Solutions Manual

to accompany

Probability, Random Variables and Stochastic Processes

Fourth Edition

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CHAPTER 2

2-1 We use De Morgan's law:

(a) $\overline{\overline{A}+\overline{B}} + \overline{\overline{A}+B} = AB + A\overline{B} = A(B+\overline{B}) = A$

(b) $(A+B)(\overline{AB}) = (A+B)(\overline{A}+\overline{B}) = A\overline{B} + B\overline{A}$

because $A\overline{A} = \{\emptyset\}$ $B\overline{B} = \{\emptyset\}$

2-2 If $A = \{2 \le x \le 5\}$ $B = \{3 \le x \le 6\}$ $S = \{-\infty < x < \infty\}$ then

$A+B = \{2 \le x \le 6\}$ $AB = \{3 \le x \le 5\}$

$(A+B)(\overline{AB}) = \{2 \le x \le 6\}\ [\{x < 3\} + \{x > 5\}]$

$= \{2 \le x < 3\} + \{5 < x \le 6\}$

2-3 If $AB = \{\emptyset\}$ then $A \subset \overline{B}$ hence

$P(A) \le P(\overline{B})$

2-4 (a) $P(A) = P(AB) + P(A\overline{B})$ $P(B) = P(AB) + P(\overline{A}B)$

If, therefore, $P(A) = P(B) = P(AB)$ then

$P(A\overline{B}) = 0$ $P(\overline{A}B) = 0$ hence

$P(\overline{A}B + A\overline{B}) = P(\overline{A}B) + P(A\overline{B}) = 0$

(b) If $P(A) = P(B) = 1$ then $1 = P(A) \le P(A+B)$ hence

$1 = P(A+B) = P(A) + P(B) - P(AB) = 2 - P(AB)$

This yields $P(AB) = 1$

2-5 From (2-13) it follows that

$P(A+B+C) = P(A) + P(B+C) - P[A(B+C)]$

$P(B+C) = P(B) + P(C) - P(BC)$

$P[A(B+C)] = P(AB) + P(AC) - P(ABC)$

because $ABAC = ABC$. Combining, we obtain the desired result.

Using induction, we can show similarly that

$$P(A_1 + A_2 + \cdots + A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$
$$- P(A_1A_2) - \cdots - P(A_{n-1}A_n)$$
$$+ P(A_1A_2A_3) + \cdots + P(A_{n-2}A_n)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\pm P(A_1A_2 \cdots A_n)$$

2-6 Any subset of S contains a countable number of elements, hence, it can be written as a countable union of elementary events. It is therefore an event.

2-7 Forming all unions, intersections, and complements of the sets {1} and {2,3}, we obtain the following sets:

{∅}, {1}, {4}, {2,3}, {1,4}, {1,2,3}, {2,3,4}, {1,2,3,4}

2-8 If $A \subset B, P(A) = 1/4$, and $P(B) = 1/3$, then

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)}{P(B)} = \frac{1/4}{1/3} = \frac{3}{4}$$

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

2-9

$$P(A|BC)P(B|C) = \frac{P(ABC)}{P(BC)}\frac{P(BC)}{P(C)} = \frac{P(ABC)}{P(C)} = P(AB|C)$$

$$P(A|BC)P(B|C)P(C) = \frac{P(ABC)}{P(BC)}\frac{P(BC)}{P(C)}P(C) = P(ABC)$$

2-10 We use induction. The formula is true for $n = 2$ because $P(A_1A_2) = P(A_2|A_1)P(A_1)$. Suppose that it is true for n. Since

$$P(A_{n+1}A_n \cdots A_1) = P(A_{n+1}|A_n \cdots A_2A_1)P(A_1 \cdots A_n)$$

we conclude that it must be true for $n + 1$.

2-11 First solution. The total number of m element subsets equals $\binom{n}{m}$ (see Probl. 2-26). The total number of m element subsets containing ζ_o equals $\binom{n-1}{m-1}$. Hence

$$p = \binom{n}{m}\Big/\binom{n-1}{m-1} = \frac{m}{n}$$

Second solution. Clearly, $P\{\zeta_o|A_m\} = m/n$ is the probability that ζ_o is in a specific A_m. Hence (total probability)

$$p = \sum P\{\zeta_o|A_m\}p(A_m) = \frac{m}{n}\sum P(A_m) = \frac{m}{n}$$

where the summation is over all sets A_m.

2-12 (a) $P\{6 \leq t \leq 8\} = \frac{2}{10}$

(b) $P\{6 \leq t \leq 8 | t > 5\} = \frac{P\{6 \leq t \leq 8\}}{P\{t > 5\}} = \frac{2}{5}$

2-13 From (2-27) it follows that

$$P\{t_o \leq t \leq t_o + t_1 | t \geq t_o\} = \int_{t_o}^{t_o + t_1} \alpha(t)dt \Big/ \int_{t_o}^{\infty} \alpha(t)dt$$

$$P\{t \leq t_1\} = \int_0^{t_1} \alpha(t)dt$$

Equating the two sides and setting $t_1 = t_0 + \Delta t$ we obtain

$$\alpha(t_o) \Big/ \int_{t_o}^{\infty} \alpha(t)dt = \alpha(0)$$

for every t_o. Hence,

$$- \ln \int_{t_o}^{\infty} \alpha(t)dt = \alpha(0)t_o \qquad \int_{t_o}^{\infty} \alpha(t)dt = e^{-\alpha(0)t_o}$$

Differentiating the setting $c = \alpha(0)$, we conclude that

$$\alpha(t_o) = c\,e^{ct} \qquad P\{t \leq t_1\} = 1 - e^{-ct_1}$$

2-14 If A and B are independent, then $P(AB) = P(A)P(B)$. If they are mutually exclusive, then $P(AB) = 0$. Hence, A and B are mutually exclusive and independent iff $P(A)P(B) = 0$.

2-15 Clearly, $A_1 = A_1A_2 + A_1\bar{A}_2$ hence

$$P(A_1) = P(A_1A_2) + P(A_1\bar{A}_2)$$

If the events A_1 and $\bar{A}_2$ are independent, then

$$P(A_1\bar{A}_2) = P(A_1) - P(A_1A_2) = P(A_1) - P(A_1)P(A_2)$$
$$= P(A_1)[1 - P(A_2)] = P(A_1)P(\bar{A}_2)$$

hence, the events A_1 and $\bar{A}_2$ are independent. Furthermore, S is independent with any A because SA = A. This yields

$$P(SA) = P(A) = P(S)P(A)$$

Hence, the theorem is true for $n = 2$. To prove it in general we use induction: Suppose that A_{n+1} is independent of $A_1, \ldots, A_n$. Clearly, A_{n+1} and $\bar{A}_{n+1}$ are independent of $B_1, \ldots, B_n$. Therefore

$$P(B_1 \cdots B_n A_{n+1}) = P(B_1 \cdots B_n)P(A_{n+1})$$
$$P(B_1 \cdots B_n \bar{A}_{n+1}) = P(B_1 \cdots B_n)P(\bar{A}_{n+1})$$

2.16 The desired probabilities are given by (a)

$$\frac{\binom{m-1}{k-1}}{\binom{n}{k}}$$

(b)

$$\frac{\binom{m}{k}}{\binom{n}{k}}$$

2.17 Let A_1, A_2 and A_3 represent the events

$A_1 =$ *"ball numbered less than or equal to m is drawn"*
$A_2 =$ *"ball numbered m is drawn"*
$A_3 =$ *"ball numbered greater than m is drawn"*

$$
\begin{aligned}
P(A_1 \text{ occurs } n_1 = k-1, \quad & A_2 \text{ occurs } n_2 = 1 \text{ and } A_3 \text{ occurs } n_3 = 0) \\
&= \frac{(n_1+n_2+n_3)!}{n_1!\,n_2!\,n_3!} p_1^{n_1} p_2^{n_2} p_3^{n_3} \\
&= \frac{k!}{(k-1)!} \left(\frac{m}{n}\right)^{k-1} \left(\frac{1}{n}\right) \\
&= \frac{k}{n} \left(\frac{m}{n}\right)^{k-1}
\end{aligned}
$$

2.18 All cars are equally likely so that the first car is selected with probability $p = 1/3$. This gives the desired probability to be

$$\binom{10}{3} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^7 = 0.26$$

2.19 $P\{\text{"}drawing\ a\ white\ ball\text{ "}\} = \dfrac{m}{m+n}$
$P(\text{"}atleat\ one\ white\ ball\ in\ k\ trials\text{ "})$

$$
\begin{aligned}
&= 1 - P(\text{"}all\ black\ balls\ in\ k\ trials\text{"}) \\
&= 1 - \frac{\binom{n}{k}}{\binom{m+n}{k}}
\end{aligned}
$$

2.20 Let $D = 2r$ represent the penny diameter. So long as the center of the penny is at a distance of r away from any side of the square, the penny will be entirely inside the square. This gives the desired probability to be

$$\frac{(1-2r)^2}{1} = \left(1 - \frac{3}{4}\right)^2 = \frac{1}{16}.$$

2.21 Refer to Example 3.14.
(a) Using (3.39), we get

$$P(\text{"}all\ one-digit\ numbers\text{"}) = \frac{\binom{9}{6}\binom{42}{0}}{\binom{51}{6}} = 5\times 10^{-6}.$$

(b)

$$P(\text{"}two\ one\text{–}digit\ and\ four\ two\text{–}digit\ numbers\text{"}) = \frac{\binom{9}{2}\binom{42}{4}}{\binom{51}{6}} = 0.224.$$

2-22 The number of equations of the form $P(A_iA_k) = P(A_i)P(A_k)$ equals $\binom{n}{2}$. The number of equations involving r sets equals $\binom{n}{r}$. Hence the total number N of such equations equals

$$N = \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n}$$

And since

$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = (1+1)^n = 2^n$$

we conclude that

$$N = 2^n - \binom{n}{0} - \binom{n}{1} = 2^n - 1 - n$$

2-23 We denote by B_1 and B_2 respectively the balls in boxes 1 and 2 and by R the set of red balls. We have (assumption)

$$P(B_1) = P(B_2) = 0.5 \qquad P(R|B_1) = 0.999 \qquad P(R|B_2) = 0.001$$

Hence (Bayes' theorem)

$$P(B_1|R) = \frac{P(R|B_1)P(B_1)}{P(R|B_1)P(B_1) + P(R|B_2)P(B_2)} = \frac{0.999}{0.999+0.001} = 0.999$$

2-24 We denote by B_1 and B_2 respectively the ball in boxes 1 and 2 and by D all pairs of defective parts. We have (assumption)

$$P(B_1) = P(B_2) = 0.5$$

To find $P(D|B_1)$ we proceed as in Example 2-10:

First solution. In box B_1 there are 1000×999 pairs. The number of pairs with both elements defective equals 100×99. Hence,

$$P(D|B_1) = \frac{100 \times 99}{1000 \times 999}$$

Second solution. The probability that the first bulb selected from B_1 is defective equals 100/1000. The probability that the second is defective assuming the first was effective equals 99/999. Hence,

$$P(D|B_1) = \frac{100}{1000} \times \frac{99}{999}$$

We similarly find

$$P(D|B_2) = \frac{100}{2000} \times \frac{99}{1999}$$

(a) $P(D) = P(D|B_1)P(B_1) + P(D|B_2)P(B_2) = 0.0062$

(b) $P(B_1|D) = \dfrac{P(D|B_1)P(B_1)}{P(D)} = 0.80$

2-25 Reasoning as in Example 2-13, we conclude that the probability that the bus and the train meet equals

$$(10+x)60 - \frac{10^2}{2} - \frac{x^2}{2}$$

Equating with 0.5, we find $x = 60 - 10\sqrt{11}$.

2-26 We wish to show that the number $N_n(k)$ of the element subsets of S equals

$$N_n(k) = \frac{n(n-1)\cdots(n-k+1)}{1 \cdot 2 \cdots\ k}$$

This is true for $k=1$ because the number of 1-element subsets equals n. Using induction in k, we shall show that

$$N_n(k+1) = N_n(k)\,\frac{n-k}{k+1} \qquad 1 < k < n \qquad \text{(i)}$$

We attach to each k-element subset of S one of the remaining $n-k$ elements of S. We, then, form $N_n(k)(n-k)$ $k+1$-element subsets. However, these subsets are not all different. They form groups each of which has $k+1$ identical elements. We must, therefore, divide by $k+1$.

2-27 In this experiment we have 8 outcomes. Each outcome is a selection of a particular coin and a specific sequence of heads or tails; for example fhh is the outcome "we selected the fair coin and we observed hh". The event F = {the selected coin is fair} consists of the four outcomes fhh, fht, fth and fhh. Its complement $\overline{F}$ is the selection of the two-headead coin. The event HH = {heads at both tosses} consists of two outcomes. Clearly,

$$P(F) = P(\overline{F}) = \frac{1}{2} \qquad P(HH|F) = \frac{1}{4} \qquad P(HH|\overline{F}) = 1$$

Our problem is to find P(F|HH). From (2-41) and (2-43) it follows that

$$P(HH) = P(HH|F)P(F) + P(HH|\overline{F})P(\overline{F}) = \frac{5}{8}$$

$$P(F|HH) = \frac{P(HH|F)P(F)}{P(HH)} = \frac{1/4 \times 1/2}{5/8} = \frac{1}{5}$$

3.1 (a) $P(\text{A } occurs\ atleast\ twice\ in\ n\ trials)$

$= 1 - P(\text{A } never\ occurs\ in\ n\ trials) - P(\text{A } occurs\ once\ in\ n\ trials)$

$= 1 - (1-p)^n - np(1-p)^{n-1}$

(b) $P(\text{A } occurs\ atleast\ thrice\ in\ n\ trials)$

$= 1 - P(\text{A never occurs in } n \text{ trials}) - P(\text{A occurs once in } n \text{ trials})$

$-P(\text{A occurs twice in } n \text{ trials})$

$= 1 - (1-p)^n - np(1-p)^{n-1} - \frac{n(n-1)}{2} p^2 (1-p)^{n-2}$

3.2

$$P(doublesix) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

$P(\text{"double six atleast three times in n trials"})$

$= 1 - \binom{50}{0}\left(\frac{1}{36}\right)^0\left(\frac{35}{36}\right)^{50} - \binom{50}{1}\left(\frac{1}{36}\right)\left(\frac{35}{36}\right)^{49} - \binom{50}{2}\left(\frac{1}{36}\right)^2\left(\frac{35}{36}\right)^{48}$

$= 0.162$

3-3 If A = {seven}, then

$$P(A) = \frac{6}{36} \qquad P(\bar{A}) = \frac{5}{6}$$

If the dice are tossed 10 times, then the probability that $\bar{A}$ will occur 10 times equals $(5/6)^{10}$. Hence, the probability p that {seven} will show at least once equals

$$1 - (5/6)^{10}$$

3-4 If k is the number of heads, then

$$P\{\text{even}\} = P\{k = 0\} + P\{k = 2\} + \cdots$$
$$= q^n + \binom{n}{2}p^2q^{n-2} + \binom{n}{4}p^4q^{n-4} + \cdots$$

But

$$1 = (q + q)^n = q^n + \binom{n}{1}p\,q^{n-1} + \binom{n}{2}p^2q^{n-2} + \cdots$$
$$(p - q)^n = q^n - \binom{n}{1}p\,q^{n-1} + \binom{n}{2}p^2\,q^{n-2} - \cdots$$

Adding, we obtain

$$1 + (p - q)^n = 2\,P\{\text{even}\}$$

3-5 In this experiment, the total number of outcomes is the number $\binom{N}{n}$ of ways of picking n out of N objects. The number of ways of picking k out of the K good components equals $\binom{K}{k}$ and the number of ways of picking n-k out of the N-K defective components equals $\binom{N-K}{n-k}$. Hence, the number of ways of picking k good components and n-k deafective components equals $\binom{K}{k}\binom{N-K}{n-k}$. From this and (2-25) it follows that

$$p = \binom{K}{k}\binom{N-K}{n-k} \Big/ \binom{N}{n}$$

3.6 (a)

$$p_1 = 1 - \left(\frac{5}{6}\right)^6 = 0.665$$

(b)

$$1 - \left(\frac{5}{6}\right)^{12} - \binom{12}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{11} = 0.619$$

(c)

$$1 - \left(\frac{5}{6}\right)^{18} - \binom{18}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{17} - \binom{18}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{16} = 0.597$$

3.7 (a) Let n represent the number of wins required in 50 games so that the net gain or loss *does not* exceed \$1. This gives the net gain to be

$$-1 < n - \frac{50-n}{4} < 1$$
$$16 < n < 17.3$$
$$n = 17$$

$$P(\text{net gain } \textit{does not} \text{ exceed \$1}) = \binom{50}{17}\left(\frac{1}{4}\right)^{17}\left(\frac{3}{4}\right)^{33} = 0.432$$
$$P(\text{net gain or loss exceeds \$1}) = 1 - 0.432 = 0.568$$

(b) Let n represent the number of wins required so that the net gain or loss *does not* exceed \$5. This gives

$$-5 < n - \frac{(50-n)}{2} < 5$$
$$13.3 < n < 20$$

$$P(\text{net gain } \textit{does not} \text{ exceed \$5}) = \sum_{n=14}^{19}\binom{50}{n}\left(\frac{1}{4}\right)^{n}\left(\frac{3}{4}\right)^{50-n} = 0.349$$
$$P(\text{net gain or loss exceeds \$5}) = 1 - 0.349 = 0.651$$

3.8 Define the events

A=" r successes in n Bernoulli trials"

B="success at the i^{th} Bernoulli trial"

C="$r-1$ successes in the remaining $n-1$ Bernoulli trials excluding the i^{th} trial"

$$P(A) = \binom{n}{r} p^r \, q^{n-r}$$

$$P(B) = p$$

$$P(C) = \binom{n-1}{r-1} p^{r-1} \, q^{n-r}$$

We need

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(BC)}{P(A)} = \frac{P(B)\,P(C)}{P(A)} = \frac{r}{n}.$$

3.9 There are $\binom{52}{13}$ ways of selecting 13 cards out of 52 cards. The number of ways to select 13 cards of any suit (out of 13 cards) equals $\binom{13}{13} = 1$. Four such (mutually exclusive) suits give the total number of favorable outcomes to be 4. Thus the desired probability is given by

$$\frac{4}{\binom{52}{13}} = 6.3 \times 10^{-12}$$

3.10 Using the hint, we obtain

$$p\,(N_{k+1} - N_k) = q\,(N_k - N_{k-1}) - 1$$

Let

$$M_{k+1} = N_{k+1} - N_k$$

so that the above iteration gives

$$\begin{aligned} M_{k+1} &= \tfrac{q}{p}\, M_k - \tfrac{1}{p} \\ &= \begin{cases} \left(\tfrac{q}{p}\right) M_1 - \dfrac{1}{p-q}\left\{1 - (\tfrac{q}{p})^i\right\}, & p \neq q \\ M_1 - \tfrac{k}{p}, & p = q \end{cases} \end{aligned}$$

This gives

$$\begin{aligned} N_i &= \sum_{k=0}^{i-1} M_{k+1} \\ &= \begin{cases} \left(M_1 + \dfrac{1}{p-q}\right) \displaystyle\sum_{k=0}^{i-1} \left(\tfrac{q}{p}\right)^k - \dfrac{i}{p-q}, & p \neq q \\ iM_1 - \dfrac{i(i-1)}{2p}, & p = q \end{cases} \end{aligned}$$

where we have used $N_o = 0$. Similarly $N_{a+b} = 0$ gives

$$M_1 + \frac{1}{p-q} = \frac{a+b}{p-q} \cdot \frac{1 - q/p}{1 - (q/p)^{a+b}}.$$

Thus

$$N_i = \begin{cases} \dfrac{a+b}{p-q} \cdot \dfrac{1 - (q/p)^i}{1 - (q/p)^{a+b}} - \dfrac{i}{p-q}, & p \neq q \\ i(a+b-i), & p = q \end{cases}$$

which gives for $i = a$

$$
\begin{aligned}
N_a &= \begin{cases} \dfrac{a+b}{p-q} \cdot \dfrac{1-(q/p)^a}{1-(q/p)^{a+b}} - \dfrac{a}{p-q}, & p \neq q \\ \qquad ab, & p = q \end{cases} \\
&= \begin{cases} \dfrac{b}{2p-1} - \dfrac{a+b}{2p-1} \cdot \dfrac{1-(q/p)^b}{1-(q/p)^{a+b}} - \dfrac{a}{p-q}, & p \neq q \\ \qquad ab, & p = q \end{cases}
\end{aligned}
$$

3.11

$$P_n = pP_{n+\alpha} + qP_{n-\beta}$$

Arguing as in (3.43), we get the corresponding iteration equation

$$P_n = P_{n+\alpha} + qP_{n-\beta}$$

and proceed as in Example 3.15.

3.12 Suppose one bet on $k = 1, 2, \cdots, 6$.
Then

$$p_1 = P(k \text{ appears on one dice}) = \binom{3}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2$$

$$p_2 = P(k \text{ appear on two dice}) = \binom{3}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)$$

$$p_3 = P(k \text{ appear on all the tree dice}) = \left(\frac{1}{6}\right)^3$$

$$p_0 = P(k \text{ appear none}) = \left(\frac{5}{6}\right)^3$$

Thus, we get

$$\text{Net gain} = 2p_1 + 3p_2 + 4p_3 - p_0 = 0.343.$$

CHAPTER 4

4-1 From the evenness of $f(x)$: $1 - F(x) = F(-x)$.

From the definition of x_u: $u = F(x_u)$, $1 - u = F(x_{1-u})$. Hence

$$1 - u = 1 - F(x_u) = F(-x_u) = F(x_{1-u}) \qquad -x_u = x_{1-u}$$

4-2 From the symmetry of $f(x)$: $1 - F(\eta+a) = F(\eta-a)$. Hence [see (4-8)]

$$P\{\eta-a < \underset{\sim}{x} < \eta+a\} = F(\eta+a) - F(\eta-a) = 2F(\eta+a) - 1$$

This yields

$$1-\alpha = 2F(\eta+a) - 1 \qquad F(\eta+a) = 1 - \alpha/2 \qquad \eta+a = x_{1-\alpha/2}$$

$$F(a-\eta) = \alpha/2 \qquad a-\eta = x_{\alpha/2}$$

4-3 (a) In a linear interpolation:

$$x_u \simeq x_a + \frac{x_b-x_a}{u_b-u_a}(u-u_a) \qquad \text{for } x_a < x_u < x_b$$

From Table 4-1 page 106

$$z_{0.9} \simeq 1.25 + \frac{0.00565}{0.00885} \times 0.05 = 1.2819$$

Proceeding simiplarly, we obtain

$u =$	0.9	0.925	0.95	0.975	0.99
$z_u =$	1.282	1.440	1.645	1.960	2.327

(b) If $\underset{\sim}{z}$ is such that $\underset{\sim}{x} = \eta + \sigma\underset{\sim}{z}$ then $\underset{\sim}{z}$ is N(0,1) and $G(z) = F_x(\eta+\sigma z)$. Hence,

$$u = G(z_u) = F_x(\eta+\sigma z_u) = F_x(x_u) \qquad x_u = \eta+\sigma z_u$$

4-4 $p_k - 2G(k) = 1 = 2\,\text{erf}k$

(a) From Table 4-1

$k =$	1	2	3
$p_k =$	0.6827	0.9545	0.9973

(b) From Table 3-1 with linear interpolation:

$p_k =$	0.9	0.99	0.999
$k =$	1.282	2.32	3.090

(c) $P\{\eta - z_u\sigma < \underset{\sim}{x} < \eta + z_u\sigma\} = 2G(z_u) - 1 = \gamma$

Hence, $G(z_u) = (1+\gamma)/2 \qquad u = (1+\gamma)/2$

4-5 (a) $F(x) = x$ for $0 \le x \le 1$; hence, $u = F(x_u) \doteq x_u$

(b) $F(x) = 1-e^{-2x}$ for $x \ge 0$; hence, $u = 1-e^{-2x_u}$

$$x_u = -\frac{1}{2}\ell n(1-u)$$

$u =$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$x_u =$	0.0527	0.1116	0.1783	0.2554	0.3466	0.4581	0.6020	0.847	1.1513

4-6 Percentage of units between 96 and 104 ohms equals 100p where $p = P\{96 < \underset{\sim}{R} < 104\} = F(104) - F(96)$

(a) $F(R) = 0.1(R-95)$ for $95 \le R \le 105$. Hence,

$p = 0.1(104-95) - 0.1(96-95) = 0.8$

(b) $p = G(2.5) - G(-2.5) = 0.9876$

4-7 From (4-34), with $\alpha = 2$ and $\beta = 1/\lambda$ we get $f(x) = c^2\, xe^{-cx}U(x)$

$$F(x) = c^2 \int_0^x ye^{-cy}dy = 1 - e^{-cx} - cxe^{-cx}$$

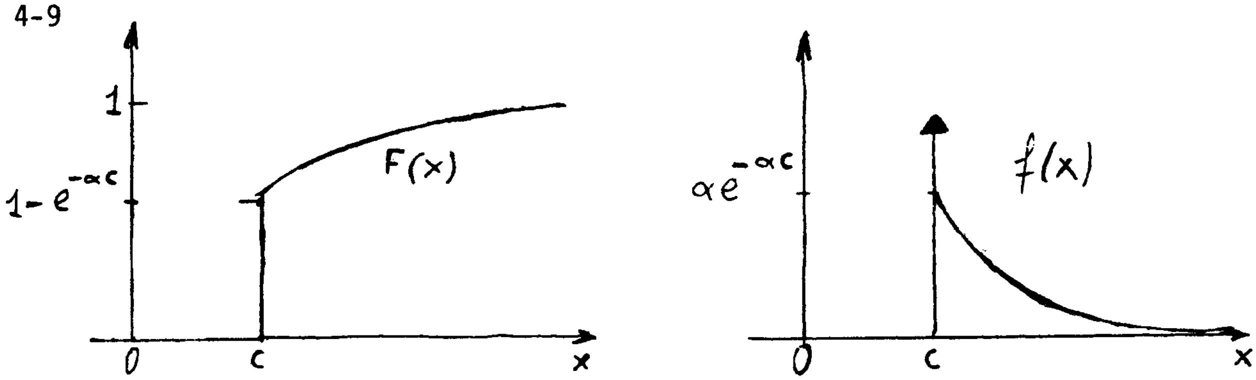
4-8 $\{(\underset{\sim}{x} - 10)^2 < 4\} = \{8 < \underset{\sim}{x} < 12\}$

$$P\{(\underset{\sim}{x} - 10)^2 < 4\} = G(12 - 10) - G(8 - 10) = 0.954$$

$$f(x|(\underset{\sim}{x} - 10)^2 < 4\} = \frac{f(x)}{P\{8 < \underset{\sim}{x} < 12\}} = \frac{1}{0.954\sqrt{2\pi}} e^{-\frac{(x-10)^2}{2}}$$

for $8 < x < 12$ and zero otherwise

4-9



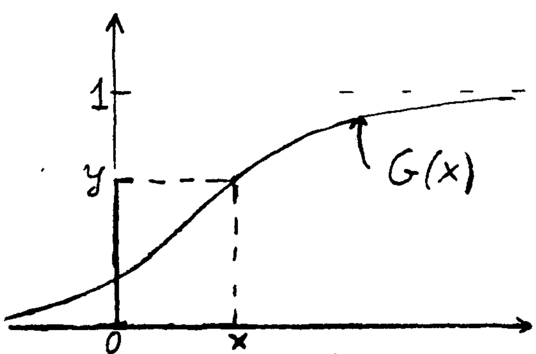
$$F(x) = (1 - e^{-\alpha x})U(x-c) \qquad f(x) = (1 - e^{-\alpha c})\delta(x-c) + e^{-\alpha x}U(x-c)$$

