Solutions Manual

to accompany

Probability, Random Variables and Stochastic Processes

Fourth Edition

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2-1 We use De Morgan's law:

(a) $\overline{\bar{A}+\bar{B}} + \overline{\bar{A}+B} = AB + A\bar{B} = A(B+\bar{B}) = A$

(b) $(A+B)(\overline{AB}) = (A+B)(\bar{A}+\bar{B}) = A\bar{B} + B\bar{A}$

because $A\bar{A} = \{\emptyset\}$ $B\bar{B} = \{\emptyset\}$

2-2 If $A = \{2 \le x \le 5\}$ $B = \{3 \le x \le 6\}$ $S = \{-\infty < x < \infty\}$ then

$A+B = \{2 \le x \le 6\}$ $AB = \{3 \le x \le 5\}$

$(A+B)(\overline{AB}) = \{2 \le x \le 6\}\ [\{x < 3\} + \{x > 5\}]$

$= \{2 \le x < 3\} + \{5 < x \le 6\}$

2-3 If $AB = \{\emptyset\}$ then $A \subset \bar{B}$ hence

$P(A) \le P(\bar{B})$

2-4 (a) $P(A) = P(AB) + P(A\bar{B})$ $P(B) = P(AB) + P(\bar{A}B)$

If, therefore, $P(A) = P(B) = P(AB)$ then

$P(A\bar{B}) = 0$ $P(\bar{A}B) = 0$ hence

$P(\bar{A}B + A\bar{B}) = P(\bar{A}B) + P(A\bar{B}) = 0$

(b) If $P(A) = P(B) = 1$ then $1 = P(A) \le P(A+B)$ hence

$1 = P(A+B) = P(A) + P(B) - P(AB) = 2 - P(AB)$

This yields $P(AB) = 1$

2-5 From (2-13) it follows that

$P(A+B+C) = P(A) + P(B+C) - P[A(B+C)]$

$P(B+C) = P(B) + P(C) - P(BC)$

$P[A(B+C)] = P(AB) + P(AC) - P(ABC)$

because $ABAC = ABC$. Combining, we obtain the desired result.

Using induction, we can show similarly that

$$P(A_1 + A_2 + \cdots + A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$
$$- P(A_1A_2) - \cdots - P(A_{n-1}A_n)$$
$$+ P(A_1A_2A_3) + \cdots + P(A_{n-2}A_n)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\pm P(A_1A_2 \cdots A_n)$$

2-6 Any subset of S contains a countable number of elements, hence, it can be written as a countable union of elementary events. It is therefore an event.

2-7 Forming all unions, intersections, and complements of the sets {1} and {2,3}, we obtain the following sets:

$\{\emptyset\}$, {1}, {4}, {2,3}, {1,4}, {1,2,3}, {2,3,4}, {1,2,3,4}

2-8 If $A \subset B, P(A) = 1/4$, and $P(B) = 1/3$, then

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)}{P(B)} = \frac{1/4}{1/3} = \frac{3}{4}$$

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

2-9
$$P(A|BC)P(B|C) = \frac{P(ABC)}{P(BC)}\frac{P(BC)}{P(C)} = \frac{P(ABC)}{P(C)} = P(AB|C)$$

$$P(A|BC)P(B|C)P(C) = \frac{P(ABC)}{P(BC)}\frac{P(BC)}{P(C)}P(C) = P(ABC)$$

2-10 We use induction. The formula is true for $n = 2$ because $P(A_1A_2) = P(A_2|A_1)P(A_1)$. Suppose that it is true for n. Since

$$P(A_{n+1}A_n \cdots A_1) = P(A_{n+1}|A_n \cdots A_2A_1)P(A_1 \cdots A_n)$$

we conclude that it must be true for $n + 1$.

2-11 First solution. The total number of m element subsets equals $\binom{n}{m}$ (see Probl. 2-26). The total number of m element subsets containing ζ_o equals $\binom{n-1}{m-1}$. Hence

$$p = \binom{n}{m}\Big/\binom{n-1}{m-1} = \frac{m}{n}$$

Second solution. Clearly, $P\{\zeta_o|A_m\} = m/n$ is the probability that ζ_o is in a specific A_m. Hence (total probability)

$$p = \sum P\{\zeta_o|A_m\}p(A_m) = \frac{m}{n}\sum P(A_m) = \frac{m}{n}$$

where the summation is over all sets A_m.

2-12 (a) $P\{6 \le t \le 8\} = \frac{2}{10}$

(b) $P\{6 \le t \le 8 | t > 5\} = \frac{P\{6 \le t \le 8\}}{P\{t > 5\}} = \frac{2}{5}$

2-13 From (2-27) it follows that

$$P\{t_o \le t \le t_o + t_1 | t \ge t_o\} = \int_{t_o}^{t_o + t_1} \alpha(t)dt \Bigg/ \int_{t_o}^{\infty} \alpha(t)dt$$

$$P\{t \le t_1\} = \int_0^{t_1} \alpha(t)dt$$

Equating the two sides and setting $t_1 = t_0 + \Delta t$ we obtain

$$\alpha(t_o) \Bigg/ \int_{t_o}^{\infty} \alpha(t)dt = \alpha(0)$$

for every t_o. Hence,

$$-\ln \int_{t_o}^{\infty} \alpha(t)dt = \alpha(0)t_o \qquad \int_{t_o}^{\infty} \alpha(t)dt = e^{-\alpha(0)t_o}$$

Differentiating the setting $c = \alpha(0)$, we conclude that

$$\alpha(t_o) = c\,e^{ct} \qquad P\{t \le t_1\} = 1 - e^{-ct_1}$$

2-14 If A and B are independent, then $P(AB) = P(A)P(B)$. If they are mutually exclusive, then $P(AB) = 0$. Hence, A and B are mutually exclusive and independent iff $P(A)P(B) = 0$.

2-15 Clearly, $A_1 = A_1A_2 + A_1\bar{A}_2$ hence

$$P(A_1) = P(A_1A_2) + P(A_1\bar{A}_2)$$

If the events A_1 and $\bar{A}_2$ are independent, then

$$P(A_1\bar{A}_2) = P(A_1) - P(A_1A_2) = P(A_1) - P(A_1)P(A_2)$$
$$= P(A_1)[1 - P(A_2)] = P(A_1)P(\bar{A}_2)$$

hence, the events A_1 and $\bar{A}_2$ are independent. Furthermore, S is independent with any A because SA = A. This yields

$$P(SA) = P(A) = P(S)P(A)$$

Hence, the theorem is true for $n = 2$. To prove it in general we use induction: Suppose that A_{n+1} is independent of $A_1,\ldots,A_n$. Clearly, A_{n+1} and $\bar{A}_{n+1}$ are independent of $B_1,\ldots,B_n$. Therefore

$$P(B_1 \cdots B_nA_{n+1}) = P(B_1 \cdots B_n)P(A_{n+1})$$
$$P(B_1 \cdots B_n\bar{A}_{n+1}) = P(B_1 \cdots B_n)P(\bar{A}_{n+1})$$

2.16 The desired probabilities are given by (a)

$$\frac{\binom{m-1}{k-1}}{\binom{n}{k}}$$

(b)

$$\frac{\binom{m}{k}}{\binom{n}{k}}$$

2.17 Let A_1, A_2 and A_3 represent the events

$$\begin{aligned} A_1 &= \text{"}\textit{ball numbered less than or equal to } m \textit{ is drawn}\text{"} \\ A_2 &= \text{"}\textit{ball numbered } m \textit{ is drawn}\text{"} \\ A_3 &= \text{"}\textit{ball numbered greater than } m \textit{ is drawn}\text{"} \end{aligned}$$

$$\begin{aligned} P(A_1 \; occurs \; n_1 = k-1, \quad & A_2 \; occurs \; n_2 = 1 \; and \; A_3 \; occurs \; n_3 = 0) \\ &= \frac{(n_1+n_2+n_3)!}{n_1!\,n_2!\,n_3!} p_1^{n_1} p_2^{n_2} p_3^{n_3} \\ &= \frac{k!}{(k-1)!} \left(\frac{m}{n}\right)^{k-1} \left(\frac{1}{n}\right) \\ &= \frac{k}{n} \left(\frac{m}{n}\right)^{k-1} \end{aligned}$$

2.18 All cars are equally likely so that the first car is selected with probability $p = 1/3$. This gives the desired probability to be

$$\binom{10}{3} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^7 = 0.26$$

2.19 $P\{\text{"}\textit{drawing a white ball }\text{"}\} = \dfrac{m}{m+n}$
$P(\text{"}\textit{atleat one white ball in k trials }\text{"})$

$$\begin{aligned} &= 1 - P(\text{"}\textit{all black balls in k trials}\text{"}) \\ &= 1 - \frac{\binom{n}{k}}{\binom{m+n}{k}} \end{aligned}$$

2.20 Let $D = 2r$ represent the penny diameter. So long as the center of the penny is at a distance of r away from any side of the square, the penny will be entirely inside the square. This gives the desired probability to be

$$\frac{(1-2r)^2}{1} = \left(1 - \frac{3}{4}\right)^2 = \frac{1}{16}.$$

2.21 Refer to Example 3.14.
(a) Using (3.39), we get

$$P(\text{"}all\ one-digit\ numbers\text{"}) = \frac{\binom{9}{6}\binom{42}{0}}{\binom{51}{6}} = 5 \times 10^{-6}.$$

(b)

$$P(\text{"}two\ one\text{–}digit\ and\ four\ two\text{–}digit\ numbers\text{"}) = \frac{\binom{9}{2}\binom{42}{4}}{\binom{51}{6}} = 0.224.$$

2-22 The number of equations of the form $P(A_iA_k) = P(A_i)P(A_k)$ equals $\binom{n}{2}$. The number of equations involving r sets equals $\binom{n}{r}$. Hence the total number N of such equations equals

$$N = \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n}$$

And since

$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = (1+1)^n = 2^n$$

we conclude that

$$N = 2^n - \binom{n}{0} - \binom{n}{1} = 2^n - 1 - n$$

2-23 We denote by B_1 and B_2 respectively the balls in boxes 1 and 2 and by R the set of red balls. We have (assumption)

$$P(B_1) = P(B_2) = 0.5 \qquad P(R|B_1) = 0.999 \qquad P(R|B_2) = 0.001$$

Hence (Bayes' theorem)

$$P(B_1|R) = \frac{P(R|B_1)P(B_1)}{P(R|B_1)P(B_1) + P(R|B_2)P(B_2)} = \frac{0.999}{0.999+0.001} = 0.999$$

2-24 We denote by B_1 and B_2 respectively the ball in boxes 1 and 2 and by D all pairs of defective parts. We have (assumption)

$$P(B_1) = P(B_2) = 0.5$$

To find $P(D|B_1)$ we proceed as in Example 2-10:

First solution. In box B_1 there are 1000×999 pairs. The number of pairs with both elements defective equals 100×99. Hence,

$$P(D|B_1) = \frac{100 \times 99}{1000 \times 999}$$

Second solution. The probability that the first bulb selected from B_1 is defective equals 100/1000. The probability that the second is defective assuming the first was effective equals 99/999. Hence,

$$P(D|B_1) = \frac{100}{1000} \times \frac{99}{999}$$

We similarly find

$$P(D|B_2) = \frac{100}{2000} \times \frac{99}{1999}$$

(a) $P(D) = P(D|B_1)P(B_1) + P(D|B_2)P(B_2) = 0.0062$

(b) $P(B_1|D) = \dfrac{P(D|B_1)P(B_1)}{P(D)} = 0.80$

2-25 Reasoning as in Example 2-13, we conclude that the probability that the bus and the train meet equals

$$(10 + x)60 - \frac{10^2}{2} - \frac{x^2}{2}$$

Equating with 0.5, we find $x = 60 - 10\sqrt{11}$.

2-26 We wish to show that the number $N_n(k)$ of the element subsets of S equals

$$N_n(k) = \frac{n(n-1) \cdots (n-k+1)}{1 \cdot 2 \cdots k}$$

This is true for $k = 1$ because the number of 1-element subsets equals n. Using induction in k, we shall show that

$$N_n(k+1) = N_n(k)\frac{n-k}{k+1} \qquad 1 < k < n \qquad \text{(i)}$$

We attach to each k-element subset of S one of the remaining $n-k$ elements of S. We, then, form $N_n(k)(n-k)$ $k+1$-element subsets. However, these subsets are not all different. They form groups each of which has $k+1$ identical elements. We must, therefore, divide by $k+1$.

2-27 In this experiment we have 8 outcomes. Each outcome is a selection of a particular coin and a specific sequence of heads or tails; for example fhh is the outcome "we selected the fair coin and we observed hh". The event F = {the selected coin is fair} consists of the four outcomes fhh, fht, fth and fhh. Its complement $\overline{F}$ is the selection of the two-headead coin. The event HH = {heads at both tosses} consists of two outcomes. Clearly,

$$P(F) = P(\overline{F}) = \frac{1}{2} \qquad P(HH|F) = \frac{1}{4} \qquad P(HH|\overline{F}) = 1$$

Our problem is to find P(F|HH). From (2-41) and (2-43) it follows that

$$P(HH) = P(HH|F)P(F) + P(HH|\overline{F})P(\overline{F}) = \frac{5}{8}$$

$$P(F|HH) = \frac{P(HH|F)P(F)}{P(HH)} = \frac{1/4 \times 1/2}{5/8} = \frac{1}{5}$$

Problem Solutions for Chapter 3

3.1 (a) $P(\text{A } occurs\ atleast\ twice\ in\ n\ trials)$

$$\begin{aligned} &= 1 - P(\text{A } never\ occurs\ in\ n\ trials) - P(\text{A } occurs\ once\ in\ n\ trials) \\ &= 1 - (1-p)^n - np(1-p)^{n-1} \end{aligned}$$

(b) $P(\text{A } occurs\ atleast\ thrice\ in\ n\ trials)$

$$\begin{aligned} &= 1 - P(\text{A never occurs in } n \text{ trials}) - P(\text{A occurs once in } n \text{ trials}) \\ &\quad -P(\text{A occurs twice in } n \text{ trials}) \\ &= 1 - (1-p)^n - np(1-p)^{n-1} - \frac{n(n-1)}{2} p^2 (1-p)^{n-2} \end{aligned}$$

3.2

$$P(doublesix) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

$$\begin{aligned} & P(\text{“double six atleast three times in n trials''}) \\ &= 1 - \binom{50}{0} \left(\tfrac{1}{36}\right)^0 \left(\tfrac{35}{36}\right)^{50} - \binom{50}{1} \left(\tfrac{1}{36}\right) \left(\tfrac{35}{36}\right)^{49} - \binom{50}{2} \left(\tfrac{1}{36}\right)^2 \left(\tfrac{35}{36}\right)^{48} \\ &= 0.162 \end{aligned}$$

3.6 (a)

$$p_1 = 1 - \left(\frac{5}{6}\right)^6 = 0.665$$

(b)

$$1 - \left(\frac{5}{6}\right)^{12} - \binom{12}{1} \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^{11} = 0.619$$

(c)

$$1 - \left(\frac{5}{6}\right)^{18} - \binom{18}{1} \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^{17} - \binom{18}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{16} = 0.597$$

3.7 (a) Let n represent the number of wins required in 50 games so that the net gain or loss *does not* exceed \$1. This gives the net gain to be

$$-1 < n - \frac{50-n}{4} < 1$$

$$16 < n < 17.3$$

$$n = 17$$

$$P(\text{net gain } \textit{does not} \text{ exceed } \$1) = \binom{50}{17}\left(\frac{1}{4}\right)^{17}\left(\frac{3}{4}\right)^{33} = 0.432$$

$$P(\text{net gain or loss exceeds } \$1) = 1 - 0.432 = 0.568$$

(b) Let n represent the number of wins required so that the net gain or loss *does not* exceed \$5. This gives

$$-5 < n - \frac{(50-n)}{2} < 5$$

$$13.3 < n < 20$$

$$P(\text{net gain } \textit{does not} \text{ exceed } \$5) = \sum_{n=14}^{19}\binom{50}{n}\left(\frac{1}{4}\right)^{n}\left(\frac{3}{4}\right)^{50-n} = 0.349$$

$$P(\text{net gain or loss exceeds } \$5) = 1 - 0.349 = 0.651$$

3.8 Define the events

A=" r successes in n Bernoulli trials"

B="success at the i^{th} Bernoulli trial"

C="$r-1$ successes in the remaining $n-1$ Bernoulli trials excluding the i^{th} trial"

$$P(A) = \binom{n}{r} p^r \, q^{n-r}$$

$$P(B) = p$$

$$P(C) = \binom{n-1}{r-1} p^{r-1} \, q^{n-r}$$

We need

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(BC)}{P(A)} = \frac{P(B)\,P(C)}{P(A)} = \frac{r}{n}.$$

3.9 There are $\binom{52}{13}$ ways of selecting 13 cards out of 52 cards. The number of ways to select 13 cards of any suit (out of 13 cards) equals $\binom{13}{13} = 1$. Four such (mutually exclusive) suits give the total number of favorable outcomes to be 4. Thus the desired probability is given by

$$\frac{4}{\binom{52}{13}} = 6.3 \times 10^{-12}$$

3.10 Using the hint, we obtain

$$p\,(N_{k+1}-N_k)=q\,(N_k-N_{k-1})-1$$

Let

$$M_{k+1}=N_{k+1}-N_k$$

so that the above iteration gives

$$\begin{aligned} M_{k+1} &= \frac{q}{p}\,M_k-\frac{1}{p} \\ &= \begin{cases} \left(\frac{q}{p}\right)^k\,M_1-\dfrac{1}{p-q}\left\{1-(\frac{q}{p})^k\right\}, & p\neq q \\ \qquad\qquad M_1-\frac{k}{p}, & p=q \end{cases} \end{aligned}$$

This gives

$$\begin{aligned} N_i &= \sum_{k=0}^{i-1} M_{k+1} \\ &= \begin{cases} \left(M_1+\dfrac{1}{p-q}\right)\sum\limits_{k=0}^{i-1}\left(\frac{q}{p}\right)^k-\dfrac{i}{p-q}, & p\neq q \\ \qquad iM_1-\dfrac{i(i-1)}{2p}, & p=q \end{cases} \end{aligned}$$

where we have used $N_o=0$. Similarly $N_{a+b}=0$ gives

$$M_1+\frac{1}{p-q}=\frac{a+b}{p-q}\cdot\frac{1-q/p}{1-(q/p)^{a+b}}.$$

Thus

$$N_i=\begin{cases} \dfrac{a+b}{p-q}\cdot\dfrac{1-(q/p)^i}{1-(q/p)^{a+b}}-\dfrac{i}{p-q}, & p\neq q \\ \qquad i(a+b-i), & p=q \end{cases}$$

which gives for $i = a$

$$
\begin{aligned}
N_a &= \begin{cases} \dfrac{a+b}{p-q} \cdot \dfrac{1-(q/p)^a}{1-(q/p)^{a+b}} - \dfrac{a}{p-q}, & p \neq q \\ ab, & p = q \end{cases} \\
&= \begin{cases} \dfrac{b}{2p-1} - \dfrac{a+b}{2p-1} \cdot \dfrac{1-(p/q)^b}{1-(p/q)^{a+b}}, & p \neq q. \\ ab, & p = q \end{cases}
\end{aligned}
$$

3.11

$$P_n = pP_{n+\alpha} + qP_{n-\beta}$$

Arguing as in (3.43), we get the corresponding iteration equation

$$P_n = P_{n+\alpha} + qP_{n-\beta}$$

and proceed as in Example 3.15.

3.12 Suppose one best on $k = 1, 2, \cdots, 6$.
Then

$$p_1 = P(k \textit{ appears on one dice}) = \binom{3}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2$$

$$p_2 = P(k \textit{ appear on two dice}) = \binom{3}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)$$

$$p_3 = P(k \textit{ appear on all the tree dice}) = \left(\frac{1}{6}\right)^3$$

$$p_0 = P(k \textit{ appear none}) = \left(\frac{5}{6}\right)^3$$

Thus, we get

$$\text{Net gain} = 2p_1 + 3p_2 + 4p_3 - p_0 = 0.343.$$

CHAPTER 4

4-1 From the evenness of $f(x)$: $1 - F(x) = F(-x)$.

From the definition of x_u: $u = F(x_u)$, $1 - u = F(x_{1-u})$. Hence

$$1 - u = 1 - F(x_u) = F(-x_u) = F(x_{1-u}) \qquad -x_u = x_{1-u}$$

4-2 From the symmetry of $f(x)$: $1 - F(\eta+a) = F(\eta-a)$. Hence [see (4-8)]

$$P\{\eta-a < \underset{\sim}{x} < \eta+a\} = F(\eta+a) - F(\eta-a) = 2F(\eta+a) - 1$$

This yields

$$1-\alpha = 2F(\eta+a) - 1 \qquad F(\eta+a) = 1 - \alpha/2 \qquad \eta+a = x_{1-\alpha/2}$$

$$F(a-\eta) = \alpha/2 \qquad a-\eta = x_{\alpha/2}$$

4-3 (a) In a linear interpolation:

$$x_u \simeq x_a + \frac{x_b-x_a}{u_b-u_a}(u-u_a) \qquad \text{for } x_a < x_u < x_b$$

From Table 4-1 page 106

$$z_{0.9} \simeq 1.25 + \frac{0.00565}{0.00885} \times 0.05 = 1.2819$$

Proceeding simiplarly, we obtain

$u =$	0.9	0.925	0.95	0.975	0.99
$z_u =$	1.282	1.440	1.645	1.960	2.327

(b) If $\underset{\sim}{z}$ is such that $\underset{\sim}{x} = \eta + \sigma\underset{\sim}{z}$ then $\underset{\sim}{z}$ is N(0,1) and $G(z) = F_x(\eta+\sigma z)$. Hence,

$$u = G(z_u) = F_x(\eta+\sigma z_u) = F_x(x_u) \qquad x_u = \eta+\sigma z_u$$

4-4 $p_k - 2G(k) = 1 = 2\ \text{erf}k$

(a) From Table 4-1

$k =$	1	2	3
$p_k =$	0.6827	0.9545	0.9973

(b) From Table 3-1 with linear interpolation:

$p_k =$	0.9	0.99	0.999
$k =$	1.282	2.32	3.090

(c) $$P\{\eta - z_u\sigma < \underset{\sim}{x} < \eta + z_u\sigma\} = 2G(z_u) - 1 = \gamma$$

Hence, $G(z_u) = (1+\gamma)/2 \qquad u = (1+\gamma)/2$

4-5 (a) $F(x) = x$ for $0 \le x \le 1$; hence, $u = F(x_u) \doteq x_u$

(b) $F(x) = 1-e^{-2x}$ for $x \ge 0$; hence, $u = 1-e^{-2x_u}$

$$x_u = -\frac{1}{2}\,\ell n(1-u)$$

$u =$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$x_u =$	0.0527	0.1116	0.1783	0.2554	0.3466	0.4581	0.6020	0.847	1.1513

4-6 Percentage of units between 96 and 104 ohms equals 100p where $p = P\{96 < \underset{\sim}{R} < 104\} = F(104) - F(96)$

(a) $F(R) = 0.1(R-95)$ for $95 \le R \le 105$. Hence,

$p = 0.1(104-95) - 0.1(96-95) = 0.8$

(b) $p = G(2.5) - G(-2.5) = 0.9876$

4-7 From (4-34), with $\alpha = 2$ and $\beta = 1/\lambda$ we get $f(x) = c^2\, xe^{-cx}U(x)$

$$F(x) = c^2 \int_0^x ye^{-cy}dy = 1 - e^{-cx} - cxe^{-cx}$$

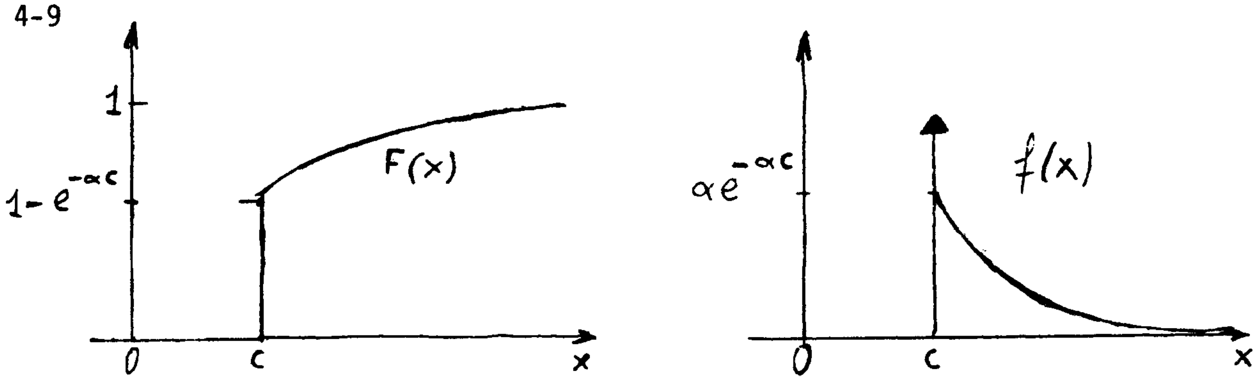
4-8 $\quad \{(\underset{\sim}{x} - 10)^2 < 4\} = \{8 < \underset{\sim}{x} < 12\}$

$$P\{(\underset{\sim}{x} - 10)^2 < 4\} = G(12 - 10) - G(8 - 10) = 0.954$$

$$f(x\,|\,(\underset{\sim}{x} - 10)^2 < 4\} = \frac{f(x)}{P\{8 < \underset{\sim}{x} < 12\}} = \frac{1}{0.954\sqrt{2\pi}}\, e^{-\frac{(x-10)^2}{2}}$$

for $8 < x < 12$ and zero otherwise

4-9



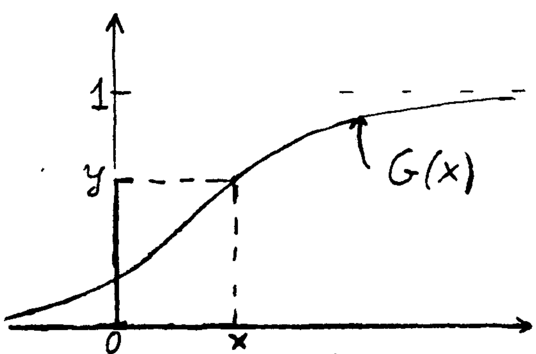
$F(x) = (1 - e^{-\alpha x})U(x-c)$ $\qquad$ $f(x) = (1 - e^{-\alpha c})\delta(x-c) + e^{-\alpha x}U(x-c)$

4-10 (a) $P\{1 \le x \le 2\} = G(\frac{2}{2}) - G(\frac{1}{2}) = 0.1499$

(b) $P\{1 \le \underset{\sim}{x} \le 2 \,|\, \underset{\sim}{x} \ge 1\} = \dfrac{G(1) - G(0.5)}{1 - G(0.5)} = \dfrac{0.1499}{0.3085} = 0.4857$

because $\{1 \le \underset{\sim}{x} \le 2,\ \underset{\sim}{x} \ge 1\} = \{1 \le \underset{\sim}{x} \le 2\}$

4-11



$x = G^{-1}(y) = H(y)$

If $\underset{\sim}{x}(t_1) \le x$

then

$t_i \le y = G(x)$

Hence,

$P\{\underset{\sim}{x} \le x\} = P\{t_i \le y\} = y = G(x)$

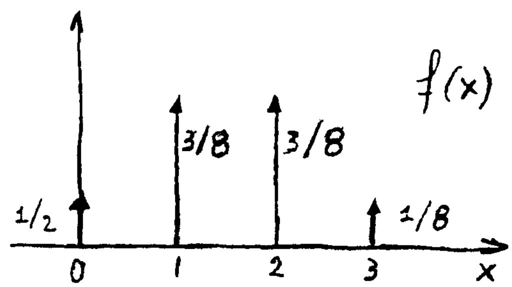
4-12 (a) $P\{\mathbf{x} < 1024\} = G\left(\frac{1024 - 1000}{20}\right) = G(1.2) = 0.8849$

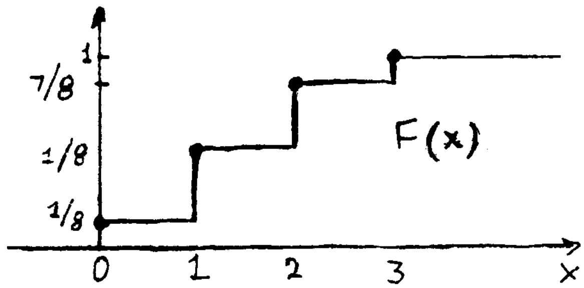
(b) $P\{\mathbf{x} < 1024 \mid \mathbf{x} > 961\} = \dfrac{P\{961 < \mathbf{x} < 1024\}}{P\{\mathbf{x} > 961\}}$

$$= \frac{G(1.2) - G(-1.95)}{1 - G(-1.95)} = 0.8819$$

(c) $P\{31 < \sqrt{\mathbf{x}} \le 32\} = P\{961 < \mathbf{x} \le 1024\} = 0.8593$

4-13 $P\{\mathbf{x} = 0\} = \frac{1}{8}$ $\quad P\{\mathbf{x} = 1\} = \frac{3}{8}$ $\quad P\{\mathbf{x} = 2\} = \frac{3}{8}$ $\quad P\{\mathbf{x} = 3\} = \frac{1}{8}$





4-14 (a) 1. $f_x(x) = \dfrac{1}{2^{900}} \sum_{k=0}^{900} \binom{900}{k} \delta(x-k)$

2. $f_x(x) = \dfrac{1}{15\sqrt{2\pi}} \sum_{k=0}^{900} e^{-(k-450)^2/450}\, \delta(x-k)$

(b) $P\{435 \le x \le 460\} = G\left(\frac{10}{15}\right) - G\left(-\frac{15}{15}\right) = 0.5888$

4-15 If $x > b$ then $\{\mathbf{x} \le x\} = S$ $\quad F(x) = 1$

If $x < a$ then $\{\mathbf{x} \le x\} = \{\emptyset\}$ $\quad F(x) = 0$

4-16 If $\mathbf{y}(\zeta_i) \le w$, then $\mathbf{x}(\zeta_i) \le w$ because $\mathbf{x}(\zeta_i) \le \mathbf{y}(\zeta_i)$.

Hence,

$$\{\mathbf{y} \le w\} \subset \{\mathbf{x} \le w\} \qquad P\{\mathbf{y} \le w\} \le P\{\mathbf{x} \le w\}$$

Therefore $\quad F_y(w) \le F_x(w)$

4-17 From (4-80)

$$f(x) = kx\, e^{-\int_0^x kt\,dt} = kx\, e^{-kx^2/2}$$

4-18 It follows from (2-41) with

$$A_1 = \{\underset{\sim}{x} \le x\} \qquad A_2 = \{\underset{\sim}{x} > x\}$$

4-19 It follows from

$$F_x(x|A) = \frac{P\{\underset{\sim}{x} \le x, A\}}{P(A)} \qquad P\{A|\underset{\sim}{x} \le x\} = \frac{P\{\underset{\sim}{x} \le x, A\}}{P\{\underset{\sim}{x} \le x\}}$$

4-20 We replace in (4-80) all probabilities with conditional probabilities assuming $\{\underset{\sim}{x} \le x_o\}$. This yields

$$\int_{-\infty}^{\infty} P(A|\underset{\sim}{x} = x, \underset{\sim}{x} \le x_o) f(x|\underset{\sim}{x} \le x_o)dx = P(A|\underset{\sim}{x} \le x_o)$$

But $f(x|\underset{\sim}{x} \le x_o) = 0$ for $x > x_o$ and

$\{\underset{\sim}{x} = x, \underset{\sim}{x} \le x_o\} = \{\underset{\sim}{x} = x\}$ for $x \le x_o$. Hence,

$$\int_{-\infty}^{x_o} P(A|\underset{\sim}{x} = x) f(x|\underset{\sim}{x} \le x_o)dx = P(A|\underset{\sim}{x} \le x_o)$$

Writing a similar equation for $P(B|\underset{\sim}{x} \le x_o)$ we conclude that, if $P(A|\underset{\sim}{x} = x) = P(B|\underset{\sim}{x} = x)$ for $x \le x_o$, then $P(A|\underset{\sim}{x} \le x_o) = P(B|\underset{\sim}{x} \le x_o)$

4-21 (a) Clearly, $f(p) = 1$ for $0 \le p \le 1$ and 0 otherwise; hence

$$P\{0.3 \le \underset{\sim}{p} \le 0.7\} = \int_{0.3}^{0.7} dp = 0.4$$

(b) We wish to find the conditional probability $P\{0.3 \le \underset{\sim}{p} \le 0.7|A\}$ where A = {6 heads in 10 tosses}. Clearly $P\{A|\underset{\sim}{p}=p\} = p^6(1-p)^4$. Hence, [see (4-81)]

$$f(p|A) = \frac{p^6(1-p)^4}{\int_0^1 p^6(1-p)^4 dp} = \frac{p^6(1-p)^4}{4329\times 10^{-7}}$$

This yields

$$P\{0.3 \le \underset{\sim}{p} \le 0.7|A\} = \int_{0.3}^{0.7} f(p|A)dp = \frac{10^7}{4329}\int_{0.3}^{0.7} p^6(1-p)^4 dp = 0.768$$

4-22 (a) In this problem, $f(p) = 5$ for $0.4 \le \underset{\sim}{p} \le 0.6$ and zero otherwise; hence [see(4-82)]

$$P(H) = 5\int_{0.4}^{0.6} pdp = 0.5$$

(b) With A = {60 heads in 100 tosses} it follows from (4-82) that

$$f(p|A) = p^{60}(1-p)^{40} \Big/ \int_{0.4}^{0.6} p^{60}(1-p)^{40}dp$$

for $0.4 \le p \le 0.6$ and 0 otherwise. Replacing $f(p)$ by $f(p|A)$ in (4-82), we obtain

$$P(H|A) = \int_{0.4}^{0.6} pf(p|A)dp = 0.56$$

4-23 $n = 900 \qquad p = q = 0.5 \qquad np = 450 \qquad \sqrt{npq} = 15$

$$k_1 = 420 \qquad k_2 = 465 \qquad \frac{k_2 - np}{\sqrt{npq}} = 1 \qquad \frac{k_1 - np}{\sqrt{npq}} = -2$$

$$P\{420 \le k \le 465\} = G(1) - [1 - G(-2)] = G(1) + G(2) - 1$$
$$= 0.819$$

4-24 For a fair coin $\sqrt{npq} = \sqrt{n}/2$. If

$k_1 = 0.49n$ and $k_2 = 0.52n$ then

$$\frac{k_2 - np}{\sqrt{npq}} = \frac{0.52n - n/2}{\sqrt{n}/2} = 0.04\sqrt{n} \qquad \frac{k_1 - np}{\sqrt{npq}} = -0.02\sqrt{n}$$

$$P\{k_1 \le k \le k_2\} = G(0.04\sqrt{n}) + G(0.02\sqrt{n}) - 1 \ge 0.9$$

From Table 4-1 (page 106) it follows that

$$0.02\sqrt{n} > 1.3 \qquad n > 65^2$$

4-25

(a) Assume $n = 1{,}000$ (Note correction to the problem)

$P(A) = 0.6 \qquad np = 600 \qquad npq = 240 \qquad k_2 = 650 \qquad k_1 = 550$

$$\frac{k_2 - np}{\sqrt{npq}} = \frac{50}{\sqrt{240}} = 3.23 \qquad\qquad \frac{k_1 - np}{\sqrt{npq}} = -\,3.23$$

$P\{550 \le k \le 650\} = 2G(3.23) - 1 = 0.999$

(b) $$P\{0.59n \le k \le 0.61n\} = 2G\left(\frac{0.01n}{\sqrt{0.24n}}\right) - 1$$

$$= 2G\left(\sqrt{\frac{n}{2400}}\right) - 1 = 0.476$$

Hence, (Table 3-1) $\quad n \simeq 9220$

4-26 With $a = 0$, $b = T/4$ it follows that

$p = 1-e^{-1/4} = 0.22 \qquad np = 220 \qquad npq = 171.6 \qquad k_2 = 100$

$$\frac{k_2 - np}{\sqrt{npq}} = -\,9.16 \quad \text{and (4-100) yields}$$

$P\{0 \le k \le 100\} \simeq G(-9.16) \simeq 0.$

4-27 The event

A = {k heads show at the first n tossings but not earlier}

occurs iff the following two events occur

B = {k-1 heads show at the first n-1 tossing}

C = {heads show at the nth tossing}

And since these two events are independent and

$$P(B) = \binom{n-1}{k-1}p^{k-1}q^{n-1-(k-1)} \qquad\qquad P(C) = p$$

we conclude that

$$P(A) = P(B)P(C) = \binom{n-1}{k-1}p^k q^{n-k}$$

4-28 $$-\frac{d}{dx}\left(\frac{1}{x}e^{-x^2/2}\right) = \left(1+\frac{1}{x^2}\right)e^{-x^2/2} > e^{-x^2/2}$$

Multiplying by $1/\sqrt{2\pi}$ and integrating from x to ∞, we obtain

$$\frac{1}{x\sqrt{2\pi}}e^{-x^2/2} > \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\zeta^2/2}\,d\zeta = 1 - G(x)$$

because

$$\frac{1}{x}e^{-x^2/2} \xrightarrow[x\to\infty]{} 0$$

The first inequality follows similarly because

$$-\frac{d}{dx}\left[\left(\frac{1}{x}-\frac{1}{x^3}\right)e^{-x^2/2}\right] = \left(1-\frac{3}{x^4}\right)e^{-x^2/2} < e^{-x^2/2}$$

4-29 If $P(A) = p$ then $P(\overline{A}) = 1-p$. Clearly $P_1 = 1-Q_1$ where Q_1 equals the probability that A does not occur at all. If $pn \ll 1$, then $Q_1 = (1-p)^n \simeq 1 - np \quad P_1 \simeq p$

4-30 With $p = 0.02$, $n = 100$, $k = 3$, it follows from (4-107) that the unknown probability equals

$$\binom{100}{3}(0.02)^3(0.98)^{97} \simeq \frac{2^3}{3!}e^{-2} = \frac{4}{3}e^{-2}$$

4-31 With $n = 3$, $r = 3$, $k_1 = 2$, $k_2 = 2$, $k_3 = 1$, $p_1 = p_2 = p_3 = 1/6$, it follows from (4-102) that the unknown probability equals

$$\frac{5!}{1!2!2!}\,\frac{1}{6^6} = 0.00386$$

4-32 With $r = 2$, $k_1 = k$, $k_2 = n-k$, $p_1 = p$, $p_2 = 1-p = q$, we obtain

$$k_1 - np_1 = k - np \qquad k_2 - np_2 = n-k-nq = np - k$$

Hence, the bracket in (4-103) equals

$$\frac{(k_1-np_1)^2}{np_1} + \frac{(k_2-np_2)^2}{np_2} = \frac{(k-np)^2}{n}\left(\frac{1}{p}+\frac{1}{q}\right) = \frac{(k-np)^2}{npq}$$

as in (4-90).

4-33 $P(M) = 2/36 \qquad P(\overline{M}) = 34/36$. The events M and $\overline{M}$ form a partition, hence, [see (2-41)]

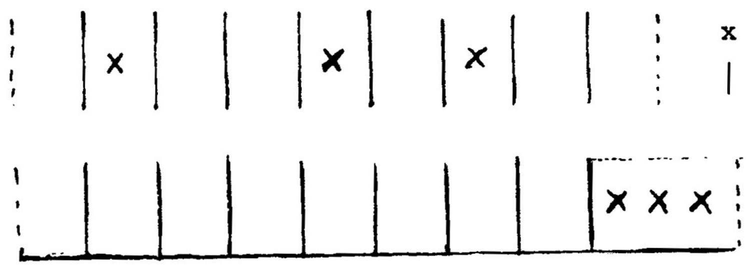
$$P(A) = P(A|M)P(M) + P(A|\overline{M})P(\overline{M}) \tag{i}$$

Clearly, $P(A|M) = 1$ because, if M occurs at first try, X wins. The probability that X wins after the first try equals $P(A|\overline{M})$. But in the experiment that starts at the second rolling, the first player is Y and the probability that he wins equals $P(\overline{A}) = 1-p$. Hence, $P(A|\overline{M}) = P(\overline{A}) = 1-p$. And since $P(M) = 1/18$ $P(\overline{M}) = 17/18$ (i) yields

$$p = \frac{1}{18} + (1-p)\frac{17}{18} \qquad\qquad p = \frac{18}{35}$$

4-34

(a) Each of the n particles can be placed in any one of the m boxes. There are n particles, hence, the number of possibilities equals $N - m^n$. In the m preselected boxes, the particles can be placed in $N_A = n!$ ways (all permutations of n objects). Hence $p = n!/m^n$.

(b) 

x n particles

| m-1 interior walls

All possibilities are obtained by permuting the n+m-1 objects consisting of the m-1 interior walls with and n particles. The (m-1)! permutations of the walls and the n! permutations of the particles must count as one. Hence

$$N = \frac{(n+m-1)!}{n!\,(m-1)!} \qquad\qquad N_A = 1$$

(c) Suppose that S is a set consisting of the m boxes. Each placing of the particles specifies a subset of S consisting of n elements (box). The number of such subsets equals $\binom{m}{n}$ (see Prob. 2-26). Hence,

$$N = \binom{m}{n} \qquad\qquad N_A = 1$$

4-35 If $k_1 + k_2 << n$, then $k_3, \simeq n$ and

$$k_3(p_1 + p_2) = [n - (k_1 + k_2)](p_1 + p_2) \simeq n(p_1 + p_2)$$

$$p_3 = 1 - (p_1 + p_2) \simeq e^{-(p_1+p_2)} \qquad p_3^{k_3} \simeq e^{-n(p_1+p_2)}$$

$$\frac{n!}{k_1!k_2!k_3!} = \frac{n(n-1)\cdots(n-k_3+1)}{k_1!k_2!} \simeq \frac{n^{k_1+k_2}}{k_1!k_2!}$$

Hence,

$$\frac{n!}{k_1!k_2!k_3!} p_1^{k_1} p_2^{k_2} p_3^{k_3} \simeq e^{-np_1} \frac{(np_1)^{k_1}}{k_1!} e^{-np_2} \frac{(np_2)^{k_2}}{k_2!}$$

4-36 The probability p that a particular point is in the interval (0,2) equals 2/100. (a) From (3-13) it follows that the probability p_1 that only one out of the 200 points is in the interval (0,2) equals

$$p_1 = \binom{200}{1} \times 0.02 \times 0.09^{199}$$

(b) With $np = 200 \times 0.02 = 4$ and $k = 1$, (3-41) yields $p_1 \simeq e^{-4} \times 4 = 0.073$

CHAPTER 5

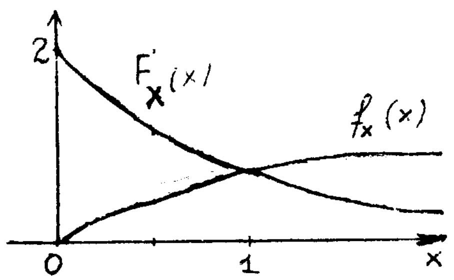
5-1 $\qquad \eta = 2\eta_x + 4 = 14 \qquad \sigma_y^2 = 4\sigma_x^2 = 16$

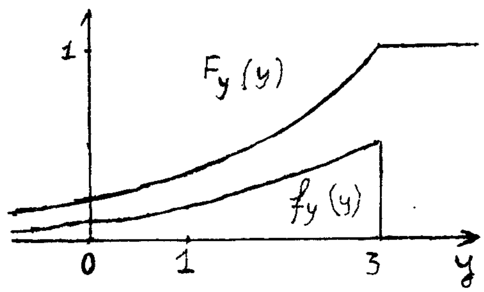
5-2 $\{\underset{\sim}{y} \le y\} = \{-4\underset{\sim}{x} + 3 \le y\}\ \{\underset{\sim}{x} \le (y-3)/4\}$. Hence

$$F_y(y) = P\left\{\underset{\sim}{x} \ge \frac{3-y}{4}\right\} = 1 - F_x\left(\frac{3-y}{4}\right) \qquad f_y(y) = \frac{1}{4} f_x\left(\frac{3-y}{4}\right)$$

Since $F_x(x) = (1-e^{-2x})U(x)$, this yields

$$F_y(y) = e^{(y-3)/2}U\left(\frac{y-3}{2}\right) \qquad f_y(y) = \frac{1}{2}e^{(y-3)/2}U\left(\frac{y-3}{2}\right)$$

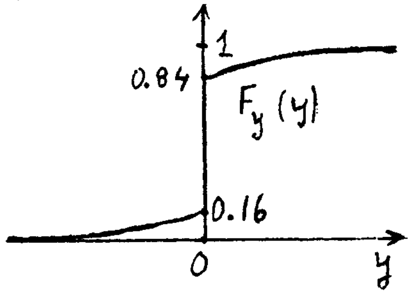


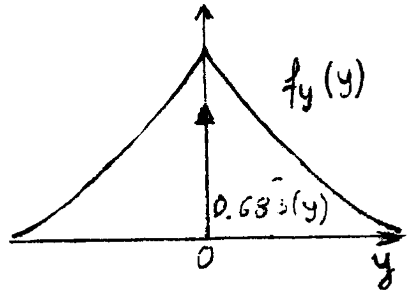


5-3 From Example 5-3 with $F_x = G(x/c)$:

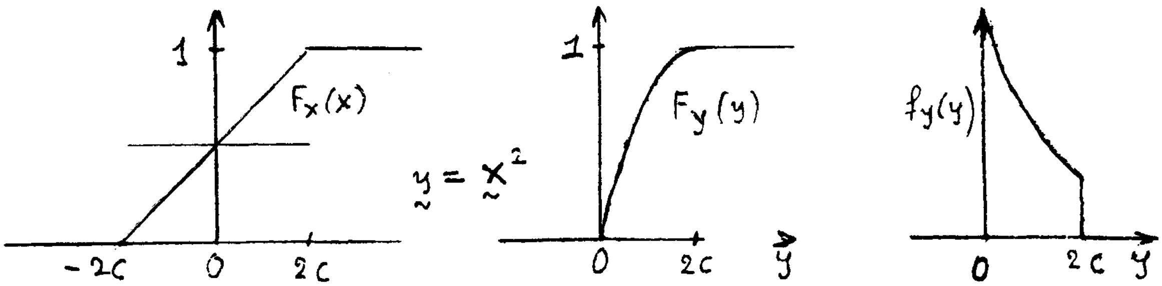
$$f_y(y) - \begin{cases} G(y/c+1) & y \ge 0 \\ G(y/c-1) & y < 0 \end{cases}$$

$$f_y(y) = 0\ 0.68\ \delta(y) + \frac{1}{c\sqrt{2\pi}}\left[e^{-(y+c)^2/2c^2}U(y) + e^{-(y-c)^2/2c^2}U(-y)\right]$$



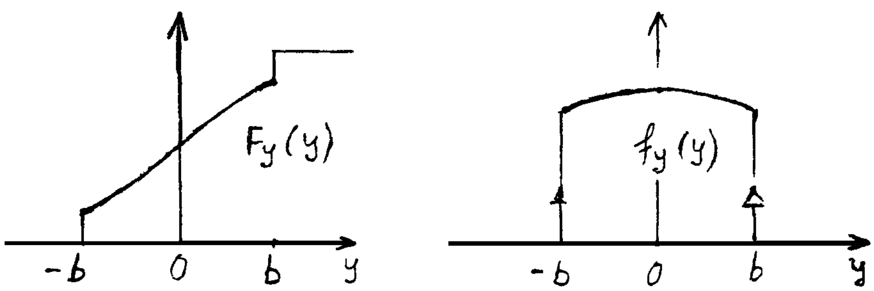


5-4 If $\underset{\sim}{y} = \underset{\sim}{x}^2$ and $F_x(x) = (x+2c)/4c$ for $|x| \le 2c$, then (see Example 5-2) $F_y(y) = \sqrt{y}\,/2c$ and $f_y(y) = 1/4\sqrt{y}$ for $0 < y < 2c$.



5-5 From Example 5-4 with $F_x(x) = G(x/b)$: For $|x| \le b$ $F_u(y) = G(y/b)$ and

$$f_y(y) = 0.16\delta(y+b) + \frac{1}{b\sqrt{2\pi}} e^{-y^2/2b^2} + 0.16\delta(y-b)$$



5-6 The equation $y = -\ln x$ has a single solution $x = e^{-y}$ for $y > 0$ and no solutions for $y < 0$. Furthermore, $g'(x) = -1/x = -e^{y}$. Hence

$$f_y(y) = \frac{f_x(e^{-y})}{e^{y}}\, U(y) = e^{-y} U(y)$$

5-7 Clearly, $\underset{\sim}{z} \le z$ iff the number $\underset{\sim}{n}(0,z)$ of the points in the interval (0,z) is at least one.

Hence,

$$F_z(z) = P\{\underset{\sim}{z} \le z\} = P\{\underset{\sim}{n}(0,z) > 0\} = 1 - P\{\underset{\sim}{n}(0,z) = 0\}$$

The probability p that a particular point is in the integral (0,z) equals z/100. With n = 200, k = 0, and p = z/100, (3-21) yields $P\{\underset{\sim}{n}(0,z) = 0\} = (1-p)^{200}$. Hence,

(a) $F_z(z) = 1 - \left(1 - \dfrac{z}{100}\right)^{100}$

(b) From (4-107) it follows thata $F_z(z) \simeq 1 - e^{-2z}$ for z << 100.

5.8

$$Y = \sqrt{X} \quad \Rightarrow \quad x_1 = y^2$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2y}$$

Thus

$$f_Y(y) = \frac{1}{\left|\frac{dy}{dx}\right|} f_X(x_1) = 2y f_X(y^2)$$

$$\frac{2y}{\lambda} e^{-y^2/\lambda} = \begin{cases} \frac{y}{\sigma^2} e^{-y^2/2\sigma^2}, & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

which represents Rayleigh density function (with $\lambda = 2\sigma^2$).

5-9 For both cases, $f_y(y) = 0$ for $y < 0$.

(a) If $y > 0$ and $|x| = y$, then $x_1 = y$, $x_2 = -y$. Hence

$$f_y(y) = [f_x(y) + f_x(-y)]U(y)$$

(b) If $y > 0$ and $e^{-x}U(x) = y$, then $x = -\ell n y$.

Furthermore, $P\{\underset{\sim}{y} = 0\} = P\{\underset{\sim}{x} \le 0\} = F_x(0)$. Hence

$$f_y(y) = F_x(0)\delta(y) + \frac{1}{y} f_x(-\ell n y)U(y)$$

5-10 (a) If $y \geq 0$ and $(x-1)U(x-1) = y$, then $\{\mathbf{y} \leq y\} = \{\mathbf{x} \leq y+1\}$.
If $y < 0$, then $\{\mathbf{y} < y\} = \{\emptyset\}$

$$F_y(y) = F_x(1+y)U(y) = [1 - e^{-2(y+1)}]U(y)$$

$$f_y(y) = (1 - e^{-2})\delta(y) + 2e^{-2(y+1)}U(y)$$

(b) If $y > 0$ and $y = x^2$, then $\{\mathbf{y} \leq y\} = \{-\sqrt{y} \leq \mathbf{x} \leq \sqrt{y}\}$

$$F_y(y) = F_x(\sqrt{y}) - F_x(-\sqrt{y}) = (1 - e^{-2\sqrt{y}})U(y)$$

$$f_y(y) = \frac{1}{\sqrt{y}} e^{-2\sqrt{y}}U(y)$$

5-11 If $y = \arctan x$, then $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$

$$f_y(y) = (1+x^2)f_x(\tan y) = \frac{1+x^2}{\pi(1+x^2)} = \frac{1}{\pi} \qquad \frac{\pi}{2} < y < \frac{\pi}{2}$$

5-12 (a) If $y = x^3$ then $x = \sqrt[3]{y}$ for any y

$$f_y(y) = \frac{1}{3\sqrt[3]{y^2}} f_x(\sqrt[3]{y}) = \frac{1}{12\pi\sqrt[3]{y^2}}$$

for $|y| < 8\pi^3$ and zero otherwise

(b) If $y = x^4$ and $y > 0$, then $x_1 = \sqrt[4]{y}$ $x_1 = -\sqrt[4]{y}$

$$f_y(y) = \frac{1}{4\sqrt[4]{y^3}} \left[f_x(\sqrt[4]{y}) + f_x(-\sqrt[4]{y}) \right] = \frac{1}{8\pi\sqrt[4]{y^3}}$$

for $0 < y < 16\pi^4$ and zero otherwise

(c) If $y = 2\sin(3x + 40°)$ and $|y| < 2$ then $x = x_i$ as shown.

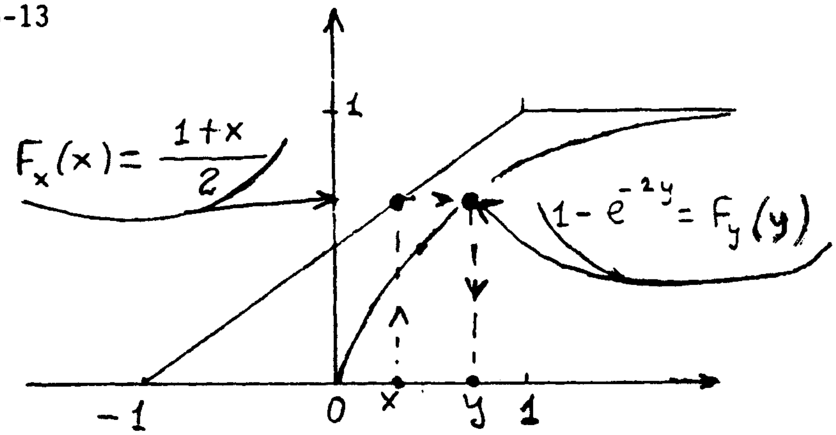
$$\frac{dy}{dx} = \frac{1}{6\sqrt{1 - y^2/4}}$$

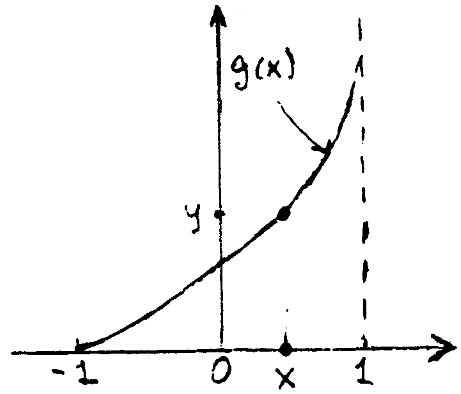
In the interval $(-2\pi, 2\pi)$ there are 12 x_i's. Hence

$$f_y(y) = \frac{1}{3\sqrt{4-y^2}} \sum_i f_x(x_i) = \frac{12}{12\pi\sqrt{4-y^2}} = \frac{1}{\pi\sqrt{4-y^2}}$$

for $|y| < 2$ and zero otherwise.

5-13





As in (5-43) $\qquad F_y[g(x)] = F_x(x)$

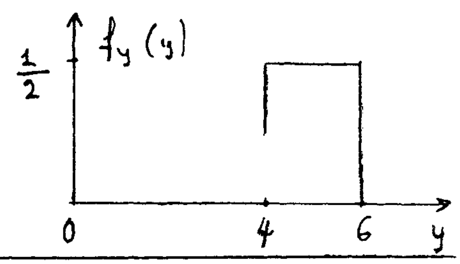
$$\frac{1+x}{2} = 1 - e^{-2y} \qquad y = g(x) = -\frac{1}{2}\ln\frac{1-x}{2}$$

for $|x| < 1$. For $x \le -1$, $g(x) = 0$; for $x \ge 1$, $g(x) = \infty$.

5-14 (a) $\quad g(x) = 2F_x(x) + 4 \qquad g'(x) = 2f_x(x)$

If $4 < y < 6$ then $y = 2F_x(x) + 4$ has a unique solution x_1 and

$$f_y(y) = \frac{f_x(x_1)}{2f_x(x_1)} = \frac{1}{2}$$



(b) $\quad$ Similarly $g(x) = 2F_x(x) + 8$

5-15 (a) $\quad$ The RV $\underset{\sim}{x}$ takes the values $k = 0,1,\ldots,10$ and

$$P\{x = k\} = p_k = \binom{10}{k}\frac{1}{2^{10}} \qquad 0 \le k \le 10$$

$F_x(x)$ is a staircase function with discontinuities at the points $x = k$ and jumps equal to p_k.

(b) $\quad$ The RY $\underset{\sim}{y} = (\underset{\sim}{x} - 3)^2$ takes the values $y = k^2$ for $k = 0,1,\ldots,7$ and probabilities $P\{y = k^2\} = q_k$.

$k =$	0	1	2	3	4	5	6	7
$q_k =$	p_3	p_2+p_4	p_1+p_5	p_0+p_6	p_7	p_8	p_9	p_{10}

5.16

$$X \sim Beta(\alpha,\beta) \quad \text{gives}$$

$$f_X(x) = \frac{1}{B(\alpha,\beta)}\, x^{\alpha-1}\,(1-x)^{\beta-1}, \quad 0 < x < 1.$$

$$Y = 1 - X \quad \Rightarrow \quad x_1 = 1 - y, \quad |\,\frac{dy}{dx}\,| = 1$$

$$\Rightarrow F_Y(y) = \frac{1}{|\frac{dy}{dx}|}\, f_X(1-y) = \begin{cases} \frac{1}{B(\beta,\alpha)}\, y^{\beta-1}\,(1-y)^{\alpha-1}, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

This gives

$$Y \sim Beta(\beta,\alpha).$$

5.17

$$X \sim \chi^2(n) \Rightarrow$$

$$f_X(x) = \frac{1}{2^{n/2}\Gamma(n/2)}\, x^{n/2-1}\, e^{-x/2}\, U(x)$$

$$y = \sqrt{x} \quad \Rightarrow \quad x_1 = y^2$$

$$\frac{dy}{dx} = \frac{1}{2y}$$

Thus

$$f_Y(y) = 2y f_X(y^2) = \frac{y^{n-1}}{2^{n/2-1}\,\Gamma(n/2)}\, e^{-y^2/2}\, U(y)$$

and it represents the chi-distribution.

5.18

$$X \sim U(0,1)$$

$$Y = -2logX \quad \Rightarrow \quad x_1 = e^{-y/2}$$

$$\frac{dy}{dx} = -\frac{2}{x} = -2e^{y/2}$$

$$\begin{aligned} f_Y(y) \quad &= \frac{1}{|\frac{dy}{dx}|}\, f_X(x_1) = \tfrac{1}{2} e^{-y/2}\, U(y) \\ &\sim \text{Exponential}(2) \equiv \chi^2(2) \end{aligned}$$

5.19

$$f_X(x) = \lambda\, e^{-\lambda x}\, u(x)$$

$$Y = X^{1/\beta} \quad \Rightarrow \quad x_1 = y^\beta$$

$$|\frac{dy}{dx}| = \frac{1}{\beta}\, x^{1/\beta-1} = \frac{1}{\beta}\, y^{1-\beta}$$

$$f_Y(y) = \frac{1}{|\,\frac{dy}{dx}\,|}\, f_X(x_1) = \lambda\,\beta\, y^{\beta-1}\, e^{-\lambda y^\beta}\, U(y)$$

and it represents Weibull distribution

5-20 For $|y| < a$ the equation $y = a \sin\omega t$ has infinitely many solutions τ_i; in each interval of length $2\pi/\omega$ there are two such solutions. Furthermore, $y'(t) = \omega\sqrt{a^2-y^2}$

$$\tau_i = \frac{1}{\omega}\sin^{-1}\frac{y}{a} \qquad\qquad \tau_{i+2} - \tau_i \quad \frac{2\pi}{\omega} \xrightarrow[\omega\to\infty]{} 0$$

Hence,

$$\frac{1}{\omega\sqrt{a^2-y^2}} \sum_{i=-\infty}^{\infty} f_t(\tau_i) \xrightarrow[\omega\to\infty]{} \frac{1}{\sqrt{a^2-y^2}} \frac{2}{2\pi} \int_{-\infty}^{\infty} f_t(\tau)d\tau = \frac{1}{\pi\sqrt{a^2-y^2}}$$

5-21 If $y > 0$ then

$$F_y(y|x \geq 0) = F_x(\sqrt{y}|x \geq 0) + F_x(-\sqrt{y}|x \geq 0) = F_x(\sqrt{y}|x \geq 0)$$

$$F_x(\sqrt{y}|x \geq 0) = \frac{P\{0 < x < \sqrt{y}\}}{P\{x \geq 0\}} = \frac{F_x(\sqrt{y}) - F_x(0)}{1 - F_x(0)}$$

$$f_y(y|x \geq 0) = \frac{d}{dy} F_y(\sqrt{y}|x \geq 0) = \frac{f_x(\sqrt{y})}{2\sqrt{y}[1-F_x(0)]}$$

5-22 (a) $\eta_y = a\,\eta_x + b \qquad \sigma_y^2 = E\{[a\,x + b - (a\,\eta_x + b)]^2\}$

$$\sigma_y^2 = E\{a^2(x - \eta_x)^2\} = a^2\sigma_x^2$$

(b) $y = \dfrac{x-\eta_x}{\sigma_x} \qquad E\{y\} = 0 \qquad \sigma_y^2 = \dfrac{\sigma_x^2}{\sigma_x^2} = 1$

5-23 If x has a Rayleigh density, then [see (5-76)]

$$E\{x^2\} = 2\alpha^2 \qquad E\{x^4\} = 8\alpha^4$$

If $y = b + cx^2$, then

$$E\{y\} = b + 2\alpha^2 c \qquad E\{y^2\} = b^2 + 4\alpha^2 bc + 8\alpha^4 c^2$$

$$\sigma_y^2 = E\{y^2\} - E^2\{y\} = 4\alpha^4 c^2$$

5-24 $\quad y = 3x^2 \qquad E\{x^2\} = \sigma_x^2 = 4 \qquad E\{x^4\} = 3\sigma_x^4 = 48$

$E\{y\} = 12 \qquad E\{y^2\} = 9 \times 48 = 432 \qquad \sigma_y^2 = 432 - 144 = 288$

If $y > 0$ then $3x^2 = y$ for $x = \pm\sqrt{y/3}$ $\qquad y' = 6x$

$$f_y(y) = \frac{\cancel{2}4}{\sqrt{12y}}\, f_x\left(\sqrt{\frac{y}{3}}\right) = \frac{1}{\sqrt{24\pi y}}\, e^{-y/24}U(y)$$

5.25

$$X \sim B(n,p) \Rightarrow P(X=k) = \binom{n}{k} p^k q^{n-k}, \quad k = 0, 1, 2, \cdots n.$$

a)

$$\begin{aligned} E(X) &= \sum_{k=0}^{n} k\, P(X=k) = \sum_{k=1}^{n} k\, \frac{n!}{k!\,(n-k)!}\, p^k\, q^{n-k} \\ &= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,(n-k)!}\, p^{k-1}\, q^{n-k} \\ &= np\,(p+q)^{n-1} = np. \end{aligned}$$

b)

$$\begin{aligned} E[X(X-1)] &= \sum_{k=2}^{n} k\,(k-1)\, \frac{n!}{k!\,(n-k)!}\, p^k\, q^{n-k} \\ &= n(n-1)p^2 \sum_{k=2}^{n} \frac{(n-2)!}{(k-2)!\,(n-k)!}\, p^{k-2}\, q^{n-k} \\ &= n(n-1)p^2\,(p+q)^{n-2} \\ &= n(n-1)p^2 \end{aligned}$$

c)

$$\begin{aligned} E[X(X-1)(X-2)] &= \sum_{k=3}^{n} k(k-1)(k-2)\, \frac{n!}{k!\,(n-k)!}\, p^k q^{n-k} \\ &= n(n-1)(n-2)\,p^3 \sum_{k=3}^{n} \frac{(n-3)!}{(k-3)!\,(n-k)!}\, p^{k-3}\, q^{n-k} \\ &= n(n-1)(n-2)\,p^3\,(p+q)^{n-3} \\ &= n(n-1)(n-2)\,p^3 \end{aligned}$$

$$E(X^2) = E\left(X(X-1)\right) + E(X) = n^2p^2 + npq$$

$$\begin{aligned} E(X^3) &= E\left(X(X-1)(X-2)\right) + 3E(X^2) - 2E(X) \\ &= n(n-1)(n-2)\,p^3 + 3(n^2p^2 + npq) - 2np \\ &= n^3p^3 + 3n^2p^2q + npq(q-p). \end{aligned}$$

5.26

$$X \sim P(\lambda) \quad \Rightarrow \quad P(X=k) = e^{\lambda}\frac{\lambda^k}{k!}, \qquad k = 0, 1, 2, \cdots$$

a)

$$E(X) = \lambda, \quad \mathrm{Var}(X) = \sigma_X^2 = \lambda$$

From Chebyshev's inequality (5-88)

$$P\left(|X-\mu| < \lambda\right) > 1 - \frac{\sigma^2}{\lambda^2} = 1 - \frac{1}{\lambda}$$

But

$$|X-\mu| < \lambda = |X-\lambda| < \lambda \quad \Rightarrow \quad 0 < X < 2\lambda$$

which gives

$$P(0 < X < 2\lambda) > 1 - \frac{1}{\lambda} = \frac{\lambda - 1}{\lambda}.$$

b)

$$\begin{aligned} E\left[X(X-1)\right] &= \textstyle\sum_{k=2}^{\infty} k(k-1)\, e^{-\lambda}\, \frac{\lambda^k}{k!} \\ &= e^{-\lambda}\,\lambda^2 \textstyle\sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} = e^{-\lambda}\,\lambda^2\, e^{\lambda} = \lambda^2. \end{aligned}$$

$$\begin{aligned} E\left[X(X-1)(X-2)\right] &= \textstyle\sum_{k=3}^{\infty} k(k-1)(k-2)\, e^{-\lambda}\, \frac{\lambda^k}{k!} \\ &= e^{-\lambda}\,\lambda^3 \textstyle\sum_{k=3}^{\infty} \frac{\lambda^{k-3}}{(k-3)!} = \lambda^3. \end{aligned}$$

5-27 Follows from (4-74)

$$E\{\underline{x}\} = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{\infty} x \sum_i f(x|A_i)P(A_i)\,dx$$

because $$E\{\underline{x}|A_i\} = \int_{-\infty}^{\infty} x f(x|A_i)\,dx$$

5-28 From (5-89) with $\alpha = \sqrt{\eta}$:

$$P\{\underline{x} \geq \sqrt{\eta}\} \leq \eta/\sqrt{\eta} = \sqrt{\eta}$$

5-29 From (5-86) with $g(x) = x^3$ $g''(x) = 6x$:

$$E\{\underset{\sim}{x}^3\} = \eta^3 + 6\eta \frac{\sigma^2}{2} = 1120$$

5-30 (a) If $y = x^3$, then $x = \sqrt[3]{y}$ $g'(x) = 3x^2 = 3\sqrt[3]{y^2}$

But $f_x(x) = 0.5$ for $10 < x < 12$, i.e., for $10^3 < y < 12^3$

and (5-16) yields

$$f_y(y) = \frac{0.5}{3\sqrt[3]{y^2}} \qquad 10^3 < y < 12^3$$

and zero otherwise.

