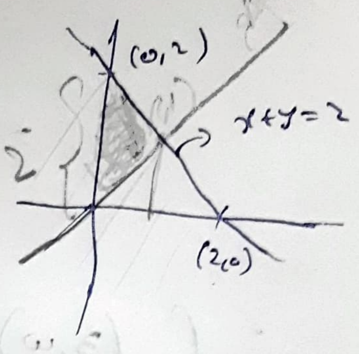
1.Sol X & Y - uniformly distributed on triangular region: $0 < x \le y \le x+y < 2$



Consider: $Z = X+Y$, $W = X-Y$

W.K.T:

$$f_{ZW}(z,w) = \sum_i |J(z,w)|_i \cdot f_{XY}(x,y)$$

given R.V's X, Y uniformly distributed over Region 'R'.

i.e. $f_{X,Y}(x,y) = k$ for some constant region.

$$\Rightarrow \iint_R f_{XY}(x,y)\,dx\,dy = \iint_R k\,dx\,dy$$

$$= k \cdot Area(R)$$

$$= k \cdot \frac{1}{2} \times 2 \times \cancel{1}$$

$$= k \cdot \cancel{1}$$

$\because \iint f_{XY}(x,y)\,dx\,dy = 1$

$$\Rightarrow \cancel{2}k = 1$$

$$\Rightarrow k = \cancel{1/2}\ 1$$

$$\therefore f_{XY}(x,y) = \begin{cases} \cancel{1/2}\ 1 ; & (x,y) \in R \\ 0, & else \end{cases}$$

$$|J(z,w)| = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix}$$

$X = \frac{Z+W}{2}$

$Y = \frac{Z-W}{2}$

$$\Rightarrow |J(z,w)| = \left|\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ +\frac{1}{2} & -\frac{1}{2} \end{bmatrix}\right| = \left| -\frac{1}{4} - \frac{1}{4} \right|$$

$$= \frac{1}{2}$$

$$\therefore f_{ZW}(z,w) = \frac{1}{2} \cdot f_{XY}\left(\frac{z+w}{2}, \frac{z-w}{2}\right)$$

$$= \frac{1}{2} \cdot \cancel{1}$$

$$= \frac{1}{2}$$

$$\therefore f_{ZW}(z,w) = \begin{cases} \frac{1}{2} ; & 0 < z < 2, \ -1 < w < 1, \\ & |w| < z. \\ 0 ; & \text{else.} \end{cases}$$

4 Sol **Given:** X & Y - independent & uniformly distributed r.v's in $(0,1)$

$W = \max(X, Y)$ & $Z = \min(X, Y)$

Consdr: $S = W + Z$

$f_X(x) = 1 \, ; \, 0 < x < 1$ $\qquad F_X(x) = x \, ; \, 0 < x < 1$

$f_Y(y) = 1 \, ; \, 0 < y < 1$ $\qquad F_Y(y) = y \, ; \, 0 < y < 1$

5 Sol $f_{XY}(x,y) = \begin{cases} e^{-x} & ; 0 \le y \le x < \infty \\ 0 & ; \text{else} \end{cases}$

$Z = X + Y$ & $W = X - Y$

$$f_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{XY}(x,y)\, dx\, dy$$

$$\cancel{f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx}$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z-x)\, dx$$

$$= \int_0^z e^{-x}\, dx$$

$$= \frac{e^{-x}}{-1}\Big|_0^z$$

$$= 1 - e^{-z} \quad ; \quad 0 < z < \infty$$

$$\boxed{\therefore f_Z(z) = 1 - e^{-z} \; ; \; 0 < z < \infty}$$

|ly $f_W(\omega) = \int_{-\infty}^{\infty} f_{XY}(x, x-\omega)\, dx$

$$= 1 - e^{-\omega} \; ; \; 0 < \omega < \infty$$

$$\boxed{\therefore f_W(\omega) = 1 - e^{-\omega} \; ; \quad 0 < \omega < \infty}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$$

$$= \int_0^x e^{-x}\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx = \int_y^{\infty} e^{-x}\, dx = \left. \frac{e^{-x}}{-1} \right|_y^{\infty}$$

$$= e^{-y}$$

$\longrightarrow$ X & Y are not independent r.v's

$$f_{ZW}(z,\omega) = \sum_i |J|_{z,\omega} \cdot f_{XY}(x,y)$$

$$= \frac{1}{2} \cdot f_{XY}\left(\frac{z+\omega}{2}, \frac{z-\omega}{2}\right)$$

$$= \frac{1}{2} \cdot e^{-(z+\omega)/2} \; ; \quad 0 < \omega < z < \infty$$

$$\boxed{\therefore f_{ZW}(z,\omega) = \frac{1}{2} \cdot e^{-(z+\omega)/2} \; ; \quad 0 < \omega < z < \infty}$$

7 Sol Given: U & V independent r.v's

'U' is uniformly distributed over (0, 1)

'V' has expotential probability with parameter λ.