4-10 (a) $P\{1 \le x \le 2\} = G(\frac{2}{2}) - G(\frac{1}{2}) = 0.1499$

(b) $P\{1 \le \underset{\sim}{x} \le 2|\underset{\sim}{x} \ge 1\} = \dfrac{G(1) - G(0.5)}{1 - G(0.5)} = \dfrac{0.1499}{0.3085} = 0.4857$

because $\{1 \le \underset{\sim}{x} \le 2,\ \underset{\sim}{x} \ge 1\} = \{1 \le \underset{\sim}{x} \le 2\}$

4-11



$x = G^{-1}(y) = H(y)$

If $\underset{\sim}{x}(t_1) \le x$

then

$t_i \le y = G(x)$

Hence,

$$P\{\underset{\sim}{x} \le x\} = P\{t_i \le y\} = y = G(x)$$

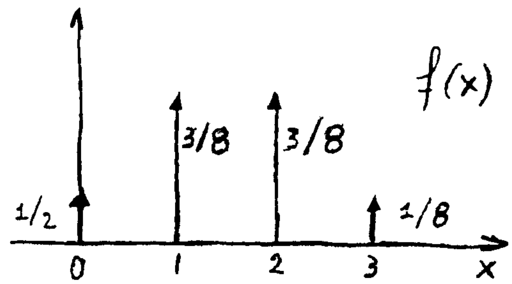
4-12 (a) $P\{\underset{\sim}{x} < 1024\} = G(\frac{1024 - 1000}{20}) = G(1.2) = 0.8849$

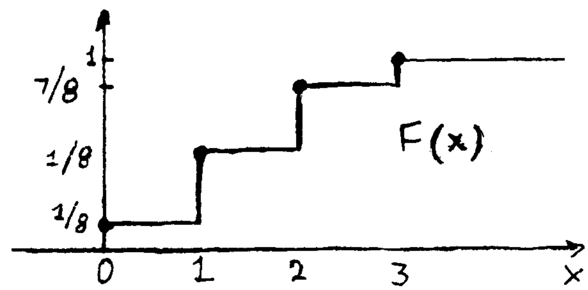
(b) $P\{\underset{\sim}{x} < 1024 | \underset{\sim}{x} > 961\} = \dfrac{P\{961 < \underset{\sim}{x} < 1024\}}{P\{\underset{\sim}{x} > 961\}}$

$$= \frac{G(1.2) - G(-1.95)}{1 - G(-1.95)} = 0.8819$$

(c) $P\{31 < \sqrt{\underset{\sim}{x}} \le 32\} = P\{961 < \underset{\sim}{x} \le 1024\} = 0.8593$

4-13 $P\{\underset{\sim}{x} = 0\} = \frac{1}{8}$ $\quad P\{\underset{\sim}{x} = 1\} = \frac{3}{8}$ $\quad P\{\underset{\sim}{x} = 2\} = \frac{3}{8}$ $\quad P\{\underset{\sim}{x} = 3\} = \frac{1}{8}$





4-14 (a) 1. $f_x(x) = \dfrac{1}{2^{900}} \sum_{k=0}^{900} \binom{900}{k} \delta(x-k)$

2. $f_x(x) = \dfrac{1}{15\sqrt{2\pi}} \sum_{k=0}^{900} e^{-(k-450)^2/450} \delta(x-k)$

(b) $P\{435 \le x \le 460\} = G(\frac{10}{15}) - G(-\frac{15}{15}) = 0.5888$

4-15 If $x > b$ then $\{\underset{\sim}{x} \le x\} = S$ $\quad F(x) = 1$

If $x < a$ then $\{\underset{\sim}{x} \le x\} = \{\emptyset\}$ $\quad F(x) = 0$

4-16 If $\underset{\sim}{y}(\zeta_i) \le w$, then $\underset{\sim}{x}(\zeta_i) \le w$ because $\underset{\sim}{x}(\zeta_i) \le \underset{\sim}{y}(\zeta_i)$.

Hence,

$\{\underset{\sim}{y} \le w\} \subset \{\underset{\sim}{x} \le w\}$ $\qquad P\{\underset{\sim}{y} \le w\} \le P\{\underset{\sim}{x} \le w\}$

Therefore $\qquad F_y(w) \le F_x(w)$

4-17 From (4-80)

$$f(x) = kx\, e^{-\int_0^x kt\,dt} = kx\, e^{-kx^2/2}$$

4-18 It follows from (2-41) with

$$A_1 = \{\underset{\sim}{x} \le x\} \qquad A_2 = \{\underset{\sim}{x} > x\}$$

4-19 It follows from

$$F_x(x|A) = \frac{P\{\underset{\sim}{x} \le x, A\}}{P(A)} \qquad P\{A|\underset{\sim}{x} \le x\} = \frac{P\{\underset{\sim}{x} \le x, A\}}{P\{\underset{\sim}{x} \le x\}}$$

4-20 We replace in (4-80) all probabilities with conditional probabilities assuming $\{\underset{\sim}{x} \le x_o\}$. This yields

$$\int_{-\infty}^{\infty} P(A|\underset{\sim}{x} = x,\ \underset{\sim}{x} \le x_o) f(x|\underset{\sim}{x} \le x_o)dx = P(A|\underset{\sim}{x} \le x_o)$$

But $f(x|\underset{\sim}{x} \le x_o) = 0$ for $x > x_o$ and

$\{\underset{\sim}{x} = x,\ \underset{\sim}{x} \le x_o\} = \{\underset{\sim}{x} = x\}$ for $x \le x_o$. Hence,

$$\int_{-\infty}^{x_o} P(A|\underset{\sim}{x} = x) f(x|\underset{\sim}{x} \le x_o)dx = P(A|\underset{\sim}{x} \le x_o)$$

Writing a similar equation for $P(B|\underset{\sim}{x} \le x_o)$ we conclude that, if $P(A|\underset{\sim}{x} = x) = P(B|\underset{\sim}{x} = x)$ for $x \le x_o$, then $P(A|\underset{\sim}{x} \le x_o) = P(B|\underset{\sim}{x} \le x_o)$

4-21 (a) Clearly, $f(p) = 1$ for $0 \le p \le 1$ and 0 otherwise; hence

$$P\{0.3 \le \underset{\sim}{p} \le 0.7\} = \int_{0.3}^{0.7} dp = 0.4$$

(b) We wish to find the conditional probability $P\{0.3 \le \underset{\sim}{p} \le 0.7|A\}$ where A = {6 heads in 10 tosses}. Clearly $P\{A|\underset{\sim}{p}=p\} = p^6(1-p)^4$. Hence, [see (4-81)]

$$f(p|A) = \frac{p^6(1-p)^4}{\int_0^1 p^6(1-p)^4 dp} = \frac{p^6(1-p)^4}{4329\times 10^{-7}}$$

This yields

$$P\{0.3 \le \underset{\sim}{p} \le 0.7|A\} = \int_{0.3}^{0.7} f(p|A)dp = \frac{10^7}{4329}\int_{0.3}^{0.7} p^6(1-p)^4 dp = 0.768$$

4-22 (a) In this problem, $f(p) = 5$ for $0.4 \le \underset{\sim}{p} \le 0.6$ and zero otherwise; hence [see(4-82)]

$$P(H) = 5\int_{0.4}^{0.6} p dp = 0.5$$

(b) With A = {60 heads in 100 tosses} it follows from (4-82) that

$$f(p|A) = p^{60}(1-p)^{40} \Big/ \int_{0.4}^{0.6} p^{60}(1-p)^{40} dp$$

for $0.4 \le p \le 0.6$ and 0 otherwise. Replacing $f(p)$ by $f(p|A)$ in (4-82), we obtain

$$P(H|A) = \int_{0.4}^{0.6} pf(p|A)dp = 0.56$$

4-23 $n = 900$ $p = q = 0.5$ $np = 450$ $\sqrt{npq} = 15$

$k_1 = 420$ $k_2 = 465$ $\dfrac{k_2 - np}{\sqrt{npq}} = 1$ $\dfrac{k_1 - np}{\sqrt{npq}} = -2$

$$P\{420 \le k \le 465\} = G(1) - [1 - G(-2)] = G(1) + G(2) - 1$$
$$= 0.819$$

4-24 For a fair coin $\sqrt{npq} = \sqrt{n}/2$. If

$k_1 = 0.49n$ and $k_2 = 0.52n$ then

$$\frac{k_2 - np}{\sqrt{npq}} = \frac{0.52n - n/2}{\sqrt{n}/2} = 0.04\sqrt{n} \qquad \frac{k_1 - np}{\sqrt{npq}} = -0.02\sqrt{n}$$

$$P\{k_1 \le k \le k_2\} = G(0.04\sqrt{n}) + G(0.02\sqrt{n}) - 1 \ge 0.9$$

From Table 4-1 (page 106) it follows that

$$0.02\sqrt{n} > 1.3 \qquad n > 65^2$$

4-25

(a) Assume n = 1,000 (Note correction to the problem)

$P(A) = 0.6 \qquad np = 600 \qquad npq = 240 \qquad k_2 = 650 \qquad k_1 = 550$

$$\frac{k_2 - np}{\sqrt{npq}} = \frac{50}{\sqrt{240}} = 3.23 \qquad \frac{k_1 - np}{\sqrt{npq}} = -3.23$$

$$P\{550 \le k \le 650\} = 2G(3.23) - 1 = 0.999$$

(b) $P\{0.59n \le k \le 0.61n\} = 2G\left(\frac{0.01n}{\sqrt{0.24n}}\right) - 1$

$$= 2G\left(\sqrt{\frac{n}{2400}}\right) - 1 = 0.476$$

Hence, (Table 3-1) $\quad n \simeq 9220$

4-26 With a = 0, b = T/4 it follows that

$p = 1-e^{-1/4} = 0.22 \qquad np = 220 \qquad npq = 171.6 \qquad k_2 = 100$

$$\frac{k_2 - np}{\sqrt{npq}} = -9.16 \quad \text{and (4-100) yields}$$

$$P\{0 \le k \le 100\} \simeq G(-9.16) \simeq 0.$$

4-27 The event

A = {k heads show at the first n tossings but not earlier}

occurs iff the following two events occur

B = {k-1 heads show at the first n-1 tossing}

C = {heads show at the nth tossing}

And since these two events are independent and

$$P(B) = \binom{n-1}{k-1} p^{k-1} q^{n-1-(k-1)} \qquad P(C) = p$$

we conclude that

$$P(A) = P(B)P(C) = \binom{n-1}{k-1} p^k q^{n-k}$$

4-28 $$-\frac{d}{dx}\left(\frac{1}{x}e^{-x^2/2}\right) = \left(1+\frac{1}{x^2}\right)e^{-x^2/2} > e^{-x^2/2}$$

Multiplying by $1/\sqrt{2\pi}$ and integrating from x to ∞, we obtain

$$\frac{1}{x\sqrt{2\pi}}e^{-x^2/2} > \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\zeta^2/2}\,d\zeta = 1 - G(x)$$

because

$$\frac{1}{x}e^{-x^2/2} \underset{x\to\infty}{\longrightarrow} 0$$

The first inequality follows similarly because

$$-\frac{d}{dx}\left[\left(\frac{1}{x}-\frac{1}{x^3}\right)e^{-x^2/2}\right] = \left(1-\frac{3}{x^4}\right)e^{-x^2/2} < e^{-x^2/2}$$

4-29 If $P(A) = p$ then $P(\bar{A}) = 1-p$. Clearly $P_1 = 1-Q_1$ where Q_1 equals the probability that A does not occur at all. If $pn \ll 1$, then $Q_1 = (1-p)^n \simeq 1 - np$ $\quad P_1 \simeq p$

4-30 With $p = 0.02$, $n = 100$, $k = 3$, it follows from (4-107) that the unknown probability equals

$$\binom{100}{3}(0.02)^3(0.98)^{97} \simeq \frac{2^3}{3!}e^{-2} = \frac{4}{3}e^{-2}$$

4-31 With $n = 3$, $r = 3$, $k_1 = 2$, $k_2 = 2$, $k_3 = 1$, $p_1 = p_2 = p_3 = 1/6$, it follows from (4-102) that the unknown probability equals

$$\frac{5!}{1!2!2!}\frac{1}{6^6} = 0.00386$$

4-32 With $r = 2$, $k_1 = k$, $k_2 = n-k$, $p_1 = p$, $p_2 = 1-p = q$, we obtain

$$k_1 - np_1 = k - np \qquad k_2 - np_2 = n-k-nq = np - k$$

Hence, the bracket in (4-103) equals

$$\frac{(k_1-np_1)^2}{np_1} + \frac{(k_2-np_2)^2}{np_2} = \frac{(k-np)^2}{n}\left(\frac{1}{p}+\frac{1}{q}\right) = \frac{(k-np)^2}{npq}$$

as in (4-90).

4-33 $P(M) = 2/36$ $P(\overline{M}) = 34/36$. The events M and $\overline{M}$ form a partition, hence, [see (2-41)]

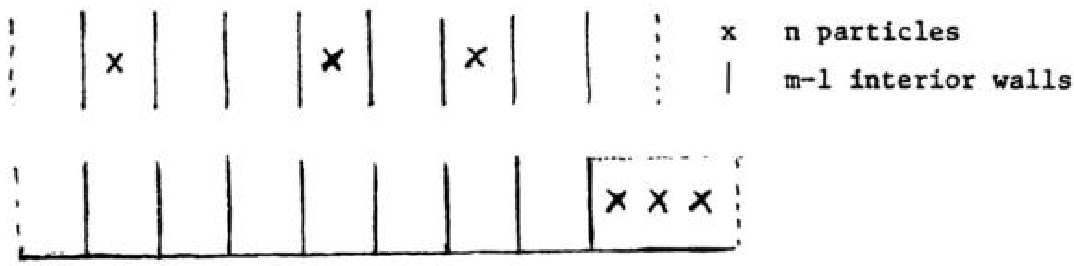
$$P(A) = P(A|M)P(M) + P(A|\overline{M})P(\overline{M}) \qquad \text{(i)}$$

Clearly, $P(A|M) = 1$ because, if M occurs at first try, X wins. The probability that X wins after the first try equals $P(A|\overline{M})$. But in the experiment that starts at the second rolling, the first player is Y and the probability that he wins equals $P(\overline{A}) = 1-p$. Hence, $P(A|\overline{M}) = P(\overline{A}) = 1-p$. And since $P(M) = 1/18$ $P(\overline{M}) = 17/18$ (i) yields

$$p = \frac{1}{18} + (1-p)\,\frac{17}{18} \qquad p = \frac{18}{35}$$

4-34

(a) Each of the n particles can be placed in any one of the m boxes. There are n particles, hence, the number of possibilities equals $N = m^n$. In the m preselected boxes, the particles can be placed in $N_A = n!$ ways (all permutations of n objects). Hence $p = n!/m^n$.

(b) 

All possibilities are obtained by permuting the n+m-1 objects consisting of the m-1 interior walls with and n particles. The (m-1)! permutations of the walls and the n! permutations of the particles must count as one. Hence

$$N = \frac{(n+m-1)!}{n!\,(m-1)!} \qquad N_A = 1$$

(c) Suppose that S is a set consisting of the m boxes. Each placing of the particles specifies a subset of S consisting of n elements (box). The number of such subsets equals $\binom{m}{n}$ (see Prob. 2-26). Hence,

$$N = \binom{m}{n} \qquad N_A = 1$$

4-35 If $k_1 + k_2 << n$, then $k_3 \simeq n$ and

$$k_3(p_1 + p_2) = [n-(k_1 + k_2)](p_1 + p_2) \simeq n(p_1 + p_2)$$

$$p_3 = 1 - (p_1 + p_2) \simeq e^{-(p_1+p_2)} \qquad p_3^{k_3} \simeq e^{-n(p_1+p_2)}$$

$$\frac{n!}{k_1!k_2!k_3!} = \frac{n(n-1)\cdots(n-k_3+1)}{k_1!k_2!} \simeq \frac{n^{k_1+k_2}}{k_1!k_2!}$$

Hence,

$$\frac{n!}{k_1!k_2!k_3!} p_1^{k_1} p_2^{k_2} p_3^{k_3} \simeq e^{-np_1} \frac{(np_1)^{k_1}}{k_1!} e^{-np_2} \frac{(np_2)^{k_2}}{k_2!}$$

4-36 The probability p that a particular point is in the interval (0,2) equals 2/100. (a) From (3-13) it follows that the probability p_1 that only one out of the 200 points is in the interval (0,2) equals

$$p_1 = \binom{200}{1} \times 0.02 \times 0.09^{199}$$

(b) With $np = 200 \times 0.02 = 4$ and $k = 1$, (3-41) yields $p_1 \simeq e^{-4} \times 4 = 0.073$

CHAPTER 5

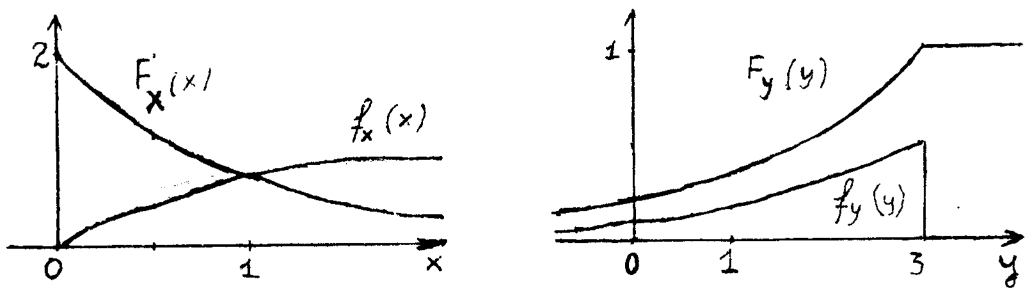
5-1 $\eta = 2\eta_x + 4 = 14 \qquad \sigma_y^2 = 4\sigma_x^2 = 16$

5-2 $\{\underset{\sim}{y} \le y\} = \{-4\underset{\sim}{x} + 3 \le \underset{\sim}{y}\}\ \{\underset{\sim}{x} \le (y-3)/4\}$. Hence

$$F_y(y) = P\left\{\underset{\sim}{x} \ge \frac{3-y}{4}\right\} = 1 - F_x\left(\frac{3-y}{4}\right) \qquad f_y(y) = \frac{1}{4} f_x\left(\frac{3-y}{4}\right)$$

Since $F_x(x) = (1-e^{-2x})U(x)$, this yields

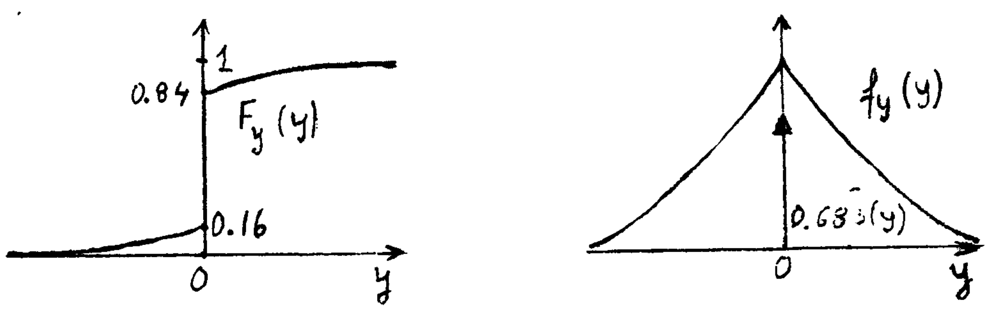
$$F_y(y) = e^{(y-3)/2} U\left(\frac{y-3}{2}\right) \qquad f_y(y) = \frac{1}{2} e^{(y-3)/2} U\left(\frac{y-3}{2}\right)$$



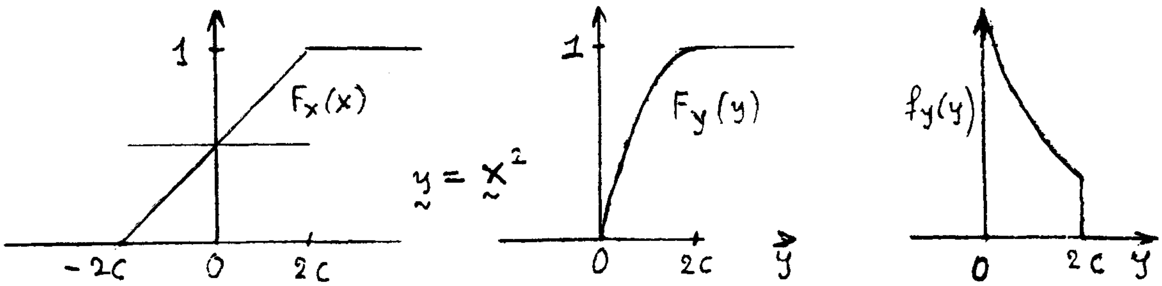
5-3 From Example 5-3 with $F_x = G(x/c)$:

$$f_y(y) = \begin{cases} G(y/c+1) & y \ge 0 \\ G(y/c-1) & y < 0 \end{cases}$$

$$f_y(y) = 0\ 0.68\ \delta(y) + \frac{1}{c\sqrt{2\pi}}\left[e^{-(y+c)^2/2c^2}U(y) + e^{-(y-c)^2/2c^2}U(-y)\right]$$

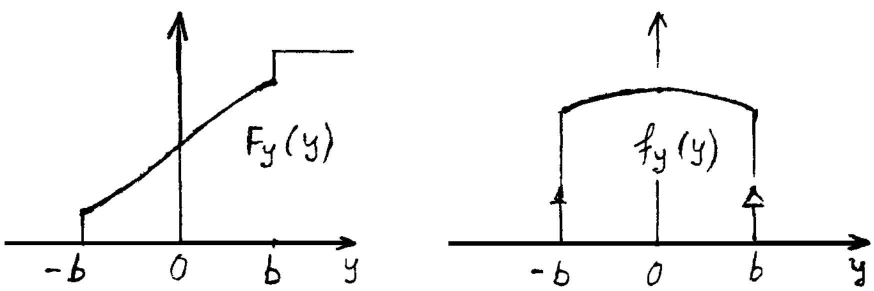


5-4 If $\underset{\sim}{y} = \underset{\sim}{x}^2$ and $F_x(x) = (x+2c)/4c$ for $|x| \le 2c$, then (see Example 5-2) $F_y(y) = \sqrt{y}\,/2c$ and $f_y(y) = 1/4\sqrt{y}$ for $0 < y < 2c$.



5-5 From Example 5-4 with $F_x(x) = G(x/b)$: For $|x| \le b$ $F_u(y) = G(y/b)$ and

$$f_y(y) = 0.16\delta(y+b) + \frac{1}{b\sqrt{2\pi}} e^{-y^2/2b^2} + 0.16\delta(y-b)$$



5-6 The equation $y = -\ell nx$ has a single solution $x = e^{-y}$ for $y > 0$ and no solutions for $y < 0$. Furthermore, $g'(x) = -1/x = -e^{y}$. Hence

$$f_y(y) = \frac{f_x(e^{-y})}{e^{y}}\, U(y) = e^{-y}U(y)$$

5-7 Clearly, $\underset{\sim}{z} \le z$ iff the number $\underset{\sim}{n}(0,z)$ of the points in the interval $(0,z)$ is at least one. Hence,

$$F_z(z) = P\{\underset{\sim}{z} \le z\} = P\{\underset{\sim}{n}(0,z) > 0\} = 1 - P\{\underset{\sim}{n}(0,z) = 0\}$$

The probability p that a particular point is in the integral $(0,z)$ equals $z/100$. With $n = 200$, $k = 0$, and $p = z/100$, (3-21) yields $P\{\underset{\sim}{n}(0,z) = 0\} = (1-p)^{200}$. Hence,

(a) $F_z(z) = 1 - \left(1 - \frac{z}{100}\right)^{100}$

(b) From (4-107) it follows thata $F_z(z) \simeq 1 - e^{-2z}$ for $z << 100$.

5.8

$$Y = \sqrt{X} \quad \Rightarrow \quad x_1 = y^2$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2y}$$

Thus

$$f_Y(y) = \frac{1}{\left|\frac{dy}{dx}\right|} f_X(x_1) = 2y f_X(y^2)$$

$$\frac{2y}{\lambda} e^{-y^2/\lambda} = \begin{cases} \frac{y}{\sigma^2} e^{-y^2/2\sigma^2}, & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

which represents Rayleigh density function (with $\lambda = 2\sigma^2$).

5-9 For both cases, $f_y(y) = 0$ for $y < 0$.

(a) If $y > 0$ and $|x| = y$, then $x_1 = y$, $x_2 = -y$. Hence

$$f_y(y) = [f_x(y) + f_x(-y)]U(y)$$

(b) If $y > 0$ and $e^{-x}U(x) = y$, then $x = -\ell n y$.