(b) 1.
$$E\{\underset{\sim}{x}^3\} = 0.5 \int_{10}^{12} x^3 dx = 1342$$

2. With $g(x) = x^3$ $E\{\underset{\sim}{x}\} = 11$ $\sigma_x^2 = 1/3$, (5-86) yields

$$E\{x^3\} \simeq 11^3 + 6 \times 11 \times \frac{1}{6} \simeq 1342$$

5-31 With $g(x)=1/x$, $g''(x)=2/x^3$, $\eta=100$, and $\sigma=3$, (5-55) yields

$$E\left\{\frac{1}{\underset{\sim}{x}}\right\} \simeq \frac{1}{100} + \frac{9}{2} \times \frac{2}{100^3} = 0.010009$$

5-32

$$\frac{\partial|x-a|}{\partial a} = \begin{cases} 1 & x < a \\ -1 & x > a \end{cases} \qquad \text{If } I(a) = E\{|\underset{\sim}{x}-a|\} \text{ then}$$

$$\frac{dI(a)}{da} = E\,\frac{\partial|\underset{\sim}{x}-a|}{\partial a} = 1\,P\{\underset{\sim}{x} < a\} - 1\,P\{\underset{\sim}{x} > a\}$$
$$= 2\,F(a) - 1$$

(a)
$$I(a) = I(m) + \int_m^a I'(\alpha)d\alpha = I(m) + \int_m^a [2\,F(\alpha) - 1]d\alpha$$

$$= E\{|\underset{\sim}{x} - m|\} - 2\int_m^a x\,f(x)dx$$

because

$$\int_m^a F(\alpha)d\alpha = a\,F(a) - m\,F(m) - \int_m^a x f(x)dx$$

$$F(m) = \frac{1}{2} \qquad\qquad \int_m^a f(x)dx = F(a) - F(m)$$

(b) $I(a) = E\{|\underset{\sim}{x} - a|\}$ is minimum if

$I'(a) = 2F(a) - 1 = 0$ i.e. if $F(a) = \frac{1}{2}$ $\quad a = m$

5-33
$$E\{|\underset{\sim}{x}|\} = \int_0^\infty x f(x)dx - \int_{-\infty}^0 x f(x)dx$$

$$\eta = E\{\underset{\sim}{x}\} = \int_0^\infty x f(x)dx + \int_{-\infty}^0 x f(x)dx$$

$$\frac{E\{|\underset{\sim}{x}|+\eta\}}{2} = \int_0^\infty x f(x)dx = \frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty x\, e^{-(x-\eta)^2/2\sigma^2}dx$$

$$\frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty (x+\eta)e^{-(x-\eta)^2/2\sigma^2}dx = \frac{1}{\sigma\sqrt{2\pi}}\int_\eta^\infty y\, e^{-y^2/2\sigma^2}dy = \frac{\sigma}{\sqrt{2\pi}}\, e^{-\eta^2/2\sigma^2}$$

$$\frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty e^{-(x-\eta)^2/2\sigma^2}dx = G(\frac{\eta}{\sigma})$$

Multiplying the last line by η and subtracting from the fourth line, we obtain

$$\frac{E\{|x|+\eta\}}{2} = \frac{\sigma}{\sqrt{2\pi}}\, e^{-\eta^2/2\sigma^2} + G(\frac{\eta}{\sigma})$$

5-34 The proof is given in sec 14-3: [see (14-100)].

5-35 (a) Follows from (5-89) (b) $e^{sx} \geq e^{sA}$ iff $x \geq A$ for $s > 0$ and $x \leq A$ for $s < 0$.

5.36 See proof for Lyapunov inequality (Ch.5, Eq.(5-92).)

5-37 (a) If $\Phi(\omega) = e^{-\alpha|\omega|}$ then [see (5-102)]

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha|\omega|}\, e^{j\omega x} d\omega = \frac{1}{\pi}\int_{0}^{\infty} \cos \omega x\, e^{-\alpha\omega} d\omega = \frac{\alpha}{\pi(\alpha^2+x^2)}$$

(b) If $f(x) = \frac{\alpha}{2}\, e^{-\alpha|x|}$, then [see (5-94)]

$$\Phi(\omega) = \frac{\alpha}{2}\int_{-\infty}^{\infty} e^{-\alpha|x|} e^{-j\omega x} dx = \alpha\int_{0}^{\infty} e^{-\alpha x} \cos \omega x dx = \frac{\alpha^2}{\alpha^2+\omega^2}$$

5.38 a) On comparing Eq.(4-34) with Eq.(5-106), Example 5-29, we get

$$X \sim G(\alpha,\beta) \quad \Rightarrow \quad \phi_X(\omega) = (1 - j\beta\omega)^{-\alpha}$$

$$\phi_X'(\omega) = -\alpha(1-j\beta\omega)^{-(\alpha+1)}\,(-j\beta)$$

so that

$$E(X) = \frac{1}{j}\,\phi_X'(0) = \alpha\,\beta.$$

Similarly

$$\phi_X''(\omega) = j\alpha\beta(\alpha+1)\,(1-j\beta\,\omega)^{-(\alpha+2)}\,(j\beta)$$

and hence

$$E(X^2) = \frac{1}{j^2}\,\phi_X''(0) = \alpha\beta^2\,(\alpha+1).$$

Thus

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \alpha\beta^2.$$

b)

$$X \sim \chi^2(n) \quad \Rightarrow \quad \alpha = \frac{n}{2}, \quad \beta = 2$$

in Gamma(α,β). This gives

$$\begin{aligned}\phi_X(\omega) &= (1-j2\omega)^{-n/2}\\ E(X) &= n\\ \mathrm{Var}(X) &= 2n.\end{aligned}$$

c)

$$X \sim \; B(n,p).$$

From Prob 5-25 (a)-(b)

$$\begin{aligned}E(X) &= np\\ \mathrm{Var}(X) &= E\,(X(X-1)) + E(X) = npq.\end{aligned}$$

and

$$\begin{aligned}\phi_X(\omega) &= \sum_{k=0}^{n} e^{jk\omega}\, P(X=k)\\ &= \sum_{k=0}^{n} \binom{n}{k}\,(p\,e^{j\omega})^k\, q^{n-k} = (pe^{j\omega}+q)^n.\end{aligned}$$

d)

$$X \sim N\,Binomial\,(r, p).$$

From (4-64)

$$\begin{aligned}\phi_X(\omega) &= \sum_{k=0}^{\infty} e^{jk\omega}\, P(X=k)\\ &= \sum_{k=0}^{\infty} \binom{r+k-1}{k} p^r\,(qe^{j\omega})^k\\ &= p^r \sum_{k=0}^{\infty} \binom{-r}{k} (-qe^{j\omega})^k\\ &= p^r\,(1-qe^{j\omega})^{-r}.\end{aligned}$$

5-39

$$\Gamma(z) = \sum_{k=0}^{\infty} p\, q^k z^k = \frac{p}{1-qz} \qquad q = 1-p$$

$$\Gamma'(z) = \frac{pq}{(1-qz)^2} \qquad \Gamma'(1) = \frac{pq}{(1-q)^2} = \frac{p}{q} = \eta_x$$

$$\Gamma''(z) = \frac{2pq^2}{(1-qz)^3} \qquad \Gamma''(1) = \frac{2q^2}{p^2} = m_2 - m_1$$

$$\sigma^2 = m_2 - m_1^2 = 2\frac{q^2}{p^2} + m_1 - m_1^2 = \frac{q}{p^2}$$

5-40

$$\Gamma(z) = p^n \sum_{k=0}^{\infty} \binom{-n}{k} (-q)^k\, z^k = p^n(1-qz)^{-n}$$

(binomial expansion with negative exponent)

$$\Gamma'(z) = \frac{n\,p^n q}{(1-qz)^{n+1}} \qquad \Gamma'(1) = \frac{nq}{p} = \eta_x$$

$$\Gamma''(z) = \frac{n(n+1)p^n q^2}{(1-qz)^{n+2}} \qquad \Gamma''(1) = \frac{n(n+1)q^2}{p^2} = m_2 - m_1$$

$$\sigma_x^2 = \Gamma''(1) + m_1 - m_1^2 = \frac{nq}{p^2}$$

5.41 We have

$$P(X=k) = \binom{k-1}{r-1} p^r\, q^{k-r}, \quad k = r, r+1, \cdots$$

Let $k = n + r$ so that

$$\begin{aligned}
P(X = n+r) &= \binom{n+r-1}{r-1} p^r\, q^n, \quad n = 0, 1, 2, \cdots \\
&= \frac{(n+r-1)!}{n!\,(r-1)!}\, p^r\, (1-p)^n \\
&= \frac{1}{n!}\, \frac{(n+r-1)(n+r-2)\cdots(r)}{r^n}\, [r(1-p)]^n\, p^r \\
&= \frac{\lambda^n}{n!} \left\{ \left(1 + \frac{n-1}{r}\right) \left(1 + \frac{n-2}{r}\right) \cdots \right\} \left(1 - \frac{r(1-p)}{r}\right)^r \\
&= \frac{\lambda^n}{n!} \left\{ \prod_{k=1}^{n} \left(1 + \frac{n-k}{r}\right) \right\} \left(1 - \frac{\lambda}{r}\right)^r,
\end{aligned}$$

where $\lambda = r(1-p)$. Thus

$$\begin{aligned}
\lim_{r\to\infty} P(X = n+r) = \frac{\lambda^n}{n!} \left\{ \lim_{r\to\infty} \prod_{k=1}^{n} \left(1 + \frac{n-k}{r}\right) \right\} \lim_{r\to\infty} \left(1 - \frac{\lambda}{r}\right)^r \\
\to \frac{\lambda^n}{n!} e^{-\lambda} \sim P(\lambda).
\end{aligned}$$

5-42 $E\{e^{s\underline{x}}\} = e^{s\eta}\, E\{e^{s(\underline{x}-\eta)}\} = e^{s\eta}\, E\left\{\sum_{n=0}^{\infty} \frac{s^n}{n!} (\underline{x}-\eta)^n\right\}$

$$= e^{s\eta} \sum_{n=0}^{\infty} \frac{s^n}{n!}\, \mu_n$$

5-43 If $\Phi(\omega_1) = 0$, then [see also (9-176]

$$\int_{-\infty}^{\infty} (1-e^{j\omega_1 x}) f(x)dx = 0, \text{ hence, } f(x) = \sum_{n=\infty}^{\infty} p_n \delta(x - \frac{2\pi n}{\omega_1})$$

5-44 (a) If $\eta = 0$, then $m_n = \mu_n \qquad \lambda_1 = \eta = 0$

$$\Phi(s) = \sum_{n=0}^{\infty} \frac{\mu_n}{n!} s^n \qquad\qquad \Psi(s) = \sum_{n=2}^{\infty} \frac{\lambda_n}{n!} s^n$$

$$1 + \frac{\mu_2}{2!}s^2 + \frac{\mu_3}{3!} s^3 + \frac{\mu_4}{4!} s^4 + \cdots = \exp\left\{\frac{\lambda_2}{2!} s^2 + \frac{\lambda_3}{3!} s^3 + \frac{\lambda_4}{4!} s^4 + \cdots\right\}$$

Expanding the exponential and equating powers of s, we obtain

$$\mu_2 = \lambda_2 \qquad \mu_3 = \lambda_3 \qquad\qquad \frac{\mu_4}{4!} = \frac{\lambda_4}{4!} + \frac{1}{2!} \left(\frac{\lambda_2}{2!}\right)^2$$

(b) If $\underline{y}$ is $N(0;\sigma_y)$ then

$$\Psi_y(s) = \frac{\lambda_2}{2} s^2 , \quad \text{hence,} \quad \lambda_n = 0 \quad \text{for} \quad n \geq 3$$

5-45 $$P\{\underset{\sim}{y} = 0\} = P\{\underset{\sim}{x} \le 1\} = p_o + p_1$$

$$P\{\underset{\sim}{y} = k\} = P\{\underset{\sim}{x} = k + 1\} = p_{k+1} \qquad k \ge 1$$

$$\Gamma_y(z) = p_o + p_1 + \sum_{k=1}^{\infty} p_{k+1} z^k = p_o + z^{-1}[\Gamma_x(z) - p_o]$$

$$\eta_y = \sum_{k=1}^{\infty} k\, p_{k+1} = \sum_{r=1}^{\infty} r\, p_r - \sum_{r=1}^{\infty} p_r = \eta_x - 1 + p_o$$

$$E\{\underset{\sim}{y}^2\} = \sum_{k=1}^{\infty} k^2 p_{k+1} = \sum_{r=1}^{\infty} (r-1)^2 p_r = E\{\underset{\sim}{x}^2\} - 2\eta_x + 1 - p_o$$

5-46 $$0 \le E\left\{\left|\sum_{i=1}^{n} a_i e^{j\omega_i \underset{\sim}{x}}\right|^2\right\} = E\left\{\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j^* e^{j(\omega_i-\omega_j)\underset{\sim}{x}}\right\}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j^* \Phi(\omega_i - \omega_j)$$

5-47 From the assumptions it follows that

$$g'(-x) = -g'(x) \qquad g''(x) \ge 0 \qquad f(x-\eta) = f(\eta-x)$$

Hence, if $I(a) = E\{g(\underset{\sim}{x}-a)\}$, then

$$I'(a) = -\int_{-\infty}^{\infty} g'(x-a)f(x)dx \qquad I'(\eta) = 0$$

$$I''(a) = \int_{-\infty}^{\infty} g''(x-a)f(x)dx \ge 0 \qquad \text{all } a$$

Hence, $I(a)$ is minimum for $a = \eta$.

5-48 $$f(x,v) = \frac{1}{\sqrt{2\pi v}} e^{-x^2/2v}$$

$$\sqrt{2\pi}\,\frac{\partial f}{\partial v} = \frac{-1 + x^2/v}{2v\,\sqrt{v}}\, e^{-x^2/2v}$$

$$\sqrt{2\pi}\,\frac{\partial^2 f}{\partial x^2} = \frac{-1 + x/.}{v\,\sqrt{v}}\, e^{-x^2/2v}$$

Hence

(see also (6-198) - (6-199)) $$\boxed{\frac{\partial f}{\partial v} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}} \qquad (1)$$

(a) Integrating by parts, using (1) and assuming that $g^{(k)}(x)f(x) \to 0$ as $|x| \to \infty$, $k = 0, 1, 2$, we obtain

$$E\{g''(x)\} = \int_{-\infty}^{\infty} \frac{d^2 g}{dx^2} f dx = \int_{-\infty}^{\infty} g \frac{\partial^2 f}{\partial x^2} dx = 2\int_{-\infty}^{\infty} g \frac{\partial f}{\partial v} dx$$

$$= 2\frac{d}{dv}\int_{-\infty}^{\infty} g\, f dx = 2\frac{d}{dv} E\{g(\underset{\sim}{x})\}$$

(b) The moments $\mu_n(u) = E\{\underset{\sim}{x}^n\}$ of $\underset{\sim}{x}$ depend on the variance v of $\underset{\sim}{x}$ and (i) yields

$$\mu_n'(v) = \frac{d}{dv} E\{\underset{\sim}{x}^n\} = \frac{1}{2} E\{n(n-1)\underset{\sim}{x}^{n-2}\} = \frac{n(n-1)}{2}\mu_{n-2}(v)$$

Furthermore, $\mu_n(0) = 0$ because, if $v = 0$, then $\underset{\sim}{x} = 0$.

Hence
$$\mu_n(v) = \frac{n(n-1)}{2}\int_0^v \mu_{n-2}(\beta)d\beta$$

5-49 The function

$$\Gamma(e^{j\omega}) = E\{e^{jx\omega}\} = \sum_{k=0}^{\infty} p_k\, e^{jk\omega}$$

is periodic with period 2π and Fourier series coefficients $p_k = E\{x = k\}$.

5.50 The event $\{X = 1\}$ is given by the disjoint union "$TH \cup HT$". Similarly, the event "$X = k$" is given by the union of the disjoint events (k "T"s followed by "H" or k "H"s followed by "T")

$$\text{“}TT\cdots TTH\text{''} \cup \text{“}HH\cdots HHT\text{''}, \qquad k = 1, 2, \cdots$$

Thus

$$\begin{aligned} P(X = k) &= P(\text{“}TT\cdots TH\text{''} \cup \text{“}HH\cdots HT\text{''}) \\ &= P(TT\cdots TH) + P(HH\cdots HT) = q^k p + p^k q, \quad k = 1, 2, \cdots \end{aligned}$$

Also

$$\begin{aligned} E(X) &= \sum_{k=1}^{\infty} kP(X = k) \\ &= \sum_{k=1}^{\infty} kq^k p + \sum_{k=1}^{\infty} kp^k q = pq\left\{\sum_{k=1}^{\infty} kq^{k-1} + \sum_{k=1}^{\infty} kp^{k-1}\right\} \\ &= pq\left\{\frac{\partial}{\partial q}\sum_{k=1}^{\infty} q^k + \frac{\partial}{\partial p}\sum_{k=1}^{\infty} p^k\right\} = pq\left\{\frac{\partial}{\partial q}\left(\frac{q}{1-q}\right) + \frac{\partial}{\partial p}\left(\frac{p}{1-p}\right)\right\} \\ &= pq\left\{\frac{1}{p^2} + \frac{1}{q^2}\right\} = \frac{p}{q} + \frac{q}{p}. \end{aligned}$$

5.51 (a) When samples are drawn with replacement, probability of each item being defective is given by

$$p = \frac{M}{N} < 1 \quad \text{(constant)}$$

and

$$q = 1 - p = \frac{N-M}{M} < 1$$

represents the constant probability that the chosen item is not defective. In that case (with replacement), there are $\binom{n}{k}$ possible ways of arranging k defective items among n chosen items, and each such arrangement has probability $p^k q^{n-k}$. This gives

$$P(X = k) = \binom{n}{k} p^k\, q^{n-k}, \quad k = 0, 1, 2, \cdots n$$

which represents the Binomial distribution.
(b) If the samples are drawn without replacement, there are $\binom{M}{k}$ possible ways of choosing k defective item from a total of M defective items, and $\binom{N-M}{n-k}$ possible ways of choosing $n-k$ "good" items from $(N-M)$ "good" items independently. This gives

$$\binom{M}{k}\binom{N-M}{n-k}$$

to be the total number of ways of selecting k defective items and $n-k$ "good" items from a subsample of M and $N-M$ items respectively (favorable ways). But there are a total of $\binom{N}{n}$ ways of selecting n items among N items. This gives

$$P(X = k) = \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}},$$

since $0 \le k \le M$ and $n - k \le N - M, n - k \ge 0$, i.e. $0 \le k \le M, k \le n, k \ge n + M - N$.
(c) From (b)

$$P(X = k) = \frac{M!}{k!(M-k)!} \frac{(N-M)!}{(n-k)!(N-M-n+k)!} \frac{n!(N-n)!}{N!}$$

$$= \binom{n}{k} \frac{M(M-1)\cdots(M-k+1)}{N(N-1)\cdots(N-k+1)} \frac{(N-M)(N-M-1)\cdots(N-M-n+k+1)}{(N-k)(N-k-1)\cdots(N-n+1)}$$

$$\simeq \binom{n}{k} \left(\frac{M}{N}\right)^k \left(\frac{N-M}{N}\right)^{n-k} = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \cdots n$$

since $N \to \infty, M \to \infty$ such that $M/N \to p$, and $n \ll N$. Thus

$$P(X = k) \to \text{Binomial}(n, p = M/N)$$

under the above conditions.

5.52 (a) Refer to discussions in problem 5.51 (a) if sampleing is done with replacement, then

$$p = \frac{n}{n+m}$$

represents the probability of selecting a white marble on any trial. The event "$X = k$" is given by "$r-1$ white mables among the first $k-1$ trials" followed by "a white marble at the k^{th} trial". But from problem 5.51 (a), the event $r-1$ white mables among the first $k-1$ trials has a binomial distribution whose probability is given by $\binom{k-1}{r-1}p^{r-1}q^{k-r}$. Thus

$$P(X=k) = \binom{k-1}{r-1}p^{r-1}q^{k-r}p = \binom{k-1}{r-1}p^r q^{k-r}, \quad k = r, r+1, \cdots$$

which represents the Negative-binomial distribution

(b) If sampling is done with replacement, then the favarable ways of choosing the white balls are given by:

(i) $\binom{k-1}{r-1}$ ways of selecting $r-1$ white balls among the first $k-1$ trials/balls.

(ii) One ways of selecting (the r^{th}) white ball at the k^{th} trial

(iii) $\binom{m+n-k}{n-r}$ ways of selecting the remaining $n-r$ white balls among the remaining $m+n-k$ balls.

This gives $\binom{k-1}{r-1}\cdot 1 \cdot \binom{m-n-k}{n-r}$ to be the total number of favorable ways of selecting the white balls. Since there are $n+m$ balls there are a total of $\binom{n+m}{n}$ ways of selecting n white balls. This gives

$$P(X=k) = \binom{k-1}{r-1}\frac{\binom{m+n-k}{n-r}}{\binom{n+m}{n}}, \qquad k = r, r+1, \cdots$$

(c) From (b)

$$P(X=k) = \binom{k-1}{r-1}\frac{(m+n-k)!}{(n-r)!\,(m-k+r)!}\frac{n!m!}{(m+n)!}$$

$$= \binom{k-1}{r-1}\left(\frac{n}{m+n}\right)\left(\frac{n-1}{m+n-1}\right)\cdots\left(\frac{n-r+1}{m+n-r+1}\right)\left(\frac{m!(m+n-k)!}{(m+n-r)!\,(m-k+r)!}\right)$$

$$\simeq \binom{k-1}{r-1}\left(\frac{n}{m+n}\right)^r\left(\frac{m}{m+n-r}\right)\left(\frac{m-1}{m+n-r-1}\right)\cdots\left(\frac{m-k+r+1}{m+n-k+1}\right)$$

$$\simeq \binom{k-1}{r-1}\left(\frac{n}{m+n}\right)^r\left(\frac{m}{m+n}\right)^{k-r} \text{ as } m+n\to\infty$$

$$= \binom{k-1}{r-1}p^r q^{k-r}, \qquad k = r, r+1, \cdots, \quad q = 1-p$$

$$\sim \mathrm{N}B\,(r, p = n/(n+m))\,.$$

CHAPTER 6

6.1 (a) Define

$$Z = X + Y$$

Note that both X and Y positive random variables hence (use Eq. (6-45))

$$\begin{aligned} f_Z(z) &= \int_0^z f_{XY}(z-y,y)dy = \int_0^z e^{-(z-y+y)}dy \\ &= z\,e^{-z}U(z). \end{aligned}$$

(b)

$$Z = X - Y$$

Z ranges over the entire real axis for the random variables X and Y (see Eq. (6-55))

$$F_Z(z) = \begin{cases} \displaystyle\int_0^\infty \int_0^{z+y} f_{XY}(x,y)dx\,dy, & z > 0 \\ \displaystyle\int_{-z}^\infty \int_0^{z+y} f_{XY}(x,y)dx\,dy, & z < 0 \end{cases}$$

Differrentiation gives

$$f_Z(z) = \begin{cases} \displaystyle\int_0^\infty f_{XY}(z+y,y)\,dy, & z > 0 \\ \displaystyle\int_{-z}^\infty f_{XY}(z+y,y)\,dy, & z < 0 \end{cases}$$

$$f_Z(z) = \begin{cases} \displaystyle\int_0^\infty e^{-(z+y+y)}\,dy = e^{-z}\int_0^\infty e^{-2y}\,dy = \tfrac{1}{2}\,e^{-z}, & z > 0 \\ \displaystyle\int_{-z}^\infty e^{-(z+y+y)}\,dy = e^{-z}\int_{-z}^\infty e^{-2y}\,dy = \tfrac{1}{2}\,e^{z}, & z < 0 \end{cases}$$

or

$$f_Z(z) = \frac{1}{2}\,e^{-|z|}, \qquad -\infty \le z \le \infty.$$

(c)

$$Z = XY.$$

$$\begin{aligned} F_Z(z) &= P\{Z \le z\} = P\{XY \le z\} \\ &= \int_0^\infty \int_0^{z/y} f_{XY}(x,y)dx\,dy \end{aligned}$$

or (see Eq. (6-148))

$$f_Z(z) = \int_0^\infty \frac{1}{y} f_{XY}(\frac{z}{y}, y)dy = \int_0^\infty \frac{1}{y} e^{-((z/y)+y)}dy$$

(d)

$$\begin{aligned} Z &= X/Y \\ F_Z(z) &= P\{Z \le z\} = P\{\tfrac{X}{Y} \le z\} \\ &= \int_0^\infty \int_0^{yz} f_{XY}(x,y)dx\,dy \end{aligned}$$

(use Eq. (6-60))

$$\begin{aligned} f_Z(z) &= \int_0^\infty y f_{XY}(yz, y)dy = \int_0^\infty y e^{y(z+1)} dy = \int_0^\infty y e^{(1+z)y} \\ &= \left[y \frac{e^{-(1+z)y}}{-(1+z)} \right]_0^\infty + \left(\frac{1}{1+z}\right) \int_0^\infty e^{(1+z)y} dy \\ &= \left(\frac{1}{1+z}\right) \left[\frac{e^{-(1+z)y}}{-(1+z)}\right]_0^\infty = \frac{1}{(1+z)^2} U(z) \end{aligned}$$

(e)

$$\begin{aligned} Z &= \min\,(X,Y) \\ F_Z(z) &= P\{\min\,(X,Y) \le z\} \\ &= 1 - P\{Z > z, Y > z\} \\ &= 1 - [1 - F_X(z)]\,[1 - F_Y(z)] \\ &= F_X(z) + F_Y(z) - F_X(z)\,F_Y(z) \end{aligned}$$

(see Eq. (6-81))

$$f_Z(z) = f_X(z) + f_Y(z) - F_X(z) f_Y(z) - f_X(z) F_Y(z).$$

We have

$$f_X(z) = f_Y(z) = e^{-z}\,U(z)$$

so that

$$F_X(z) = \int_0^z e^{-x} dx = (1 - e^{-z})\,U(z) = F_Y(z)$$

$$\begin{aligned} f_Z(z) &= [e^{-z} + e^{-z} - 2(1 - e^{-z})\,e^{-z}]U(z) \\ &= 2e^{-z}\,[1 - 1 + e^{-z}]\,U(z) \\ &= 2e^{-2z}\,U(z) \sim \text{Exponential } (2). \end{aligned}$$

(f)

$$Z = \max(X,Y)$$

$$\begin{aligned} F_Z(z) &= P\{\max(X,Y) \le z\} = P\{X \le z, Y \le z\} \\ &= P\{X \le z\}\,P\{Y \le z\} = F_X(z)\,F_Y(z) \end{aligned}$$

$$\begin{aligned} f_Z(z) &= F_X(z)\,f_Y(z) + f_X(z)\,F_Y(z) \\ &= e^{-z}\,(1 - e^{-z}) + e^{-z}\,(1 - e^{-z}) \\ &= 2e^{-z}\,(1 - e^{-z})\,U(z) \end{aligned}$$

(g)

$$Z = \frac{\min(X,Y)}{\max(X,Y)}, \quad 0 < z < 1$$

$$\begin{aligned}
F_Z(z) &= P\left\{\left(\frac{\min(X,Y)}{\max(X,Y)} \le z\right) \bigcap ((X \le Y) \bigcup (X > Y))\right\} \\
&= P\left\{\left(\frac{\min(X,Y)}{\max(X,Y)} \le z\right) \bigcap (X \le Y)\right\} + P\left\{\left(\frac{\min(X,Y)}{\max(X,Y)} \le z\right) \bigcap (X > Y)\right\} \\
&= P\{\tfrac{X}{Y} \le z, X \le Y\} + P\{\tfrac{Y}{X} \le z, X > Y\} \\
&= P\{X \le Yz, X \le Y\} + P\{Y \le Xz, X > Y\} \\
&= \int_0^\infty \int_0^{yz} f_{XY}(x,y)\, dx\, dy + \int_0^\infty \int_0^{xz} f_{XY}(x,y)\, dy\, dx
\end{aligned}$$

$$\begin{aligned}
f_Z(z) &= \int_0^\infty y\, f_{XY}(yz, y) dy + \int_0^\infty x f_{XY}(x, xz) dx \\
&= \int_0^\infty y\, f_{XY}(yz, y) dy + \int_0^\infty y\, f_{XY}(y, yz) dy \\
&= \int_0^\infty y \left(e^{-(yz+y)} + e^{-(y+yz)}\right) dy \\
&= 2\int_0^\infty y e^{-y(1+z)} dz = \begin{cases} \dfrac{2}{(1+z)^2}, & 0 \le z \le 1 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

6.2

$$f_{XY}(x,y) = f_X(x)\, f_Y(y) = \frac{1}{a^2}, \qquad 0 < x \le a, \quad 0 < y \le a$$

(a)

$$F_Z(z) = P\left\{\frac{X}{Y} \le z\right\} = P\{X \le zY\}$$

(i) $z < 1$

$$\begin{aligned}
F_Z(z) &= P\{X \le zY\} \\
&= \int_0^a \int_0^{zy} \tfrac{1}{a} \cdot \tfrac{1}{a}\, dx\, dy = \tfrac{z}{2}, \quad z \le 1
\end{aligned}$$

(ii) $z \ge 1$

$$\begin{aligned}
F_Z(z) &= P\{X \le zY\} \\
&= 1 - \int_0^a \int_0^{x/z} \tfrac{1}{a} \cdot \tfrac{1}{a}\, dy\, dx \\
&= 1 - \int_0^1 \tfrac{x}{z}\, dx = 1 - \tfrac{1}{2z} \quad z > 1
\end{aligned}$$

$$f_Z(z) = \begin{cases} \tfrac{1}{2}, & z \le 1 \\ \dfrac{1}{2z^2}, & z > 1 \end{cases}$$

(b)

$$\begin{aligned} F_Z(z) &= P(Z \le z) = P\left\{\frac{Y}{X+Y} \le z\right\} \\ &= P\left\{\frac{X}{Y} \ge \frac{1}{z} - 1\right\} = 1 - P\left(\frac{X}{Y} \le \frac{1-z}{z}\right) \\ &= \begin{cases} \frac{1}{2}\left(\frac{z}{1-z}\right), & 0 < z \le 1/2 \\ 1 - \frac{1}{2}\left(\frac{1-z}{z}\right), & 1/2 < z < 1 \end{cases} \end{aligned}$$

$$f_Z(z) = \begin{cases} \dfrac{1}{2(1-z)^2}, & 0 < z \le 1/2 \\ \dfrac{1}{2z^2}, & 1/2 < z < 1 \end{cases}$$

(c)

$$\begin{aligned} F_Z(z) &= P\{Z \le z\} = P\{|X-Y| \le z\} \\ &= P\{\{|X-Y| \le z\} \bigcap (X \ge Y)\} + P\{\{|X-Y| \le z\} \bigcap (X < Y)\} \\ &= P\{X - Y \le z, X \ge Y\} + P\{Y - X \le z, X < Y\} \\ &= \int_0^\infty \int_y^{y+z} f_{XY}(x,y)\,dx\,dy + \int_0^\infty \int_x^{x+z} f_{XY}(x,y)\,dy\,dx \\ &= \int_0^\infty \int_y^{y+z} f_{XY}(x,y)\,dx\,dy + \int_0^\infty \int_y^{y+z} f_{XY}(y,x)\,dx\,dy \\ &= \int_0^\infty \int_y^{y+z} \{f_{XY}(x,y) + f_{XY}(y,x)\}\,dx\,dy. \end{aligned}$$

In general

$$\begin{aligned} f_Z(z) &= \int_0^\infty \frac{d}{dz} \int_y^{y+z} f_{XY}(x,y) + f_{XY}(y,x)\,dx\,dy \\ &= \int_0^\infty \{f_{XY}(y+z,y) + f_{XY}(y,y+z)\}\,dy. \end{aligned}$$

Here

$$X \sim U(0,a), \qquad Y \sim U(0,a)$$

$$F_Z(z) = 1 - \frac{1}{a^2} \cdot 2 \cdot \frac{(a-z)^2}{2} = 1 - \left(1 - \frac{z}{a}\right)^2$$

and

$$f_Z(z) = \frac{2}{a}\left(1 - \frac{z}{a}\right) \qquad 0 \le z \le a.$$

6.3

$$\begin{aligned} F_Z(z) &= P\{Z \le z\} = P\{X+Y \le z\} \\ &= \tfrac{1}{2} - \tfrac{z^2}{2}, \quad -1 < z < 0, \end{aligned}$$

(which represents the area below the line $X + Y = z$.)

$$\begin{aligned} F_Z(z) &= P\{Z \le z\} = P\{X+Y \le z\} \\ &= \tfrac{1}{2} + \tfrac{z^2}{2}, \quad 0 \le z < 1 \end{aligned}$$

$$f_Z(z) = \begin{cases} -z, & -1 \le z < 0 \\ z, & 0 \le z < 1 \end{cases}$$

6.4

$$Z = X - Y$$

For $z < 0$

$$\begin{aligned} F_Z(z) &= P\{Z \le z\} \\ &= \int_0^{(1+z)/2} \int_{x-z}^{1-x} f_{XY}(x,y)\, dy\, dx = \int_0^{(1+z)/2} \int_{x-z}^{1-x} 6x\, dy\, dx \\ &= \int_0^{(1+z)/2} 6x\, [y]_{x-z}^{1-x}\, dx = \int_0^{(1+z)/2} 6x(1-x-x+z)\, dx \\ &= 6\left[(1+z)\frac{x^2}{2} - \frac{2x^3}{3}\right]_0^{(1+z)/2} = 6\left[\frac{(1+z)^3}{8} - \frac{(1+z)^3}{12}\right] \\ &= \frac{(1+z)^3}{4}, \quad z \le 0. \end{aligned}$$

For $z > 0$

$$\begin{aligned} F_Z(z) &= P\{Z \le z\} = 1 - P\{Z > z\} \\ &= 1 - \int_0^{(1-z)/2} \int_{z+y}^{1-y} f_{XY}(x,y)\, dx\, dy = 1 - \int_0^{(1-z)/2} \int_{z+y}^{1-y} 6x\, dy \\ &= 1 - \int_0^{(1-z)/2} \left[\frac{6x^2}{2}\right]_{z+y}^{1-y} dy = 1 - 3\int_0^{(1-z)/2} \left[(1-y)^2 - (z-y)^2\right] dy \\ &= 1 - 3\,(1+z)\left[\frac{(1-z)^2}{2} - \frac{(1-z)^2}{4}\right] = 1 - \tfrac{3}{4}\,(1+z)(1-z)^2 \quad z \le 0. \end{aligned}$$

$$f_Z(z) = \begin{cases} \tfrac{3}{4}\,(1-z)(1+3z), & 0 \le z \le 1 \\ \tfrac{3}{4}\,(1+z)^2, & -1 < z < 0 \end{cases}$$

6.5 (a) See Example 6-15 for solutions
(b) See Example 6-14 for solutions
(c)

$$U = X - Y \sim N(0, 2\sigma^2)$$

since linear combinations of jointly Gaussian random variables are Gaussian random variables (see Eq. (6-120) Text.). Here $Var(U) = Var(X) + Var(Y) = 2\sigma^2$.

6.6

$$Z = XY$$

$$\begin{aligned}
F_Z(z) &= P(XY \le z) = 1 - P(XY > z) \\
&= 1 - \int_z^1 \int_{z/y}^1 f_{XY}(x,y)\,dx\,dy \\
f_Z(z) &= 1 + \int_z^1 \frac{1}{y}\, f_{XY}(z/y, y)\,dy = 1 + \int_z^1 \left\{\frac{2}{y} - \frac{2z}{y^2}\right\} dy \\
&= 1 - 2\ln z\ + 2z, \quad 0 \le z \le 1
\end{aligned}$$

6.7 (a)

$$Z_1 = X + Y$$

$$F_{Z_1}(z) = P(X+Y \le z) = \begin{cases} \displaystyle\int_0^z \int_0^{z-y} f_{XY}(x,y)\,dx\,dy, & 0 < z < 1 \\ \displaystyle 1 - \int_{z-1}^1 \int_{z-y}^1 f_{XY}(x,y)\,dx\,dy, & 1 < z < 2 \end{cases}$$

$$f_{Z_1}(z) = \begin{cases} \displaystyle\int_0^z f_{XY}(z-y, y)\,dy, & 0 < z < 1 \\ \displaystyle\int_{z-1}^1 f_{XY}(z-y, y)\,dy, & 1 < z < 2 \end{cases}$$

$$= \begin{cases} z^2, & 0 < z < 1 \\ z(2-z), & 1 < z < 2 \\ 0, & \text{otherwise} \end{cases}$$

(b)

$$Z_2 = XY$$

$$F_{Z_2}(z) = P(XY \le z) = 1 - \int_z^1 \int_{z/y}^1 f_{XY}(x,y)\,dx\,dy$$

$$\begin{aligned}
f_{Z_2}(z) &= \int_z^1 \frac{1}{y}\, f_{XY}(z/y, y)\,dy = \int_z^1 \frac{1}{y}\left(\frac{z}{y} + y\right) dy \\
&= 2(1-z), \quad 0 < z < 1
\end{aligned}$$

(c)

$$Z_3 = \frac{Y}{X}$$

$$F_{Z_3}(z) = P(Y/X \le z) = \begin{cases} \displaystyle\int_0^1 \int_0^{zx} f_{XY}(x,y)\,dy\,dx, & 0 < z < 1 \\ \displaystyle 1 - \int_0^1 \int_0^{y/z} f_{XY}(x,y)\,dx\,dy, & z > 1 \end{cases}$$

$$f_{Z_3}(z) = \begin{cases} \displaystyle\int_0^1 x\, f_{XY}(x, zx)\, dx, & 0 < z < 1 \\ \displaystyle\int_0^1 \frac{y}{z^2}\, f_{XY}(y/z, y)\, dy, & z > 1 \end{cases}$$

$$= \begin{cases} \dfrac{1+z}{3}, & 0 < z < 1 \\ \dfrac{1+z}{3z^3}, & z > 1 \end{cases}$$

(d)

$$Z_4 = Y - X$$

$$F_{Z_4}(z) = P(Y - X \le z) = \begin{cases} 1 - \displaystyle\int_z^1 \int_0^{y-z} f_{XY}(x, y)\, dx\, dy & 0 < z < 1 \\ \displaystyle\int_0^{z+1} \int_{y-z}^1 f_{XY}(x, y)\, dx\, dy, & -1 < z < 0 \end{cases}$$

$$f_{Z_4}(z) = \begin{cases} \displaystyle\int_z^1 f_{XY}(y - z, y)\, dy, & 0 < z < 1 \\ \displaystyle\int_0^{z+1} f_{XY}(y - z, y)\, dy, & -1 < z < 0 \end{cases}$$

$$= \begin{cases} 1 - z, & 0 < z < 1 \\ 1 + z, & -1 < z < 0 \end{cases} = 1 - |z|, \quad |z| < 1$$

6.8

$$F_Z(z) = P(X + Y \le z)$$

$$= \begin{cases} \displaystyle\int_0^{z/3} \int_{2y}^{z-y} f_{XY}(x, y)\, dx\, dy = \frac{z^2}{6}, & 0 < z < 2 \\ 1 - \displaystyle\int_{2z/3}^2 \int_{z-x}^{x/2} f_{XY}(x, y)\, dy\, dx = 2z - \frac{z^2}{3} - 2, & 2 < z < 3 \end{cases}$$

Thus

$$f_Z(z) = \begin{cases} \displaystyle\int_0^{z/3} f_{XY}(z - y, y)\, dy & 0 < z < 2 \\ \displaystyle\int_{2z/3}^2 f_{XY}(x, z - x)\, dx & 2 < z < 3 \end{cases}$$

$$f_Z(z) = \begin{cases} \frac{1}{3} z, & 0 < z < 2 \\ 2 - \frac{2z}{3}, & 2 < z < 3 \\ 0, & \text{otherwise} \end{cases}$$

6.9 (a)

$$Z = \frac{X}{Y}, \qquad z \geq 1$$

$$F_Z(z) = P(X \leq Yz) = \int_0^1 \int_{x/z}^{x} f_{XY}(x,y)\,dy\,dx$$

$$f_Z(z) = \int_0^1 \frac{x}{z^2} f_{XY}(x, x/z)\,dx = \frac{1}{z^2}, \quad z \geq 1$$

(b)

$$W = XY$$

$$\begin{aligned} F_W(w) &= P(W \leq w) = P(XY \leq w) = 1 - P(XY > w) \\ &= 1 - \int_{\sqrt{w}}^1 \int_{w/x}^{x} f_{XY}(x,y)\,dy\,dx \end{aligned}$$

Hence

$$\begin{aligned} f_W(w) &= \int_{\sqrt{w}}^1 \frac{1}{x} f_{XY}(x, w/x)\,dx = \int_{\sqrt{w}}^1 \frac{2}{x}\,dx \\ &= \ln(1/w), \quad 0 < w \leq 1 \end{aligned}$$

6.10 (a)

$$Z = X + Y$$

$$F_Z(z) = \int_0^{z/2} \int_x^{2-x} f_{XY}(x,y)\,dx = \frac{z^2}{4}, \quad 0 < z < 2$$

$$f_Z(z) = \frac{z}{2}, \qquad 0 < z < 2$$

(b)

$$W = X - Y$$

$$F_W(w) = \frac{1}{2}(2+w)(1+\frac{w}{2}) = \left(1 + \frac{w}{2}\right)^2$$

$$f_W(w) = \begin{cases} 1 + \frac{w}{2}, & -2 < w < 0 \\ 0, & \text{otherwise} \end{cases}$$

6.11 (a) The charaterristic function of $X+Y$ is given by

$$\begin{aligned}\phi_{X+Y}(\omega) &= \phi_X(\omega)\,\phi_Y(\omega) = \tfrac{1}{(1-j\omega\beta)^\alpha}\cdot\tfrac{1}{(1-j\omega\beta)^\alpha}\\ &= \tfrac{1}{(1-j\omega\beta)^{2\alpha}} \sim \text{Gamma}\,(2\alpha,\beta)\end{aligned}$$

(b)

$$f_{XY}(x,y) = f_X(x)\,f_Y(y) = \frac{(xy)^{(\alpha-1)}}{(\Gamma(\alpha)\beta^\alpha)^2}\,e^{(x+y)/\beta},\quad x>0,\ y>0$$

Let

$$Z = \frac{X}{Y}$$

Using (Eq. 6-60) we get

$$\begin{aligned}f_Z(z) &= \int_0^\infty y\,\frac{(y^2 z)^{(\alpha-1)}}{(\Gamma(\alpha)\beta^\alpha)^2}\,e^{-(1+z)y/\beta}\,dy\\ &= \frac{z^{(\alpha-1)}}{(\Gamma(\alpha)\beta^\alpha)^2}\int_0^\infty y^{(2\alpha-1)}\,e^{-(1+z)y/\beta}\,dy\\ &= \frac{z^{(\alpha-1)}}{(\Gamma(\alpha))^2\,\beta^{2\alpha}}\,\frac{\beta^{(2\alpha-1)}}{(1+z)^{2\alpha-1}}\,\frac{\beta}{(1+z)}\int_0^\infty u^{2\alpha-1}\,e^{-u}\,du\\ &= \frac{(\Gamma(2\alpha))\;z^{\alpha-1}}{(\Gamma(\alpha))^2\,(1+z)^{2\alpha}},\quad z>0\end{aligned}$$

(see also Example 6-27 for the answer).

(c)

$$W = \frac{X}{X+Y} = \frac{X/Y}{X/Y+1} = \frac{Z}{Z+1}$$

$$F_W(w) = P\left(\frac{Z}{Z+1}\le w\right) = P\left(Z\le\frac{w}{1-w}\right) = F_Z\left(\frac{w}{1-w}\right)$$

This gives

$$\begin{aligned}f_W(w) &= \frac{1}{(1-w)^2}\,f_Z\left(\tfrac{w}{1-w}\right)\\ &= \frac{\Gamma(2\alpha)}{(\Gamma(\alpha))^2}\,w^{\alpha-1}\,(1-w)^{\alpha-1}\\ &\sim\ \mathit{Beta}\,(\alpha,\alpha)\end{aligned}$$

where we have used results from (b) above.

6.12

$$X\sim U(0,1),\quad Y\sim U(0,1),\quad X,Y \text{ are independent, and}$$

$$U = X+Y,\qquad V = X-Y\ \Rightarrow\ |v|<u<2.$$

U and V have one pair of solutions given by

$$x_1 = \frac{u+v}{2},\ y_1 = \frac{u-v}{2}.$$

Also the Jacobian is given by

$$J = \begin{vmatrix} 1 & 1\\ 1 & -1\end{vmatrix} = -2$$

so that

$$f_{UV}(u,v) = \frac{1}{|J|}\,f_{XY}(x_1,y_1) = \frac{1}{2},\quad 0<|v|<u<2$$

6.13

$$f_{XY}(x,y) = \frac{xy}{\sigma^4}\, e^{-(x^2+y^2)/2\sigma^2}, \quad x, y \geq 0$$

$$Z = \frac{X}{Y}$$

$$F_Z(z) = P(Z \leq z) = P(X/Y \leq z) = \int_0^\infty \int_0^{zy} f_{XY}(x,y)\, dx\, dy.$$

This gives the density function of z to be

$$\begin{aligned} f_Z(z) &= \int_0^\infty y\, f_{XY}(zy, y)\, dy = \int_0^\infty \frac{zy^3}{\sigma^4}\, e^{-(z^2y^2+y^2)/2\sigma^2}\, dy \\ &= \frac{z}{\sigma^4}\int_0^\infty y^3\, e^{-y^2(z^2+1)/2\sigma^2}\, dy \quad \text{Let, } t = y^2(z^2+1)/2\sigma^2 \\ &= \frac{2z}{(z^2+1)^2}\int_0^\infty t\, e^{-t}\, dt = \frac{2z}{(z^2+1)^2}, \quad 0 \leq z \leq \infty. \end{aligned}$$

6-14 $\quad \underset{\sim}{z} = \underset{\sim}{x} + \underset{\sim}{y} \qquad f_z(z) = f_x(z) * f_y(z)$

For $z > 0$

$$c^2 z\, e^{-cz} = \int_0^z c\, e^{-c(z-y)}\, f_y(y)\, dy$$

$$c\, z = \int_0^z e^{cy}\, f_y(y)\, dy \qquad c = e^{cz}\, f_y(z)$$

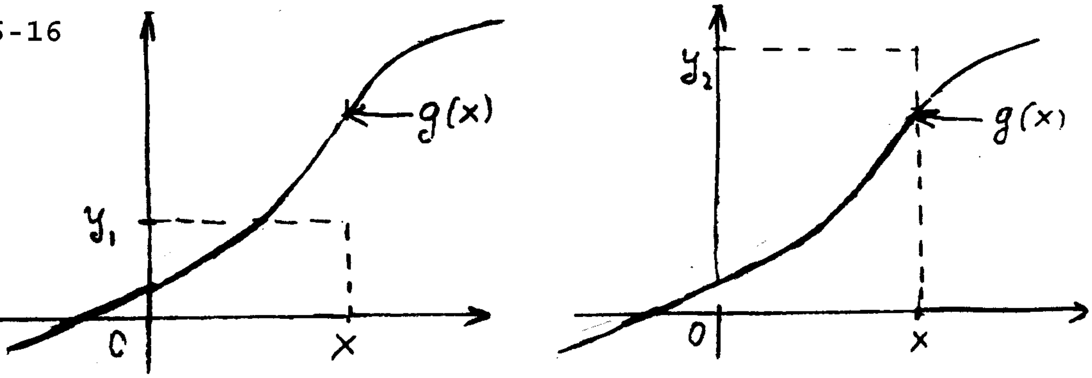
(differentiation). Hence, $f_y(z) = c\, e^{-cz}$; and zero for $z < 0$.

6-15

$$f_z(z) = \int_{-\infty}^{\infty} f_x(x) f_y(z-x)\, dx = \int_{z-1}^{z} f_x(x)\, dx = F_x(z) - F_x(z-1)$$

because $f_y(z-x) = 1$ for $z - 1 < x < z$ and zero otherwise.

6-16



All probability masses are on the line $y = g(x)$.

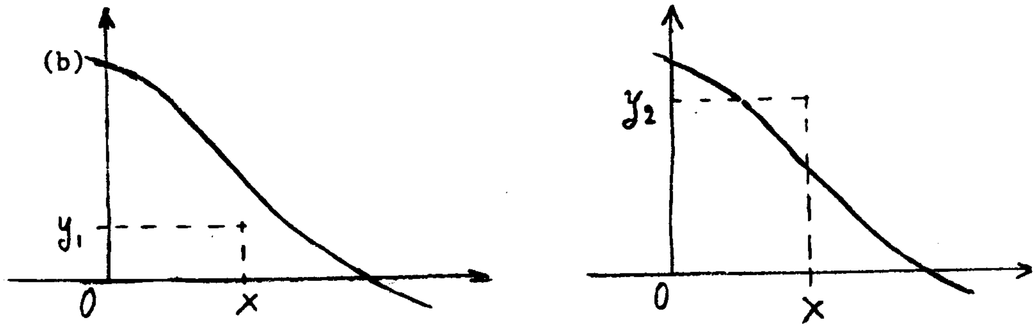
(a) If $y = y_1 < g(x)$ then

$$F(x,y) = P\{\mathbf{x} \le x,\ \mathbf{y} \le y_1\} = P\{\mathbf{y} \le y_1\} = F_y(y_1).$$

If $y = y_2 > g(x)$ then

$$F(x,y) = P\{\mathbf{x} \le x,\ \mathbf{y} \le y_2\} = P\{\mathbf{x} \le x\} = F_x(x)$$

(b)



If $y = y_1 < g(x)$ then

$$F(x,y) = P\{\mathbf{x} \le x,\ \mathbf{y} \le y_1\} = 0$$

If $y = y_2 > g(x)$ then

$$F(x,y) = P\{\mathbf{x} \le x,\ \mathbf{y} \le y_2\} = P\{\mathbf{x} \le x\} - P\{\mathbf{y} > y_2\}$$

$$= F_x(x) - [1 - F_y(y_2)]$$

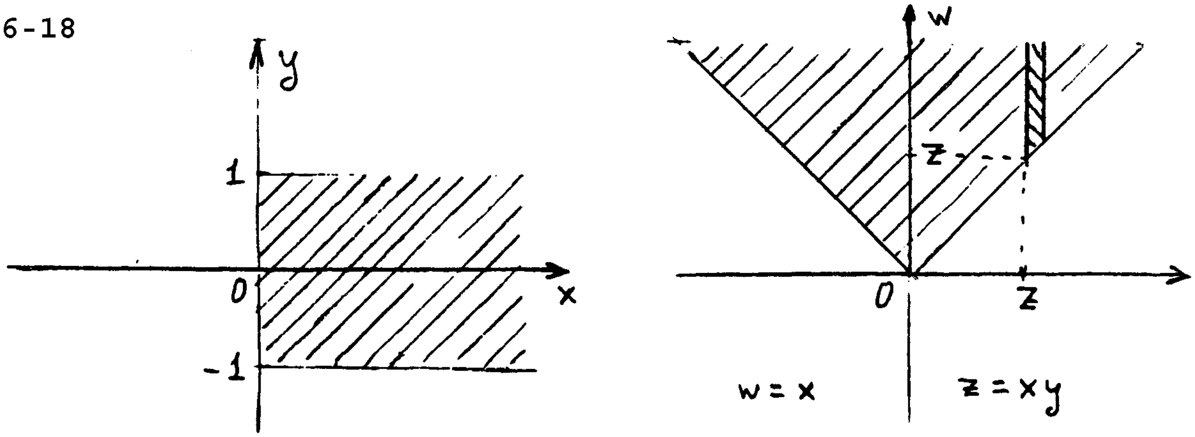
6-17 (a) If $\underset{\sim}{z} = 2\underset{\sim}{x} + 3\underset{\sim}{y}$ then $E\{\underset{\sim}{z}\} = 0$ $\qquad \sigma_z^2 = 4\sigma_x^2 + 9\sigma_y^2 = 5^2$

Hence, $\underset{\sim}{z}$ is $N(0;\sqrt{52})$

(b) If $\underset{\sim}{z} = \underset{\sim}{x}/\underset{\sim}{y}$, then from (6-63) with $\sigma_1 = \sigma_2 = 2$, $r = 0$

$$F_z(z) = \frac{1}{2} + \frac{1}{\pi}\arctan z \qquad f_z(z) = \frac{1}{\pi(1+z^2)}$$

6-18



$$f_{zw}(z,w) = \frac{1}{|x|} f_{xy}(x,y) \qquad x = w \qquad y = z/w$$

The function $f_{zw}(z,w)$ is different from zero in the shaded areas shown. Hence, with $w^2 - z^2 = s^2$

$$f_z(z) = \frac{1}{\pi\alpha^2}\int_{|z|}^{\infty} e^{-w^2/2\alpha^2} \frac{dw}{\sqrt{1 - z^2/w^2}}$$

$$= \frac{1}{\pi\alpha^2}\int_0^{\infty} e^{-(z^2+s^2)/2\alpha^2}\, ds = \frac{1}{\alpha\sqrt{2\pi}}\, e^{-z^2/2\alpha^2}$$

6-19 (a) $\underset{\sim}{z} = \underset{\sim}{x}/\underset{\sim}{y} \qquad \underset{\sim}{w} = \underset{\sim}{y} \qquad J = 1/y$

$$f_z(z) = \int_{-\infty}^{\infty} |w| f_x(zw) f_y(w)\,dw \qquad z > 0$$

$$= \frac{z}{\alpha^2\beta^2} \int_0^{\infty} w^3\, e^{-cw^2} dw = \frac{z}{2\alpha^2\beta^2 c^2} \qquad c = \frac{z^2}{2\alpha^2} + \frac{1}{2\beta^2}$$

$$= \frac{2\alpha^2}{\beta^2} \frac{z}{(z^2 + \alpha^2/\beta^2)^2} \qquad \text{for } z > 0 \text{ and zero otherwise}$$

(b) $$F_z(z) = \int_0^z \frac{2\alpha^2 z\,dz}{\beta^2(z^2 + \alpha^2/\beta^2)^2} = \frac{\alpha^2}{\beta^2} \int_{\alpha^2/\beta^2}^{z^2 + \alpha^2/\beta^2} \frac{dt}{t^2}$$

$$= \frac{z^2}{z^2 + \alpha^2/\beta^2} = P\{\underset{\sim}{z} \le z\} = P\{\underset{\sim}{x} \le zy\}$$

6-20 1. The density of $2\underset{\sim}{x}$ equals $\frac{1}{2} f_x(\frac{x}{2})$. Hence, if $\underset{\sim}{z} = 2\underset{\sim}{x} + \underset{\sim}{y}$, then

$$f_z(z) = \int_0^z \frac{\alpha}{2} e^{-\alpha x/2} \beta e^{-\beta(z-x)} dx = \frac{\alpha\beta}{\alpha - 2\beta}(e^{-\beta z} - e^{-\alpha z/2})U(z)$$

2. The density of $\underset{\sim}{y}$ equals $f_y(-y)$. Hence, if $\underset{\sim}{z} = \underset{\sim}{x} - \underset{\sim}{y}$, then

$$f_z(z) = f_x(z) * f_y(-z)$$

$$= \alpha\beta \begin{cases} \displaystyle\int_z^\infty e^{-\alpha x} e^{-\beta(x-z)} dx = \frac{\alpha\beta}{\alpha+\beta} e^{-\alpha z} & z > 0 \\ \displaystyle\int_0^\infty e^{-\alpha x} e^{-\beta(x-z)} dx = \frac{\alpha\beta}{\alpha+\beta} e^{\beta z} & z < 0 \end{cases}$$

3. $\underset{\sim}{z} = \underset{\sim}{x}/\underset{\sim}{y}$ $\qquad$ $\underset{\sim}{w} = \underset{\sim}{y}$ $\qquad$ $J = 1/y$

$$f_z(z) = \alpha\beta \int_0^\infty w e^{-\alpha z w} e^{-\beta w} dw = \frac{\alpha\beta}{(\alpha z + \beta)^2} U(z)$$

4. $\underset{\sim}{z} = \max(\underset{\sim}{x},\underset{\sim}{y})$ $\qquad$ $F_z(z) = F_{xy}(z,z) = F_x(z)F_y(z)$

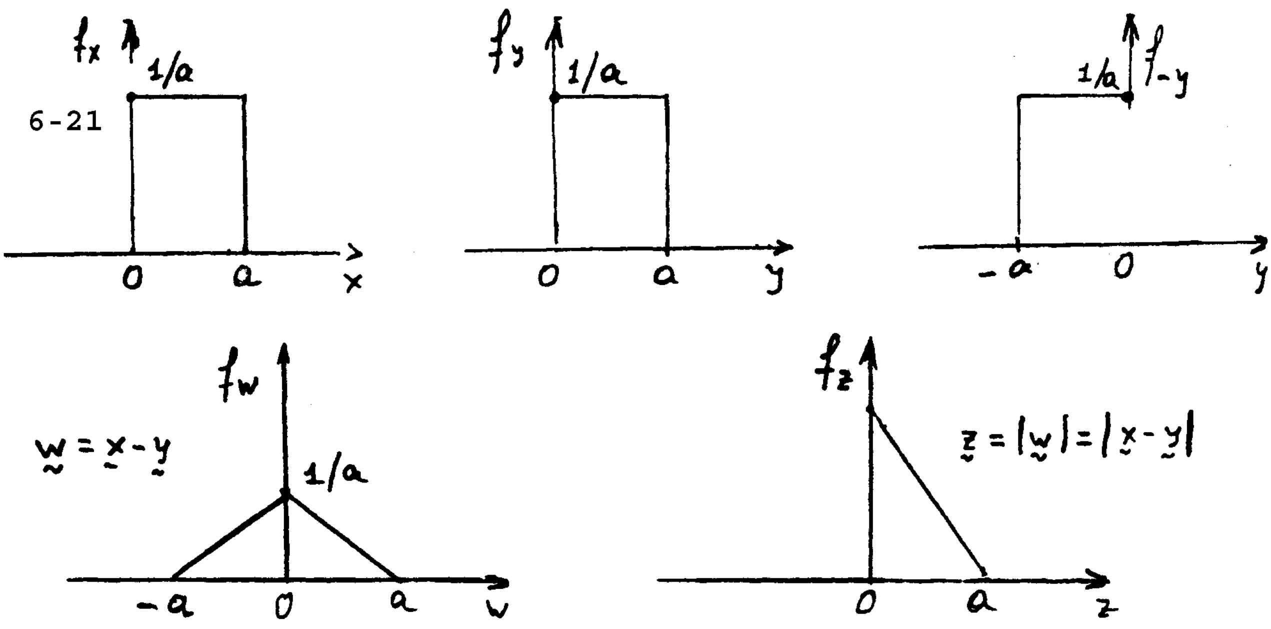
$$f_z(z) = f_x(z)F_y(z) + f_y(z)F_x(z)$$

$$= \left[\alpha e^{-\alpha z}(1 - e^{-\beta z}) + \beta e^{-\beta z}(1 - e^{-\alpha z})\right] U(z)$$

5. $\underset{\sim}{z} = \min(\underset{\sim}{x},\underset{\sim}{y})$ $\qquad$ $F_z(z) = F_x(z) + F_y(z) - F_x(z)F_y(z)$

$$f_z(z) = f_x(z)[1 - F_y(z)] + f_y(z)[1 - F_x(z)] = (\alpha+\beta)e^{-(\alpha+\beta)z}U(z)$$

6-21



6-22 (a)
$$\alpha y^2 + \beta(x-y)^2 = (\alpha+\beta)\left(y - \frac{\beta x}{\alpha+\beta}\right)^2 + \frac{\alpha\beta}{\alpha+\beta}x^2$$

$$e^{-\alpha x^2} * e^{-\beta x^2} = \int_{-\infty}^{\infty} e^{-\alpha y^2 - \beta(x-y)^2}\,dy$$

$$= e^{-\alpha\beta x^2/(\alpha+\beta)} \int_{-\infty}^{\infty} e^{-(\alpha+\beta)\left(y - \frac{\beta x}{\alpha+\beta}\right)^2}\,dy = \sqrt{\frac{\pi}{\alpha+\beta}}\; e^{\frac{-\alpha\beta x^2}{\alpha+\beta}}$$

(b)
$$\frac{\alpha/\pi}{x^2+\alpha^2} * \frac{\beta/\pi}{x^2+\beta^2} = \frac{\alpha\beta}{\pi^2}\int_{-\infty}^{\infty} \frac{dy}{(y^2+\alpha^2)[(x-y)^2+\beta^2]} = \frac{(\alpha+\beta)/\pi}{x^2+(\alpha+\beta)^2}$$

Characteristic functions lead to a simpler derivation of the above [see (6-192)]

6-23 We introduce the auxiliary variable w=y. The Jacobian of the transformation z=nx/my, w=y equals n/my. Since x=mzw/n, y=w and the RVs $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are independent, (6-113) yields

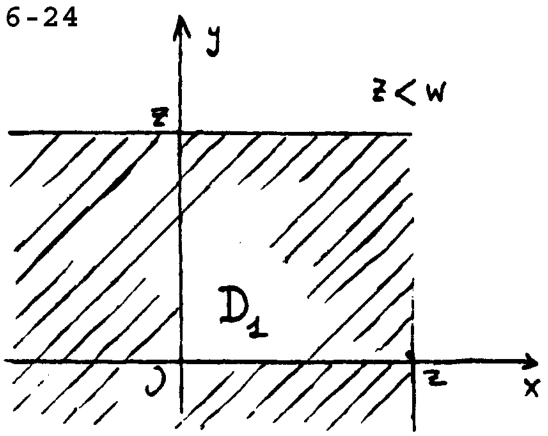
$$f_{zw}(z,w) = \frac{mw}{n} f_x\left(\frac{m}{n}zw\right) f_y(w) \sim w(zw)^{m/2-1}e^{-mzw/2}w^{n/2-1}e^{-w/2}$$

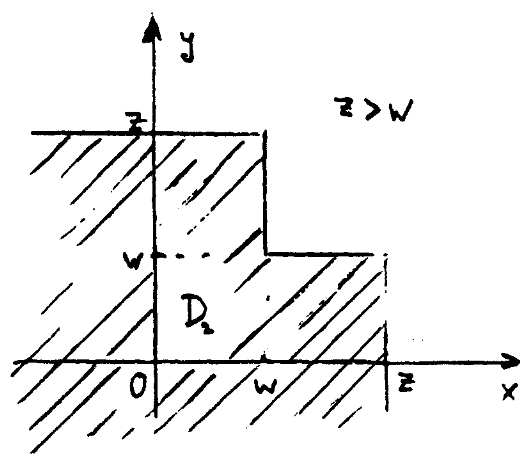
for z>0, w>0 and 0 othrwise. Integrating with respect to w, we obtain

$$f_z(z) \sim z^{m/2-1}\int_0^\infty w^{(m+n)/2-1}\exp\left\{-\frac{w}{2}\left(1+\frac{m}{n}z\right)\right\}dw$$

$$\sim \frac{z^{m/2-1}}{(1+mz/n)^{(m+n/2)}}\int_0^\infty q^{(m+n)/2}e^{-q}dq$$

6-24





If $z \le w$ then

$$P\{\underset{\sim}{z} \le z,\ \underset{\sim}{w} \le w\} = P\{\underset{\sim}{z} \le z\} = P\{(\underset{\sim}{x},\underset{\sim}{y}) \in D_1\} = F_{xy}(z,z)$$

If $z > w$ then

$$P\{\underset{\sim}{z} \le z,\ \underset{\sim}{w} \le w\} = P\{(\underset{\sim}{x},\underset{\sim}{y}) \in D_2\}$$

$$= F_{xy}(z,w) + F_{xy}(w,z) - F_{xy}(w,w)$$

6.25

$$X \sim \text{Exponential}(\lambda), \quad Y \sim \text{Exponential}(\lambda)$$

X and Y are independent so that

$$f_{XY}(x,y) = f_X(x)\, f_Y(y) = \frac{1}{\lambda^2}\, e^{-(x+y)/\lambda}\, U(x)\, U(y)$$

$$Z = X + Y$$

$$\phi_Z(\omega) = \phi_X(\omega)\, \phi_Y(\omega) \frac{1}{(1 - j\omega\lambda)^2}$$

$$Z \sim \text{ Gamma } (2, \lambda)$$

This gives

$$f_Z(z) = \frac{z}{\lambda^2}\, e^{-z/\lambda} U(z)$$

$$P(Z > 2\lambda) = \int_{2\lambda}^{\infty} \frac{z}{\lambda^2}\, e^{-z/\lambda}\, dz = \int_2^{\infty} x e^{-x}\, dx = 3e^{-2} = 0.406$$

Let,

$$W = Y - X$$

Then

$$P(Y - X > \lambda) = P(W > \lambda) = \int_{\lambda}^{\infty} f_W(w) dw$$

Notice that $F_W(w)$ is given by (6-55).
For $w > 0$, this gives

$$\begin{aligned} f_W(w) &= \int_0^{\infty} \frac{1}{\lambda^2}\, e^{-(w+2y)/\lambda}\, dy = \frac{1}{\lambda^2}\, e^{-w/\lambda} \int_0^{\infty} e^{-2y/\lambda}\, dy \\ &= \frac{1}{2\lambda}\, e^{-w/\lambda}, \quad w > 0 \end{aligned}$$

Hence

$$P(Y - X > \lambda) = P(W > \lambda) = \int_{\lambda}^{\infty} \frac{1}{2\lambda}\, e^{-w/\lambda}\, dw = \frac{1}{2e}$$

6.26 (a)

$$\begin{aligned} R &= W - Z \\ &= \max(\mathrm{X}, \mathrm{Y}) \; - \; \min(\mathrm{X}, \mathrm{Y}) \\ &= \begin{cases} X - Y, & X \geq Y \\ Y - X, & X < Y \end{cases} \end{aligned}$$

$$\begin{aligned} F_R(r) &= P\{R \leq r\} \\ &= P\{R \leq r, X \geq Y\} + P\{R \leq r, X < Y\} \\ &= P\{X - Y \leq r, X \geq Y\} + P\{Y - X \leq r, X < Y\} \\ &= 1 - 2\frac{(1-r)^2}{2} = 1 - (1-r)^2, \quad 0 \leq r \leq 1 \end{aligned}$$

$$f_R(r) = \begin{cases} 2(1-r), & 0 \leq r \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(b)

$$\begin{aligned} S &= W + Z \\ &= \max(\mathrm{X}, \mathrm{Y}) \; + \; \min(\mathrm{X}, \mathrm{Y}) = X + Y \end{aligned}$$

Case 1: $0 < s < 1$

$$F_S(s) = P\{S \leq s\} = P\{X + Y \leq s\} = \frac{s^2}{2}, \quad 0 < s < 1$$

Case 2: $1 \leq s \leq 2$

$$F_S(s) = P\{S \leq s\} = P\{X + Y \leq s\} = 1 - \frac{(2-s)^2}{2}, \quad 1 \leq s \leq 2$$

$$F_S(s) = \begin{cases} s, & 0 \leq s \leq 1 \\ (2-s), & 1 \leq s \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

6.27 (a) X, Y are independent, identically distributed exponential random variables.

$$Z = \frac{Y}{\max(X,Y)} = \begin{cases} \frac{Y}{X}, & X \geq Y \\ 1, & X < Y \end{cases} \quad \Rightarrow \; 0 < z \leq 1.$$

$0 < z < 1$

$$\begin{aligned} F_Z(z) &= P(Z \leq z) = P\left\{\tfrac{Y}{X} \leq z,\, X > Y\right\} \\ &= P\{Y \leq Xz,\, X > Y\} = \int_0^\infty \int_0^{xz} f_{XY}(x,y)\, dy\, dx \end{aligned}$$

$$f_Z(z) = \int_0^\infty x\, f_{XY}(x, xz)\, dx = \int_0^\infty \frac{x}{\lambda^2}\, e^{-(1+z)x/\lambda}\, dx = \frac{1}{(1+z)^2}, \quad 0 < z < 1.$$

Also

$$P(Z = 1) = P(X < Y) = \int_0^\infty \int_0^y \frac{1}{\lambda^2}\, e^{-(x+y)/\lambda}\, dx\, dy = \frac{1}{2}$$

(b)

$$W = \frac{X}{\min(X, 2Y)} = \begin{cases} \frac{X}{2Y}, & X \geq 2Y \\ 1, & X < 2Y \end{cases} \quad \Rightarrow \; 1 \leq w < \infty$$

$$F_W(w) = P(X \leq 2Yw,\, X > 2Y) = \int_0^\infty \int_{2y}^{2wy} f_{XY}(x,y)\, dx\, dy$$

This gives

$$\begin{aligned} f_W(w) &= \int_0^\infty 2y\, f_{XY}(2wy, y) dy = \int_0^\infty \frac{2y}{\lambda^2}\, e^{-(1+2w)y/\lambda} dy \\ &= \frac{2}{(1+2w)^2}, \quad w > 1 \end{aligned}$$

Also

$$P(W = 1) = P(X < 2Y) = \int_0^\infty \int_0^{2y} \frac{1}{\lambda^2}\, e^{-(x+y)/\lambda}\, dx\, dy = \frac{2}{3}$$

Note that the p.d.f. of Z as well as W has an impulse at $z = 1$ and $w = 1$ respectively.

