g_j'(x) = 0$

$$L(x, \lambda, u) = f(x) - \sum_i \lambda_i h_i(x) - \sum_j u_j g_j'(x)$$

$$\boxed{\frac{\partial L}{\partial x} = 0}$$

$$\Rightarrow \frac{\partial f}{\partial x} + \sum_i \lambda_i \frac{\partial h_i}{\partial x} + \sum_j u_j \frac{\partial}{\partial x}\left[g_j(x) + S_j^2 - b_j\right] = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} + \sum_i \lambda_i \frac{\partial h_i}{\partial x} + \sum_j u_j \frac{\partial g_j}{\partial x} = 0$$

$$\boxed{\Delta f + \sum_i \lambda_i \Delta h_i + \sum_j u_j \nabla g_j = 0}$$ → 1st KKT Condition

→ let us first consider there are no equality constraints

$$\frac{\partial L}{\partial u_i} = 0 \Rightarrow g_j(x) + S_j^2 - b_j = 0$$

$$\Rightarrow S_j^2 = b_j - g_j(x)$$

$$\frac{\partial L}{\partial S_j} = 0 \Rightarrow \sum 2 S_j u_j = 0$$

$$\Rightarrow S_j^2 u_j = 0$$

$$\Rightarrow [b_j - g_j(x)] u_j = 0$$

$$\boxed{u_j g_j = 0}$$ → complementary slackless condition

→ 2nd KKT condition

$u_j$ → Lagrange's Multiplier

$g_j$ → constraint

11/04/2023

NCPP

Ex: Max: $Z = 3.6x_1 - 0.4x_1^2 + 1.6x_2 - 0.2x_2^2$

s.t: $2x_1 + x_2 \le 10$

$x_1, x_2 \ge 0$

KKT conditions: $\frac{\partial L}{\partial x} = 0$, $\lambda_i g_i = 0$, $\lambda_i \ge 0$ (max)

$\le 0$ (min)

$h_i(x) = 0$; $g_j(x) \ge 0$ / $\le 0$

Sol: $L = f(x) - \lambda g(x)$

$= (3.6x_1 - 0.4x_1^2 + 1.6x_2 - 0.2x_2^2)$

$- \lambda(2x_1 + x_2 - 10)$

Necessary conditions: $\frac{\partial L}{\partial x} = 0$, $\lambda g = 0$, $g \le 0$, $\lambda \ge 0$, $x \ge 0$

$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 3.6 - 0.8x_1 - 2\lambda = 0 \longrightarrow (1)$

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 1.6 - 0.4x_2 - \lambda = 0 \longrightarrow (2)$

$\lambda(2x_1 + x_2 - 10) = 0 \longrightarrow (3)$

$\lambda \ge 0 \longrightarrow (4)$

$2x_1 + x_2 \le 10 \longrightarrow (5)$

$x_1, x_2 \ge 0 \longrightarrow (6)$

Case I : for $\lambda = 0$, $x_1 = 4.5$, $x_2 = 4$ — not a feasible solution

$2(4.5) + 4 \leq 10$ (False)

Case II : $\lambda \neq 0$ : $x_1 = \frac{3.6 - 2\lambda}{0.8}$ ; $x_2 = \frac{1.6 - \lambda}{0.4}$

$$2\left(\frac{3.6 - 2\lambda}{0.8}\right) + \frac{1.6 - \lambda}{0.4} = 10$$

$$\Rightarrow \lambda = 0.4$$

Hence, $x_1 = 3.5$, $x_2 = 3$ — optimal solution.