Furthermore, $P\{\underset{\sim}{y} = 0\} = P\{\underset{\sim}{x} \le 0\} = F_x(0)$. Hence

$$f_y(y) = F_x(0)\delta(y) + \frac{1}{y} f_x(-\ell n y)U(y)$$

5-10 (a) If $y \geq 0$ and $(x-1)U(x-1) = y$, then $\{\underset{\sim}{y} \leq y\} = \{\underset{\sim}{x} \leq y+1\}$.
If $y < 0$, then $\{\underset{\sim}{y} < y\} = \{\emptyset\}$

$$F_y(y) = F_x(1+y)U(y) = [1 - e^{-2(y+1)}]U(y)$$

$$f_y(y) = (1 - e^{-2})\delta(y) + 2e^{-2(y+1)}U(y)$$

(b) If $y > 0$ and $y = x^2$, then $\{\underset{\sim}{y} \leq y\} = \{-\sqrt{y} \leq \underset{\sim}{x} \leq \sqrt{y}\}$

$$F_y(y) = F_x(\sqrt{y}) - F_x(-\sqrt{y}) = (1 - e^{-2\sqrt{y}})U(y)$$

$$f_y(y) = \frac{1}{\sqrt{y}} e^{-2\sqrt{y}}U(y)$$

5-11 If $y = \arctan x$, then $\frac{dy}{dx} = \frac{1}{1+x^2}$

$$f_y(y) = (1+x^2)f_x(\tan y) = \frac{1+x^2}{\pi(1+x^2)} = \frac{1}{\pi} \qquad \frac{\pi}{2} < y < \frac{\pi}{2}$$

5-12 (a) If $y = x^3$ then $x = \sqrt[3]{y}$ for any y

$$f_y(y) = \frac{1}{3\sqrt[3]{y^2}} f_x(\sqrt[3]{y}) = \frac{1}{12\pi\sqrt[3]{y^2}}$$

for $|y| < 8\pi^3$ and zero otherwise

(b) If $y = x^4$ and $y > 0$, then $x_1 = \sqrt[4]{y}$ $x_1 = -\sqrt[4]{y}$

$$f_y(y) = \frac{1}{4\sqrt[4]{y^3}}\left[f_x(\sqrt[4]{y}) + f_x(-\sqrt[4]{y})\right] = \frac{1}{8\pi\sqrt[4]{y^3}}$$

for $0 < y < 16\pi^4$ and zero otherwise

(c) If $y = 2\sin(3x + 40°)$ and $|y| < 2$ then $x = x_i$ as shown.

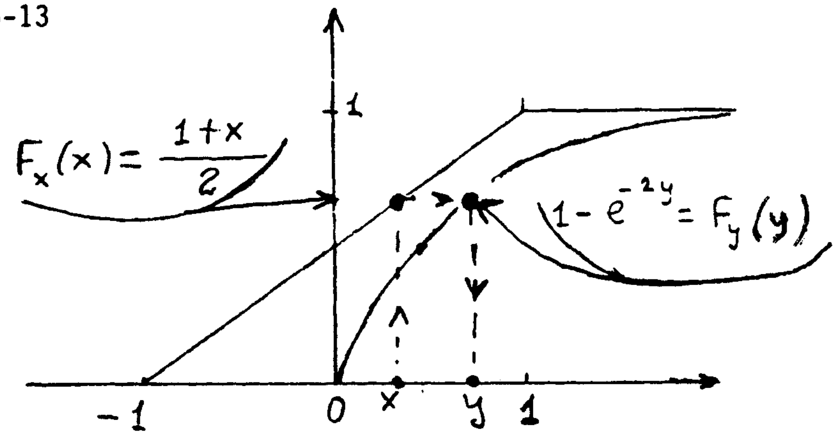
$$\frac{dy}{dx} = \frac{1}{6\sqrt{1 - y^2/4}}$$

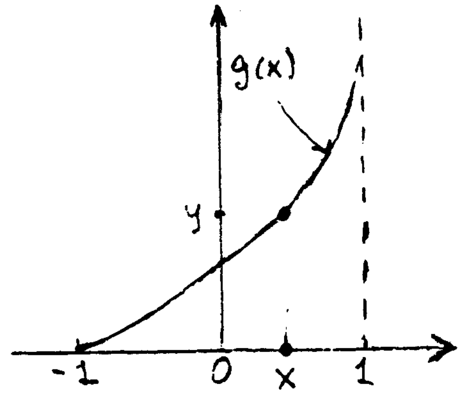
In the interval $(-2\pi, 2\pi)$ there are 12 x_i's. Hence

$$f_y(y) = \frac{1}{3\sqrt{4-y^2}} \sum_i f_x(x_i) = \frac{12}{12\pi\sqrt{4-y^2}} = \frac{1}{\pi\sqrt{4-y^2}}$$

for $|y| < 2$ and zero otherwise.

5-13





As in (5-43) $\qquad F_y[g(x)] = F_x(x)$

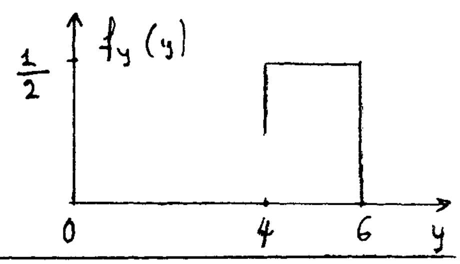
$$\frac{1+x}{2} = 1 - e^{-2y} \qquad y = g(x) = -\frac{1}{2}\ln\frac{1-x}{2}$$

for $|x| < 1$. For $x \le -1$, $g(x) = 0$; for $x \ge 1$, $g(x) = \infty$.

5-14 (a) $\quad g(x) = 2F_x(x) + 4 \qquad g'(x) = 2f_x(x)$

If $4 < y < 6$ then $y = 2F_x(x) + 4$ has a unique solution x_1 and

$$f_y(y) = \frac{f_x(x_1)}{2f_x(x_1)} = \frac{1}{2}$$



(b) Similarly $g(x) = 2F_x(x) + 8$

5-15 (a) The RV $\underset{\sim}{x}$ takes the values $k = 0,1,\ldots,10$ and

$$P\{x = k\} = p_k = \binom{10}{k}\frac{1}{2^{10}} \qquad 0 \le k \le 10$$

$F_x(x)$ is a staircase function with discontinuities at the points $x = k$ and jumps equal to p_k.

(b) The RY $\underset{\sim}{y} = (\underset{\sim}{x} - 3)^2$ takes the values $y = k^2$ for $k = 0,1,\ldots,7$ and probabilities $P\{y = k^2\} = q_k$.

k =	0	1	2	3	4	5	6	7
q_k =	p_3	p_2+p_4	p_1+p_5	p_0+p_6	p_7	p_8	p_9	p_{10}

5.16

$$X \sim Beta(\alpha, \beta) \quad \text{gives}$$

$$f_X(x) = \frac{1}{B(\alpha,\beta)}\, x^{\alpha-1}\,(1-x)^{\beta-1}, \quad 0 < x < 1.$$

$$Y = 1 - X \quad \Rightarrow \quad x_1 = 1 - y, \quad |\frac{dy}{dx}| = 1$$

$$\Rightarrow F_Y(y) = \frac{1}{|\frac{dy}{dx}|}\, f_X(1-y) = \begin{cases} \frac{1}{B(\beta,\alpha)}\, y^{\beta-1}\,(1-y)^{\alpha-1}, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

This gives

$$Y \sim Beta(\beta, \alpha).$$

5.17

$$X \sim \chi^2(n) \Rightarrow$$

$$f_X(x) = \frac{1}{2^{n/2}\Gamma(n/2)}\, x^{n/2-1}\, e^{-x/2}\, U(x)$$

$$y = \sqrt{x} \quad \Rightarrow \quad x_1 = y^2$$

$$\frac{dy}{dx} = \frac{1}{2y}$$

Thus

$$f_Y(y) = 2y f_X(y^2) = \frac{y^{n-1}}{2^{n/2-1}\,\Gamma(n/2)}\, e^{-y^2/2}\, U(y)$$

and it represents the chi-distribution.

5.18

$$X \sim U(0,1)$$

$$Y = -2logX \quad \Rightarrow \quad x_1 = e^{-y/2}$$

$$\frac{dy}{dx} = -\frac{2}{x} = -2e^{y/2}$$

$$\begin{aligned} f_Y(y) \; &= \frac{1}{|\frac{dy}{dx}|}\, f_X(x_1) = \tfrac{1}{2} e^{-y/2}\, U(y) \\ &\sim \text{ Exponential}(2) \equiv \chi^2(2) \end{aligned}$$

5.19

$$f_X(x) = \lambda\, e^{-\lambda x}\, u(x)$$

$$Y = X^{1/\beta} \quad \Rightarrow \quad x_1 = y^\beta$$

$$|\frac{dy}{dx}| = \frac{1}{\beta}\, x^{1/\beta - 1} = \frac{1}{\beta}\, y^{1-\beta}$$

$$f_Y(y) = \frac{1}{|\frac{dy}{dx}|}\, f_X(x_1) = \lambda\, \beta\, y^{\beta-1}\, e^{-\lambda y^\beta}\, U(y)$$

and it represents Weibull distribution

5-20 For $|y| < a$ the equation $y = a \sin\omega t$ has infinitely many solutions τ_i; in each interval of length $2\pi/\omega$ there are two such solutions. Furthermore, $y'(t) = \omega\sqrt{a^2-y^2}$

$$\tau_i = \frac{1}{\omega}\sin^{-1}\frac{y}{a} \qquad \tau_{i+2} - \tau_i \quad \frac{2\pi}{\omega} \xrightarrow[\omega\to\infty]{} 0$$

Hence,

$$\frac{1}{\omega\sqrt{a^2-y^2}} \sum_{i=-\infty}^{\infty} f_t(\tau_i) \xrightarrow[\omega\to\infty]{} \frac{1}{\sqrt{a^2-y^2}} \frac{2}{2\pi} \int_{-\infty}^{\infty} f_t(\tau)d\tau = \frac{1}{\pi\sqrt{a^2-y^2}}$$

5-21 If $y > 0$ then

$$F_y(y|\underset{\sim}{x} \geq 0) = F_x(\sqrt{y}|\underset{\sim}{x} \geq 0) + F_x(-\sqrt{y}|\underset{\sim}{x} \geq 0) = F_x(\sqrt{y}|\underset{\sim}{x} \geq 0)$$

$$F_x(\sqrt{y}|x \geq 0) = \frac{P\{0 < x < \sqrt{y}\}}{P\{x \geq 0\}} = \frac{F_x(\sqrt{y}) - F_x(0)}{1 - F_x(0)}$$

$$f_y(y|x \geq 0) = \frac{d}{dy} F_y(\sqrt{y}|x \geq 0) = \frac{f_x(\sqrt{y})}{2\sqrt{y}[1-F_x(0)]}$$

5-22 (a) $\eta_y = a\,\eta_x + b \qquad \sigma_y^2 = E\{[a\,\underset{\sim}{x} + b - (a\,\eta_x + b)]^2\}$

$$\sigma_y^2 = E\{a^2(\underset{\sim}{x} - \eta_x)^2\} = a^2\sigma_x^2$$

(b) $\underset{\sim}{y} = \dfrac{\underset{\sim}{x}-\eta_x}{\sigma_x} \qquad E\{\underset{\sim}{y}\} = 0 \qquad \sigma_y^2 = \dfrac{\sigma_x^2}{\sigma_x^2} = 1$

5-23 If $\underset{\sim}{x}$ has a Rayleigh density, then [see (5-76)]

$$E\{\underset{\sim}{x}^2\} = 2\alpha^2 \qquad E\{\underset{\sim}{x}^4\} = 8\alpha^4$$

If $\underset{\sim}{y} = b + c\underset{\sim}{x}^2$, then

$$E\{\underset{\sim}{y}\} = b + 2\alpha^2 c \qquad E\{\underset{\sim}{y}^2\} = b^2 + 4\alpha^2 bc + 8\alpha^4 c^2$$

$$\sigma_y^2 = E\{\underset{\sim}{y}^2\} - E^2\{\underset{\sim}{y}\} = 4\alpha^4 c^2$$

5-24 $\underset{\sim}{y} = 3\underset{\sim}{x}^2$ $E\{\underset{\sim}{x}^2\} = \sigma_x^2 = 4$ $E\{\underset{\sim}{x}^4\} = 3\sigma_x^4 = 48$

$E\{\underset{\sim}{y}\} = 12$ $E\{y^2\} = 9 \times 48 = 432$ $\sigma_y^2 = 432 - 144 = 288$

If $y > 0$ then $3x^2 = y$ for $x = \pm\sqrt{y/3}$ $y' = 6x$

$$f_y(y) = \frac{\cancel{2}4}{\sqrt{12y}}\, f_x\left(\sqrt{\frac{y}{3}}\right) = \frac{1}{\sqrt{24\pi y}}\, e^{-y/24} U(y)$$

5.25

$$X \sim B(n,p) \Rightarrow P(X=k) = \binom{n}{k} p^k q^{n-k}, \quad k = 0, 1, 2, \cdots n.$$

a)

$$\begin{aligned} E(X) &= \sum_{k=0}^{n} k\, P(X=k) = \sum_{k=1}^{n} k \frac{n!}{k!\,(n-k)!} p^k q^{n-k} \\ &= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,(n-k)!} p^{k-1} q^{n-k} \\ &= np\,(p+q)^{n-1} = np. \end{aligned}$$

b)

$$\begin{aligned} E[X(X-1)] &= \sum_{k=2}^{n} k\,(k-1) \frac{n!}{k!\,(n-k)!} p^k q^{n-k} \\ &= n(n-1)p^2 \sum_{k=2}^{n} \frac{(n-2)!}{(k-2)!\,(n-k)!} p^{k-2} q^{n-k} \\ &= n(n-1)p^2\,(p+q)^{n-2} \\ &= n(n-1)p^2 \end{aligned}$$

c)

$$\begin{aligned} E[X(X-1)(X-2)] &= \sum_{k=3}^{n} k(k-1)(k-2) \frac{n!}{k!\,(n-k)!} p^k q^{n-k} \\ &= n(n-1)(n-2)\,p^3 \sum_{k=3}^{n} \frac{(n-3)!}{(k-3)!\,(n-k)!} p^{k-3} q^{n-k} \\ &= n(n-1)(n-2)\,p^3\,(p+q)^{n-3} \\ &= n(n-1)(n-2)\,p^3 \end{aligned}$$

$$E(X^2) = E\left(X(X-1)\right) + E(X) = n^2p^2 + npq$$

$$\begin{aligned} E(X^3) &= E\left(X(X-1)(X-2)\right) + 3E(X^2) - 2E(X) \\ &= n(n-1)(n-2)\,p^3 + 3(n^2p^2 + npq) - 2np \\ &= n^3p^3 + 3n^2p^2q + npq(q-p). \end{aligned}$$

5.26

$$X \sim P(\lambda) \quad \Rightarrow \quad P(X=k) = e^{\lambda}\frac{\lambda^k}{k!}, \qquad k = 0, 1, 2, \cdots$$

a)

$$E(X) = \lambda, \quad \mathrm{Var}(X) = \sigma_X^2 = \lambda$$

From Chebyshev's inequality (5-88)

$$P\left(|X-\mu| < \lambda\right) > 1 - \frac{\sigma^2}{\lambda^2} = 1 - \frac{1}{\lambda}$$

But

$$|X-\mu| < \lambda = |X-\lambda| < \lambda \quad \Rightarrow \quad 0 < X < 2\lambda$$

which gives

$$P(0 < X < 2\lambda) > 1 - \frac{1}{\lambda} = \frac{\lambda - 1}{\lambda}.$$

b)

$$\begin{aligned} E\left[X(X-1)\right] &= \textstyle\sum_{k=2}^{\infty} k(k-1)\, e^{-\lambda}\,\frac{\lambda^k}{k!} \\ &= e^{-\lambda}\,\lambda^2 \textstyle\sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} = e^{-\lambda}\,\lambda^2\, e^{\lambda} = \lambda^2. \end{aligned}$$

$$\begin{aligned} E\left[X(X-1)(X-2)\right] &= \textstyle\sum_{k=3}^{\infty} k(k-1)(k-2)\, e^{-\lambda}\,\frac{\lambda^k}{k!} \\ &= e^{-\lambda}\,\lambda^3 \textstyle\sum_{k=3}^{\infty} \frac{\lambda^{k-3}}{(k-3)!} = \lambda^3. \end{aligned}$$

5-27 Follows from (4-74)

$$E\{\underline{x}\} = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{\infty} x \sum_i f(x|A_i)P(A_i)\,dx$$

because
$$E\{\underline{x}|A_i\} = \int_{-\infty}^{\infty} x f(x|A_i)\,dx$$

5-28 From (5-89) with $\alpha = \sqrt{\eta}$:

$$P\{\underline{x} \geq \sqrt{\eta}\} \leq \eta/\sqrt{\eta} = \sqrt{\eta}$$

5-29 From (5-86) with $g(x) = x^3$ $\quad g''(x) = 6x$:

$$E\{\underset{\sim}{x}^3\} = \eta^3 + 6\eta\,\frac{\sigma^2}{2} = 1120$$

5-30 (a) If $y = x^3$, then $x = \sqrt[3]{y}$ $\quad g'(x) = 3x^2 = 3\sqrt[3]{y^2}$

But $f_x(x) = 0.5$ for $10 < x < 12$, i.e., for $10^3 < y < 12^3$

and (5-16) yields

$$f_y(y) = \frac{0.5}{3\sqrt[3]{y^2}} \qquad 10^3 < y < 12^3$$

and zero otherwise.

(b) 1.

$$E\{\underset{\sim}{x}^3\} = 0.5\int_{10}^{12} x^3 dx = 1342$$

2. With $g(x) = x^3$ $\quad E\{\underset{\sim}{x}\} = 11$ $\quad \sigma_x^2 = 1/3$, (5-86) yields

$$E\{x^3\} \simeq 11^3 + 6 \times 11 \times \frac{1}{6} \simeq 1342$$

5-31 With $g(x)=1/x$, $g''(x)=2/x^3$, $\eta=100$, and $\sigma=3$, (5-55) yields

$$E\left\{\frac{1}{\underset{\sim}{x}}\right\} \simeq \frac{1}{100} + \frac{9}{2} \times \frac{2}{100^3} = 0.010009$$

5-32

$$\frac{\partial|x-a|}{\partial a} = \begin{cases} 1 & x < a \\ -1 & x > a \end{cases} \qquad \text{If } I(a) = E\{|\underset{\sim}{x}-a|\} \text{ then}$$

$$\frac{dI(a)}{da} = E\,\frac{\partial|\underset{\sim}{x}-a|}{\partial a} = 1\,P\{\underset{\sim}{x} < a\} - 1\,P\{\underset{\sim}{x} > a\}$$
$$= 2\,F(a) - 1$$

(a)
$$I(a) = I(m) + \int_m^a I'(\alpha)d\alpha = I(m) + \int_m^a [2\,F(\alpha) - 1]d\alpha$$

$$= E\{|\underset{\sim}{x} - m|\} - 2\int_m^a x\,f(x)dx$$

because

$$\int_m^a F(\alpha)d\alpha = a\,F(a) - m\,F(m) - \int_m^a x f(x)dx$$

$$F(m) = \frac{1}{2} \qquad \int_m^a f(x)dx = F(a) - F(m)$$

(b) $I(a) = E\{|\underset{\sim}{x} - a|\}$ is minimum if

$I'(a) = 2F(a) - 1 = 0$ i.e. if $F(a) = \frac{1}{2}$ $a = m$

5-33

$$E\{|\underset{\sim}{x}|\} = \int_0^\infty xf(x)dx - \int_{-\infty}^0 xf(x)dx$$

$$\eta = E\{\underset{\sim}{x}\} = \int_0^\infty xf(x)dx + \int_{-\infty}^0 xf(x)dx$$

$$\frac{E\{|\underset{\sim}{x}|+\eta\}}{2} = \int_0^\infty xf(x)dx = \frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty x\,e^{-(x-\eta)^2/2\sigma^2}dx$$

$$\frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty (x+\eta)e^{-(x-\eta)^2/2\sigma^2}dx = \frac{1}{\sigma\sqrt{2\pi}}\int_\eta^\infty y\, e^{-y^2/2\sigma^2}dy = \frac{\sigma}{\sqrt{2\pi}}e^{-\eta^2/2\sigma^2}$$

$$\frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty e^{-(x-\eta)^2/2\sigma^2}dx = G(\frac{\eta}{\sigma})$$

Multiplying the last line by η and subtracting from the fourth line, we obtain

$$\frac{E\{|x|+\eta\}}{2} = \frac{\sigma}{\sqrt{2\pi}}e^{-\eta^2/2\sigma^2} + G(\frac{\eta}{\sigma})$$

5-34 The proof is given in sec 14-3: [see (14-100)].

5-35 (a) Follows from (5-89) (b) $e^{sx} \geq e^{sA}$ iff $x \geq A$ for $s > 0$ and $x \leq A$ for $s < 0$.

5.36 See proof for Lyapunov inequality (Ch.5, Eq.(5-92).)

5-37 (a) If $\Phi(\omega) = e^{-\alpha|\omega|}$ then [see (5-102)]

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha|\omega|}\, e^{j\omega x} d\omega = \frac{1}{\pi}\int_0^{\infty} \cos\omega x\, e^{-\alpha\omega} d\omega = \frac{\alpha}{\pi(\alpha^2+x^2)}$$

(b) If $f(x) = \frac{\alpha}{2} e^{-\alpha|x|}$, then [see (5-94)]

$$\Phi(\omega) = \frac{\alpha}{2}\int_{-\infty}^{\infty} e^{-\alpha|x|} e^{-j\omega x} dx = \alpha\int_0^{\infty} e^{-\alpha x}\cos\omega x dx = \frac{\alpha^2}{\alpha^2+\omega^2}$$

5.38 a) On comparing Eq.(4-34) with Eq.(5-106), Example 5-29, we get

$$X \sim G(\alpha,\beta) \quad \Rightarrow \quad \phi_X(\omega) = (1-j\beta\omega)^{-\alpha}$$

$$\phi'_X(\omega) = -\alpha(1-j\beta\omega)^{-(\alpha+1)}(-j\beta)$$

so that

$$E(X) = \frac{1}{j}\phi'_X(0) = \alpha\,\beta.$$

Similarly

$$\phi''_X(\omega) = j\alpha\beta(\alpha+1)\,(1-j\beta\,\omega)^{-(\alpha+2)}\,(j\beta)$$

and hence

$$E(X^2) = \frac{1}{j^2}\phi''_X(0) = \alpha\beta^2\,(\alpha+1).$$

Thus

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \alpha\beta^2.$$

b)

$$X \sim \chi^2(n) \quad \Rightarrow \quad \alpha = \frac{n}{2}, \quad \beta = 2$$

in Gamma(α,β). This gives

$$\begin{aligned}\phi_X(\omega) &= (1-j2\omega)^{-n/2}\\ E(X) &= n\\ \mathrm{Var}(X) &= 2n.\end{aligned}$$

c)

$$X \sim B(n,p).$$

From Prob 5-25 (a)-(b)

$$\begin{aligned}E(X) &= np\\ \mathrm{Var}(X) &= E\,(X(X-1)) + E(X) = npq.\end{aligned}$$

and

$$\begin{aligned}\phi_X(\omega) &= \sum_{k=0}^{n} e^{jk\omega}\, P(X=k)\\ &= \sum_{k=0}^{n}\binom{n}{k}\,(p\,e^{j\omega})^k\, q^{n-k} = (pe^{j\omega}+q)^n.\end{aligned}$$

d)

$$X \sim N\,Binomial\,(r,p).$$

From (4-64)

$$\begin{aligned}\phi_X(\omega) &= \sum_{k=0}^{\infty} e^{jk\omega}\, P(X=k)\\ &= \sum_{k=0}^{\infty} \binom{r+k-1}{k} p^r\,(qe^{j\omega})^k\\ &= p^r \sum_{k=0}^{\infty} \binom{-r}{k} (-qe^{j\omega})^k\\ &= p^r\,(1-qe^{j\omega})^{-r}.\end{aligned}$$

5-39

$$\Gamma(z) = \sum_{k=0}^{\infty} p\, q^k z^k = \frac{p}{1-qz} \qquad q = 1-p$$

$$\Gamma'(z) = \frac{pq}{(1-qz)^2} \qquad \Gamma'(1) = \frac{pq}{(1-q)^2} = \frac{p}{q} = \eta_x$$

$$\Gamma''(z) = \frac{2pq^2}{(1-qz)^3} \qquad \Gamma''(1) = \frac{2q^2}{p^2} = m_2 - m_1$$

$$\sigma^2 = m_2 - m_1^2 = 2\frac{q^2}{p^2} + m_1 - m_1^2 = \frac{q}{p^2}$$

5-40

$$\Gamma(z) = p^n \sum_{k=0}^{\infty} \binom{-n}{k} (-q)^k\, z^k = p^n (1-qz)^{-n}$$