6.28 X, Y are independent identically distributed exponential random variables.

$$Z = \frac{X}{X+Y}$$

$$\begin{aligned} F_Z(z) &= P\left(\frac{X}{X+Y} \le z\right) = P\left(\frac{X}{Y} \le \frac{z}{1-z}\right) \\ &= P\left\{X \le \frac{zY}{1-z}\right\} = \int_0^\infty \int_0^{(zy)/(1-z)} f_{XY}(x,y)\,dx\,dy \end{aligned}$$

$$\begin{aligned} f_Z(z) &= \int_0^\infty \frac{y}{(1-z)^2} f_{XY}(zy/(1-z), y)\,dy \\ &= \frac{1}{(1-z)^2}\int_0^\infty y\,\frac{1}{\lambda^2}\,e^{-(z/(1-z)+1)(y/\lambda)}\,dy \\ &= \frac{1}{(1-z)^2}\int_0^\infty \frac{y}{\lambda^2}\,e^{-[y/(1-z)\lambda]}\,dy \\ &= \int_0^\infty u\,e^{-u}\,du = 1, \quad 0 < z < 1 \\ &\Rightarrow \frac{X}{X+Y} \sim U(0,1) \end{aligned}$$

6.29 Let

$$f_X(x) = \frac{1}{\lambda}\,e^{-x/\lambda}\,U(x), \quad f_Y(y) = \frac{1}{\lambda}\,e^{-y/\lambda}\,U(y).$$

$$Z = \min(X, Y)$$

$$W = \max(X, Y) - \min(X, Y)$$

$$Z = \begin{cases} Y, & X \ge Y \\ X, & X < Y \end{cases}$$

$$W = \begin{cases} X - Y, & X \ge Y \\ Y - X, & X < Y \end{cases}$$

$Z = \min(X, Y)$. See Example 6-18, Eq.(6-82) for solution. From there (replace λ by $1/\lambda$ in (6-82))

$$f_Z(z) = \frac{2}{\lambda}\,e^{-2z/\lambda}\,U(z).$$

$$\begin{aligned} F_W(w) &= P(X - Y \le w, X \ge Y) + P(Y - X \le w, X < Y) \\ &= \int_0^\infty \int_y^{y+w} f_{XY}(x,y)\,dx\,dy \\ &+ \int_0^\infty \int_x^{x+w} f_{XY}(x,y)\,dy\,dx, \quad w > 0 \end{aligned}$$

This gives

$$\begin{aligned} F_W(w) &= \int_0^\infty f_{XY}(y+w, y)\,dy + \int_0^\infty f_{XY}(x, x+w)\,dx \\ &= 2\int_0^\infty \frac{1}{\lambda^2}\,e^{(2y+w)/\lambda}\,dy \\ &= \frac{2}{\lambda^2}\,e^{-w/\lambda}\,\left.\frac{e^{-2y/\lambda}}{-2/\lambda}\right|_0^\infty = \frac{1}{\lambda}\,e^{-w/\lambda}, \quad w > 0 \end{aligned}$$

Also

$$\begin{aligned}
F_{ZW}(z,w) &= P\{Z \le z, W \le w\} \\
&= P\{Y \le z, X - Y \le w, X \ge Y\} \\
&+ P\{X \le z, Y - X \le w, X < Y\} \\
&= \int_0^z \int_y^{y+w} f_{XY}(x,y)\,dx\,dy + \int_0^z \int_x^{x+w} f_{XY}(x,y)\,dy\,dx
\end{aligned}$$

Repeated use of (6-39)-(6-40) gives

$$\begin{aligned}
f_{ZW}(z,w) &= f_{XY}(z+w,z) + f_{XY}(z,z+w) \\
&= \frac{2}{\lambda^2}\,e^{-(2z+w)/\lambda} = \frac{2}{\lambda}\,e^{-2z/\lambda}\frac{1}{\lambda}\,e^{-w/\lambda} \\
&= f_Z(z)\,f_W(w)
\end{aligned}$$

Thus Z and W are independent exponential random variables.

6.30 (a) Let

$$U = X + Y, \qquad 0 < u < 2\beta.$$

The probability density function of U can be computed as in (6-48)-(6-50). Using Fig. 6-11, for $0 < u \le \beta$, we have

$$F_U(u) = \int_0^u \int_0^{u-x} f_{XY}(x,y)\,dy\,dx$$

which gives

$$\begin{aligned}
f_U(u) &= \int_0^u f_{XY}(x, u-x)dx = \alpha^2\beta^{-2\alpha}\int_0^u x^{\alpha-1}\,(u-x)^{\alpha-1}\,dx \\
&= \alpha^2\beta^{-2\alpha}u^{2\alpha-1}\int_0^1 y^{\alpha-1}\,(1-y)^{\alpha-1}\,dy \\
&= B(\alpha,\alpha)\,\alpha^2\beta^{-2\alpha}u^{2\alpha-1} \quad 0 < u \le \beta
\end{aligned}$$

where we have substituted $y = ux$ and made use of the beta function defied in (4-49)-(4-51). Similarly for $\beta < u \le 2\beta$, we get (see (6-49))

$$F_U(u) = 1 - \int_{u-\beta}^{\beta}\int_{u-x}^{\beta} f_{XY}(x,y)\,dydx$$

and hence

$$\begin{aligned}
f_U(u) &= \int_{u-\beta}^{\beta} f_{XY}(x,u-x)dx = \alpha^2\beta^{-2\alpha}\int_{u-\beta}^{\beta} x^{\alpha-1}(u-x)^{\alpha-1}\,dx \\
&= \alpha^2\beta^{-2\alpha}u^{2\alpha-1}\int_{1-\beta/u}^{\beta/u} y^{\alpha-1}(1-y)^{\alpha-1}\,dy, \quad \beta < u \le 2\beta
\end{aligned}$$

(b)

$$Z = \min(X,Y), \qquad W = \max(X,Y)$$

We can proceed as in Example 6-21 to complete this problem. From (6-92) and (6-93), we get

$$F_{ZW}(z,w) = \begin{cases} F_{XY}(z,w) + F_{XY}(w,z) - F_{XY}(z,z), & w \ge z \\ F_{XY}(w,w), & w < z \end{cases}$$

which gives

$$f_{ZW}(z,w) = f_X(z)f_Y(w) + f_X(w)f_Y(z), \quad 0 < z \le w < \beta$$

$$f_{ZW}(z,w) = \begin{cases} 2\alpha^2\beta^{-2\alpha}z^{\alpha-1}w^{\alpha-1}, & 0 < z \le w < \beta \\ 0, & \text{otherwise} \end{cases}$$

check:

$$\begin{aligned} \int_0^\beta \int_0^w f_{ZW}(z,w)dzdw &= 2\alpha^2\beta^{-2\alpha}\int_0^\beta w^{\alpha-1}\left(\frac{z^\alpha}{\alpha}\bigg|_0^w\right)dw \\ &= 2\alpha\beta^{-2\alpha}\int_0^\beta w^{2\alpha-1}dw = 1 \end{aligned}$$

Note: Z and W are not independent random variables, since

$$f_Z(z) = 2\alpha\beta^{-2\alpha}\, z^{\alpha-1}\,(\beta^\alpha - z^\alpha), \quad 0 < z < \beta$$

and

$$f_W(w) = 2\alpha\beta^{-2\alpha}\, w^{2\alpha-1}, \quad 0 < w < \beta$$

(c) Let

$$V = \frac{Z}{W} = \frac{\min(X,Y)}{\max(X,Y)} = \begin{cases} \frac{Y}{X}, & X \ge Y \\ \frac{X}{Y}, & X < Y \end{cases}$$

and

$$W = \max(X,Y) = \begin{cases} X, & X \ge Y \\ Y, & X < Y \end{cases}$$

For $0 < v < 1,\ 0 < w < \beta$

$$\begin{aligned} F_{VW}(v,w) &= P(V \le v,\, W \le w) \\ &= P\{V \le v,\, W \le w,\ (X \ge Y)\textstyle\bigcup(X < Y)\} \\ &= P\{Y \le Xv,\, X \le w,\, X \ge Y\} \\ &+ P\{X < Yv,\, Y \le w,\, X < Y\} \\ &= \int_0^w \int_0^{xv} f_{XY}(x,y)\,dy\,dx + \int_0^w \int_0^{yv} f_{XY}(x,y)\,dx\,dy \end{aligned}$$

Hence

$$
\begin{aligned}
f_{VW}(v,w) &= \frac{\partial^2 F_{VW}(v,w)}{\partial v\,\partial w} \\
&= \frac{\partial}{\partial v}\left\{\int_0^{vw} f_{XY}(w,y)\,dy + \int_0^{vw} f_{XY}(x,w)\,dx\right\} \\
&= w\,\{f_{XY}(w,vw) + f_{XY}(vw,w)\} \\
&= 2\alpha^2\,\beta^{-2\alpha}\,w^{2\alpha-1}\,v^{\alpha-1}, \qquad 0<v<1,\ 0<w<\beta
\end{aligned}
$$

Hence

$$
f_V(v) = \int_0^{\beta} f_{VW}(v,w)dw = \alpha\,v^{\alpha-1}, \quad 0<v<1
$$

$$
f_W(w) = \int_0^{1} f_{VW}(v,w)dv = 2\alpha\,\beta^{-2\alpha}\,w^{2\alpha-1}, \quad 0<w<\beta
$$

and

$$
f_{VW}(v,w) = f_V(v)\,f_W(w).
$$

Thus V and W are independent random variables.

6.31 (a) Solved in Examples 6-27 and 6-12.
(b) Solved in Example 6-27.
(c)

$$
Z = X+Y, \qquad W = \frac{X}{X+Y}
$$

$$
x_1 = zw, \qquad y_1 = z - x_1 = z(1-w)
$$

$$
J = \begin{vmatrix} 1 & 1 \\ \dfrac{y}{(x+y)^2} & -\dfrac{x}{(x+y)^2} \end{vmatrix} = \frac{1}{x+y} = \frac{1}{z}
$$

$$
\begin{aligned}
f_{ZW}(z,w) &= \frac{z}{\alpha^{m+n}\Gamma(m)\Gamma(n)}\,(zw)^{m-1}\,\{z(1-w)\}^{n-1} \\
&= \left(\frac{z^{m+n-1}}{\alpha^{m+n}\Gamma(\alpha+\beta)}\;e^{-z/\alpha}\right)\left(\frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)}\;w^{m-1}\,(1-w)^{n-1}\right) \\
&= f_Z(z)\,f_W(w)
\end{aligned}
$$

Thus Z and W are independent random variables.

6.32 (a)

$$Z = \frac{X}{|Y|}, \qquad W = \frac{|X|}{|Y|} = |Z|$$

$$\begin{aligned} F_Z(z) &= P(Z \le z) = P(X \le |Y|z) = \int_{-\infty}^{\infty} \int_0^{|y|z} f_{XY}(x,y)\,dx\,dy \\ &= 2\int_0^{\infty} |y| f_{XY}(|y|z, y) dy = \frac{2}{2\pi\sigma^2} \int_0^{\infty} y e^{-(z^2+1)y^2/2\sigma^2}\,dy \\ &= \frac{1/\pi}{1+z^2}, \qquad -\infty < z < \infty \end{aligned}$$

Thus Z is a Cauchy random variable. Interestingly, the random variable X/Y is also a Cauchy random variable (see Example 6-11).

$$W = |Z|$$

so that

$$\begin{aligned} F_W(w) &= P(W \le w) = P(|Z| \le w) \\ &= P(-w < Z < w) = F_Z(w) - F_Z(-w) \end{aligned}$$

and hence

$$f_W(w) = f_Z(w) + f_Z(-w) = \frac{2/\pi}{1+w^2}, \quad w > 0.$$

(b)

$$U = X + Y \sim N(0, 2)$$
$$V = X^2 + Y^2 \sim \text{Exponential } (2)$$

(see Example 6-14). Here U, V are *not* independent, since

$$J(x,y) = \begin{vmatrix} 1 & 1 \\ 2x & 2y \end{vmatrix} = -2(x-y) = 2\sqrt{2v - u^2}$$

and

$$\begin{aligned} f_{UV}(u,v) &= \frac{1}{2\sqrt{2v-u^2}} \frac{1}{2\pi\sigma^2} e^{-v/2\sigma^2} \\ &\ne f_U(u)\, f_V(v), \qquad -\infty < u < \infty, \;\; v > 0. \end{aligned}$$

6.33

$$Z = X + Y, \qquad W = X - Y$$

are jointly normal random variables. Hence if they are uncorrelated, then they are also independent.

$$\begin{aligned} Cov(Z, W) &= E[(Z - \mu_Z)(W - \mu_W)] \\ &= E\left[\{(X - \mu_X) + (Y - \mu_Y)\}\{(X - \mu_X) - (Y - \mu_Y)\}\right] \\ &= \text{Var}(X) - \text{Var}(Y) = \sigma_X^2 - \sigma_Y^2. \end{aligned}$$

The random variables Z and W are uncorrelated implies that $Cov(Z, W) = 0$. Hence $\sigma_X^2 = \sigma_Y^2$ is the necessary and sufficient condition for the independence of $X + Y$ and $X - Y$.

6.34 (a)-(b) Let

$$R = \sqrt{X^2 + Y^2}, \qquad \theta = \tan^{-1}\left(\frac{Y}{X}\right)$$

Form Example 6-22, R and θ are independent random variables with joint p.d.f. as in (6-128). (see (6-131)). In term of R and θ, we have $X = R\cos\theta, Y = R\sin\theta$ and hence we obtain

$$U = \frac{X^2 - Y^2}{\sqrt{X^2 + Y^2}} = R\cos 2\theta$$

$$V = \frac{2XY}{\sqrt{X^2 + Y^2}} = R\sin 2\theta$$

This gives

$$J = \begin{vmatrix} \cos 2\theta & -2r\sin 2\theta \\ \sin 2\theta & 2r\cos 2\theta \end{vmatrix} = 2r = 2\sqrt{u^2 + v^2}$$

$$r = \sqrt{u^2 + v^2}, \quad \theta_1 = \frac{1}{2}\tan^{-1}\left(\frac{v}{u}\right), \quad 2\theta_2 = \pi + 2\theta_1.$$

There are two sets of solutions (r, θ_1) and (r, θ_2). Substituting into (6-128) we get

$$\begin{aligned} f_{UV}(u, v) &= \frac{1}{|J|}\{f_{r,\theta}(r, \theta_1) + f_{r,\theta}(r, \theta_2)\} = \frac{2}{|J|}\, f_{r,\theta}(r, \theta_1) \\ &= \frac{2}{2\sqrt{u^2 + v^2}}\, \frac{\sqrt{u^2 + v^2}}{2\pi\sigma^2}\, e^{-(u^2+v^2)/2\sigma^2} \\ &= \frac{1}{2\pi\sigma^2}\, e^{-(u^2+v^2)/2\sigma^2} = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-u^2/2\sigma^2} \cdot \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-v^2/2\sigma^2} \\ &= f_U(u) f_V(v) \end{aligned}$$

Thus U and V are independent normal random variables. Hence it follows that $U = \dfrac{X^2 - Y^2}{\sqrt{X^2 + Y^2}}$ and $V/2 = \dfrac{XY}{\sqrt{X^2 + Y^2}}$ are independent random variables.

(c)

$$\begin{aligned} Z &= \frac{(X - Y)^2 - 2Y^2}{\sqrt{X^2 + Y^2}} = \frac{(X^2 - Y^2) - 2XY}{\sqrt{X^2 + Y^2}} \\ &= \frac{X^2 - Y^2}{\sqrt{X^2 + Y^2}} - \frac{2XY}{\sqrt{X^2 + Y^2}} \\ &= U - V \sim N(0, 2\sigma^2). \end{aligned}$$

6.35 (a) $Z \sim F(m,n)$ is given by (6-157) Let

$$Y = \frac{1}{Z}$$

Then

$$\begin{aligned} F_Y(y) &= \frac{1}{|dy/dz|}\, f_Z(1/y) \\ &= \frac{1}{y^2}\,\frac{(m/n)^{m/2}}{\beta(m/2,n/2)}\,\frac{1}{y^{m/2-1}}\,\frac{1}{(1+m/ny)^{m+n/2}} \\ &= \frac{(n/m)^{n/2}}{\beta(n/2,m/2)}\, y^{n/2-1}\left(1+\frac{n}{my}\right)^{-(m+n)/2} \\ &\sim F(n,m). \end{aligned}$$

(b)

$$W = \frac{Zm}{Zm+n}$$

$$\begin{aligned} F_W(w) &= P(W \le w) = P\left(\frac{Zm}{Zm+n} \le w\right) \\ &= P\left(Z \le \frac{nw}{m(1-w)}\right) = F_Z\left(\frac{nw}{m(1-w)}\right) \end{aligned}$$

which gives

$$\begin{aligned} f_W(w) &= \frac{n}{m(1-w)^2}\, f_Z\left(\frac{nw}{m(1-w)}\right) \\ &= \frac{n}{m(1-w)^2}\,\frac{(m/n)^{m/2}}{\beta(m/2,n/2)}\left(\frac{nw}{m(1-w)}\right)^{m/2-1}\left(1+\frac{w}{(1-w)}\right)^{-(m+n)/2} \\ &= \frac{1}{\beta(m/2,n/2)}\, w^{m/2-1}\,(1-w)^{n/2-1}, \quad 0<w<1. \end{aligned}$$

Thus W has Beta distribution.

6.36

$$Z = X+Y > 0, \qquad W = X-Y > 0$$

$$x_1 = \frac{z+w}{2}, \qquad y_1 = \frac{z-w}{2}$$

is the only solution. Moreover

$$J = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

so that

$$f_{ZW}(z,w) = \frac{1}{|J|} f_{XY}(x_1,y_1) = \frac{1}{2}\, e^{-(z+w)/2}, \quad 0<w<z<\infty$$

$$\begin{aligned} F_Z(z) &= \int_0^z f_{ZW}(z,w)\,dw = \frac{1}{2}\, e^{-z/2}\,\left.\frac{e^{-w/2}}{(-1/2)}\right|_0^z \\ &= \frac{1}{2}\, e^{-z/2}\,\left.\frac{e^{-w/2}}{(-1/2)}\right|_0^z = e^{-z/2}(1-e^{-z/2}), \quad z>0 \end{aligned}$$

6.37

$$Z = X + Y > 0, \qquad W = \frac{Y}{X} > 1$$

$$y = xw, \quad x(1+w) = z, \quad x_1 = \frac{z}{1+w}, \quad y_1 = \frac{zw}{1+w}$$

is the only solution. Also

$$J = \begin{vmatrix} 1 & 1 \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{x+y}{x^2} = \frac{(1+w)^2}{z}$$

This gives

$$\begin{aligned} f_{ZW}(z,w) &= \frac{1}{|J|} f_{XY}(x_1, y_1) \\ &= \frac{z}{(1+w)^2} \, 2\, e^{-z}, \quad z > 0, \ w > 1 \\ &= z\, e^{-z} \frac{2}{(1+w)^2} = f_Z(z)\, f_W(w) \end{aligned}$$

since

$$\begin{aligned} f_Z(z) &= \int_1^\infty f_{ZW}(z,w)\, dw \\ &= 2ze^{-z} \int_1^\infty \frac{1}{(1+w)^2}\, dw = z\, e^{-z}, \quad z > 0 \end{aligned}$$

and

$$\begin{aligned} f_w(w) &= \int_0^\infty f_{ZW}(z,w)\, dz \\ &= \frac{2}{(1+w)^2} \int_0^\infty z e^{-z}\, dz = \frac{2}{(1+w)^2}, \quad w > 1. \end{aligned}$$

Thus Z and W are independent random variables.

6-38 $\quad z = xy \qquad y = \cos(\omega t + \theta)$

$$w = y \qquad J = |y| \qquad f_y(y) = \begin{cases} \dfrac{1}{\pi\sqrt{1-y^2}} & |y| < 1 \\ 0 & |y| > 1 \end{cases}$$

The RVs x and y are independent. Hence,

$$f_{zw}(z,w) = \frac{1}{|w|} f_x\left(\frac{z}{w}\right) f_y(w)$$

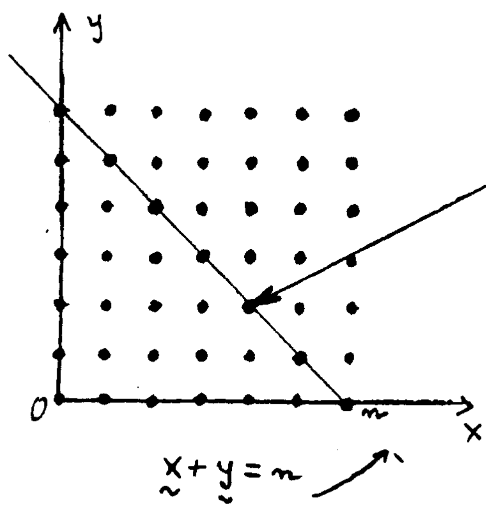
$$f_z(z) = \frac{1}{\pi}\int_{-1}^{1} \frac{f_x(z/w)}{|w|\sqrt{1-w^2}}\, dw = \frac{1}{\pi}\int_{|x|>z} \frac{f_x(x)}{\sqrt{x^2-z^2}}\, dx$$

6-39 $\quad z = x + s \qquad s = a\cos y$

$$f_z(z) = f_x(z) * f_s(z) \qquad f_s(s) = \begin{cases} \dfrac{1}{\pi\sqrt{a^2-s^2}} & |s| < a \\ 0 & |s| > a \end{cases}$$

$$f_z(z) = \frac{1}{\pi\sigma\sqrt{2\pi}}\int_{-a}^{a} \frac{e^{-(z-s)^2/2\sigma^2}}{\sqrt{a^2-s^2}}\, ds = \frac{1}{\pi\sigma\sqrt{2\pi}}\int_{-\pi}^{\pi} e^{-(z-a\cos y)^2/2\sigma^2}\, dy$$

6-40



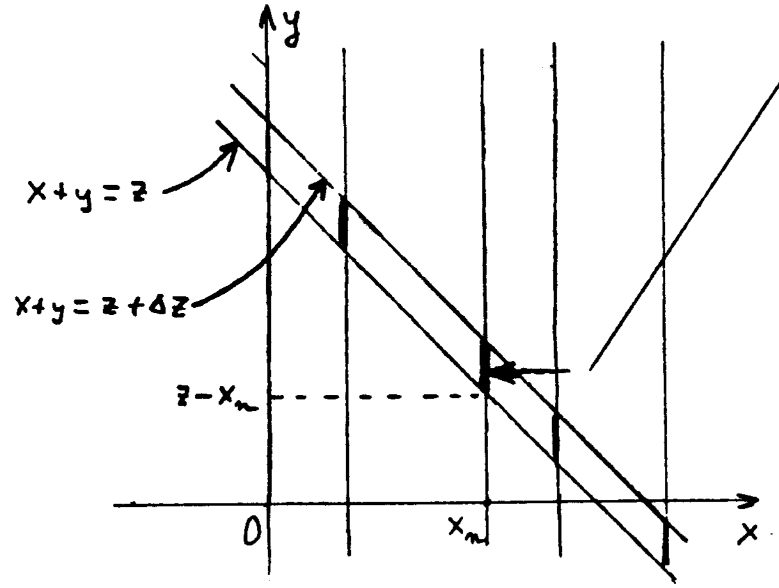
Point masses

$$P\{x = k,\ y = n-k\} = a_k b_{n-k}$$

$$\{z = n\} = \sum_{k=0}^{n} \{x = k,\ y = n-k\}$$

$$P\{z = n\} = \sum_{k=0}^{n} P\{x = k,\ y = n-k\}$$

6-41 (a) **Line masses**



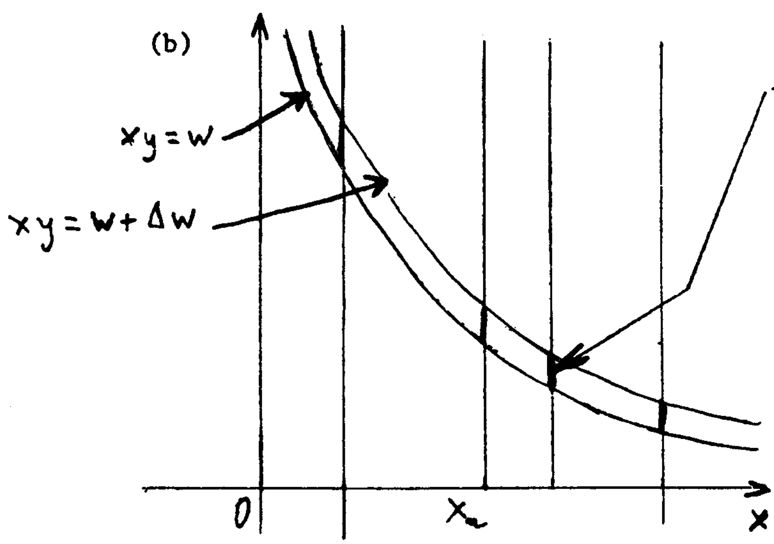
$P\{\underset{\sim}{x} = x_n,\ z - x_n < \underset{\sim}{y} \le z - x_n + \Delta z\}$

$= p_n f_y(z - x_n)\Delta z$

$$\{z < \underset{\sim}{z} \le z + \Delta z\} = \sum_n \{\underset{\sim}{x} = x_n,\ z - x_n < \underset{\sim}{y} \le z - x_n + \Delta z\}$$

$$f_z(z)\Delta z = \sum_n p_n f_y(z - x_n)\Delta z$$

(b)



$P\{\underset{\sim}{x} = x_n,\ \frac{w}{x_n} < \underset{\sim}{y} \le \frac{w+\Delta w}{x_n}\}$

$= p_n f_y\left(\frac{w}{x_n}\right)\frac{\Delta w}{x_n}$

$$\{w < \underset{\sim}{w} \le w + \Delta w\} = \sum_n \{\underset{\sim}{x} = x_n,\ \frac{w}{x_n} < \underset{\sim}{y} \le \frac{w+\Delta w}{x_n}\}$$

$$f_w(w)\Delta w = \sum_n p_n f_y\left(\frac{w}{x_n}\right)\Delta w$$

6.42 X, Y are independent geometric random variables. Thus

$$\begin{aligned} P\{X=k, Y=m\} &= P\{X=k\}\, P\{Y=m\} \\ &= (pq^k)\,(pq^m) = p^2 q^{k+m}, \qquad k, m = 0, 1, 2, \cdots \end{aligned}$$

(a) Let

$$Z = X + Y$$

$$\begin{aligned} P\{Z=n\} &= P\{X+Y=n\} = \textstyle\sum_k P\{X=k, Y=n-k\} \\ &= \textstyle\sum_{k=0}^{n} P\{X=k, Y=n-k\} \\ &= \textstyle\sum_{k=0}^{n} P\{X=k\}\, P\{Y=n-k\} \\ &= \textstyle\sum_{k=0}^{n} pq^k\, pq^{n-k} = \sum_{k=0}^{n} p^2\, q^n \\ &= (n+1)\, p^2\, q^n, \quad n = 0, 1, 2, \cdots \end{aligned}$$

(b) Let

$$W = X - Y$$

Case 1: $\quad W \geq 0 \Rightarrow X \geq Y$. Thus for $m \geq 0$

$$\begin{aligned} P\{W=m\} &= P\{X-Y=m\} = \textstyle\sum_{k=0}^{\infty} P\{X=m+k, Y=k\} \\ &= \textstyle\sum_{k=0}^{\infty} P\{X=m+k, Y=k\} \\ &= \textstyle\sum_{k=0}^{\infty} P\{X=m+k\}\, P\{Y=k\} \\ &= \textstyle\sum_{k=0}^{\infty} (pq^{m+k})\,(pq^k) = p^2 q^m \sum_{k=0}^{\infty} q^{2k} \\ &= p^2 q^m\,(1 + q^2 + q^4 + \cdots) = \frac{p^2 q^m}{(1-q^2)} \\ &= \frac{pq^m}{1+q}, \qquad m = 0, 1, 2, \cdots \qquad (1) \end{aligned}$$

Case 2: $\quad W < 0 \Rightarrow X < Y$. Thus for $m < 0$

$$\begin{aligned} P\{W=m\} &= P\{X-Y=m\} = \textstyle\sum_k P\{X=k, Y=k-m\} \\ &= \textstyle\sum_{k=0}^{\infty} P\{X=k, Y=k-m\} \\ &= \textstyle\sum_{k=0}^{\infty} P\{X=k\}\, P\{Y=k-m\} \\ &= \textstyle\sum_{k=0}^{\infty} (pq^k)\,(pq^{k-m}) = p^2 q^{-m} \sum_{k=0}^{\infty} q^{2k} \\ &= \frac{p^2 q^{-m}}{(1-q^2)} = \frac{pq^{-m}}{1+q}, \quad m = -1, -2, \cdots \qquad (2) \end{aligned}$$

Thus combining (1) and (2) we can write

$$P\{W=m\} = \frac{pq^{|m|}}{1+q}, \qquad m = 0, \pm 1, \pm 2, \cdots$$

6.43 We have X and Y are independent and $P(X=k)=P(Y=k)=p_k$. Also

$$\begin{aligned} P(X=k|X+Y=k) &= \frac{P(X=k,Y=0)}{P(X+Y=k)} \\ &= \frac{p_k p_0}{\sum\limits_{i=0}^{k} p_i p_{k-i}} = \frac{1}{k+1}. \end{aligned} \tag{1}$$

Also

$$P(X=k-1|X+Y=k)$$

$$= \frac{P(X=k-1,Y=1)}{P(X+Y=k)} \frac{p_{k-1}p_1}{\sum_{i=0}^{k} p_i p_{k-i}} = \frac{1}{k+1}. \tag{2}$$

From (1) and (2),

$$\frac{p_k}{p_{k-1}} = \frac{p_1}{p_0} \Rightarrow p_k = \lambda p_{k-1} = \lambda^k p_0$$

where $\lambda \triangleq p_1/p_0$. Since $\sum_{k=0}^{\infty} p_k = 1$, we must have $\lambda < 1$, and this gives

$$\sum_{k=0}^{\infty} p_k = \frac{p_0}{1-\lambda} = 1 \rightarrow p_0 = 1-\lambda.$$

Thus

$$p_k = p_0\,\lambda^k = (1-\lambda)\lambda^k, \quad k=0,1,2,\cdots,\; 0<\lambda<1$$

represents a geometric distribution. Thus X and Y are geometric random variables.

6.44 The moment generating functions of X and Y are given by (see (5-117))

$$\Gamma_X(z) = (pz+q)^n, \qquad \Gamma_Y(z) = (pz+q)^n$$

Also

$$\Gamma_{X+Y}(z) = E[z^{X+Y}] = \Gamma_X(z)\Gamma_Y(z) = (pz+q)^{2n} \sim \text{Binomial}\,(2n,p)$$

6.45 (a) Let

$$Z = \min(X,Y), \qquad W = X - Y$$

$$\begin{aligned}
&P\{Z=k, W=m\} \\
&= P\{\min(X,Y)=k, X-Y=m\} \\
&= P\{(\min(X,Y)=k, X-Y=m) \bigcap (X \geq Y \bigcup X < Y)\} \\
&= P\{Y=k, X-Y=m, X \geq Y\} + P\{X=k, X-Y=m, X<Y\} \\
&= P\{X=m+k, Y=k, X \geq Y\} + P\{X=k, Y=k-m, X<Y\}
\end{aligned}$$

Note that $k \geq 0$, and m takes both positive, zero and negative values. Hence

$$\begin{aligned}
P\{Z=k, W=m\} &= \begin{cases} P\{X=k+m, Y=k, X \geq Y\}, & k \geq 0,\ m \geq 0 \\ P\{X=k, Y=k-m, X<Y\}, & k \geq 0,\ m < 0 \end{cases} \\
&= \begin{cases} pq^{k+m}\, pq^{k}, & k \geq 0,\ m \geq 0 \\ pq^{k}\, pq^{k-m}, & k \geq 0,\ m < 0 \end{cases}
\end{aligned}$$

$$P\{Z=k, W=m\} = p^2 q^{2k+|m|}, \qquad k = 0,1,2,\cdots, \quad m = 0, \pm 1, \pm 2, \cdots$$

Also

$$\begin{aligned}
P\{Z=k\} &= \textstyle\sum_{m=-\infty}^{\infty} P\{Z=k, W=m\} \\
&= p^2 q^{2k} \textstyle\sum_{m=-\infty}^{\infty} q^{|m|} = p^2 q^{2k}\left(1 + 2\sum_{m=1}^{\infty} q^m\right) \\
&= p^2 q^{2k}\left(1 + \frac{2q}{p}\right) = p(1+q)q^{2k}, \qquad k = 0,1,2,\cdots
\end{aligned}$$

and

$$\begin{aligned}
P\{W=m\} &= \textstyle\sum_{k=0}^{\infty} P\{Z=k, W=m\} \\
&= p^2 q^{|m|} \textstyle\sum_{k=0}^{\infty} q^{2k} \\
&= \frac{p}{1+q} q^{|m|}, \qquad m = 0, \pm 1, \pm 2, \cdots
\end{aligned}$$

Note that

$$P\{Z = k, W = m\} = P\{Z = k\}\, P\{W = m\}$$

and hence Z and W are independent random variables.
(b) Let

$$Z = \min(X, Y), \quad W = \max(X, Y) - \min(X, Y)$$

Proceeding as in (a), we obtain

$$\begin{aligned}
&P\{Z = k, W = m\} \\
&= P(Y = k, X - Y = m, X \geq Y) + P(X = k, Y - X = m, X < Y) \\
&= P(X = k + m, Y = k, X \geq Y) + P(X = k, Y = k + m, X < Y) \\
&= \begin{cases} pq^{k+m}\, pq^k + pq^k\, pq^{k+m}, & k = 0, 1, 2, \cdots, \ m = 1, 2, \cdots \\ pq^{k+m}\, pq^k, & k = 0, 1, 2, \cdots, \ m = 0 \end{cases} \\
&= \begin{cases} 2p^2 q^{2k+m}, & k = 0, 1, 2, \cdots, \ m = 1, 2, \cdots \\ p^2 q^{2k}, & k = 0, 1, 2, \cdots, \ m = 0 \end{cases}
\end{aligned}$$

This gives

$$\begin{aligned}
P\{Z = k\} &= \sum_{m=0}^{\infty} P\{Z = k, W = m\} \\
&= p^2 q^{2k} \left(1 + 2\sum_{m=1}^{\infty} q^m\right) = p^2 q^{2k} \left(1 + \frac{2q}{p}\right) \\
&= p(1+q) q^{2k}, \quad k = 0, 1, 2, \cdots
\end{aligned}$$

Also

$$\begin{aligned}
P\{W = m\} &= \sum_{k=0}^{\infty} P\{Z = k, W = m\} \\
&= \begin{cases} \dfrac{p}{1+q}, & m = 0 \\ \dfrac{2p}{1+q}\, q^m, & m = 1, 2, \cdots \end{cases}
\end{aligned}$$

Notice that

$$P\{Z = k, W = m\} = P\{Z = k\}\, P\{W = m\}$$

and hence Z and W are also indepedndent random variables in this case also.

6.46 The moment generating function of X and Y are given by (see (5-119))

$$\Gamma_X(z) = e^{\lambda_1(z-1)}, \qquad \Gamma_Y(z) = e^{\lambda_2(z-1)}$$

Also

$$\Gamma_{X+Y}(z) = \Gamma_X(z)\Gamma_Y(z) = e^{(\lambda_1+\lambda_2)(z-1)}$$

so that

$$Z \sim P(\lambda_1 + \lambda_2)$$

Thus

$$P(X+Y=k) = e^{-(\lambda_1+\lambda_2)}\,\frac{(\lambda_1+\lambda_2)^k}{k!}$$

and

$$\begin{aligned}
&P(X=k|X+Y=n)\\
&\quad = \frac{P(X=k, X+Y=n)}{P(X+Y=n)} = \frac{P(X=k)P(Y=n-k)}{P(X+Y=n)}\\
&\quad = \frac{e^{-\lambda_1}(\lambda_1^k/k!)\,e^{-\lambda_2}(\lambda_2^{n-k}/(n-k)!)}{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n/n!}\\
&\quad = \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}, \quad k = 0, 1, 2, \cdots n\\
&\quad \sim \text{Binomial}(n,p), \;\text{ where } p = \frac{\lambda_1}{\lambda_1+\lambda_2}.
\end{aligned}$$

See also (6-222). From there the converse is also true (proceed as in Example 6-43).

6-47

$$C = \begin{bmatrix} \sigma_1^2 & r\sigma_1\sigma_2 \\ r\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \qquad \Delta = \sigma_1^2\sigma_2^2(1-r^2)$$

$$C^{-1} = \begin{bmatrix} \dfrac{1}{(1-r^2)\sigma_1^2} & \dfrac{r}{(1-r^2)\sigma_1\sigma_2} \\[2ex] \dfrac{r}{(1-r^2)\sigma_1\sigma_2} & \dfrac{1}{(1-r^2)\sigma^2} \end{bmatrix}$$

$$XC^{-1}X^t = \frac{1}{(1-r^2)}\left(\frac{x_1^2}{\sigma_1^2} - 2r\,\frac{x_1x_2}{\sigma_1\sigma_2} + \frac{x_2^2}{\sigma_2^2}\right)$$

6-48

$$\{\mathbf{x}\,\mathbf{y} < 0\} = \{\mathbf{x} < 0,\ \mathbf{y} > 0\} + \{\mathbf{x} > 0,\ \mathbf{y} < 0\}$$

$$P\{\mathbf{x}\,\mathbf{y} < 0\} = F_x(0)[1-F_y(0)] + [1-F_x(0)]F_y(0)$$

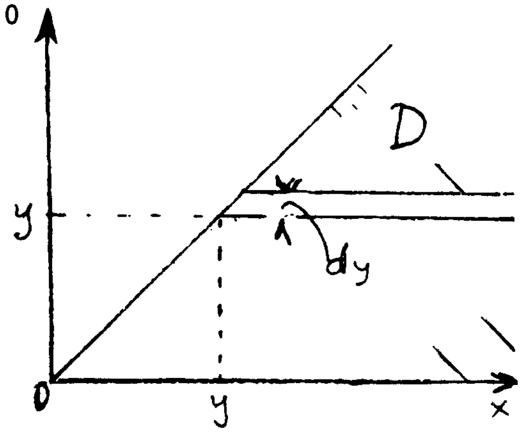
$$F_x(0) = 1 - G\left(\frac{\eta_x}{\sigma_x}\right) \qquad F_y(0) = 1 - G\left(\frac{\eta_y}{\sigma_y}\right)$$

6-49 If $w = x-y$, then $E\{w\} = 0$ $\qquad \sigma_w^2 = \sigma_x^2 + \sigma_y^2 = 2\sigma^2$

Thus, $w = 1$, $N(0; \sigma\sqrt{2})$ and [see (5-74)]

$$E\{z\} = E\{|w|\} = \sqrt{2}\,\sigma\sqrt{\frac{2}{\pi}} \qquad E\{z^2\} = E\{w^2\} = 2\sigma^2$$

6-50



$$E\{z\} = \iint_D (x-y)f(x,y)\,dxdy$$

$$= \int_0^\infty\int_y^\infty (x-y)e^{-x}\,e^{-y}\,dxdy = \frac{1}{2}$$

6-51 Since $|E\{x\,y\}| \le E\{|x||y|\}$, we can assume that the RVs x and y are real

(a) $D \le E\{[z\,x - y]^2\} = z^2\,E\{x^2\} - 2z\,E\{x\,y\} + E\{y^2\}$

The above is a non-negative quadratic in z for any z. Hence, its discriminant is non-positive.

(b) Using (a), we obtain

$$E\{x^2\} + E\{y^2\} + 2\sqrt{E\{x^2\}E\{y^2\}}$$

$$\ge E\{x^2\} + E\{y^2\} + 2\,E\{x\,y\} = E\{(x + y)^2\}$$

6-52 If $r_{xy} = 1$ then

$$E^2\{(x - \eta_x)(y - \eta_y)\} = E\{(x - \eta_x)^2\}E\{(y - \eta_y)^2\}$$

i.e., the discriminant of the quadratic

$$E\{[z(x - \eta_x) - (y - \eta_y)]^2\}$$

is zero. This is possible only if the quadratic is zero for some $z = z_o$. This shows that $z(x - \eta_x) - (y - \eta_y) = 0$ in the MS sense.

6-53 If $E\{\underset{\sim}{x}\} = E\{\underset{\sim}{y}^2\} = E\{\underset{\sim}{x}\,\underset{\sim}{y}\}$, then

$$E\{(\underset{\sim}{x} - \underset{\sim}{y})^2\} = E\{\underset{\sim}{x}^2\} + E\{\underset{\sim}{y}^2\} - 2\,E\{\underset{\sim}{x}\,\underset{\sim}{y}\} = 0.$$

Hence, $\underset{\sim}{x} = \underset{\sim}{y}$ in the MS sense.

6-54 If $\underset{\sim}{x}$ has a Cauchy density, then (Prob. 5-31)

$$E\{e^{j\omega\underset{\sim}{x}}\} = e^{-\alpha|\omega|} \qquad E\{e^{j\omega k\underset{\sim}{x}}\} = e^{-\alpha k|\omega|}$$

Hence, [see (6-240)]

$$\Phi_z(\omega) = E\{e^{j\omega\underset{\sim}{n}\underset{\sim}{x}}\} = E\{E\{e^{j\omega\underset{\sim}{n}\underset{\sim}{x}}|\underset{\sim}{n}\}\} =$$

$$\sum_{k=0}^{\infty} E\{e^{j\omega k\underset{\sim}{x}}\}e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty} e^{-\alpha k|\omega|}\frac{k}{k!} = e^{-\lambda}e^{-\lambda e^{-\alpha|\omega|}}$$

6.55 If $X = k$, then

$$Y = n - k$$

and

$$Z = X - Y = 2X - n,$$

where Z takes the values $-n, -(n-2), \cdots n-2, n$.

$$\begin{aligned} P\{Z = z\} &= P\{2X - n = z\}P\{X = \tfrac{n+z}{2}\} \\ &= \binom{n}{n + z/2} p^{(n+z)/2}\, q^{(n-z)/2}. \end{aligned}$$

Also

$$E(Z) = E[2X - n] = 2np - n = n(2p - 1).$$

$$\mathrm{Var}(Z) = E[(z - \mu_z)^2] = 4E[(X - np)^2] = 4\mathrm{Var}(X) = 4npq$$

6.56 (a)

$$\begin{aligned}\phi_Z(\omega) &= E[e^{j\omega Z}] = E[e^{j\omega(aX+bY+c)}] \\ &= \phi_X(a\omega)\,\phi_Y(b\omega)e^{j\omega c} = e^{j\omega c-(a^2\sigma_1^2+b^2\sigma_2^2)\omega^2/2}\end{aligned}$$

(see (5-100)).
(b) On comparing with (5-100) we obtain

$$Z \sim \mathrm{N}(c, a^2\,\sigma_1^2 + b^2\,\sigma_2^2)$$

(c)

$$E[Z] = c, \qquad \mathrm{Var}(Z) = a^2\,\sigma_1^2 + b^2\,\sigma_2^2$$

6.57

$$P(X = k|Y = n) = \binom{n}{k} p_1^k\,q_1^{n-k}, \quad k = 0, 1, 2, \cdots n$$

$$E[e^{j\omega X}|Y = n] = \sum_{k=0}^{n} e^{j\omega k}\,P(X = k|Y = n) = \left(p_1 e^{j\omega} + q_1\right)^n$$

use (5-117). Also

$$\begin{aligned}\phi_X(\omega) &= E[e^{j\omega X}] = E\left\{E[e^{j\omega X}|Y = n]\right\} \\ &= \sum\nolimits_{n=0}^{M} E[e^{j\omega X}|Y = n]\,P(Y = n) \\ &= \sum\nolimits_{n=0}^{\infty} (p_1 e^{j\omega} + q_1)^n \binom{M}{n} p_2^n\,q_2^{M-n} \\ &= \sum\nolimits_{n=0}^{M} \binom{M}{n} [p_2(p_1\,e^{j\omega} + q_1)]^n\,q_2^{M-n} \\ &= (p_2p_1e^{j\omega} + q_1p_2 + q_2)^M\end{aligned}$$

But

$$1 - p_1p_2 = 1 - (1 - q_1)(1 - q_2) = q_1p_2 + q_2$$

Hence

$$\phi_X(\omega) = \left(pe^{j\omega} + q\right)^M$$

where $p = p_1p_2$. Thus

$$X \sim \text{Binomial}(M, p_1p_2).$$

6.58

$$\int\int f_{XY}(x,y)\,dx\,dy = \int_0^1 \int_x^1 kx\,dy\,dx = k\int_0^1 x(1-x)\,dx$$

$$\frac{k}{6} = 1 \quad \Rightarrow \quad k = 6.$$

$$f_X(x) = \int_x^1 6x\,dy = 6x(1-x), \quad 0 < x < 1.$$

$$f_Y(y) = \int_0^y 6x\,dy = 3y^2, \quad 0 < y < 1.$$

$$E[X] = \int_0^1 x\,f_X(x)\,dx = 6\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Bigg|_0^1 = \frac{1}{2}.$$

$$E[X^2] = \int_0^1 x^2\,f_X(x)\,dx = 6\left(\frac{x^4}{4} - \frac{x^5}{5}\right)\Bigg|_0^1 = \frac{3}{10}.$$

$$\text{Var}(X) = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}.$$

$$E[Y] = \int_0^1 y\,f_Y(y)\,dy = 3\left.\frac{y^4}{4}\right|_0^1 = \frac{3}{4}.$$

$$E[Y^2] = \int_0^1 y^2\,f_Y(y)\,dy = 3\left.\frac{y^5}{5}\right|_0^1 = \frac{3}{5}.$$

$$\text{Var}(Y) = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}.$$

$$\begin{aligned} E[XY] &= \int\int xy\,f_{XY}(x,y)\,dy\,dx \\ &= \int_0^1\int_x^1 xy\,6x\,dy\,dx = \int_0^1 3x^2\,(1-x^2)\,dx \\ &= 3\left(\tfrac{x^3}{3} - \tfrac{x^5}{5}\right)\Big|_0^1 = 3\left(\tfrac{1}{3} - \tfrac{1}{5}\right) = \tfrac{2}{5} \end{aligned}$$

$$\begin{aligned} Cov(X,Y) &= E(XY) - E(X)E(Y) \\ &= \tfrac{2}{5} - \tfrac{1}{2}\tfrac{3}{4} = \tfrac{1}{40} \end{aligned}$$

6.59 (a)

$$\begin{aligned} \phi_{X,Y}(\omega_1,\omega_2) &= E[e^{j(\omega_1 X + \omega_2 Y)}] \\ &= E[e^{j\omega_1 X}]\;E[e^{j\omega_2 Y}] = \phi_X(\omega_1)\,\phi_Y(\omega_2) \\ &= e^{\lambda(e^{j\omega_1}-1)}\;e^{(j\mu\omega_2 - \sigma^2\omega_2^2/2)} \end{aligned}$$

(b)

$$\begin{aligned} \phi_Z(\omega) &= E[e^{j\omega Z}] \\ &= E[e^{j\omega(X+Y)}] = \phi_{X,Y}(\omega,\omega) \\ &= e^{\{\lambda(e^{j\omega}-1)+(j\mu\omega - \sigma^2\omega^2/2)\}} \end{aligned}$$

6.60 (a)

$$Z = \min(X, Y)$$

From Example 6-18, we have

$$f_Z(z) = 2\lambda\, e^{-2\lambda z}, \qquad z \geq 0$$

and hence

$$E[Z] = E[\min(X, Y)] = \frac{1}{2\lambda}$$

(b)

$$\begin{aligned}
E[\max(2X, Y)] &= \int\int \max(2x, y)\, f_{XY}(x, y)\, dx\, dy \\
&= \int\int_{2x \geq y} 2x\, f_{XY}(x, y)\, dx\, dy + \int\int_{2x < y} y\, f_{XY}(x, y)\, dx\, dy \\
&= \int_0^\infty \int_0^{2x} 2x\, \lambda^2\, e^{-\lambda x}\, e^{-\lambda y}\; dy\, dx + \int_0^\infty \int_0^{y/2} y\, \lambda^2\, e^{-\lambda x}\, e^{-\lambda y}\; dx\, dy \\
&= \lambda \int_0^\infty 2x\, e^{-\lambda x}(1 - e^{-2\lambda x})\, dx + \lambda \int_0^\infty y\, e^{-\lambda y}(1 - e^{-\lambda y/2})\, dy \\
&= 2\lambda \int_0^\infty \left(x e^{-\lambda x} + 2x e^{-2\lambda x} - 3x e^{-3\lambda x}\right) dx \\
&= \frac{2}{\lambda} \int_0^\infty \left(u e^{-u} + 2u e^{-2u} - 3u e^{-3u}\right) du \\
&= \frac{2}{\lambda}\left(1 + \frac{2}{4} - \frac{3}{9}\right) = \frac{7}{3\lambda}.
\end{aligned}$$

6.61 (a)

$$Z = X - Y \quad \rightarrow \quad -1 < z < 1.$$

$z > 0$

$$\begin{aligned}
F_Z(z) &= P(X - Y \leq z) = 1 - P(X - Y > z) \\
&= 1 - \int_0^{(1-z)/2} \int_{y+z}^{1-y} f_{XY}(x, y)\, dx\, dy \\
&= 1 - \int_0^{(1-z)/2} \left(\int_{y+z}^{1-y} 6x dx\right) dy \\
&= 1 - 3\int_0^{(1-z)/2} \{(1 - z^2) - 2(1 + z)y\}\, dy \\
&= 1 - \frac{3}{4}(1 + z)(1 - z)^2, \quad z \geq 0.
\end{aligned}$$

$z < 0$

$$\begin{aligned}
F_Z(z) &= P(X - Y \leq z) \\
&= \int_0^{(1+z)/2} \int_{x-z}^{1-x} 6x\, dy\, dx = \int_0^{(1+z)/2} 6x\,(1 + z - 2x)\, dx \\
&= \frac{(1 + z)^3}{4}, \quad z < 0.
\end{aligned}$$

This gives

$$f_Z(z) = \begin{cases} \frac{3}{4}(1-z)(1+3z), & 0 < z < 1 \\ \dfrac{3(1+z)^2}{4}, & -1 < z < 0 \end{cases}$$

(b)

$$f_X(x) = \int_0^{1-x} 6x\,dy = 6x(1-x), \quad 0 < x < 1$$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{F_X(x)} = \frac{1}{1-x}, \quad 0 < y \le 1-x$$

(c)

$$W = X + Y$$

we have

$$F_W(w) = P(X+Y \le w) = \int_0^w \left(\int_0^{w-x} 6x\,dy \right) dx = w^3,$$

and

$$f_W(w) = \int_0^w 6x dx = 3w^2, \quad 0 < w < 1$$

$$E[W] = \frac{3}{4}$$

$$E[W^2] = \frac{3}{5}$$

$$\mathrm{Var}(X+Y) = \mathrm{Var}(W) = E(W^2) - (E(W))^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}.$$

6.62

$$X = \frac{1}{Z}.$$

where Z represents a Chi-square random variable. Thus (see (4-39))

$$f_Z(z) = \frac{z^{-1/2}}{\sqrt{2}\Gamma(1/2)}\, e^{-z/2} = \frac{z^{-1/2}}{\sqrt{2\pi}}\, e^{-z/2}$$

or

$$f_X(x) = \frac{1}{\left|\frac{dx}{dz}\right|} f_Z(1/x) = \frac{1}{x^2} \frac{x^{1/2}}{\sqrt{2\pi}} e^{-1/2x} = \frac{1}{\sqrt{2\pi} x^{3/2}}\, e^{-1/2x}, \quad x > 0$$

Also it is given that

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi x}}\, e^{-y^2/2x}$$

so that

$$f_{XY}(x,y) = f_{Y|X}(y|x)\, f_X(x) = \frac{1}{2\pi x^2}\, e^{-(1+y^2)/2x}$$

and hence

$$\begin{aligned} f_Y(y) &= \int_0^\infty f_{XY}(x,y)\,dx \\ &= \frac{1}{2\pi} \int_0^\infty \frac{1}{x^2}\, e^{-(1+y^2)/2x}\, dx \\ &= \frac{1}{2\pi} \frac{2}{1+y^2} \int_0^\infty e^{-u}\, du = \frac{1/\pi}{1+y^2}, \quad -\infty < y < \infty. \end{aligned}$$

Thus Y represents a Cauchy random variable.

6.63 (a) For any two random variables X and Y we have

$$\begin{aligned}\sigma_{X+Y}^2 &= \text{Var}(X+Y) = E[\{(X-\mu_X)+(Y-\mu_Y)\}^2] \\ &= \text{Var}(X)+\text{Var}(Y)+2\text{Cov}(X,Y) = \sigma_X^2+\sigma_Y^2+2\sigma_X\sigma_Y\rho_{XY} \\ &\leq (\sigma_X+\sigma_Y)^2\end{aligned}$$

since $|\rho_{XY}| \leq 1$. Thus

$$\sigma_{X+Y} \leq \sigma_X + \sigma_Y,$$

and hence it easily follows that

$$\frac{\sigma_{X+Y}}{\sigma_X+\sigma_Y} \leq 1.$$

(However, (b) is not so easy!)
(b) We shall prove this result in three parts by making use of Holder's inequality.
(i) **Holder's inequality:** The function $\log x$ is concave, for $0 < \alpha < 1$, and hence we have

$$\log[\alpha x_1 + (1-\alpha)x_2] \geq \alpha \log x_1 + (1-\alpha)\log x_2$$

or

$$x_1^{\alpha}\, x_2^{1-\alpha} \leq \alpha x_1 + (1-\alpha)x_2, \quad 0 < \alpha < 1. \qquad (6.63-1)$$

Let

$$x_1 = |x|^p, \quad \alpha = \frac{1}{p}, \text{ so that } 1-\alpha = 1-\frac{1}{p} \triangleq \frac{1}{q}, \quad x_2 = |y|^q \quad (6.63-2)$$

so that (6.63-1) becomes

$$|xy| \leq \frac{|x|^p}{p} + \frac{|y|^q}{q}, \quad p > 1, \qquad (6.63-3)$$

the Holder's inequality. From (6.63-2), note that

$$\frac{1}{p} + \frac{1}{q} = 1, \quad p > 1, \quad q > 1 \qquad (6.63-4)$$

(ii) Define

$$x = X\left(E\{|X|^p\}\right)^{-1/p}, \quad y = Y\left(E\{|Y|^q\}\right)^{-1/q}$$

where p and q are as in (6.63-4). Substituting these into the Holder's inequality in (6.63-3), we get

$$\begin{aligned}|XY| &\leq p^{-1}\,|X|^p\,\left(E\{|X|^p\}\right)^{1/p-1}\left(E\{|Y|\}\right)^{1/q} \\ &+q^{-1}\,|Y|^q\,\left(E\{|Y|^q\}\right)^{1/q-1}\left(E\{|X|^p\}\right)^{1/p}. \qquad (6.63-5)\end{aligned}$$

Taking expected values on both sides of (6.63-5), we get

$$E\{|XY|\} \leq \left(E\{|X|^p\}\right)^{1/p}\left(E\{|Y|^q\}\right)^{1/q} \qquad (6.63-6)$$

which represents the generalization of the Cauchy-Schwarz inequality. (Note $p = q = 2$ corresponds to Cauchy-Schwarz inequality)
(iii) To prove the desired inequality, notice that

$$\begin{aligned}|X+Y|^p &= |X+Y||X+Y|^{p-1} \\ &\leq |X||X+Y|^{p-1} + |Y||X+Y|^{p-1}, \quad p > 1\end{aligned}$$

and taking expected values on both sides we get

$$E\{|X+Y|^p\} \leq E\{|X||X+Y|^{p-1}\} + E\{|Y||X+Y|^{p-1}\}. \quad (6.63-7)$$

Applying (6.63-6) to each term on the right side of (6.63-7) we get

$$E\{|X||X+Y|^{p-1}\} \leq \left(E\{|X|^p\}\right)^{1/p}\left(E\{|X+Y|^{(p-1)q}\}\right)^{1/q} \quad (6.63-8)$$

and

$$E\{|Y||X+Y|^{p-1}\} \leq \left(E\{|Y|^p\}\right)^{1/p}\left(E\{|X+Y|^{(p-1)q}\}\right)^{1/q} \quad (6.63-9)$$

Using (6.63-8) and (6.63-9) together with $(p-1)q = p$ in (6.63-7) we get

$$E\{|X+Y|^p\} \leq \left[\left(E\{|X|^p\}\right)^{1/p} + \left(E\{|Y|^p\}\right)^{1/p}\right]\cdot\left(E\{|X+Y|^p\}\right)^{1/q}$$

or for $p > 1$

$$\left(E\{|X+Y|^p\}\right)^{1/p} \leq \left(E\{|X|^p\}\right)^{1/p} + \left(E\{|Y|^p\}\right)^{1/p}.$$

the desired inequality. Since $p = 1$ follows trivially, we get

$$\frac{\left(E\{|X+Y|^p\}\right)^{1/p}}{\left(E\{|X|^p\}\right)^{1/p} + \left(E\{|Y|^p\}\right)^{1/p}} \leq 1, \quad p \geq 1.$$

6.64 (a) See Example 6-41. From there

$$E(Y|X=x) = \mu_Y + \frac{\rho_{XY}\sigma_Y(x-\mu_X)}{\sigma_X}$$

(b) Similarly

$$f_{X|Y}(X|Y=y) \sim N(\mu, \sigma^2)$$

where

$$\mu = \mu_X + \frac{\rho_{XY}\sigma_X(y-\mu_Y)}{\sigma_Y}$$

and

$$\sigma^2 = \sigma_X^2(1-\rho_{XY}^2).$$

Since

$$E(X^2|Y=y) = \mathrm{Var}(X|Y=y) + (E[X|Y=y])^2$$

we obtain

$$E(X^2|Y=y) = \sigma^2 + \mu^2$$

6.65 (a) See footnote 4, Chapter 8, Page 337. From there (or directly) we have

$$\mathrm{Var}(X|Y) \triangleq E(X^2|Y) - (E\{X|Y\})^2$$

$$\mathrm{Var}\left(E\{X|Y\}\right) \triangleq E\left[E\{X|Y\}\right]^2 - (E[E\{X|Y\}])^2$$

so that

$$\begin{aligned} E[\mathrm{Var}(X|Y)] + \mathrm{Var}\left(E\{X|Y\}\right) &= E\left[E\{X^2|Y\}\right] - (E[E\{X|Y\}])^2 \\ &= E(X^2) - [E(X)]^2 = \mathrm{Var}(X) \qquad (1) \end{aligned}$$

or

$$\mathrm{Var}(X) \geq E[\mathrm{Var}\{X|Y\}]$$

Also

$$\mathrm{Var}(X) \geq \mathrm{Var}[E\{X|Y\}]$$

(b) See (1).