$f_U(u) = 1 \,;\ 0 < u < 1$

$f_V(v) = \lambda e^{-\lambda v} \,;\ 0 < v < 1$

(a) $E\left[v^2/1+u\right] = ?$

$f_{UV}(u,v) = f_U(u) \cdot f_V(v)$

$= 1 \cdot \lambda e^{-\lambda v}$

$f_{V/U}(v/u) = \dfrac{f_{UV}(u,v)}{f_U(u)} = \lambda e^{-\lambda v}$

$\therefore E\left[v^2/1+u\right] = \int_{-\infty}^{\infty} v^2 f_{V/U}(v/u)\, dv$

$= \int_0^1 v^2 \cdot \lambda e^{-\lambda v} dv$

$$\Rightarrow \mathbb{E}\left[v^2 \mid 1+v\right] = \lambda \int_0^1 v^2 e^{-\lambda v}\, dv$$

$$= \lambda \cdot \left[\frac{e^{-\lambda v} v^2}{-\lambda} \Big|_0^1 + \int_0^1 2v \cdot \frac{e^{-\lambda v}}{\lambda}\, dv \right]$$

$$= \lambda \left[-\frac{e^{-\lambda}}{\lambda} + \frac{2}{\lambda} \cdot \int_0^1 v \cdot e^{-\lambda v}\, dv \right]$$

$$= -e^{-\lambda} + 2 \left[\frac{e^{-\lambda v}}{-\lambda} \cdot v \Big|_0^1 + \int_0^1 \frac{e^{-\lambda v}}{+\lambda}\, dv \right]$$

$$= -e^{-\lambda} + 2 \left[\frac{-e^{-\lambda}}{\lambda} - \frac{e^{-\lambda v}}{\lambda^2} \Big|_0^1 \right]$$

$$= -e^{-\lambda} - \frac{2e^{-\lambda}}{\lambda} - \frac{2e^{-\lambda}}{\lambda^2} + \frac{2}{\lambda^2}$$

$$= \frac{2}{\lambda^2} - e^{-\lambda} \left(\frac{2}{\lambda^2} + \frac{2}{\lambda} + 1 \right)$$

(9) Given Pair of Stationary Process $X(t)$ & $Y(t)$

P.T.P: (a) $R_{XY}(\tau) = R_{YX}(-\tau)$

(b) $|R_{XY}(\tau)| \leq \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$

Proof:

$$R_{XY}(\tau) = E[X(t+\tau) \cdot Y(t)]$$

$$R_{YX}(\tau) = E[Y(t+\tau) \cdot X(t)]$$

$$= E[Y(t) \cdot X(t-\tau)]$$

$$R_{YX}(-\tau) = E[Y(t-\tau) \cdot X(t)]$$

$$= E[Y(t) \; X(t+\tau)]$$

$$= E[X(t+\tau) \cdot Y(t)]$$

$$= R_{XY}(\tau)$$

$$\boxed{\therefore R_{XY}(\tau) = R_{YX}(-\tau)}$$

Hence, Proved.

~~W.k.T~~

$$\cancel{R_{xy}(\tau) = E[x(t+\tau) \cdot y(t)]}$$

$$\cancel{E\{[x(t+\tau) \cdot y(t)]^2\} \geq 0}$$

$$\Rightarrow \cancel{E\{x^2(t+\tau) + 2x(t+\tau)y(t) + y^2(t)\} \geq 0}$$

$$\Rightarrow$$

W.k.T:

$$E\left[\{x(t+\tau) - y(t)\}^2\right] \geq 0$$

$$\Rightarrow E\left[x^2(t+\tau) - 2x(t+\tau)\,y(t) + y^2(t)\right] \geq 0$$

$$\Rightarrow R_{xx}(0) + R_{yy}(0) - 2R_{xy}(\tau) \geq 0$$

$$\Rightarrow |R_{xy}(\tau)| \leq \frac{1}{2}\left[R_{xx}(0) + R_{yy}(0)\right]$$

$$\boxed{\therefore R_{xy}(\tau) \leq \frac{1}{2}\left[R_{xx}(0) + R_{yy}(0)\right]}$$

Hence, proved