(binomial expansion with negative exponent)

$$\Gamma'(z) = \frac{n\,p^n q}{(1-qz)^{n+1}} \qquad \Gamma'(1) = \frac{nq}{p} = \eta_x$$

$$\Gamma''(z) = \frac{n(n+1)p^n q^2}{(1-qz)^{n+2}} \qquad \Gamma''(1) = \frac{n(n+1)q^2}{p^2} = m_2 - m_1$$

$$\sigma_x^2 = \Gamma''(1) + m_1 - m_1^2 = \frac{nq}{p^2}$$

5.41 We have

$$P(X=k)=\binom{k-1}{r-1}p^r\,q^{k-r},\quad k=r,r+1,\cdots$$

Let $k=n+r$ so that

$$\begin{aligned}
P(X=n+r) &= \binom{n+r-1}{r-1}p^r\,q^n,\quad n=0,1,2,\cdots\\
&= \frac{(n+r-1)!}{n!\,(r-1)!}p^r\,(1-p)^n\\
&= \frac{1}{n!}\,\frac{(n+r-1)(n+r-2)\cdots(r)}{r^n}\,[r(1-p)]^n\,p^r\\
&= \frac{\lambda^n}{n!}\left\{\left(1+\frac{n-1}{r}\right)\left(1+\frac{n-2}{r}\right)\cdots\right\}\left(1-\frac{r(1-p)}{r}\right)^r\\
&= \frac{\lambda^n}{n!}\left\{\prod_{k=1}^{n}\left(1+\frac{n-k}{r}\right)\right\}\left(1-\frac{\lambda}{r}\right)^r,
\end{aligned}$$

where $\lambda=r(1-p)$. Thus

$$\lim_{r\to\infty}P(X=n+r)=\frac{\lambda^n}{n!}\left\{\lim_{r\to\infty}\prod_{k=1}^{n}\left(1+\frac{n-k}{r}\right)\right\}\lim_{r\to\infty}\left(1-\frac{\lambda}{r}\right)^r$$

$$\to\frac{\lambda^n}{n!}e^{-\lambda}\sim P(\lambda).$$

5-42 $$E\{e^{s\underline{x}}\} = e^{s\eta} E\{e^{s(\underline{x}-\eta)}\} = e^{s\eta} E\left\{\sum_{n=0}^{\infty} \frac{s^n}{n!} (\underline{x}-\eta)^n\right\}$$

$$= e^{s\eta} \sum_{n=0}^{\infty} \frac{s^n}{n!} \mu_n$$

5-43 If $\Phi(\omega_1) = 0$, then [see also (9-176)]

$$\int_{-\infty}^{\infty} (1-e^{j\omega_1 x}) f(x)dx = 0, \text{ hence, } f(x) = \sum_{n=\infty}^{\infty} p_n \delta(x - \frac{2\pi n}{\omega_1})$$

5-44 (a) If $\eta = 0$, then $m_n = \mu_n \qquad \lambda_1 = \eta = 0$

$$\Phi(s) = \sum_{n=0}^{\infty} \frac{\mu_n}{n!} s^n \qquad\qquad \Psi(s) = \sum_{n=2}^{\infty} \frac{\lambda_n}{n!} s^n$$

$$1 + \frac{\mu_2}{2!}s^2 + \frac{\mu_3}{3!} s^3 + \frac{\mu_4}{4!} s^4 + \cdots = \exp\left\{\frac{\lambda_2}{2!} s^2 + \frac{\lambda_3}{3!} s^3 + \frac{\lambda_4}{4!} s^4 + \cdots\right\}$$

Expanding the exponential and equating powers of s, we obtain

$$\mu_2 = \lambda_2 \qquad \mu_3 = \lambda_3 \qquad\qquad \frac{\mu_4}{4!} = \frac{\lambda_4}{4!} + \frac{1}{2!} \left(\frac{\lambda_2}{2!}\right)^2$$

(b) If $\underline{y}$ is $N(0;\sigma_y)$ then

$$\Psi_y(s) = \frac{\lambda_2}{2} s^2 , \quad \text{hence,} \quad \lambda_n = 0 \quad \text{for} \quad n \geq 3$$

5-45 $$P\{\underset{\sim}{y} = 0\} = P\{\underset{\sim}{x} \le 1\} = p_o + p_1$$

$$P\{\underset{\sim}{y} = k\} = P\{\underset{\sim}{x} = k + 1\} = p_{k+1} \qquad k \ge 1$$

$$\Gamma_y(z) = p_o + p_1 + \sum_{k=1}^{\infty} p_{k+1}\, z^k = p_o + z^{-1}[\Gamma_x(z) - p_o]$$

$$\eta_y = \sum_{k=1}^{\infty} k\, p_{k+1} = \sum_{r=1}^{\infty} r\, p_r - \sum_{r=1}^{\infty} p_r = \eta_x - 1 + p_o$$

$$E\{\underset{\sim}{y}^2\} = \sum_{k=1}^{\infty} k^2\, p_{k+1} = \sum_{r=1}^{\infty} (r-1)^2\, p_r = E\{\underset{\sim}{x}^2\} - 2\eta_x + 1 - p_o$$

5-46 $$0 \le E\left\{\left| \sum_{i=1}^{n} a_i\, e^{j\omega_i \underset{\sim}{x}}\right|^2\right\} = E\left\{\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j^*\, e^{j(\omega_i - \omega_j)\underset{\sim}{x}}\right\}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j^*\, \Phi(\omega_i - \omega_j)$$

5-47 From the assumptions it follows that

$$g'(-x) = -g'(x) \qquad g''(x) \ge 0 \qquad f(x-\eta) = f(\eta-x)$$

Hence, if $I(a) = E\{g(\underset{\sim}{x}-a)\}$, then

$$I'(a) = -\int_{-\infty}^{\infty} g'(x-a)f(x)\,dx \qquad I'(\eta) = 0$$

$$I''(a) = \int_{-\infty}^{\infty} g''(x-a)f(x)\,dx \ge 0 \qquad \text{all } a$$

Hence, I(a) is minimum for $a = \eta$.

5-48 $$f(x,v) = \frac{1}{\sqrt{2\pi v}} e^{-x^2/2v}$$

$$\sqrt{2\pi}\,\frac{\partial f}{\partial v} = \frac{-1 + x^2/v}{2v\,\sqrt{v}}\, e^{-x^2/2v}$$

$$\sqrt{2\pi}\,\frac{\partial^2 f}{\partial x^2} = \frac{-1 + x/.}{v\,\sqrt{v}}\, e^{-x^2/2v}$$

Hence

(see also (6-198) - (6-199)) $$\boxed{\frac{\partial f}{\partial v} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}} \qquad (1)$$

(a) Integrating by parts, using (1) and assuming that $g^{(k)}(x)f(x) \to 0$ as $|x| \to \infty$, $k = 0, 1, 2$, we obtain

$$E\{g''(x)\} = \int_{-\infty}^{\infty} \frac{d^2g}{dx^2}\, f dx = \int_{-\infty}^{\infty} g\, \frac{\partial^2 f}{\partial x^2}\, dx = 2\int_{-\infty}^{\infty} g\, \frac{\partial f}{\partial v}\, dx$$

$$= 2\,\frac{d}{dv}\int_{-\infty}^{\infty} g\, f dx = 2\,\frac{d}{dv}\, E\{g(\underset{\sim}{x})\}$$

(b) The moments $\mu_n(u) = E\{\underset{\sim}{x}^n\}$ of $\underset{\sim}{x}$ depend on the variance v of $\underset{\sim}{x}$ and (i) yields

$$\mu_n'(v) = \frac{d}{dv}\, E\{\underset{\sim}{x}^n\} = \frac{1}{2}\, E\{n(n-1)\underset{\sim}{x}^{n-2}\} = \frac{n(n-1)}{2}\, \mu_{n-2}(v)$$

Furthermore, $\mu_n(0) = 0$ because, if $v = 0$, then $\underset{\sim}{x} = 0$.

Hence

$$\mu_n(v) = \frac{n(n-1)}{2} \int_0^v \mu_{n-2}(\beta)\, d\beta$$

5-49 The function

$$\Gamma(e^{j\omega}) = E\{e^{jx\omega}\} = \sum_{k=0}^{\infty} p_k\, e^{jk\omega}$$

is periodic with period 2π and Fourier series coefficients $p_k = E\{\underline{x} = k\}$.

5.50 The event $\{X = 1\}$ is given by the disjoint union "$TH \cup HT$". Similarly, the event "$X = k$" is given by the union of the disjoint events (k "T"s followed by "H" or k "H"s followed by "T")

$$\text{“}TT\cdots TTH\text{”} \cup \text{“}HH\cdots HHT\text{”}, \qquad k = 1, 2, \cdots$$

Thus

$$\begin{aligned} P(X = k) &= P(\text{“}TT\cdots TH\text{”} \cup \text{“}HH\cdots HT\text{”}) \\ &= P(TT\cdots TH) + P(HH\cdots HT) = q^k p + p^k q, \quad k = 1, 2, \cdots \end{aligned}$$

Also

$$\begin{aligned} E(X) &= \sum_{k=1}^{\infty} kP(X = k) \\ &= \sum_{k=1}^{\infty} kq^k p + \sum_{k=1}^{\infty} kp^k q = pq\left\{\sum_{k=1}^{\infty} kq^{k-1} + \sum_{k=1}^{\infty} kp^{k-1}\right\} \\ &= pq\left\{\frac{\partial}{\partial q}\sum_{k=1}^{\infty} q^k + \frac{\partial}{\partial p}\sum_{k=1}^{\infty} p^k\right\} = pq\left\{\frac{\partial}{\partial q}\left(\frac{q}{1-q}\right) + \frac{\partial}{\partial p}\left(\frac{p}{1-p}\right)\right\} \\ &= pq\left\{\frac{1}{p^2} + \frac{1}{q^2}\right\} = \frac{p}{q} + \frac{q}{p}. \end{aligned}$$

5.51 (a) When samples are drawn with replacement, probability of each item being defective is given by

$$p = \frac{M}{N} < 1 \quad \text{(constant)}$$

and

$$q = 1 - p = \frac{N-M}{M} < 1$$

represents the constant probability that the chosen item is not defective. In that case (with replacement), there are $\binom{n}{k}$ possible ways of arranging k defective items among n chosen items, and each such arrangement has probability $p^k q^{n-k}$. This gives

$$P(X = k) = \binom{n}{k} p^k \, q^{n-k}, \qquad k = 0, 1, 2, \cdots n$$

which represents the Binomial distribution.
(b) If the samples are drawn without replacement, there are $\binom{M}{k}$ possible ways of choosing k defective item from a total of M defective items, and $\binom{N-M}{n-k}$ possible ways of choosing $n-k$ "good" items from $(N-M)$ "good" items independently. This gives

$$\binom{M}{k}\binom{N-M}{n-k}$$

to be the total number of ways of selecting k defective items and $n-k$ "good" items from a subsample of M and $N-M$ items respectively (favorable ways). But there are a total of $\binom{N}{n}$ ways of selecting n items among N items. This gives

$$P(X = k) = \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}},$$

since $0 \le k \le M$ and $n - k \le N - M, n - k \ge 0$, i.e. $0 \le k \le M, k \le n, k \ge n + M - N$.
(c) From (b)

$$P(X = k) = \frac{M!}{k!(M-k)!} \, \frac{(N-M)!}{(n-k)!(N-M-n+k)!} \, \frac{n!(N-n)!}{N!}$$

$$= \binom{n}{k} \frac{M(M-1)\cdots(M-k+1)}{N(N-1)\cdots(N-k+1)} \, \frac{(N-M)(N-M-1)\cdots(N-M-n+k+1)}{(N-k)(N-k-1)\cdots(N-n+1)} \, \frac{(I}{(I}$$

$$\simeq \binom{n}{k} \left(\tfrac{M}{N}\right)^k \left(\tfrac{N-M}{N}\right)^{n-k} = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, 1, 2, \cdots n$$

since $N \to \infty, M \to \infty$ such that $M/N \to p$, and $n \ll N$. Thus

$$P(X = k) \to \text{Binomial}(n, p = M/N)$$

under the above conditions.

5.52 (a) Refer to discussions in problem 5.51 (a) if sampleing is done with replacement, then

$$p = \frac{n}{n+m}$$

represents the probability of selecting a white marble on any trial. The event "$X = k$" is given by "$r-1$ white mables among the first $k-1$ trials" followed by "a white marble at the k^{th} trial". But from problem 5.51 (a), the event $r-1$ white mables among the first $k-1$ trials has a binomial distribution whose probability is given by $\binom{k-1}{r-1}p^{r-1}q^{k-r}$. Thus

$$P(X = k) = \binom{k-1}{r-1}p^{r-1}q^{k-r}p = \binom{k-1}{r-1}p^r q^{k-r}, \quad k = r, r+1, \cdots$$

which represents the Negative-binomial distribution

(b) If sampling is done with replacement, then the favarable ways of choosing the white balls are given by:

(i) $\binom{k-1}{r-1}$ ways of selecting $r-1$ white balls among the first $k-1$ trials/balls.

(ii) One ways of selecting (the r^{th}) white ball at the k^{th} trial

(iii) $\binom{m+n-k}{n-r}$ ways of selecting the remaining $n-r$ white balls among the remaining $m+n-k$ balls.

This gives $\binom{k-1}{r-1} \cdot 1 \cdot \binom{m-n-k}{n-r}$ to be the total number of favorable ways of selecting the white balls. Since there are $n+m$ balls there are a total of $\binom{n+m}{n}$ ways of selecting n white balls. This gives

$$P(X = k) = \binom{k-1}{r-1}\frac{\binom{m+n-k}{n-r}}{\binom{n+m}{n}}, \qquad k = r, r+1, \cdots$$

(c) From (b)

$$P(X = k) = \binom{k-1}{r-1}\frac{(m+n-k)!}{(n-r)!\,(m-k+r)!}\frac{n!m!}{(m+n)!}$$

$$= \binom{k-1}{r-1}\left(\frac{n}{m+n}\right)\left(\frac{n-1}{m+n-1}\right)\cdots\left(\frac{n-r+1}{m+n-r+1}\right)\left(\frac{m!(m+n-k)!}{(m+n-r)!\,(m-k+r)!}\right)$$

$$\simeq \binom{k-1}{r-1}\left(\frac{n}{m+n}\right)^r\left(\frac{m}{m+n-r}\right)\left(\frac{m-1}{m+n-r-1}\right)\cdots\left(\frac{m-k+r+1}{m+n-k+1}\right)$$

$$\simeq \binom{k-1}{r-1}\left(\frac{n}{m+n}\right)^r\left(\frac{m}{m+n}\right)^{k-r} \text{ as } m+n \to \infty$$

$$= \binom{k-1}{r-1}p^r q^{k-r}, \qquad k = r, r+1, \cdots, \quad q = 1-p$$

$$\sim \mathrm{N}B\,(r, p = n/(n+m))\,.$$

CHAPTER 6

6.1 (a) Define

$$Z = X + Y$$

Note that both X and Y positive random variables hence (use Eq. (6-45))

$$\begin{aligned} f_Z(z) &= \int_0^z f_{XY}(z-y, y)dy = \int_0^z e^{-(z-y+y)}dy \\ &= z\, e^{-z}U(z). \end{aligned}$$

(b)

$$Z = X - Y$$

Z ranges over the entire real axis for the random variables X and Y (see Eq. (6-55))

$$F_Z(z) = \begin{cases} \int_0^\infty \int_0^{z+y} f_{XY}(x,y)dx\,dy, & z > 0 \\ \int_{-z}^\infty \int_0^{z+y} f_{XY}(x,y)dx\,dy, & z < 0 \end{cases}$$

Differrentiation gives

$$f_Z(z) = \begin{cases} \int_0^\infty f_{XY}(z+y, y)\,dy, & z > 0 \\ \int_{-z}^\infty f_{XY}(z+y, y)\,dy, & z < 0 \end{cases}$$

$$f_Z(z) = \begin{cases} \int_0^\infty e^{-(z+y+y)}\,dy = e^{-z}\int_0^\infty e^{-2y}\,dy = \frac{1}{2}\,e^{-z}, & z > 0 \\ \int_{-z}^\infty e^{-(z+y+y)}\,dy = e^{-z}\int_{-z}^\infty e^{-2y}\,dy = \frac{1}{2}\,e^{z}, & z < 0 \end{cases}$$

or

$$f_Z(z) = \frac{1}{2}\,e^{-|z|}, \qquad -\infty \le z \le \infty.$$

(c)

$$Z = XY.$$

$$\begin{aligned} F_Z(z) &= P\{Z \le z\} = P\{XY \le z\} \\ &= \int_0^\infty \int_0^{z/y} f_{XY}(x,y)dx\,dy \end{aligned}$$

or (see Eq. (6-148))

$$f_Z(z) = \int_0^\infty \frac{1}{y} f_{XY}(\frac{z}{y}, y)dy = \int_0^\infty \frac{1}{y} e^{-((z/y)+y)}dy$$

(d)

$$Z = X/Y$$

$$\begin{aligned} F_Z(z) &= P\{Z \le z\} = P\{\tfrac{X}{Y} \le z\} \\ &= \int_0^\infty \int_0^{yz} f_{XY}(x,y)dx\,dy \end{aligned}$$

(use Eq. (6-60))

$$\begin{aligned} f_Z(z) &= \int_0^\infty y f_{XY}(yz,y)dy = \int_0^\infty ye^{y(z+1)}dy = \int_0^\infty ye^{(1+z)y} \\ &= \left[y\frac{e^{-(1+z)y}}{-(1+z)}\right]_0^\infty + \left(\frac{1}{1+z}\right)\int_0^\infty e^{(1+z)y}dy \\ &= \left(\frac{1}{1+z}\right)\left[\frac{e^{-(1+z)y}}{-(1+z)}\right]_0^\infty = \frac{1}{(1+z)^2}\,U(z) \end{aligned}$$

(e)

$$Z = \min\,(X,Y)$$

$$\begin{aligned} F_Z(z) &= P\{\min\,(X,Y) \le z\} \\ &= 1 - P\{Z > z, Y > z\} \\ &= 1 - [1 - F_X(z)]\,[1 - F_Y(z)] \\ &= F_X(z) + F_Y(z) - F_X(z)\,F_Y(z) \end{aligned}$$

(see Eq. (6-81))

$$f_Z(z) = f_X(z) + f_Y(z) - F_X(z)f_Y(z) - f_X(z)F_Y(z).$$

We have

$$f_X(z) = f_Y(z) = e^{-z}\,U(z)$$

so that

$$F_X(z) = \int_0^z e^{-x}dx = (1 - e^{-z})\,U(z) = F_Y(z)$$

$$\begin{aligned} f_Z(z) &= [e^{-z} + e^{-z} - 2(1 - e^{-z})\,e^{-z}]U(z) \\ &= 2e^{-z}\,[1 - 1 + e^{-z}]\,U(z) \\ &= 2e^{-2z}\,U(z) \sim \text{Exponential }(2). \end{aligned}$$

(f)

$$Z = \max(X,Y)$$

$$\begin{aligned} F_Z(z) &= P\{\max(X,Y) \le z\} = P\{X \le z, Y \le z\} \\ &= P\{X \le z\}\,P\{Y \le z\} = F_X(z)\,F_Y(z) \end{aligned}$$

$$\begin{aligned} f_Z(z) &= F_X(z)\,f_Y(z) + f_X(z)\,F_Y(z) \\ &= e^{-z}\,(1 - e^{-z}) + e^{-z}\,(1 - e^{-z}) \\ &= 2e^{-z}\,(1 - e^{-z})\,U(z) \end{aligned}$$

(g)

$$Z = \frac{\min(X,Y)}{\max(X,Y)}, \quad 0 < z < 1$$

$$\begin{aligned}
F_Z(z) &= P\left\{\left(\frac{\min(X,Y)}{\max(X,Y)} \le z\right) \bigcap ((X \le Y) \bigcup (X > Y))\right\} \\
&= P\left\{\left(\frac{\min(X,Y)}{\max(X,Y)} \le z\right) \bigcap (X \le Y)\right\} + P\left\{\left(\frac{\min(X,Y)}{\max(X,Y)} \le z\right) \bigcap (X > Y)\right\} \\
&= P\{\tfrac{X}{Y} \le z, X \le Y\} + P\{\tfrac{Y}{X} \le z, X > Y\} \\
&= P\{X \le Yz, X \le Y\} + P\{Y \le Xz, X > Y\} \\
&= \int_0^\infty \int_0^{yz} f_{XY}(x,y)\, dx\, dy + \int_0^\infty \int_0^{xz} f_{XY}(x,y)\, dy\, dx
\end{aligned}$$

$$\begin{aligned}
f_Z(z) &= \int_0^\infty y\, f_{XY}(yz, y) dy + \int_0^\infty x f_{XY}(x, xz) dx \\
&= \int_0^\infty y\, f_{XY}(yz, y) dy + \int_0^\infty y\, f_{XY}(y, yz) dy \\
&= \int_0^\infty y \left(e^{-(yz+y)} + e^{-(y+yz)}\right) dy \\
&= 2\int_0^\infty y e^{-y(1+z)} dz = \begin{cases} \dfrac{2}{(1+z)^2}, & 0 \le z \le 1 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

6.2

$$f_{XY}(x,y) = f_X(x)\, f_Y(y) = \frac{1}{a^2}, \qquad 0 < x \le a, \quad 0 < y \le a$$

(a)

$$F_Z(z) = P\left\{\frac{X}{Y} \le z\right\} = P\{X \le zY\}$$

(i) $z < 1$

$$\begin{aligned}
F_Z(z) &= P\{X \le zY\} \\
&= \int_0^a \int_0^{zy} \tfrac{1}{a} \cdot \tfrac{1}{a}\, dx\, dy = \tfrac{z}{2}, \quad z \le 1
\end{aligned}$$

(ii) $z \ge 1$

$$\begin{aligned}
F_Z(z) &= P\{X \le zY\} \\
&= 1 - \int_0^a \int_0^{x/z} \tfrac{1}{a} \cdot \tfrac{1}{a}\, dy\, dx \\
&= 1 - \int_0^1 \tfrac{x}{z}\, dx = 1 - \tfrac{1}{2z} \quad z > 1
\end{aligned}$$

$$f_Z(z) = \begin{cases} \tfrac{1}{2}, & z \le 1 \\ \dfrac{1}{2z^2}, & z > 1 \end{cases}$$

(b)

$$\begin{aligned}
F_Z(z) &= P(Z \le z) = P\left\{\frac{Y}{X+Y} \le z\right\} \\
&= P\left\{\frac{X}{Y} \ge \frac{1}{z} - 1\right\} = 1 - P\left(\frac{X}{Y} \le \frac{1-z}{z}\right) \\
&= \begin{cases} \frac{1}{2}\left(\frac{z}{1-z}\right), & 0 < z \le 1/2 \\ 1 - \frac{1}{2}\left(\frac{1-z}{z}\right), & 1/2 < z < 1 \end{cases}
\end{aligned}$$

$$f_Z(z) = \begin{cases} \dfrac{1}{2(1-z)^2}, & 0 < z \le 1/2 \\ \dfrac{1}{2z^2}, & 1/2 < z < 1 \end{cases}$$

(c)

$$\begin{aligned}
F_Z(z) &= P\{Z \le z\} = P\{|X-Y| \le z\} \\
&= P\{\{|X-Y| \le z\} \bigcap (X \ge Y)\} + P\{\{|X-Y| \le z\} \bigcap (X < Y)\} \\
&= P\{X - Y \le z, X \ge Y\} + P\{Y - X \le z, X < Y\} \\
&= \int_0^\infty \int_y^{y+z} f_{XY}(x,y)\,dx\,dy + \int_0^\infty \int_x^{x+z} f_{XY}(x,y)\,dy\,dx \\
&= \int_0^\infty \int_y^{y+z} f_{XY}(x,y)\,dx\,dy + \int_0^\infty \int_y^{y+z} f_{XY}(y,x)\,dx\,dy \\
&= \int_0^\infty \int_y^{y+z} \{f_{XY}(x,y) + f_{XY}(y,x)\}\,dx\,dy.
\end{aligned}$$

In general

$$\begin{aligned}
f_Z(z) &= \int_0^\infty \frac{d}{dz} \int_y^{y+z} f_{XY}(x,y) + f_{XY}(y,x)\,dx\,dy \\
&= \int_0^\infty \{f_{XY}(y+z,y) + f_{XY}(y,y+z)\}\,dy.
\end{aligned}$$

Here

$$X \sim U(0,a), \qquad Y \sim U(0,a)$$

$$F_Z(z) = 1 - \frac{1}{a^2} \cdot 2 \cdot \frac{(a-z)^2}{2} = 1 - \left(1 - \frac{z}{a}\right)^2$$

and

$$f_Z(z) = \frac{2}{a}\left(1 - \frac{z}{a}\right) \qquad 0 \le z \le a.$$

6.3

$$\begin{aligned} F_Z(z) &= P\{Z \le z\} = P\{X+Y \le z\} \\ &= \tfrac{1}{2} - \tfrac{z^2}{2}, \quad -1 < z < 0, \end{aligned}$$

(which represents the area below the line $X + Y = z$.)

$$\begin{aligned} F_Z(z) &= P\{Z \le z\} = P\{X+Y \le z\} \\ &= \tfrac{1}{2} + \tfrac{z^2}{2}, \quad 0 \le z < 1 \end{aligned}$$

$$f_Z(z) = \begin{cases} -z, & -1 \le z < 0 \\ z, & 0 \le z < 1 \end{cases}$$

6.4

$$Z = X - Y$$

For $z < 0$

$$\begin{aligned} F_Z(z) &= P\{Z \le z\} \\ &= \int_0^{(1+z)/2} \int_{x-z}^{1-x} f_{XY}(x,y)\,dy\,dx = \int_0^{(1+z)/2} \int_{x-z}^{1-x} 6x\,dy\,dx \\ &= \int_0^{(1+z)/2} 6x\,[y]_{x-z}^{1-x}\,dx = \int_0^{(1+z)/2} 6x(1-x-x+z)\,dx \\ &= 6\left[(1+z)\frac{x^2}{2} - \frac{2x^3}{3}\right]_0^{(1+z)/2} = 6\left[\frac{(1+z)^3}{8} - \frac{(1+z)^3}{12}\right] \\ &= \frac{(1+z)^3}{4}, \quad z \le 0. \end{aligned}$$

For $z > 0$

$$\begin{aligned} F_Z(z) &= P\{Z \le z\} = 1 - P\{Z > z\} \\ &= 1 - \int_0^{(1-z)/2} \int_{z+y}^{1-y} f_{XY}(x,y)\,dx\,dy = 1 - \int_0^{(1-z)/2} \int_{z+y}^{1-y} 6x\,dy \\ &= 1 - \int_0^{(1-z)/2} \left[\frac{6x^2}{2}\right]_{z+y}^{1-y} dy = 1 - 3\int_0^{(1-z)/2} \left[(1-y)^2 - (z-y)^2\right] dy \\ &= 1 - 3\,(1+z)\left[\frac{(1-z)^2}{2} - \frac{(1-z)^2}{4}\right] = 1 - \tfrac{3}{4}\,(1+z)(1-z)^2 \quad z \le 0. \end{aligned}$$

$$f_Z(z) = \begin{cases} \frac{3}{4}\,(1-z)(1+3z), & 0 \le z \le 1 \\ \frac{3}{4}\,(1+z)^2, & -1 < z < 0 \end{cases}$$

6.5 (a) See Example 6-15 for solutions
(b) See Example 6-14 for solutions
(c)

$$U = X - Y \sim N(0, 2\sigma^2)$$

since linear combinations of jointly Gaussian random variables are Gaussian random variables (see Eq. (6-120) Text.). Here $Var(U) = Var(X) + Var(Y) = 2\sigma^2$.