6.66

$$Z = aX + (1-a)Y, \qquad 0 < a < 1$$

$$\sigma_Z^2 = \mathrm{Var}(Z) = a^2\sigma_1^2 + (1-a)^2\sigma_2^2$$

$$\frac{\partial \sigma_Z^2}{\partial a} = 2a\sigma_1^2 + 2(1-a)(-1)\sigma_2^2 = 0$$

or

$$a(\sigma_1^2 + \sigma_2^2) = \sigma_2^2$$

$$a = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} < 1$$

minimizes Var(Z).

6-67 From (6-240)

$$E\{g(\underset{\sim}{x},\underset{\sim}{y})\} = E\{E\{g(\underset{\sim}{x},\underset{\sim}{y})|\underset{\sim}{y}\}\} = E\{g(x_n,\underset{\sim}{y})P\{\underset{\sim}{x} = x_n\}\} .$$

From (4-74) **with** $A_n = \{\underset{\sim}{x} = x_n\}$

$$f_z(z) = \sum_n f_z(z|\underset{\sim}{x} = x_n)P\{\underset{\sim}{x} = x_n\}$$

6-68 (a) The conditional density f(y|x) is $N(rx;\sigma\sqrt{1-r^2})$ [see (7-42)]. Hence

$$E\{f_y(\underset{\sim}{y}|x)\} = \int_{-\infty}^{\infty} f_y(y|x)f_y(y)dy$$

$$= \frac{1}{2\pi\sigma^2\sqrt{1-r^2}}\int_{-\infty}^{\infty} \exp\left\{\frac{-(y-rx)^2}{2\sigma^2(1-r^2)}\right\}\exp\left\{\frac{-y^2}{2\sigma^2}\right\}dy = \frac{1}{\sigma\sqrt{2\pi(2-r^2)}}\exp\left\{\frac{-r^2x^2}{2\sigma^2(2-r^2)}\right\}$$

(b) From (6-241) it follows that

$$E\{f_x(\underset{\sim}{x})f_y(\underset{\sim}{y})\} = E\{f_x(\underset{\sim}{x})E\{f_y(\underset{\sim}{y}|\underset{\sim}{x})\}\} = \int_{-\infty}^{\infty} f_x(x)\,E\{f_y(y|x)\}f_x(x)dx$$

$$= \frac{1}{2\pi\sigma^3\sqrt{2\pi(2-r^2)}}\int_{-\infty}^{\infty}\exp\left\{-\frac{x^2}{\sigma^2}\right\}\exp\left\{\frac{-r^2x^2}{2\sigma^2(2-r^2)}\right\}dx = \frac{1}{2\pi\sigma^2\sqrt{4-r^2}}$$

6-69 We shall use (6-64) and Price's theorem (10-94):

$$\frac{\partial E\{|\underset{\sim}{x}\underset{\sim}{y}|\}}{\partial \mu} = E\left\{\frac{d|\underset{\sim}{x}|}{d\underset{\sim}{x}}\ \frac{d|\underset{\sim}{y}|}{\partial \underset{\sim}{y}}\right\} = E\{\text{sgn}\ \underset{\sim}{x}\ \text{sgn}\ \underset{\sim}{y}\}$$

$$= P\{\underset{\sim}{x}\underset{\sim}{y} > 0\} - P\{\underset{\sim}{x}\underset{\sim}{y} < 0\} = \frac{2\alpha}{\pi} = \frac{2}{\pi}\ \text{arc sin}\ \frac{\mu}{\sigma_1\sigma_2}$$

If $\mu = 0$, then the RVs $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are independent, hence,

$$E\{|\underset{\sim}{x}\ \underset{\sim}{y}|\}\Big|_{\mu=0} = E\{|\underset{\sim}{x}|\}E\{|\underset{\sim}{y}|\} = \frac{2}{\pi}\sigma_1\sigma_2$$

[see (5-74)]. Integrating (i) and using the above, we obtain

$$E\{|\underset{\sim}{x}\ \underset{\sim}{y}|\} = \frac{2}{\pi}\int_0^{\mu} \text{arc sin}\ \frac{C}{\sigma_1\sigma_2}\, dC + \frac{2}{\pi}\sigma_1\sigma_2 = \frac{2\sigma_1\sigma_2}{\pi}(\cos\alpha + \alpha\sin\alpha)$$

6-70 From Example 6-41

$$f(y|x) : N(\eta_2 + \frac{r\sigma_2}{\sigma_1}x\,;\ \sigma_2\sqrt{1 - r^2}) = N(4+x;\ \sqrt{3})$$

$$f(x|y) : N(\eta_1 + \frac{r\sigma_1}{\sigma_2}y\,;\ \sigma_1\sqrt{1 - r^2}) = N(3+\frac{y}{4};\ \sqrt{3}/2)$$

6-71 The mass density in the square $|\underset{\sim}{x}|\le 1$, $|\underset{\sim}{y}|\le 1$ of the xy plane equals 1/4; hence, $P\{\underset{\sim}{r}\le 1\}=\pi/4$ and $P\{\underset{\sim}{r}\le r\} = \pi r^2/4$ for $r<1$. This yields

$$P\{\underset{\sim}{r}\le r, \underset{\sim}{r}\le 1\} = \begin{cases} P\{\underset{\sim}{r}\le r\} = \pi r^2/4 & r\le 1 \\ P\{\underset{\sim}{r}\le 1\} = \pi/4 & r>1 \end{cases}$$

$$F_r(r|M) = \frac{P\{\underset{\sim}{r}\le r, M\}}{P(M)} = \begin{cases} r^2 & r\le 1 \\ 1 & r>1 \end{cases} \qquad f_r(r|m) = \begin{cases} 2r, & r<+1 \\ 0 & \text{otherwise} \end{cases}$$

6-72 $\underset{\sim}{z} = \underset{\sim}{x} + \underset{\sim}{y} \qquad \underset{\sim}{w} = \underset{\sim}{x} \qquad f_{xz}(x,z) = f_{xy}(x, z-x)$

If $f_{xy}(x,y) = f_x(x)f_y(y)$, then

$$f_z(z|x) = \frac{f_{xz}(x,z)}{f_x(x)} = f_y(z-x)$$

6-73 The system $z = F_x(x) \qquad w = F_y(y|x)$ has a solution only if $z \leq z \leq 1$ and $0 \leq w \leq 1$. Furthermore,

$$\frac{\partial z}{\partial x} = f_x(x) \qquad \frac{\partial z}{\partial y} = 0$$

$$J = f_x(x)f_y(y|x)$$

$$\frac{\partial w}{\partial x} \qquad \frac{\partial w}{\partial y} = f_y(y|x)$$

$$f_{zw}(z,w) = \frac{f_{xy}(x,y)}{f_x(x)f_y(y|x)} = 1 \quad \text{for} \quad 0 \leq z,w \leq 1$$

6-74 We introduce the events C_r = {we selected the rth coin} and A_k = {heads in a specific order}. From the assumptions it follows that

$$P(C_r) = \frac{1}{m} \qquad P(A_k|C_r) = p_r^k(1-p_r)^{n-k}$$

We wish to find the probability $P(C_r|A_k)$. The events C_r form a pertition; hence,

$$P(C_r|A_k) - \frac{\frac{1}{m}P(a_k|C_r)}{\frac{1}{m}\sum_{i-1}^{m} P(A_k|C_i)}$$

6-75 We wish to show that

$$E\{x^2\} = \frac{n}{n-1}$$

From page 207: $x^2 = ny^2/z$ where y is N(0,1) and z is $\chi^2(n)$. Hence, $E\{y^2\} = 1$ and (also (4-35) and (4-39))

$$E\left\{\frac{1}{z}\right\} = \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^{\infty} z^{n/2-2} e^{-z/2} dz = \frac{2^{m/2-1}\Gamma(n/2-1)}{2^{n/2}\Gamma(n/2)}$$

From this and the independence of y and z it follows that

$$E\{x^2\} = n\, E\{y^2\}\, E\left\{\frac{1}{z}\right\} = \frac{n}{n-2}$$

6-76 From (6-222) :

$$R_x(x) = \exp\left\{ - \int_0^x \beta_x(t)dt \right\} = \exp\left\{ -k \int_0^x \beta_y(t)dt \right\} = R_y^k(t)$$

6-77 From (5-89) it follows with $x = |z|^2$ and $\alpha = \varepsilon^2$ that

$$E\{|z|^2 > \varepsilon^2\} \le \frac{E\{|z|^2\}}{\varepsilon^2}$$

for any z. And the result follows with $z = x - y$.

6-78 $$E\{U(a-\underset{\sim}{x})\} = \int_{-\infty}^{\infty} U(a-x)f(x)dx = \int_{-\infty}^{a} f(x)dx = F_x(a)$$

$$E\{U(b-\underset{\sim}{y})\} = F_y(b)$$

$$E\{U(a-\underset{\sim}{x})U(b-\underset{\sim}{y}) = \int_{-\infty}^{a}\int_{-\infty}^{b} f(x,y)dxdy = F_{xy}(a,b)$$

Hence

$$F_{xy}(a,b) = F_x(a)F_y(b)$$

6-79 From Example 6-38

$$E\{\underset{\sim}{y}|\underset{\sim}{x} \le 0\} = \int_{-\infty}^{\infty} y\ f_y(y|\underset{\sim}{x} \le 0\}dy = \frac{1}{F_x(0)} \int_{-\infty}^{\infty} y\ \frac{\partial F(0,y)}{\partial y}\ dy$$

From (7-41) and (7-57)

$$\int_{-\infty}^{\infty} E\{\underset{\sim}{y}|x\}f_x(x)dx = \int_{-\infty}^{\infty} y \int_{-\infty}^{0} f(x,y)dxdy = \int_{-\infty}^{\infty} y\ \frac{\partial F(0,y)}{\partial y}\ dy$$

CHAPTER 7

7-1 $$0 \le P\{x_1 < \mathbf{x} \le x_2,\ y_1 < \mathbf{y} \le y_2,\ z_1 < \mathbf{z} \le z_2\} =$$

$$= P\{\mathbf{x} \le x_2,\ y_1 < \mathbf{y} \le y_2,\ z_1 < \mathbf{z} \le z_2\} - P\{\mathbf{x} \le x_1,\ y_1 < \mathbf{y} \le y_2,\ z_1 < \mathbf{z} \le z_2\} =$$

$$= P\{\mathbf{x} \le x_2,\ \mathbf{y} \le y_2,\ z_1 < \mathbf{z} \le z_2\} - P\{\mathbf{x} \le x_2,\ \mathbf{y} \le y_1,\ z_1 < \mathbf{z} \le z_2\}$$

$$- P\{\mathbf{x} \le x_1,\ \mathbf{y} \le y_2,\ z_1 < \mathbf{z} \le z_2\} + P\{\mathbf{x} \le x_1,\ \mathbf{y} \le y_1,\ z_1 < \mathbf{z} \le z_2\} =$$

$$= P\{\mathbf{x} \le x_2,\ \mathbf{y} \le y_2,\ \mathbf{z} \le z_2\} - P\{\mathbf{x} \le x_2,\ \mathbf{y} \le y_2,\ \mathbf{z} \le z_1\}$$

$$- P\{\mathbf{x} \le x_2,\ \mathbf{y} \le y_1,\ \mathbf{z} \le z_2\} + P\{\mathbf{x} \le x_2,\ \mathbf{y} \le y_1,\ \mathbf{z} \le z_1\}$$

$$- P\{\mathbf{x} \le x_1,\ \mathbf{y} \le y_2,\ \mathbf{z} \le z_2\} + P\{\mathbf{x} \le x_1,\ \mathbf{y} \le y_2,\ \mathbf{z} \le z_1\}$$

$$+ P\{\mathbf{x} \le x_1,\ \mathbf{y} \le y_1,\ \mathbf{z} \le z_2\} - P\{\mathbf{x} \le x_1,\ \mathbf{y} \le y_1,\ \mathbf{z} \le z_1\}$$

7-2 $P\{\mathbf{x}_A = 1,\ \mathbf{x}_B = 1,\ \mathbf{x}_C = 1\} = P(ABC) = 1/4$

$P\{\mathbf{x}_A = 1\} = P(A) = 1/2$ $\qquad$ $P\{\mathbf{x}_B = 1\} = P(B) = 1/2$

$P\{\mathbf{x}_C = 1\} = P(C) = 1/2$ hence

$$P\{\mathbf{x}_A = 1,\ \mathbf{x}_B = 1,\ \mathbf{x}_C = 1\} \ne P\{\mathbf{x}_A = 1\}P\{\mathbf{x}_B = 1\}P\{\mathbf{x}_C = 1\}$$

hence $\mathbf{x}_A$, $\mathbf{x}_B$, $\mathbf{x}_C$ are not independent. But

$P\{\mathbf{x}_A = 1,\ \mathbf{x}_B = 1\} = P(AB) = 1/4 = P\{\mathbf{x}_A = 1\}P\{\mathbf{x}_B = 1\}$

Similarly for any other combination, e.g.,
since $P(A) = P(AB) + P(A\bar{B})$, we conclude that
$P(A\bar{B}) = 1/2 - 1/4 = 1/4$ $\qquad$ $P(\bar{B}) = 1 - P(B) = 1/2$
$P\{\mathbf{x}_A = 1,\ \mathbf{x}_B = 0\} = P(A\bar{B}) = 1/4$
$P\{\mathbf{x}_B = 0\} = P(\bar{B}) = 1/2$ hence
$P\{\mathbf{x}_A = 1,\ \mathbf{x}_B = 0\} = P\{\mathbf{x}_A = 1\}P\{\mathbf{x}_B = 0\}$

7-3 If $\underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z}$ are independent in pairs, then

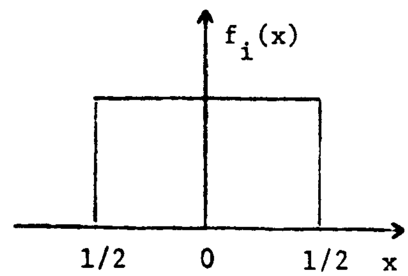
$$r_{xy} = r_{xz} = r_{yz} = 0$$

and (7-60) yields (we assume $\eta_x = \eta_y = \eta_z = 0$)

$$\Phi(\omega_1,\omega_2,\omega_3) = \exp\left\{-\frac{1}{2}\left(\sigma_1^2\omega_1^2 + \sigma_2^2\omega_2^2 + \sigma_3^2\omega_3^2\right)\right\}$$

$$f(x_1,x_2,x_3) = f(x_1)f(x_2)f(x_3)$$

7-4 $\underset{\sim}{x} = \underset{\sim}{x}_1 + \underset{\sim}{x}_2 + \underset{\sim}{x}_3$. To determine $E\{\underset{\sim}{x}^4\}$ we shall use char. functions



$$\Phi_i(\omega) = \int_{-1/2}^{1/2} e^{j\omega x}dx = \frac{2\sin(\omega/2)}{\omega}$$

$$\Phi(\omega) = \left[\frac{2\sin(\omega/2)}{\omega}\right]^3 = \left(1 - \frac{\omega^2}{24} + \frac{\omega^4}{1920} - \cdots\right)^3$$

The coefficient of ω^4 in this expansion equals

$$\frac{13}{1920} \text{ hence } \frac{1}{4!}\ \frac{d^4\Phi(0)}{d\omega^4} = \frac{13}{1920}$$

and [see (5-103)]

$$E\{\underset{\sim}{x}^4\} = m_4 = \frac{13\times 4!}{1920} = \frac{13}{80}$$

7-5 (a) The joint density $f(x,y)$ has circular symmetry because

$$f(x,y) = \int_{-\infty}^{\infty} f(\sqrt{x^2 + y^2 + z^2})\, dz$$

depends only on $x^2 + y^2$. The same holds for $f(x,z)$ and $f(y,z)$. And since the RVs $\underset{\sim}{x}, \underset{\sim}{y}$, and $\underset{\sim}{z}$ are independent, they must be normal [see (6-29)].

(b) From (a) it follows that the RVs $\underset{\sim}{v}_x, \underset{\sim}{v}_y, \underset{\sim}{v}_z$ are $N(0; \sqrt{kT/m})$. With $\sigma^2 = kT/m$ and $n = 3$ it follows from (7-62)-(7-63) and (5-25) that

$$f_v(v) = \sqrt{\frac{2m^3}{\pi k^3 T^3}}\; v^2\, e^{-mv^2/2kT}\, U(v)$$

$$E\{\underset{\sim}{v}\} = 2\sqrt{\frac{2kT}{\pi m}} \qquad E\{\underset{\sim}{v}^{2n}\} = 1 \times 3 \cdots (2n+1)\left(\frac{kT}{m}\right)^n$$

7-6 From Prob. 6-52: $\underset{\sim}{y} = a\underset{\sim}{x} + b$, $\underset{\sim}{z} = c\underset{\sim}{y} + d$, hence,

$$\underset{\sim}{z} = A\underset{\sim}{x} + B \qquad \eta_z = A\eta_x + B \qquad \sigma_z = A\sigma_x$$

$$E\{(\underset{\sim}{z} - \eta_z)(\underset{\sim}{x} - \eta_x\} = E\{A(\underset{\sim}{x} - \eta_x)(\underset{\sim}{x} - \eta_x)\} = A\sigma_x^2 = \sigma_x\sigma_z$$

7-7 It follows from (6-241) with $g_1(x) = x$, $g_2(y) = y$ if we replace all densities with conditional densities assuming $\underset{\sim}{x}_3$.

7-8 Reasoning as in (7-82), we conclude that

$E\{[y-(a_1x_1+a_2x_1)]^2\}$ is minimum if

$$E\{[y-(a_1x_1+a_2x_2)]x_i\}=0 \qquad i=1,2$$

With $R_{0i}=E\{y\,x_i\}$, $R_{ij}=E\{x_ix_j\}$, the above yields

$$R_{01}=a_1R_{11}+a_2R_{12} \qquad\qquad R_{02}=a_1R_{12}+a_2R_{22}$$

But $\hat{E}\{y|x_1\}=A\,x_1$ $\qquad\qquad A=R_{01}/R_{11}=a_1+a_2R_{12}/R_{11}$

$$\hat{E}\{\hat{E}\{y|x_1,x_2\}|x_1\}=\hat{E}\{a_1x_1+a_2x_2|x_1\}$$

$$=a_1x_1+a_2\hat{E}\{x_2|x_1\}=\left(a_1+a_2\frac{R_{12}}{R_{11}}\right)x_1=A\,x_1$$

7-9 As in Probl. 6-51

$E^2\{x_ix_j\}\le E^2\{x_i\}E^2\{x_j\}=M^2 \qquad\qquad |E\{x_ix_j\}|\le M$

$$E\{s^2|n=n\}=E\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}x_ix_j\right\}\le Mn^2$$

Hence [see (6-240)]

$$E\{s^2\}=E\{E\{s^2|n\}\}<E\{Mn^2\}$$

7-10 As we know,

$$1 + x + \cdots + x^n + \cdots = \frac{1}{1-x} \qquad |x| < 1$$

Differentiating, we obtain

$$1 + 2x + \cdots + n\,x^{n-1} + \cdots = \sum_{k=1}^{\infty} k\,x^{k-1} = \frac{1}{(1-x)^2} \tag{i}$$

The RV $\underset{\sim}{x}_1$ equals the number of tosses until heads shows for the first time, Hence, $\underset{\sim}{x}_1$ takes the values 1,2,... with $P\{\underset{\sim}{x}_1 = k\} = pq^{k-1}$. Hence, [see (3-12) and (i)]

$$E\{\underset{\sim}{x}_1\} = \sum_{k=1}^{\infty} k\, P\{\underset{\sim}{x}_1 = k\} = \sum_{k=1}^{\infty} k\,p\,q^{k-1} = \frac{p}{(1-q)^2} = \frac{1}{p}$$

Starting the count after the first head shows, we conclude that the RV $\underset{\sim}{x}_2 - \underset{\sim}{x}_1$ has the same statistics as the RV $\underset{\sim}{x}_1$. Hence,

$$E\{\underset{\sim}{x}_2 - \underset{\sim}{x}_1\} = E\{\underset{\sim}{x}_1\} \qquad E\{\underset{\sim}{x}_2\} = 2E\{\underset{\sim}{x}_1\} = \frac{2}{p}$$

Reasoning similarly, we conclude that

$E\{\underset{\sim}{x}_n - \underset{\sim}{x}_{n-1}\} = E\{\underset{\sim}{x}_1\}$. Hence (induction)

$$E\{\underset{\sim}{x}_n\} = E\{\underset{\sim}{x}_{n-1}\} + E\{\underset{\sim}{x}_1\} = \frac{n-1}{p} + \frac{1}{p} = \frac{n}{p}$$

7-11 If n accidents occur in a day, the probability that m of them will be fatal equals $\binom{n}{m} p^m q^{n-m}$ for $m \le n$ and zero for $m > n$. Hence,

$$P\{\underset{\sim}{m} = m \mid \underset{\sim}{n} = n\} = \begin{cases} 0 & m > n \\ \binom{n}{m} p^m q^{n-m} & m \le n \end{cases}$$

This yields

$$E\{e^{j\omega \underset{\sim}{m}} \mid \underset{\sim}{n} = n\} = \sum_{m=0}^{n} e^{j\omega m} \binom{n}{m} p^m q^{n-m} = (p\,e^{j\omega} + q)^n$$

But

$$P\{\underset{\sim}{n} = n\} = e^{-a} \frac{a^n}{n!} \qquad n = 0,1,\ldots$$

Hence,

$$E\{e^{j\omega \mathbf{m}}\} = E\{E\{e^{j\omega \mathbf{m}} \mid \mathbf{n}\}\} = E\{(p\, e^{j\omega} + q)^{\mathbf{n}}\}$$

$$\sum_{n=o}^{\infty} (p\, e^{j\omega} + q)^n \, e^{-a} \frac{a^n}{n!} = e^{a(p\, e^{j\omega} + q)} e^{-a} = e^{a\, p\, (e^{j\omega} - 1)}$$

This shows that the RV $\mathbf{m}$ is Poisson distributed with parameter $a\,p$ [see (5-119)].

7-12 We shall determine first the conditional distribution

$$F_s(s \mid \mathbf{n} = n) = \frac{P\{\mathbf{s} \le s,\ \mathbf{n} = n\}}{P\{\mathbf{n} = n\}}$$

The event $\{\mathbf{s} < s,\ \mathbf{n} = n\}$ consists of all outcomes such that $\mathbf{n} = n$ and $\sum_{k=1}^{n} \mathbf{x}_k \le s$. Since the RV $\mathbf{n}$ is independent of the RVs $\mathbf{x}_k$, this yields

$$F_s(s \mid \mathbf{n} = n) = P\{\sum_{k=1}^{n} \mathbf{x}_k \le s\} P\{\mathbf{n} = n\}/P\{\mathbf{n} = n\}$$

From the above and the independence of the RVs $\mathbf{x}_k$ it follows that [see (7-51)]

$$f_s(s \mid \mathbf{n} = n) = f_1(s) * f_2(s) * \cdots * f_n(s)$$

Setting $A_k = \{\mathbf{n} = k\}$ in (4-74), we obtain

$$f_s(s) = \sum_k p_k \, [f_1(s) * \cdots * f_k(s)]$$

7-13 From the independence of the RVs n and $\underset{\sim}{x}_i$ it follows that

$$E\{e^{s\underset{\sim}{y}} \mid \underset{\sim}{n} = k\} = E\{e^{s(\underset{\sim}{x}_1 + \cdots + \underset{\sim}{x}_k)}\}$$

$$= E\{e^{s\underset{\sim}{x}_1}\} \cdots E\{e^{s\underset{\sim}{x}_k}\} = \Phi_x^k(s)$$

Hence,

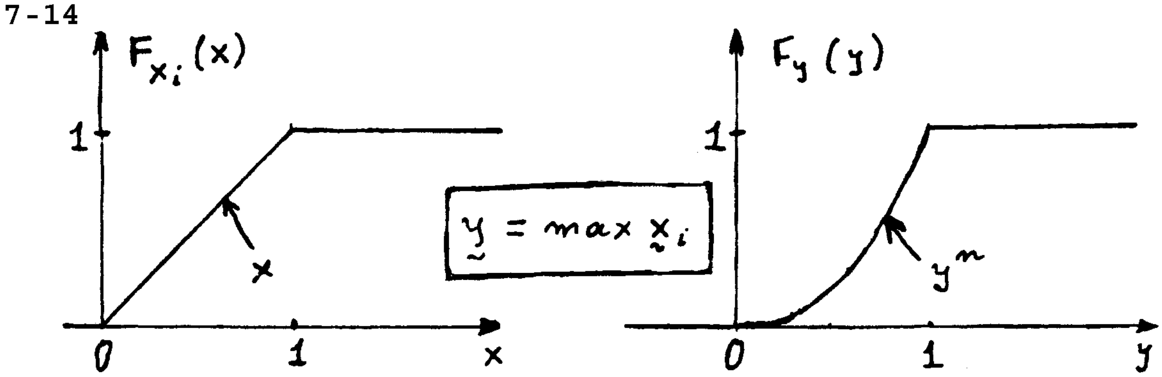
$$\Phi_y(s) = E\{e^{s\underset{\sim}{y}}\} = E\{E\{e^{s\underset{\sim}{y}} \mid \underset{\sim}{n}\}\} = E\{\Phi_x^{\underset{\sim}{n}}(s)\}$$

$$= \Gamma_n[\Phi_x(s)] \text{ because } E\{z^{\underset{\sim}{n}}\} = \Gamma_n(z)$$

<u>Special case</u>. If $\underset{\sim}{n}$ is Poisson with parameter a, then [see (5-119)]

$$\Gamma_n(z) = e^{az - a} \qquad \Phi_y(s) = e^{a\Phi_x(s) - a}$$

7-14



$$\{\underset{\sim}{y} \le y\} = \{\underset{\sim}{x}_1 \le y, \underset{\sim}{x}_2 \le y, \ldots, \underset{\sim}{x}_n \le y\}$$

From the independence of $\underset{\sim}{x}_i$ and the abcve it follows that

$$F_y(y) = P\{\underset{\sim}{y} \le y\} = P\{\underset{\sim}{x}_1 \le y\} \cdots P\{\underset{\sim}{x}_n \le y\}$$

$$= F_1(y) \cdots F_n(y)$$

where $F_i(y) = y$ for $0 \le y \le 1$.

7-15 The RV $\underset{\sim}{x}$ is defined in the space S. The set

$$C = \{z < \underset{\sim}{z} \le z + dz,\ w < \underset{\sim}{w} \le w + dz\} \qquad z > w$$

is an event in the space S_n of repeated trials and its probability equals

$$P(C) = f_{zw}(z,w)dzdw$$

We introduce the events

$$D_1 = \{\underset{\sim}{x} \le w\} \qquad D_2 = \{w < \underset{\sim}{x} \le w + dw\} \qquad D_3 = \{w + dw < \underset{\sim}{x} \le z\}$$

$$D_4 = \{z < \underset{\sim}{x} \le z + dz\} \qquad D_5 = \{z + dz < \underset{\sim}{x}\}$$

These events form a partition of S and their probabilities $p_i = P(D_i)$ equal

$$F_x(w) \qquad f_x(w)dw \qquad F_x(z)-F_x(w+dw) \qquad f_z(z)dz \qquad 1-F_x(z+dz)$$

respectively. The event C occurs iff the smallest of the RVs $\underset{\sim}{x}_i$ is in the interval (w, w+dw), the largest is in the interval (z, z+dz), and, consequently, all others are between w+dw and z. This is the case iff D_1 does not occur at all, D_2 occurs once, D_3 occurs n-2 times, D_4 occurs once, and D_5 does not occur at all. With

$$k_1=0 \qquad k_2=1 \qquad k_3=n-2 \qquad k_4=1 \qquad k_5=0$$

it follows from (4-102) that

$$P(C) = \frac{n!}{(n-2)!} p_2\, p_3^{n-2}\, p_4 = n(n-1)f_x(w)dw\,[f_x(z)-F_x(w+dw)]^{n-1}f_x(z)dz$$

for z > w, and 0 otherwise.

7-16 If $\underset{\sim}{z}$ is $N(\eta,1)$ then

$$E\{e^{s\underset{\sim}{z}^2}\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{sz}e^{-(z-\eta)^2/2}dz$$

$$sz^2 - \frac{(z-\eta)^2}{2} = \left(s - \frac{1}{2}\right)\left(z - \frac{\eta}{1-2s}\right)^2 + \frac{\eta^2 s}{1-1s}$$

Since

$$\frac{1}{\sqrt{2\pi}}\int_{-\eta}^{\infty} e^{-a(z-b)^2}dz = \frac{1}{\sqrt{2a}}$$

the above yields

$$E\{e^{s\underset{\sim}{z}^2}\} = \frac{1}{\sqrt{2(1/2-S)}}\exp\left\{\frac{\eta^2 S}{1-2S}\right\}$$

$$\Phi_w(s) = \frac{1}{\sqrt{1-2s}}\exp\left\{\frac{\eta_1 s}{1-2s}\right\}\cdots\frac{1}{\sqrt{1-2s}}\exp\left\{\frac{\eta_n s}{1-2s}\right\}$$

7-17 We wish to show that the RVs

$$\bar{\underset{\sim}{x}} = \frac{1}{n}\sum_{i=1}^{n}\underset{\sim}{x}_i \qquad \underset{\sim}{s}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(\underset{\sim}{x}_i-\bar{x})^2$$

are independent. Since $\underset{\sim}{s}^2$ is a function of the n RVs $\underset{\sim}{x}_i-\bar{\underset{\sim}{x}}$, it suffices to show that each of these RVs is independent of $\bar{\underset{\sim}{x}}$. We assume for simplicity that $E\{\underset{\sim}{x}_i\}=0$. Clearly,

$$E\{\underset{\sim}{x}_i\bar{x}\} = \frac{1}{n}E\{\underset{\sim}{x}_i^2\} = \frac{\sigma^2}{n} \qquad E\{\bar{\underset{\sim}{x}}\bar{\underset{\sim}{x}}\} = \frac{1}{n^2}\sum_{i=1}^{n}\underset{\sim}{x}_i^2 = \frac{\sigma^2}{n}$$

because $E\{\underset{\sim}{x}_i\underset{\sim}{x}_j\}=0$ for $i\neq i$. Hence,

$$E\{(\underset{\sim}{x}_i-\bar{\underset{\sim}{x}})\bar{\underset{\sim}{x}}\} = 0$$

Thus, the RVs $\underset{\sim}{x}_i-\bar{\underset{\sim}{x}}$ and $\bar{\underset{\sim}{x}}$ are orthogonal; and since they are jointly normal, they are independent.

7-18 Since $\eta_s = \alpha_0 + \alpha_1\eta_1 + \alpha_2\eta_2$ [see (7-87)], the mean of the error

$$\varepsilon = s - (\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2) = (s - \eta_s) - [\alpha_1(x_1 - \eta_1) + \alpha_2(x_2 - \eta_2)]$$

is zero. Furthermore, ε is orthogonal to x_i, hence, it is also orthogonal to $x_i - \eta_i$.

7-19 From the orthogonality principle:

$$\hat{E}\{y|x_1,x_2\} = a_1x_1 + a_2x_2 \qquad y - \{a_1x_1 + a_2x_2\} \perp x_1, x_2$$

$$\hat{E}\{y|x_1\} = A\,x_1 \qquad y - A\,x_1 \perp x_1$$

Hence

$$y - (a_1x_1 + a_2x_2) - (y - A\,x_1) = a_1x_1 + a_2x_2 - A\,x_1 \perp x_1$$

From this it follows that

$$\hat{E}\{a_1x_1 + a_2x_2|x_1\} = A\,x_1$$

$$\hat{E}\{\hat{E}\{y|x_1,x_2\}|x_1\} = \hat{E}\{y|x_1\}$$

7-20 The event $\{\mathbf{x} \le x\}$ occurs if there is at least one point in the interval $(0,x)$; the event $\{\mathbf{y} \le y\}$ occurs if all the points are in the interval $(0,y)$:

$$A_x = \{\text{at least one point in } (0,x)\} = \{\mathbf{x} \le x\}$$

$$B_y = \{\text{no points in } (y,1)\}$$

$$= \{\text{all points in } (0,y)\} = \{\mathbf{y} \le y\}$$

Hence, for $0 \le x \le 1, \quad 0 \le y \le 1$

$$F_x(x) = P(A_x) = 1 - P(\bar{A}_x) = 1 - (1 - x)^n$$

$$F_y(y) = P(B_y) = y^n$$

Furthermore,

$$\{\mathbf{x} \le x,\ \mathbf{y} \le y\} = A_x B_y \qquad\qquad A_x B_y + \bar{A}_x B_y = B_y$$

If $x \le y$ then

$$\bar{A}_x B_y = \{\text{all points in } (x,y)\}$$

$$P(\bar{A}_x B_y) = (y - x)^n$$

If $x > y$, then $\bar{A}_x B_y = \{\emptyset\}$. Hence

$$F_{xy}(x,y) = P(A_x B_y) = \begin{cases} y^n - (y-x)^n & x \le y \\ y^n & x > y \end{cases}$$

7-21 Suppose that $E\{x_i\} = 0$, $E\{x_i^2\} = \sigma^2$, $E\{x_i^4\} = \mu_4$

If $A = \sum_{i=1}^{n} x_i^2$, then $E\{A\} = n\sigma^2$

$$E\{A^2\} = \sum_{i,j=1}^{n} E\{x_i^2 x_j^2\} = n\mu_4 + (n^2 - n)\sigma^4$$

because

$$E\{x_i^2 x_j^2\} = \begin{cases} \mu_4 & i = j \\ \sigma^4 & i \neq j \end{cases}$$

Furthermore

$$E\{\bar{x}^2 x_j^2\} = \frac{1}{n^2} E\left\{\left(\sum_{i=1}^{n} x_i\right)^2 x_j^2\right\} = \frac{1}{n^2}[\mu_4 + (n-1)\sigma^4]$$

$$E\{\bar{x}^2 A\} = \frac{1}{n}[\mu_4 + (n-1)\sigma^4]$$

$$E\{\bar{x}^4\} = \frac{1}{n^4} E\left\{\left(\sum_{i=1}^{n} x_i\right)^4\right\} = \frac{1}{n^4}[n\mu_4 + 3n(n-1)\sigma^4]$$

because

$$E\{x_i x_j x_k x_r\} = \begin{cases} \mu_4 & i = j = k = r \quad \text{[n such terms]} \\ \sigma^4 & i = j \neq k = r \quad \text{[3n(n-1) such terms]} \\ 0 & \text{otherwise} \end{cases}$$

Clearly, $(n-1)\bar{V} = \sum_{i=1}^{n} (x_i - \bar{x})^2 = A - n\bar{x}^2$, $E\{\bar{V}\} = \sigma^2$. Hence

$$(n-1)^2 E\{V^2\} = E\{A^2\} - 2nE\{\bar{x}^2 A\} + n^2 E\{\bar{x}^4\}$$

$$= n\mu_4 + (n^2 - n)\sigma^4 - 2[\mu_4 + (n-1)\sigma^4] + \frac{1}{n}[\mu_4 + 3(n-1)\sigma^4]$$

This yields

$$E\{\bar{V}^2\} = \frac{\mu_4}{n} + \frac{n^2 - 2n + 3}{n(n-1)}\sigma^4 = \sigma^4 + \sigma_{\bar{V}}^2$$

<u>Note</u> If the RVs x_i are $N(0,\sigma^2)$, then $\mu_4 = 3\sigma^4$

$$\sigma_{\bar{V}}^2 = \frac{1}{n}\left(3\sigma^4 - \frac{n-3}{n-1}\sigma^4\right) = \frac{2}{n-1}\sigma^4$$

7-22 From Prob. 6-49:

$$E\{|\underset{\sim}{x}_{2i} - \underset{\sim}{x}_{2i-1}|\} = \frac{2\sigma}{\sqrt{\pi}} \qquad E\{|\underset{\sim}{x}_{2i} - \underset{\sim}{x}_{2i-1}|^2\} = 2\sigma^2$$

Hence,

$$E\{|\underset{\sim}{x}_{2i} - \underset{\sim}{x}_{2i-1}|\,|\underset{\sim}{x}_{2j} - \underset{\sim}{x}_{2j-1}|\} = \begin{cases} 2\sigma^2 & i = j \\ 4\sigma^2/\pi & i \neq j \end{cases}$$

$$E\{\underset{\sim}{z}\} = \frac{\sqrt{\pi}}{2n}\,\frac{2\sigma n}{\sqrt{\pi}} = \sigma$$

$$E\{\underset{\sim}{z}^2\} = \frac{\pi}{4n^2}\,[2n\sigma^2 + \frac{4\sigma^2}{\pi}(n^2 - n)]$$

$$\sigma_z^2 = \frac{\pi}{2n}\sigma^2 + (1 - \frac{1}{n})\sigma^2 - \sigma^2 = \frac{\pi - 2}{2n}\sigma^2$$

7-23 If $R^{-1} = \begin{bmatrix} a_{11} \cdots a_{1n} \\ \\ a_{n1} \cdots a_{nn} \end{bmatrix}$ then $\sum\limits_j a_{ij}R_{ji} = 1$

Hence,

$$E\{\underset{\sim}{X}R^{-1}\underset{\sim}{X}^t\} = E\{\sum_{i=1}^{n}\sum_{j=1}^{n} \underset{\sim}{x}_i a_{ij} \underset{\sim}{x}_j\}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}R_{ji} = \sum_{i=1}^{n} 1 = n$$

7-24 The density $f_z(z)$ of the sum $\underset{\sim}{z} = \underset{\sim}{x}_1 + \cdots + \underset{\sim}{x}_n$ tends to a normal curve with variance $\sigma_1^2 + \cdots + \sigma_n^2 \to \infty$ as $n \to \infty$ (we assume $\sigma_i > c > 0$). Hence, $f_z(z)$ tends to a constant in any interval of length 2π. The result follows as in (5-37) and Prob. 5-20.

7-25 Since $a_n - a \to 0$, we conclude that

$$E\{(\underset{\sim}{x}_n - a)^2\} = E\{[(\underset{\sim}{x}_n - a_n) + (a_n - a)]^2\}$$

$$= E\{(\underset{\sim}{x}_n - a_n)^2\} + 2(a_n - a)E\{\underset{\sim}{x}_n - a_n\} + (a_n - a)^2 \to 0$$

as $n \to \infty$.

7-26 If $E\{\underset{\sim}{x}_n\underset{\sim}{x}_m\} \to a$ as $n,m \to \infty$, then, given $\varepsilon > 0$, we can find a number n_o such that

$$E\{\underset{\sim}{x}_n\underset{\sim}{x}_n\} = a + \theta(n,m) \qquad |\theta| < \varepsilon \qquad \text{if } n,m > 0$$

Hence,

$$E\{(\underset{\sim}{x}_n - \underset{\sim}{x}_m)^2\} = E\{\underset{\sim}{x}_n^2\} + E\{\underset{\sim}{x}_m^2\} - 2E\{\underset{\sim}{x}_n\underset{\sim}{x}_m\}$$

$$= a + \theta_1 + a + \theta_2 - 2(a + \theta_3) = \theta_1 + \theta - 2\theta_3$$

and since $|\theta_1 + \theta_2 - 2\theta_3| < 4\,\varepsilon$ for any ε, it follows that

$E\{(\underset{\sim}{x}_n - \underset{\sim}{x}_m)^2\} \to 0$, hence (Cauchy) $\underset{\sim}{x}_n$ tends to a limit.

<u>Conversely</u> If $\underset{\sim}{x}_n \to \underset{\sim}{x}$ in the MS sense, then

$E\{(\underset{\sim}{x}_n - \underset{\sim}{x})^2\} \to 0$. Furthermore,

$$E\{\underset{\sim}{x}_n^2\} \to E\{\underset{\sim}{x}^2\} \qquad E\{\underset{\sim}{x}\,\underset{\sim}{x}_n\} \to E\{\underset{\sim}{x}^2\}$$

because (see Prob. 6-51)

$$E^2\{\underset{\sim}{x}_n^2 - \underset{\sim}{x}^2\} = E^2\{(\underset{\sim}{x}_n - \underset{\sim}{x})(\underset{\sim}{x}_n + \underset{\sim}{x})\}$$

$$\le E\{(\underset{\sim}{x}_n - \underset{\sim}{x})^2\}E\{(\underset{\sim}{x}_n + \underset{\sim}{x})^2\} \to 0$$

$$E^2\{\underset{\sim}{x}(\underset{\sim}{x}_n - \underset{\sim}{x})\} \le E\{\underset{\sim}{x}^2\}E\{(\underset{\sim}{x}_n - \underset{\sim}{x})^2\} \to 0$$

Similarly, $E\{(x_n - x)(x_m - x)\} \to 0$. Hence,

$$E\{x_n x_m\} + E\{x^2\} - E\{x\,x_n\} - E\{x\,x_m\} \to 0$$

Combining, we conclude that $E\{x_n x_m\} \to E\{x^2\}$.

7-27 $$E\{x_k\} = 0 \qquad\qquad E\{x_k^2\} = \sigma_k^2$$

$$E\left\{\left(\sum_{k=n_1}^{n_2} x_k\right)^2\right\} = \sum_{k=n_1}^{n_2} E\{x_k^2\}$$

If $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$, then given $\varepsilon > 0$, we can find n_o such that $\sum_{k=n+1}^{n+m} \sigma_k^2 < \varepsilon$

for any m and $n > n_o$. Thus

$$E\{(y_{n+m} - y_n)^2\} = E\left\{\left(\sum_{k=n+1}^{n+m} x_k\right)^2\right\} = \sum_{k=n+1}^{n+m} \sigma_k^2 < \varepsilon$$

This shows that (Cauchy), y_k converges in the MS sense. The proof of the converse is similar.

7-28 If $f_i(x) = c\,e^{-cx}U(x)$ then $\Phi_i(s) = \dfrac{c}{c-s}$

$$\Phi(s) = \Phi_1(s) \cdots \Phi_n(s) = \frac{c^n}{(c-s)^n}$$

Hence (see Example 5-29) $f(x) = \dfrac{c^n x^{n-1}}{(n-1)!} e^{-cx}\,U(x)$

7-29 From Prob. 7-28 it follows that $f(x)$ is the density of the sum $x = x_1 + \cdots + x_n$. Furthermore,

$$E\{x\} = \frac{n}{c} \qquad\qquad \sigma_x^2 = \frac{n}{c^2}$$

From the central limit theorem it follows, therefore, that for large n, the Erlang density is nearly equal to a normal curve with mean n/c and variance n/c^2.

7-30 $\quad E\{\underset{\sim}{r}_i\} = 500 \qquad \sigma_i^2 = 50^2/3$

$$\underset{\sim}{r} = \underset{\sim}{r}_1 + \underset{\sim}{r}_2 + \underset{\sim}{r}_3 + \underset{\sim}{r}_4 \qquad E\{\underset{\sim}{r}\} = 2{,}000 \qquad \sigma_r^2 = 10^4/3$$

Thus, $\underset{\sim}{r}$ is approximately $N(2000;10^2/\sqrt{3})$

$$P\{1900 \le \underset{\sim}{r} \le 2100\} = 2\ G\left(\frac{100\sqrt{3}}{100}\right) - 1 = 0.9169.$$

7-31 The RVs $\underset{\sim}{x}_i$ are independent with (see Prob. 5-37)

$$f_i(x) = \frac{c_i}{\pi(c_i^2 + x^2)} \qquad \Phi_i(\omega) = e^{-c_i|\omega|}$$

In that case, (7-104) does not hold because

$$\int_{-\infty}^{\infty} x^{\alpha}\ f(x)\,dx = \frac{c_i}{\pi}\int_{-\infty}^{\infty} \frac{x^{\alpha}}{c_i^2 + x^2}\ dx = \infty \qquad \alpha > 2$$

In fact, the density of $\underset{\sim}{x} = \underset{\sim}{x}_1 + \cdots + \underset{\sim}{x}_n$ is Cauchy with parameter $c = c_1 + \cdots + c_n$ because

$$\Phi(\omega) = e^{-c_1|\omega|} \cdots e^{-c_n|\omega|} = e^{-(c_1 + \cdots + c_n)|\omega|}$$

7-32 In this problem, $\sigma_z^2 = E\{|\underset{\sim}{z}|^2\} = E\{\underset{\sim}{x}^2 + \underset{\sim}{y}^2\} = 2\sigma^2$

$$f_z(x) = f_x(x)f_y(y) = \frac{1}{2\pi\sigma^2}e^{-(x^2+y^2)/2\sigma^2} = \frac{1}{2\pi\sigma_z^2}e^{-|z|^2/\sigma_z^2}$$

$$\Phi_z(\Omega) = \Phi_x(u)\Phi_y(v) = \exp\left\{-\frac{1}{2}\sigma^2(u^2+v^2)\right\} = \exp\left\{-\frac{1}{4}\sigma_z^2|\Omega|^2\right\}$$

CHAPTER 8

8-1 (a) From (8-11) with γ=.95, u=.975, $z_{.975} \simeq 2$, σ=0.1, and n=9 we obtain

$$c = \frac{z_u \sigma}{\sqrt{n}} = 0.066$$

(b) From (8-11) with c=91.01-91=0.05mm:

$$z_u = \frac{c\sqrt{n}}{\sigma} = 1.5 \qquad u = .933 \qquad \gamma = .866$$

8-2 (a) From (8-11) with σ=1 and n=4: $\bar{x} \pm \sigma z_u/\sqrt{n} \simeq 203 \pm 1$mm

(b) From (8-12) with δ=.05: $c = \sigma/\sqrt{n\delta} = 2.236$mm

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8-3 From (8-4)with γ=.9, u=.95: $\bar{x} \pm z_u\sigma/\sqrt{n} = 25{,}000 \pm 1{,}028$ miles

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8-4 We wish to find n such that $P\{|\bar{x}-a|<0.2\} = 0.95$ where $a=E\{\bar{x}\}$.From (8-4) it follows with u=.975 and σ=0.1mm that

$$\frac{z_u \sigma}{\sqrt{n}} \le 0.2, \text{ hence, } n=1$$

8-5 In this problem, x is uniform with $E\{x\}=\theta$ and $\sigma^2=4/3$.We can use, however, the normal approximation for $\bar{x}$ because n=100. With γ=.95, (8-11) yields the interval

$$\bar{x} \pm z_{.975}\ \sigma\sqrt{n} = 30 \pm 0.227$$

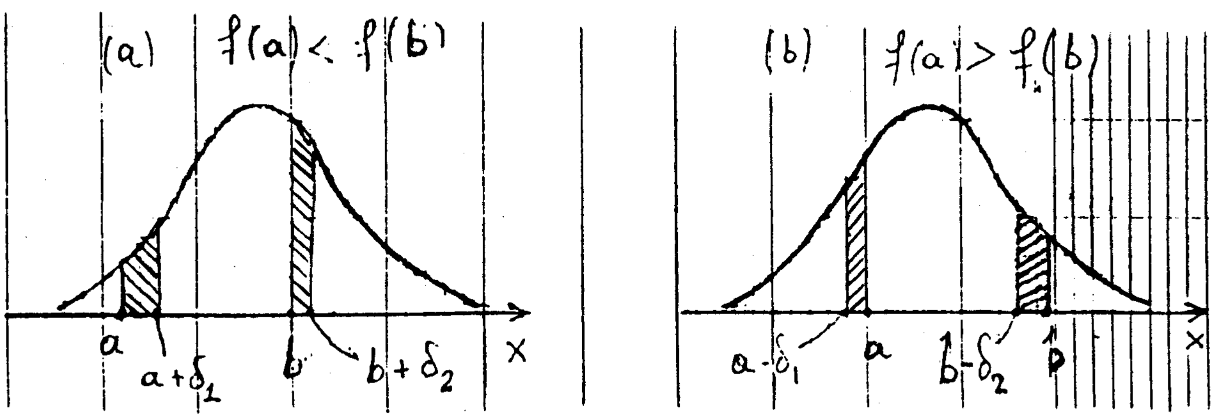
8-6 We shall show that if $f(x)$ is a density with a single maximum and $P\{a<x<b\}=\gamma$, then $b-a$ is minimum if $f(a)=f(b)$. The density $xe^{-x}U(x)$ is a special case. It suffices to show that $b-a$ is not minimum if $f(a)< f(b)$ or $f(a)>f(b)$.

Suppose first that $f(a)< f(b)$ as in figure (a). Clearly. $f'(a)> 0$ and $f'(b)< 0$, hence, we can find two constants $\delta_1>0$ and$\delta_2>0$ such that $P\{a+\delta_1 < x < b+\delta_2\} = \gamma$ and

$$f(a) < f(a+\delta_1) < f(b+\delta_2) < f(b)$$

From this it follows that $\delta_1 > \delta_2$, hence, the length of thenew interval $(a+\delta_1, b+\delta_2)$ is smaller than $b-a$.

If $f(a) > f(b)$, we form the interval $(a-\delta_1, b-\delta_2)$ (Fig. 8-6b) and proceed similarly.



Special case. If $f(x)=xe^{-x}$ then (see Problem 4-9) $F(x)=1-e^{-x}-xe^{-x}$, hence,

$$P\{a < x < b\} = e^{-a} + ae^{-a} - e^{-b} - be^{-b} = .95$$

And since $f(a)=f(b)$, the system

$$ae^{-a} = be^{-b} \qquad e^{-a} - e^{-b} = .95$$

results. Solving, we obtain $a \simeq 0.04 \quad b \simeq 5.75$.

A numerically simpler solution results if we set

$$0.025 = P\{x \le a\} = F(a) \qquad 0.025 = P\{x > b\} = 1 - F(b)$$

as in (9-5). This yields the system

$$0.025 = 1 - e^{-a} - ae^{-a} \qquad 0.025 = e^{-b} + be^{-b}$$

Solving, we obtain $a=0.242$, $b=5.572$. However, the length $5.572-0.242=5.33$ of the resulting interval is larger than the length $4.75-0.04=4.71$ of the optimum interval.

8-7 We start with the general problem: We observe the n samples x_i of an $N(\eta,10)$ RV x and we wish to predict the value x of x at a future trial in terms of the average $\bar{x}$ of the observations. If η is known, we have an ordinary prediction problem. If it is unknown, we must first estimate it. To do so, we form the RV $w=x-\bar{x}$. This RV is

$N(0,\sigma_w)$ where $\sigma_w^2=\sigma_x^2+\sigma_{\bar{x}}^2=\sigma^2+\sigma^2/n$. With $c=z_{.975}\sigma_w$$ it follows that $P\{|w| < c\}=.95$. Hence

$$P\{\bar{x} - c < x < \bar{x} + c\} = 0.95$$

For $n=20$ and $\sigma=10$ the above yields $\sigma_w=10.25$ and $c \simeq 20.5$. Thus, we can expect with .95 confidence coefficient that our bulb will last at least $80-20.5=59.5$ and at most $80+20=100.5$ hours.

8-8 The time of arrival of the 40th patient is the sum $x_1 + \cdots + x_n$ of $n=39$ RVs with exponential distribution. We shall estimate the mean $\eta=1/\theta$ of x in terms of its sample mean $\bar{x}=240/39=6.15$ minutes using two methods. The first is approximate (large n) and is based on (8-11).

<u>Normal approximation</u>. With $\lambda=\eta$ and $z_{.975}/\sqrt{39}=0.315$:

$$P\left\{\frac{\bar{x}}{1.315} < \eta < \frac{\bar{x}}{0.685}\right\} = .95 \qquad 4.68 < \eta < 8.98 \text{ minutes}$$

<u>Exact solution</u>. The RVs $\underset{\sim}{x}_i$ are i.i.d. with exponential distribution. From this and (7-52) it follows that their sum $\underset{\sim}{y}=\underset{\sim}{x}_1 + \cdots + \underset{\sim}{x}_n = n\underset{\sim}{\bar{x}}$ has an Erlang distribution:

$$\Phi_y(s) = \frac{\theta^n}{(\theta-s)^n} \qquad f_y(y) = \frac{\theta^n}{(n-1)!} y^{n-1} e^{-\theta y} U(y)$$

and the RV $\underset{\sim}{z}=2\theta\underset{\sim}{y} = 2n\theta\underset{\sim}{\bar{x}}$ has a $\chi^2(2n)$ distribution:

$$f_z(z) = \frac{1}{2\theta} f_y\left(\frac{z}{2\theta}\right)U(z) = \frac{z^{n-1}}{2^n(n-1)!} e^{-z/2} U(z)$$

Hence,

$$P\left\{\chi^2_{\delta/2}(2n) < \frac{2n\bar{x}}{\eta} < \chi^2_{1-\delta/2}(2n)\right\} = \gamma = 1-\delta$$

Since $\chi^2_{.025}(78) = 54.6$, $\chi^2_{.975}(78)=104.4$, and $2n\bar{x}=480$, this yields the interval

$$4.60 < \eta < 8.79 \text{ minutes}$$

8-9 From (8-19) with $\bar{x}=2{,}550/200=12.75$ $n=200$ and $z_u \simeq 2$

$$\lambda^2 - 25.52\,\lambda + 12.75^2 = 0 \qquad \lambda_1 = 12.255 < \lambda < 13.265 = \lambda_2$$

8-10 From (8-21) with $\bar{x}=2{,}080/4000=0.52$, $n=4{,}000$ and $z_u \simeq 2.326$.

$$p_{1,2} \simeq \bar{x} \pm z_u \sqrt{\frac{\bar{x}(1-\bar{x})}{n}} = .52 \pm .018$$

Hence, $.502 < p < .538$.

8-11 (a) In this problem, $\bar{x}=0.40$, $n=900$ and $z_u \simeq 2$. From (8-21): Margin of error

$$\pm\ 100\ z_u\sqrt{\frac{\bar{x}(1-\bar{x})}{n}} = \pm\ 3.27\%$$

(b) We wish to find z_u. From (9-21) and Table 1a:

$$100 z_u\sqrt{\frac{\bar{x}(1-\bar{x})}{n}} = 2 \qquad z_u = 1.225 \qquad u = .89$$

This yields the confidence coefficient $\gamma = 2u - 1 = .78$

8-12 From (8-21) with $\bar{x}=0.29$ and $z_u=2$:

$$z_u\sqrt{\frac{\bar{x}(1-\bar{x})}{n}} = 0.04 \qquad\qquad n > \frac{\bar{x}(1-\bar{x})}{.04^2}\, z^2_u = 515$$

8-13 The problem is to find n such that [see (8-20)] $z_u\sqrt{\frac{p(1-p)}{n}} \le .02$

for every p. Since $z_u \simeq 2$ and $p(1-p) \le 1/4$, this is the case if

$$z_u\sqrt{1/4n} \le .02 \qquad n \ge 2{,}500$$

8-14 From (8-36) with k=1

$$f(p) = \begin{cases} 5 & .4 < p < .6 \\ 0 & \text{otherwise} \end{cases} \qquad P\{k=1\} = 5\int_{.4}^{.6} p\,dp = .5 = \frac{1}{\gamma}$$

$$f_p(p|1) = \begin{cases} 10p & .4 < p < .6 \\ 0 & \text{otherwise} \end{cases} \qquad \hat{p} = 10\int_{.4}^{.6} p^2 dp = .5067$$

8-15 From Prob. 8-8: $f_{\bar{x}}(\bar{x}|\theta) = \dfrac{(\theta n)^n}{(n-1)!}\,\bar{x}^{n-1}e^{-n\theta\bar{x}}$

From (8-32):

$$f_\theta(\theta\,|\bar{x}) = \frac{(c+n\bar{x})^{n+1}}{n!}\,\theta^n e^{-(c+n\bar{x})\theta}$$

From (8-31):

$$\hat{\theta} = \frac{(c+n\bar{x})^{n+1}}{n!}\int_0^\infty \theta^{n+1}e^{-(c+n\bar{x})\theta}d\theta = \frac{n+1}{c+n\bar{x}}$$

8-16 The sum $n\bar{x}$ is a Poisson RV with mean $n\theta$ (see Prob. 8-8). In the context of Bayesian estimation, this means that

$$f_{\bar{x}}(\bar{x}|\theta) = e^{-n\theta}\frac{(n\theta)^k}{k!} \qquad k = n\bar{x} = 0,1,\ldots$$

Inserting into (8-32), we obtain [see (4-76)]

$$f_\theta(\theta|(\bar{x}) = \frac{(n+c)^{n\bar{x}+b+1}}{\Gamma(n\bar{x}+b+a)}\,\theta^{n\bar{x}+b}e^{-(n+c)\theta}$$

and (8-31) yields

$$\hat{\theta} = \frac{(n+c)^{n\bar{x}+b+1}}{\Gamma(n\bar{x}+b+1)}\,\frac{\Gamma(n\bar{x}+b+2)}{(n+c)^{n\bar{x}+b+2}} = \frac{n\bar{x}+b+1}{n+c} \xrightarrow[n\to\infty]{} \bar{x}$$

8-17 From (8-17) with $t_{.95}(9)=2.26$

$$\bar{x} \pm \frac{t_u s}{\sqrt{n}} = 90 \pm 3.57 \qquad 86.43 < \eta \cdot 93.57$$

From (8-24) with $\chi^2_{.975}(9)=19.02$, $\chi^2_{.025}(9)=2.70$.

$$\frac{9 \times 5^2}{19.02} = 11.83 < \sigma^2 < \frac{9 \times 5^2}{2.70} = 83.33 \qquad 3.44 < \sigma < 9.13$$

8-18 The RVs x_i/σ are N(0,1), hence, the sum $z=(x_1^2 + \cdots + x^2_{10})/\sigma^2$ has a $\chi^2(10)$ distribution. This yields

$$P\{\chi^2_{.025}(10) < z < \chi^2_{.975}(10)\} = .95$$

$$\chi^2_{.025}(10) = 3.25 < \frac{4}{\sigma^2} < \chi^2_{.975}(10) = 20.48$$

$$0.442 < \sigma < 1.109$$

8-19 From (8-23) with $n=4, \chi^2_{.025}(4)=0.48,\ \chi^2_{.975}(4)=11.14$

$$n\hat{v} = .1^2 + .15^2 + .05^2 + .04^2 = .0366$$

$$\frac{.0366}{.048} > \sigma^2 > \frac{.0366}{11.14} \qquad 0.276 > \sigma > 0.057$$

8-20 In this problem $n=3$, $x_1+x_2+x_3=9.8$

$$f(x,c) \sim c^4x^3e^{-cx} \qquad f(X,c) = c^{4n}(x_1\ldots x_n)^{3n}e^{-cn\bar{x}}$$

$$\frac{\partial f(X,c)}{\partial c} = \left(\frac{4n}{c} - n\bar{x}\right)f(X,\theta) = 0 \qquad \hat{c} = \frac{4}{\bar{x}} = 1.224$$

8-21 The joint density

$$f(X,c) = c^n e^{-cn(\bar{x}-x_0)} \qquad x_i > x_0$$

has an interior maximum if

$$\frac{\partial f(X,c)}{\partial c} = 0 \qquad \hat{c} = \frac{1}{\bar{x}-x_0}$$

8-22 The probability

$$p = 1 - F_x(200) = e^{-200c}$$

of the event $\{x > 200\}$ is a monoton decreasing function of c. To find the ML estimate $\hat{c}$ of c it suffices to find the ML estimate $\hat{p}$ of p. From Example 8-28 it follows with k=62 and n=80 that

$$\hat{p} = \frac{62}{80} = .775 \text{ hence}$$

$$\hat{c} = -\frac{1}{200} \ln \hat{p} = 0.0013$$

8-23 The samples of x are the integers x_i and the joint density of the RVs x_i equals

$$f(X,\theta) = e^{-n\theta} \prod \frac{\theta^{x_i}}{x_i!} = e^{-n\theta} \frac{\theta^{n\bar{x}}}{\Pi x_i!}$$

Hence, $f(X,\theta)$ is maximum if $-n + n\bar{x}/\theta = 0$. This yields $\hat{\theta} = \bar{x}$

8-24 If $L = \ln f(x,\theta)$ then

$$\frac{\partial L}{\partial \theta} = \frac{1}{f}\frac{\partial f}{\partial \theta} \qquad \frac{\partial^2 L}{\partial \theta^2} = \frac{1}{f}\frac{\partial^2 f}{\partial \theta^2} - \frac{1}{f^2}\left(\frac{\partial f}{\partial \theta}\right)^2 \qquad \frac{\partial^2 L}{\partial \theta^2} + \left(\frac{\partial L}{\partial \theta}\right)^2 = \frac{1}{f}\frac{\partial^2 f}{\partial \theta^2}$$

But

$$E\left\{\frac{1}{f}\frac{\partial^2 f}{\partial \theta^2}\right\} = \int_R \frac{1}{f}\frac{\partial^2 f}{\partial \theta^2} f dX = 0 \quad \text{hence} \quad E\left\{\frac{\partial^2 L}{\partial \theta^2} + \left(\frac{\partial L}{\partial \theta}\right)^2\right\} = 0$$

8-25 (a) From (8-307): Critical region

$$\bar{x} > c = \eta_0 + z_{1-\alpha} \frac{\sigma}{\sqrt{n}} = 8+2.326 \times \frac{2}{8} = 8.58$$

If η=8.7, then $\eta_q = \frac{8.7-8}{218} = 2.8$

$$\beta(\eta) = G(2.36 - 2.8) = .32$$

(b) We assume that α=.01, $\beta(8.7)$=.05 and wish to find n and c.

$$G(z_{1-\alpha}-\eta_q) = \beta \qquad z_{1-\alpha}-\eta_q = z_\beta$$

$$\eta_q = z_{.99} - z_{.05} = 4.97 = \frac{8.7-8}{2/\sqrt{n}}$$

$$n = 129 \qquad c = 8 + \frac{2}{\sqrt{129}} z_{.99} = 8.41$$

8-26 Our objective is to test the composite null hypothesis $\eta > \eta_0 = 28$ against the hypothesis $\eta < \eta_0$. Consider first the simple null hypothesis $\eta = \eta_0 = 28$. In this case, we can use (8-301) with

$$q = \frac{\bar{x}-\eta_0}{s/\sqrt{n}} \qquad \bar{x} = \frac{1}{17}\sum x_i = 27.67 \qquad s^2 = \frac{1}{16}\sum(x_i-\bar{x}) = 17.6$$

This yields s=4.2 and q=-0.33. Since

$$q_u = t_u\,(n-1) = t_{0.05}(16) = -\ 1.95 < -0.33$$

we conclude that the evidence does not support the rejection of the hypothesis η=28. The resulting OC function $\beta_0(\eta)$ is determined from (9-60c).

If η_0>28, then the corresponding value of q is larger than -0.33. From this it follows that the evidence does not support the hypothesis η_0 for any $\eta_0>28$. We note, however, that the corresponding OC function $\beta(\eta)$ is smaller than the function $\beta_0(\eta)$ obtained from (8-301) with $\eta_0=28$.

8-27 From (8-297) with $q_u=t_u(n-1)$: Critical region $|\bar{x}-\eta_0| > t_{1-\alpha/2}(n-1)s/\sqrt{n}$

1. $\underline{\alpha = .1}$ $\quad t_{.95}(63) = 1.67 \quad |\bar{x}-8| > 1.67 \times 1.5/8 = 0.313$

Since $\bar{x}=7.7$ is in the interval 8 ± 0.317, we accept H_0

2. $\underline{\alpha = .01}$ $\quad t_{.995}(63) = 2.62 \quad |\bar{x}-8| > 2.62 \times 1.5/8 = 0.49$

Since $\bar{x}=7.7$ is outside the interval 8 ± 0.49, we reject H_0.

8-28 We assume that the RVs $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are normal and independent. We form the difference $\underset{\sim}{w}=\underset{\sim}{\bar{x}}-\underset{\sim}{\bar{y}}$ of their sample means

$$\underset{\sim}{\bar{x}} = \frac{1}{16}\sum_{i=1}^{16} \underset{\sim}{x}_1 \qquad \underset{\sim}{\bar{y}} = \frac{1}{26}\sum_{i=1}^{26} \underset{\sim}{y}_i$$

and use as test statistic the ratio

$$\underset{\sim}{q} = \frac{\underset{\sim}{w}}{\sigma_w} \qquad \sigma_w^2 = \frac{\sigma_x^2}{16} + \frac{\sigma_y^2}{26}$$

The RV $\underset{\sim}{q}$ is normal with $\sigma_q=1$ and under hypothesis H_0, $E\{\underset{\sim}{q}\}=0$. We can, therefore, use (8-307) because $q_u = z_u$. To find q, we must determine σ_w.