$\therefore (x_1, x_2 ; \lambda) = (3.5, 3, 0.4)$

$\therefore Z = 10.7$

---

NLPP

Ex: ~~M~~Min: $Z = -\log x_1 - \log x_2$

s.t: $x_1 + x_2 \leq 2$ ; $x_1, x_2 \geq 0$

Sol: $L(x, \lambda) = -\log x_1 - \log x_2 - \lambda(x_1 + x_2 - 2)$

$\frac{\partial L}{\partial x_1} = 0 \Rightarrow \frac{-1}{x_1} - \lambda = 0 \longrightarrow (1)$

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow \frac{-1}{x_2} - \lambda = 0 \longrightarrow (2)$

$\lambda(x_1 + x_2 - 2) = 0 \longrightarrow (3)$

$x_1 + x_2 \leq 2 \longrightarrow (4)$

$\lambda \leq 0 \longrightarrow (5)$

$x_1, x_2 \geq 0 \longrightarrow (6)$

Case 1: $\lambda = 0 \Rightarrow x_1 = \infty, x_2 = \infty$

- case is discarded

Case 2: $\lambda \neq 0$

$\Rightarrow x_1 = \frac{-1}{\lambda}$ & $x_2 = \frac{-1}{\lambda}$

$$\left(\frac{-1}{\lambda}\right) + \left(\frac{-1}{\lambda}\right) = 2$$

$$\Rightarrow \lambda = -1$$

Here, $x_1 = 1, x_2 = 1$

$\therefore (x_1, x_2, \lambda) = (1, 1, -1)$

$\therefore Z = 0$

---

Ex: Max: $Z = 8x_1 + 10x_2 - x_1^2 - x_2^2$

S.t: $3x_1 + 2x_2 \leq 6$

$x_1, x_2 \geq 0$

Part 2

Ex & obj

Cond with

2 constraint

For NLPP:

$$\boxed{L = f(x) - \sum \lambda_i g_i(x)}$$

Lagrangian function

(1) Max $f(x)$ & S.t: $g_i(x) \leq 0$

Necessary conditions: $\frac{\partial L}{\partial x} = 0$, $\lambda_i g_i = 0$, $g_i(x) \leq 0$,

$\lambda_i \geq 0$

(2) Min $f(x)$ & S.t: $g_i(x) \leq 0$

Necessary conditions: $\frac{\partial L}{\partial x} = 0$, $\lambda_i g_i = 0$, $g_i(x) \leq 0$,

$\lambda_i \leq 0$

---

Ex: Max: $Z = 12x_1 + 21x_2 + 2x_1x_2 - 2x_1^2 - 2x_2^2$

S.t: $x_1 + x_2 \leq 10$, $x_2 \leq 8$, $x_1, x_2 \geq 0$

# Advanced Optimization Techniques

## Genetic Algorithm (GA):

→ A GA is a search technique used in computing to find true or approximate solutions to optimization and search problems.

→ GAs are categorized as global search heuristics..

Ex: A population of $n$ random strings.

Suppose that $k=10$ & $n=6$, fitness fn. $f$ is

toss a fair coin 60 times & get following initial population.

$S_1 = 1111010101$ $f(S_1)=7$, $S_2 = 0111000101$ $f(S_2)=5$

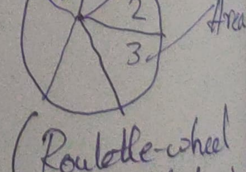
$S_3 = 1110110101$ $f(S_3)=7$, $S_4 = 0100010011$ $f(S_4)=4$

$S_5 = 1110111101$ $f(S_5)=8$, $S_6 = 0100110000$ $f(S_6)=3$

- $\dfrac{f(i)}{\sum_i f(i)}$ - Individual $i$ will have a probability to be chosen.