6.6

$$Z = XY$$

$$\begin{aligned}
F_Z(z) &= P(XY \le z) = 1 - P(XY > z) \\
&= 1 - \int_z^1 \int_{z/y}^1 f_{XY}(x,y)\,dx\,dy \\
f_Z(z) &= 1 + \int_z^1 \frac{1}{y} f_{XY}(z/y, y)\,dy = 1 + \int_z^1 \left\{\frac{2}{y} - \frac{2z}{y^2}\right\} dy \\
&= 1 - 2\ln z + 2z, \quad 0 \le z \le 1
\end{aligned}$$

6.7 (a)

$$Z_1 = X + Y$$

$$F_{Z_1}(z) = P(X+Y \le z) = \begin{cases} \displaystyle\int_0^z \int_0^{z-y} f_{XY}(x,y)\,dx\,dy, & 0 < z < 1 \\ \displaystyle 1 - \int_{z-1}^1 \int_{z-y}^1 f_{XY}(x,y)\,dx\,dy, & 1 < z < 2 \end{cases}$$

$$f_{Z_1}(z) = \begin{cases} \displaystyle\int_0^z f_{XY}(z-y, y)\,dy, & 0 < z < 1 \\ \displaystyle\int_{z-1}^1 f_{XY}(z-y, y)\,dy, & 1 < z < 2 \end{cases}$$

$$= \begin{cases} z^2, & 0 < z < 1 \\ z(2-z), & 1 < z < 2 \\ 0, & \text{otherwise} \end{cases}$$

(b)

$$Z_2 = XY$$

$$F_{Z_2}(z) = P(XY \le z) = 1 - \int_z^1 \int_{z/y}^1 f_{XY}(x,y)\,dx\,dy$$

$$\begin{aligned}
f_{Z_2}(z) &= \int_z^1 \frac{1}{y} f_{XY}(z/y, y)\,dy = \int_z^1 \frac{1}{y}\left(\frac{z}{y} + y\right) dy \\
&= 2(1-z), \quad 0 < z < 1
\end{aligned}$$

(c)

$$Z_3 = \frac{Y}{X}$$

$$F_{Z_3}(z) = P(Y/X \le z) = \begin{cases} \displaystyle\int_0^1 \int_0^{zx} f_{XY}(x,y)\,dy\,dx, & 0 < z < 1 \\ \displaystyle 1 - \int_0^1 \int_0^{y/z} f_{XY}(x,y)\,dx\,dy, & z > 1 \end{cases}$$

$$f_{Z_3}(z) = \begin{cases} \int_0^1 x\, f_{XY}(x, zx)\, dx, & 0 < z < 1 \\ \int_0^1 \frac{y}{z^2} f_{XY}(y/z, y)\, dy, & z > 1 \end{cases}$$

$$= \begin{cases} \frac{1+z}{3}, & 0 < z < 1 \\ \frac{1+z}{3z^3}, & z > 1 \end{cases}$$

(d)

$$Z_4 = Y - X$$

$$F_{Z_4}(z) = P(Y - X \le z) = \begin{cases} 1 - \int_z^1 \int_0^{y-z} f_{XY}(x, y)\, dx\, dy & 0 < z < 1 \\ \int_0^{z+1} \int_{y-z}^1 f_{XY}(x, y)\, dx\, dy, & -1 < z < 0 \end{cases}$$

$$f_{Z_4}(z) = \begin{cases} \int_z^1 f_{XY}(y - z, y)\, dy, & 0 < z < 1 \\ \int_0^{z+1} f_{XY}(y - z, y)\, dy, & -1 < z < 0 \end{cases}$$

$$= \begin{cases} 1 - z, & 0 < z < 1 \\ 1 + z, & -1 < z < 0 \end{cases} = 1 - |z|, \quad |z| < 1$$

6.8

$$F_Z(z) = P(X + Y \le z)$$

$$= \begin{cases} \int_0^{z/3} \int_{2y}^{z-y} f_{XY}(x, y)\, dx\, dy = \frac{z^2}{6}, & 0 < z < 2 \\ 1 - \int_{2z/3}^{2} \int_{z-x}^{x/2} f_{XY}(x, y)\, dy\, dx = 2z - \frac{z^2}{3} - 2, & 2 < z < 3 \end{cases}$$

Thus

$$f_Z(z) = \begin{cases} \int_0^{z/3} f_{XY}(z - y, y)\, dy & 0 < z < 2 \\ \int_{2z/3}^{2} f_{XY}(x, z - x)\, dx & 2 < z < 3 \end{cases}$$

$$f_Z(z) = \begin{cases} \frac{1}{3} z, & 0 < z < 2 \\ 2 - \frac{2z}{3}, & 2 < z < 3 \\ 0, & \text{otherwise} \end{cases}$$

6.9 (a)

$$Z = \frac{X}{Y}, \qquad z \geq 1$$

$$F_Z(z) = P(X \leq Yz) = \int_0^1 \int_{x/z}^x f_{XY}(x,y)\,dy\,dx$$

$$f_Z(z) = \int_0^1 \frac{x}{z^2} f_{XY}(x, x/z)\,dx = \frac{1}{z^2}, \quad z \geq 1$$

(b)

$$W = XY$$

$$\begin{aligned} F_W(w) &= P(W \leq w) = P(XY \leq w) = 1 - P(XY > w) \\ &= 1 - \int_{\sqrt{w}}^1 \int_{w/x}^x f_{XY}(x,y)\,dy\,dx \end{aligned}$$

Hence

$$\begin{aligned} f_W(w) &= \int_{\sqrt{w}}^1 \frac{1}{x} f_{XY}(x, w/x)\,dx = \int_{\sqrt{w}}^1 \frac{2}{x}\,dx \\ &= \ln(1/w), \quad 0 < w \leq 1 \end{aligned}$$

6.10 (a)

$$Z = X + Y$$

$$F_Z(z) = \int_0^{z/2} \int_x^{2-x} f_{XY}(x,y)\,dx = \frac{z^2}{4}, \quad 0 < z < 2$$

$$f_Z(z) = \frac{z}{2}, \qquad 0 < z < 2$$

(b)

$$W = X - Y$$

$$F_W(w) = \frac{1}{2}(2+w)(1+\frac{w}{2}) = \left(1 + \frac{w}{2}\right)^2$$

$$f_W(w) = \begin{cases} 1 + \frac{w}{2}, & -2 < w < 0 \\ 0, & \text{otherwise} \end{cases}$$

6.11 (a) The charaterristic function of $X+Y$ is given by

$$\begin{aligned}\phi_{X+Y}(\omega) &= \phi_X(\omega)\,\phi_Y(\omega) = \tfrac{1}{(1-j\omega\beta)^{\alpha}} \cdot \tfrac{1}{(1-j\omega\beta)^{\alpha}} \\ &= \tfrac{1}{(1-j\omega\beta)^{2\alpha}} \sim \text{Gamma}\,(2\alpha, \beta)\end{aligned}$$

(b)

$$f_{XY}(x,y) = f_X(x)\,f_Y(y) = \frac{(xy)^{(\alpha-1)}}{(\Gamma(\alpha)\beta^{\alpha})^2}\; e^{(x+y)/\beta}, \quad x>0,\ y>0$$

Let

$$Z = \frac{X}{Y}$$

Using (Eq. 6-60) we get

$$\begin{aligned}f_Z(z) &= \int_0^{\infty} y\,\frac{(y^2 z)^{(\alpha-1)}}{(\Gamma(\alpha)\beta^{\alpha})^2}\; e^{-(1+z)y/\beta}\,dy \\ &= \frac{z^{(\alpha-1)}}{(\Gamma(\alpha)\beta^{\alpha})^2}\int_0^{\infty} y^{(2\alpha-1)}\,e^{-(1+z)y/\beta}\,dy \\ &= \frac{z^{(\alpha-1)}}{(\Gamma(\alpha))^2\;\beta^{2\alpha}}\,\frac{\beta^{(2\alpha-1)}}{(1+z)^{2\alpha-1}}\,\frac{\beta}{(1+z)}\int_0^{\infty} u^{2\alpha-1}\,e^{-u}\,du \\ &= \frac{(\Gamma(2\alpha))\;z^{\alpha-1}}{(\Gamma(\alpha))^2\,(1+z)^{2\alpha}}, \qquad z>0\end{aligned}$$

(see also Example 6-27 for the answer).
(c)

$$W = \frac{X}{X+Y} = \frac{X/Y}{X/Y+1} = \frac{Z}{Z+1}$$

$$F_W(w) = P\left(\frac{Z}{Z+1} \le w\right) = P\left(Z \le \frac{w}{1-w}\right) = F_Z\left(\frac{w}{1-w}\right)$$

This gives

$$\begin{aligned}f_W(w) &= \frac{1}{(1-w)^2}\,f_Z\left(\tfrac{w}{1-w}\right) \\ &= \frac{\Gamma(2\alpha)}{(\Gamma(\alpha))^2}\,w^{\alpha-1}\,(1-w)^{\alpha-1} \\ &\sim \ \mathit{Beta}\,(\alpha,\alpha)\end{aligned}$$

where we have used results from (b) above.

6.12

$$X \sim U(0,1), \quad Y \sim U(0,1), \quad X, Y \text{ are independent, and}$$

$$U = X+Y, \qquad V = X-Y \ \Rightarrow\ |v| < u < 2.$$

U and V have one pair of solutions given by

$$x_1 = \frac{u+v}{2},\, y_1 = \frac{u-v}{2}.$$

Also the Jacobian is given by

$$J = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

so that

$$f_{UV}(u,v) = \frac{1}{|J|}\,f_{XY}(x_1,y_1) = \frac{1}{2}, \quad 0<|v|<u<2$$

6.13

$$f_{XY}(x,y) = \frac{xy}{\sigma^4}\, e^{-(x^2+y^2)/2\sigma^2}, \quad x, y \geq 0$$

$$Z = \frac{X}{Y}$$

$$F_Z(z) = P(Z \leq z) = P(X/Y \leq z) = \int_0^\infty \int_0^{zy} f_{XY}(x,y)\,dx\,dy.$$

This gives the density function of z to be

$$\begin{aligned} f_Z(z) &= \int_0^\infty y\, f_{XY}(zy, y)\,dy = \int_0^\infty \frac{zy^3}{\sigma^4}\, e^{-(z^2y^2+y^2)/2\sigma^2}\,dy \\ &= \frac{z}{\sigma^4}\int_0^\infty y^3\, e^{-y^2(z^2+1)/2\sigma^2}\,dy \quad \text{Let, } t = y^2(z^2+1)/2\sigma^2 \\ &= \frac{2z}{(z^2+1)^2}\int_0^\infty t\, e^{-t}\,dt = \frac{2z}{(z^2+1)^2}, \quad 0 \leq z \leq \infty. \end{aligned}$$

6-14 $\quad \underset{\sim}{z} = \underset{\sim}{x} + \underset{\sim}{y} \qquad f_z(z) = f_x(z) * f_y(z)$

For $z > 0$

$$c^2 z\, e^{-cz} = \int_0^z c\, e^{-c(z-y)}\, f_y(y)\,dy$$

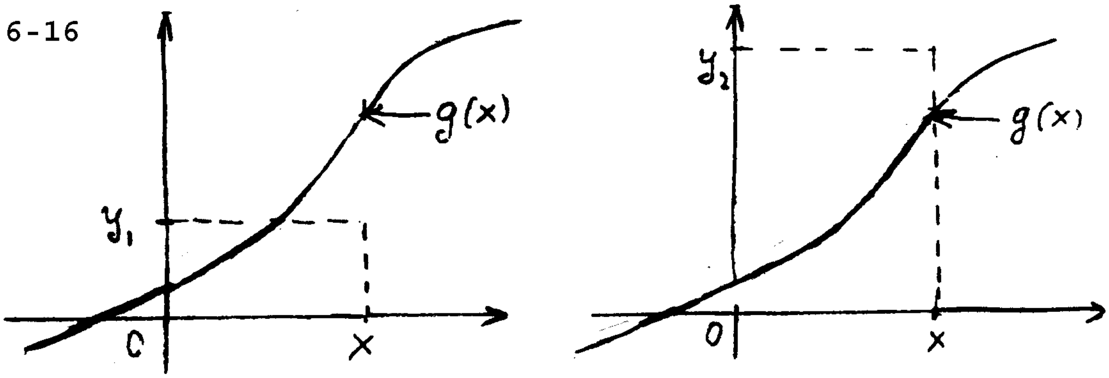
$$c\,z = \int_0^z e^{cy}\, f_y(y)\,dy \qquad c = e^{c\,z} f_y(z)$$

(differentiation). Hence, $f_y(z) = c\,e^{-cz}$; and zero for $z < 0$.

6-15 $$f_z(z) = \int_{-\infty}^{\infty} f_x(x) f_y(z-x)\,dx = \int_{z-1}^{z} f_x(x)\,dx = F_x(z) - F_x(z-1)$$

because $f_y(z-x) = 1$ for $z - 1 < x < z$ and zero otherwise.

6-16



All probability masses are on the line $y = g(x)$.

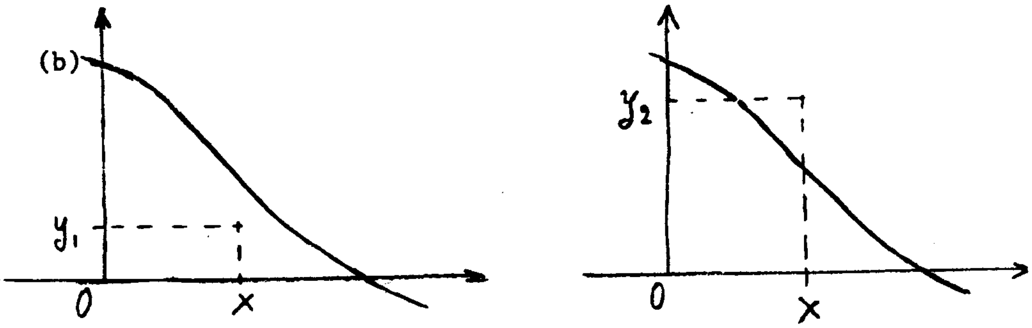
(a) If $y = y_1 < g(x)$ then

$$F(x,y) = P\{\underset{\sim}{x} \le x,\ \underset{\sim}{y} \le y_1\} = P\{\underset{\sim}{y} \le y_1\} = F_y(y_1).$$

If $y = y_2 > g(x)$ then

$$F(x,y) = P\{\underset{\sim}{x} \le x,\ \underset{\sim}{y} \le y_2\} = P\{\underset{\sim}{x} \le x\} = F_x(x)$$

(b)



If $y = y_1 < g(x)$ then

$$F(x,y) = P\{\underset{\sim}{x} \le x,\ \underset{\sim}{y} \le y_1\} = 0$$

If $y = y_2 > g(x)$ then

$$F(x,y) = P\{\underset{\sim}{x} \le x,\ \underset{\sim}{y} \le y_2\} = P\{\underset{\sim}{x} \le x\} - P\{\underset{\sim}{y} > y_2\}$$

$$= F_x(x) - [1 - F_y(y_2)]$$

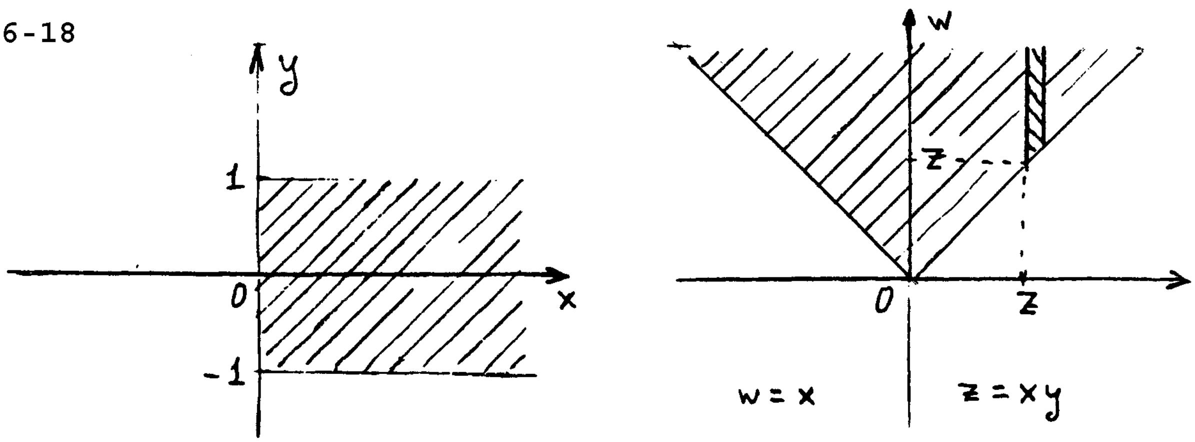
6-17 (a) If $\underset{\sim}{z} = 2\underset{\sim}{x} + 3\underset{\sim}{y}$ then $E\{\underset{\sim}{z}\} = 0$ $\quad \sigma_z^2 = 4\sigma_x^2 + 9\sigma_y^2 = 5^2$

Hence, $\underset{\sim}{z}$ is $N(0;\sqrt{52})$

(b) If $\underset{\sim}{z} = \underset{\sim}{x}/\underset{\sim}{y}$, then from (6-63) with $\sigma_1 = \sigma_2 = 2$, $r = 0$

$$F_z(z) = \frac{1}{2} + \frac{1}{\pi}\arctan z \qquad f_z(z) = \frac{1}{\pi(1+z^2)}$$

6-18



$$f_{zw}(z,w) = \frac{1}{|x|} f_{xy}(x,y) \qquad x = w \qquad y = z/w$$

The function $f_{zw}(z,w)$ is different from zero in the shaded areas shown. Hence, with $w^2 - z^2 = s^2$

$$f_z(z) = \frac{1}{\pi\alpha^2}\int_{|z|}^{\infty} e^{-w^2/2\alpha^2} \frac{dw}{\sqrt{1 - z^2/w^2}}$$

$$= \frac{1}{\pi\alpha^2}\int_0^{\infty} e^{-(z^2+s^2)/2\alpha^2}\, ds = \frac{1}{\alpha\sqrt{2\pi}} e^{-z^2/2\alpha^2}$$

6-19 (a) $\underset{\sim}{z} = \underset{\sim}{x}/\underset{\sim}{y} \qquad \underset{\sim}{w} = \underset{\sim}{y} \qquad J = 1/y$

$$f_z(z) = \int_{-\infty}^{\infty} |w| f_x(zw) f_y(w)\,dw \qquad z > 0$$

$$= \frac{z}{\alpha^2\beta^2} \int_0^{\infty} w^3\, e^{-cw^2} dw = \frac{z}{2\alpha^2\beta^2 c^2} \qquad c = \frac{z^2}{2\alpha^2} + \frac{1}{2\beta^2}$$

$$= \frac{2\alpha^2}{\beta^2} \frac{z}{(z^2 + \alpha^2/\beta^2)^2} \qquad \text{for } z > 0 \text{ and zero otherwise}$$

(b) $$F_z(z) = \int_0^z \frac{2\alpha^2 z\,dz}{\beta^2(z^2+\alpha^2/\beta^2)^2} = \frac{\alpha 2}{\beta^2} \int_{\alpha^2/\beta^2}^{z^2+\alpha^2/\beta^2} \frac{dt}{t^2}$$

$$= \frac{z^2}{z^2 + \alpha^2/\beta^2} = P\{\underset{\sim}{z} \le z\} = P\{\underset{\sim}{x} \le zy\}$$

6-20 1. The density of $2\underset{\sim}{x}$ equals $\frac{1}{2} f_x(\frac{x}{2})$. Hence, if $\underset{\sim}{z} = 2\underset{\sim}{x} + \underset{\sim}{y}$, then

$$f_z(z) = \int_0^z \frac{\alpha}{2} e^{-\alpha x/2} \beta e^{-\beta(z-x)} dx = \frac{\alpha\beta}{\alpha - 2\beta}(e^{-\beta z} - e^{-\alpha z/2})U(z)$$

2. The density of $\underset{\sim}{y}$ equals $f_y(-y)$. Hence, if $\underset{\sim}{z} = \underset{\sim}{x} - \underset{\sim}{y}$, then

$$f_z(z) = f_x(z) * f_y(-z)$$

$$= \alpha\beta \begin{cases} \displaystyle\int_z^\infty e^{-\alpha x} e^{-\beta(x-z)} dx = \frac{\alpha\beta}{\alpha+\beta} e^{-\alpha z} & z > 0 \\ \displaystyle\int_0^\infty e^{-\alpha x} e^{-\beta(x-z)} dx = \frac{\alpha\beta}{\alpha+\beta} e^{\beta z} & z < 0 \end{cases}$$

3. $\underset{\sim}{z} = \underset{\sim}{x}/\underset{\sim}{y} \qquad \underset{\sim}{w} = \underset{\sim}{y} \qquad J = 1/y$

$$f_z(z) = \alpha\beta \int_0^\infty w e^{-\alpha z w} e^{-\beta w} dw = \frac{\alpha\beta}{(\alpha z + \beta)^2} U(z)$$

4. $\underset{\sim}{z} = \max(\underset{\sim}{x},\underset{\sim}{y}) \qquad F_z(z) = F_{xy}(z,z) = F_x(z)F_y(z)$

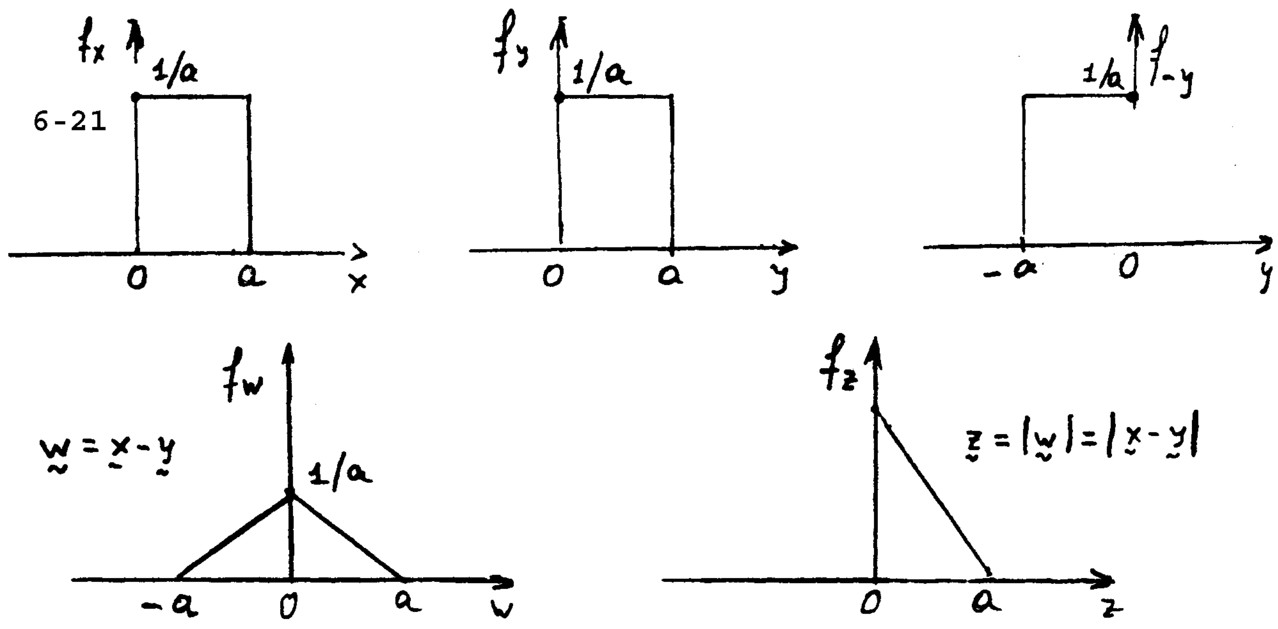
$$f_z(z) = f_x(z)F_y(z) + f_y(z)F_x(z)$$

$$= \left[\alpha e^{-\alpha z}(1 - e^{-\beta z}) + \beta e^{-\beta z}(1 - e^{-\alpha z})\right] U(z)$$

5. $\underset{\sim}{z} = \min(\underset{\sim}{x},\underset{\sim}{y}) \qquad F_z(z) = F_x(z) + F_y(z) - F_x(z)F_y(z)$

$$f_z(z) = f_x(z)[1 - F_y(z)] + f_y(z)[1 - F_x(z)] = (\alpha+\beta)e^{-(\alpha+\beta)z}U(z)$$

6-21



6-22 (a)

$$\alpha y^2 + \beta(x-y)^2 = (\alpha+\beta)\left(y - \frac{\beta x}{\alpha+\beta}\right)^2 + \frac{\alpha\beta}{\alpha+\beta}x^2$$

$$e^{-\alpha x^2} * e^{-\beta x^2} = \int_{-\infty}^{\infty} e^{-\alpha y^2 - \beta(x-y)^2}\,dy$$

$$= e^{-\alpha\beta x^2/(\alpha+\beta)} \int_{-\infty}^{\infty} e^{-(\alpha+\beta)\left(y - \frac{\beta x}{\alpha+\beta}\right)^2}\,dy = \sqrt{\frac{\pi}{\alpha+\beta}}\; e^{\frac{-\alpha\beta x^2}{\alpha+\beta}}$$

(b) $$\frac{\alpha/\pi}{x^2+\alpha^2} * \frac{\beta/\pi}{x^2+\beta^2} = \frac{\alpha\beta}{\pi^2}\int_{-\infty}^{\infty} \frac{dy}{(y^2+\alpha^2)[(x-y)^2+\beta^2]} = \frac{(\alpha+\beta)/\pi}{x^2+(\alpha+\beta)^2}$$

Characteristic functions lead to a simpler derivation of the above [see (6-192)]