Since σ_x and σ_y are not specified, we shall use the approximations $\sigma_x \simeq s_x = 1.1$ and $\sigma_y \simeq s_y = 0.9$. This yields

$$\sigma_w^2 \simeq \frac{1.1^2}{16} + \frac{0.9^2}{26} = 0.107 \qquad q = \frac{\bar{x}-\bar{y}}{\sigma_w} = \frac{0.4}{0.327} = 1.223$$

Since $z_{0.95} = 1.645 > 1.223$, we accept H_0.

8-29 (a) In this problem, n=64, k=22, $p_0 = q_0 = 0.5$

$$q = \frac{k - np_0}{\sqrt{np_0 q_0}} = 2.5 \qquad z_{\alpha/2} = -z_{1-\alpha/2} \simeq -2$$

Since 2.5 is outside the interval (2, -2), we reject the fair coin hypothesis [see (8-313)].

(b) From (8-313) with n=16, $p_0 = q_0 = 0.5$:

$$\frac{k_1 - np_0}{\sqrt{np_0 q_0}} = z_{\alpha/2} \qquad \frac{k_2 - np_0}{\sqrt{np_0 q_0}} = -z_{\alpha/2}$$

This yields $k_1 = 8 - 2\times 2 = 4$, $k_2 = 8 + 2\times 2 = 12$

8-30 We shall use as test statistic the sum

$$\underset{\sim}{q} = \underset{\sim}{x}_1 + \cdots + \underset{\sim}{x}_m \qquad n = 22$$

The critical region of the test is $q<q_\alpha$ where $q=x_1+\cdots+x_n=90$ [see (8-301)].

The RV $\underset{\sim}{q}$ is Poisson distributed with parameter $n\lambda$. Under hypothesis H_0, $\lambda=\lambda_0=5$; hence, $\eta_q=n\lambda_0=110=\sigma_q^2$. To find q_α we shall use the normal approximation. With $\alpha=0.05$ this yields

$$q_\alpha = n\lambda_0 + z_\alpha \sqrt{n\lambda_0} = 90-17.25 = 72.75$$

Since 90 > 72.75, we accapt the hypothesis that $\lambda=5$.

8-31 From (9-75) with n=102 and $p_{0i}=1/6$

$$q = \sum_{i=1}^{6} \frac{(k_i-17)^2}{17} = 2 \qquad \chi^2_{.95}(5) \simeq 11$$

Since 2<11, we accept the fair die hypothesis.

8-32 Uniformly distributed integers from 0 to 9 means that they have the same probability of appearing. With m=10, $p_{01}=.1$, and n=1,000, it follows from (8-325) that

$$q = \sum_{j=0}^{9} \frac{(n_j-100)^2}{100} = 17.76 \qquad \chi^2_{.95}(9) = 16.92$$

Since 17.76 > 16.92, we reject the uniformity hypothesis.

8-33 In this problem

$$f(x,\theta) = e^{-\theta}\frac{\theta^x}{x!} \qquad f(X,\theta) = \frac{e^{-n\theta}\,\theta^{n\bar{x}}}{x_1!\cdots x_n!}$$

$f(X,\theta)$ is maximum for $\theta=\theta_m=\bar{x}$. And since $\theta_{m0}=\theta_0$ we conclude that

$$\lambda(X) = \frac{e^{-n\theta_0}\theta_0^{\,n\bar{x}}}{e^{-n\bar{x}}\bar{x}^{n\bar{x}}} \qquad w = -2\, ln\,\lambda = 2n(\theta_0-\bar{x}) + \bar{x}\, ln(\bar{x}/\theta_0)$$

With $n=50$, $\theta_0=20$, $\bar{x}=1{,}058/50=21.16$, this yields $w=3$. Since $m_0=1$, $m=1$, and $\chi^2_{.95}(1)=3.84>3$, we accept H_0.

8-34 We form the RVs

$$\underset{\sim}{z} = \sum_{i=1}^{m}\left(\frac{x_i-\eta_x}{\sigma_x}\right)^2 \qquad \underset{\sim}{w} = \sum_{i=1}^{n}\left(\frac{y_i-\eta_y}{\sigma_y}\right)^2$$

These RVs are $\chi^2(m)$ and $\chi^2(n$ respectively. If $\sigma_x=\sigma_y$, then

$$\underset{\sim}{q} = \frac{z/m}{\underset{\sim}{w}/n}$$

Hence (see Prob. 6-23), $\underset{\sim}{q}$ has a Snedecor distribution. To test the hypothesis $\sigma_x=\sigma_y$, we use (8-297) where $q_u=F_u(m,n)$ is the tabulated u percentile of the Snedecor distribution. This yields the following test:

$$\text{Accept } H_o \text{ iff} \qquad F_{\alpha/2}(m,n) < q < F_{1-\alpha/2}(m,n).$$

8-35 If $\underset{\sim}{x}$ has a student-t distribution, then $f(-x)=f(x)$, hence (see Prob. 6-75)

$$E\{\underset{\sim}{x}\} = 0 \qquad \sigma_x^2 = E\{\underset{\sim}{x}^2\} = \frac{n}{n-2}$$

8-36 (a) Suppose that the probability $P(A)$ that player A wins a set equals $p=1-q$. He wins the match in five sets if he wins two of the first four sets and the fifth set. Hence, the probability $p_5(A)$ that he wins in five equals $6p^3q^2$. Similarly, the probability $p_5(B)$ that player B wins in five equals $6p^2q^3$. Hence,

$$p_5 = p_5(A) + p_5(B) = 6p^3q^2 + 6p^2q^3 = 6p^2q^2$$

is the probability that the match lasts five sets. If $p=q=1/2$, then $p_5=3/8$.

(b) Suppose now that $P(A) = \underset{\sim}{p}$ is an RV with density $f(p)$. In this case,

$$\underset{\sim}{p}_5 = 6\underset{\sim}{p}^2(1-\underset{\sim}{p}^2)$$

is an RV. We wish to find its best bayesian estimate. Using the MS criterion, we obtain

$$\hat{p}_5 = E\{\underset{\sim}{p}_5\} = \int_0^1 6p^2(1-p^2)f(p)dp$$

If $f(p)=1$, then $\hat{p}_5 = 1/5$.

8-37 Given

$$f_v(v) \sim e^{-v^2/2\sigma^2} \qquad f_\theta(\theta) \sim e^{-(\theta-\theta_o)^2/2\sigma_o^2}$$

To show that

$$f_\theta(\theta|X) \sim e^{-(\theta-\theta_1)^2/2\sigma_1^2}$$

where

$$\frac{1}{\sigma_1^2} \equiv \frac{1}{\sigma_o^2} + \frac{n}{\sigma^2} \qquad \theta_1 \equiv \frac{\sigma_1^2}{\sigma_o^2}\theta_o + \frac{n\sigma_1^2}{\sigma^2}\bar{x}$$

<u>Proof</u>

$$f_x(x|\theta) = f_v(x-\theta) \sim \exp\left\{-\frac{(x-\theta)^2}{2\sigma^2}\right\}$$

$$f(X)|\theta \sim \exp\left\{-\frac{1}{2\sigma^2}\sum (x_i-\theta)^2\right\}$$

Since $\sum (x_i-\theta)^2 = \sum (x_i-\bar{x})^2 + n(\bar{x}-\theta)^2$, we conclude from (8-32) omitting factors that do not depend on θ that

$$f(\theta|X) \sim \exp\left\{-\frac{1}{2}\left[\frac{(\theta-\theta_o)^2}{\sigma_o^2} + \frac{n(\bar{x}-\theta)^2}{\sigma^2}\right]\right\}$$

The above bracket equals

$$\left(\frac{1}{\sigma_o^2} + \frac{n}{\sigma^2}\right)\theta^2 - 2\left(\frac{\theta_o}{\sigma_o^2} + \frac{n\bar{x}}{\sigma^2}\right)\theta + \cdots = \frac{1}{\sigma_1^2}(\theta^2 - 2\theta\theta_1) + \cdots$$

and (i) follows.

8-38 The likelihood function of X equals

$$f(X,\theta) = \frac{1}{(\sqrt{2\pi\theta})^n} \exp\left\{-\frac{1}{2\theta}\sum (x_i-\eta)^2\right\}$$

where $\theta = \sigma^2$ is the unknown parameter. Hence

$$L(X,\theta) = -\frac{n}{2}\ln(2\pi\theta) - \frac{1}{2\theta}\sum (x_i-\eta)^2$$

$$\frac{\partial L(X,\theta)}{\partial\theta} = -\frac{n}{2\theta} + \frac{1}{2\theta^2}\sum (x_i-\theta)^2 = 0 \qquad \hat{\theta} = \frac{1}{n}\sum (x_i-\eta)^2$$

8-39 The estimators $\hat{\underset{\sim}{\theta}}_1$ and $\hat{\underset{\sim}{\theta}}_2$ have the same variance because otherwise one or the other would not be best. Thus

$$E\{\hat{\underset{\sim}{\theta}}_1\} = E\{\hat{\underset{\sim}{\theta}}_2\} = \theta \qquad \text{var}\,\hat{\underset{\sim}{\theta}}_1 = \text{var}\,\hat{\underset{\sim}{\theta}}_2 = \sigma^2$$

If $\hat{\underset{\sim}{\theta}} = \frac{1}{2}(\hat{\underset{\sim}{\theta}}_1 + \hat{\underset{\sim}{\theta}}_2)$, then

$$E\{\hat{\underset{\sim}{\theta}}\} = \theta \qquad \sigma_{\hat{\theta}}^2 = \frac{1}{2}(\sigma^2 + \sigma^2 + 2r\sigma^2) = \frac{1}{2}(1+r)\sigma^2$$

where σ is the correlation coefficient of $\hat{\underset{\sim}{\theta}}_1$ and $\hat{\underset{\sim}{\theta}}_2$. If r<1 then $\sigma_{\hat{\theta}}<\sigma$ which is impossible. Hence, r=1 and $\hat{\underset{\sim}{\theta}}_1 = \hat{\underset{\sim}{\theta}}_2$ (see Prob. 6-53).

8-40 $k_1+k_2-np_1-np_2 = n-n(p_1+p_2) = 0$; Hence, $|k_1-np_1| = |k_2-np_2|$

$$\frac{(k_1-np_1)^2}{np_1} + \frac{(k_2-np_2)^2}{np_2} = (k_1-np_1)^2\left(\frac{1}{np_1} + \frac{1}{np_2}\right) = \frac{(k_1-np_1)^2}{np_1p_2}$$

8.41 It is given that

$$E\{T(X)\} = \int_{-\infty}^{\infty} T(X)\, f(X;\theta)\, dx = \psi(\theta),$$

so that after differentiating and making use of (8-81) we get

$$\int_{-\infty}^{\infty} T(X)\, \frac{\partial f(X;\theta)}{\partial \theta}\, dx = \psi^{'}(\theta) \qquad (8.41-1)$$

Also using (8-80)

$$\int_{-\infty}^{\infty} \psi(\theta)\, \frac{\partial f(X;\theta)}{\partial \theta}\, dx = 0, \qquad (8.41-2)$$

and the above two expressions give

$$\int_{-\infty}^{\infty} [T(X) - \psi(\theta)]\, \frac{\partial f(X;\theta)}{\partial \theta}\, dx = \psi^{'}(\theta) \qquad (8.41-3)$$

But

$$\frac{\partial f(X;\theta)}{\partial \theta} = \frac{1}{f(X;\theta)}\, \frac{\partial \log f(X;\theta)}{\partial \theta}$$

so that (8.41-3) simplifies to

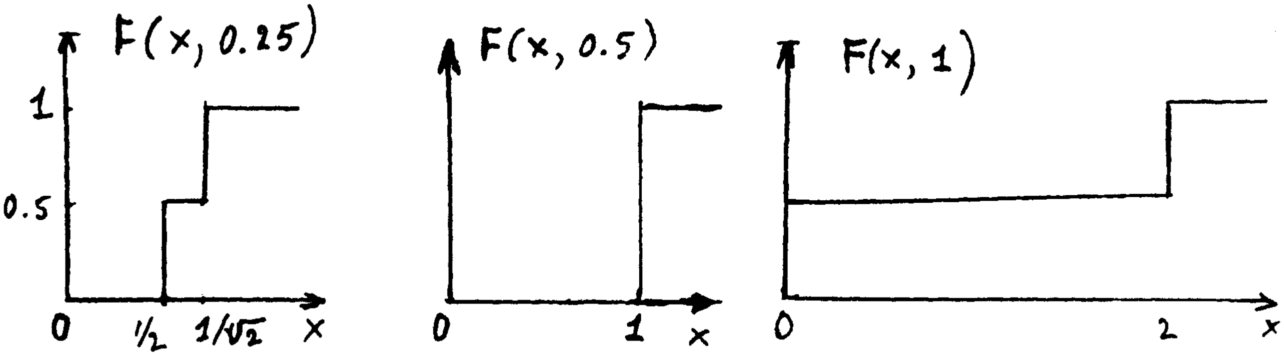
$$\int_{-\infty}^{\infty} \left[\{T(X) - \psi(\theta)\}\sqrt{f(X;\theta)}\right] \left[\sqrt{f(X;\theta)}\ \frac{\partial \log f(X;\theta)}{\partial \theta}\right] dx = \psi^{'}(\theta)$$

and application of Cauchy-Schwarz inequality as in (8-89)-(8-92), Text gives

$$E\left[\{T(X) - \psi(\theta)\}^2\right] \geq \frac{[\psi^{'}(\theta)]^2}{E\left\{\left(\frac{\partial \log f(X;\theta)}{\partial \theta}\right)^2\right\}}$$

9-1 (a) $E\{\mathbf{x}(t)\} = t + 0.5 \sin \pi t$

$$\mathbf{x}(t, \text{heads}) = \sin \pi t = \begin{cases} 1/\sqrt{2} & t = 0.25 \\ 1 & t = 0.5 \\ 0 & t = 1 \end{cases} \qquad \mathbf{x}(t, \text{tails}) = 2t = \begin{cases} 0.5 \\ 1 \\ 2 \end{cases}$$



9-2

$$\mathbf{x}(t) = e^{\mathbf{a}t}$$

$$\eta(t) = \int_{-\infty}^{\infty} e^{at} f_a(a)\,da \qquad R(t_1,t_2) = \int_{-\infty}^{\infty} e^{at_1} e^{at_2} f_a(a)\,da$$

From (5-16) with $x = g(a) = e^{ta}$ $\qquad g'(a) = t\,e^{ta} = tx$

$$f(x,t) = \frac{1}{x|t|} f_a\left(\frac{1}{t} \ln x\right) U(x)$$

9-3 As we know, $E\{\underset{\sim}{x}(t)\} = \lambda t$ and var $\underset{\sim}{x}(t) = \lambda^2 t^2$ [see (9-18)]. But $E\{\underset{\sim}{x}(9) = 6\}$ by assumption, hence, $\lambda = 2/3$

(a) $E\{\underset{\sim}{x}(8)\} = 24$ var $\underset{\sim}{x}^2(t) = 24^2$

(b) The RV $\underset{\sim}{x}(2)$ is Poisson distributed with parameter $2\lambda = 6$. Hence,

$$P\{\underset{\sim}{x}(2) \le 3\} = e^{-2\lambda} \sum_{k=0}^{3} \frac{(2\lambda)^k}{k!}$$

(c) The RVs $\underset{\sim}{z} = \underset{\sim}{x}(2)$ and $\underset{\sim}{w} = \underset{\sim}{x}(4) - \underset{\sim}{x}(2)$ are independent and Poisson distributed with parameter 2λ. Hence,

$$P\{\underset{\sim}{z}=k\} = e^{-2\lambda} \frac{(2\lambda)^k}{k!} \qquad P\{\underset{\sim}{z} = k, \underset{\sim}{w} = m\} = e^{-4\lambda} \frac{(2\lambda)^k}{k!} \frac{(2\lambda)^m}{m!}$$

$$P\{\underset{\sim}{x}(4) \le 5 \mid \underset{\sim}{x}(2) \le 3\} = \frac{P\{z \le 3,\ w \le 5 - z\}}{P\{\underset{\sim}{z} \le 3\}} \qquad P\{\underset{\sim}{z} \le 3\} = \sum_{k=0}^{3} p\{\underset{\sim}{z}=k\}$$

$$P\{\underset{\sim}{z} \le 3, \underset{\sim}{w} \le 5 - \underset{\sim}{z}\} = \sum_{k=0}^{3} \sum_{m=0}^{5-k} P\{\underset{\sim}{z} = k, \underset{\sim}{w} = m\}$$

9-4 $$\underset{\sim}{x}(t) = U(t - \underset{\sim}{c}) \qquad \underset{\sim}{y}(t) = \delta(t - \underset{\sim}{c}) = \underset{\sim}{x}'(t)$$

For t_1 or $t_2 < 0$, $R(t_1,t_2) = 0$; for t_1 and $t_2 > T$, $R(t_1,t_2) = 1$. Otherwise,

$$R_x(t_1,t_2) = \frac{1}{T}\min(t_1,t_2) \qquad \frac{\partial R_x}{\partial t_1} = \frac{1}{T}U(t_1 - t_2) \qquad -\frac{\partial^2 R_x}{\partial t_1 \partial t_2} = \frac{1}{T}\delta(t_1 - t_2)$$

From this and (9-105) it follows that $T R_y(t_1 - t_2) = \delta(t_1 - t_2)$ for $0 < t_1, t_2 < T$ and 0 otherwise.

9-5 $\underset{\sim}{a} - \underset{\sim}{b}t = 0$ iff $t = \underset{\sim}{t}_1 = \underset{\sim}{a}/\underset{\sim}{b}$. Setting $\sigma_1 = \sigma_2 = \sigma$ and $r = 0$ in (6-63), we obtain

$$P\{0 < \underset{\sim}{t}_1 < T\} = \frac{1}{2} + \frac{1}{\pi}\text{arc tan}\,T - \left(\frac{1}{2} + \frac{1}{\pi}\text{arc tan } 0\right)$$

9-6 **The equations**

$$w''(t) = \nu(t)U(t) \qquad w(0) = w'(0) = 0$$

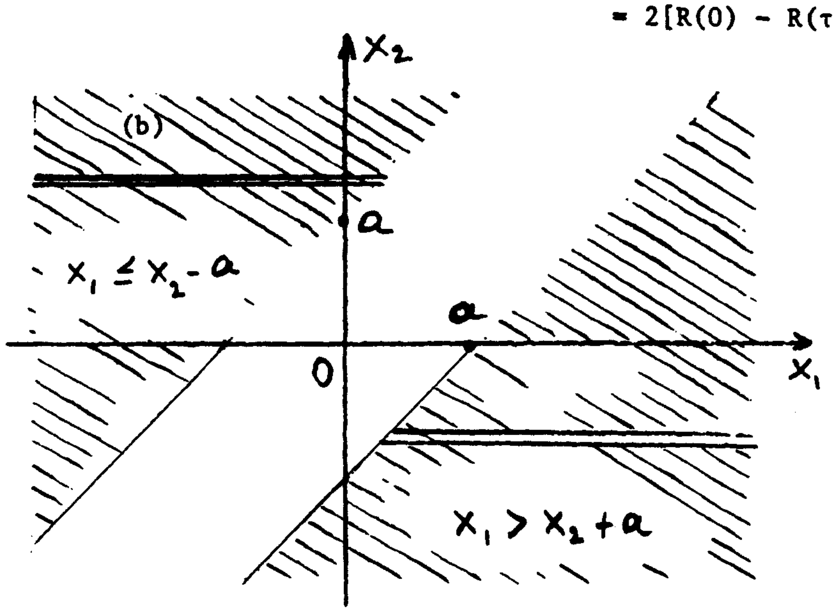
specify a system with input $\nu(t)U(t)$ and impulse response $h(t) = t\,U(t)$. Hence [see (9-100)]

$$E\{w^2(t)\} = q(t)U(t) * t^2U(t) = \int_0^t (t-\tau)^2 q(\tau)d\tau$$

9-7 (a) From (5-88) with $x = x(t+\tau) - x(t)$, and (8-101):

$$P\{|x(t+\tau) - x(t)| \geq a\} \leq \frac{E\{[x(t+\tau) - x(t)]^2\}}{a^2}$$

$$= 2[R(0) - R(\tau)]/a^2$$



The above probability equals the mass in the regions (shaded) $x_2 - x_1 > a$ and $x_2 - x_1 < -a$

Hence,

$$P\{|x(t+\tau) - x(t)| \geq a\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{x_2 - a} f(x_1,x_2;\tau)dx_1dx_2 + \int_{-\infty}^{\infty}\int_{x_2+a}^{\infty} f(x_1,x_2;\tau)dx_1dx_2$$

9-8 (a) The RV $\underset{\sim}{x}(t)$ is normal with zero mean and variance $E\{\underset{\sim}{x}^2(t)\} = R(0)=4$, hence it is $N(0,2)$ and $P\{\underset{\sim}{x}(t)\le 3\} = F(3) = G(1.5) = 0.933$

(b) $E\{[\underset{\sim}{x}(t+1) - \underset{\sim}{x}(t-1)]\} = 2[R(0)-R(2)] = 8(1-e^{-4})$

9-9 If $\underset{\sim}{x}(t) = \underset{\sim}{c}e^{j(\omega t+\theta)}$ and $\eta_c = 0$ then

$$\eta_x(t) = \eta_c e^{j(\omega t+\theta)} = 0 \qquad R_{xx}(t+\tau,t) = \sigma_c^2 e^{j\omega\tau}$$

hence, $\underset{\sim}{x}(t)$ is WSS. We shall prove the converse:

If the process $\underset{\sim}{x}(t) = \underset{\sim}{c}\,w(t)$ is WSS, then $\eta_c=0$ and $w(t) = e^{j(\omega t+\theta)}$ within a constant factor.

<u>Proof</u> $\eta_x(t) = \eta_c w(t)$ is independent of t; hence, $\eta_c=0$. The function $R_{xx}(t_1,t_2) = \sigma_c^2 w(t_1)w^*(t_2)$ depends only on $\tau=t_1-t_2$; hence, $w(t+\tau)w^*(t)=g(\tau)$. With $\tau=0$ this yields

$$|w(t)|^2 = g(0) = \text{constant} \qquad w(t) = ae^{j\phi(t)}$$

$$w(t+\tau)w^*(t) = a^2 e^{j[\phi(t+\tau)-\phi(t)]}$$

Hence the difference $\phi(t+\tau)-\phi(t)$ depends only on τ:

$$\phi(t+\tau)-\phi(t) = f(\tau) \tag{i}$$

From this it follows that, if $\phi(t)$ is continuous then, $\phi(t)$ is a linear function of t. To simplify the proof, we shall assume that $\phi(t)$ is differentiable. Differentiating with respect to t, we obtain $\phi'(t+\tau) = \phi'(t)$ for every τ. With t=0 this yields

$$\phi'(\tau) = \phi'(0) = \text{constant} \qquad \phi(\tau) = a\tau+b$$

9-10 We shall show that if $\underset{\sim}{x}(t)$ is a normal process with zero mean and $\underset{\sim}{z}(t) = \underset{\sim}{x}^2(t)$, then $C_{zz}(\tau) = 2C_{xx}^2(\tau)$.

From (7-61): If the RVs $\underset{\sim}{x}_k$ are normal and $E\{\underset{\sim}{x}_k\}=0$, then

$$E\{x_1x_2x_3x_4\} = E\{x_1x_2\}\,E\{x_3x_4\} + E\{x_1x_3\}\,E\{x_2x_4\} + E\{x_1x_4\}\,E\{x_2x_3\}$$

With $x_1=x_2=x(t+\tau)$ and $x_3 = x_4 = x(t)$, we conclude that the autocorrelation of $z(t)$ equals

$$E\{x^2(t+\tau)x^2(t)\} = E^2\{x^2(t+\tau)\} + 2E^2\,\{x(t+\tau)x(t)\} = R_{xx}{}^2(0) + 2R_{xx}{}^2(\tau)$$

And since $R_{xx}(\tau)=C_{xx}(\tau)$, and $E\{z(t)\} = R_{xx}(0)$, the above yields

$$C_{zz}(\tau) = R_{zz}(\tau) - E^2\{z(t)\} = 2C_{xx}{}^2(\tau)$$

9-11 $$y''(t) + 4y'(t) + 13y(t) = x(t) \text{ all } t$$

The process $y(t)$ is the response of a system with input $x(t) = 26 + \nu(t)$ and

$$H(s) = \frac{1}{s^2+45+13} \qquad h(t) = \frac{1}{3}\,e^{-2t}\sin 3tU(t)$$

Since $\eta_x = 26$, this yields $\eta_y = \eta_x H(0) = 2$. The centered process $\bar{y}(t) = y(t)-\eta_y$ is the response due to $\nu(t)$. Hence [see (9-100)]

$$E\{\bar{y}^2(t)\} = q\int_0^\infty h^2(t)dt = \frac{10}{104}$$

With b=4 and c=13 it follows that (see Example 9-276)

$$R_{yy}(\tau) = \frac{10}{104}e^{-2|\tau|}\left[\cos 3\tau - \frac{2}{3}\sin 3|\tau|\right] + 4$$

If ν is normal, then $y(t)$ is normal with mean 2 and variance $R_{yy}(0) - 4 = 10/104$; hence,

$$P\{y(t) \le 3\} = G\left(\frac{3-2}{0.31}\right) = G(3.24)$$

9-12 $$E\{y(t)\} = 0 \qquad R_{yy}(t_1,t_2) = \frac{R_{xx}(t_1,t_2)}{f(t_1)f(t_2)} = w(t_1-t_2)$$

$$E\{z(t)\} = 0 \qquad R_{zz}(t_1,t_2) = \frac{R_{xx}(t_1,t_2)}{\sqrt{q(t_1)}\,\sqrt{q(t_2)}} = \delta(t_1-t_2)$$

because $q(t_1)\delta(t_1-t_2) = \sqrt{q(t_1)}\,\sqrt{q(t_2)}\,\delta(t_1-t_2)$.

9-13 From (9-181) and the identity $4ab \le (a+b)^2$ it follows that

$$|R_{xy}(\tau)|^2 \le R_{xx}(0)R_{yy}(0) \le \frac{1}{4}[R_{xx}(0)+R_{yy}(0)]^2$$

9-14 Clearly (stationarity assumption)

$$E\{|\underset{\sim}{x}^*(t) - \underset{\sim}{y}^*(t)|^2\} = E\{|\underset{\sim}{x}(0) - \underset{\sim}{y}(0)|^2\} = 0$$

Furthermore,

$$E\{\underset{\sim}{x}(t+\tau)[\underset{\sim}{x}^*(t) - \underset{\sim}{y}^*(t)]\} = R_{xx}(\tau) - R_{xy}(\tau)$$

and [see (9-177)]

$$|E\{\underset{\sim}{x}(t+\tau)[\underset{\sim}{x}^*(t) - \underset{\sim}{y}^*(t)]\}|^2 \le E\{|\underset{\sim}{x}(t+\tau)|^2\}E\{|\underset{\sim}{x}^*(t) - \underset{\sim}{y}^*(t)|^2\} = 0$$

Hence, $R_{xx}(\tau) - R_{xy}(\tau) = 0$; similarly, $R_{yy}(\tau) = R_{xy}(\tau)$

9-15 $E\{|\underset{\sim}{x}(t+\tau) - \underset{\sim}{x}(t)|^2\} = E\{[\underset{\sim}{x}(t+\tau) - \underset{\sim}{x}(t)][\underset{\sim}{x}^*(t+\tau) - \underset{\sim}{x}^*(t)]\}$

$= R(0) - R(\tau) - R^*(\tau) + R(0) = 2R(0) - 2\ \underline{\mathrm{Re}}\ R(\tau)$

9-16 From $\Phi(1) = \Phi(2) = 0$ it follows that

$$E\{\cos \underset{\sim}{\phi}\} = E\{\sin \underset{\sim}{\phi}\} = E\{\cos 2\underset{\sim}{\phi}\} = E\{\sin 2\underset{\sim}{\phi}\} = 0$$

Hence, $E\{\underset{\sim}{x}(t)\} = \cos\omega t\ E\{\cos\underset{\sim}{\phi}\} - \sin\omega t\ E\{\sin\underset{\sim}{\phi}\} = 0$

and as in Example 9-14

$$2\cos[\omega(t+\tau)+\underset{\sim}{\phi}]\cos(\omega t+\underset{\sim}{\phi}) = \cos\omega\tau + \cos(2\omega t + \omega\tau + 2\underset{\sim}{\phi})$$

$$2R_x(\tau) = \cos\omega\tau$$

If $\underset{\sim}{\phi}$ is uniform in $(-\pi,\pi)$, then

$$\Phi(\lambda) = \frac{\sin\pi\omega}{\pi\omega} \qquad \Phi(1) = \Phi(2) = 0$$

9-17 (a) $\underline{x}(t_1)\underline{x}(t_2) = [\underline{x}(t_1) - \underline{x}(0)][\underline{x}(t_2) - \underline{x}(t_1) + \underline{x}(t_1) - x(0)]$

$$R(t_1,t_2) = E\{[\underline{x}(t_1) - \underline{x}(0)]^2\} = E\{\underline{x}^2(t_1)\} = R(t_1,t_1)$$

(b) If $t_1 + \varepsilon < t_2$, then $R_y(t_1,t_2) = 0$; if

$t_1 < t_2 < t_1 + \varepsilon$ then

$$E\{[\underline{x}(t_1+\varepsilon) - \underline{x}(t_1)][\underline{x}(t_2+\varepsilon) - \underline{x}(t_2)]\} = q(t_1 + \varepsilon - t_2)$$

Hence, $\varepsilon^2 R_y(\tau) = q(\varepsilon - |\tau|)$ for $|\tau| = |t_2 - t_1| \le \varepsilon$

9-18

$$E\{\underline{x}(t)\underline{y}(t)\} = \int_{-\infty}^{\infty} E\{\underline{x}(t)\underline{x}(t-\tau)\}h(\tau)d\tau$$

$$= \int_{-\infty}^{\infty} R_{xx}(t,t-\tau)h(\tau)d\tau = \int_{-\infty}^{\infty} q(t)\delta(\tau)h(\tau)d\tau = h(0)q(t)$$

9-19 As in Prob. 5-14, $g(x) = 6 + 3\,F_x(x)$. In this case,

$E\{\underline{x}^2(t)\} = 4$, hence, $\underline{x}(t)$ is $N(0,2)$ and $F_x(x) = G(x/2)$

9-20 $\underline{x}(t)$ is SSS, hence, $P\{x(t) \le y\} = F_x(y)$ does not depend on t. The RVs $\underline{\varepsilon}$ and $\underline{x}(t)$ are independent, hence, [see (6-238)]

$$F_y(y) = P\{\underline{x}(t-\underline{\varepsilon}) \le y \mid \underline{\varepsilon} = \varepsilon\} = P\{\underline{x}(t-\varepsilon) \le y \mid \underline{\varepsilon} = \varepsilon\}$$

$$= P\{\underline{x}(t-\varepsilon) < y\} = F_x(y)$$

is independent of t. Similarly for higher order distributions.

9-21 $E\{\underset{\sim}{x}(t)\} = \eta =$ constant, hence, [see (9-102)] $E\{\underset{\sim}{x}'(t)\} = 0$

Furthermore, $R_{xx}(-\tau) = R_{xx}(\tau)$. hence, $R'_{xx}(0) = 0$ and (10-97) yields

$$E\{\underset{\sim}{x}(t)\underset{\sim}{x}'(t)\} = R_{xx'}(0) = 0$$

9-22 (a) $E\{\underset{\sim}{z}\,\underset{\sim}{w}\} = R_x(2) = 4e^{-4}$ $\qquad E\{\underset{\sim}{z}^2\} = E\{\underset{\sim}{w}^2\} = R_x(0) = 4$

$$E\{(\underset{\sim}{z}+\underset{\sim}{w})^2\} = R_x(0) + R_x(0) + 2R_x(2) = 8(1+e^{-4})$$

(b) $\underset{\sim}{z}$ is $N(0,2)$ $\qquad P\{\underset{\sim}{z} < 1\} = F_z(1) = G(1/2)$

$r_{zw} = e^{-4}$, $\quad f_{zw}(z,w) : N(0,0;2,2;e^{-4})$

9-23 The RV $\underset{\sim}{x}'(t)$ is normal with zero mean and variance

$$E\{|\underset{\sim}{x}'(t)|^2\} = R_{x'x'}(0) = -R''(0)$$

Hence, $P\{\underset{\sim}{x}'(t) \le a\} = F_{x'}(a) = G\left[a/\sqrt{|R''(0)|}\right]$

9-24 The function arc sin x is odd, hence, it can be expanded into a sine series in the interval (-1,1):

$$\alpha(x) \equiv \arcsin x = \sum_{n=1}^{\infty} b_n \sin n\pi x \qquad |x| \le 1$$

$$b_n = \int_{-1}^{1} \alpha(x)\sin n\pi x\,dx = -\frac{1}{n\pi}\int_{-1}^{1}\alpha(x)\,d\cos n\pi x$$

$$= -\frac{\alpha(x)\cos n\pi x}{n\pi}\Bigg|_{-1}^{1} + \frac{1}{n\pi}\int_{-1}^{1}\cos n\pi x\,d\alpha(x)$$

$$= -\frac{\cos n\pi}{n} + \frac{1}{n\pi}\int_{-\pi/2}^{\pi/2}\cos(n\pi\sin x)\,dx$$

and the result follows because [see (9-81)]

$$R_y(\tau) = \frac{2}{\pi}\arcsin\frac{R_x(\tau)}{R_x(0)} \qquad J_o(z) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos(z\sin x)\,dx$$

9-25 As we know [see (5-100) and (6-193)]

$$E\{e^{j\omega \underset{\sim}{x}(t)}\} = \exp\{-\frac{R(0)}{2}\ \omega^2\}$$

$$E\{e^{j[\omega_1 \underset{\sim}{x}(t+\tau)+\omega_2 \underset{\sim}{x}(t)]}\} = \exp\{-\frac{1}{2}[R(0)\omega_1^2 + 2R(\tau)\omega_1\omega_2 + R(0)\omega_2^2]\}$$

Hence, with $j\omega = a$

$$E\{I\,e^{a\underset{\sim}{x}(t)}\} = \exp\{\frac{a^2}{2}R_x(0)\}\,I$$

$$E\{I\,e^{a\underset{\sim}{x}(t+\tau)}I\,e^{a\underset{\sim}{x}(t)}\} = I^2\exp\{a\,[R_x(0) + R_x(\tau)]\}$$

9-26 (a) $R_y(\tau) = a^2E\{\underset{\sim}{x}[c(t+\tau)]\underset{\sim}{x}(ct)\} = a^2R(c\tau)$

(b) If $\underset{\sim}{z}_\varepsilon(t) = \sqrt{\varepsilon}\ \underset{\sim}{x}(\varepsilon t)$ then $R_{z_\varepsilon}(\tau) = \varepsilon R_x(\varepsilon\tau)$ [as in (a)].

If $\delta > 0$ is sufficiently small and $\phi(t)$ is continuous at the origin, then

$$\int_{-\delta}^{\delta} R_{z_\varepsilon}(\tau)\phi(\tau)d\tau \simeq \phi(0)\int_{-\delta}^{\delta}\varepsilon R_x(\varepsilon\tau)\,d\tau$$

$$= \phi(0)\int_{-\varepsilon\delta}^{\varepsilon\delta} R(\tau)d\tau \xrightarrow[\varepsilon\to\infty]{} \phi(0)\int_{-\infty}^{\infty} R(\tau)d\tau = q\ \phi(0)$$

Hence, $R_{z_\varepsilon}(\tau) \longrightarrow q\ \delta(\tau)$ as $\varepsilon \to \infty$.

9-27

$$\underset{\sim}{y}(t) = \int_{t-T}^{t} \underset{\sim}{x}(\tau)h(t-\tau)d\tau$$

Hence, $\underset{\sim}{y}(t_1)$ and $\underset{\sim}{y}(t_2)$ depend linearly on the values of $\underset{\sim}{x}(t)$ in the intervals $(t_1 - T,\ t_1)$ and $(t_2 - T,\ t_2)$ respectively. If $|t_1 - t_2| > T$ then these intervals do not overlap and since $E\{\underset{\sim}{x}(\tau_1)\underset{\sim}{x}(\tau_2)\} = 0$ for $\tau_1 \neq \tau_2$, it follows that $E\{\underset{\sim}{y}(t_1)\underset{\sim}{y}(t_2)\} = 0$.

9-28 (a)

$$I(t) = E\left\{\int_0^t \int_0^t h(t,\alpha)\underset{\sim}{x}(\alpha)h(t,\beta)\underset{\sim}{x}(\beta)\ d\alpha d\beta\right\}$$

$$= \int_0^t \int_0^t h(t,\alpha)h(t,\alpha)q(\alpha)\delta(\alpha-\beta)d\alpha d\beta = \int_0^t h^2(t,\alpha)q(\alpha)d\alpha$$

(b) If $\underset{\sim}{y}'(t) + c(t)\underset{\sim}{y}(t) = \underset{\sim}{x}(t)$, then $\underset{\sim}{y}(t)$ is the output of a linear time-varying system as in (a) with impulse response $h(t,\alpha)$ such that

$$\frac{\partial h(t,\alpha)}{\partial t} + c(t)h(t,\alpha) = \delta(t-\alpha) \qquad h(\alpha^-,\alpha) = 0$$

or equivalently

$$\frac{\partial h(t,\alpha)}{\partial t} + c(t)h(t,\alpha) = 0 \qquad t > 0 \quad h(\alpha^+,\alpha) = 1$$

This yields

$$h(t,\alpha) = e^{-\int_\alpha^t c(\tau)d\tau}$$

Hence, if

$$I(t) = \int_0^t h^2(t,\alpha)q(\alpha)d\alpha \quad \text{then} \quad I'(t) + 2c(t)I(t) = q(t)$$

because the impulse response of this equation equals

$$e^{-2\int_\alpha^t c(\tau)d\tau} = h^2(t,\alpha)$$

9-29 (a) If $\underset{\sim}{y}'(t) + 2\underset{\sim}{y}(t) = \underset{\sim}{x}(t)$, then $\underset{\sim}{y}(t) = \underset{\sim}{x}(t)*h(t)$

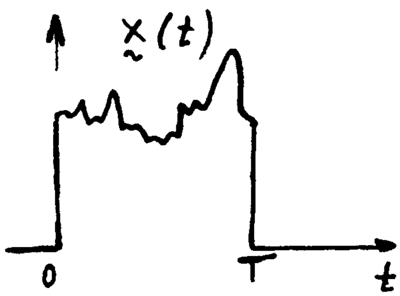
where $h(t) = e^{-2t}U(t)$ and with $q(t) = 5$, (10-90) yields

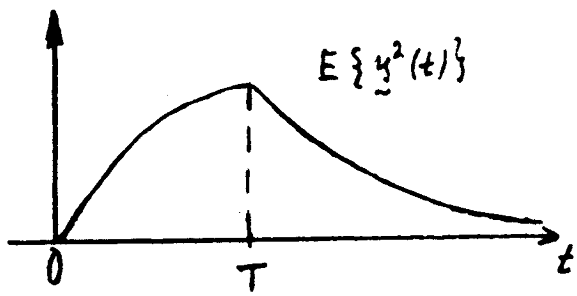
$$E\{\underset{\sim}{y}^2(t)\} = 5*e^{-4t}U(t) = 5\int_0^\infty e^{-4\tau}d\tau = \frac{5}{4}$$

(b) As in (a) with $q(t) = 5U(t)$. Hence, for $t > 0$

$$E\{\underset{\sim}{y}^2(t)\} = 5U(t)*e^{-4t}U(t) = 5\int_0^t e^{-4\tau}d\tau = \frac{5}{4}(1 - e^{-4t})$$

9-30





From (9-90) with $q(t) = N[U(t) - U(t-T)]$

$$E\{\underset{\sim}{y}^2(t)\} = \begin{cases} AN\displaystyle\int_0^t e^{-2\alpha(t-\tau)}d\tau = \frac{AN}{2\alpha}(1-e^{-2\alpha t}) & 0 \le t < T \\ AN\displaystyle\int_0^T e^{-2\alpha(t-\tau)}d\tau = \frac{AN}{2\alpha}(e^{2\alpha T} - 1)e^{-2\alpha t} & t > T \end{cases}$$

9-31 Since $\underset{\sim}{x}(t)$ is WSS, the moments of S equal the moments of

$$\underset{\sim}{z} = \int_{-5}^{5} \underset{\sim}{x}(t)dt$$

Hence, (see Fig. 9-5)

$$E\{\underset{\sim}{s}^2\} = \int_{-5}^{5}\int_{-5}^{5} R_x(t_1-t_2)dt_1dt_2 = \int_{-10}^{10}(10-|\tau|)R_x(\tau)d\tau$$

$$E\{\underset{\sim}{s}\} = 80 \qquad \sigma_s^2 = 2\int_0^{10}(10-\tau)10e^{-2\tau}d\tau$$

9-32 $$\underset{\sim}{y}(t) = \underset{\sim}{x}(t)*h(t) \qquad h(t) = e^{-2t}U(t)$$

(a) $$E\{\underset{\sim}{y}^2(t)\} = 5*e^{-4t}\,U(t) = 5/4$$

$$R_{xy}(t_1,t_2) = 5\,\delta(t_1-t_2)*e^{-2t_2}\,U(t_2) = 5e^{-2(t_2-t_1)}U(t_2-t_1)$$

$$R_{yy}(t_1,t_2) = 5e^{-2(t_2-t_1)}U(t_2-t_1)*e^{-2t_1}\,U(t_1)$$

$$= \frac{5}{4}\,e^{-2|t_1-t_2|}$$

The first equation follows from (9-100) with $q(t) = 5$; the second from (9-94) with $R_{xx}(t_1,t_2) = 5\delta(t_1-t_2)$, and the third from (9-96).

(b) With $R_{xx}(t_1,t_2) = 5\delta(t_1-t_2)U(t_1)U(t_2)$, (9-94) and (9-96) yield the following: For t_1 or $t_2 < 0$, $R_{xy}(t_1,t_2) = R_{yy}(t_1,t_2) = 0$. For $0 < t_1 < t_2$

$$R_{xy}(t_1,t_2) = 5\delta(t_1-t_2)*e^{-2t_2} = 5\,e^{-2t_2}$$

$$R_{yy}(t_1,t_2) = \int_0^{t_1} 5\,e^{-2(t_1-\tau)}\,e^{-2(t_1-\tau)}\,d\tau = \frac{5}{4}\,e^{-2(t_2-t_1)}(1-e^{-4t_1})$$

9-33 $$\int_{-\infty}^{\infty} e^{-\alpha\tau^2} e^{-s\tau} d\tau = e^{s^2/4\alpha} \int_{-\infty}^{\infty} e^{-\alpha(\tau + s/2\alpha)^2} d\tau = \sqrt{\frac{\pi}{\alpha}}\, e^{s^2/4\alpha}$$

This yields

$$e^{-\alpha\tau^2} \longleftrightarrow \sqrt{\frac{\pi}{\alpha}}\, e^{-\omega^2/4\alpha}$$

$$e^{-\alpha\tau^2} \cos \omega_o\tau \longleftrightarrow \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \left[e^{-(\omega-\omega_o)^2/4\alpha} + e^{-(\omega+\omega_o)^2/4\alpha} \right]$$

9-34 $$G(x_1,x_2;\omega) = \int_{-\infty}^{\infty} f(x_1,x_2;\tau) e^{-j\omega\tau} d\tau$$

$$R(\tau) = E\{\mathbf{x}(t+\tau)\mathbf{x}(t)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2\, f(x_1,x_2;\tau) dx_1 dx_2$$

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} e^{-j\omega\tau} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2\, f(x_1,x_2;\tau) dx_1\, dx_2 d\tau$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2 \int_{-\infty}^{\infty} e^{-j\omega\tau}\, f(x_1,x_2;\tau) d\tau dx_1 dx_2$$

9-35 The process $\mathbf{y}(t) = \mathbf{x}(t+a) - \mathbf{x}(t-a)$ is the output of a system with input $\mathbf{x}(t)$ and system function

$$H(\omega) = e^{ja\omega} - e^{-ja\omega} = 2j \sin a\omega$$

Hence [see (9-150)]

$$S_y(\omega) = 4 \sin^2 a\omega\, S_x(\omega) = (2 - e^{j2a\omega} - e^{-j2a\omega}) S_x(\omega)$$

$$R_y(\tau) = 2\, R_x(\tau) - R_x(\tau + 2a) - R_x(\tau - 2a)$$

9-36 Since $S(\omega) \geq 0$, we conclude with (9-136) that

$$R(0) - R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)(1 - \cos \omega\tau)\, d\omega$$

$$\geq \frac{1}{8\pi} \int_{-\infty}^{\infty} S(\omega)(1 - \cos 2\omega\tau)\, d\omega = \frac{1}{4}[R(0) - R(2\tau)]$$

and the result follows for $n = 1$. Repeating the above, we obtain the general result.

9-37 **From** (6-197)

$$E\{\underset{\sim}{x}^2(t+\tau)\underset{\sim}{x}^2(t)\} = E\{\underset{\sim}{x}^2(t+\tau)\}E\{\underset{\sim}{x}^2(t)\} + 2E^2\{\underset{\sim}{x}^2(t+\tau)\underset{\sim}{x}^2(t)\}$$

Hence,

$$R_y(\tau) = R_x^2(0) + 2\, R_x^2(\tau) = I^2(1 + e^{-2\alpha|\tau|} + e^{-2\alpha|\tau|}\cos 2\beta\tau)$$

$$S_y(\omega) = \left[2\pi\delta(\omega) + \frac{4\alpha}{4\alpha^2 + \omega^2} + \frac{2\alpha}{4\alpha^2 + (\omega - 2\beta)^2} + \frac{2\alpha}{4\alpha^2 + (\omega + 2\beta)^2}\right]$$

Furthermore,

$$\eta_y = E\{\underset{\sim}{x}^2(t)\} = R_x(0) \qquad C_y(\tau) = 2R_x^2(\tau)$$

9-38

$$\int_{-\infty}^{\infty} S(\omega)\left|\sum_i a_i\, e^{j\omega\tau_i}\right|^2 d\omega = \int_{-\infty}^{\infty} S(\omega) \sum_{i,k} a_i a_k^* \, e^{j\omega(\tau_i - \tau_k)}\, d\omega$$

$$= \sum_{i,k} a_i a_k^* \, R(\tau_i - \tau_k) \geq 0$$

9-39 (a) $S(s) = \dfrac{1}{1+s^4} = \dfrac{1}{(s^2+\sqrt{2}s+1)(s^2-\sqrt{2}s+1)}$

A special case of example 9-27b with $b = \sqrt{2}$, $c = 1$. Hence,

$$R(\tau) = \frac{1}{2\sqrt{2}}\, e^{-|\tau|/\sqrt{2}} \quad \left(\cos\frac{\tau}{\sqrt{2}} + \sin\frac{|\tau|}{\sqrt{2}}\right)$$

(b) From the pair $e^{-2|\tau|} \leftrightarrow 4/(4+\omega^2)$ and the convolution theorem it follows that

$$e^{-2|\tau|} * e^{-2|\tau|} \leftrightarrow \frac{16}{(4+\omega^2)^2}$$

Hence, for $\tau > 0$

$$16\, R(\tau) = \int_{\infty}^{\infty} e^{-2|x|}\, e^{-2|\tau - x|} dx = \int_{-\infty}^{0} e^{2x}\, e^{-2(\tau - x)} dx$$

$$+ \int_{0}^{\tau} e^{-2x}\, e^{-2(\tau-x)}\, dx + \int_{\tau}^{\infty} e^{-2x}\, e^{2(\tau-x)}\, dx = \frac{1}{2} e^{-2\tau}(1 + 2\tau)$$

And since $R(-\tau) = R(\tau)$, the above yields

$$e^{-2|\tau|}\, \frac{1+2|\tau|}{32} \longleftrightarrow \frac{1}{(4+\omega^2)^2}$$

9-40 $H^*(-s^*)\Big|_{s=j\omega} = H^*(j\omega) \qquad H^*(1/z^*)\Big|_{z=e^{j\omega T}} = H^*(e^{j\omega T})$

Hence

$$H(s)H^*(-s^*)\Big|_{s=j\omega} = |H(j\omega)|^2 \qquad H(z)H^*(1/z^*)\Big|_{z=j\omega T} = |H(e^{j\omega T})|^2$$

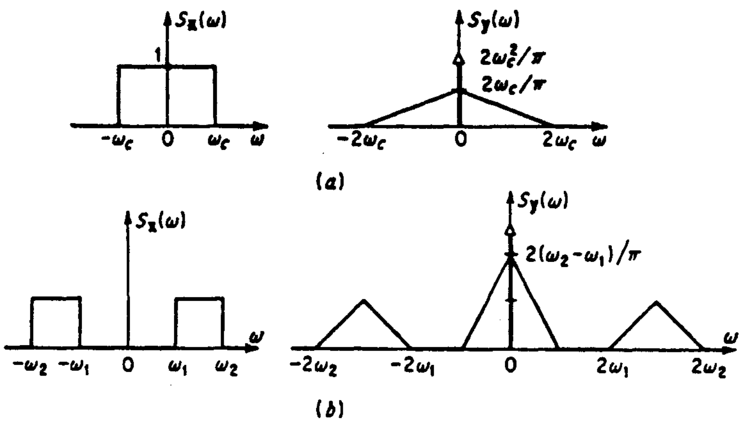
9-41 From (6-197)

$$R_y(\tau) = E\{\underset{\sim}{x}^2(t+\tau)\underset{\sim}{x}^2(t)\}$$

$$= E\{\underset{\sim}{x}^2(t+\tau)\}E\{\underset{\sim}{x}^2(t)\} + 2\, E^2\{\underset{\sim}{x}(t+\tau)\underset{\sim}{x}(t)\} = R_x^2(0) + 2\, R_x^2(\tau)$$

From the above and the frequency convolution theorem it follows that

$$S_y(\omega) = 2\pi R_x^2(0)\delta(\omega) + \frac{1}{\pi} S_x(\omega) * S_x(\omega)$$



9-42 $\quad \underset{\sim}{y}(t) = 2\underset{\sim}{x}(t) + 3\underset{\sim}{x}'(t) \qquad \eta_x = 5 \qquad C_{xx}(\tau) = 4e^{-2|\sigma|}$

The process $\underset{\sim}{y}(t)$ is the output of the system H(s) = 2+3s with input $\underset{\sim}{x}(t)$. Hence, η_y=5H(0)=10

$$S_{yy}{}^c(\omega) = S_{xx}{}^c(\omega)|2+3j\omega|^2 = \frac{16}{4+\omega^2}(4+9\omega^2) = 144 - \frac{512}{4+\omega^2} = S_{yy}(\omega) - 2\pi\eta_y{}^2\delta(\omega)$$

9-43 (a) $y'(t) + 3y(t) = x(t)$, $R_{xx}(\tau) = 5\delta(\tau)$. The process $y(t)$ is the output of the system

$$H(s) = \frac{1}{s+3} \qquad h(t) = e^{-3t}U(t)$$

Hence, [see (9-100) and (9-150)]

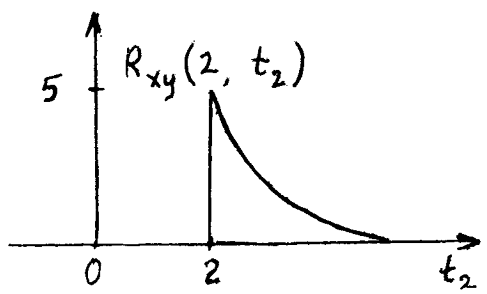
$$E\{y^2(t)\} = 5\int_0^{\infty} e^{-6t}dt = \frac{5}{6}$$

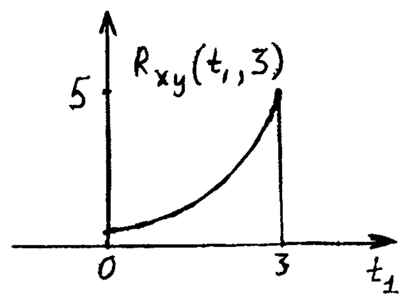
$$S_{yy}(\omega) = \frac{5}{\omega^2+9} \qquad R_{yy}(\tau) = \frac{5}{6}e^{-3|\tau|}$$

(b) As in Example 9-18:

$$E\{y^2(t)\} = 5\int_0^{t} e^{-6\alpha}d\alpha = \frac{5}{6}(1-e^{-6t}) \qquad t > 0$$

$$R_{xy}(t_1,t_2) = 5e^{-2|t_2-t_1|}U(t_1)U(t_2)U(t_2-t_1)$$





9-44 We shall show that: If $x(t)$ is a complex process with autocorrelation $R(\tau)$ and $|R(\tau_1)|=R(0)$ for some τ_1, then $R(\tau)=e^{j\omega_o\tau}w(\tau)$ where $w(\tau)$ is a periodic function with period τ_1. Furthermore, the process $y(t) = e^{-j\omega_o t}x(t)$ is MS periodic.

<u>Proof</u> Clearly, $R(\tau_1) = R(0)e^{j\phi}$. With $\omega_o = \phi/\tau_1$,

$$R_{yy}(\tau) = E\{x(t+\tau)e^{-j\omega_o(t+\tau)}x^*(t)e^{j\omega_o t}\} = R(\tau)e^{-j\omega\tau}$$

Hence, $R_{yy}(\tau_1)=e^{-j\omega_o\tau_1}R(\tau_1) = R(0) = R_{yy}(0)$. From this and (10-168) it follows that the function $w(\tau) = R_{yy}(\tau)$ is periodic.

9-45 (a) The cross spectrum $S_{\check{x}x}(\omega) = -j\,\mathrm{sgn}\,\omega\; S_{xx}(\omega)$ is an odd function. Hence,

$$E\{x(t)\check{x}'(t)\} = \frac{-j}{2\pi}\int_{\infty}^{\infty} \mathrm{sgn}\,\omega S_{xx}(\omega)\,d\omega = 0$$

(b) The process $\check{\check{x}}(t)$ is the output of the system

$$(-j\,\mathrm{sgn}\,\omega)(-j\,\mathrm{sgn}\,\omega) = -1$$

with input $x(t)$. Hence, $\check{\check{x}}(t) = -\,x(t)$.

9-46 In general

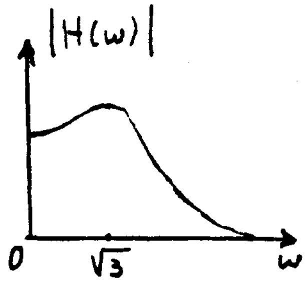
$$E\{y^2(t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\,|H(\omega)|^2 d\omega$$

$$\le |H(\omega_m)|^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\,d\omega = E\{x^2(t)\}\,|H(\omega_m)|^2$$

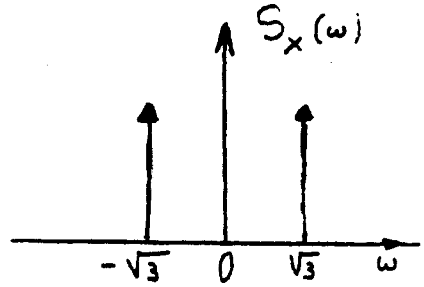
where $|H(\omega_m)|$ is the maximum of $|H(\omega)|$. In our case,

$$|H(\omega)|^2 = \frac{1}{(5-\omega)^2 + 4\omega^2} \quad \text{is maximum for } \omega = \sqrt{3}$$

and $|H(\omega_m)|^2 = 1/16$. Hence $E\{y^2(t)\} \le 10/16$ with equality if $R_x(10) = 10\cos\sqrt{3}\,\tau$ (Fig. b).



(a)



(b)

9-47 If $R_x(\tau) = e^{j\omega_o\tau}$, then $S_x(\omega) = 2\pi\delta(\omega-\omega_o)$, hence, the integral of $S_x(\omega)$ equals zero in any interval not including the point $\omega = \omega_o$. From (9-182) it follows that the same is true for the integral of $S_{xy}(\omega)$. This shows that $S_{xy}(\omega)$ is a line at $\omega = \omega_o$ for any $\underset{\sim}{y}(t)$.

9-48 (a) As in (9-147) and (9-149)

$$R_{yx}(\tau) = R_{xx}(\tau) * h(\tau) = \int_{-\infty}^{\infty} e^{j\alpha(\tau-\gamma)} h(\gamma)\, d\gamma = e^{j\alpha\tau} H(\alpha)$$

$$R_{yy}(\tau) = R_{xx}(\tau) * \rho(\tau) = \int_{-\infty}^{\infty} e^{j\alpha(\tau-\gamma)} \rho(\gamma) d\gamma = e^{j\alpha\tau} |H(\alpha)|^2$$

(b) As in (9-94) and (9-95)

$$R_{yx}(t_1,t_2) = e^{-j\beta t_2} \int_{-\infty}^{\infty} e^{j\alpha(t_1-\gamma)} h(\gamma) d\gamma = e^{j(\alpha t_1-\beta t_2)} H(\alpha)$$

$$R_{yy}(t_1,t_2) = e^{-j\alpha t_1} H(\alpha) \int_{-\infty}^{\infty} e^{-j\beta(t_2-\gamma)} h(\gamma) d\gamma = e^{j(\alpha t_1-\beta t_2)} H(\alpha) H^*(\beta)$$

because $h(t)$ is real and $H(-\beta) = H^*(\beta)$.

9-49 If $S_{xx}(\omega) S_{yy}(\omega) \equiv 0$ then $S_{xx}(\omega) = 0$ or $S_{yy}(\omega) = 0$ in any interval (a,b). From this and (10-168) it follows that the integral of $S_{xy}(\omega)$ in any interval equals zero, hence, $S_{xy}(\omega) \equiv 0$.

9-50 This is the discrete-time version of theorem (9-162). From (9-163)

$$E^2\{(\mathbf{x}[n+m+1] - \mathbf{x}[n+m])\mathbf{x}[n]\} \le E\{|\mathbf{x}[n+m+1] - \mathbf{x}[n+m]|^2\}E\{|\mathbf{x}[n]|^2\}$$

$$(R[m+1] - R[m])^2 \le 2(R[0] - R[1])R[0] = 0$$

Hence, $R[m+1] = R[m]$ for any m.

9-51 We shall show that

$$2\,\frac{R^2[1]}{R[0]} - R[0] \le R[2] \le R[0] \qquad \text{(i)}$$

The covariance matrix of the RVs $\mathbf{x}[n]$, $\mathbf{x}[n+1]$, and $\mathbf{x}[n+2]$ is non-negative [see (7-29)]:

$$\begin{vmatrix} R[0] & R[1] & R[2] \\ R[1] & R[0] & R[1] \\ R[2] & R[1] & R[0] \end{vmatrix} \ge 0$$

This yields

$$R[0]R^2[2] - 2\,R^2[1]R[2] - R^3[0] + 2R[0]R^2[1] \le 0$$

The above is a quadratic in $R[2]$ with roots

$$R[0] \text{ and } -R[0] + 2\,R^2[1]/R[0]$$

Since it is nonpositive, $R[2]$ must be between the roots as in (i)

9-52 If $\mathbf{x}[n] = Ae^{jn\boldsymbol{\omega}T}$ then

$$R_x[m] = A^2E\{e^{j(m+n)\boldsymbol{\omega}T}e^{-jn\boldsymbol{\omega}T}\} = A^2\int_{-\sigma}^{\sigma} e^{jm\omega T} f(\omega)\,d\omega$$

But [see (9-194)]

$$R[m] = \frac{1}{2\sigma}\int_{\sigma}^{\sigma} S_x(\omega)e^{jm\omega T}d\omega$$

hence, $A^2 f(\omega) = S_x(\omega)/2\sigma$

9-53 (a) If $\underset{\sim}{y}(0) = \underset{\sim}{y}'(0) = 0$, then $\underset{\sim}{y}(t)$ is the output of a system with input $\underset{\sim}{x}(t)U(t)$ and impulse response $h(t)$ such that

$$h''(t) + 7h'(t) + 10h(t) = \delta(t) \qquad h(0^-) = h'(0^-) = 0$$

$$h(t) = \frac{1}{3}(e^{-2t} - e^{-5t})U(t)$$

and with $q(t) = 5\,U(t)$, (9-100) yields

$$E\{\underset{\sim}{y}^2(t)\} = \frac{5}{9}\int_0^t (e^{-2\tau} - e^{-5\tau})^2 d\tau$$

(b) If $\underset{\sim}{y}[-1] = \underset{\sim}{y}[-2] = 0$, then $\underset{\sim}{y}[n]$ is the output of a system with input $\underset{\sim}{x}[n]U[n]$ and delta response $h[n]$ such that

$$8h[n] - 6h[n-1] + h[n-2] = \delta[n] \qquad h[-1] = h[-2] = 0$$

$$h[n] = \left(\frac{1}{2^{n+2}} - \frac{1}{2^{2n+3}}\right) U[n]$$

and with $q[n] = 5\,U[n]$, (10-176) yields

$$E\{\underset{\sim}{y}^2[n]\} = 5\sum_{k=0}^{n}\left(\frac{1}{2^{k+2}} - \frac{1}{2^{2k+3}}\right)^2$$

9-54 $$\underset{\sim}{y}[n] = \underset{\sim}{x}[n]*h[n] \qquad h[n] = 2^{-n}U[n]$$

(a) $$E\{\underset{\sim}{y}^2[n]\} = 5*2^{-2n}U[n] = 0$$

$$R_{xy}[m_1,m_2] = 5\delta[m_1-m_2]*2^{-m_2}U[m_2] = 5\,2^{-(m_2-m_1)}U[m_2-m_1]$$

$$R_{yy}[m_1,m_2] = 5\times 2^{-(m_2-m_1)}U[m_2-m_1]*2^{-m_1}U[m_1]$$

$$= \frac{20}{3}\times 2^{-|m_1-m_2|}$$

The first equation follows from (9-190) with $q[n] = 5$; the second and third from (9-191) with $R_{xx}[m_1,m_2] = 5\,\delta[m_1-m_2]$.

(b) With $R_{xx}[m_1,m_2] = 5\,\delta[m_1-m_2]U[m_1]U[m_2]$, Prob. 9-25a yields the following: For m_1 or $m_2 < 0$, $R_{xy}[m_1,m_2] = R_{yy}[m_1,m_2] = 0$. For $0 < m_1 < m_2$

$$R_{xy}[m_1,m_2] = 5\,\delta[m_1-m_2]*2^{-m_2} = 5\times 2^{-m_2}$$

$$R_{yy}[m_1,m_2] = \sum_{k=0}^{m_1} 5\times 2^{-(m_2-k)}\,2^{-(m_1-k)} = \frac{5}{3}\,2^{-(m_2-m_1)}(4 - 2^{-2m_1})$$

9-55

(a) $R_x[m_1,m_2] = q[m_1]\delta[m_1-m_2]$

$$E\{\underset{\sim}{s}^2\} = \sum_{n=0}^{N} \sum_{k=0}^{N} a_n a_k \; E\{\underset{\sim}{x}[n]\underset{\sim}{x}[k]\}$$

$$= \sum_{n=0}^{N} \sum_{k=0}^{N} a_n a_k \; q[n]\delta[n-k] = \sum_{n=0}^{N} a_n^2 \; q[n]$$

(b) $R_x(t_1,t_2) = q(t_1)\delta(t_1-t_2)$

$$E\{s^2\} = \int_0^T \int_0^T a(t)a(\tau) \; E\{x(t)x(\tau)\}d\tau dt$$

$$= \int_0^T \int_0^T a(t)a(\tau)q(t)\delta(t-\tau)d\tau dt = \int_0^T a^2(t)q(t)dt$$

CHAPTER 10

10-1

(a) If $\underset{\sim}{x}(t)$ is a Poisson process as in Fig. 9-3a, then for a fixed t, $\underset{\sim}{x}(t)$ is a Poisson RV with parameter λt. Hence [see (5-119)] its characteristic function equals $\exp\{\lambda t(e^{j\omega} - 1)\}$.

(b) If $\underset{\sim}{x}(t)$ is a Wiener process then $f(x,t)$ is $N(0,\sqrt{\alpha t})$. Hence [see (5-100)] its first order characteristic function equals $\exp\{-\alpha t\omega^2/2\}$.

10-2 For large $\underset{\sim}{t}$, $\underset{\sim}{x}(t)$ and $\underset{\sim}{y}(t)$ can be approximated by two independent Wiener processes as in (10-52):

$$f_x(x,t) = \frac{1}{\sqrt{2\pi\alpha t}} e^{-x^2/2\alpha t} \qquad f_y(y,t) = \frac{1}{\sqrt{2\pi\alpha t}} e^{-y^2/2\alpha t}$$

Hence, $\underset{\sim}{z}(t)$ has a Rayleigh density [see (6-70)]. [Note. Exactly, $\underset{\sim}{z}(t)$ is a discrete-type RV taking the values $s\sqrt{m^2+n^2}$ where m and n are integers]. The product $f_z(z,t)dz$ equals approximately the probability that $\underset{\sim}{z}(t)$ is between z and $z+dz$ provided that $dz >> T$.