(Ex:

| Binary representation | $x$ | $f(x)=x^2$ |
|---|---|---|
| $00101 \leftarrow$ | 5 | 25 ← has more chance $\dfrac{f(x_i)}{\sum_i f(x_i)}$ ) |
| | 1 | 1 |
| $10011 \leftarrow$ | $-3$ | 9 |
| | $-0.1$ | $0.01$ |




(Roulette-wheel Selection)

Step1: Selection

Step2: Crossover

Step3: Mutations - intentionally making errors

→ Methods of Representation: Arrays of strings, integers, characters (binary bits)

→ Methods of Selection:

- Roulette - Wheel selection
- Elitist Selection
- Fitness - pre
- Scaling Selection
- Rank "
- Cut-off "

→ Methods of Reproduction: (Single-point, Multi-point)

- Crossover
- Mutation

13/04/2023

GA

1. Initialization of the population.
2. Calculation of the fitness)
   Evaluating the function at given character/trait
3. Selection 4. Pairing

Roulette-wheel method

Ex:

New offspring

| | | |
|---|---|---|
| $P_1$ | $S1 = P_3$ | ← $P_4$ |
| $P_2$ | $S2 = P_1$ | |
| $P3$ | $S3 = P_4$ | |
| $P4$ – max. probability | ⋮ | |
| ⋮ | ⋮ | |
| | $S9 = P_4$ | |
| $P_{10}$ | $S10 =$ | |

## Particle Swarm Optimization (PSO)

- Proposed by James Kennedy
- 
- Inspired from nature
- PSO uses a no. of agents (particles) that constitute a swarm moving around in search space looking for the best solution.
- Each particle in search space adjusts its 'flying' according to its own flying experience as well as flying experience of other particles.

· Each particle has 3 parameters position, velocity and previous best position, particle with best fitness value is called a global best position.

· Particle best : $P_i$ best – the best position the particle seen so far

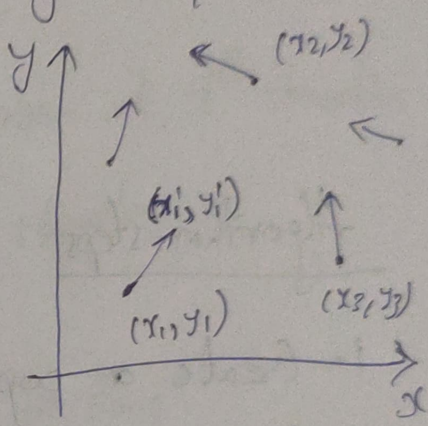
· Global best : Best position among all particles.

$(x_1, y_1)$ $f_1 \longleftarrow f_1'$

$(x_2, y_2)$ $f_2 \longleftarrow f_2'$

$(x_3, y_3)$ $f_3 \longleftarrow f_3'$

$f_4 \longleftarrow f_4'$




· Particle tries to move towards the particle best.

· Overall, all particles tries to move towards the global best.

· If $f_1'$ - fitness value is more than $f_1$ - fitness value, then $f_1'$ is considered, if it is not a large value compared to previous value than the previous value & only considered.

→ Movement towards a promising area to get global optimum

→

## Algorithm-steps

1. Create a population of agents uniformly distributed over X

2. Evaluate ~~[illegible]~~ each particle's position according to objective function.

3. If a particle's current position is better than its previous

4.

5. Update particle's velocities : $(C_1, C_2)$ → constants $(U_1, U_2)$ → some random value

$$V_i^{t+1} = w V_i^t + C_1 U_1^t (pb_i^t - p_i^t) + C_2 U_2^t (gb^t - p_i$$

weight velocity

$t+1^{th}$ iteration

$pb_i^t$ - $i^{th}$ particle best position at $t^{th}$ iteration

$v_i^t$ - inertia of $i^{th}$ particle at $t^{th}$ iteration

$c_1 u_1^t (pb_i^t - P_i^t)$ - personal influence

$c_2 u_2^t (gb^t - P_i^t)$ - Social influence

$P_i^t$ - $i^{th}$ particle present position at $t^{th}$ iteration

$gb^t$ - global best at $t^{th}$ iteration.

6. Move particles to their new positions

$$P_i^{t+1} = P_i^t + V_i^{t+1}$$ constant which gives a value proportional to velocity

7. go to step 2 until stopping criteria are satisfied.

<u>Inertia</u>: Makes particle move in same direction & with same velocity.