6-23 We introduce the auxiliary variable w=y. The Jacobian of the transformation z=nx/my, w=y equals n/my. Since x=mzw/n, y=w and the RVs $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are independent, (6-113) yields

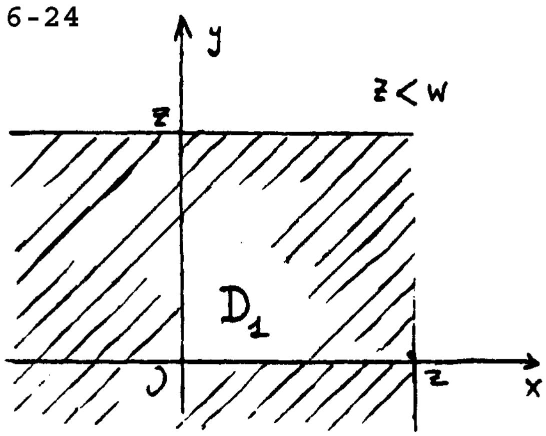
$$f_{zw}(z,w) = \frac{mw}{n} f_x\left(\frac{m}{n} zw\right) f_y(w) \sim w(zw)^{m/2-1} e^{-mzw/2} w^{n/2-1} e^{-w/2}$$

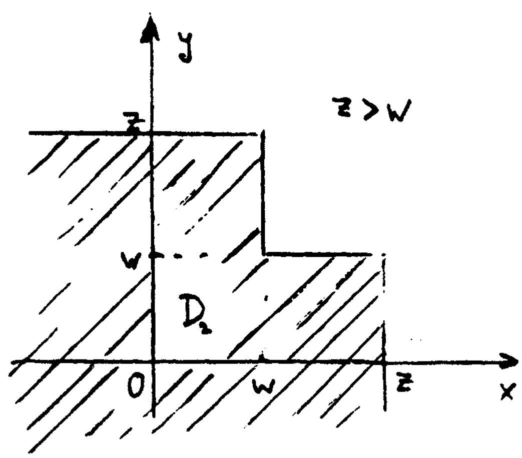
for z>0, w>0 and 0 othrwise. Integrating with respect to w, we obtain

$$f_z(z) \sim z^{m/2-1} \int_0^\infty w^{(m+n)/2-1} \exp\left\{-\frac{w}{2}\left(1 + \frac{m}{n} z\right)\right\} dw$$

$$\sim \frac{z^{m/2-1}}{(1+mz/n)^{(m+n/2)}} \int_0^\infty q^{(m+n)/2} e^{-q} dq$$

6-24





If $z \le w$ then

$$P\{\underset{\sim}{z} \le z, \underset{\sim}{w} \le w\} = P\{\underset{\sim}{z} \le z\} = P\{(\underset{\sim}{x},\underset{\sim}{y}) \in D_1\} = F_{xy}(z,z)$$

If $z > w$ then

$$P\{\underset{\sim}{z} \le z, \underset{\sim}{w} \le w\} = P\{(\underset{\sim}{x},\underset{\sim}{y}) \in D_2\}$$

$$= F_{xy}(z,w) + F_{xy}(w,z) - F_{xy}(w,w)$$

6.25

$$X \sim \text{Exponential}(\lambda), \quad Y \sim \text{Exponential}(\lambda)$$

X and Y are independent so that

$$f_{XY}(x,y) = f_X(x)\, f_Y(y) = \frac{1}{\lambda^2}\, e^{-(x+y)/\lambda}\, U(x)\, U(y)$$

$$Z = X + Y$$

$$\phi_Z(\omega) = \phi_X(\omega)\, \phi_Y(\omega) \frac{1}{(1 - j\omega\lambda)^2}$$

$$Z \sim \text{ Gamma } (2, \lambda)$$

This gives

$$f_Z(z) = \frac{z}{\lambda^2}\, e^{-z/\lambda} U(z)$$

$$P(Z > 2\lambda) = \int_{2\lambda}^{\infty} \frac{z}{\lambda^2}\, e^{-z/\lambda}\, dz = \int_{2}^{\infty} x e^{-x}\, dx = 3e^{-2} = 0.406$$

Let,

$$W = Y - X$$

Then

$$P(Y - X > \lambda) = P(W > \lambda) = \int_{\lambda}^{\infty} f_W(w) dw$$

Notice that $F_W(w)$ is given by (6-55).
For $w > 0$, this gives

$$\begin{aligned} f_W(w) &= \int_0^{\infty} \frac{1}{\lambda^2}\, e^{-(w+2y)/\lambda}\, dy = \frac{1}{\lambda^2}\, e^{-w/\lambda} \int_0^{\infty} e^{-2y/\lambda}\, dy \\ &= \frac{1}{2\lambda}\, e^{-w/\lambda}, \quad w > 0 \end{aligned}$$

Hence

$$P(Y - X > \lambda) = P(W > \lambda) = \int_{\lambda}^{\infty} \frac{1}{2\lambda}\, e^{-w/\lambda}\, dw = \frac{1}{2e}$$

6.26 (a)

$$\begin{aligned} R &= W - Z \\ &= \max(\mathrm{X},\mathrm{Y}) \;-\; \min(\mathrm{X},\mathrm{Y}) \\ &= \begin{cases} X - Y, & X \geq Y \\ Y - X, & X < Y \end{cases} \end{aligned}$$

$$\begin{aligned} F_R(r) &= P\{R \leq r\} \\ &= P\{R \leq r, X \geq Y\} + P\{R \leq r, X < Y\} \\ &= P\{X - Y \leq r, X \geq Y\} + P\{Y - X \leq r, X < Y\} \\ &= 1 - 2\frac{(1-r)^2}{2} = 1 - (1-r)^2, \quad 0 \leq r \leq 1 \end{aligned}$$

$$f_R(r) = \begin{cases} 2(1-r), & 0 \leq r \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(b)

$$\begin{aligned} S &= W + Z \\ &= \max(\mathrm{X},\mathrm{Y}) \;+\; \min(\mathrm{X},\mathrm{Y}) = X + Y \end{aligned}$$

Case 1: $0 < s < 1$

$$F_S(s) = P\{S \leq s\} = P\{X + Y \leq s\} = \frac{s^2}{2}, \quad 0 < s < 1$$

Case 2: $1 \leq s \leq 2$

$$F_S(s) = P\{S \leq s\} = P\{X + Y \leq s\} = 1 - \frac{(2-s)^2}{2}, \quad 1 \leq s \leq 2$$

$$F_S(s) = \begin{cases} s, & 0 \leq s \leq 1 \\ (2-s), & 1 \leq s \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

6.27 (a) X, Y are independent, identically distributed exponential random variables.

$$Z = \frac{Y}{\max(X,Y)} = \begin{cases} \frac{Y}{X}, & X \geq Y \\ 1, & X < Y \end{cases} \quad \Rightarrow \ 0 < z \leq 1.$$

$0 < z < 1$

$$\begin{aligned} F_Z(z) &= P(Z \leq z) = P\left\{ \tfrac{Y}{X} \leq z, \, X > Y \right\} \\ &= P\{Y \leq Xz, \, X > Y\} = \int_0^\infty \int_0^{xz} f_{XY}(x,y)\, dy\, dx \end{aligned}$$

$$f_Z(z) = \int_0^\infty x\, f_{XY}(x, xz)\, dx = \int_0^\infty \frac{x}{\lambda^2} e^{-(1+z)x/\lambda}\, dx = \frac{1}{(1+z)^2}, \quad 0 < z < 1.$$

Also

$$P(Z = 1) = P(X < Y) = \int_0^\infty \int_0^y \frac{1}{\lambda^2} e^{-(x+y)/\lambda}\, dx\, dy = \frac{1}{2}$$

(b)

$$W = \frac{X}{\min(X, 2Y)} = \begin{cases} \frac{X}{2Y}, & X \geq 2Y \\ 1, & X < 2Y \end{cases} \quad \Rightarrow \ 1 \leq w < \infty$$

$$F_W(w) = P(X \leq 2Yw, \, X > 2Y) = \int_0^\infty \int_{2y}^{2wy} f_{XY}(x,y)\, dx\, dy$$

This gives

$$\begin{aligned} f_W(w) &= \int_0^\infty 2y\, f_{XY}(2wy, y) dy = \int_0^\infty \frac{2y}{\lambda^2} e^{-(1+2w)y/\lambda} dy \\ &= \frac{2}{(1+2w)^2}, \quad w > 1 \end{aligned}$$

Also

$$P(W = 1) = P(X < 2Y) = \int_0^\infty \int_0^{2y} \frac{1}{\lambda^2} e^{-(x+y)/\lambda}\, dx\, dy = \frac{2}{3}$$

Note that the p.d.f. of Z as well as W has an impulse at $z = 1$ and $w = 1$ respectively.

6.28 X, Y are independent identically distributed exponential random variables.

$$Z = \frac{X}{X+Y}$$

$$\begin{aligned} F_Z(z) &= P\left(\frac{X}{X+Y} \le z\right) = P\left(\frac{X}{Y} \le \frac{z}{1-z}\right) \\ &= P\left\{X \le \frac{zY}{1-z}\right\} = \int_0^\infty \int_0^{(zy)/(1-z)} f_{XY}(x,y)\,dx\,dy \end{aligned}$$

$$\begin{aligned} f_Z(z) &= \int_0^\infty \frac{y}{(1-z)^2} f_{XY}(zy/(1-z), y)\,dy \\ &= \frac{1}{(1-z)^2}\int_0^\infty y\,\frac{1}{\lambda^2}\,e^{-(z/(1-z)+1)(y/\lambda)}\,dy \\ &= \frac{1}{(1-z)^2}\int_0^\infty \frac{y}{\lambda^2}\,e^{-[y/(1-z)\lambda]}\,dy \\ &= \int_0^\infty u\,e^{-u}\,du = 1, \quad 0 < z < 1 \\ &\Rightarrow \frac{X}{X+Y} \sim U(0,1) \end{aligned}$$

6.29 Let

$$f_X(x) = \frac{1}{\lambda}\,e^{-x/\lambda}\,U(x), \quad f_Y(y) = \frac{1}{\lambda}\,e^{-y/\lambda}\,U(y).$$

$$Z = \min(X, Y)$$

$$W = \max(X,Y) - \min(X,Y)$$

$$Z = \begin{cases} Y, & X \ge Y \\ X, & X < Y \end{cases}$$

$$W = \begin{cases} X - Y, & X \ge Y \\ Y - X, & X < Y \end{cases}$$

$Z = \min(X, Y)$. See Example 6-18, Eq.(6-82) for solution. From there (replace λ by $1/\lambda$ in (6-82))

$$f_Z(z) = \frac{2}{\lambda}\,e^{-2z/\lambda}\,U(z).$$

$$\begin{aligned} F_W(w) &= P(X - Y \le w, X \ge Y) + P(Y - X \le w, X < Y) \\ &= \int_0^\infty \int_y^{y+w} f_{XY}(x,y)\,dx\,dy \\ &+ \int_0^\infty \int_x^{x+w} f_{XY}(x,y)\,dy\,dx, \quad w > 0 \end{aligned}$$

This gives

$$\begin{aligned} F_W(w) &= \int_0^\infty f_{XY}(y+w, y)\,dy + \int_0^\infty f_{XY}(x, x+w)\,dx \\ &= 2\int_0^\infty \frac{1}{\lambda^2}\,e^{(2y+w)/\lambda}\,dy \\ &= \frac{2}{\lambda^2}\,e^{-w/\lambda}\,\frac{e^{-2y/\lambda}}{-2/\lambda}\bigg|_0^\infty = \frac{1}{\lambda}\,e^{-w/\lambda}, \quad w > 0 \end{aligned}$$

Also

$$\begin{aligned} F_{ZW}(z,w) &= P\{Z \le z, W \le w\} \\ &= P\{Y \le z, X - Y \le w, X \ge Y\} \\ &+ P\{X \le z, Y - X \le w, X < Y\} \\ &= \int_0^z \int_y^{y+w} f_{XY}(x,y)\,dx\,dy + \int_0^z \int_x^{x+w} f_{XY}(x,y)\,dy\,dx \end{aligned}$$

Repeated use of (6-39)-(6-40) gives

$$\begin{aligned} f_{ZW}(z,w) &= f_{XY}(z+w,z) + f_{XY}(z,z+w) \\ &= \frac{2}{\lambda^2}\,e^{-(2z+w)/\lambda} = \frac{2}{\lambda}\,e^{-2z/\lambda}\frac{1}{\lambda}\,e^{-w/\lambda} \\ &= f_Z(z)\,f_W(w) \end{aligned}$$

Thus Z and W are independent exponential random variables.

6.30 (a) Let

$$U = X + Y, \qquad 0 < u < 2\beta.$$

The probability density function of U can be computed as in (6-48)-(6-50). Using Fig. 6-11, for $0 < u \le \beta$, we have

$$F_U(u) = \int_0^u \int_0^{u-x} f_{XY}(x,y)\,dy\,dx$$

which gives

$$\begin{aligned} f_U(u) &= \int_0^u f_{XY}(x, u-x)dx = \alpha^2\beta^{-2\alpha}\int_0^u x^{\alpha-1}\,(u-x)^{\alpha-1}\,dx \\ &= \alpha^2\beta^{-2\alpha}u^{2\alpha-1}\int_0^1 y^{\alpha-1}\,(1-y)^{\alpha-1}\,dy \\ &= B(\alpha,\alpha)\,\alpha^2\beta^{-2\alpha}u^{2\alpha-1} \quad 0 < u \le \beta \end{aligned}$$

where we have substituted $y = ux$ and made use of the beta function defied in (4-49)-(4-51). Similarly for $\beta < u \le 2\beta$, we get (see (6-49))

$$F_U(u) = 1 - \int_{u-\beta}^{\beta}\int_{u-x}^{\beta} f_{XY}(x,y)\,dydx$$

and hence

$$\begin{aligned} f_U(u) &= \int_{u-\beta}^{\beta} f_{XY}(x,u-x)dx = \alpha^2\beta^{-2\alpha}\int_{u-\beta}^{\beta} x^{\alpha-1}(u-x)^{\alpha-1}\,dx \\ &= \alpha^2\beta^{-2\alpha}u^{2\alpha-1}\int_{1-\beta/u}^{\beta/u} y^{\alpha-1}(1-y)^{\alpha-1}\,dy, \quad \beta < u \le 2\beta \end{aligned}$$

(b)

$$Z = \min(X,Y), \qquad W = \max(X,Y)$$

We can proceed as in Example 6-21 to complete this problem. From (6-92) and (6-93), we get

$$F_{ZW}(z,w) = \begin{cases} F_{XY}(z,w) + F_{XY}(w,z) - F_{XY}(z,z), & w \ge z \\ F_{XY}(w,w), & w < z \end{cases}$$

which gives

$$f_{ZW}(z,w) = f_X(z)f_Y(w) + f_X(w)f_Y(z), \quad 0 < z \le w < \beta$$

$$f_{ZW}(z,w) = \begin{cases} 2\alpha^2\beta^{-2\alpha}z^{\alpha-1}w^{\alpha-1}, & 0 < z \le w < \beta \\ 0, & \text{otherwise} \end{cases}$$

check:

$$\begin{aligned} \int_0^\beta \int_0^w f_{ZW}(z,w)dzdw &= 2\alpha^2\beta^{-2\alpha}\int_0^\beta w^{\alpha-1}\left(\frac{z^\alpha}{\alpha}\bigg|_0^w\right)dw \\ &= 2\alpha\beta^{-2\alpha}\int_0^\beta w^{2\alpha-1}dw = 1 \end{aligned}$$

Note: Z and W are not independent random variables, since

$$f_Z(z) = 2\alpha\beta^{-2\alpha}\, z^{\alpha-1}\,(\beta^\alpha - z^\alpha), \quad 0 < z < \beta$$

and

$$f_W(w) = 2\alpha\beta^{-2\alpha}\, w^{2\alpha-1}, \quad 0 < w < \beta$$

(c) Let

$$V = \frac{Z}{W} = \frac{\min(X,Y)}{\max(X,Y)} = \begin{cases} \frac{Y}{X}, & X \ge Y \\ \frac{X}{Y}, & X < Y \end{cases}$$

and

$$W = \max(X,Y) = \begin{cases} X, & X \ge Y \\ Y, & X < Y \end{cases}$$

For $0 < v < 1,\ 0 < w < \beta$

$$\begin{aligned} F_{VW}(v,w) &= P(V \le v,\, W \le w) \\ &= P\{V \le v,\, W \le w,\ (X \ge Y) \bigcup (X < Y)\} \\ &= P\{Y \le Xv,\, X \le w,\, X \ge Y\} \\ &+ P\{X < Yv,\, Y \le w,\, X < Y\} \\ &= \int_0^w \int_0^{xv} f_{XY}(x,y)\,dy\,dx + \int_0^w \int_0^{yv} f_{XY}(x,y)\,dx\,dy \end{aligned}$$

Hence

$$\begin{aligned}
f_{VW}(v,w) &= \frac{\partial^2 F_{VW}(v,w)}{\partial v\,\partial w} \\
&= \frac{\partial}{\partial v}\left\{\int_0^{vw} f_{XY}(w,y)\,dy + \int_0^{vw} f_{XY}(x,w)\,dx\right\} \\
&= w\left\{f_{XY}(w,vw) + f_{XY}(vw,w)\right\} \\
&= 2\alpha^2\,\beta^{-2\alpha}\,w^{2\alpha-1}\,v^{\alpha-1}, \qquad 0<v<1,\ 0<w<\beta
\end{aligned}$$

Hence

$$f_V(v) = \int_0^{\beta} f_{VW}(v,w)dw = \alpha\, v^{\alpha-1}, \quad 0<v<1$$

$$f_W(w) = \int_0^{1} f_{VW}(v,w)dv = 2\alpha\,\beta^{-2\alpha}\,w^{2\alpha-1}, \quad 0<w<\beta$$

and

$$f_{VW}(v,w) = f_V(v)\,f_W(w).$$

Thus V and W are independent random variables.

6.31 (a) Solved in Examples 6-27 and 6-12.
(b) Solved in Example 6-27.
(c)

$$Z = X+Y, \qquad W = \frac{X}{X+Y}$$

$$x_1 = zw, \qquad y_1 = z - x_1 = z(1-w)$$

$$J = \begin{vmatrix} 1 & 1 \\ \dfrac{y}{(x+y)^2} & -\dfrac{x}{(x+y)^2} \end{vmatrix} = \frac{1}{x+y} = \frac{1}{z}$$

$$\begin{aligned}
f_{ZW}(z,w) &= \frac{z}{\alpha^{m+n}\Gamma(m)\Gamma(n)}\,(zw)^{m-1}\left\{z(1-w)\right\}^{n-1} \\
&= \left(\frac{z^{m+n-1}}{\alpha^{m+n}\Gamma(\alpha+\beta)}\;e^{-z/\alpha}\right)\left(\frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)}\;w^{m-1}\,(1-w)^{n-1}\right) \\
&= f_Z(z)\,f_W(w)
\end{aligned}$$

Thus Z and W are independent random variables.

6.32 (a)

$$Z = \frac{X}{|Y|}, \qquad W = \frac{|X|}{|Y|} = |Z|$$

$$\begin{aligned} F_Z(z) &= P(Z \le z) = P(X \le |Y|z) = \int_{-\infty}^{\infty} \int_0^{|y|z} f_{XY}(x, y)\, dx\, dy \\ &= 2\int_0^{\infty} |y| f_{XY}(|y|z, y) dy = \frac{2}{2\pi\sigma^2} \int_0^{\infty} y e^{-(z^2+1)y^2/2\sigma^2}\, dy \\ &= \frac{1/\pi}{1+z^2}, \qquad -\infty < z < \infty \end{aligned}$$

Thus Z is a Cauchy random variable. Interestingly, the random variable X/Y is also a Cauchy random variable (see Example 6-11).

$$W = |Z|$$

so that

$$\begin{aligned} F_W(w) &= P(W \le w) = P(|Z| \le w) \\ &= P(-w < Z < w) = F_Z(w) - F_Z(-w) \end{aligned}$$

and hence

$$f_W(w) = f_Z(w) + f_Z(-w) = \frac{2/\pi}{1+w^2}, \quad w > 0.$$

(b)

$$U = X + Y \sim N(0, 2)$$
$$V = X^2 + Y^2 \sim \text{Exponential } (2)$$

(see Example 6-14). Here U, V are *not* independent, since

$$J(x, y) = \begin{vmatrix} 1 & 1 \\ 2x & 2y \end{vmatrix} = -2(x - y) = 2\sqrt{2v - u^2}$$

and

$$\begin{aligned} f_{UV}(u, v) &= \frac{1}{2\sqrt{2v-u^2}} \frac{1}{2\pi\sigma^2} e^{-v/2\sigma^2} \\ &\neq f_U(u)\, f_V(v), \qquad -\infty < u < \infty, \;\; v > 0. \end{aligned}$$

6.33

$$Z = X + Y, \qquad W = X - Y$$

are jointly normal random variables. Hence if they are uncorrelated, then they are also independent.

$$\begin{aligned} Cov(Z, W) &= E[(Z - \mu_Z)(W - \mu_W)] \\ &= E\left[\{(X - \mu_X) + (Y - \mu_Y)\}\{(X - \mu_X) - (Y - \mu_Y)\}\right] \\ &= \text{Var}(X) - \text{Var}(Y) = \sigma_X^2 - \sigma_Y^2. \end{aligned}$$

The random variables Z and W are uncorrelated implies that $Cov(Z, W) = 0$. Hence $\sigma_X^2 = \sigma_Y^2$ is the necessary and sufficient condition for the independence of $X + Y$ and $X - Y$.

6.34 (a)-(b) Let

$$R = \sqrt{X^2+Y^2}, \qquad \theta = \tan^{-1}\left(\frac{Y}{X}\right)$$

Form Example 6-22, R and θ are independent random variables with joint p.d.f. as in (6-128). (see (6-131)). In term of R and θ, we have $X = R\cos\theta, Y = R\sin\theta$ and hence we obtain

$$U = \frac{X^2-Y^2}{\sqrt{X^2+Y^2}} = R\cos 2\theta$$

$$V = \frac{2XY}{\sqrt{X^2+Y^2}} = R\sin 2\theta$$

This gives

$$J = \begin{vmatrix} \cos 2\theta & -2r\sin 2\theta \\ \sin 2\theta & 2r\cos 2\theta \end{vmatrix} = 2r = 2\sqrt{u^2+v^2}$$

$$r = \sqrt{u^2+v^2}, \quad \theta_1 = \frac{1}{2}\tan^{-1}\left(\frac{v}{u}\right), \quad 2\theta_2 = \pi + 2\theta_1.$$

There are two sets of solutions (r,θ_1) and (r,θ_2). Substituting into (6-128) we get

$$\begin{aligned} f_{UV}(u,v) &= \frac{1}{|J|}\{f_{r,\theta}(r,\theta_1) + f_{r,\theta}(r,\theta_2)\} = \frac{2}{|J|} f_{r,\theta}(r,\theta_1) \\ &= \frac{2}{2\sqrt{u^2+v^2}} \frac{\sqrt{u^2+v^2}}{2\pi\sigma^2} e^{-(u^2+v^2)/2\sigma^2} \\ &= \frac{1}{2\pi\sigma^2} e^{-(u^2+v^2)/2\sigma^2} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-u^2/2\sigma^2} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-v^2/2\sigma^2} \\ &= f_U(u) f_V(v) \end{aligned}$$

Thus U and V are independent normal random variables. Hence it follows that $U = \dfrac{X^2-Y^2}{\sqrt{X^2+Y^2}}$ and $V/2 = \dfrac{XY}{\sqrt{X^2+Y^2}}$ are independent random variables.

(c)

$$\begin{aligned} Z &= \frac{(X-Y)^2 - 2Y^2}{\sqrt{X^2+Y^2}} = \frac{(X^2-Y^2) - 2XY}{\sqrt{X^2+Y^2}} \\ &= \frac{X^2-Y^2}{\sqrt{X^2+Y^2}} - \frac{2XY}{\sqrt{X^2+Y^2}} \\ &= U - V \sim N(0, 2\sigma^2). \end{aligned}$$

6.35 (a) $Z \sim F(m,n)$ is given by (6-157) Let

$$Y = \frac{1}{Z}$$

Then

$$\begin{aligned} F_Y(y) &= \frac{1}{|dy/dz|}\, f_Z(1/y) \\ &= \frac{1}{y^2}\,\frac{(m/n)^{m/2}}{\beta(m/2,n/2)}\,\frac{1}{y^{m/2-1}}\,\frac{1}{(1+m/ny)^{m+n/2}} \\ &= \frac{(n/m)^{n/2}}{\beta(n/2,m/2)}\, y^{n/2-1}\left(1+\frac{n}{my}\right)^{-(m+n)/2} \\ &\sim F(n,m). \end{aligned}$$

(b)

$$W = \frac{Zm}{Zm+n}$$

$$\begin{aligned} F_W(w) &= P(W \le w) = P\left(\frac{Zm}{Zm+n} \le w\right) \\ &= P\left(Z \le \frac{nw}{m(1-w)}\right) = F_Z\left(\frac{nw}{m(1-w)}\right) \end{aligned}$$

which gives

$$\begin{aligned} f_W(w) &= \frac{n}{m(1-w)^2}\, f_Z\left(\frac{nw}{m(1-w)}\right) \\ &= \frac{n}{m(1-w)^2}\,\frac{(m/n)^{m/2}}{\beta(m/2,n/2)}\left(\frac{nw}{m(1-w)}\right)^{m/2-1}\left(1+\frac{w}{(1-w)}\right)^{-(m+n)/2} \\ &= \frac{1}{\beta(m/2,n/2)}\, w^{m/2-1}\,(1-w)^{n/2-1}, \quad 0 < w < 1. \end{aligned}$$

Thus W has Beta distribution.