10-3 The voltage $\underline{v}(t)$ is the output of a system with input $\underline{n}_e(t)$ and system function

$$H_1(s) = \frac{1}{LCs^2 + RCs + 1}$$

Hence,

$$S_v(\omega) = S_{n_e}(\omega)|H_1(j\omega)|^2 = \frac{2kTR}{(1-\omega^2 LC)^2 + R^2C^2\omega^2}$$

Furthermore,

$$Z_{ab}(s) = \frac{R + Ls}{LCs^2 + RCs + 1} \qquad \underline{Re}\ Z_{ab}(j\omega) = \frac{R}{(1-\omega^2 LC)^2 + R^2C^2\omega^2}$$

in agreement with (10-75).

The current $\underline{i}(t)$ is the output of a system with input $\underline{n}_e(t)$ and system function

$$H_2(s) = \frac{1}{R + Ls}$$

Hence,

$$S_i(\omega) = S_{n_e}(\omega)|H_2\ (j\omega)|^2 = \frac{2kTR}{R^2 + \omega^2 L^2}$$

Furthermore (short circuit admittance)

$$Y_{ab}(s) = \frac{1}{R + LS} \qquad \underline{Re}Y_{ab}(j\omega) = \frac{2kTR}{R^2 + L^2\omega^2}$$

in agreement with (10-78).

10-4 The equation $m\underline{x}''(t) + f\underline{x}'(t) = \underline{F}(t)$ specifies a system with

$$H(s) = \frac{1}{ms^2 + fs} \qquad h(t) = \frac{1}{f}(1 - e^{-ft/m})U(t)$$

and (9-100) yields

$$E\{\underline{x}^2(t)\} = \frac{2kTf}{f^2}\int_0^t (1 - e^{-2\alpha\tau})^2\, d\tau \qquad \alpha = \frac{f}{2m}$$

10-5 As in Example 12-2, a and b are such that

$$\underset{\sim}{x}(\tau) - a\,\underset{\sim}{x}(0) - b\underset{\sim}{v}(0) \perp \underset{\sim}{x}(0),\ \underset{\sim}{v}(0)$$

This yields

$$R_{xx}(\tau) = aR_{xx}(0) + b\,R_{xv}(0) \qquad \text{(i)}$$

$$R_{xv}(\tau) = aR_{xv}(0) + b\,R_{vv}(0)$$

where [see (10-163)]

$$R_{xx}(\tau) = A\,e^{-\alpha\tau}(\cos\beta\tau + \frac{\alpha}{\beta}\sin\beta\tau) \qquad \tau > 0$$

$$R_{xv}(\tau) = -R'_{xx}(\tau) = A\quad e^{-\alpha\tau}(\sin\beta\tau)\,\frac{\alpha^2+\beta^2}{\beta}$$

$$R_{vv}(\tau) = R'_{xv}(\tau) = A\quad e^{-\alpha\tau}(\cos\beta\tau - \frac{\alpha}{\beta}\sin\beta\tau)\,\frac{\alpha^2+\beta^2}{\beta^2}$$

Inserting into (i) and solving, we obtain

$$a = e^{-\alpha\tau}(\cos\beta\tau + \frac{\alpha}{\beta}\sin\beta\tau)$$

$$b = \frac{1}{\beta}e^{-\alpha\tau}\sin\beta\tau$$

Finally,

$$P = E\{[\underset{\sim}{x}(t) - a\,\underset{\sim}{x}(0) - b\,\underset{\sim}{v}(0)]\underset{\sim}{x}(t)\} = R_{xx}(0) - a\,R_{xx}(t) - b\,R_{xv}(t)$$

$$= \frac{2kTf}{m^2}\left[1 - e^{-2\alpha t}(1 + \frac{2\alpha^2}{\beta}\sin^2\beta t + \frac{\alpha}{\beta}\sin 2\beta t)\right]$$

10-6 If $\underset{\sim}{x}(t) = \underset{\sim}{w}(t^2)$ then [see (10-70)]

$$R_x(t_1,t_2) = E\{\underset{\sim}{w}(t_1^2)\underset{\sim}{w}(t_2^2)\} = \alpha t_1^2$$

If $\underset{\sim}{y}(t) = \underset{\sim}{w}^2(t)$ then [see (6-197)]

$$R_y(t_1,t_2) = E\{\underset{\sim}{w}^2(t_1)\underset{\sim}{w}^2(t_2)\}$$

$$= E\underset{\sim}{w}^2(t_1)E\{\underset{\sim}{w}^2(t_2)\} + 2\,E^2\{\underset{\sim}{w}(t_1)\underset{\sim}{w}(t_2)\} = \alpha^2 t_1 t_2 + 2\alpha^2 t_1^2$$

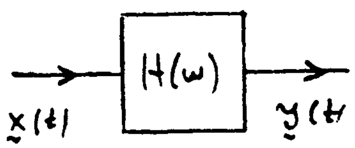
10-7 From (10-112):

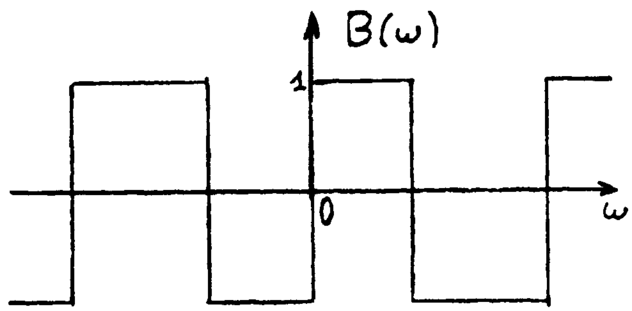
$$\eta_s = 3\int_0^{10} 2\,dt = 60 \qquad \sigma_s^2 = 3\int_0^{10} 4dt = 120 \qquad E\{\underset{\sim}{s}^2\} = 3720$$

$\underset{\sim}{s}(7) = 0$ if there are no points in the interval (7-10, 7). The number of points in this interval is a Poission RV with parameter $10\lambda = 30$. Hence, $P\{\underset{\sim}{s}(7) = 0\} = e^{-30}$.

10-8

$H(\omega) = j\,B(\omega)$





From the assumption: $S_{xx}(\omega) = S_{yy}(\omega) \qquad S_{xy}(-\omega) = -S_{xy}(\omega)$

From (9-148): $S_{yy}(\omega) = S_{xx}(\omega)|H(\omega)|^2 \qquad S_{xy}(\omega) = S_{xx}(\omega)H^*(\omega)$

Combining, we obtain

$$|H(\omega)|^2 = 1 \qquad H(-\omega) = -H(\omega)$$

Since h(t) is real, the second equation yields $H(\omega) = jB(\omega)$ and from the first it follows that

$$|B(\omega)| = 1$$

as in the figure.

10-9 With $\underset{\sim}{i}(t) = \underset{\sim}{a}(t)$, $\underset{\sim}{q}(t) = \underset{\sim}{b}(t)$, (11-63) yields

$$S_i(\omega) = S_q(\omega) \qquad S_{iq}(\omega) = -S_{qi}(\omega) = S_{qi}(-\omega)$$

Hence [see (11-75) and (11-82)],

$$S_w(\omega) = 2\,S_i(\omega) + 2j\,S_{qi}(\omega)$$

$$S_w(-\omega) = 2\,S_i(\omega) - 2j\,S_{qi}(\omega)$$

Adding and subtracting, we obtain

$$4\,S_i(\omega) = S_w(\omega) + S_w(-\omega) \qquad 4j\,S_{iq}(\omega) = S_w(-\omega) - S_w(\omega)$$

10-10 From (10-133)

$$\underset{\sim}{x}(t) = \underset{\sim}{\mathrm{Re}}\,[\underset{\sim}{w}(t)e^{j\omega_0 t}]$$

$$\underset{\sim}{x}(t-\tau) = \underset{\sim}{\mathrm{Re}}[\underset{\sim}{w}_{\tau}(t)e^{j\omega_0 t}] = \underset{\sim}{\mathrm{Re}}[\underset{\sim}{w}(t-\tau)e^{j\omega_0(t-\tau)}]$$

$$\underset{\sim}{w}_{\tau}(t) = \underset{\sim}{w}(t-\tau)e^{-j\omega_0\tau}$$

10-11 $$R''_x(\tau) \longleftrightarrow -\omega^2 S_x(\omega)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2 S_x(\omega)\,d\omega = -R''_x(0)$$

and with ω_0 the optimum carrier frequency, (10-150) yields

$$E\{|\underset{\sim}{w}'(t)|^2\} = \frac{M}{2\pi} = -2R''_x(0) - 2\omega_0^2 R_x(0)$$

10-12 From the stationarity of the process $\underset{\sim}{x}(t)\cos\omega t + \underset{\sim}{y}(t)\sin\omega t$ it follows that [see (10-130)]

$$C_{xx}(\tau) = C_{yy}(\tau) \qquad C_{xy} = -C_{yx}(\tau) \tag{i}$$

Using these identities, we shall express the joint density f(X,Y) of the 2n RVs

$$\underset{\sim}{X} = [\underset{\sim}{x}(t_1), \ldots, \underset{\sim}{x}(t_n)] \qquad \underset{\sim}{Y} = [\underset{\sim}{y}(t_1), \ldots, \underset{\sim}{y}(t_n)]$$

in terms of the covariance matrix C_{ZZ} of the complex vector $\underset{\sim}{Z} = \underset{\sim}{X} + j\underset{\sim}{Y}$. From (i) it follows that

$$E\{x(t_i)\underset{\sim}{x}(t_j)\} = E\{\underset{\sim}{y}(t_i)\underset{\sim}{y}(t_j)\} \qquad E\{\underset{\sim}{x}(t_i)\underset{\sim}{y}(t_j)\} = -E\{\underset{\sim}{y}(t_i)\underset{\sim}{x}(t_j)\}$$

This yields

$C_{XX} = C_{YY}$, and $C_{XY} = -C_{YX}$; hence, f(X,Y) is given by (8-62).

10-13 The signal $\underset{\sim}{c}(t) = f(t)$ is an extreme case of a cyclostationary process as in (10-178) with

$$h(t) = \begin{cases} f(t) & 0 \le t < T \\ 0 & \text{otherwise} \end{cases} \longleftrightarrow H(\omega) = \int_0^T f(t)e^{-j\omega t}dt$$

and $\underset{\sim}{c}_m = 1$, $R[m] = 1$. Hence [see (10A-2)]

$$\sum_{m=-\infty}^{\infty} R_m e^{-jm\omega T} = \sum_{m=-\infty}^{\infty} e^{-jm\omega T} = T\sum_{m=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{T}m)$$

From the above and (10-180) it follows that the process $\underset{\sim}{x}(t) = f(t - \underset{\sim}{\theta})$ is stationary with power spectrum

$$S(\omega) = \left| \int_0^T f(t)e^{-j\omega t}dt \right|^2 \sum_{m=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{T}m)$$

10-14
The process

$$\mathbf{y}_N(t) = \mathbf{x}(t+\tau) - \sum_{n=-N}^{N} \mathbf{x}(t+nT)\,\frac{\sin\sigma(\tau-nT)}{\sigma(\tau-nT)}$$

is the output of a system with input $\mathbf{x}(t)$ and system function

$$H_N(\omega) = e^{j\omega\tau} - \sum_{n=-N}^{N} \frac{\sin\sigma(\tau-nT)}{\sigma(\tau-nT)}\, e^{jnT\omega}$$

Furthermore, $\boldsymbol{\varepsilon}_N(\tau) = \mathbf{y}_N(0)$, hence [see (9-153)]

$$E\{\boldsymbol{\varepsilon}_N^2(\tau)\} = E\{\mathbf{y}^2(0)\} = \frac{1}{2\pi}\int_{\infty}^{\infty} S(\omega)\,|H_N(\omega)|^2 d\omega \qquad \text{(i)}$$

The function $H_N(\omega)$ is the truncation error in the Fourier series expansion of $e^{j\omega\tau}$ in the interval $(-\sigma,\sigma)$. Hence, for $N > N_0$

$$|H_N(\omega)| < \varepsilon \qquad\qquad |\omega| < \sigma$$

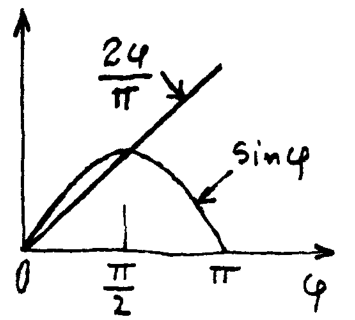
From this and (i) it follows that, if $S(\omega) = 0$ for $|\omega| < \sigma$, then

$$E\{\boldsymbol{\varepsilon}_N^2(\tau)\} = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S(\omega)\,|H_N(\omega)|^2 d\omega < \varepsilon\, R(0) \qquad N > N_0$$

10-15 [see after (10-195)]

$$R(0) - R(\tau) = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S(\omega)(1-\cos\omega\tau)\,d\omega$$

$$\le \frac{\tau^2}{4\pi}\int_{-\sigma}^{\sigma}\omega^2 S(\omega)\,d\omega = \frac{-\tau^2}{2}R''(0)$$



Furthermore, since

$$\sin\phi \ge \frac{2\phi}{\pi} \qquad 0 \le \phi \le \frac{\pi}{2}$$

we obtain

$$R(0) - R(\tau) = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S(\omega)\, 2\sin^2\frac{\omega\tau}{2}\, d\omega$$

$$\ge \frac{2\tau^2}{\pi^2}\frac{1}{2\pi}\int_{-\sigma}^{\sigma}\omega^2 S(\omega)\,d\omega = \frac{-2\tau^2}{\pi^2}R''(0)$$

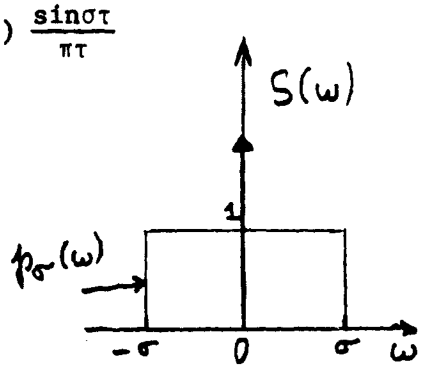
10-16 With $T = \pi/\sigma$

$$R(mT) = E\{\underset{\sim}{x}(nT+mT)\underset{\sim}{x}(nT)\} = \begin{cases} I & m = 0 \\ \eta^2 & m \ne 0 \end{cases}$$

Hence [see (10-196)]

$$R(\tau) = \sum_{m=-\infty}^{\infty} R(mT)\frac{\sin\sigma(\tau-mT)}{\sigma(\tau-mT)} = \eta^2 + (I-\eta^2)\frac{\sin\sigma\tau}{\pi\tau}$$

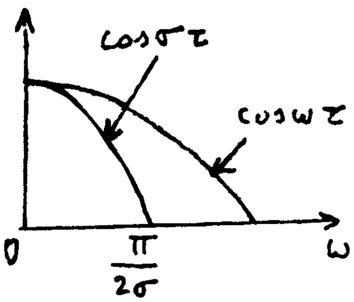
$$S(\omega) = 2\pi\eta^2\delta(\omega) + 2\pi(I-\eta^2)p_\sigma(\omega)$$



10-17 Given $E\{\underset{\sim}{x}(n+m)\underset{\sim}{x}(n)\} = N\delta[m]$

This is a special case of Prob. 10-16 with $\eta = 0$, $I = N$.

10-18 If $|\tau| < \pi/2\sigma$, then



$$\cos\omega\tau \geq \cos\sigma\tau \qquad |\omega| \leq \sigma$$

$$R(\tau) = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S(\omega)\cos\omega\tau\, d\omega$$

$$\geq \frac{\cos\sigma\tau}{2\pi}\int_{-\sigma}^{\sigma} S(\omega)\, d\omega = R(0)\cos\sigma\tau$$

10-19 From (10-133) with $c = \sigma$

$$P_1(\omega,\tau) + j\omega\, P_2(\omega,\tau) = 1$$

$$P_1(\omega,\tau) + j(\omega+\tau)P_2(\omega,\tau) = e^{j\sigma\tau}$$

Hence,

$$P_1(\omega,\tau) = 1 - \frac{\omega}{\sigma}(e^{j\sigma\tau} - 1) \qquad P_2(\omega,\tau) = \frac{1}{j\sigma}(e^{j\sigma\tau} - 1)$$

Inserting into (11-141), we obtain

$$p_1(\tau) = \frac{4\sin^2(\sigma\tau/2)}{\sigma^2\tau^2} \qquad p_2(\tau) = \frac{4\sin^2(\sigma\tau/2)}{\sigma^2\tau}$$

and with $t = 0$, the desired result follows from (10-206) because $\bar{T} = 2T$ and

$$\sin^2\frac{\sigma(\tau-2nT)}{2} = \sin^2\left(\frac{\sigma\tau}{2} - n\pi\right) = \sin^2\frac{\sigma\tau}{2}$$

10-20 As in (10-213)

$$\underset{\sim}{P}(\omega) = \frac{1}{\lambda}\int_{-a}^{a} \cos\omega t \; \underset{\sim}{z}(t) \cos\omega_c t \, dt$$

$$E\{\underset{\sim}{P}(\omega)\} = \int_{-a}^{a} \cos\omega t \cos\omega_c t \, dt$$

$$\sigma^2_{P(\omega)} = \frac{1}{\lambda}\int_{-a}^{a} \cos^2\omega_c t_2 \cos^2 \omega t_2 \, dt_2$$

10-21 We shall show that if

$$\underset{\sim}{X}_c(\omega) = \frac{1}{\lambda}\sum_{|t_i|<c} \underset{\sim}{x}(t_i)e^{-j\omega t_i} = \frac{1}{\lambda}\int_{-a}^{a} \underset{\sim}{x}(t)\underset{\sim}{z}(t)e^{-j\omega t}dt$$

where $\underset{\sim}{z}(t) = \sum \delta(t-\underset{\sim}{t}_i)$ is a Poisson impulse train, then

$$E\{|\underset{\sim}{X}_c(\omega)|^2\} \simeq 2cS_x(\omega) + \frac{2c}{\lambda} R_x(0)$$

Proof

Since $R_z(\tau) = \lambda^2 + \lambda\delta(\tau)$, it follows that

$$E\left\{|\underset{\sim}{X}_c(\omega)|^2\right\} = \frac{1}{\lambda^2}\int_{-c}^{c}\int_{-c}^{c} R_x(t_1-t_2)e^{-j\omega(t_1-t_2)}dt_1dt_2$$

$$= \int_{-c}^{c} e^{j\omega t_2}\int_{-c}^{c} R_x(t_1-t_2)e^{-j\omega t_1}dt_1dt_2 + \frac{1}{\lambda}\int_{-c}^{c} R_x(0)dt_2$$

If $\int_{-\infty}^{\infty} |R_x(\tau)| < \infty$ then for sufficient large c, the inner integral on the right is nearly equal to $S_x(\omega)^{-j\omega t_2}$ and (i) follows.

10-22 $E\{\underset{\sim}{z}(t)\} = g(t) \qquad E\{\underset{\sim}{w}(t)\} = g(t) - g(T)t/T = g(t)$

$$\underset{\sim}{w}(t) = (1 - \frac{t}{T}) \int_0^t \underset{\sim}{x}(\alpha)d\alpha - \frac{t}{T} \int_t^T \underset{\sim}{x}(\alpha)d\alpha$$

The above two integrals are uncorrelated because $\underset{\sim}{n}(t)$ is white noise. Hence, as in Example 9-5

$$\sigma_w^2 = (1 - \frac{t}{T})^2 Nt + \frac{t^2}{T^2} N(T - t) = Nt(1 - \frac{t}{T})$$

Note The above shows that the information that $g(T) = 0$ can be used to improve the estimate of $g(t)$. Indeed, if we use $\underset{\sim}{w}(t)$ instead of $\underset{\sim}{z}(t)$ for the estimate of $g(t)$ in terms of the data $\underset{\sim}{x}(t)$, the variance is reduced from Nt to $Nt(1-t/T)$.

10-23 (a) Since $|\sum_i a_i b_i| \le \sum |a_i||b_i|$, it suffices to assume that the numbers a_i and b_i are real. The quadratic

$$I(z) = \sum_i (a_i - z\, b_i)^2 = z^2 \sum_i b_i^2 - 2z \sum_i a_i b_i + \sum_i a_i^2$$

is nonnegative for every real z, hence, its discriminant cannot be positive. This yields (i).

(b) With $f[n]$ and $R_\nu[m] = S_o\delta[m]$ as in Prob. 10-24a (white noise)

$$y_f[n_0] = \sum h[n] f[n_0 - n] \qquad \underset{\sim}{y}_\nu[n] = \sum h[n] \underset{\sim}{\nu}[n]$$

$$E\{\underset{\sim}{y}_\nu^2[n]\} = S_0\, \rho[0] = S_o \sum |h[n]|^2$$

[see (9-213)] And (i) yields

$$\frac{y_f^2[n_0]}{E\{\underset{\sim}{y}_\nu^2[n]\}} = \frac{|\sum h[n] f[n_0 - n]|^2}{S_0 \sum h^2[n]} \le \frac{1}{S_0} \sum |h[n]|^2$$

with equality iff $h[n] = kf^*[n_0 - n]$.

10-24 (a) Given $F(z)$ and $S_{\nu}(\omega) = S_0 \equiv$ constant. The z transform of $y_f[n]$ equals $F(z)H(z)$. Hence, [see (9-109)]

$$y_f[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{j\omega T})H(e^{j\omega T})e^{jn\omega T}d\omega$$

$$\frac{y_f^2[n]}{E\{\mathbf{y}_\nu^2[n]\}} = \frac{\left|\int_{-\pi}^{\pi} F(e^{j\omega T})H(e^{j\omega T})d\omega\right|^2}{S_0 \int_{-\pi}^{\pi}\left|H(e^{j\ \ T})\right|^2 d\omega}$$

$$\leq \frac{1}{S_0}\int_{\pi}^{\pi}\left|F(e^{j\omega T})\right|^2 d\omega$$

The last inequality follows from Schwarz's inequality with equality iff

$$H(e^{j\omega T}) = kF^*(e^{j\omega T}) = kF(e^{-j\omega T}), \text{ i.e., iff } H(z) = kF(z^{-1})$$

(b) Given arbitrary $R_\nu[m]$, $F(z)$, and the form of $H(z)$ (FIR); to find the coefficients a_m of $H(z)$. In this case

$$y_f[n] = a_0 f[n] + a_1 f[n-1] + \cdots + a_N f[n-N]$$

$$\mathbf{y}_\nu[n] = a_0\boldsymbol{\nu}[n] + a_1\boldsymbol{\nu}[n-1] + \cdots + a_N\boldsymbol{\nu}[n-N]$$

To maximize the signal-to-noise ratio it suffices to minimize

$$E\{\mathbf{y}_\nu^2[n]\} = \sum_{k,r=0}^{N} a_k a_r R_\nu[k-r]$$

subject to the constraint that the sum

$$y_f[0] = a_0 f[0] + a_1 f[-1] + \cdots + a_N f[-N]$$

is constant. With λ a constant (Lagrange multiplier), we minimize the sum

$$I = \sum_{k,r=0}^{N} a_k a_r R\,[k-r] - \lambda\left[\sum_{k=0}^{N} a_k f[-k] - y_f[0]\right]$$

this yields the system

$$\frac{\partial I}{\partial a_k} = 0 = \sum_{r=0}^{N}\left[a_r R_\nu[k-r] - \lambda f[-k]\right] \qquad k = 0,..,N$$

whose solution yields a_k.

10-25 $B = A|H(\omega_0)| = \dfrac{A}{\sqrt{\alpha^2+\omega_0^2}}$ $\qquad S_{y_n}(\omega) = \dfrac{N}{\alpha^2+\omega^2}$

$$R_{y_n}(\tau) = \frac{N}{2\alpha}\; e^{-\alpha|\tau|} \qquad E\{\underset{\sim}{y}_n^2(t)\} = R_{y_n}(0) = \frac{N}{2\alpha}$$

$$\frac{B^2}{E\{\underset{\sim}{y}_n^2(t)\}} = \frac{2A^2}{N}\,\frac{\alpha}{\alpha^2+\omega_0^2} \qquad \text{Max. if } \alpha = \omega_0$$

10-26 Since $H(\omega)$ is determined within a constant factor, we can sssume that the response $y_f(t_o)$ of the optimum $H(\omega)$ due to $f(t)$ is constant:

$$y_f(t_o) = \sum_{i=0}^{m} a_i\, f(t_o - iT) = c \qquad \text{(i)}$$

Our problem is to minimize the variance

$$V = E\{\underset{\sim}{y}_\nu^2(t)\} = \sum_{n=0}^{m} a_n \sum_{i=0}^{m} a_i\, R(nT-iT) \qquad \text{(ii)}$$

of $\underset{\sim}{y}_\nu(t)$ subject to the constraint (i). This yields the system

$$\frac{\partial V}{\partial a_n} = \sum_{i=0}^{m} a_i\, R(nT-iT) - kf(t_o - nT) = 0$$

where k is a constant (lagrange multiplier). With a_n so determined, we conclude from (ii) that

$$V = \sum_{n=0}^{m} ka_n\, f(t_o-nT) = ky_f(t_o) \qquad r^2 = \frac{y_f^2(t_o)}{ky_f(t_o)}$$

10-27 $R_{yyy}(\mu,\nu) = E\{\underset{\sim}{x}(t+\mu)+c[\,[\underset{\sim}{x}(t+\nu)+c]\,[\underset{\sim}{x}(t)+c]\} = R(\mu,\nu) + cR(\mu) + cR(\nu) + cR(\mu-\nu) + c^3$

because $E\{\underset{\sim}{x}(t)\} = 0$. Furthermore,

$$R(\mu) \leftrightarrow 2\pi S(u)\delta(v) \qquad R(\nu) = 2\pi\delta(u)S(v) \qquad c^3 \leftrightarrow 4\pi^2\delta(u)\delta(v)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R(\mu-\nu)e^{-j(u\mu+v\nu)}d\mu d\nu = \int_{-\infty}^{\infty} R(\tau)e^{-ju\tau}d\tau \int_{-\infty}^{\infty} e^{-j(u+v)\nu}d\nu = 2\pi S(u)\delta(u+v)$$

10-28 We shall use the equations $E\{\underset{\sim}{x}(t)\} = 0$, $E\{\underset{\sim}{x}^2(t)\} = \lambda t$. Suppose that $t_1 < t_2 < t_3$. Clearly,

$$\underset{\sim}{x}(t_2) = \underset{\sim}{x}(t_1) + [\underset{\sim}{x}(t_2)-\underset{\sim}{x}(t_1)]$$

$$\underset{\sim}{x}(t_3) = \underset{\sim}{x}(t_1) + [\underset{\sim}{x}(t_2) - \underset{\sim}{x}(t_1)] + [\underset{\sim}{x}(t_3)-\underset{\sim}{x}(t_2)] \qquad \text{(i)}$$

Inserting into the product $\underset{\sim}{x}(t_1)\underset{\sim}{x}(t_2)\underset{\sim}{x}(t_3)$ and using the identity $E\{\underset{\sim}{x}(t_i)-\underset{\sim}{x}(t_j)\} = 0$ and the independence of the three terms on the right of (i), we obtain

$$E\{\underset{\sim}{x}(t_1)\underset{\sim}{x}(t_2)\underset{\sim}{x}(t_3)\} = E\{\underset{\sim}{x}^3(t_1)\} = \lambda t_1 = \lambda \min(t_1, t_2, t_3)$$

Since $\underset{\sim}{z}(t) = \underset{\sim}{x}'(t)$, we conclude from (9-120)-(9-122) that

$$R_{zzz}(t_1,t_2,t_3) = \frac{\partial^3 R_{xxx}(t_1,t_2,t_3)}{\partial t_1 \partial t_2 \partial t_3} = \lambda \frac{\partial^3 \min(t_1,t_2,t_3)}{\partial t_1 \partial t_2 \partial t_3}$$

It suffices therefore to show that the right side equals $\lambda\delta(t_1-t_2)\delta(t_1-t_2)$. This is a consequence of the following:

$$\frac{\partial \min(t_1,t_2,t_3)}{\partial t_3} = t_1 U(t_2-t_1)\delta(t_3-t_1) + t_2\, U(t_1-t_2)\, \delta(t_3-t_2)$$

$$+ U(t_1-t_3)U(t_2-t_3) - t_3\delta(t_1-t_3)U(t_2-t_3) - t_3 U(t_1-t_3)\delta(t_2-t_3)$$

$$= U(t_1-t_3)U(t_2-t_3)$$

because $t_i\delta(t_i-t_j) = t_j\delta(t_j-t_i)$. Hence,

$$\frac{\partial^2 \min(t_1,t_2,t_3)}{\partial t_2 \partial t_3} = U(t_1-t_3)\delta(t_2-t_3) \qquad \frac{\partial^2 \min(t_1,t_2,t_3)}{\partial t_1 \delta t_2 \partial t_3} = \delta(t_1-t_2)\delta(t_1-t_3)$$

10-29 See outline given in text.

CHAPTER 11

11-1 $$S_x(z) = \frac{5-2(z+1/z)}{10-3(z+1/z)} = \frac{2}{3} + \frac{5/9}{10/3 - (z+1/z)}$$

$$R[m] = \frac{2}{3} + \frac{5}{18}\,3^{-|m|} \qquad \Gamma(z) = \frac{3z-1}{2z-1}$$

11-2 $$S_x(s) = \frac{s^4+64}{s^4-10s^2+9} = \frac{s^2+4s+8}{s^2+4s+3}\,\frac{s^2-4s+8}{s^2-4s+3}$$

$$L(s) = \frac{s^2+4s+8}{s^2+4s+3}$$

11-3 First proof

$$\mathbf{s}[n] = \sum_{k=0}^{\infty} \ell[n]\mathbf{i}[n-k] \qquad E\{\mathbf{x}^2[n]\} = \sum_{k=0}^{\infty} \ell^2[k]$$

Second proof

$$S(z) = L(z)L(1/z) \qquad R[m] = \ell[m] * \ell[-m] = \sum_{k=0}^{\infty} \ell[k]\ell[k-m]$$

$$R[0] = \sum_{k=0}^{\infty} \ell^2[k]$$

11-4 (a) This is a special case of (11-22) and (11-23).

(b) From (a) it follows that

$$R''_{yx}(\tau) + 3\,R'_{yx}(\tau) + 2\,R_{yx}(\tau) = q\delta(\tau)$$

Since $R_{xx}(\tau) = 0$ for $\tau < 0$, the above shows that

$$R_{yx}(\tau) = 0 \text{ for } \tau \le 0^- \qquad R'_{yx}(0^-) = 0$$

Furthermore,

$$S_{yx}(s) = \frac{q}{s^2 + 3s + 2}$$

hence (initial value theorem)

$$R_{yx}(0^+) = \lim_{s\to\infty} s\ S_{yx}(s) = 0 \qquad R'_{yx}(0^+) = \lim_{s\to\infty} s^2\ S_{yx}(s) = q$$

Similarly,

$$R''_{yy}(\tau) + 3\,R'_{yy}(\tau) + 2\,R_{yy}(\tau) = R_{xy}(\tau) = R_{yx}(-\tau) = 0 \quad \text{for } \tau > 0$$

$$S_{yy}(s) = \frac{q}{(s^2 + 3s + 2)(s^2 - 3s + 2)} = \frac{qs/12 + q/4}{s^2 + 3s + 2} + \frac{-qs/12 + q/4}{s^2 - 3s + 2}$$

$$S^+_{yy}(s) = \frac{qs/12 + q/4}{s^2 + 3s + 2}$$

$$R^+_{yy}(0^+) = R_{yy}(0) = \lim_{s\to\infty} s^2\ S^+_{yy}(s) = \frac{q}{12}$$

$$R'_{yy}(0) = \lim_{s\to\infty} s\ [s\ S^+_{yy}(s) - \frac{q}{12}] = 0$$

11-5 $$S_x(z) = S_s(z) + S_\nu(z) = \frac{1}{D(z)} + q = \frac{1 + qD(z)}{D(z)}$$

If $R_s[m] = 2^{-|m|}$ and $S_\nu(z) = 5$, then (see Example 9-31)

$$S_s(z) = \frac{1.5}{2.5 - (z^{-1} + z)} \qquad S_x(z) = \frac{5 - 14z^{-1} + 5z^{-2}}{1 - 2.5z^{-1} + z^{-2}}$$

11-6 The process

$$\underset{\sim}{y}[n] = \frac{1}{n}\sum_{k=1}^{n} \underset{\sim}{x}(nT+kT)$$

is the output of a system with input $\underset{\sim}{x}[n]$ and system function

$$H(z) = \frac{1}{n}\sum_{k=1}^{n} z^k$$

Furthermore, $\underset{\sim}{s} = \underset{\sim}{y}[0]$ and

$$n^2\left|H(e^{j\omega T})\right|^2 = \left|\sum_{k=1}^{n} e^{jk\omega T}\right|^2$$

$$= \left|\frac{e^{j\omega T} - e^{j(n+1)\omega T}}{1 - e^{j\omega T}}\right|^2 = \frac{\sin^2 n\omega T/2}{\sin^2 \omega T/2}$$

Hence [see (9-51)]

$$E\{\underset{\sim}{s}^2\} = R_y[0] = \frac{1}{2\pi n^2}\int_{-\infty}^{\infty} S_x(\omega)\,\frac{\sin^2 n\omega T/2}{\sin^2 \omega T/2}\,d\omega$$

11-7 Since $R(t_1,t_2) = e^{-c|t_1-t_2|}$, (12-58) yields

$$\int_{-a}^{t_1} e^{-c(t_1-t_2)}\phi(t_2)dt_2 + \int_{t_1}^{a} e^{c(t_1-t_2)}\phi(t_2)dt_2 = \lambda\phi(t_1) \qquad \text{(i)}$$

Differentiating twice and using (i) we obtain (omitting details)

$$\lambda\phi''(t) + (2c - \lambda c^2)\phi(t) = 0$$

Hence,

$$\phi(t) = \beta\cos \omega t \quad \text{and} \quad \phi(t) = \beta'\cos\omega' t$$

To determine ω, we insert into (i). This yields

$$\frac{2c}{c^2+\omega^2} + \frac{\omega\sin a\omega - c\cos a\omega}{c^2+\omega^2}\; e^{-ac}\,(e^{ct} + e^{-ct}) = 2c\,\lambda\cos\omega t$$

This yields

$$\omega_n \sin a\omega_n - c\cos a\omega_n = 0 \qquad\qquad \lambda_n = \frac{2c}{c^2+\omega_n^2}$$

The constants β_n are determined from (normalization)

$$1 = \int_{-a}^{a} \beta_n^2 \cos^2\omega_n t\,dt \qquad\qquad \beta_n^2 = \frac{1}{a+c\lambda_n}$$

Similarly for $\beta_n' \sin\omega_n' t$.

11-8 As in (9-60)

$$E\{|\underset{\sim}{X}_T(\omega)|^2\} = \int_{-T/2}^{T/2} R(t_1-t_2)e^{-j\omega(t_1-t_2)}dt_1 dt_2$$

$$= \int_{-T}^{T} (T-|\tau|)R(\tau)\, e^{-j\omega\tau}d\tau$$

Differentiating with respect to T and using the fact that if

$$\phi(t) = \int_{-t}^{t} f(x;t)dx$$

then

$$\frac{d\phi(t)}{dt} = f(t;t) - f(-t,t) + \int_{-t}^{t} \frac{\partial f}{\partial t}(x,t)dx$$

we obtain

$$\frac{\partial E\{|\underset{\sim}{X}_T(\omega)|^2\}}{\partial T} = \int_{-T}^{T} R(\tau)e^{-j\omega\tau}d\tau = E\{\frac{\partial}{\partial T}|\underset{\sim}{X}_T(\omega)|^2\}$$

The above approaches $S(\omega)$ as $T \to \infty$.

11-9 $$E\{\underset{\sim}{X}(\omega)\} = \int_{-a}^{a} 5\cos 3t\, e^{-j\omega t}dt = \frac{5\sin a(\omega-3)}{\omega-3} + \frac{5\sin a(\omega+3)}{\omega+3}$$

Var. $\underset{\sim}{X}(\omega) = 2qa = 4a.$

11-10 $$E\{\underset{\sim}{X}(u)\underset{\sim}{X}(v)\} = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \sigma_n^2\delta[n-k]\; e^{-j(nu-kv)T}$$

$$= \sum_{n=-\infty}^{\infty} \sigma_n^2\; e^{-jn(u-v)T}$$

11-11 Shifting the origin, we set

$$\underset{\sim}{c}_n = \frac{1}{T}\int_{-T/2}^{T/2} \underset{\sim}{x}(t)e^{-jn\omega_o t}dt \qquad \beta_n(\alpha) = \frac{1}{T}\int_{-T/2}^{T/2} R(\tau-\alpha)e^{-jn\omega_o\tau}d\tau$$

(a) We shall show that if

$$\hat{\underset{\sim}{x}}(t) = \sum_{n=-\infty}^{\infty} \underset{\sim}{c}_n e^{jn\omega_o t} \text{ then } E\{|\underset{\sim}{x}(t)-\hat{\underset{\sim}{x}}(t)|^2\} = 0 \text{ for } |t| < T/2 \tag{i}$$

Proof $$E\{\underset{\sim}{c}_n\underset{\sim}{x}^*(\alpha)\} = \frac{1}{T}\int_{-T/2}^{T/2} E\{\underset{\sim}{x}(t)\underset{\sim}{x}^*(\alpha)\}e^{-jn\omega_o t}dt = \beta_n(\alpha)$$

The functions $\beta_n(\alpha)$ are the coefficients of the Fourier expansion of $R(\tau-\alpha)$:

$$R(\tau-\alpha) = \sum_{n=-\infty}^{\infty} \beta_n(\alpha)e^{jn\omega_o\tau} \qquad |\tau| < T/2 \tag{ii}$$

Hence

$$E\{\hat{\underset{\sim}{x}}(t)\underset{\sim}{x}^*(t)\} = \sum_{n=-\infty}^{\infty} E\{\underset{\sim}{c}_n\underset{\sim}{x}^*(t)\}^{jn\omega_o t} = \sum_{n=-\infty}^{\infty} \beta_n(t)e^{jn\omega_o t}$$

From (ii) it follows with $\tau = \alpha = t$ that the last sum equals R(0). Similarly, $E\{\hat{\underset{\sim}{x}}^*(t)\underset{\sim}{x}(t)\} =$ R(0) and (i) results.

(b) $E\{\underset{\sim}{c}_n\underset{\sim}{c}_m^*\} = \frac{1}{T}\int_{-T/2}^{T/2} E\{c_n\underset{\sim}{x}^*(t)\}e^{jn\omega_o t}dt = \frac{1}{T}\int_{-T/2}^{T/2} \beta_n(t)e^{jn\omega_o t}dt$

(c) If T is sufficiently large, then

$$T\beta_n(\alpha) = \int_{-T/2}^{T/2} R(\tau-\alpha)e^{-jn\omega_o\tau}d\tau \simeq S(n\omega_o)e^{-jn\omega_o\alpha}$$

$$E\{c_n c_m^*\} = \frac{S(n\omega_o)}{T^2}\int_{-T/2}^{T/2} e^{j(m-n)\omega_o\alpha}d\alpha = \begin{cases} S(n\omega_o)/T & m=n \\ 0 & m\neq n \end{cases}$$

Thus, for large T, the coefficients $\underset{\sim}{c}_n$ of an arbitrary WSS process are nearly orthogonal.

11-12 $$E\{\underset{\sim}{x}(t_1)\underset{\sim}{x}^*(t_2)\} = \frac{1}{4\pi^2} E\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\{\underset{\sim}{X}(u)\underset{\sim}{X}^*(v)\}e^{j(ut_1-vt_2)}dudv\right.$$

$$= \frac{1}{4\pi^2} E\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q(u)\delta(u-v)e^{j(ut_1-vt_2)}dudv = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} Q(u)e^{ju(t_1-t_2)}du\right.$$

This depends only on $\tau = t_1-t_2$:

$$R_{xx}(\tau) = \frac{1}{x_2}\int_{-\infty}^{\infty} Q(u)e^{ju\tau}du \qquad S_{xx}(\omega) = \frac{Q(\omega)}{2\pi}$$

11-13 Equations (11-79) can be written in the following form:

$$E\{\underset{\sim}{A}(u)\underset{\sim}{A}(v)\} = Q(u)\delta(u-v) = E\{\underset{\sim}{B}(u)\underset{\sim}{B}(v)\} \qquad E\{\underset{\sim}{A}(u)\underset{\sim}{B}(v)\} = 0$$

for $u \geq 0$, $v \geq 0$. We shall show that if the above is true and $E\{\underset{\sim}{A}(\omega)\} = E\{\underset{\sim}{B}(\omega)\} = 0$, then the process

$$\underset{\sim}{x}(t) = \frac{1}{\pi}\int_0^{\infty}\left[\underset{\sim}{A}(\omega)\cos\omega t - \underset{\sim}{B}(\omega)\sin\omega t\right] d\omega$$

is WSS.

<u>Proof</u> Clearly, $E\{\underset{\sim}{x}(t)\} = 0$ and

$E\{\underset{\sim}{x}(t+\tau)\underset{\sim}{x}(t)\}$

$$= \frac{1}{\pi^2}\int_0^\infty\int_0^\infty E\{\underset{\sim}{A}(u)\cos u(t+\tau) - \underset{\sim}{B}(u)\sin u(t+\tau)\}\ [\underset{\sim}{A}(v)\cos vt - B(v)\sin vt]\ du dv$$

$$= \frac{1}{\pi}\int_0^\infty\int_0^\infty Q(u)\delta(u-v)\ [\cos u(t+\tau)\cos vt + \sin u(t+\tau)\sin v(t)\ du dv\ ]\ dv du$$

$$= \frac{1}{\pi^2}\int_0^\infty Q(u)\ [\cos u(t+\tau)\cos u\, t + \sin u(t+\tau)\sin ut]\ du$$

$$= \frac{1}{\pi^2}\int_0^\infty Q(u)\cos u\tau du$$

From this and (9-136) it follows that $\underset{\sim}{x}(t)$ is WSS with $S_{xx}(\omega) = Q(\omega)/\pi$.

11-14 $\quad E\{\underset{\sim}{v}(t)\} = 0 \qquad E\{\underset{\sim}{X}_T(\omega)\} = \int_{-T}^{T} f(t)e^{-j\omega t}dt$

The above integral is the transform of the product $f(t)p_T(t)$, hence (frequency convolution theorem), it equals $F(\omega)*\sin T\omega/\pi\omega$.

$$\text{Var}\ \underset{\sim}{X}_T(\omega) = E\left\{ \left| \int_{-T}^{T} \underset{\sim}{\nu}(t)e^{-j\omega t}dt \right|^2 \right\}$$

The integral is the transform of the nonstationary white noise $\underset{\sim}{\nu}(t)p_T(t)$. The autocorrelation of this process equals $q(t_1)\delta(t_1-t_2)$ where $q(t) = qp_T(t)$. Hence, [see (11-69)]

$$\text{Var}\ \underset{\sim}{X}_T(\omega) = Q(0) = \int_{-T}^{T} q dt = 2qT$$

CHAPTER 12

12-1 $\underset{\sim}{x}(t) = 10 + \underset{\sim}{\nu}(t) \qquad R_\nu(\tau) = 2\,\delta(\tau) \qquad E\{\underset{\sim}{\nu}(t)\} = 0$

$$E\{\underset{\sim}{n}_T\} = E\{\underset{\sim}{x}(t)\} = 10 \qquad C_x(\tau) = 2\delta(\tau)$$

From (12-5)

$$\sigma^2_{n_T} = \frac{1}{2T}\int_{-T}^{T} C_x(\tau)\left(1 - \frac{|\tau|}{2T}\right) d\tau = \frac{1}{2T}\int_{-T}^{T} 2\delta(\tau)\;\left(1 - \frac{|\tau|}{2T}\right) d\tau = \frac{1}{T}$$

12-2 The process $\underset{\sim}{x}(t)$ is normal (note correction) and such that

$$F(x,x;\tau) \longrightarrow F^2(x) \quad \text{as} \quad \tau \to \infty \qquad (i)$$

We shall show that it is mean-ergodic. It suffices to show that [see (12-10)]

$$C(\tau) \longrightarrow 0 \qquad \text{as} \qquad \tau \to \infty$$

<u>Proof</u>. We can assume (scaling and centering) that $\eta = 0 \quad C(0) = 1$. With this assumption, the RVs $\underset{\sim}{x}(t+\tau)$ and $\underset{\sim}{x}(t)$ are $N(0,0;1,1;r)$ where $r = r(\tau) = C(\tau)$ is the autocovariance of $\underset{\sim}{x}(t)$. Hence,

$$f(x_1,x_2;\tau) = \frac{1}{2\pi\sqrt{1-r^2}} \exp\left\{-\frac{1}{2(1-r^2)}\left(x_1^2 - 2rx_1x_2 + x_2^2\right)\right\}$$

$$= \frac{1}{2\pi\sqrt{1-r^2}} \exp\left\{-\frac{1}{2(1-r^2)}(x_1 - rx_2)^2\right\} e^{-x_2^2/2}$$

Clearly, $f(x,y) = f(y,x)$, hence, (see figure)

$$F(x+dx,\; x+dx;\tau) - F(x,x,\tau) = 2\int_{-\infty}^{x} f(\xi,x)\, d\xi\, dx$$

$$= \frac{1}{\pi\sqrt{1-r^2}} \int_{-\infty}^{x} \exp\left\{-\frac{1}{2(1-r^2)}(\xi - xr)^2\right\} d\xi\; e^{-x^2/2}\, dx$$

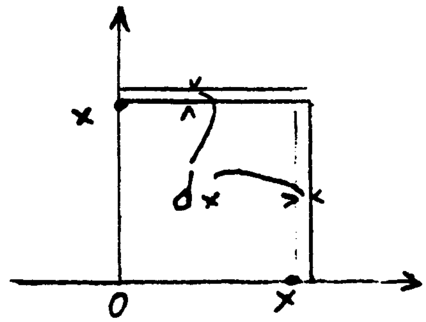
Furthermore,

$$F^2(x+dx) - F^2(x) = 2\,F(x)f(x)dx$$

From the above and (i) it follows that

$$G\left(\frac{x-rx}{\sqrt{1-r^2}}\right) \xrightarrow[\tau\to\infty]{} G(x)$$

Hence, $r(\tau) \longrightarrow 0$ as $\tau \to \infty$



12-3 If $\underset{\sim}{x}(t)$ is normal, then [see (12-27)]

$$C_{zz}(\tau) = R_x(\lambda+\tau)R_x(\lambda-\tau) + R_x^2(\tau) \qquad \underset{\sim}{z}(t) = \underset{\sim}{x}(t+\lambda)\underset{\sim}{x}(t)$$

If, therefore, $R_x(\tau) = 0$ for $|\tau| > a$, then $C_{zz}(\tau) = 0$ for $|\tau| > \lambda + a$.

12-4 If $\underset{\sim}{x}(t) = \underset{\sim}{a}\, e^{j(\omega t + \underset{\sim}{\phi})}$ then the time-average

$$\frac{1}{2T}\int_{-T}^{T} \underset{\sim}{x}(t+\tau)\underset{\sim}{x}^*(t)dt = e^{j\omega\tau}\,|\underset{\sim}{a}|^2$$

12-5 If $\underset{\sim}{z}(t) = \underset{\sim}{x}(t+\lambda)\underset{\sim}{y}(t)$, then

$$C_{zz}(\tau) = E\{\underset{\sim}{x}(t+\lambda+\tau)\underset{\sim}{y}(t+\tau)\underset{\sim}{x}(t+\lambda)\underset{\sim}{y}(t)\} - R_{xy}^2(\lambda)$$

and the result follows from (12-5).

12-6 The process $\bar{\underset{\sim}{x}}(t) = \underset{\sim}{x}(t - \underset{\sim}{\theta})$ is stationary with mean $\bar{\eta}$ and covariance $\bar{C}(\tau)$ given by [see (10-176) and (10-177)]

$$\bar{\eta} = \frac{1}{T}\int_0^T \eta(t)dt \qquad \bar{C}(\tau) = \frac{1}{T}\int_0^T C(t+\tau,t)dt$$

If $R(t+\tau,t) \to \eta^2(t)$ as $\tau \to \infty$ (note correction), then

$$C(t+\tau,t) \xrightarrow[\tau\to\infty]{} 0 \qquad \text{hence} \qquad \bar{C}(\tau) \xrightarrow[\tau\to\infty]{} 0$$

This shows that [see (12-10)], $\bar{\underset{\sim}{x}}(t)$ is ergodic, therefore,

$$\frac{1}{2c}\int_{-c}^{c} \bar{\underset{\sim}{x}}(t)dt = \frac{1}{2c}\int_{-c+\underset{\sim}{\theta}}^{c+\underset{\sim}{\theta}} \underset{\sim}{x}(t)dt \longrightarrow \bar{\eta}$$

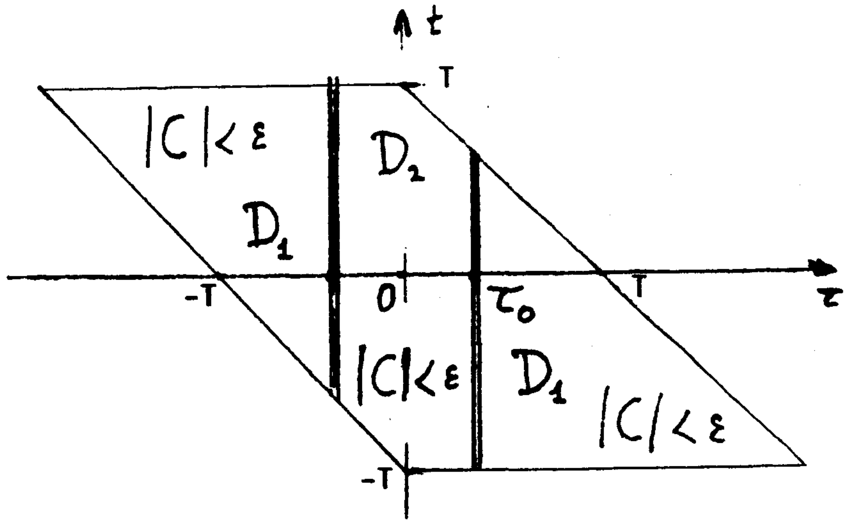
This yields the desired result because for a specific outcome, $\underset{\sim}{\theta}(\zeta) = \theta$ is a constant and

$$\lim_{c\to\infty} \frac{1}{2c}\int_{-c+\theta}^{c+\theta} \underset{\sim}{x}(t)dt = \lim_{c\to\infty} \frac{1}{2c}\int_{-c}^{c} \underset{\sim}{x}(t)dt$$

12-7 From (9-38) it follows that

$$4T^2\sigma_T^2 = \int_{-T}^{T}\int_{-T}^{T} C(t_1,t_2)dt_1,dt_2 = \iint_D C(t+\tau,t)d\tau dt$$

where D is the parallelogram in the figure. Given $\varepsilon > 0$, we can find a constant τ_o such that



$|C(t+\tau,t)| < \varepsilon$ for $|\tau| > \tau_o$ (uniform continuity). Furthermore, if $C(t,t) < P$ then

$$C^2(t_1,t_2) \le C(t_1,t_1)C(t_2,t_2) < P^2$$

Thus,

$|C| < \varepsilon$ in D_1 and $|C| < P$ in D_2

The area of D_1 is less than $4T^2$; the area of D_2 is less than $4\tau_o T$. Hence

$$\sigma_T^2 < \varepsilon + \frac{\tau_o}{T} \xrightarrow[T\to\infty]{} \varepsilon$$

And since ε is arbitrary, we conclude that $\sigma_T \to 0$.

12-8 It follows from (6-234) with $\underset{\sim}{x}(t) = \underset{\sim}{x}$, $\underset{\sim}{x}(t+\lambda) = \underset{\sim}{y}$

$$\eta_1 = \eta_2 = 0 \qquad \sigma_1^2 = \sigma_2^2 = R(0) \qquad r\sigma_1,\sigma_2 = R(\lambda)$$

(b) The proof is based on the identity

$$E\{\underset{\sim}{y}|M\} = E\,\{E\{\underset{\sim}{y}|\underset{\sim}{x}\}|M\} \qquad M = \{\underset{\sim}{x}(t)\in D\} \tag{i}$$

Proof Suppose first that D consists of the union of open intervals. In this case, if $x\in D$, then for small δ the interval $(x,x+\delta)$ is a subset of D, hence

$$\{x \le \underset{\sim}{x} < x + dx,\, M\} = \{x \le \underset{\sim}{x} < x + dx\}$$

for $x\in D$ and $\{\emptyset\}$ otherwise. This yields

$$f(x|M)\,dx = \frac{P\{x\le \underset{\sim}{x}<x+dx\}}{P(M)} = \frac{1}{p}\,f(x)dx \qquad p = P(M)$$

for $x\in D$ and 0 otherwise. Similarly, $f(x,y|M) = f(x,y)/p$ for $x\in D$ and 0 otherwise. From the above it follows that

$$E\{E\{\underset{\sim}{y}|\underset{\sim}{x}|M\}\} = \int_D \left(\int_{-\infty}^{\infty} yf(y|x)dy\right) f_x(x|M)dx$$

$$= \int_D \int_{-\infty}^{\infty} \frac{yf(x,y)f(x)}{f(x)\,p}\,dydx = \int_D \int_{-\infty}^{\infty} yf(x,y|M)dydx = E\{\underset{\sim}{y}|M\}$$

If D has isolated points, we replace each $x \in D$ by an open interval $(x-\varepsilon, x+\varepsilon)$ forming an open set D_ε. Clearly, $D_\varepsilon \to D$ as $\varepsilon \to 0$ and (i) follows if at the isolated points x_i of D, $E\{\underset{\sim}{y}|x_i\}$ is interpreted as a limit.

Since $E\{\underset{\sim}{x}(t+\lambda)|\underset{\sim}{x}(t)\} = R(\lambda)\underset{\sim}{x}(t)/R(0)$, (i) yields

$$E\{\underset{\sim}{x}(t+\lambda)|M\} = E\{E\{\underset{\sim}{x}(t+\lambda)|\underset{\sim}{x}(t)\}|M\} = E\left\{\frac{R(\lambda)}{R(0)}\underset{\sim}{x}(t)|M\right\} = \frac{R(\lambda)}{R(0)}\ \bar{x}$$

(c) We select for D the interval (a,b) and we form the samples $\underset{\sim}{x}(nT)$, $\underset{\sim}{x}(nT+\lambda)$ of a single realization of $\underset{\sim}{x}(t)$ retaining only the pairs $\underset{\sim}{x}(t_i)$, $\underset{\sim}{x}(t_i+\lambda)$ such that $a< \underset{\sim}{x}(t_i)<b$. Using (5-51), we obtain

$$E\{\underset{\sim}{x}(t+\lambda)|\ a < \underset{\sim}{x}(t) < b\} = \frac{R(\lambda)}{R(0)}\bar{x} \simeq \frac{1}{N}\sum_{i=1}^{N}\underset{\sim}{x}(t_i+\lambda)$$

where $\bar{x} = E\{\underset{\sim}{x}(t)|a<\underset{\sim}{x}(t)<b\}$. This approximation is satisfactory if N is large and $R(\tau)\simeq 0$ for $\tau > T$.

12-9 (a) From (7-61) with $E\{w(t)\} = C_{xy}(\lambda)$:

$$R_{ww}(\tau) = C_{xy}(\lambda+\tau)C_{xy}(\lambda-\tau) + C_{xx}(\tau)C_{yy}(\tau) + C_{xy}^2(\lambda) = C_{ww}(\tau) + C_{xy}^2(\lambda)$$

(b) It follows from (a) that if

$$C_{xx}(\tau) \to 0 \qquad C_{yy}(\tau) \to 0 \qquad C_{xy}(\sigma) \to 0$$

then $C_{ww}(\tau) \to 0$ as $|\tau| \to \infty$; hence [see (12-10)] the process $\underset{\sim}{x}(t)$ and $\underset{\sim}{y}(t)$ are covariance ergodic.

12-10 From (10B-1) with $g(x) = 1$:

$$\left| \int_a^b f(x)dx \right|^2 \le \int_a^b |f(x)|^2 dt \int_a^b 1^2 \times dx = (b-a)\int_a^b |f(x)|^2 dx$$

12-11 We use as estimate of η the time average $\hat{\eta}_T$ in (12-1): As we know (see Example 12-4)

$$E\{\eta_T\} = \eta \qquad \sigma_T^2 = \frac{5}{2T}$$

We wish to find ε such that

$$P\{\eta - \varepsilon < \eta_T < \eta + \varepsilon\} = 0.95$$

(a) From (5-88):

$$0.95 = P\{|\eta_T - \eta| \le \varepsilon\} \le 1 - \frac{\sigma_T^2}{\varepsilon^2} \qquad \varepsilon = \varepsilon_a \le \frac{\sigma_T}{\sqrt{0.05}} = \frac{50}{T}$$

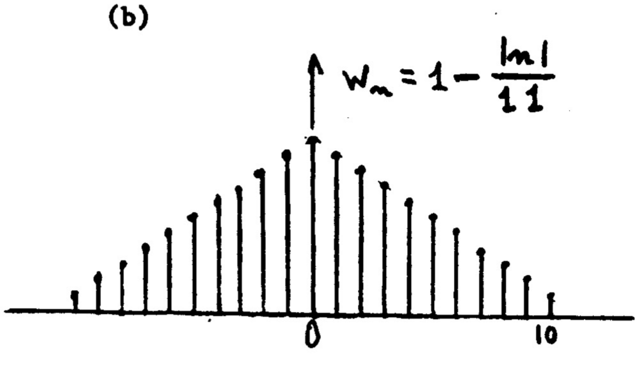
(b) If $\nu(t)$ is normal, then η_T is normal; hence,

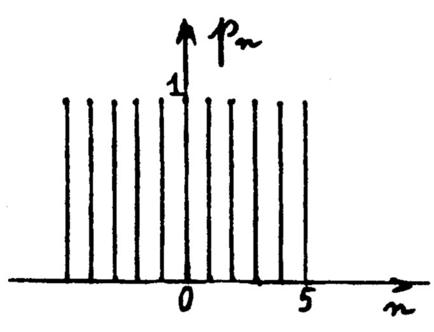
$$0.95 = 2G\left(\frac{\varepsilon}{\sigma_T}\right) - 1 \qquad G\left(\frac{\varepsilon}{\sigma_T}\right) = 0.975 \qquad \frac{\varepsilon}{\sigma_T} = z_{0.975}$$

This yields $\varepsilon = \varepsilon_b \simeq \sqrt{10/5} = \varepsilon_a\sqrt{5}$

12-12 (a) It follows from the convolution theorem for Fourier series

(b)





With p_n as above, $w_n = \frac{1}{11} p_n * p_n$

$$P(\omega) = \sum_{n=-5}^{5} e^{-jnT\omega} = \frac{\sin 5.5\omega T}{\sin 0.5\omega T} \qquad W(\omega) = \frac{1}{11} P^2(\omega)$$

12-13

$$\underset{\sim}{X}_T(\omega) = \frac{1}{\sqrt{2T}} \int_{-T}^{T} x(t)e^{-j\omega t}dt \qquad \underset{\sim}{S}_T(\omega) = |X_T(\omega)|^2$$

and

$$\Gamma(u,v) = \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} R(t_1 - t_2)e^{-j(u\, t_1 + v\, t_2)}\, dt_1 dt_2 \qquad \text{(i)}$$

as in (9-173) and (9-174) yield

$$E\{\underset{\sim}{S}_T(\omega)\} = \Gamma(\omega, -\omega)$$

$$\text{Var}\ \underset{\sim}{S}_T(\omega) = |\Gamma(\omega, -\omega)|^2 + |\Gamma(\omega,\omega)|^2 \geq E^2\{\underset{\sim}{S}_T(\omega)\}$$

$$\text{Var}\ \underset{\sim}{S}_T(0) = 2|\Gamma(0,0)|^2 = 2\{E^2\{\underset{\sim}{S}_T(0)\}$$

The remaining part of the problem is more difficult. We outline the proof (For details see Papoulis, Signal Analysis). From (i) and the convolution theorem it follows that

$$\Gamma(u,v) = \int_{-\infty}^{\infty} \frac{\sin T(u + v - \alpha)}{\pi\ T(u + v - \alpha)}\ \frac{\sin T\alpha}{\alpha}\ S(v - \alpha)d\alpha$$

If $S(\omega)$ is nearly constant in an interval of length $1/T$, then it can be taken outside the integral sign. Hence,

$$\Gamma(u,v) \simeq S(v) \int_{-\infty}^{\infty} \frac{\sin T(u + v - \alpha)}{\pi\ T(u + v - \alpha)}\ \frac{\sin T\alpha}{\alpha} d\alpha = S(v)\ \frac{\sin T(u + v)}{T(u + v)}$$

This yields

$$\Gamma(\omega,-\omega) \simeq S(\omega) < \Gamma(\omega,\omega) \simeq S(\omega)\ \frac{\sin 2\, T\omega}{T\omega} \xrightarrow[\omega T\to\infty]{} 0$$

$$\text{Var}\ \underset{\sim}{S}_T(\omega) \leq 2\ E^2\{\underset{\sim}{S}_T(\omega)\} \xrightarrow[\omega T\to\infty]{} E^2\{\underset{\sim}{S}_T(\omega)\}$$

12-14 The function

$$\underset{\sim}{X}_c(\omega) = \int_{-T}^{T} c(t)\underset{\sim}{x}(t)e^{-j\omega t}dt$$

is the Fourier transform of the product

$$c(t)\underset{\sim}{x}_T(t) \qquad\qquad \underset{\sim}{x}_T(t) = \begin{cases} 1 & |t| < T \\ 0 & |t| > T \end{cases}$$

Hence, the function

$$2\,T\,\underset{\sim}{S}_T(\omega) = |\underset{\sim}{X}_c(\omega)|^2$$

is the Fourier transform of

$$\underset{\sim}{c}(t)\underset{\sim}{x}_T(t) * c(-t)\underset{\sim}{x}_T(-t)$$

$$= \int_{-T+|\tau|/2}^{T-|\tau|/2} c(t+\tfrac{\tau}{2})\underset{\sim}{x}_T(t+\tfrac{\tau}{2})c(t-\tfrac{\tau}{2})\underset{\sim}{x}(t-\tfrac{\tau}{2})dt$$

12-15 Since $C(-\tau) = C(\tau)$, it follows from (12-28) that for large T,

$$\text{Var } \underset{\sim}{R}_T(\lambda) \simeq \frac{1}{2T}\int_{-\infty}^{\infty} [C(\lambda+\tau)C(\lambda-\tau) + C^2(\tau)]\, d\tau$$

Since $S(\omega)$ is real, it follows from Parseval's formula and the pairs

$$C(\lambda+\tau) \leftrightarrow e^{j\lambda\omega}S(\omega) \qquad\qquad C(\lambda-\tau) \leftrightarrow e^{-j\lambda\omega}S(\omega)$$

that the above integral equals

$$\int_{-\infty}^{\infty} \left[e^{j\lambda\omega}\, S(\omega)e^{j\lambda\omega}S(\omega) + S^2(\omega)\right]\, d\omega$$

12-16 With $c = T - |\tau|/2$

$$\underset{\sim}{z}(t) = \underset{\sim}{x}(t + \tfrac{\tau}{2})\underset{\sim}{x}(t - \tfrac{\tau}{2}) \qquad E\{\underset{\sim}{R}_T(\tau)\} = R(\tau)(1 - \tfrac{|\tau|}{T})$$

(7-37) yields

$$E\{\underset{\sim}{z}(t_1)\underset{\sim}{z}(t_2)\} - E\{\underset{\sim}{z}(t_1)\}E\{\underset{\sim}{z}(t_2)\}$$

$$= R^2(t_1 - t_2) + R(t_1 - t_2 + \tau)R(t_1 - t_2 - \tau)$$

$$4\,T^2 \operatorname{Var} \underset{\sim}{R}_T(\tau) = \int_{-c}^{c}\int_{-c}^{c} [R^2(t_1 - t_2) + R(t_1 - t_2 + \tau)R(t_1 - t_2 - \tau)]dt_1 dt_2$$

$$= \int_{-2c}^{2c} [R^2(\alpha) + R(\alpha + \tau)R(\alpha - \tau)](2T - |\tau| - |\alpha|)d\alpha$$

12-17 Equating coefficients of z^k in (12-98), we obtain

$$(1-K_N^2)\,\alpha_k^{N-1} = \alpha_k^N + K_N\,\alpha_{N-k}^N$$

12-18 $R[0] = 8 \qquad R[1] = 4$

From (13-67)

$$P_0 = 8 \qquad a_1^1 = K_1 = 0.5 \qquad P_1 = (1 - K_1^2)P_0 = 6$$

$$E_1(z) = 1 - 0.5z^{-1} \qquad S(\omega) = \frac{6}{|E_1(e^{j\omega})|^2}$$

12-19 From page (13-67)

$$P_0 = 13 \qquad a_1^1 = K_1 = \frac{5}{13} \qquad P_1 = \frac{144}{13}$$

$$P_1K_2 = R[2] - a_1^1R[1] \qquad K_2 = \frac{1}{144}$$

$$a_1^2 = \frac{55}{144} \qquad a_2^2 = \frac{1}{144} \qquad P_2 = \frac{1595}{144}$$

$$S_{MEM}(\omega) = \frac{1595 \times 144}{\left|144 - 55e^{-j\omega T} - e^{-j2\omega T}\right|^2}$$

From (12-119)

$$\begin{vmatrix} 13-q & 5 & 2 \\ 5 & 13-q & 5 \\ 2 & 5 & 13-q \end{vmatrix} = 0 \qquad q_o = 14-\sqrt{51} \simeq 6.86$$

Inserting the modified data 6.14, 5, 2 into the Yule-Walker equations (12-82), we obtain

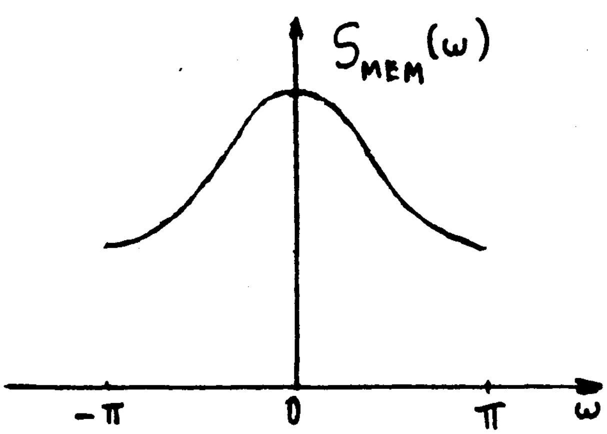
$$a_1^2 \simeq 4.07 \qquad a_2^2 = -1 \qquad E_2(z) = 1 - 4.07z^{-1} + z^{-2}$$

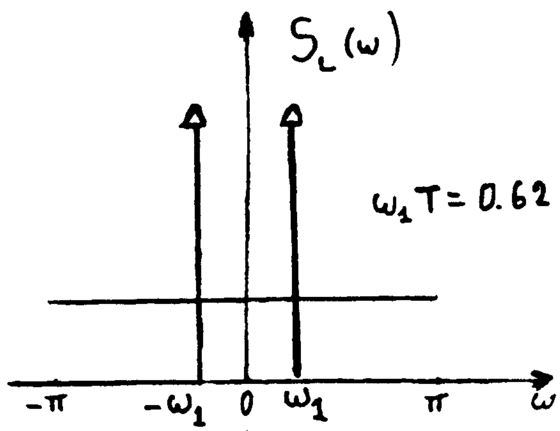
$$E_2(z) = 1 - 4.07z^{-1} + z^{-2} \qquad z_{1,2} \simeq e^{\pm j0.62}$$

Solving (12-91) we obtain

$$R_L[m] = 6.86\ \delta[m] + 3.07 \cos 0.62m$$

$$S_L(\omega) = 6.86 + \frac{2\pi}{T} \times 3.07\ [\delta(\omega - 0.62) + \delta(\omega + 0.62)]$$





12.20 (a) Let $z = e^{j\theta_1}$ represent one of the roots of the Levinsion Polynomial $P_n(z)$ that lie on the unit circle. In that case

$$P_n\left(e^{j\theta_1}\right) = 0$$

and substituting this into the recursion equation (12-177) we get

$$|s_n| = \left|\frac{P_{n-1}\left(e^{j\theta_1}\right)}{\tilde{P}_{n-1}\left(e^{j\theta_1}\right)}\right| = 1$$

so that

$$s_n = e^{j\alpha}.$$

Let

$$P_{n-1}\left(e^{j\theta}\right) = R(\theta)\, e^{j\psi(\theta)}$$

and since $P_{n-1}(z)$ is free of zeros in $|z| \leq 1$, we have $R(\theta) > 0, 0 < \theta < 2\pi$, and once again substituting these into (12-177) we obtain

$$\begin{aligned}
\sqrt{1-s_n^2}\, P_n\left(e^{j\theta}\right) &= R(\theta)\, e^{j\psi(\theta)} - e^{j(\theta+\alpha)}\, e^{j(n-1)\theta}\, R(\theta)\, e^{-j\psi(\theta)} \\
&= R(\theta)\left[e^{j\psi(\theta)} - e^{j(n\theta+\alpha)}\, e^{-j\psi(\theta)}\right] \\
&= 2j\, R(\theta)\, e^{j(n\theta+\alpha)/2}\, sin\left(\psi(\theta) - \frac{n\theta}{2} - \frac{\alpha}{2}\right).
\end{aligned}$$

Due to the strict Hurwitz nature of $P_{n-1}(z)$, as θ varies from 0 to 2π, there is no net increment in the phase term $\psi(\theta)$, and the entire argument of the sine term above increases by $n\pi$. Consequently $P_n(e^{j\theta})$ equals zero atleast at n distinct points $\theta_1, \theta_2, \cdots \theta_n, \quad 0 < \theta_i < 2\pi$. However $P_n(z)$ is a polynomial od degree n in z and can have atmost n zeros. Thus all the above zeros are simple and they all lie on the unit circle.

(b) Suppose $P_n(z)$ and $P_{n-1}(z)$ has a common zero at $z = z_0$. Then $|z_0| > 1$ and from (12-137), we get

$$z_0\, s_n\, \tilde{P}_{n-1}(z_0) = 0$$

which gives $s_n = 0$, since $\tilde{P}_{n-1}(z_0) \neq 0$, ($\tilde{P}_{n-1}(z)$ has all its zeros in $|z| < 1$). Hence $s_n \neq 0$ implies $P_n(z)$ and $P_{n-1}(z)$ do not have a common zero.

12.21 Substituting $s_n = \rho^n, |\rho| < 1$ in (12-177) we get

$$\sqrt{1-\rho^{2n}}\, P_n(z) = P_{n-1}(z) - (z\rho)^n\, P^*_{n-1}(1/z^*)$$

Let $x = z\rho$ and

$$P_n(z) = P_n(x/\rho) \triangleq A_n(x)$$

so that the above iteration reduces to

$$\begin{aligned}\sqrt{1-\rho^{2n}}\, A_n(x) &= A_{n-1}(x) - x^n\, A^*_{n-1}(1/x^*) \\ &= A_{n-1}(x) - x\, \tilde{A}_{n-1}(x)\end{aligned}$$

From problem (12-20), the polynomial $A_n(x)$ has all its zeros on the unit circle (since $s_n = 1$). i.e.,

$$x_k = e^{j\theta_k} = z_k\, \rho.$$

Hence the zeros $z_k = (1/\rho)e^{j\theta_k}$ or $|z_k| = 1/\rho$. (The zeros of $P_n(z)$ lie on a circle of radius $1/\rho$).

12.22 The Levinson Polynomials $P_n(z)$ satisfy the recusion in (12-177). Define $s'_n = \lambda^n s_n$, $|\lambda| = 1$, and replacing s_n by s'_n and $P_n(z)$ by $P'_n(z)$ in (12-177) we get

$$\begin{aligned}P'_n(z) &= P'_{n-1}(z) - zs'_n\, \tilde{P'}_{n-1}(z) \\ &= P'_{n-1}(z) - (z\lambda)^n\, s_n\, P'^*_{n-1}(z)\,(1/z^*)\end{aligned}$$

Let $y = z\lambda$ and define $P'_n(y/\lambda) = A_n(y)$ so that the above recursion simplifies to

$$\begin{aligned}A_n(y) &= A_{n-1}(y) - y^n\, s_n\, A^*_{n-1}(1/y^*) \\ &= A_{n-1}(y) - ys_n\, A^*_{n-1}(y)\end{aligned}$$

and on comparing with (12-177), we notice that $A_n(y) = P_n(y) = P_n(\lambda z)$. Thus $P_n(\lambda z)$ represents the new set of Levinson Polynomials.

12.23 (a) In this case

$$S(\theta) = \left|H(e^{j\theta})\right|^2 = |1 - e^{j\theta}|^2 = 2 - e^{j\theta} - e^{-j\theta} = 2(1 - cos\theta)$$

so that $r_0 = 2, r_1 = -1, r_k = 0, |k| \geq 2$. Substituing these values into (9-196) and taking the determinant of the tridiagonal matrix $\mathbf{T}_n$ we obtain the recursion

$$|\mathbf{T}_n| = \Delta_n = 2\Delta_{n-1} - \Delta_{n-2}$$

where $\Delta_0 = 2, \Delta_1 = 3$. Let $D(z) = \sum_{n=0}^{\infty} \Delta_n \, z^n$ so that the above recursion gives

$$D(z) = \frac{2 - z}{(1 - z)^2} = \frac{1}{1 - z} + \frac{1}{(1 - z)^2} = \sum_{n=0}^{\infty} (n + 2) \, z^n$$

and hence we get

$$\Delta_n = n + 2, \qquad n \geq 0.$$

Using (12-192) and (9-196) we get

$$s_n = (-1)^{n-1} \, \frac{\Delta_n^{(1)}}{\Delta_{n-1}} = \frac{(-1)^{n-1}(-1)^n}{n + 1} = -\frac{1}{n + 1}, \quad k \geq 1.$$

(b) The new set of reflection coefficient $s'_k = -s_k$ switches around the Levinson Polynomials $P_n(z)$ and $Q(z)$, and hence it follows that they correspond to the positive-real function

$$Z'(z) = \frac{2}{1 - z}$$

which gives $r'_0 = 2, r'_k = 1, \, k \geq 1$.

CHAPTER 13

13-1 $$\hat{\underline{s}}(t-\tfrac{T}{2}) = a\,\underline{s}(t) + b\,\underline{s}(t-T)$$

$$\underline{s}(t-\tfrac{T}{2}) - [a\,\underline{s}(t) + b\,\underline{s}(t-T)] \perp \underline{s}(t),\ \underline{s}(t-T)$$

$$R(T/2) = a\,R(0) + b\,R(T)$$

$$a = b = \frac{R(T/2)}{R(0)+R(T)} = \frac{e^{-1/2}}{1+e^{-1}}$$

$$R(T/2) = a\,R(T) + b\,R(0)$$

$$P = E\{[\underline{s}(t-\tfrac{T}{2}) - \hat{\underline{s}}(t-\tfrac{T}{2})]\underline{s}(t-\tfrac{T}{2})\}$$

$$= R(0) - aR(T/2) - bR(T/2) = R(0) - \frac{R^2(T/2)}{R(0)+R(T)} = \frac{I}{1+e^{-1}}$$

13-2 $$\int_0^T \underline{s}(t)dt - [a\,\underline{s}(0) + b\,\underline{s}(T)] \perp \underline{s}(0),\ \underline{s}(T)$$

$$\int_0^T R(t)dt = aR(0) + bR(T) \qquad \int_0^T R(T-t)dt = aR(T) + bR(0)$$

The above two integrals are equal. Hence,

$$a = b = \frac{\int_0^T R(t)dt}{R(0)+R(T)}$$

13-3 $$\hat{s}'(t) = a\,x(t) + b\,x(t-\tau)$$

$$s'(t) - [a\,x(t) + b\,x(t-\tau)] \perp x(t),\ x(t-\tau)$$

$$R_{s'x}(0) = a\,R_{xx}(0) + b\,R_{xx}(\tau) \qquad R_{s'x}(0) = R_{s's}(0) = 0$$

$$R_{s'x}(\tau) = a\,R_{xx}(\tau) + b\,R_{xx}(0) \qquad R_{xx}(\tau) = R_{ss}(\tau) + R_{\nu\nu}(\tau)$$

For small τ

$$R_{s'x}(\tau) = R_{s's}(\tau) = R'_{ss}(\tau) \simeq \tau\,R''_{ss}(0) \qquad R_{xx}(\tau) \simeq R_{xx}(0) + \tau^2 R''_{xx}(0)/2$$

Hence,

$$a = -b + O(\tau^2) \qquad \tau R''_{ss}(0) = a\,\tau^2\,R''_{xx}(0)/2 + O(\tau^3)$$

13-4 It suffices to show that, for any m,

$$E\{\left[x(t) - \sum_{n=-\infty}^{\infty} \frac{\sin(\sigma t - n\pi)}{\sigma t - n\pi}\,x(nT)\right] x(mT)\} = 0$$

The left side equals

$$R(t-mT) - \sum_{n=-\infty}^{\infty} \frac{\sin(\sigma t - n\pi)}{\sigma t - n\pi}\,R(nT-mT)$$

From the sampling theorem (10-140) it follows that this is zero because the Fourier transform

$$e^{-jmT}S(\omega)$$

of $R(t-mT)$ is zero for $|\omega| > \sigma$.

13-5 Since

$$\hat{E}\{\mathbf{x}(t+\lambda)|\mathbf{s}(t)\} = a\mathbf{s}(t) \qquad a = R(\lambda)/R(0)$$

it follows from the assumption that

$$\mathbf{s}(t+\lambda) - a\,\mathbf{s}(t) \perp \mathbf{s}(t-\tau)$$

Hence

$$R(\lambda+\tau) = \frac{R(\lambda)}{R(0)} R(\tau) \tag{i}$$

The only continuous function satisfying the above is an exponential. This is easily shown if we assume that $R(\lambda)$ is differentiable for $\lambda > 0$. Differentiating (i) with respect to λ and setting $\lambda = 0^+$, we obtain

$$R'(\tau) + \alpha R(\tau) = 0 \qquad \alpha = -R'(0^+)/R(0) \qquad \tau > 0$$

This yields $R(\tau) = Ie^{-\alpha\tau}$ for $\tau > 0$.

13-6 Given:

$$E\{\mathbf{y}_n\} = 0 \qquad \mathbf{x}_n = \mathbf{y}_1 + \cdots + \mathbf{y}_n = \mathbf{x}_{n-1} + \mathbf{y}_n$$

Furthermore the RVs $\mathbf{y}_n$ are independent. Hence, $\mathbf{y}_n$ is independent of $\mathbf{x}_{n-1}, \ldots, \mathbf{x}_1$. This yields

$$E\{\mathbf{x}_n|\mathbf{x}_{n-1},\ldots,\mathbf{x}_1\} = E\{\mathbf{x}_{n-1} + \mathbf{y}_n|\mathbf{x}_{n-1},\ldots,\mathbf{x}_1\}$$

$$= E\{\mathbf{x}_{n-1}|\mathbf{x}_{n-1},\ldots,\mathbf{x}_1\} + E\{\mathbf{y}_n\} = \mathbf{x}_{n-1}$$

13-7 (a) If $\hat{E}\{x_n | x_{n-1},\ldots,x_1\} = x_{n-1}$, then

$$= x_n - x_{n-1}, \perp x_{n-1},\ldots,x_1$$

From this it follows that the RVs $y_n = x_n - x_{n-1}$ are orthogonal and

$$x_n = y_n + x_{n-1} = y_n + y_{n-1} + \cdots + y_1 \qquad (i)$$

Conversely, if (i) is true and the RVs y_n are orthogonal, then

$$x_n - x_{n-1} = y_n \perp x_{n-1},\ldots,x_1$$

(b) $E\{x_n^2\} = E\{[(x_n - x_{n-1}) + x_{n-1}]^2\}$

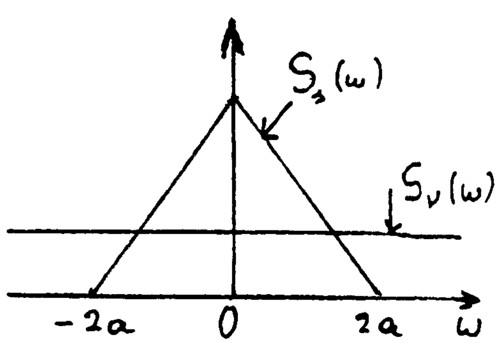
$$= E\{(x_n - x_{n-1})^2\} + E\{x_{n-1}^2\} \geq E\{x_{n-1}^2\}$$

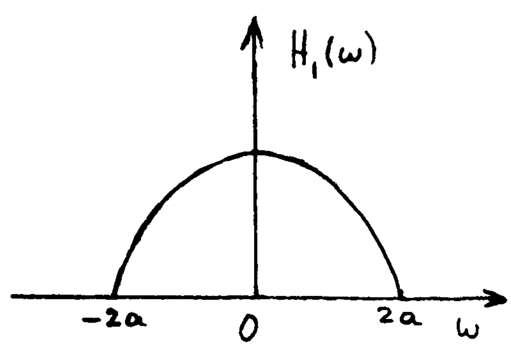
for any n.

13-8 The Fourier transform $S_s(\omega)$ of the function

$$R_s(\tau) = A\,\frac{\sin^2 a\tau}{\tau^2}$$

is a triangle as shown





And since $S_\nu(\omega) = N$, (13-16) yields

$$H_1(\omega) = \frac{S_s(\omega)}{S_s(\omega) + S_\nu(\omega)} = \frac{Aa\pi(1 - |\omega|/2a)}{Aa\pi(1 - |\omega|/2a) + N}$$

We show next that

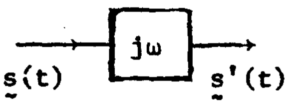
$$H_2(\omega) = j\omega H_1(\omega)$$

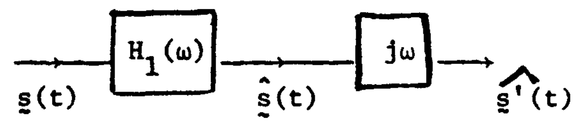
Proof

$$\varepsilon(t) = s(t) \quad \hat{s}'(t) \perp x(\xi) \quad \text{all } t,\xi. \quad \text{Hence } R_{\varepsilon x}(\tau) = 0.$$

This yields [see (9-131)]

$$R_{\varepsilon' x}(\tau) = R'_{\varepsilon x}(\tau) = 0, \text{ hence } s'(t) - \hat{s}'(t) \perp x(\xi)$$





In other words, the estimate of $s'(t)$ equals the derivative of the estimate $\hat{s}(t)$ of $s(t)$. This follows from Prob. 13-9 with $T(\omega) = j\omega$.

13-9 We wish to show that the estimator of

$$y(t) = \int_{-\infty}^{\infty} p(t-\alpha)s(\alpha)\,d\alpha \qquad p(t) \leftrightarrow T(\omega)$$

equals

$$\hat{y}(t) = \int_{-\infty}^{\infty} p(t-\alpha)\hat{s}(\alpha)\,d\alpha$$

where $\hat{s}(t)$ is the estimator of $s(t)$.

Proof. Clearly

$$E\{[s(t) - \hat{s}(t)]x(\xi)\} = 0 \qquad \text{all } t,\xi$$

Hence

$$E\{[y(t) - \hat{y}(t)]x(\xi)\}$$

$$= \int_{-\infty}^{\infty} p(t-\alpha)E\{[s(\alpha) - \hat{s}(\alpha)]x(\xi)\}\,d\alpha = 0$$

13-10 [See (13-46) and beyond]

(a) $S(s) = \dfrac{1}{s^4+1} = \dfrac{1}{(s^2+\sqrt{2}\,s+1)\,(s^2-\sqrt{2}\,s+1)}$

(b) $L(s) = \dfrac{1}{(s+\alpha)^2+\beta^2}$ $\quad \ell(t) = \dfrac{1}{\beta}\, e^{-\alpha t} \sin\beta t U(t) \qquad \alpha = \beta = \dfrac{1}{\sqrt{2}}$

(c) $h_i(t) = \dfrac{1}{\beta}\, e^{-\alpha\lambda}\, e^{-\alpha t} \sin\beta(t+\lambda)U(t)$

$$H_i(s) = \frac{1}{\beta}\, e^{-\alpha\lambda}\, \frac{(s+\alpha)\sin\beta\lambda + \beta\cos\beta\lambda}{(s+\alpha)^2+\beta^2}$$

$$H(s) = \frac{H_i(s)}{L(s)} = b_0 + b_1 s \qquad b_0 = e^{-\alpha\lambda}(\cos\beta\lambda + \frac{\alpha}{}\sin\beta\lambda) \qquad b_1 = \frac{\sin\beta\lambda}{\beta}\, e^{-\alpha\lambda}$$

13-11 **(a)** The given equation is the Wiener-Hopf equation (13-40) for the prediction problem with $\lambda = \ell n2$. We can, therefore, use the method described after (13-46):

$$S(s) = \frac{3}{1-s^2} + \frac{22}{9-s^2} = \frac{49-25s^2}{(1-s^2)(9-s^2)}$$

$$L(s) = \frac{7+5s}{(1+s)(3+s)} \qquad \ell(t) = (e^{-t}+4e^{-3t})U(t)$$

$$h_i(t) = (e^{-\ell n2}e^{-t}+4e^{-3\ell n2}e^{-3t})U(t)$$

$$H_i(s) = \frac{1/2}{1+s} + \frac{4/8}{3+s} = \frac{2+s}{(1+s)(3+s)} \qquad H(s) = \frac{2+s}{7+5s}$$

(b)

$$H(s) = \frac{N(s)}{D(s)} \qquad \frac{N(s)-2^sD(s)}{D(s)}\,L(s)L(-s) = Y(s)$$

Since $Y(s)$ is analytic for $\underline{\mathrm{Re}}\, s < 0$, all roots of $D(s)$ must be cancelled by the zeros of $L(s)$, hence, $D(s) = 7+5s$. Similarly the poles $s = -1$ and $s = -3$ of $L(s)$ must be cancelled by the zeros of the term $N(s) - 2^sD(s)$. With $N(s) = As+B$, this yields

$$N(-1) - 2^{-1}D(-1) = -A+B-2^{-1}(7-5) = 0 \qquad A = 1$$

$$N(-3) - 2^{-3}D(-3) = -3A+B-2^{-3}(7-15) = 0 \qquad B = 2$$

$$H(s) = \frac{2+s}{7+5s}$$

(c) The Laplace transform of the function $R(\tau)$ in (a) equals

$$\frac{49-25s^2}{9-10s^2+s^4}$$

Hence (convolution theroem), the inverse transform $y(t)$ of $Y(s)$ equals

$$y(t) = \int_0^\infty h(\alpha)R(t-\alpha)d\alpha - R(t+\ \ell n2)$$

From the analyticity of $Y(s)$ for $\underline{\mathrm{Re}}\, s < 0$ it follows that $y(t) = 0$ for $t > 0$. Therefore, (b) gives a direct method for solving the Wiener-Hopf equation (13-40).

13-12 (a) The given equation is identical with equation (13-22) for the prediction problem with $r=1$. We can, therefore, use the method in (13-31)-(13-33):

$$S(z) = \frac{3}{5-2w} + \frac{8}{10-3w} = \frac{70-25w}{(5-2w)(10-3w)} \qquad w = z+\frac{1}{z}$$

$$L(z) = \frac{a - bz^{-1}}{(2-z^{-1})(3-z^{-1})} \qquad a = \sqrt{30}+\sqrt{5} \simeq 7.75 \qquad b = \sqrt{30}-\sqrt{5} \simeq 3.25$$

$$\ell[0] = \frac{a}{6} \simeq 1.3 \qquad H(z) = 1 - \frac{\ell[0]}{L(z)} \simeq \frac{0.41\, z^{-1} - 0.167\, z^{-2}}{1-0.42z^{-1}}$$

(b)

$$H(z) = \frac{N(z)}{D(z)} \qquad \frac{N(z) - zD(z)}{D(z)} L(z)L(z^{-1}) = Y(z)$$

Since $Y(z)$ is analytic for $|z|<1$, all roots of $D(z)$ must be cancelled by the zeros of $L(z)$, hence, $D(z) = 1-0.42\, z^{-1}$. Similarly, the poles $z=1/2$ and $z=1/3$ of $L(z)$ must be cancelled by the zeros of the term $N(z)-zD(z)$. With $N(z) = A+Bz^{-1}$, this yields

$$N(\tfrac{1}{2}) - \tfrac{1}{2}D(\tfrac{1}{2}) \simeq A+2B-0.08 = 0 \qquad A \simeq 0.42$$

$$N(\tfrac{1}{3}) - \tfrac{1}{3}D(\tfrac{1}{3}) \simeq A+3B+0.09 = 0 \qquad B = -0.17$$

$$H(z) \simeq \frac{0.42-0.17z^{-1}}{1-0.42z^{-1}}$$

The z transform of the sequence R_m in (a) equals

$$\frac{70-25w}{6w^2-35w+50} \qquad w = z+z^{-1}$$

Hence, the inverse transform y_n of $Y(z)$ equals

$$y_n = \sum_{k=0}^{n} h_k R_{n-k} - R_{n+1}$$

From the analyticity of $Y(z)$ for $|z|<1$ it follows that $y_n=0$ for $n\geq 0$. Therefore, (b) gives a direct method for solving (13-22).

13-13. A predictor is a stable function H(z) vanishing at ∞. Since $H(z) \to 0$ as $z \to \infty$, we conclude that $E_N(z) = 1-H(z) \to 1$ and $E_N(z)H_a(z) \to 1$ as $z \to \infty$. From this and (13-25) it follows that the difference $1 - E_N(z)H_a(z)$ is a predictor and the MS error equals

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} | E_N(e^{j\omega}) |^2 S(\omega)d\omega = P$$

because $|A_a(e^{j\omega})| = 1$.

13-14 As we know, if

$$\underset{\sim}{s}[n] = a_1\underset{\sim}{s}[n-1] + \cdots + a_m\underset{\sim}{s}[n-m] + \underset{\sim}{\varepsilon}[n]$$

where $\underset{\sim}{\varepsilon}[n]$ is white noise, then the one-step predictor of $\underset{\sim}{s}[n]$ equals

$$\hat{\underset{\sim}{s}}_1[n] = a_1\underset{\sim}{s}[n-1] + \cdots + a_m\underset{\sim}{s}[n-m]$$

We wish to show that the sum

$$\hat{\underset{\sim}{s}}_2[n] = a_1\hat{\underset{\sim}{s}}_1[n-1] + a_2\underset{\sim}{s}[n-2] + \cdots + a_m\underset{\sim}{s}[n-m]$$

is its two-step predictor. It suffices to show that

$$\underset{\sim}{s}[n] - \hat{\underset{\sim}{s}}_2[n] \perp \underset{\sim}{s}[n-k] \qquad k \geq 2$$

<u>Proof</u>

$$\underset{\sim}{s}[n] - \hat{\underset{\sim}{s}}_2[n] = a_1\left(\underset{\sim}{s}_1[n-1] - \hat{\underset{\sim}{s}}_1[n-1]\right) + \underset{\sim}{\varepsilon}[n]$$

This completes the proof because

$$\underset{\sim}{s}_1[n-1] - \hat{\underset{\sim}{s}}_1[n-1] \perp \underset{\sim}{s}(n-k], \quad k \geq 2 \text{ and } \underset{\sim}{\varepsilon}[n] \perp \underset{\sim}{s}[n-k] \quad k \geq 1.$$

13-15 The Nth order MS estimation error P_N equals [see (13-66)]

$$P_N = \frac{\Delta_{N+1}}{\Delta_N}$$

This tends to the MS estimation error in (13-34). Hence,

$$\lim_{N\to\infty} \ln P_N = \frac{1}{2\sigma}\int_{-\sigma}^{\sigma} \ln S(\omega)d\omega = \lim_{N\to\infty} \ln \frac{\Delta_{N+1}}{\Delta_N}$$

To complete the proofs, we use (14-129)

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \ln \frac{\Delta_{n+1}}{\Delta_n} = \lim_{N\to\infty} \ln \frac{\Delta_{N+1}}{\Delta_N}$$

and the result follows because

$$\frac{1}{N}\sum_{n=1}^{N} (\ln \Delta_{n+1} - \ln \Delta_n) = \frac{\ln\Delta_{N+1}}{N} - \frac{\ln\Delta_1}{N}$$

and the last term tends to zero as $N\to\infty$.

13-16 $P_0 = R[0] = 15 \qquad R[1] = 10 \qquad R[2] = 5 \qquad R[3] = 0$

We use Levinson's algorithm [see (13-67)]

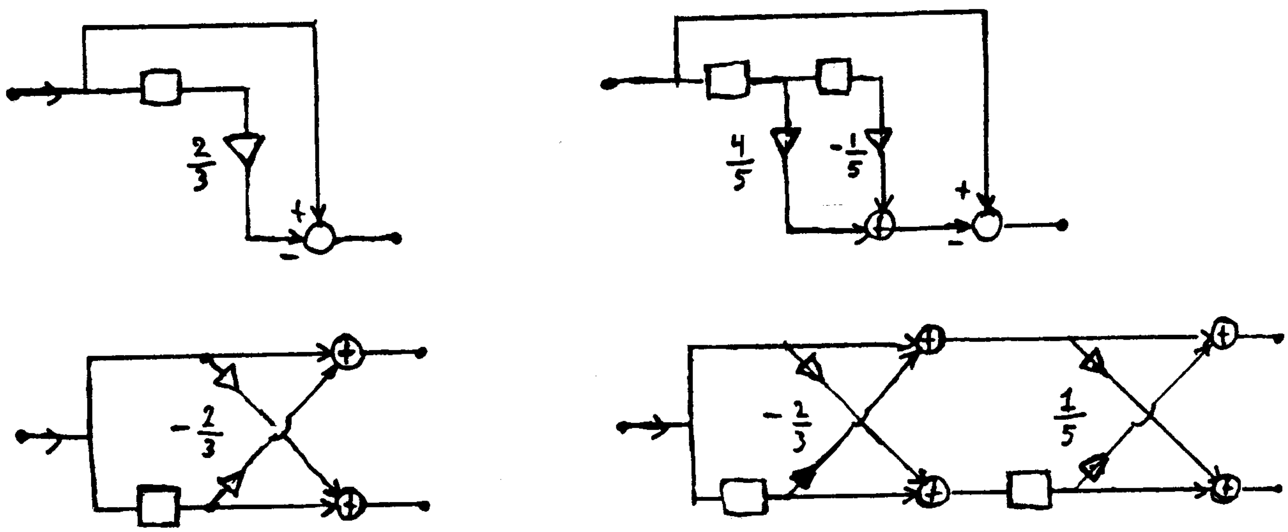
$P_0 K_1 = R[1] \qquad K_1 = a_1^1 = \frac{2}{3} \qquad P_1 = (1 - k_1^2)P_0 = \frac{25}{3}$

$P_1 K_2 = R[2] - R[1]a_1^1 = -\frac{5}{3} \qquad K_2 = -\frac{1}{5}$

$a_1^2 = a_1^1 - K_2 a_1^1 = \frac{4}{5} \qquad a_2^2 = -\frac{1}{5} \qquad P_2 = 8$

$P_2 K_3 = R[3] - R[2]a_1^2 - R[1]a_2^2 \qquad K_3 = -\frac{1}{4}$

$a_1^3 = \frac{3}{4} \qquad a_2^3 = 0 \qquad a_3^3 = -\frac{1}{4} \qquad P_3 = 7.5$



13-17 $P_0 = R[0] = 5 \qquad K_1 = 0.4 \qquad K_2 = 0.6 \qquad K_3 = 0.8$

$R[1] = P_0K_1 = 2 \qquad a_1^1 = 0.4 \qquad P_1 = 4.2$

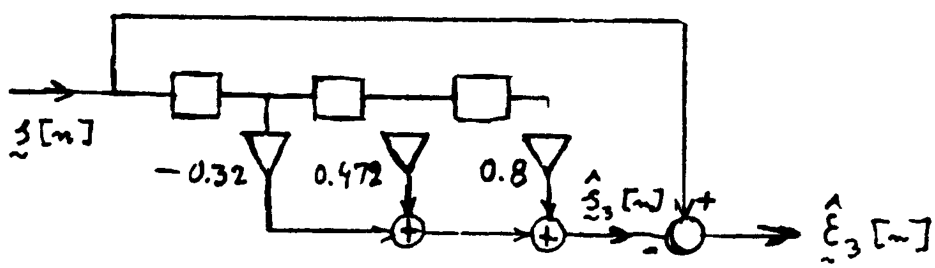
$R[2] = R[1]a_1^1 + P_1K_2 = 3.32$

$a_1^2 = 0.16 \qquad a_2^2 = 0.6 \qquad P_2 = 2.688$

$R[3] = R[2]a_1^2 + R[1]a_2^2 + P_2K_3 = 3.8816$

$a_1^3 = a_1^2 - K_3a_2^2 = -0.32 \qquad a_2^3 = a_2^2 - K_3a_1^2 = 0.472 \qquad a_3^3 = 0.8$

$a_3^3 = 0.8 \qquad P_3 = 0.968$



13-18

$$S_x(s) = \frac{4\lambda}{-s^2 + 4\lambda^2} + N = \frac{N(-s^2 + c^2)}{-s^2 + 4\lambda^2}$$

$$\Gamma_x(s) = \frac{s + 2\lambda}{\sqrt{N}\,(s + c)} \qquad c = 2\lambda\sqrt{1 + \frac{1}{\lambda N}}$$

and (13-104) yields

$$H_x(s) = \frac{c - 2\lambda}{s + c} \qquad h_x(t) = (c - 2\lambda)e^{-ct}U(t)$$

13-19 (a) $\hat{\varepsilon}_{N+m}[n+m] \perp \underset{\sim}{s}[n-k] \qquad k = -m+1,\ldots,0,\ldots,N$

$$\hat{\varepsilon}_N[n] = \underset{\sim}{s}[n] - a_1\underset{\sim}{s}[n-1] - \cdots - a_N\underset{\sim}{s}[n-N]$$

(b) $\check{\underset{\sim}{\varepsilon}}_{N+m}[n-m] \perp \underset{\sim}{s}[n+k] \qquad k = -m+1,\ldots,0,\ldots,N$

$$\check{\varepsilon}_N[n] = \underset{\sim}{s}[n] - a_1\underset{\sim}{s}[n+1] - \cdots - a_N\underset{\sim}{s}[n+N]$$

(c) $\check{\underset{\sim}{\varepsilon}}_{N+m}[n-N-m] \perp \underset{\sim}{s}[n+k] \qquad k = -N-m+1,\ldots,-N,\ldots,0$

$$\hat{\underset{\sim}{\varepsilon}}_N[n] = \underset{\sim}{s}[n] - a_1\underset{\sim}{s}[n-1] - \cdots - a_N\underset{\sim}{s}[n-N]$$

13-20

$$S_s(\omega) = \frac{2}{\omega^2+0.04} \qquad S_x(\omega) = \frac{5\omega^2+2.2}{\omega^2+0.04} \qquad L_x(s) = \sqrt{5}\,\frac{s+0.66}{s+0.2}$$

(a) From (13-16)

$$H(\omega) = \frac{2}{5\omega^2+2.2}$$

(b) From (13-104)

$$H_x(s) = 1 - \sqrt{5}\,\Gamma_x(s) = \frac{0.46}{s+0.66}$$

(c) Using (13-48)

$$L_s(s) = \frac{\sqrt{2}}{s+0.2} \qquad \ell(t) = \sqrt{2}\, e^{-0.2t}\, U(t)$$

$$h_i(t) = \sqrt{2}\, e^{-0.2(t+2)} U(t) \qquad H_i(s) = \frac{\sqrt{2}\, e^{-0.4}}{s+0.2}$$

$$H(s) = e^{-0.4} \qquad \hat{\underset{\sim}{s}}(t+2) = e^{-0.4}\underset{\sim}{s}(t)$$

(d) [see (13-99) and beyond]

$$S_{sx}(s) = \frac{1}{0.04 - s^2} \qquad \Gamma_x(s) = \frac{s+0.2}{\sqrt{5}(s+0.66)} \qquad S_{si_x}(s) = \sqrt{20}\,\frac{0.66-s}{s+0.2}$$

$$R_{si_x}(\tau) = \sqrt{20}\left[\delta(\tau) + \frac{0.86}{s+0.2}\right] \qquad h_{i_x}(\tau) = 0.86\sqrt{20}\; e^{-0.2(t+2)}U(t)$$

$$H_{i_x}(s) = \frac{0.86\ 20\ e^{-0.4}}{s+0.2} \qquad H_x(s) = \frac{1.72\ e^{-0.4}}{s+0.66}$$

13-21 As in Example 13-2 with $N_0 = 1.8$, $N = 5$, $a = 0.8$

$$S_s(z) = \frac{1.8}{(1-0.8\ z^{-1})(1-0.8z)} \qquad L_s(z) = \frac{\sqrt{1.8}}{1-0.8\ z^{-1}}$$

$$S_x(z) = \frac{8(1-0.5\ z^{-1})(1-0.5z)}{(1-0.8\ z^{-1})(1-0.8z)} \qquad L_x(z) = \frac{\sqrt{8}(1-0.5\ z^{-1}}{1-0.8\ z^{-1}}$$

(a)

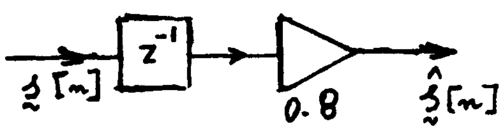
$$H(z) = \frac{9}{40(1-0.52\ z^{-1})(1-0.5z)} \qquad h[n] = 2\times 2^{-|n|}$$

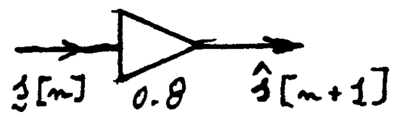
(b) From (13-114) with $\ell_x[0] = \sqrt{8}$

$$H_x(z) = \frac{3/8}{1-0.5\ z^{-1}} \qquad h_x[n] = \frac{3}{8}\times 0.5\ U[n]$$

(c) From (13-33) with $\ell[0] = \sqrt{1.8}$

$$H\ (z) = 1 - \frac{\ell[0]}{L_s(z)} = 0.8\ z^{-1}$$





(d) The power spectrum of the estimate $\hat{s}_0[n]$ of $s[n]$ obtained in (b) equals

$$S_{\hat{s}_0}(z) = S_x(z)H_x(z)H_x(z^{-1}) = \frac{9/8}{(1-0.8\ z^{-1})(1-0.8\ z)}$$

Hence, $$L_{\hat{s}_0}(z) = \frac{\sqrt{9/8}}{1-0.8\ z^{-1}}$$

Therefore, the pure predictor of $\hat{s}_0[n]$ equals [see (13-33)]

$$\hat{H}^1(z) = 1 - \frac{\ell[0]}{L(z)} = 0.8\ z^{-1}$$

And (13-117) yields

$$H_x^1(z) = H_x^0(z)\hat{H}_1(z) = \frac{0.3\,z^{-1}}{1-0.5\ z^{-1}}$$

13-22

$$R_s[m] = 5 \times 0.8^{|m|} \longleftrightarrow \frac{1.8}{(1-0.8\,z^{-1})(1-0.8\,z)}$$

Hence, as in (13-135) with $V_n = 1.8$, $N_n = 5$. And (13-143) yields

$$F_n = 0.64\,F_{n-1} + V_n\,G_{n-1} \qquad F_0 = V_0 N_0 = 9$$

$$5\,G_n = 0.64\,F_{n-1} + 6.5\,G_{n-1} \qquad G_0 = V_0 + N_0 = 6.8$$

Solving, we obtain

$$F_n = 12(1.6)^n - 3(0.4)^n \qquad G_n = 6.4(1.6)^n + 0.4(0.4)^n$$

$$P_n = \frac{F_n}{G_n} \xrightarrow[n \to \infty]{} \frac{12}{6.4} = 1.875$$

This agrees with Prob. 13-21c because the MS error of the Wiener filter equals

$$P = R_s(0) - \sum_{k=0}^{\infty} R_s[k]h_x[k] = 5 - \sum_{k=0}^{\infty} 5 \times 0.8^m \times \frac{3}{8} \times 0.5^m = 1.875$$

13-23 $R_s(\tau) = 5\, e^{-0.2|\tau|}$ $\qquad R\ (\tau) = \frac{10}{3}\,\delta(\tau)$

$$S_s(\omega) = \frac{2}{\omega^2 + 0.2^2} \qquad A(t) = 0.2 \qquad V(t) = 2 \qquad N(t) = \frac{10}{3}$$

From (13-159)

$$F'(t) = 0.2\ F(t) + 2G(t) \qquad G'(t) = 0.3\ F(t) + 0.2G(t)$$

Case 1. If $\underset{\sim}{s}(0) = 0$, then $P(0) = F(0) = 0$, $G(0) = 1$

Solving, we obtain

$$P(t) = \frac{F(t)}{G(t)} = \frac{1.25\ e^{0.8t} - 1.25\ e^{-0.8t}}{0.625\ e^{0.8t} + 0.375\ e^{-0.8t}}$$

Case 2. If $\underset{\sim}{s}(t)$ is stationary, then $F(0) = P(0) = R_s(0) = 5$

$$P(t) = \frac{F(t)}{G(t)} = \frac{5\ e^{0.8t} + 3\ e^{-0.8t}}{2.5\ e^{0.8t} - 0.9\ e^{-0.8t}}$$

13-24 The sequences $\hat{q}_N[n]$ and $\check{q}_N[n]$ are the responses of the filters

$$\hat{E}_N(z) = 1 - \sum_{k=1}^{N} a_k^N z^{-k} \qquad \check{E}_N(z) = z^{-N}\hat{E}_N(1/z)$$

respectively, with input R[m] (see Fig. 13-11a). Hence,

$$\hat{q}_N[m] = R[m] - \sum_{k=1}^{N} R[m-k]a_k^N$$

$$\check{q}_N[m] = \hat{q}_N[N-m] = R[m-N] - \sum_{k=1}^{N} R[m-N+k]a_k^N$$

From this and the Yule-Walker equation (13-65) it follows that

$$\hat{q}_N[m] = \check{q}_N[N-m] = 0 \text{ for } 1 \le m \le N-1$$

$$\hat{q}_N[0] = \check{q}_N[N] = P_N$$

This completes the proof.

CHAPTER 14

14-1 It suffices to show that [see (14-41)]

$$H(A \cdot B|B_j) = H(A|B_j)$$

Since

$$A_iB_kB_j = \begin{cases} A_iB_j & k = j \\ \{\emptyset\} & k \neq j \end{cases} \qquad \text{and } P(A_iB_j|B_j) = P(A_i|B_j)$$

(14-40) yields

$$H(A \cdot B|B_j) = -\sum_{i,k} P(A_iB_k|B_j) \log P(A_iB_k|B_j)$$

$$= -\sum_i P(A_i|B_j) \log P(A_i|B_j) = H(A|B_j)$$

14-2 If $\alpha < \beta$, then $\phi'(\alpha) > \phi'(\beta)$ because

$$\phi'(\alpha) - \phi'(\beta) = \log(\beta/\alpha) > 0. \quad \text{Hence,}$$

$$\int_a^b \phi'(\alpha)d\alpha > \int_{a+c}^{b+c} \phi'(\alpha)d\alpha \qquad c > 0$$

This yields

$$\phi(p_1 + p_2) - \phi(p_1) = \int_{p_1}^{p_1+p_2} \phi'(\alpha)d\alpha < \int_0^{p_2} \phi'(\alpha)d\alpha = \phi(p_2)$$

Similarly

$$\phi(p_1 + \varepsilon) - \phi(p_1) - \phi(p_2) + \phi(p_2 - \varepsilon)$$

$$= \int_{p_1}^{p_1+\varepsilon} \phi'(\alpha)d\alpha - \int_{p_2-\varepsilon}^{p_2} \phi'(\alpha)d\alpha > 0$$

14-3 Applying the identity

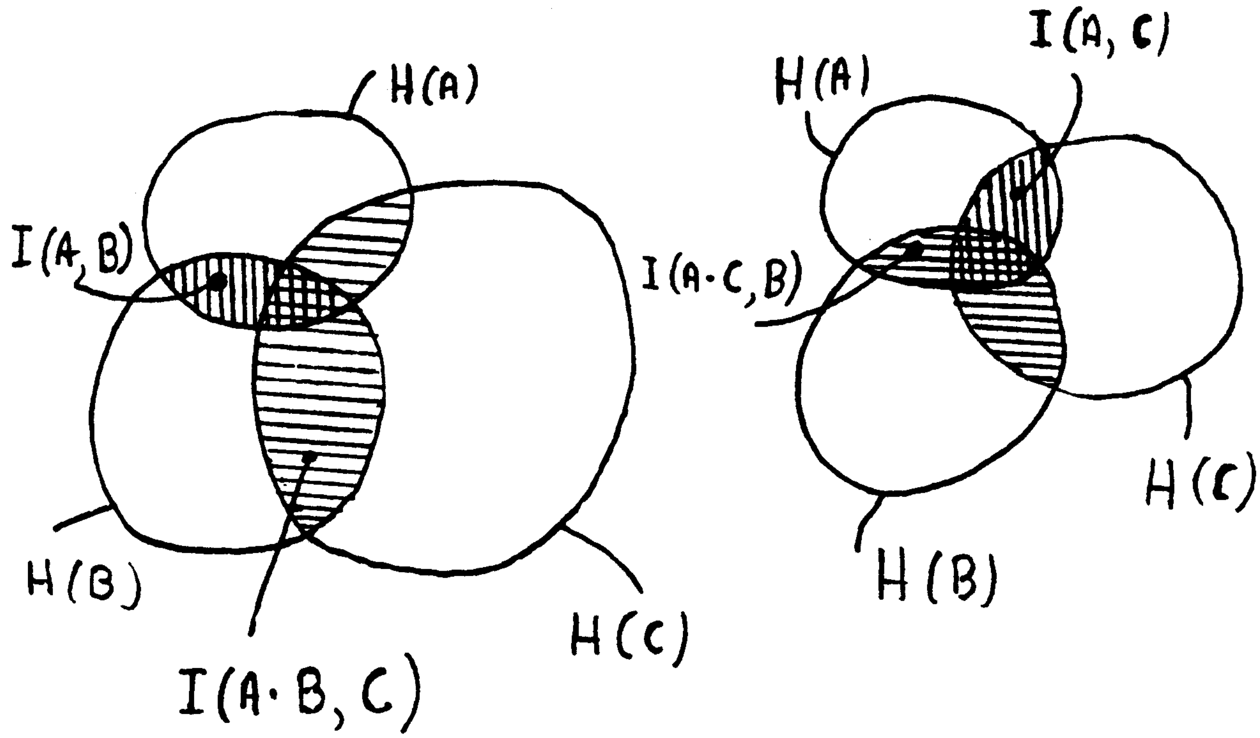
$$H(A_1 \cdot A_2) = H(A_1) + H(A_2|A_1) \tag{i}$$

to the partitions $A_1 = A$, $A_2 = B \cdot C$ and $A_1 = A \cdot B$, $A_2 = C$, we obtain the first line. The second line follows from the first [see (i)]. The third line is a consequence of the first two.

14-4 It follows if we apply the identity

$$I(A_1,A_2) = H(A_1) + H(A_2) - H(A_1 \cdot A_2)$$

to the partitions $A_1 = A \cdot B$, $A_2 = C$.



14-5 **(a)** **From** (14-53)

$$I(A,B\cdot C) = H(A) + H(B\cdot C) - H(A\cdot B\cdot C)$$

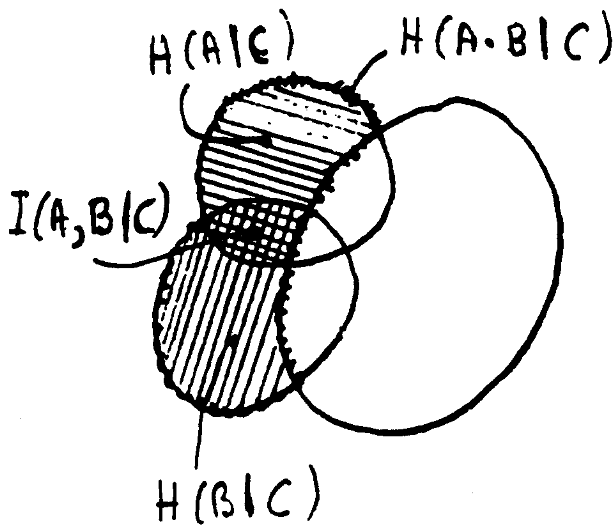
$$I(A,C) = H(A) + H(C) - H(A\cdot C)$$

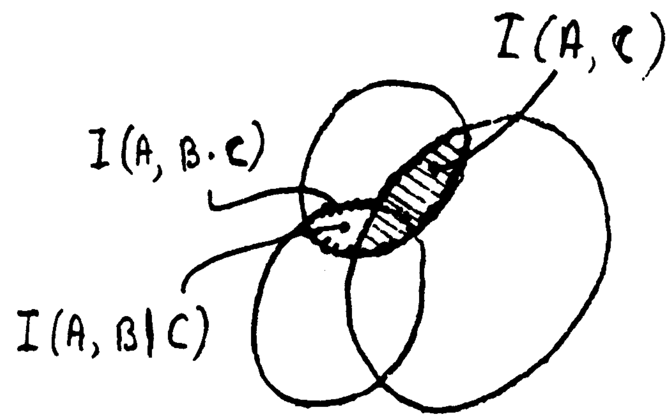
and since (see Prob. 14-4)

$$H(A\cdot B\cdot C) - H(A\cdot C) = H(A\cdot B|C) - H(A|C)$$

we conclude with (14-49) **that**

$$I(A,B\cdot C) - I(A\cdot C) = H(B|C) + H(A|C) - H(A\cdot B|C)$$





(b) If $B\cdot C$ is observed, then the resulting prediction in the uncertainty of A equals $I(A, B\cdot C)$. But, if $B\cdot C$ is observed, then C is observed, hence, the reduction in the uncertainty of A is at least $I(A,C)$. Hence

$$I(A,B\cdot C) \geq I(A,C)$$

with equality only if $I(A,B|C) = 0$, i.e., if in the subsequence of trials in which C occurred, knowledge of the occurrence of B gives no information about A.

14-6 The partition $H(A^3)$ has eight elements with respective probabilities

$$p^3, p^2q, p^2q, p^2q, pq^2, pq^2, pq^2, q^3$$

Hence

$$H(A^3) = -p^3\log p^3 - 3p^2q\log p^2q - 3pq^2\log pq^2 - q^3\log q^3$$

$$= -3p(p^2+2pq+q^2)\log p - 3q(p^2+2pq+q^2)\log q$$

$$= -3p\log p - 3q\log q = 3H(A)$$

14-7 The density of the RV $\underset{\sim}{w} = \underset{\sim}{x} + a$ equals $f_x(w-a)$. Hence,

$$H(\underset{\sim}{x}+a) = -\int_{-\infty}^{\infty} f_x(w-a)\log f_x(w-\alpha)\,dw$$

$$= -\int_{-\infty}^{\infty} f_x(x)\log f_x(x)\,dx = H(\underset{\sim}{x})$$

The joint density of the RVs $\underset{\sim}{x}$ and $\underset{\sim}{z} = \underset{\sim}{x} + \underset{\sim}{y}$ equals $f_{xy}(x,z-x)$. Hence [see (14-90)]

$$H(\underset{\sim}{z}|\underset{\sim}{x}) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,z-x)\log f_{xy}(x,z-x)/f_x(x)\,dx\,dz$$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y)\log f_{xy}(x,y)/f_x(x)\,dx\,dy = H(\underset{\sim}{y}|\underset{\sim}{x})$$

14-8 The RVs $\underset{\sim}{x}$ and $\underset{\sim}{y}$ take the values x_i and y_j respectively then $\underset{\sim}{z} = x_i + y_j$ iff $\underset{\sim}{x} = x_i$ and $\underset{\sim}{y} = y_j$ (assumption). Hence,

$$\{\underset{\sim}{z} = x_i + y_j\} = \{\underset{\sim}{x} = x_i\} \cap \{\underset{\sim}{y} = y_j\}$$

This shows that $A_z = A_x \cdot B_y$. Furthermore, since the RVs $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are independent, the events $\{\underset{\sim}{x} = x_i\}$ and $\{\underset{\sim}{y} = y_j\}$ are also independent. This shows that the partitions A_x and B_y are independent and [see (14-44) and Prob. 14-1]

$$H(A_z|A_x) = H(A_x \cdot A_y|A_x) = H(A_y|A_x) = H(A_y)$$

From this it follows that $H(\underset{\sim}{z}|\underset{\sim}{x}) = H(\underset{\sim}{y})$ because [see (14-88) and (14-41)]

$$H(\underset{\sim}{z}|\underset{\sim}{x}) = H(A_z|A_x)$$

14-9 As we see from (14-80)

$H(\underset{\sim}{x}) = \ln a$ where we assume that $a = N\delta$. The RV $\underset{\sim}{y}$ takes the values $0, \delta, \ldots, (N-1)\delta$ with probability $1/N$. The conditional density of $\underset{\sim}{x}$ assuming $\underset{\sim}{y} = k\delta$ is uniform in the interval $(k\delta, k\delta + \delta)$. Hence,

$$H(\underset{\sim}{x}|\underset{\sim}{y} = k\delta) = -\int_{k\delta}^{k\delta+\delta} f(x|\underset{\sim}{y} = k\delta) \ln f(x|\underset{\sim}{y} = k\delta)\,dx = \ln \delta$$

And as in (14-41)

$$H(\underset{\sim}{x}|\underset{\sim}{y}) = \sum_{k=0}^{N} H(\underset{\sim}{x}|\underset{\sim}{y} = k\delta) P\{\underset{\sim}{y} = k\delta\} = \ln \delta$$

Finally [see (14-95)]

$$I(\underset{\sim}{x},\underset{\sim}{y}) = H(\underset{\sim}{x}) - H(\underset{\sim}{x}|\underset{\sim}{y}) = \ln a - \ln \delta$$

14-10 If $y_i = g(x_i)$, $y_j = g(x_j)$ and $y_i = y_j$ then $x_i = x_j$. Hence,

$$p_{ij} = \begin{cases} p_i & i = j \\ 0 & i \neq j \end{cases} \qquad p_i = P\{\underset{\sim}{x} = x_i\}$$

and

$$H(\underset{\sim}{x},\underset{\sim}{y}) = -\sum_{i,j} p_{ij} \log p_{ij} = -\sum_i p_i \log p_i = H(\underset{\sim}{x})$$

14-11 From Prob. 10-10 it follows with $g(x) = x$ that $H(\underset{\sim}{x},\underset{\sim}{x}) = H(\underset{\sim}{x})$. And since [see (14-103)] $H(\underset{\sim}{x},\underset{\sim}{x}) = H(\underset{\sim}{x}|\underset{\sim}{x}) + H(\underset{\sim}{x})$ we conclude that $H(\underset{\sim}{x}|\underset{\sim}{x}) = 0$. From Prob. 14-3 it follows that

$$H(\underset{\sim}{y},\underset{\sim}{x}|\underset{\sim}{x}) = H(A_y \cdot A_x|A_x) = H(A_x \cdot A_x) + H(A_y|A_x \cdot A_x)$$

$$= H(A_y|A_x) = H(\underset{\sim}{y}|\underset{\sim}{x})$$

because $A_x \cdot A_x = A_x$ and $H(A_x \cdot A_x) = H(\underset{\sim}{x},\underset{\sim}{x}) = 0$.

14-12 $$E\{\underset{\sim}{x}_n\} = 0 \qquad E\{\underset{\sim}{x}_n^2\} = 5 \qquad E\{\underset{\sim}{y}_n\} = 0$$

$$E\{\underset{\sim}{y}_n^2\} = \sum_{k=0}^{\infty} 2^{-2k}\, E\{\underset{\sim}{x}_{n-k}^2\} = \frac{20}{3} \qquad E\{\underset{\sim}{x}_n\underset{\sim}{y}_n\} = E\{\underset{\sim}{x}_n^2\} = 5$$

(a) From (14-95), (14-84), and (15-86) with $\mu_{11} = 5$, $\mu_{22} = 20/3$, and $\mu_{12} = 5$

$$H(\underset{\sim}{x}) = \ln\sqrt{10\pi e} \qquad H(\underset{\sim}{y}) = \ln\sqrt{40\pi e/3} \qquad H(\underset{\sim}{x},\underset{\sim}{y}) = \ln 10\pi e/\sqrt{3}$$

$$I(\underset{\sim}{x},\underset{\sim}{y}) = \ln 2$$

(b) The process $\underset{\sim}{y}(t)$ is the output of the system

$$L(z) = \frac{1}{1-0.5\,z^{-1}} \qquad \ell_o = 1$$

with input $\underset{\sim}{x}_n$. Since $\bar{H}(\underset{\sim}{x}) = H(\underset{\sim}{x})$ and [see (12A-1)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln|L(e^{j\phi})|\,d\phi = \ln\ell_o = 0$$

(14-133) yields $\bar{H}(\underset{\sim}{y}) = \bar{H}(\underset{\sim}{x}) = H(\underset{\sim}{x}) = \ln\sqrt{10\pi e}$.

14-13 $$\bar{H}(\underset{\sim}{x}) = H(\underset{\sim}{x}) = -\frac{1}{2}\int_4^6 \ln\frac{1}{2}\,dx = \ln 2$$

And as in Prob. 14-12 with $\ell_o = 5$,

$$\bar{H}(\underset{\sim}{y}) = \bar{H}(x) + \ln 5 = \ln 10$$

14-14 Given that $f(x) = 0$ for $|x|>1$ and $E\{\underset{\sim}{x}\} = 0.3$, find $f(x)$. With $g(x) = x$, (14-143) yields $f(x) = Ae^{-\lambda x}$ where

$$A\int_{-1}^{1} e^{-\lambda x}\,dx = \frac{A}{\lambda}(e^{\lambda} - e^{-\lambda}) = 1$$

$$A\int_{-1}^{1} xe^{-\lambda x}\,dx = \frac{A}{\lambda^2}(e^{\lambda} - e^{-\lambda}) - \frac{A}{\lambda}(e^{\lambda} - e^{-\lambda}) = 0.31$$

Solving, we obtain $A \simeq 0.425$, $\lambda \simeq -1$

14-15 $f(x) = Ae^{-\lambda x}$ for $1<x<5$ and 0 otherwise,

$$A\int_1^5 e^{-\lambda x}\,dx = 0.31 \qquad A\int_1^5 xe^{-\lambda x}dx = 3\,\frac{37}{60}$$

Hence, $A \simeq 1.06$, $\lambda \simeq 0.5$

14-16 From (14-151) with x_k=k, $g_1(x_k) = g_1(k) = k$, k=1, ..., 6

$$g_2(x_k)=\begin{cases}0 & k=1,3,5\\ 1 & k=2,4,6\end{cases} \quad p_k=\begin{cases}Ae^{-\lambda_1 k} & k=1,3,5\\ Ae^{-\lambda_1 x-\lambda_2} & k=2,4,6\end{cases}$$

Since $p_1 + p_3 + p_5 = 0.5$ and $E\{\underset{\sim}{x}\} = 4.44$, we conclude with $z = e^{-\lambda_2}$ and $w = e^{-\lambda_2}$ that

$$A(z+z^3+z^5) = Aw(z^2+z^4z^6)$$

$$A(\underset{\sim}{z}+3z^3+5z^5) + Aw(2z^2+4z^4+6z^6) = 4.44$$

This yields $A \simeq 0.0437$, $\underset{\sim}{z} = 1/w \simeq 1.468$

14-17 (a) The transformation $\underset{\sim}{y} = 3\underset{\sim}{x}$ is one-to-one, hence, $H(\underset{\sim}{y}) = H(\underset{\sim}{x})$

(b) From (14-113) with $g(x) = 3x$: $H(\underset{\sim}{y}) = H(\underset{\sim}{x}) + \ln 3$

14-18 (a) For fair dice, $P\{7\} = \frac{1}{6}$, $P(11) = \frac{1}{18}$, $P\{\text{neither 7 nor 11}\} = \frac{14}{18}$

$$H(A) = -\left(\frac{1}{6}\ln\frac{1}{6} + \frac{1}{18}\ln\frac{1}{18} + \frac{14}{18}\ln\frac{14}{18}\right) = 0.655$$

(b) From (14-10) with n=100 and N=3:

$$n_T \simeq e^{nH(A)} \simeq 2.79 \times 10^{28} \qquad n_a \simeq N^n \simeq 5.16 \times 10^{47}$$

14-19 The process x_n is WSS with entropy rate $\bar{H}(x)$. Show that, if

$$w_n = \sum_{k=0}^{n} x_{n-k}\,\ell_k$$

then

$$\lim_{n\to\infty} \frac{1}{n+1} H(w_o,\ldots,w_n) = \bar{H}(x) + \ln|\ell_o| \qquad \text{(i)}$$

<u>Proof.</u> The RVs $w_o,\ldots,w_n$ are linear transformations of the RVs $x_o,\ldots,x_n$ and the transformation matrix equals

$$\begin{bmatrix} \ell_0 & 0 & \cdots & 0 \\ \ell_1 & \ell_0 & \cdots & 0 \\ \hline \ell_n & \ell_{n-1} & \cdots & 0 \end{bmatrix}$$

Since the determinant of this transformation equals $|\ell_o|^{n+1}$, (14-115) yields

$$H(w_o,\ldots,w_n) = H(x_o,\ldots x_n) + (n+1)\ln|\ell_o|$$

Dividing by (n+1) we obtain (i) as $n\to\infty$.

14-20 As in Example 14-19, $f(p) = A\,e^{-\lambda p}$. To find λ, we use the λ-η curve of Fig. 14-16. This yields

$$\lambda \simeq -1.23 \qquad f(p) \simeq 0.51\,e^{1.23p}$$

14-21 As in Example 14-22, $p_k = A\, e^{-\lambda k}$. To find λ, we use the w-η curve of Fig. 14-17. This yields (see also Jaynes)

$$w \simeq 1.449 \qquad \lambda \simeq -0.371$$

p_1	p_2	p_3	p_4	p_5	p_6
0.054	0.079	0.114	0.165	0.240	0.348

14-22 The unknown density is normal as in (14-157) where

$$\Delta = \begin{vmatrix} 4 & 1 & 1 \\ 1 & 4 & m_{23} \\ 1 & m_{23} & 4 \end{vmatrix} = -4m_{23}^2 + 2m_{23} + 56$$

The moment $m_{23} = E\{\mathbf{x}_2\mathbf{x}_3\}$ must be such as to maximize Δ. This yields $m_{23} = 0.25$.

14-23

	p_i	0.3	0.2	0.15	0.15	0.1	0.06	0.04	
		$\frac{1}{4} \le p_i < \frac{1}{2}$	$\frac{1}{8} \le p_i < \frac{1}{4}$			$\frac{1}{16} \le p_i < \frac{1}{8}$	$\frac{1}{32} \le p_i < \frac{1}{6}$		$\sum_{i=1}^{7} \frac{1}{2^{n_i}}$
Shannon	n_i	2	3	3	3	4	5	5	0.75
		2	3	3	3	4	4	4	0.8125
		2	3	3	3	3	4	4	0.875
L = 2.7		2	3	3	3	3	3	4	0.9375
		2	3	3	3	3	3	3	1
	x_i	00	010	011	100	101	110	111	

Fano

$L = 2.6$

<table>
<tr><td>p_i</td><td>0.3</td><td>0.2</td><td>0.15</td><td>0.15</td><td>0.1</td><td>0.06</td><td>0.04</td></tr>
<tr><td rowspan="4"></td><td colspan="2">A_0 0.5</td><td colspan="5">A_1 0.5</td></tr>
<tr><td rowspan="3">A_{00}
0.3</td><td rowspan="3">A_{01}
0.2</td><td colspan="2">A_{10} 0.3</td><td colspan="3">A_{11} 0.2</td></tr>
<tr><td rowspan="2">A_{100}
0.15</td><td rowspan="2">A_{101}
0.15</td><td rowspan="2">A_{110}
0.1</td><td colspan="2">A_{111} 0.1</td></tr>
<tr><td>A_{1110}</td><td>A_{1111}</td></tr>
<tr><td>x_i</td><td>00</td><td>01</td><td>100</td><td>101</td><td>110</td><td>1110</td><td>1111</td></tr>
</table>

Huffman

$L = 2.6$

<table>
<tr><td></td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td></tr>
<tr><td></td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td colspan="2">6 7
0 1</td></tr>
<tr><td></td><td>1</td><td>2</td><td colspan="3">5 6 7
0 10 11</td><td colspan="2">3 4</td></tr>
<tr><td></td><td>1</td><td colspan="2">3 4
0 1</td><td>2</td><td colspan="3">5 6 7
0 10 11</td></tr>
<tr><td></td><td colspan="4">2 5 6 7
0 10 110 111</td><td>1</td><td colspan="2">3 4
0 1</td></tr>
<tr><td></td><td colspan="3">1 3 4
0 10 11</td><td colspan="4">2 5 6 7
0 10 110 111</td></tr>
<tr><td></td><td colspan="7">1 3 4 2 5 6 7
00 010 011 10 110 1110 1111</td></tr>
<tr><td>x_i</td><td>00</td><td>10</td><td>010</td><td>011</td><td>110</td><td>1110</td><td>1111</td></tr>
</table>

14-24 If $\underset{\sim}{x}_n = 0$, then $\bar{x}_n = 000$ and $\underset{\sim}{y}_n = 1$ iff $\bar{y}_n$ consists of one 0 or no zeros. The probability of one and only one zero equals $3\beta^2(1-\beta)$ [see (3-13)]; the probability of no zeros equals β^3. Hence,

$$P\{\underset{\sim}{y}_n = 1 | \underset{\sim}{x}_n = 0\} = 3\beta^2(1-\beta) + \beta^3$$

Thus, the redundantly coded channel of Example 14-29 is symmetrical as in (14-191) with probability of error $\beta_1 = \beta^2$.