6.36

$$Z = X + Y > 0, \qquad W = X - Y > 0$$

$$x_1 = \frac{z+w}{2}, \qquad y_1 = \frac{z-w}{2}$$

is the only solution. Moreover

$$J = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

so that

$$f_{ZW}(z,w) = \frac{1}{|J|} f_{XY}(x_1,y_1) = \frac{1}{2}\, e^{-(z+w)/2}, \quad 0 < w < z < \infty$$

$$\begin{aligned} F_Z(z) &= \int_0^z f_{ZW}(z,w)\,dw = \frac{1}{2}\, e^{-z/2}\,\left.\frac{e^{-w/2}}{(-1/2)}\right|_0^z \\ &= \frac{1}{2}\, e^{-z/2}\,\left.\frac{e^{-w/2}}{(-1/2)}\right|_0^z = e^{-z/2}(1-e^{-z/2}), \quad z > 0 \end{aligned}$$

6.37

$$Z = X + Y > 0, \qquad W = \frac{Y}{X} > 1$$

$$y = xw, \quad x(1+w) = z, \quad x_1 = \frac{z}{1+w}, \quad y_1 = \frac{zw}{1+w}$$

is the only solution. Also

$$J = \begin{vmatrix} 1 & 1 \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{x+y}{x^2} = \frac{(1+w)^2}{z}$$

This gives

$$\begin{aligned} f_{ZW}(z,w) &= \frac{1}{|J|} f_{XY}(x_1, y_1) \\ &= \frac{z}{(1+w)^2} \, 2\, e^{-z}, \quad z > 0, \; w > 1 \\ &= z\, e^{-z} \, \frac{2}{(1+w)^2} = f_Z(z)\, f_W(w) \end{aligned}$$

since

$$\begin{aligned} f_Z(z) &= \int_1^\infty f_{ZW}(z,w)\, dw \\ &= 2ze^{-z} \int_1^\infty \frac{1}{(1+w)^2}\, dw = z\, e^{-z}, \quad z > 0 \end{aligned}$$

and

$$\begin{aligned} f_w(w) &= \int_0^\infty f_{ZW}(z,w)\, dz \\ &= \frac{2}{(1+w)^2} \int_0^\infty ze^{-z}\, dz = \frac{2}{(1+w)^2}, \quad w > 1. \end{aligned}$$

Thus Z and W are independent random variables.

6-38 $\underset{\sim}{z} = \underset{\sim}{x}\,\underset{\sim}{y}$ $\qquad \underset{\sim}{y} = \cos(\omega t + \underset{\sim}{\theta})$

$\underset{\sim}{w} = \underset{\sim}{y}$ $\qquad J = |y|$ $\qquad f_y(y) = \begin{cases} \dfrac{1}{\pi\sqrt{1-y^2}} & |y| < 1 \\ 0 & |y| > 1 \end{cases}$

The RVs $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are independent. Hence,

$$f_{zw}(z,w) = \frac{1}{|w|} f_x\left(\frac{z}{w}\right) f_y(w)$$

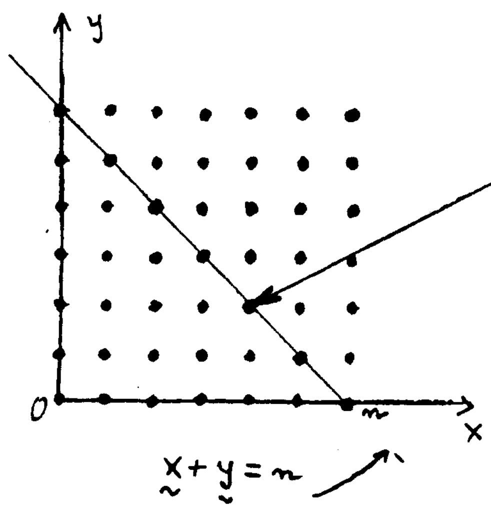
$$f_z(z) = \frac{1}{\pi}\int_{-1}^{1} \frac{f_x(z/w)}{|w|\sqrt{1-w^2}}\,dw = \frac{1}{\pi}\int_{|x|>z} \frac{f_x(x)}{\sqrt{x^2-z^2}}\,dx$$

6-39 $\underset{\sim}{z} = \underset{\sim}{x} + \underset{\sim}{s}$ $\qquad \underset{\sim}{s} = a\cos\underset{\sim}{y}$

$f_z(z) = f_x(z) * f_s(z)$ $\qquad f_s(s) = \begin{cases} \dfrac{1}{\pi\sqrt{a^2-s^2}} & |s| < a \\ 0 & |s| > a \end{cases}$

$$f_z(z) = \frac{1}{\pi\sigma\sqrt{2\pi}}\int_{-a}^{a} \frac{e^{-(z-s)^2/2\sigma^2}}{\sqrt{a^2-s^2}}\,ds = \frac{1}{\pi\sigma\sqrt{2\pi}}\int_{-\pi}^{\pi} e^{-(z-a\cos y)^2/2\sigma^2}\,dy$$

6-40



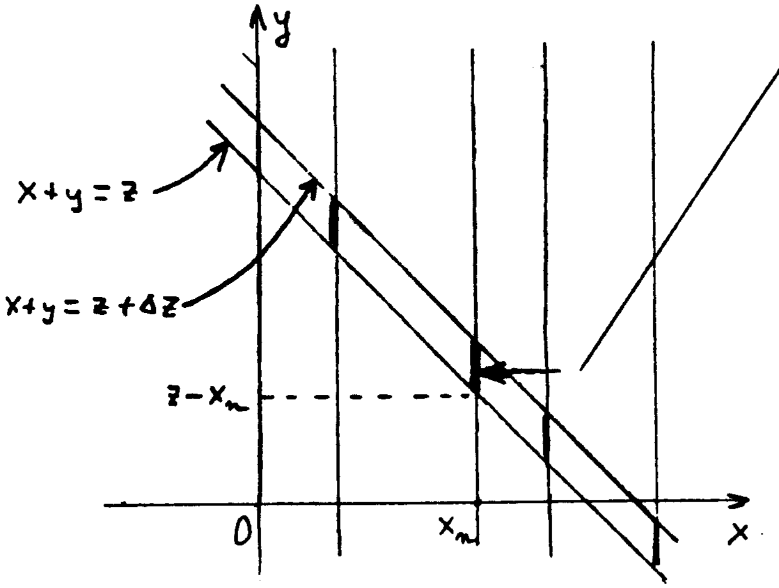
Point masses

$P\{\underset{\sim}{x} = k,\ \underset{\sim}{y} = n-k\} = a_k b_{n-k}$

$$\{\underset{\sim}{z} = n\} = \sum_{k=0}^{n} \{\underset{\sim}{x} = k,\ \underset{\sim}{y} = n-k\}$$

$$P\{\underset{\sim}{z} = n\} = \sum_{k=0}^{n} P\{\underset{\sim}{x} = k,\ \underset{\sim}{y} = n-k\}$$

6-41 (a) **Line masses**



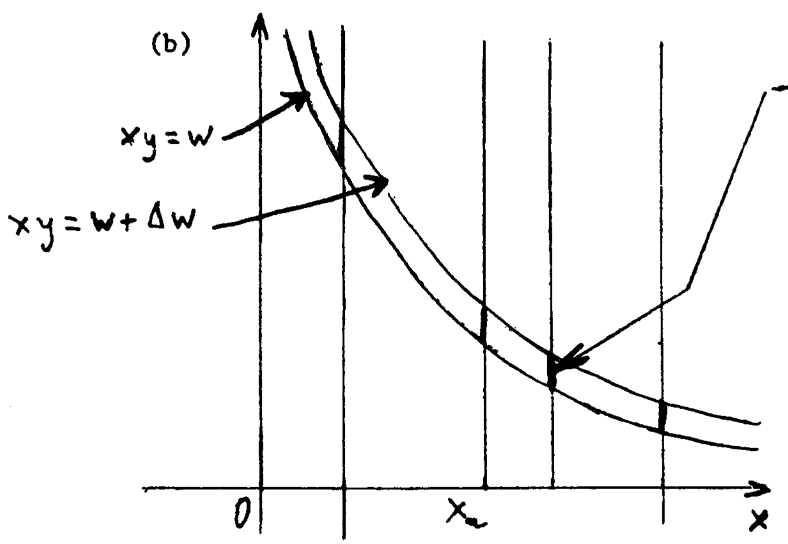
$P\{\underset{\sim}{x} = x_n,\ z - x_n < \underset{\sim}{y} \le z - x_n + \Delta z\}$

$= p_n f_y(z - x_n)\Delta z$

$$\{z < \underset{\sim}{z} \le z + \Delta z\} = \sum_n \{\underset{\sim}{x} = x_n,\ z - x_n < \underset{\sim}{y} \le z - x_n + \Delta z\}$$

$$f_z(z)\Delta z = \sum_n p_n f_y(z - x_n)\Delta z$$

(b)



$P\{\underset{\sim}{x} = x_n,\ \frac{w}{x_n} < \underset{\sim}{y} \le \frac{w+\Delta w}{x_n}\}$

$= p_n f_y(\frac{w}{x_n}) \frac{\Delta w}{x_n}$

$$\{w < \underset{\sim}{w} \le w + \Delta w\} = \sum_n \{\underset{\sim}{x} = x_n,\ \frac{w}{x_n} < \underset{\sim}{y} \le \frac{w+\Delta w}{x_n}\}$$

$$f_w(w)\Delta w = \sum_n p_n f_y(\frac{w}{x_n})\Delta w$$

6.42 X, Y are independent geometric random variables. Thus

$$\begin{aligned} P\{X=k, Y=m\} &= P\{X=k\}\,P\{Y=m\} \\ &= (pq^k)\,(pq^m) = p^2 q^{k+m}, \qquad k, m = 0, 1, 2, \cdots \end{aligned}$$

(a) Let

$$Z = X + Y$$

$$\begin{aligned} P\{Z=n\} &= P\{X+Y=n\} = \textstyle\sum_k P\{X=k, Y=n-k\} \\ &= \textstyle\sum_{k=0}^n P\{X=k, Y=n-k\} \\ &= \textstyle\sum_{k=0}^n P\{X=k\}\,P\{Y=n-k\} \\ &= \textstyle\sum_{k=0}^n pq^k\,pq^{n-k} = \sum_{k=0}^n p^2\,q^n \\ &= (n+1)\,p^2\,q^n, \quad n = 0, 1, 2, \cdots \end{aligned}$$

(b) Let

$$W = X - Y$$

Case 1: $W \geq 0 \Rightarrow X \geq Y$. Thus for $m \geq 0$

$$\begin{aligned} P\{W=m\} &= P\{X-Y=m\} = \textstyle\sum_{k=0}^\infty P\{X=m+k, Y=k\} \\ &= \textstyle\sum_{k=0}^\infty P\{X=m+k, Y=k\} \\ &= \textstyle\sum_{k=0}^\infty P\{X=m+k\}\,P\{Y=k\} \\ &= \textstyle\sum_{k=0}^\infty (pq^{m+k})\,(pq^k) = p^2 q^m \sum_{k=0}^\infty q^{2k} \\ &= p^2 q^m\,(1+q^2+q^4+\cdots) = \frac{p^2 q^m}{(1-q^2)} \\ &= \frac{pq^m}{1+q}, \qquad m = 0, 1, 2, \cdots \qquad (1) \end{aligned}$$

Case 2: $W < 0 \Rightarrow X < Y$. Thus for $m < 0$

$$\begin{aligned} P\{W=m\} &= P\{X-Y=m\} = \textstyle\sum_k P\{X=k, Y=k-m\} \\ &= \textstyle\sum_{k=0}^\infty P\{X=k, Y=k-m\} \\ &= \textstyle\sum_{k=0}^\infty P\{X=k\}\,P\{Y=k-m\} \\ &= \textstyle\sum_{k=0}^\infty (pq^k)\,(pq^{k-m}) = p^2 q^{-m} \sum_{k=0}^\infty q^{2k} \\ &= \frac{p^2 q^{-m}}{(1-q^2)} = \frac{pq^{-m}}{1+q}, \quad m = -1, -2, \cdots \qquad (2) \end{aligned}$$

Thus combining (1) and (2) we can write

$$P\{W=m\} = \frac{pq^{|m|}}{1+q}, \qquad m = 0, \pm 1, \pm 2, \cdots$$

6.43 We have X and Y are independent and $P(X=k)=P(Y=k)=p_k$. Also

$$P(X=k|X+Y=k) = \frac{P(X=k,Y=0)}{P(X+Y=k)} = \frac{p_k p_0}{\sum_{i=0}^{k} p_i p_{k-i}} = \frac{1}{k+1}. \qquad (1)$$

Also

$$P(X=k-1|X+Y=k) = \frac{P(X=k-1,Y=1)}{P(X+Y=k)} \frac{p_{k-1}p_1}{\sum_{i=0}^{k} p_i p_{k-i}} = \frac{1}{k+1}. \qquad (2)$$

From (1) and (2),

$$\frac{p_k}{p_{k-1}} = \frac{p_1}{p_0} \Rightarrow p_k = \lambda p_{k-1} = \lambda^k p_0$$

where $\lambda \triangleq p_1/p_0$. Since $\sum_{k=0}^{\infty} p_k = 1$, we must have $\lambda < 1$, and this gives

$$\sum_{k=0}^{\infty} p_k = \frac{p_0}{1-\lambda} = 1 \to p_0 = 1-\lambda.$$

Thus

$$p_k = p_0\,\lambda^k = (1-\lambda)\lambda^k, \quad k=0,1,2,\cdots, \; 0<\lambda<1$$

represents a geometric distribution. Thus X and Y are geometric random variables.

6.44 The moment generating functions of X and Y are given by (see (5-117))

$$\Gamma_X(z) = (pz+q)^n, \qquad \Gamma_Y(z) = (pz+q)^n$$

Also

$$\Gamma_{X+Y}(z) = E[z^{X+Y}] = \Gamma_X(z)\Gamma_Y(z) = (pz+q)^{2n} \sim \text{Binomial}\,(2n,p)$$

6.45 (a) Let

$$Z = \min(X, Y), \qquad W = X - Y$$

$$\begin{aligned}
&P\{Z = k, W = m\} \\
&= P\{\min(X, Y) = k, X - Y = m\} \\
&= P\{(\min(X, Y) = k, X - Y = m) \bigcap (X \geq Y \bigcup X < Y)\} \\
&= P\{Y = k, X - Y = m, X \geq Y\} + P\{X = k, X - Y = m, X < Y\} \\
&= P\{X = m + k, Y = k, X \geq Y\} + P\{X = k, Y = k - m, X < Y\}
\end{aligned}$$

Note that $k \geq 0$, and m takes both positive, zero and negative values. Hence

$$\begin{aligned}
P\{Z = k, W = m\} &= \begin{cases} P\{X = k + m, Y = k, X \geq Y\}, & k \geq 0,\ m \geq 0 \\ P\{X = k, Y = k - m, X < Y\}, & k \geq 0,\ m < 0 \end{cases} \\
&= \begin{cases} pq^{k+m}\, pq^k, & k \geq 0,\ m \geq 0 \\ pq^k\, pq^{k-m}, & k \geq 0,\ m < 0 \end{cases}
\end{aligned}$$

$$P\{Z = k, W = m\} = p^2 q^{2k+|m|}, \qquad k = 0, 1, 2, \cdots, \quad m = 0, \pm 1, \pm 2, \cdots$$

Also

$$\begin{aligned}
P\{Z = k\} &= \textstyle\sum_{m=-\infty}^{\infty} P\{Z = k, W = m\} \\
&= p^2 q^{2k} \textstyle\sum_{m=-\infty}^{\infty} q^{|m|} = p^2 q^{2k} \left(1 + 2\sum_{m=1}^{\infty} q^m\right) \\
&= p^2 q^{2k} \left(1 + \frac{2q}{p}\right) = p(1 + q) q^{2k}, \qquad k = 0, 1, 2, \cdots
\end{aligned}$$

and

$$\begin{aligned}
P\{W = m\} &= \textstyle\sum_{k=0}^{\infty} P\{Z = k, W = m\} \\
&= p^2 q^{|m|} \textstyle\sum_{k=0}^{\infty} q^{2k} \\
&= \frac{p}{1 + q} q^{|m|}, \qquad m = 0, \pm 1, \pm 2, \cdots
\end{aligned}$$

Note that

$$P\{Z = k, W = m\} = P\{Z = k\}\, P\{W = m\}$$

and hence Z and W are independent random variables.
(b) Let

$$Z = \min(X, Y), \quad W = \max(X, Y) - \min(X, Y)$$

Proceeding as in (a), we obtain

$$\begin{aligned}
&P\{Z = k, W = m\} \\
&= P(Y = k, X - Y = m, X \geq Y) + P(X = k, Y - X = m, X < Y) \\
&= P(X = k + m, Y = k, X \geq Y) + P(X = k, Y = k + m, X < Y) \\
&= \begin{cases} pq^{k+m}\, pq^k + pq^k\, pq^{k+m}, & k = 0, 1, 2, \cdots,\ m = 1, 2, \cdots \\ pq^{k+m}\, pq^k, & k = 0, 1, 2, \cdots,\ m = 0 \end{cases} \\
&= \begin{cases} 2p^2 q^{2k+m}, & k = 0, 1, 2, \cdots,\ m = 1, 2, \cdots \\ p^2 q^{2k}, & k = 0, 1, 2, \cdots,\ m = 0 \end{cases}
\end{aligned}$$

This gives

$$\begin{aligned}
P\{Z = k\} &= \sum_{m=0}^{\infty} P\{Z = k, W = m\} \\
&= p^2 q^{2k} \left(1 + 2\sum_{m=1}^{\infty} q^m\right) = p^2 q^{2k} \left(1 + \frac{2q}{p}\right) \\
&= p(1 + q) q^{2k}, \quad k = 0, 1, 2, \cdots
\end{aligned}$$

Also

$$\begin{aligned}
P\{W = m\} &= \sum_{k=0}^{\infty} P\{Z = k, W = m\} \\
&= \begin{cases} \dfrac{p}{1+q}, & m = 0 \\ \dfrac{2p}{1+q}\, q^m, & m = 1, 2, \cdots \end{cases}
\end{aligned}$$

Notice that

$$P\{Z = k, W = m\} = P\{Z = k\}\, P\{W = m\}$$

and hence Z and W are also indepedndent random variables in this case also.

6.46 The moment generating function of X and Y are given by (see (5-119))

$$\Gamma_X(z) = e^{\lambda_1(z-1)}, \qquad \Gamma_Y(z) = e^{\lambda_2(z-1)}$$

Also

$$\Gamma_{X+Y}(z) = \Gamma_X(z)\Gamma_Y(z) = e^{(\lambda_1+\lambda_2)(z-1)}$$

so that

$$Z \sim P(\lambda_1 + \lambda_2)$$

Thus

$$P(X+Y=k) = e^{-(\lambda_1+\lambda_2)}\,\frac{(\lambda_1+\lambda_2)^k}{k!}$$

and

$$\begin{aligned}
&P(X=k|X+Y=n)\\
&= \frac{P(X=k, X+Y=n)}{P(X+Y=n)} = \frac{P(X=k)P(Y=n-k)}{P(X+Y=n)}\\
&= \frac{e^{-\lambda_1}(\lambda_1^k/k!)\,e^{-\lambda_2}(\lambda_2^{n-k}/(n-k)!)}{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n/n!}\\
&= \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}, \quad k = 0, 1, 2, \cdots n\\
&\sim \text{Binomial}(n,p), \text{ where } p = \frac{\lambda_1}{\lambda_1+\lambda_2}.
\end{aligned}$$

See also (6-222). From there the converse is also true (proceed as in Example 6-43).

6-47

$$C = \begin{bmatrix} \sigma_1^2 & r\sigma_1\sigma_2 \\ r\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \qquad \Delta = \sigma_1^2\sigma_2^2(1-r^2)$$

$$C^{-1} = \begin{bmatrix} \dfrac{1}{(1-r^2)\sigma_1^2} & \dfrac{r}{(1-r^2)\sigma_1\sigma_2} \\ \dfrac{r}{(1-r^2)\sigma_1\sigma_2} & \dfrac{1}{(1-r^2)\sigma^2} \end{bmatrix}$$

$$XC^{-1}X^t = \frac{1}{(1-r^2)}\left(\frac{x_1^2}{\sigma_1^2} - 2r\,\frac{x_1x_2}{\sigma_1\sigma_2} + \frac{x_2^2}{\sigma_2^2}\right)$$

6-48

$$\{\underset{\sim}{x}\,\underset{\sim}{y} < 0\} = \{\underset{\sim}{x} < 0,\ \underset{\sim}{y} > 0\} + \{\underset{\sim}{x} > 0,\ \underset{\sim}{y} < 0\}$$

$$P\{\underset{\sim}{x}\,\underset{\sim}{y} < 0\} = F_x(0)[1 - F_y(0)] + [1 - F_x(0)]F_y(0)$$

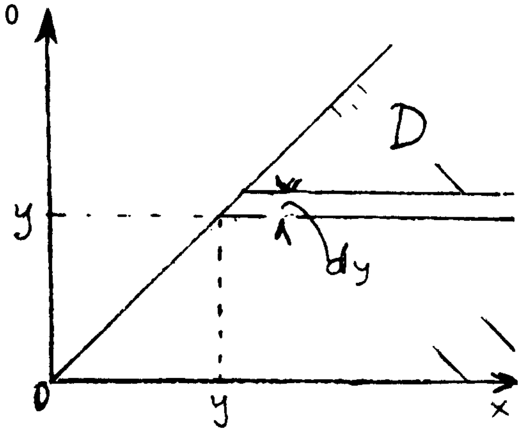
$$F_x(0) = 1 - G\left(\frac{\eta_x}{\sigma_x}\right) \qquad F_y(0) = 1 - G\left(\frac{\eta_y}{\sigma_y}\right)$$

6-49 If $w = x - y$, then $E\{w\} = 0$ $\quad \sigma_w^2 = \sigma_x^2 + \sigma_y^2 = 2\sigma^2$

Thus, $w = 1$, $N(0;\ \sigma\sqrt{2})$ and [see (5-74)]

$$E\{z\} = E\{|w|\} = \sqrt{2}\,\sigma\sqrt{\frac{2}{\pi}} \qquad E\{z^2\} = E\{w^2\} = 2\sigma^2$$

6-50



$$E\{z\} = \iint_D (x-y) f(x,y)\,dxdy$$

$$= \int_0^\infty \int_y^\infty (x-y) e^{-x} e^{-y}\,dxdy = \frac{1}{2}$$

6-51 Since $|E\{x\,y\}| \le E\{|x||y|\}$, we can assume that the RVs x and y are real

(a) $D \le E\{[z\,x - y]^2\} = z^2\,E\{x^2\} - 2z\,E\{x\,y\} + E\{y^2\}$

The above is a non-negative quadratic in z for any z. Hence, its discriminant is non-positive.

(b) Using (a), we obtain

$$E\{x^2\} + E\{y^2\} + 2\sqrt{E\{x^2\}E\{y^2\}}$$

$$\ge E\{x^2\} + E\{y^2\} + 2\,E\{x\,y\} = E\{(x + y)^2\}$$

6-52 If $r_{xy} = 1$ then

$$E^2\{(x - \eta_x)(y - \eta_y)\} = E\{(x - \eta_x)^2\}E\{(y - \eta_y)^2\}$$

i.e., the discriminant of the quadratic

$$E\{[z(x - \eta_x) - (y - \eta_y)]^2\}$$

is zero. This is possible only if the quadratic is zero for some $z = z_o$. This shows that $z(x - \eta_x) - (y - \eta_y) = 0$ in the MS sense.

6-53 If $E\{x\} = E\{y^2\} = E\{x\ y\}$, then

$$E\{(x - y)^2\} = E\{x^2\} + E\{y^2\} - 2\ E\{x\ y\} = 0.$$

Hence, $x = y$ in the MS sense.