14-25 If the received information is always wrong, then

$$P\{\underset{\sim}{y}_n = 1 | \underset{\sim}{x}_n = 0\} = \beta = 1, \text{ hence } C = 1 - r(\beta) = 1$$

Chapter 15

15.1 The chain represented by

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

is irreducible and aperiodic.

The second chain is also irreducible and aperiodic.

The third chain has two aperiodic closed sets $\{e_1, e_2\}$ and $\{e_3, e_4\}$ and a transient state e_5.

15.2 Note that both the row sums and column sums are unity in this case. Hence P represents a doubly stochastic matrix here, and

$$P^n = \frac{1}{m+1}\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$$

$$\lim_{n\to\infty} P\{\mathbf{x}_n = e_k\} = \frac{1}{m+1}, \qquad k = 0, 1, 2, \cdots m.$$

15.3 This is the "success runs" problem discussed in Example 15-11 and 15-23. From Example 15-23, we get

$$u_{i+1} = p_{i,i+1} u_i = \frac{1}{i+1} u_i = \frac{u_o}{(i+1)!}$$

so that from (15-206)

$$\sum_{k=1}^{\infty} u_k = u_0 \sum_{k=1}^{\infty} \frac{1}{k!} = e \cdot u_0 = 1$$

gives $u_0 = 1/e$ and the steady state probabilities are given by

$$u_k = \frac{1/e}{k!}, \qquad k = 1, 2, \cdots$$

15.4 If the zeroth generation has size m, then the overall process may be considered as the sum of m independent and identically distributed branching processes $\mathbf{x}_n^{(k)}$, $k = 1, 2, \cdots m$, each corresponding to unity size at the zeroth generation. Hence if π_0 represents the probability of extinction for any one of these individual processes, then the overall probability of extinction is given by

$$\begin{aligned}
&\lim_{n\to\infty} P[\mathbf{x}_n = 0|\mathbf{x}_0 = m] = \\
&= P[\{\mathbf{x}_n^{(1)} = 0|\mathbf{x}_0^{(1)} = 1\} \bigcap \{\mathbf{x}_n^{(2)} = 0|\mathbf{x}_0^{(2)} = 1\} \bigcap \cdots \{\mathbf{x}_n^{(m)} = 0|\mathbf{x}_0^{(m)} = 1\}] \\
&= \textstyle\prod_{k=1}^m P[\mathbf{x}_n^{(k)} = 0|\mathbf{x}_0^{(k)} = 1] \\
&= \pi_0^m
\end{aligned}$$

15.5 From (15-288)-(15-289),

$$P(z) = p_0 + p_1 z + p_2 z^2, \quad \text{since } p_k = 0, \quad k \geq 3.$$

Also $p_0 + p_1 + p_2 = 1$, and from (15-307) the extinction probability is given by sloving the equation

$$P(z) = z.$$

Notice that

$$\begin{aligned}
P(z) - z &= p_0 - (1 - p_1)z + p_2 z^2 \\
&= p_0 - (p_0 + p_2)z + p_2 z^2 \\
&= (z - 1)(p_2 z - p_0)
\end{aligned}$$

and hence the two roots of the equation $P(z) = z$ are given by

$$z_1 = 1, \qquad z_2 = \frac{p_0}{p_2}.$$

Thus if $p_2 < p_0$, then $z_2 > 1$ and hence the smallest positive root of $P(z) = z$ is 1, and it represents the probability of extinction. It follows that such a tribe which does not produce offspring in abundence is bound to extinct.

15.6 Define the branching process $\{\mathbf{x}_n\}$

$$\mathbf{x}_{n+1} = \sum_{k=1}^{\mathbf{x}_n} \mathbf{y}_k$$

where $\mathbf{y}_k$ are i.i.d random variables with common moment generating function $P(z)$ so that (see (15-287)-(15-289))

$$P'(1) = E\{\mathbf{y}_k\} = \mu.$$

Thus

$$\begin{aligned} E\{\mathbf{x}_{n+1}|\mathbf{x}_n\} &= E\{\textstyle\sum_{k=1}^{\mathbf{x}_n} \mathbf{y}_k|\mathbf{x}_n = m\} \\ &= E\{\textstyle\sum_{k=1}^{m} \mathbf{y}_k|\mathbf{x}_n = m\} \\ &= E\{\textstyle\sum_{k=1}^{m} \mathbf{y}_k\} = mE\{\mathbf{y}_k\} = \mathbf{x}_n\,\mu \end{aligned}$$

Similarly

$$\begin{aligned} E\{\mathbf{x}_{n+2}|\mathbf{x}_n\} &= E\{E\{\mathbf{x}_{n+2}|\mathbf{x}_{n+1}, \mathbf{x}_n\}\} \\ &= E\{E\{\mathbf{x}_{n+2}|\mathbf{x}_{n+1}\}|\mathbf{x}_n\} \\ &= E\{\mu\mathbf{x}_{n+1}|\mathbf{x}_n\} = \mu^2\,\mathbf{x}_n \end{aligned}$$

and in general we obtain

$$E\{\mathbf{x}_{n+r}|\mathbf{x}_n\} = \mu^r\,\mathbf{x}_n. \tag{i}$$

Also from (15-310)-(15-311)

$$E\{\mathbf{x}_n\} = \mu^n. \tag{ii}$$

Define

$$\mathbf{w}_n = \frac{\mathbf{x}_n}{\mu^n}. \tag{iii}$$

This gives

$$E\{\mathbf{w}_n\} = 1.$$

Dividing both sider of (i) with μ^{n+r} we get

$$E\{\frac{\mathbf{x}_{n+r}}{\mu^{n+r}}|\mathbf{x}_n = x\} = \mu^r \cdot \frac{\mathbf{x}_n}{\mu^{n+r}} = \frac{\mathbf{x}_n}{\mu^n} = \mathbf{w}_n$$

or

$$E\{\mathbf{w}_{n+r}|\mathbf{w}_n = \frac{x}{\mu^n} \triangleq w\} = \mathbf{w}_n$$

which gives

$$E\{\mathbf{w}_{n+r}|\mathbf{w}_n\} = \mathbf{w}_n,$$

the desired result.

15.7

$$\mathbf{s}_n = \mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_n$$

where $\mathbf{x}_n$ are i.i.d. random variables. We have

$$\mathbf{s}_{n+1} = \mathbf{s}_n + \mathbf{x}_{n+1}$$

so that

$$E\{\mathbf{s}_{n+1}|\mathbf{s}_n\} = E\{\mathbf{s}_n + \mathbf{x}_{n+1}|\mathbf{s}_n\} = \mathbf{s}_n + E\{\mathbf{x}_{n+1}\} = \mathbf{s}_n.$$

Hence $\{\mathbf{s}_n\}$ represents a Martingale.

15.8 (a) From Bayes' theorem

$$\begin{aligned} P\{\mathbf{x_n} = j|\mathbf{x_{n+1}} = i\} &= \frac{P\{\mathbf{x}_{n+1} = i|\mathbf{x}_n = j\}\, P\{\mathbf{x}_n = j\}}{P\{\mathbf{x}_{n+1} = i\}} \\ &= \frac{q_j\, p_{ji}}{q_i} = p^*_{ij}, \end{aligned} \tag{i}$$

where we have assumed the chain to be in steady state.

(b) Notice that time-reversibility is equivalent to

$$p^*_{ij} = p_{ij}$$

and using (i) this gives

$$p^*_{ij} = \frac{q_j\, p_{ji}}{q_i} = p_{ij} \tag{ii}$$

or, for a time-reversible chain we get

$$q_j\, p_{ji} = q_i\, p_{ij}. \tag{iii}$$

Thus using (ii) we obtain by direct substitution

$$\begin{aligned} p_{ij}\,p_{jk}\,p_{ki} &= \left(\frac{q_j}{q_i}\,p_{ji}\right)\left(\frac{q_k}{q_j}\,p_{kj}\right)\left(\frac{q_i}{q_k}\,p_{ik}\right) \\ &= p_{ik}\,p_{kj}\,p_{ji}, \end{aligned}$$

the desired result.

15.9 (a) It is given that $A = A^T$, $(a_{ij} = a_{ji})$ and $a_{ij} > 0$. Define the i^{th} row sum

$$r_i = \sum_k a_{ik} > 0, \qquad i = 1, 2, \cdots$$

and let

$$p_{ij} = \frac{a_{ij}}{\sum_k a_{ik}} = \frac{a_{ij}}{r_i}.$$

Then

$$\begin{aligned} p_{ji} &= \frac{a_{ji}}{\sum\limits_m a_{jm}} = \frac{a_{ji}}{r_j} = \frac{a_{ij}}{r_j} \\ &= \frac{r_i}{r_j}\,\frac{a_{ij}}{r_i} = \frac{r_i}{r_j}\,p_{ij} \end{aligned} \tag{i}$$

or

$$r_i\,p_{ij} = r_j\,p_{ji}.$$

Hence

$$\sum_i r_i\,p_{ij} = \sum_i r_j\,p_{ji} = r_j \sum_i p_{ji} = r_j, \tag{ii}$$

since

$$\sum_i p_{ji} = \frac{\sum_i a_{ji}}{r_j} = \frac{r_j}{r_j} = 1.$$

Notice that (ii) satisfies the steady state probability distribution equation (15-167) with

$$q_i = c\,r_i, \qquad i = 1, 2, \cdots$$

where c is given by

$$c \sum_i r_i = \sum_i q_i = 1 \Longrightarrow c = \frac{1}{\sum_i r_i} = \frac{1}{\sum_i \sum_j a_{ij}}.$$

Thus

$$q_i = \frac{r_i}{\sum_i r_i} = \frac{\sum_j a_{ij}}{\sum_i \sum_j a_{ij}} > 0 \tag{iii}$$

represents the stationary probability distribution of the chain.

With (iii) in (i) we get

$$p_{ji} = \frac{q_i}{q_j}\, p_{ij}$$

or

$$p_{ij} = \frac{q_j\, p_{ji}}{q_i} = p_{ij}^*$$

and hence the chain is time-reversible.

15.10 (a) $M = (m_{ij})$ is given by

$$M = (I - W)^{-1}$$

or

$$(I - W)M = I$$
$$M = I + WM$$

which gives

$$\begin{aligned} m_{ij} &= \delta_{ij} + \textstyle\sum_k w_{ik}\, m_{kj}, \quad e_i, e_j \in T \\ &= \delta_{ij} + \textstyle\sum_k p_{ik}\, m_{kj}, \quad e_i, e_j \in T \end{aligned}$$

(b) The general case is solved in pages 743-744. From page 744, with $N = 6$ (2 absorbing states; 5 transcient states), and with $r = p/q$ we obtain

$$m_{ij} = \begin{cases} \dfrac{(r^j - 1)(r^{6-i} - 1)}{(p-q)(r^6 - 1)}, & j \le i \\[2ex] \dfrac{(r^i - 1)(r^{6-i} - r^{j-i})}{(p-q)(r^6 - 1)}, & j \ge i. \end{cases}$$

15.11 If a stochastic matrix $A = (a_{ij}), a_{ij} > 0$ corresponds to the two-step transition matrix of a Markov chain, then there must exist another stochastic matrix P such that

$$A = P^2, \qquad P = (p_{ij})$$

where

$$p_{ij} > 0, \quad \sum_j p_{ij} = 1,$$

and this may not be always possible. For example in a two state chain, let

$$P = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{pmatrix}$$

so that

$$A = P^2 = \begin{pmatrix} \alpha^2 + (1-\alpha)(1-\beta) & (\alpha+\beta)(1-\alpha) \\ (\alpha+\beta)(1-\beta) & \beta^2 + (1-\alpha)(1-\beta) \end{pmatrix}.$$

This gives the sum of this its diagonal entries to be

$$\begin{aligned} a_{11} + a_{22} &= \alpha^2 + 2(1-\alpha)(1-\beta) + \beta^2 \\ &= (\alpha+\beta)^2 - 2(\alpha+\beta) + 2 \\ &= 1 + (\alpha+\beta-1)^2 \;\; \geq 1. \end{aligned} \tag{i}$$

Hence condition (i) necessary. Since $0 < \alpha < 1$, $0 < \beta < 1$, we also get $1 < a_{11} + a_{22} \leq 2$. Futher, the condition (i) is also sufficient in the 2×2 case, since $a_{11} + a_{22} > 1$, gives

$$(\alpha + \beta - 1)^2 = a_{11} + a_{22} - 1 > 0$$

and hence

$$\alpha + \beta = 1 \pm \sqrt{a_{11} + a_{22} - 1}$$

and this equation may be solved for all admissible set of values $0 < \alpha < 1$ and $0 < \beta < 1$.

15.12 In this case the chain is irreducible and aperiodic and there are no absorption states. The steady state distribution $\{u_k\}$ satisfies (15-167),and hence we get

$$u_k = \sum_j u_j \, p_{jk} = \sum_{j=0}^{N} u_j \binom{N}{k} p_j^k \, q_j^{N-k}.$$

Then if $\alpha > 0$ and $\beta > 0$ then "fixation to pure genes" does not occur.

15.13 The transition probabilities in all these cases are given by (page 765) (15A-7) for specific values of $A(z) = B(z)$ as shown in Examples 15A-1, 15A-2 and 15A-3. The eigenvalues in general satisfy the equation

$$\sum_j p_{ij}\, x_j^{(k)} = \lambda_k\, x_i^{(k)}, \quad k = 0, 1, 2, \cdots N$$

and trivially $\sum_j p_{ij} = 1$ for all i implies $\lambda_0 = 1$ is an eigenvalue in all cases.

However to determine the remaining eigenvalues we can exploit the relation in (15A-7). From there the corresponding conditional moment generating function in (15-291) is given by

$$G(s) = \sum_{j=0}^{N} p_{ij}\, s^j \tag{i}$$

where from (15A-7)

$$\begin{aligned} p_{ij} &= \frac{\{A^i(z)\}_j\, \{B^{N-i}(z)\}_{N-j}}{\{A^i(z)\, B^{N-i}(z)\}_N} \\ &= \frac{\text{coefficient of } s^j\, z^N \text{ in } \{A^i(sz)\, B^{N-i}(z)\}}{\{A^i(z)\, B^{N-i}(z)\}_N} \end{aligned} \tag{ii}$$

Substituting (ii) in (i) we get the compact expression

$$G(s) = \frac{\{A^i(sz)\, B^{N-i}(z)\}_N}{\{A^i(z)\, B^{N-i}(z)\}_N}. \tag{iii}$$

Differentiating $G(s)$ with respect to s we obtain

$$\begin{aligned} G'(s) &= \sum_{j=0}^{N} P_{ij}\, j\, s^{j-1} \\ &= \frac{\{iA^{i-1}(sz)\, A'(sz) z\, B^{N-i}(z)\}_N}{\{A^i(z)\, B^{N-i}(z)\}_N} \\ &= i \cdot \frac{\{A^{i-1}(sz)\, A'(sz)\, B^{N-i}(z)\}_{N-1}}{\{A^i(z)\, B^{N-i}(z)\}_N}. \end{aligned} \tag{iv}$$

Letting $s = 1$ in the above expression we get

$$G'(1) = \sum_{j=0}^{N} p_{ij}\, j = i\, \frac{\{A^{i-1}(z)\, A'(z)\, B^{N-i}(z)\}_{N-1}}{\{A^{i}(z)\, B^{N-i}(z)\}_{N}}. \tag{v}$$

In the special case when $A(z) = B(z)$, Eq.(v) reduces to

$$\sum_{j=0}^{N} p_{ij}\, j = \lambda_1\, i \tag{vi}$$

where

$$\lambda_1 = \frac{\{A^{N-1}(z)\, A'(z)\}_{N-1}}{\{A^{N}(z)\}_{N}}. \tag{vii}$$

Notice that (vi) can be written as

$$P x_1 = \lambda_1\, x_1, \quad x_1 = [0, 1, 2, \cdots N]^T$$

and by direct computation with $A(z) = B(z) = (q + pz)^2$ (Example 15A-1) we obtain

$$\begin{aligned} \lambda_1 &= \frac{\{(q+pz)^{2(N-1)}\, 2p(q+pz)\}_N}{\{(q+pz)^{2N}\}_N} \\ &= \frac{2p\{(q+pz)^{2N-1}\}_{N-1}}{\{(q+pz)^{2N}\}_N} = \frac{2p\begin{pmatrix} 2N \\ N-1 \end{pmatrix} q^N\, p^{N-1}}{\begin{pmatrix} 2N \\ N \end{pmatrix} q^N\, p^N} = 1. \end{aligned}$$

Thus $\sum_{j=0}^{N} p_{ij}\, j = i$ and from (15-224) these chains represent Martingales. (Similarly for Examples 15A-2 and 15A-3 as well).

To determine the remaining eigenvalues we differentiate $G'(s)$ once more. This gives

$$\begin{aligned} G''(s) &= \sum_{j=0}^{N} p_{ij}\, j(j-1)\, s^{j-2} \\ &= \frac{\{i(i-1)A^{i-2}\,(sz)[A'(sz)]^2\, z\, B^{N-i}(z) + iA^{i-1}(sz)\, A''(sz) z\, B^{N-i}(z)\}_{N-1}}{\{A^{i}(z)\, B^{N-i}(z)\}_N} \quad . \\ &= \frac{\{i\, A^{i-2}(sz)\, B^{N-i}(z)[(i-1)\,(A'(sz))^2 + A(sz)\, A''(sz)]\}_{N-2}}{\{A^{i}(z)\, B^{N-i}(z)\}_N}. \end{aligned}$$

With $s = 1$, and $A(z) = B(z)$, the above expression simplifies to

$$\sum_{j=0}^{N} p_{ij}\, j(j-1) = \lambda_2\, i(i-1) + i\mu_2 \qquad \text{(viii)}$$

where

$$\lambda_2 = \frac{\{A^{N-2}(z)\,[A'(z)]^2\}_{N-2}}{\{A^N(z)\}_N}$$

and

$$\mu_2 = \frac{\{A^{N-1}(z)\,A''(z)\}_{N-2}}{\{A^N(z)\}_N}.$$

Eq. $(viii)$ can be rewritten as

$$\sum_{j=0}^{N} p_{ij}\, j^2 = \lambda_2\, i^2 + (\text{polynomial in } i \text{ of degree } \le 1)$$

and in general repeating this procedure it follows that (show this)

$$\sum_{j=0}^{N} p_{ij}\, j^k = \lambda_k\, i^k + (\text{polynomial in } i \text{ of degree } \le k-1) \qquad \text{(ix)}$$

where

$$\lambda_k = \frac{\{A^{N-k}(z)\,[A'(z)]^k\}_{N-k}}{\{A^N(z)\}_N}, \quad k = 1, 2, \cdots N. \qquad \text{(x)}$$

Equations $(viii)$–(x) motivate to consider the identities

$$P\, q_k = \lambda_k\, q_k \qquad \text{(xi)}$$

where q_k are polynomials in i of degree $\le k$, and by proper choice of constants they can be chosen in that form. It follows that λ_k, $k = 1, 2, \cdots N$ given by (ix) represent the desired eigenvalues.

(a) The transition probabilities in this case follow from Example 15A-1 (page 765-766) with $A(z) = B(z) = (q + pz)^2$. Thus using (ix) we obtain the desired eigenvalues to be

$$\begin{aligned}
\lambda_k &= \frac{\{(q+pz)^{2(N-k)}[2p(q+pz)]^k\}_{N-k}}{\{(q+pz)^{2N}\}_N} \\
&= 2^k\, p^k\, \frac{\{(q+pz)^{2N-k}\}_{N-k}}{\{(q+pz)^{2N}\}_N\}} \\
&= 2^k\, \frac{\binom{2N-k}{N-k}}{\binom{2N}{N}}, \quad k = 1, 2, \cdots N.
\end{aligned}$$

(b) The transition probabilities in this case follows from Example 15A-2 (page 766) with

$$A(z) = B(z) = e^{\lambda(z-1)}$$

and hence

$$\begin{aligned}
\lambda_k &= \frac{\{e^{\lambda(N-k)(z-1)}\,\lambda^k\, e^{\lambda k(z-1)}\}_{N-k}}{\{e^{\lambda N(z-1)}\}_N} \\
&= \frac{\lambda^k\,\{e^{\lambda N z}\}_{N-k}}{\{e^{\lambda N z}\}_N} = \frac{\lambda^k\,(\lambda N)^{N-k}/(N-k)!}{(\lambda N)^N/N!} \\
&= \frac{N!}{(N-k)!\,N^k} = \left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\cdots\left(1-\frac{k-1}{N}\right), \quad k = 1, 2, \cdots N
\end{aligned}$$

(c) The transition probabilities in this case follow from Example 15A-3 (page 766-767) with

$$A(z) = B(z) = \frac{q}{1-pz}.$$

Thus

$$\begin{aligned}
\lambda_k &= p^k\, \frac{\{1/(1-pz)^{N+k}\}_{N-k}}{\{1/(1-pz)^N\}_N} \\
&= (-1)^k\, \frac{\binom{-(N+k)}{N-k}}{\binom{-N}{N}} = \frac{\binom{2N-1}{N-k}}{\binom{2N-1}{N}}, \quad r = 2, 3, \cdots N
\end{aligned}$$

15.14 From (15-240), the mean time to absorption vector is given by

$$m = (I - W)^{-1} E, \quad E = [1, 1, \cdots 1]^T,$$

where

$$W_{ik} = p_{jk}, \quad j, k = 1, 2, \cdots N-1,$$

with p_{jk} as given in (15-30) and (15-31) respectively.

15.15 The mean time to absorption satisfies (15-240). From there

$$\begin{aligned} m_i &= 1 + \sum_{k \in T} p_{ik}\, m_k = 1 + p_{i,i+1}\, m_{i+1} + p_{i,i-1}\, m_{i-1} \\ &= 1 + p\, m_{i+1} + q\, m_{i-1}, \end{aligned}$$

or

$$m_k = 1 + p\, m_{k+1} + q\, m_{k-1}.$$

This gives

$$p\,(m_{k+1} - m_k) = q\,(m_k - m_{k-1}) - 1$$

Let

$$M_{k+1} = m_{k+1} - m_k$$

so that the above iteration gives

$$\begin{aligned} M_{k+1} &= \frac{q}{p}\, M_k - \frac{1}{p} \\ &= \left(\frac{q}{p}\right)^k M_1 - \frac{1}{p}\left[1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{k-1}\right] \\ &= \begin{cases} \left(\frac{q}{p}\right)^k M_1 - \dfrac{1}{p-q}\left\{1 - (\frac{q}{p})^k\right\}, & p \neq q \\ M_1 - \dfrac{k}{p}, & p = q \end{cases} \end{aligned}$$

This gives

$$
\begin{aligned}
m_i &= \sum_{k=0}^{i-1} M_{k+1} \\
&= \begin{cases} \left(M_1 + \frac{1}{p-q}\right) \sum\limits_{k=0}^{i-1} \left(\frac{q}{p}\right)^k - \frac{i}{p-q}, & p \neq q \\ iM_1 - \frac{i(i-1)}{2p}, & p = q \end{cases} \\
&= \begin{cases} \left(M_1 + \frac{1}{p-q}\right) \frac{1-(q/p)^i}{1-q/p} - \frac{i}{p-q}, & p \neq q \\ iM_1 - \frac{i(i-1)}{2p}, & p = q \end{cases}
\end{aligned}
$$

where we have used $m_o = 0$. Similarly $m_{a+b} = 0$ gives

$$
M_1 + \frac{1}{p-q} = \frac{a+b}{p-q} \cdot \frac{1-q/p}{1-(q/p)^{a+b}}.
$$

Thus

$$
m_i = \begin{cases} \frac{a+b}{p-q} \cdot \frac{1-(q/p)^i}{1-(q/p)^{a+b}} - \frac{i}{p-q}, & p \neq q \\ i(a+b-i), & p = q \end{cases}
$$

which gives for $i = a$

$$
\begin{aligned}
m_a &= \begin{cases} \frac{a+b}{p-q} \cdot \frac{1-(q/p)^a}{1-(q/p)^{a+b}} - \frac{a}{p-q}, & p \neq q \\ ab, & p = q \end{cases} \\
&= \begin{cases} \frac{b}{2p-1} - \frac{a+b}{2p-1} \cdot \frac{1-(p/q)^b}{1-(p/q)^{a+b}}, & p \neq q \\ ab, & p = q \end{cases}
\end{aligned}
$$

by writing

$$
\frac{1-(q/p)^a}{1-(q/p)^{a+b}} = 1 - \frac{(q/p)^a - (q/p)^{a+b}}{1-(q/p)^{a+b}} = 1 - \frac{1-(p/q)^b}{1-(p/q)^{a+b}}
$$

(see also problem 3-10).

Chapter 16

16.1 Use (16-132) with $r = 1$. This gives

$$p_n = \begin{cases} \dfrac{\rho^n}{n!}\, p_0, & n \leq 1 \\ \rho^n\, p_0, & 1 < n \leq m \end{cases}$$

$$= \rho^n\, p_0, \qquad 0 \leq n \leq m$$

Thus

$$\sum_{n=0}^{m} p_n = p_0 \sum_{n=0}^{m} \rho^n = p_0\, \frac{(1-\rho^{m+1})}{1-\rho} = 1$$

$$\Longrightarrow \quad p_0 = \frac{1-\rho}{1-\rho^{m+1}}$$

and hence

$$p_n = \frac{1-\rho}{1-\rho^{n+1}}\, \rho^n, \qquad 0 \leq n \leq m, \quad \rho \neq 1$$

and $lim\, \rho \to 1$, we get

$$p_n = \frac{1}{m+1}, \qquad \rho = 1.$$

16.2 (a) Let $n_1(t) = X + Y$, where X and Y represent the two queues. Then

$$\begin{aligned} p_n &= P\{n_1(t) = n\} = P\{X + Y = n\} \\ &= \sum_{k=0}^{n} P\{X = k\}\, P\{Y = n-k\} \\ &= \sum_{k=0}^{n} (1-\rho)\rho^k\, (1-\rho)\rho^{n-k} \\ &= (n+1)(1-\rho)^2 \rho^n, \qquad n = 0, 1, 2, \cdots \end{aligned} \tag{i}$$

where $\rho = \lambda/\mu$.

(b) When the two queues are merged, the new input rate $\lambda' = \lambda + \lambda = 2\lambda$. Thus from (16-102)

$$p_n = \begin{cases} \dfrac{(\lambda'/\mu)^n}{n!} p_0 = \dfrac{(2\rho)^n}{n!}\, p_0, & n < 2 \\ \dfrac{2^2}{2!}\, (\dfrac{\lambda'}{2\mu})^n\, p_0 = 2\rho^n p_0, & n \geq 2. \end{cases}$$

Hence

$$\begin{aligned} \sum_{k-0}^{\infty} p_k &= p_0(1 + 2\rho + 2\sum_{k=2}^{\infty} \rho^k) \\ &= p_0(1 + 2\rho + \frac{2\rho^2}{1-\rho}) \\ &= \frac{p_0}{1-\rho}((1+2\rho)\,(1-\rho) + 2\rho^2) \\ &= \frac{p_0}{1-\rho}\,(1+\rho) = 1 \end{aligned}$$

$$\Longrightarrow p_0 = \frac{1-\rho}{1+\rho}, \qquad (\rho = \lambda/\mu). \tag{ii}$$

Thus

$$p_n = \begin{cases} 2\,(1-\rho)\,\rho^n/(1+\rho), & n \leq 1 \\ (1-\rho)/(1+\rho), & n = 0 \end{cases} \tag{iii}$$

(c) For an $M/M/1$ queue the average number of items waiting is given by (use (16-106) with $r = 1$)

$$E\{X\} = L_1' = \sum_{n=2}^{\infty} (n-1)\, p_n$$

where p_n is an in (16-88). Thus

$$\begin{aligned} L_1' &= \sum_{n=2}^{\infty} (n-1)(1-\rho)\,\rho^n \\ &= (1-\rho)\,\rho^2 \sum_{n=2}^{\infty} (n-1)\,\rho^{n-2} \\ &= (1-\rho)\,\rho^2 \sum_{k=1}^{\infty} k\,\rho^{k-1} \\ &= (1-\rho)\,\rho^2 \frac{1}{(1-\rho)^2} = \frac{\rho^2}{(1-\rho)}. \end{aligned} \tag{iv}$$

Since $n_1(t) = X + Y$ we have

$$\begin{aligned} L_1 &= E\{n_1(t)\} = E\{X\} + E\{Y\} \\ &= 2L_1' = \frac{2\rho^2}{1-\rho} \end{aligned} \tag{v}$$

For L_2 we can use (16-106)-(16-107) with $r = 2$. Using (iii), this gives

$$\begin{aligned} L_2 &= p_r \frac{\rho}{(1-\rho)^2} \\ &= 2\,\frac{(1-\rho)\,\rho^2}{1+\rho}\,\frac{\rho}{(1-\rho)^2} = \frac{2\,\rho^3}{1-\rho^2} \\ &= \frac{2\,\rho^2}{1-\rho}\left(\frac{\rho}{1+\rho}\right) < L_1 \end{aligned} \tag{vi}$$

From (vi), a single queue configuration is more efficient then two separate queues.

16.3 The only non-zero probabilities of this process are

$$\lambda_{0,0} = -\lambda_0 = -m\lambda, \quad \lambda_{0,1} = \mu$$

$$\lambda_{i,i+1} = (m-i)\,\lambda, \quad \lambda_{i,i-1} = i\mu$$

$$\lambda_{i,i} = [(m-i)\,\lambda + i\mu], \quad i = 1, 2, \cdots, m-1$$

$$\lambda_{m,m} = -\lambda_{m,m-1} = -m\mu.$$

Substituting these into (16-63) text, we get

$$m\,\lambda\,p_0 = \mu\,p_1 \tag{i}$$

$$[(m-i)\lambda + i\mu]\,p_i = (m-i+1)\,p_{i-1} + (i+1)\,\mu\,p_{i+1}, \quad i = 1, 2, \cdots, m-1 \tag{ii}$$

and

$$m\,\mu\,p_m = \lambda\,p_{m-1}. \tag{iii}$$

Solving (i)-(iii) we get

$$p_i = \binom{m}{i}\left(\frac{\lambda}{\lambda+\mu}\right)^i\left(\frac{\mu}{\lambda+\mu}\right)^{m-i}, \quad i = 0, 1, 2, \cdots, m$$

16.4 (a) In this case

$$p_n = \begin{cases} \dfrac{\lambda}{\mu_1}\,\dfrac{\lambda}{\mu_1}\,\cdots\,\dfrac{\lambda}{\mu_1} = \left(\dfrac{\lambda}{\mu_1}\right)^n p_0, & n < m \\ \dfrac{\lambda}{\mu_1}\,\dfrac{\lambda}{\mu_1}\,\cdots\,\dfrac{\lambda}{\mu_1}\,\dfrac{\lambda}{\mu_2}\,\cdots\,\dfrac{\lambda}{\mu_2}\,p_0, & n \geq m \end{cases}$$

$$= \begin{cases} \rho_1^n\,p_0, & n < m \\ \rho_1^{m-1}\,\rho_2^{n-m+1}\,p_0, & n \geq m, \end{cases}$$

where

$$\begin{aligned} \sum_{n=0}^{\infty} p_n &= p_0\left[\sum_{k=0}^{m-1}\rho_1^k + \rho_1^{m-1}\rho_2\sum_{n=0}^{\infty}\rho_2^n\right] \\ &= p_0\left[\frac{1-\rho_1^m}{1-\rho_1} + \frac{\rho_2\rho_1^{m-1}}{1-\rho_2}\right] = 1 \end{aligned}$$

gives

$$p_0 = \left(\frac{1-\rho_1^m}{1-\rho_1} + \frac{\rho_2 \rho_1^{m-1}}{1-\rho_2} \right)^{-1}.$$

(b)

$$\begin{aligned}
L &= \sum_{n=0}^{\infty} n\, p_n \\
&= p_0 \left[\sum_{n=0}^{m-1} n\, \rho_1^n + \sum_{n=m}^{\infty} n\, \rho_1^{m-1}\, \rho_2^{n-m+1} \right] \\
&= p_0 \left[\rho_1 \sum_{n=0}^{m-1} n\, \rho_1^{n-1} + \rho_1 \, \left(\frac{\rho_1}{\rho_2} \right)^{m-2} \sum_{n=m}^{\infty} n\, \rho_2^{n-1} \right] \\
&= p_0 \left[\rho_1 \frac{d}{d\rho_1} \left(\sum_{n=0}^{m-1} \rho_1^n \right) + \rho_1 \, \left(\frac{\rho_1}{\rho_2} \right)^{m-2} \frac{d}{d\rho_2} \sum_{n=m}^{\infty} \rho_2^n \right] \\
&= p_0 \left[\rho_1 \frac{d}{d\rho_1} \left(\frac{1-\rho_1^m}{1-\rho} \right) + \rho_1 \, \left(\frac{\rho_1}{\rho_2} \right)^{m-2} \frac{d}{d\rho} \left(\frac{\rho^m}{1-\rho} \right) \right] \\
&= p_0 \left[\frac{\rho_1 [1 + (m-1)\rho_1^m - m\rho_1^{m-1}]}{(1-\rho_1)^2} + \frac{\rho_2\, \rho_1^{m-1} + [m - (m-1)\rho_2]}{(1-\rho_2)^2} \right].
\end{aligned}$$

16.5 In this case

$$\lambda_i = \begin{cases} \lambda, & j < r \\ p\lambda, & j \geq r \end{cases} \qquad \mu_i = \begin{cases} j\mu, & j < r \\ r\mu, & j \geq r. \end{cases}$$

Using (16-73)-(16-74), this gives

$$p_n = \begin{cases} \dfrac{(\lambda/\mu)^n}{n!}\, p_0, & n < r \\ \dfrac{(\lambda/\mu)^r}{r!}\, (p\lambda/r\mu)^{n-r}, & n \geq r. \end{cases}$$

16.6

$$\begin{aligned} P\{w>t\} &= \sum_{n=r}^{m-1} p_n \, P(w>t|n) \\ &= \sum_{n=r}^{m-1} p_n \, (1-F_w(t|n)) = \sum p_r \, \left(\frac{\lambda}{r\mu}\right)^{n-r} \, (1-F_w(t|n)) \end{aligned}$$

$$f_w(t|n) = e^{-\gamma\mu t} \, \frac{(\gamma\mu)^{n-r+1} \, t^{n-r}}{(n-r)!)} \quad (\textit{see } 16.116)$$

and

$$F_w(t|n) = 1 - \sum_{k=0}^{n-r} \frac{(\gamma\mu t)^k}{k!} \, e^{-\gamma\mu t} \quad (\textit{see } 4.)$$

so that

$$1 - F_w(t|n) = \sum_{k=0}^{n-r} \frac{(\gamma\mu t)^k}{k!} \, e^{-\gamma\mu t}$$

$$\begin{aligned} P\{w>t\} &= \sum_{n=r}^{m-1} p_r \, \left(\frac{\lambda}{\gamma\mu}\right)^{n-r} \sum_{k=0}^{n-r} \frac{(\gamma\mu t)^k}{k!} \, e^{-\gamma\mu t} \\ &= \sum_{i=0}^{m-r-1} p_r \, \rho^i \sum_{k=0}^{i} \frac{(\gamma\mu t)^k}{k!} \, e^{-\gamma\mu t}, \quad n-r=i \\ &= p_r \, e^{-\gamma\mu t} \sum_{k=0}^{m-r-1} \rho^k \sum_{i=0}^{k} \frac{(\gamma\mu t)^i}{i!} \\ &= \sum_{k=0}^{m-r-1} \sum_{i=0}^{k} = \sum_{i=0}^{m-r-1} \sum_{k=i}^{m-r-1} \end{aligned}$$

$$\begin{aligned} P\{w>t\} &= p_r \, e^{-\gamma\mu t} \sum_{i=0}^{m-r-1} \frac{(\gamma\mu t)^i}{i!} \sum_{k=i}^{m-r-1} \rho^k \\ &= \frac{p_r}{1-\rho} \, e^{-\gamma\mu t} \sum_{i=0}^{m-r-1} \frac{(\gamma\mu t)^i}{i!} \, (\rho^i - \rho^{m-r}), \quad \rho = \lambda/\gamma\mu. \end{aligned}$$

Note that $m \to \infty \Longrightarrow M/M/r/m \Longrightarrow M/M/r$ and

$$\begin{aligned} P(w > t) &= \frac{p_r}{1-\rho}\, e^{-\gamma\mu t} \sum_{i=0}^{\infty} \frac{(\gamma\mu\rho t)^i}{i!} \\ &= \frac{p_r}{1-\rho}\, e^{-\gamma\mu(1-\rho)t} \quad t > 0. \end{aligned}$$

and it agrees with (16.119)

16.7 (a) Use the hints

(b)

$$-\sum_{n=1}^{\infty} (\lambda+\mu)\, p_n\, z^n + \frac{\mu}{z} \sum_{n=1}^{\infty} p_{n+1}\, z^{n+1} + \lambda \sum_{n=1}^{\infty} \sum_{k=1}^{n} p_{n-k}\, c_k\, z^n = 0$$

$$-(\rho+1)\,(P(z) - p_0) + \frac{\mu}{z}\,(P(z) - p_0 - p_1 z) + \lambda \sum_{k=1}^{\infty} c_k z^k \sum_{m=0} p_m\, z^m = 0$$

which gives

$$P(z)[1 - z - \rho z\,(1 - C(z))] = p_0(1-z)$$

or

$$P(z) = \frac{p_0(1-z)}{1 - z - \rho z\,(1 - C(z))}.$$

$$1 = P(1) = \frac{-p_0}{-1 - \rho + \rho_z\, C'(z) + \rho C(z)} = \frac{-p_0}{-1 + \rho C'(1)}$$

$$\Longrightarrow p_0 = 1 - \rho_0, \quad \rho_0 = \rho C'(1).$$

Let

$$D(z) = \frac{1 - C(z)}{1 - z}.$$

Then

$$P(z) = \frac{1 - \rho_L}{1 - \rho z D(z)}.$$

(c) This gives

$$P'(z) = \frac{(1 - \rho_c)}{(1 - \rho z D(z))^2}\; (\rho D(z) + \rho z D'(z))$$

$$L = P'(1) \quad = \frac{(1-\rho_c)}{(1-\rho_c)^2}\,\rho\,(D(1) + D'(1))$$

$$= \frac{1}{(1-\rho_c)}\;(C'(1) + D'(1))$$

$$C'(1) = E(x)$$

$$D(z) = \frac{1 - C(z)}{1 - z}$$

$$D'(z) \quad = \frac{(1-z)\,(-C'(z)) - (1 - C(z))\,(-1)}{(1-z)^2}$$

$$= \frac{1 - C(z) - (1-z)C'(z)}{(1-z)^2}$$

By L-Hopital's Rule

$$D'(1) \quad = \lim_{z\to 1} \frac{-C'(z) - (-1)C'(z) - (1-z)C''(z)}{-2(1-z)}$$

$$= \lim_{z\to 1} = 1/2C''(z) = \frac{C''(z)}{2}$$

$$= 1/2 \sum k(k-1)\,C_k = \frac{E(X^2) - E(X)}{2}$$

$$L = \frac{\rho\,(E(X) + E(X^2))}{2\,(1 - \rho E(X))}.$$

(d)

$$C(z)z^m \quad E(X) = m$$

$$P(z) = \frac{1-\rho}{1 - \rho \sum_{k=1}^{m} z^k}$$

$$D(z) = \frac{1 - z^m}{1 - z} = \sum_{k=0}^{m-1} z^k$$

$$E(X) = m, \quad E(X^2) = m^2$$

$$L = \frac{\rho(m + m^2)}{2(1 - \rho m)}$$

(e)

$$C(z) = \frac{qz}{1-Pz}$$

$$P(z) = \frac{1-\rho_0}{1-\rho z D(z)}, \quad C(z) = \frac{qz}{1-pz}$$

$$D(z) = \frac{1-C(z)}{1-z} = \frac{1-\frac{qz}{1-P(z)}}{1-z} = \frac{1-Pz-(1-P)_z}{(1-z)(1-Pz)} = \frac{1-z}{(1-z)(1-Pz)} = \frac{1}{1-Pz}$$

$$P(z) = \frac{(1-\rho_0)(1-pz)}{1-pz-\rho z} = \frac{(1-\rho_0)(1-pz)}{1-(p+\rho)z}$$

$$C'(1) = \frac{(1-pz)q-qz(-p)}{(1-Pz)^2} = \frac{q}{q^2} = \frac{1}{q}$$

$$D(z) = \frac{1-C(z)}{1-z}$$

$$D(1) = C'(1)$$

$$L = P'(1) = \frac{1-\rho_c}{(1-\rho_c)^2}\ (\rho \cdot C'(1) + \rho \cdot D'(1))$$

$$D'(z) = \frac{-(1-z)C'(z)-(1-C(z))\,(\rho-1)}{(1-z)^2} = \frac{1-C(z)-(1-z)C'(1)}{(1-z)^2}$$

$$\begin{aligned} \lim_{z\to 1} D'(z) \quad &= \lim_{z\to 1} \frac{-C'(z)-(-1)C'(z)-(1-z)C''(z)}{2(1-z)} \\ &= \frac{-(1-z)C''(z)}{-2(1-z)} = \frac{\rho''(z)}{2} \end{aligned}$$

$$D'(1) = \frac{C''(1)}{2}$$

$$L = \frac{1}{(1-\rho_c)}\left(\rho E(X) + \frac{\rho\,(E(X^2)-E(X))}{2}\right) = \frac{\rho E(X) + \rho E(X^2)}{2(1-\rho_c)}.$$

16.8 (a) Use the hints.
(b)

$$-\sum_{n=1}^{\infty}(\lambda+\mu)\,p_n\,z^n + \frac{\mu}{z^n}\sum_{n=1}^{\infty} p_{n+m}\,z^{n+m} + \lambda z\sum_{n=1}^{\infty} p_{n-1}\,z^{n-1} = 0$$

or

$$-(1+\rho)\,(P(z)-p_0)+\frac{1}{z^m}\left(P(z)-\sum_{k=0}^{m}p_k\,z^k\right)+\rho zP(z)=0$$

which gives

$$P(z)\left[\rho\,z^{m+1}-(\rho+1)\,z^m+1\right]=\sum_{k=0}^{m}p_k\,z^k-p_0\,(1+\rho)\,z^m$$

or

$$P(z)=\frac{\displaystyle\sum_{k=0}^{m}p_k\,z^k-p_0\,(1+\rho)\,z^m}{\rho\,z^{m+1}-(\rho+1)\,z^m+1}=\frac{N(z)}{M(z)}. \tag{i}$$

(c) Consider the denominator polynomial $M(z)$ in (i) given by

$$M(z)=\rho\,z^{m+1}-(1+\rho)\,z^m+1=f(z)+g(z)$$

where

$$f(z)=-(1+\rho)\,z^m,$$
$$g(z)=1+\rho\,z^{m+1}.$$

Notice that $|f(z)|>|g(z)|$ in a circle defined by $|z|=1+\varepsilon$, $\varepsilon>0$. Hence by Rouche's Theorem $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside the unit circle ($|z|=1+\varepsilon$). But $f(z)$ has m zeros inside the unit circle. Hence $f(z)+g(z)=M(z)$ also has m zeros inside the unit circle. Hence

$$M(z)=M_1(z)\,(z-z_0) \tag{ii}$$

where $|z_0|>1$ and $M_1(z)$ is a polynomial of degree m whose zeros are all *inside* or on the unit circle. But the moment generating function $P(z)$ is analytic inside and on the unit circle. Hence all the m zeros of $M(z)$ that are inside or on the unit circle must cancel out with the zeros of the numerator polynomial of $P(z)$. Hence

$$N(z)=M_1(z)\,a. \tag{iii}$$

Using (ii) and (iii) in (i) we get

$$P(z) = \frac{N(z)}{M(z)} = \frac{a}{z - z_0}.$$

But $P(1) = 1$ gives $a = 1 - z_0$
or

$$\begin{aligned} P(z) &= \frac{z_0 - 1}{z_0 - z} \\ &= \left(1 - \frac{1}{z_0}\right) \sum_{n=0}^{\infty} (z/z_0)^n \end{aligned}$$

$$\Longrightarrow p_n = \left(1 - \frac{1}{z_0}\right) \left(\frac{1}{z_0}\right)^n = (1 - r)\, r^n, \quad n \geq 0 \tag{iv}$$

where $r = 1/z_0$.

(d) Average system size

$$L = \sum_{n=0}^{\infty} n\, p_n = \frac{r}{1 - r}.$$

16.9 (a) Use the hints in the previous problem.
(b)

$$\begin{gathered} -\sum_{n=m}^{\infty} (\lambda + \mu)\, p_n\, z^n + \mu \sum_{n=m}^{\infty} p_{n+m}\, z^n + \lambda \sum_{n=m}^{\infty} p_{n-1}\, z^n \\ -(1 + \rho)\left(P(z) - \sum_{k=0}^{m-1} p_k\, z^k\right) + \frac{1}{z^m}\left(P(z) - \sum_{k=0}^{2m-1} p_k\, z^k\right) \\ +\rho\, z\left(P(z) - \sum_{k=0}^{m-2} p_k\, z^k\right) = 0. \end{gathered}$$

After some simplifications we get

$$P(z)\left[\rho\, z^{m+1} - (\rho + 1)\, z^m + 1\right] = (1 - z^m) \sum_{k=0}^{m-1} p_k\, z^k$$

or

$$P(z) = \frac{(1 - z^m) \sum_{k=0}^{m-1} p_k\, z^k}{\rho\, z^{m+1} - (\rho + 1)\, z^m + 1} = \frac{(z_0 - 1) \sum_{k=0}^{m-1} z^k}{m\,(z_0 - z)}$$

where we have made use of Rouche's theerem and $P(z) \equiv 1$ as in problem 16-8.
(c)

$$P(z) = \sum_{n=0}^{\infty} p_n \, z^n = \frac{1-r}{m} \, \frac{\sum_{k=0}^{m-1} z^k}{1 - rz}$$

gives

$$p_n = \begin{cases} (1 + r + \cdots + r^k) \, p_0, & k \le m-1 \\ r^{n-m+1} \, (1 + r + \cdots + r^{m-1}) \, p_0, & k \ge m \end{cases}$$

where

$$p_0 = \frac{1-r}{m}, \qquad r = \frac{1}{z_0}.$$

Finally

$$L = \sum_{n=0}^{\infty} n \, p_n = P_n'(1).$$

But

$$P'(z) = \left(\frac{1-r}{m}\right) \frac{\sum_{k=1}^{m-1} k \, z^{k-1} \, (1 - rz) - \sum_{k=0}^{m-1} z^k \, (-r)}{(1 - rz)^2}$$

so that

$$\begin{aligned} L &= P'(1) = \frac{1-r}{m} \, \frac{m - 1 + r}{(1-r)^2} = \frac{m - (1-r)}{m \, (1-r)} \\ &= \frac{1}{1-r} - \frac{1}{m}. \end{aligned}$$

16.10 Proceeding as in (16-212),

$$\begin{aligned} \psi_A(u) &= \int_0^{\infty} e^{-u\tau} dA(\tau) \\ &= \left(\frac{\lambda \, m}{u + \lambda \, m \, z}\right)^m. \end{aligned}$$

This gives

$$
\begin{aligned}
B(z) &= \psi_A(\psi(1-z)) \\
&= \left(\frac{\lambda\, m}{\mu\,(1-z)+\lambda\, m}\right)^m \\
&= \left(\frac{1}{1+\frac{1}{\rho}(1-z)}\right)^m \\
&= \left(\frac{\rho}{(1+\rho)-z}\right)^m, \qquad \rho=\frac{\lambda}{m\,\mu}.
\end{aligned}
\tag{i}
$$

Thus the equation $B(z) = z$ for π_0 reduce to

$$\left(\frac{\rho}{(1+\rho)-z}\right)^m = z$$

or

$$\frac{\rho}{(1+\rho)-z} = z^{1/m},$$

which is the same as

$$\rho\, z^{-1/m} = (1+\rho) - z \tag{ii}$$

Let $x = z^{-1/m}$. Sustituting this into (ii) we get

$$\rho\, x = (1+\rho) - x^{-m}$$

or

$$\rho\, x^{m+1} - (1+\rho)\, x^m + 1 = 0 \tag{iii}$$

16.11 From Example 16.7, Eq.(16-214), the characteristic equation for $Q(z)$ is given by $(\rho = \lambda/m\,\mu)$

$$1 - z[1+\rho\,(1-z)]^m = 0$$

which is equivalent to

$$1 + \rho\,(1 - z) = z^{-1/m}. \tag{i}$$

Let $x = z^{1/m}$ in this case, so that (i) reduces to

$$[(1 + \rho) - \rho\, x^m]\, x = 1$$

or the characteristic equation satisfies

$$\rho\, x^{m+1} - (1 + \rho)\, x + 1 = 0. \tag{ii}$$

16.12 Here the service time distribution is given by

$$\frac{dB(t)}{dt} = \sum_{i=1}^{k} d_i\, \delta(t - T_i)$$

and this Laplace transform equals

$$\Phi_s(s) = \sum_{i=1}^{k} d_i\, e^{-s\,T_i} \tag{i}$$

substituting (i) into (15.219), we get

$$\begin{aligned} A(z) &= \Phi_s\,(\lambda\,(1 - z)) \\ &= \sum_{i=1}^{k} d_i\, e^{-\lambda\,T_i\,(1-z)} \\ &= \sum_{i=1}^{k} d_i\, e^{-\lambda\,T_i}\, e^{\lambda\,T_i\,z} \\ &= \sum_{i=1}^{k} d_i\, e^{-\lambda\,T_i} \sum_{j=0}^{\infty} \frac{(\lambda\,T_i)^j\, z^j}{j!} = \sum_{j=0}^{\infty} a_j\, z^j. \end{aligned}$$

Hence

$$a_j = \sum_{i=1}^{k} d_i\, e^{-\lambda\,T_i}\, \frac{(\lambda\,T_i)^j}{j!}, \qquad j = 0, 1, 2, \cdots. \tag{i}$$

To get an explicit formula for the steady state probabilities $\{q_n\}$, we can make use of the analysis in (16.194)-(16.204) for an $M/G/1$ queue. From (16.203)-(16.204), let

$$c_0 = 1 - a_0, \qquad c_n = 1 - \sum_{k=0}^{n} a_k, \qquad n \geq 1$$

and let $\{c_k^{(m)}\}$ represent the $m-$fold convolution of the sequence $\{c_k\}$ with itself. Then the steady-state probabilities are given by (16.203) as

$$q_n = (1-\rho) \sum_{m=0}^{\infty} \sum_{k=0}^{n} a_k \, c_{n-k}^{(m)}.$$

(b) *State-Dependent Service Distribution*
Let $B_i(t)$ represent the service-time distribution for those customers entering the system, where the most recent departure left i customers in the queue. In that case, (15.218) modifies to

$$a_{k,i} = P\{A_k | B_i\}$$

where

$$A_k = \text{"}k \text{ customers arrive during a service time"}$$

and

$$B_i = \text{"}i \text{ customers in the system at the most recent departure."}$$

This gives

$$\begin{aligned} a_{k,i} &= \int_0^\infty e^{-\lambda t} \, \frac{(\lambda t)^k}{k!} \, dB_i(t) \\ &= \begin{cases} \displaystyle\int_0^\infty e^{-\lambda t} \, \frac{(\lambda t)^k}{k!} \, \mu_1 \, e^{-\mu_1 t} \, dt = \frac{\mu_1 \lambda^k}{(\lambda+\mu_1)^{k+1}}, & i = 0 \\ \displaystyle\int_0^\infty e^{-\lambda t} \, \frac{(\lambda t)^k}{k!} \, \mu_2 \, e^{-\mu_2 t} \, dt = \frac{\mu_2 \lambda^k}{(\lambda+\mu_2)^{k+1}}, & i \geq 1 \end{cases} \end{aligned} \tag{i}$$

This gives

$$A_i(z) = \sum_{k=0}^{\infty} a_{k,i}\, z^k = \begin{cases} \dfrac{1}{1+\rho_1(1-z)}, & i = 0 \\ \dfrac{1}{1+\rho_2(1-z)}, & i \geq 1 \end{cases} \tag{ii}$$

where $\rho_1 = \lambda/\mu_1$, $\rho_2 = \lambda/\mu_2$. Proceeding as in Example 15.24, the steady state probabilities satisfy [(15.210) gets modified]

$$q_j = q_0\, a_{j,0} + \sum_{i=1}^{j+1} q_i\, a_{j-i+1,i} \tag{iii}$$

and (see(15.212))

$$\begin{aligned} Q(z) &= \sum_{j=0}^{\infty} q_j\, z^j \\ &= q_0 \sum_{j=0}^{\infty} a_{j,0}\, z^j + \sum_{j=0}^{\infty} q_i\, a_{j-i+1,i} \\ &= q_0\, A_0(z) + \sum_{i=1}^{\infty} q_i\, z^i \sum_{m=0}^{\infty} a_{m,i}\, z^m\, z^{-1} \\ &= q_0\, A_0(z) + (Q(z) - q_0)\, A_1(z)/z \end{aligned} \tag{iv}$$

where (see (ii))

$$A_0(z) = \frac{1}{1+\rho_1(1-z)} \tag{v}$$

and

$$A_1(z) = \frac{1}{1+\rho_2(1-z)}. \tag{vi}$$

From (iv)

$$Q(z) = \frac{q_0\,(z\, A_0(z) - A_1(z))}{z - A_1(z)}. \tag{vii}$$

Since

$$Q(1) \quad = 1 = \frac{q_0\,\left[A_0'(1) + A_0(1) - A_1'(1)\right]}{1 - A_1'(1)}$$
$$= \frac{q_0\,(1 + \rho_1 - \rho_2)}{1 - \rho_2}$$

we obtain

$$q_0 = \frac{1 - \rho_2}{1 + \rho_1 - \rho_2}. \tag{viii}$$

Substituting (viii) into (vii) we can rewrite $Q(z)$ as

$$\begin{aligned} Q(z) \quad &= (1 - \rho_2)\,\frac{(1 - z)\,A_1(z)}{A_1(z) - z} \cdot \frac{1}{1 + \rho_1 - \rho_2}\,\frac{1 - z\,A_0(z)/A_1(z)}{1 - z} \\ &= \left(\frac{1 - \rho_2}{1 - \rho_2\,z}\right)\frac{1}{1 + \rho_1 - \rho_2}\,\frac{1 - \frac{\rho_2}{1+\rho_1}\,z}{1 - \frac{\rho_1}{1+\rho_1}\,z} \\ &= Q_1(z)\,Q_2(z) \end{aligned} \tag{ix}$$

where

$$Q_1(z) = \frac{1 - \rho_2}{1 - \rho_2\,z} = (1 - \rho_2)\sum_{k=0}^{\infty} \rho_2^k\,z^k$$

and

$$Q_2(z) = \frac{1}{1 + \rho_1 - \rho_2}\,\left(1 - \frac{\rho_2}{1 + \rho_1}\,z\right)\sum_{i=0}^{\infty}\left(\frac{\rho_1}{1 + \rho_1}\right)^i\,z^i.$$

Finally substituting. $Q_1(z)$ and $Q_2(z)$ into (ix) we obtain

$$q_n = q_0\left[\sum_{i=0}^{n}\left(\frac{\rho_1}{1 + \rho_1}\right)^{n-i}\,\rho_2^i - \sum_{i=0}^{n-1}\rho_2^{i+1}\,\frac{\rho_1^{n-i-1}}{(1 + \rho_1)^{n-i}}\right]. \quad n = 1, 2, \cdots$$

with q_0 as in (viii).

16.13 From (16-209), the Laplace transform of the waiting time distribution is given by

$$\begin{aligned}\Psi_w(s) &= \frac{1-\rho}{1-\lambda\left(\frac{1-\Phi_s(s)}{s}\right)} \\ &= \frac{1-\rho}{1-\rho\,\mu\left(\frac{1-\Phi_s(s)}{s}\right)}.\end{aligned} \tag{i}$$

Let

$$\begin{aligned}F_r(t) &= \mu\int_0^t [1-B(\tau)]d\tau \\ &= \mu\left[t-\int_0^t B(\tau)d\tau\right].\end{aligned} \tag{ii}$$

represent the residual service time distribution. Then its Laplace transform is given by

$$\begin{aligned}\Phi_F(s) &= L\left\{F_r(t)\right\} = \mu\left(\frac{1}{s}-\frac{\Phi_s(s)}{s}\right) \\ &= \mu\left(\frac{1-\Phi_s(s)}{s}\right).\end{aligned} \tag{iii}$$

Substituting (iii) into (i) we get

$$\Psi_w(s) = \frac{1-\rho}{1-\rho\,\Phi_F(s)} = (1-\rho)\sum_{n=0}^{\infty}[\rho\,\Phi_F(s)]^n, \quad |\Phi_F(s)|<1. \tag{iv}$$

Taking inverse transform of (iv) we get

$$F_w(t) = (1-\rho)\sum_{n=0}^{\infty}\rho^n\,F_r^{(n)}(t),$$

where $F_r^{(n)}(t)$ is the n^{th} convolution of $F_r(t)$ with itself.

16.14 Let ρ in (16.198) that represents the average number of customers that arrive during any service period be greater than one. Notice that

$$\rho = A^{'}(1) > 1$$

where

$$A(z) = \sum_{k=0}^{\infty} a_k \, z^k$$

From Theorem 15.9 on Extinction probability (pages 759-760) it follows that if $\rho = A^{'}(1) > 1$, the eqution

$$A(z) = z \tag{i}$$

has a unique positive root $\pi_0 < 1$. On the other hand, the transient state probabilities $\{\sigma_i\}$ satisfy the equation (15.236). By direct substitution with $x_i = \pi_0^i$ we get

$$\sum_{j=1}^{\infty} p_{ij} \, x_j = \sum_{j=1}^{\infty} a_{j-i+1} \, \pi_0^j \tag{ii}$$

where we have made use of $p_{ij} = a_{j-i+1}$, $i \geq 1$ in (15.33) for an $M/G/1$ queue. Using $k = j - i + 1$ in (ii), it reduces to

$$\sum_{k=2-i}^{\infty} a_k \, \pi_0^{k+i-1} = \pi_0^{i-1} \sum_{k=0}^{\infty} a_k \, \pi_0^k$$

$$= \pi_0^{i-1} \, \pi_0 = \pi_0^i = x_i \tag{iii}$$

since π_0 satisfies (i). Thus if $\rho > 1$, the $M/G/1$ system is transient with probabilities $\sigma_i = \pi_0^i$.

16.15 (a) The transition probability matrix here is the truncated version of (15.34) given by

$$P = \begin{pmatrix} a_0 & a_1 & a_2 & \cdot & \cdot & \cdot & \cdot & a_{m-2} & 1 - \sum_{k=0}^{m-2} a_k \\ a_0 & a_1 & a_2 & \cdot & \cdot & \cdot & \cdot & a_{m-2} & 1 - \sum_{k=0}^{m-2} a_k \\ 0 & a_0 & a_1 & \cdot & \cdot & \cdot & \cdot & a_{m-3} & 1 - \sum_{k=0}^{m-3} a_k \\ \vdots & \vdots & \vdots & & & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & a_0 & a_1 & 1-(a_0+a_1) \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & a_0 & 1 - a_0 \end{pmatrix} \tag{i}$$

and it corresponds to the upper left hand block matrix in (15.34) followed by an m^{th} column that makes each row sum equal to unity.

(b) By direct sybstitution of (i) into (15-167), the steady state probabilities $\{q_j^*\}_{j=0}^{m-1}$ satisfy

$$q_j^* = q_0^*\, a_j + \sum_{i=1}^{j+1} q_i^*\, a_{j-i+1}, \qquad j = 0, 1, 2, \cdots, m-2 \tag{ii}$$

and the normalization condition gives

$$q_{m-1}^* = 1 - \sum_{i=0}^{m-2} q_i^*. \tag{iii}$$

Notice that (ii) in the same as the first $m-1$ equations in (15-210) for an $M/G/1$ queue. Hence the desired solution $\{q_j^*\}_{j=0}^{m-1}$ must satisfy the first $m-1$ equations in (15-210) as well. Since the unique solution set to (15.210) is given by $\{q_j\}_{j=0}^{\infty}$ in (16.203), it follows that the desired probabilities satisfy

$$q_j^* = c\, q_j, \qquad j = 0, 1, 2, \cdots, m-1 \tag{iv}$$

where $\{q_j\}_{j=0}^{m-1}$ are as in (16.203) for an $M/G/1$ queue. From (iii) we also get the normalization constant c to be

$$c = \frac{1}{\sum_{i=0}^{m-1} q_i}. \tag{v}$$

16.16 (a) The event $\{X(t) = k\}$ can occur in several mutually exclusive ways, *viz.*, in the interval $(0, t)$, n customers arrive and k of them continue their service beyond t. Let A_n = "n arrivals in $(0, t)$", and $B_{k,n}$ ="exactly k services among the n arrivals continue beyond t", then by the theorem of total probability

$$P\{X(t) = k\} = \sum_{n=k}^{\infty} P\{A_n \cap B_{k,n}\} = \sum_{n=k}^{\infty} P\{B_{k,n}|A_n\}P(A_n).$$

But $P(A_n) = e^{-\lambda t}(\lambda t)^n/n!$, and to evaluate $P\{B_{k,n}|A_n\}$, we argue as follows: From (9.28), under the condition that there are n arrivals in $(0, t)$, the joint distribution of the arrival instants agrees with the joint distribution of n independent random variables arranged in increasing order and distributed uniformly in $(0, t)$. Hence the probability that a service time S does not terminate by t, given that its starting time $\mathbf{x}$ has a uniform distribution in $(0, t)$ is given by

$$p_t = \int_0^t P(S > t - x|\mathbf{x} = x) f_{\mathbf{x}}(x) dx$$

$$= \int_0^t [1 - B(t - x)] \frac{1}{t} dx = \frac{1}{t} \int_0^t (1 - B(\tau))\, d\tau = \frac{\alpha(t)}{t}$$

Thus $B_{k,n}$ given A_n has a Binomial distribution, so that

$$P\{B_{k,n}|A_n\} = \binom{n}{k} p_t^k (1 - p_t)^{n-k}, \quad k = 0, 1, 2, \cdots n,$$

and

$$
\begin{aligned}
P\{X(t)=k\} &= \sum_{n=k}^{\infty} e^{-\lambda t}\,\frac{(\lambda t)^n}{n!}\binom{n}{k}\left(\frac{\alpha(t)}{t}\right)^k\,\left(\frac{1}{t}\int_0^t B(\tau)d\tau\right)^{n-k} \\
&= e^{-\lambda t}\,\frac{[\lambda\alpha(t)]^k}{k!}\sum_{n=k}^{\infty}\frac{\left(\lambda t\frac{1}{t}\int_0^t B(\tau)d\tau\right)^{n-k}}{(n-k)!} \\
&= \frac{[\lambda\alpha(t)]^k}{k!}\,e^{-\lambda\left[t-\int_0^t B(\tau)d\tau\right]} \\
&= \frac{[\lambda\alpha(t)]^k}{k!}\,e^{-\lambda\int_0^t[1-B(\tau)]d\tau} \\
&= \frac{[\lambda\alpha(t)]^k}{k!}\,e^{-\lambda\,\alpha(t)}, \qquad k=0,1,2,\cdots
\end{aligned}
\tag{i}
$$

(b)

$$
\begin{aligned}
\lim_{t\to\infty}\alpha(t) &= \int_0^\infty [1-B(\tau)]d\tau \\
&= E\{\mathbf{s}\}
\end{aligned}
\tag{ii}
$$

where we have made use of (5-52)-(5-53). Using (ii) in (i), we obtain

$$
\lim_{t\to\infty} P\{x(t)=k\} = e^{-\rho}\,\frac{\rho^k}{k!} \tag{iii}
$$

where $\rho = \lambda\, E\{\mathbf{s}\}$.