6-54 If x has a Cauchy density, then (Prob. 5-31)

$$E\{e^{j\omega x}\} = e^{-\alpha|\omega|} \qquad E\{e^{j\omega kx}\} = e^{-\alpha k|\omega|}$$

Hence, [see (6-240)]

$$\Phi_z(\omega) = E\{e^{j\omega nx}\} = E\{E\{e^{j\omega nx}|n\}\} =$$

$$\sum_{k=0}^{\infty} E\{e^{j\omega kx}\}e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty} e^{-\alpha k|\omega|}\frac{k}{k!} = e^{-\lambda}e^{-\lambda e^{-\alpha|\omega|}}$$

6.55 If $X = k$, then

$$Y = n - k$$

and

$$Z = X - Y = 2X - n,$$

where Z takes the values $-n, -(n-2), \cdots n-2, n$.

$$\begin{aligned} P\{Z = z\} &= P\{2X - n = z\}P\{X = \tfrac{n+z}{2}\} \\ &= \binom{n}{n + z/2} p^{(n+z)/2}\ q^{(n-z)/2}. \end{aligned}$$

Also

$$E(Z) = E[2X - n] = 2np - n = n(2p - 1).$$

$$\text{Var}(Z) = E[(z - \mu_z)^2] = 4E[(X - np)^2] = 4\text{Var}(X) = 4npq$$

6.56 (a)

$$\begin{aligned}\phi_Z(\omega) &= E[e^{j\omega Z}] = E[e^{j\omega(aX+bY+c)}] \\ &= \phi_X(a\omega)\,\phi_Y(b\omega)e^{j\omega c} = e^{j\omega c-(a^2\sigma_1^2+b^2\sigma_2^2)\omega^2/2}\end{aligned}$$

(see (5-100)).
(b) On comparing with (5-100) we obtain

$$Z \sim \mathrm{N}(c, a^2\,\sigma_1^2 + b^2\,\sigma_2^2)$$

(c)
$$E[Z] = c, \qquad \mathrm{Var}(Z) = a^2\,\sigma_1^2 + b^2\,\sigma_2^2$$

6.57
$$P(X=k|Y=n) = \binom{n}{k} p_1^k\, q_1^{n-k}, \quad k = 0, 1, 2, \cdots n$$

$$E[e^{j\omega X}|Y=n] = \sum_{k=0}^{n} e^{j\omega k}\, P(X=k|Y=n) = \left(p_1 e^{j\omega} + q_1\right)^n$$

use (5-117). Also

$$\begin{aligned}\phi_X(\omega) &= E[e^{j\omega X}] = E\left\{E[e^{j\omega X}|Y=n]\right\} \\ &= \sum\nolimits_{n=0}^{M} E[e^{j\omega X}|Y=n]\,P(Y=n) \\ &= \sum\nolimits_{n=0}^{\infty} (p_1e^{j\omega}+q_1)^n \binom{M}{n} p_2^n\, q_2^{M-n} \\ &= \sum\nolimits_{n=0}^{M} \binom{M}{n} [p_2(p_1\, e^{j\omega}+q_1)]^n\, q_2^{M-n} \\ &= (p_2p_1e^{j\omega}+q_1p_2+q_2)^M\end{aligned}$$

But
$$1 - p_1p_2 = 1 - (1-q_1)(1-q_2) = q_1p_2 + q_2$$

Hence
$$\phi_X(\omega) = \left(pe^{j\omega} + q\right)^M$$

where $p = p_1p_2$. Thus

$$X \sim \mathrm{Binomial}(M, p_1p_2).$$

6.58

$$\int\int f_{XY}(x,y)\,dx\,dy = \int_0^1 \int_x^1 kx\,dy\,dx = k\int_0^1 x(1-x)\,dx$$

$$\frac{k}{6} = 1 \quad \Rightarrow \quad k = 6.$$

$$f_X(x) = \int_x^1 6x\,dy = 6x(1-x), \quad 0 < x < 1.$$

$$f_Y(y) = \int_0^y 6x\,dy = 3y^2, \quad 0 < y < 1.$$

$$E[X] = \int_0^1 x\,f_X(x)\,dx = 6\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Bigg|_0^1 = \frac{1}{2}.$$

$$E[X^2] = \int_0^1 x^2\,f_X(x)\,dx = 6\left(\frac{x^4}{4} - \frac{x^5}{5}\right)\Bigg|_0^1 = \frac{3}{10}.$$

$$\mathrm{Var}(X) = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}.$$

$$E[Y] = \int_0^1 y\,f_Y(y)\,dy = 3\left.\frac{y^4}{4}\right|_0^1 = \frac{3}{4}.$$

$$E[Y^2] = \int_0^1 y^2\,f_Y(y)\,dy = 3\left.\frac{y^5}{5}\right|_0^1 = \frac{3}{5}.$$

$$\mathrm{Var}(Y) = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}.$$

$$\begin{aligned} E[XY] &= \int\int xy\,f_{XY}(x,y)\,dy\,dx \\ &= \int_0^1\int_x^1 xy\,6x\,dy\,dx = \int_0^1 3x^2\,(1-x^2)\,dx \\ &= 3\left(\tfrac{x^3}{3} - \tfrac{x^5}{5}\right)\Big|_0^1 = 3\left(\tfrac{1}{3} - \tfrac{1}{5}\right) = \tfrac{2}{5} \end{aligned}$$

$$\begin{aligned} Cov(X,Y) &= E(XY) - E(X)E(Y) \\ &= \tfrac{2}{5} - \tfrac{1}{2}\tfrac{3}{4} = \tfrac{1}{40} \end{aligned}$$

6.59 (a)

$$\begin{aligned} \phi_{X,Y}(\omega_1,\omega_2) &= E[e^{j(\omega_1 X + \omega_2 Y)}] \\ &= E[e^{j\omega_1 X}]\;E[e^{j\omega_2 Y}] = \phi_X(\omega_1)\,\phi_Y(\omega_2) \\ &= e^{\lambda(e^{j\omega_1}-1)}\;e^{(j\mu\omega_2 - \sigma^2\omega_2^2/2)} \end{aligned}$$

(b)

$$\begin{aligned} \phi_Z(\omega) &= E[e^{j\omega Z}] \\ &= E[e^{j\omega(X+Y)}] = \phi_{X,Y}(\omega,\omega) \\ &= e^{\{\lambda(e^{j\omega}-1) + (j\mu\omega - \sigma^2\omega^2/2)\}} \end{aligned}$$

6.60 (a)

$$Z = \min(X, Y)$$

From Example 6-18, we have

$$f_Z(z) = 2\lambda\, e^{-2\lambda z}, \qquad z \geq 0$$

and hence

$$E[Z] = E[\min(X, Y)] = \frac{1}{2\lambda}$$

(b)

$$\begin{aligned}
E[\max(2X, Y)] &= \int\int \max(2x, y)\, f_{XY}(x, y)\, dx\, dy \\
&= \int\int_{2x \geq y} 2x\, f_{XY}(x, y)\, dx\, dy + \int\int_{2x<y} y\, f_{XY}(x, y)\, dx\, dy \\
&= \int_0^\infty \int_0^{2x} 2x\, \lambda^2\, e^{-\lambda x}\, e^{-\lambda y}\; dy\, dx + \int_0^\infty \int_0^{y/2} y\, \lambda^2\, e^{-\lambda x}\, e^{-\lambda y}\; dx\, dy \\
&= \lambda \int_0^\infty 2x\, e^{-\lambda x}(1 - e^{-2\lambda x})\, dx + \lambda \int_0^\infty y\, e^{-\lambda y}(1 - e^{-\lambda y/2})\, dy \\
&= 2\lambda \int_0^\infty \left(xe^{-\lambda x} + 2xe^{-2\lambda x} - 3xe^{-3\lambda x}\right)\, dx \\
&= \frac{2}{\lambda} \int_0^\infty \left(ue^{-u} + 2ue^{-2u} - 3ue^{-3u}\right)\, du \\
&= \frac{2}{\lambda}\left(1 + \frac{2}{4} - \frac{3}{9}\right) = \frac{7}{3\lambda}.
\end{aligned}$$

6.61 (a)

$$Z = X - Y \quad \rightarrow \quad -1 < z < 1.$$

$z > 0$

$$\begin{aligned}
F_Z(z) &= P(X - Y \leq z) = 1 - P(X - Y > z) \\
&= 1 - \int_0^{(1-z)/2} \int_{y+z}^{1-y} f_{XY}(x, y)\, dx\, dy \\
&= 1 - \int_0^{(1-z)/2} \left(\int_{y+z}^{1-y} 6x dx\right)\, dy \\
&= 1 - 3\int_0^{(1-z)/2} \{(1 - z^2) - 2(1 + z)y\}\, dy \\
&= 1 - \frac{3}{4}(1 + z)(1 - z)^2, \quad z \geq 0.
\end{aligned}$$

$z < 0$

$$\begin{aligned}
F_Z(z) &= P(X - Y \leq z) \\
&= \int_0^{(1+z)/2} \int_{x-z}^{1-x} 6x\, dy\, dx = \int_0^{(1+z)/2} 6x\,(1 + z - 2x)\, dx \\
&= \frac{(1 + z)^3}{4}, \quad z < 0.
\end{aligned}$$

This gives

$$f_Z(z) = \begin{cases} \frac{3}{4}(1-z)(1+3z), & 0 < z < 1 \\ \dfrac{3(1+z)^2}{4}, & -1 < z < 0 \end{cases}$$

(b)

$$f_X(x) = \int_0^{1-x} 6x\,dy = 6x(1-x), \quad 0 < x < 1$$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{F_X(x)} = \frac{1}{1-x}, \quad 0 < y \le 1-x$$

(c)

$$W = X + Y$$

we have

$$F_W(w) = P(X+Y \le w) = \int_0^w \left(\int_0^{w-x} 6x\,dy \right) dx = w^3,$$

and

$$f_W(w) = \int_0^w 6x dx = 3w^2, \quad 0 < w < 1$$

$$E[W] = \frac{3}{4}$$

$$E[W^2] = \frac{3}{5}$$

$$\text{Var}(X+Y) = \text{Var}(W) = E(W^2) - (E(W))^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}.$$

6.62

$$X = \frac{1}{Z}.$$

where Z represents a Chi-square random variable. Thus (see (4-39))

$$f_Z(z) = \frac{z^{-1/2}}{\sqrt{2}\Gamma(1/2)}\, e^{-z/2} = \frac{z^{-1/2}}{\sqrt{2\pi}}\, e^{-z/2}$$

or

$$f_X(x) = \frac{1}{\left|\frac{dx}{dz}\right|} f_Z(1/x) = \frac{1}{x^2}\,\frac{x^{1/2}}{\sqrt{2\pi}}\, e^{-1/2x} = \frac{1}{\sqrt{2\pi}x^{3/2}}\, e^{-1/2x}, \quad x > 0$$

Also it is given that

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi x}}\, e^{-y^2/2x}$$

so that

$$f_{XY}(x,y) = f_{Y|X}(y|x)\, f_X(x) = \frac{1}{2\pi x^2}\, e^{-(1+y^2)/2x}$$

and hence

$$\begin{aligned} f_Y(y) &= \int_0^\infty f_{XY}(x,y)\,dx \\ &= \frac{1}{2\pi} \int_0^\infty \frac{1}{x^2}\, e^{-(1+y^2)/2x}\, dx \\ &= \frac{1}{2\pi}\,\frac{2}{1+y^2} \int_0^\infty e^{-u}\,du = \frac{1/\pi}{1+y^2}, \quad -\infty < y < \infty. \end{aligned}$$

Thus Y represents a Cauchy random variable.

6.63 (a) For any two random variables X and Y we have

$$\begin{aligned}\sigma^2_{X+Y} &= \text{Var}(X+Y) = E[\{(X-\mu_X)+(Y-\mu_Y)\}^2] \\ &= \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X,Y) = \sigma^2_X + \sigma^2_Y + 2\sigma_X\sigma_Y\rho_{XY} \\ &\leq (\sigma_X+\sigma_Y)^2\end{aligned}$$

since $|\rho_{XY}| \leq 1$. Thus

$$\sigma_{X+Y} \leq \sigma_X + \sigma_Y,$$

and hence it easily follows that

$$\frac{\sigma_{X+Y}}{\sigma_X+\sigma_Y} \leq 1.$$

(However, (b) is not so easy!)
(b) We shall prove this result in three parts by making use of Holder's inequality.
(i) **Holder's inequality:** The function $\log x$ is concave, for $0 < \alpha < 1$, and hence we have

$$\log[\alpha x_1 + (1-\alpha)x_2] \geq \alpha \log x_1 + (1-\alpha)\log x_2$$

or

$$x_1^\alpha x_2^{1-\alpha} \leq \alpha x_1 + (1-\alpha)x_2, \quad 0 < \alpha < 1. \qquad (6.63-1)$$

Let

$$x_1 = |x|^p, \quad \alpha = \frac{1}{p}, \text{ so that } 1-\alpha = 1-\frac{1}{p} \stackrel{\Delta}{=} \frac{1}{q}, \quad x_2 = |y|^q \quad (6.63-2)$$

so that (6.63-1) becomes

$$|xy| \leq \frac{|x|^p}{p} + \frac{|y|^q}{q}, \quad p > 1, \qquad (6.63-3)$$

the Holder's inequality. From (6.63-2), note that

$$\frac{1}{p} + \frac{1}{q} = 1, \quad p > 1, \quad q > 1 \qquad (6.63-4)$$

(ii) Define

$$x = X\left(E\{|X|^p\}\right)^{-1/p}, \quad y = Y\left(E\{|Y|^q\}\right)^{-1/q}$$

where p and q are as in (6.63-4). Substituting these into the Holder's inequality in (6.63-3), we get

$$\begin{aligned} |XY| &\le p^{-1}\,|X|^p\,\left(E\{|X|^p\}\right)^{1/p-1}\,\left(E\{|Y|\}\right)^{1/q} \\ &+q^{-1}\,|Y|^q\,\left(E\{|Y|^q\}\right)^{1/q-1}\,\left(E\{|X|^p\}\right)^{1/p}. \end{aligned} \qquad (6.63-5)$$

Taking expected values on both sides of (6.63-5), we get

$$E\{|XY|\} \le \left(E\{|X|^p\}\right)^{1/p}\left(E\{|Y|^q\}\right)^{1/q} \qquad (6.63-6)$$

which represents the generalization of the Cauchy-Schwarz inequality. (Note $p = q = 2$ corresponds to Cauchy-Schwarz inequality)
(iii) To prove the desired inequality, notice that

$$\begin{aligned} |X+Y|^p &= |X+Y||X+Y|^{p-1} \\ &\le |X||X+Y|^{p-1} + |Y||X+Y|^{p-1}, \quad p > 1 \end{aligned}$$

and taking expected values on both sides we get

$$E\{|X+Y|^p\} \le E\{|X||X+Y|^{p-1}\} + E\{|Y||X+Y|^{p-1}\}. \quad (6.63-7)$$

Applying (6.63-6) to each term on the right side of (6.63-7) we get

$$E\{|X||X+Y|^{p-1}\} \le \left(E\{|X|^p\}\right)^{1/p}\left(E\{|X+Y|^{(p-1)q}\}\right)^{1/q} \quad (6.63-8)$$

and

$$E\{|Y||X+Y|^{p-1}\} \le \left(E\{|Y|^p\}\right)^{1/p}\left(E\{|X+Y|^{(p-1)q}\}\right)^{1/q} \quad (6.63-9)$$

Using (6.63-8) and (6.63-9) together with $(p-1)q = p$ in (6.63-7) we get

$$E\{|X+Y|^p\} \le \left[\left(E\{|X|^p\}\right)^{1/p} + \left(E\{|Y|^p\}\right)^{1/p}\right]\cdot\left(E\{|X+Y|^p\}\right)^{1/q}$$

or for $p > 1$

$$\left(E\{|X+Y|^p\}\right)^{1/p} \le \left(E\{|X|^p\}\right)^{1/p} + \left(E\{|Y|^p\}\right)^{1/p}.$$

the desired inequality. Since $p = 1$ follows trivially, we get

$$\frac{\left(E\{|X+Y|^p\}\right)^{1/p}}{\left(E\{|X|^p\}\right)^{1/p} + \left(E\{|Y|^p\}\right)^{1/p}} \le 1, \quad p \ge 1.$$

6.64 (a) See Example 6-41. From there

$$E(Y|X=x) = \mu_Y + \frac{\rho_{XY}\sigma_Y(x-\mu_X)}{\sigma_X}$$

(b) Similarly

$$f_{X|Y}(X|Y=y) \sim N(\mu, \sigma^2)$$

where

$$\mu = \mu_X + \frac{\rho_{XY}\sigma_X(y-\mu_Y)}{\sigma_Y}$$

and

$$\sigma^2 = \sigma_X^2(1-\rho_{XY}^2).$$

Since

$$E(X^2|Y=y) = \mathrm{Var}(X|Y=y) + (E[X|Y=y])^2$$

we obtain

$$E(X^2|Y=y) = \sigma^2 + \mu^2$$

6.65 (a) See footnote 4, Chapter 8, Page 337. From there (or directly) we have

$$\mathrm{Var}(X|Y) \triangleq E(X^2|Y) - (E\{X|Y\})^2$$

$$\mathrm{Var}\left(E\{X|Y\}\right) \triangleq E\left[E\{X|Y\}\right]^2 - (E[E\{X|Y\}])^2$$

so that

$$\begin{aligned} E[\mathrm{Var}(X|Y)] + \mathrm{Var}\left(E\{X|Y\}\right) &= E\left[E\{X^2|Y\}\right] - (E[E\{X|Y\}])^2 \\ &= E(X^2) - [E(X)]^2 = \mathrm{Var}(X) \qquad (1) \end{aligned}$$

or

$$\mathrm{Var}(X) \geq E[\mathrm{Var}\{X|Y\}]$$

Also

$$\mathrm{Var}(X) \geq \mathrm{Var}[E\{X|Y\}]$$

(b) See (1).

6.66

$$Z = aX + (1-a)Y, \qquad 0 < a < 1$$
$$\sigma_Z^2 = \mathrm{Var}(Z) = a^2\sigma_1^2 + (1-a)^2\sigma_2^2$$
$$\frac{\partial \sigma_Z^2}{\partial a} = 2a\sigma_1^2 + 2(1-a)(-1)\sigma_2^2 = 0$$

or

$$a(\sigma_1^2 + \sigma_2^2) = \sigma_2^2$$
$$a = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} < 1$$

minimizes Var(Z).

6-67 **From** (6-240)

$$E\{g(\underset{\sim}{x},\underset{\sim}{y})\} = E\{E\{g(\underset{\sim}{x},\underset{\sim}{y})\,|\,\underset{\sim}{y}\}\} = E\{g(x_n,\underset{\sim}{y})P\{\underset{\sim}{x} = x_n\}\} .$$

From (4-74) **with** $A_n = \{\underset{\sim}{x} = x_n\}$

$$f_z(z) = \sum_n f_z(z|\underset{\sim}{x} = x_n)P\{\underset{\sim}{x} = x_n\}$$

6-68 (a) The conditional density f(y|x) is $N(rx;\sigma\sqrt{1-r^2})$ [see (7-42)]. Hence

$$E\{f_y(\underset{\sim}{y}|x)\} = \int_{-\infty}^{\infty} f_y(y|x)f_y(y)dy$$

$$= \frac{1}{2\pi\sigma^2\sqrt{1-r^2}}\int_{-\infty}^{\infty} \exp\left\{\frac{-(y-rx)^2}{2\sigma^2(1-r^2)}\right\}\exp\left\{\frac{-y^2}{2\sigma^2}\right\}dy = \frac{1}{\sigma\sqrt{2\pi(2-r^2)}}\exp\left\{\frac{-r^2x^2}{2\sigma^2(2-r^2)}\right\}$$

(b) From (6-241) it follows that

$$E\{f_x(\underset{\sim}{x})f_y(\underset{\sim}{y})\} = E\{f_x(\underset{\sim}{x})E\{f_y(\underset{\sim}{y}|\underset{\sim}{x})\}\} = \int_{-\infty}^{\infty} f_x(x)\,E\{f_y(y|x)\}f_x(x)dx$$

$$= \frac{1}{2\pi\sigma^3\sqrt{2\pi(2-r^2)}}\int_{-\infty}^{\infty}\exp\left\{-\frac{x^2}{\sigma^2}\right\}\exp\left\{\frac{-r^2x^2}{2\sigma^2(2-r^2)}\right\}dx = \frac{1}{2\pi\sigma^2\sqrt{4-r^2}}$$

6-69 We shall use (6-64) and Price's theorem (10-94):

$$\frac{\partial E\{|\underset{\sim}{x}\underset{\sim}{y}|\}}{\partial \mu} = E\left\{\frac{d|\underset{\sim}{x}|}{d\underset{\sim}{x}}\ \frac{d|\underset{\sim}{y}|}{\partial \underset{\sim}{y}}\right\} = E\{\operatorname{sgn} \underset{\sim}{x} \operatorname{sgn} \underset{\sim}{y}\}$$

$$= P\{\underset{\sim}{x}\underset{\sim}{y} > 0\} - P\{\underset{\sim}{x}\underset{\sim}{y} < 0\} = \frac{2\alpha}{\pi} = \frac{2}{\pi} \arcsin \frac{\mu}{\sigma_1\sigma_2}$$

If $\mu = 0$, then the RVs $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are independent, hence,

$$E\{|\underset{\sim}{x}\,\underset{\sim}{y}|\}\Big|_{\mu=0} = E\{|\underset{\sim}{x}|\}E\{|\underset{\sim}{y}|\} = \frac{2}{\pi}\sigma_1\sigma_2$$

[see (5-74)]. Integrating (i) and using the above, we obtain

$$E\{|\underset{\sim}{x}\,\underset{\sim}{y}|\} = \frac{2}{\pi}\int_0^{\mu} \arcsin \frac{C}{\sigma_1\sigma_2}\, dC + \frac{2}{\pi}\sigma_1\sigma_2 = \frac{2\sigma_1\sigma_2}{\pi}(\cos\alpha + \alpha\sin\alpha)$$

6-70 From Example 6-41

$$f(y|x) : N(\eta_2 + \frac{r\sigma_2}{\sigma_1}x; \sigma_2\sqrt{1 - r^2}) = N(4+x;\ \sqrt{3})$$

$$f(x|y) : N(\eta_1 + \frac{r\sigma_1}{\sigma_2}y; \sigma_1\sqrt{1 - r^2}) = N(3+\frac{y}{4};\ \sqrt{3}/2)$$

6-71 The mass density in the square $|\underset{\sim}{x}|\le 1$, $|\underset{\sim}{y}|\le 1$ of the xy plane equals 1/4; hence, $P\{\underset{\sim}{r}\le 1\}=\pi/4$ and $P\{\underset{\sim}{r}\le r\} = \pi r^2/4$ for $r<1$. This yields

$$P\{r\le r, r\le 1\} = \begin{cases} P\{r\le r\} = \pi r^2/4 & r\le 1 \\ P\{r\le 1\} = \pi/4 & r>1 \end{cases}$$

$$F_r(r|M) = \frac{P\{r\le r, M\}}{P(M)} = \begin{cases} r^2 & r\le 1 \\ 1 & r>1 \end{cases} \qquad f_r(r|m) = \begin{cases} 2r, & r<+1 \\ 0 & \text{otherwise} \end{cases}$$

6-72 $\quad \underset{\sim}{z} = \underset{\sim}{x} + \underset{\sim}{y} \qquad \underset{\sim}{w} = \underset{\sim}{x} \qquad f_{xz}(x,z) = f_{xy}(x, z-x)$

If $f_{xy}(x,y) = f_x(x)f_y(y)$, then

$$f_z(z|x) = \frac{f_{xz}(x,z)}{f_x(x)} = f_y(z-x)$$

6-73 The system $z = F_x(x) \qquad w = F_y(y|x)$ has a solution only if $z \le z \le 1$ and $0 \le w \le 1$. Furthermore,

$$\frac{\partial z}{\partial x} = f_x(x) \qquad \frac{\partial z}{\partial y} = 0$$

$$J = f_x(x)f_y(y|x)$$

$$\frac{\partial w}{\partial x} \qquad \frac{\partial w}{\partial y} = f_y(y|x)$$

$$f_{zw}(z,w) = \frac{f_{xy}(x,y)}{f_x(x)f_y(y|x)} = 1 \quad \text{for} \quad 0 \le z,w \le 1$$

6-74 We introduce the events C_r = {we selected the rth coin} and A_k = {heads in a specific order}. From the assumptions it follows that

$$P(C_r) = \frac{1}{m} \qquad P(A_k|C_r) = p_r^k(1-p_r)^{n-k}$$

We wish to find the probability $P(C_r|A_k)$. The events C_r form a pertition; hence,

$$P(C_r|A_k) - \frac{\frac{1}{m}P(a_k|C_r)}{\frac{1}{m}\sum_{i-1}^{m} P(A_k|C_i)}$$

6-75 We wish to show that

$$E\{x^2\} = \frac{n}{n-1}$$

From page 207: $x^2 = ny^2/z$ where y is N(0,1) and z is $\chi^2(n)$. Hence, $E\{y^2\} = 1$ and (also (4-35) and (4-39))

$$E\left\{\frac{1}{z}\right\} = \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty z^{n/2-2}e^{-z/2}dz = \frac{2^{m/2-1}\Gamma(n/2-1)}{2^{n/2}\Gamma(n/2)}$$

From this and the independence of y and z it follows that

$$E\{x^2\} = n\, E\{y^2\}\, E\left\{\frac{1}{z}\right\} = \frac{n}{n-2}$$

6-76 From (6-222) :

$$R_x(x) = \exp\left\{-\int_0^x \beta_x(t)dt\right\} = \exp\left\{-k\int_0^x \beta_y(t)dt\right\} = R_y^k(t)$$

6-77 From (5-89) it follows with $x = |z|^2$ and $\alpha = \varepsilon^2$ that

$$E\{|z|^2 > \varepsilon^2\} \le \frac{E\{|z|^2\}}{\varepsilon^2}$$

for any z. And the result follows with $z = x - y$.

6-78 $$E\{U(a-\underset{\sim}{x})\} = \int_{-\infty}^{\infty} U(a-x)f(x)dx = \int_{-\infty}^{a} f(x)dx = F_x(a)$$

$$E\{U(b-\underset{\sim}{y})\} = F_y(b)$$

$$E\{U(a-\underset{\sim}{x})U(b-\underset{\sim}{y}) = \int_{-\infty}^{a}\int_{-\infty}^{b} f(x,y)dxdy = F_{xy}(a,b)$$

Hence

$$F_{xy}(a,b) = F_x(a)F_y(b)$$

6-79 From Example 6-38

$$E\{\underset{\sim}{y}|\underset{\sim}{x} \le 0\} = \int_{-\infty}^{\infty} y\, f_y(y|\underset{\sim}{x} \le 0\}dy = \frac{1}{F_x(0)} \int_{-\infty}^{\infty} y \frac{\partial F(0,y)}{\partial y}\, dy$$

From (7-41) and (7-57)

$$\int_{-\infty}^{\infty} E\{\underset{\sim}{y}|x\}f_x(x)dx = \int_{-\infty}^{\infty} y \int_{-\infty}^{0} f(x,y)dxdy = \int_{-\infty}^{\infty} y \frac{\partial F(0,y)}{\partial y}\, dy$$

7-1 $$0 \le P\{x_1 < \mathbf{x} \le x_2,\ y_1 < \mathbf{y} \le y_2,\ z_1 < \mathbf{z} \le z_2\} =$$

$$= P\{\mathbf{x} \le x_2,\ y_1 < \mathbf{y} \le y_2,\ z_1 < \mathbf{z} \le z_2\} - P\{\mathbf{x} \le x_1,\ y_1 < \mathbf{y} \le y_2,\ z_1 < \mathbf{z} \le z_2\} =$$

$$= P\{\mathbf{x} \le x_2,\ \mathbf{y} \le y_2,\ z_1 < \mathbf{z} \le z_2\} - P\{\mathbf{x} \le x_2,\ \mathbf{y} \le y_1,\ z_1 < \mathbf{z} \le z_2\}$$

$$- P\{\mathbf{x} \le x_1,\ \mathbf{y} \le y_2,\ z_1 < \mathbf{z} \le z_2\} + P\{\mathbf{x} \le x_1,\ \mathbf{y} \le y_1,\ z_1 < \mathbf{z} \le z_2\} =$$

$$= P\{\mathbf{x} \le x_2,\ \mathbf{y} \le y_2,\ \mathbf{z} \le z_2\} - P\{\mathbf{x} \le x_2,\ \mathbf{y} \le y_2,\ \mathbf{z} \le z_1\}$$

$$- P\{\mathbf{x} \le x_2,\ \mathbf{y} \le y_1,\ \mathbf{z} \le z_2\} + P\{\mathbf{x} \le x_2,\ \mathbf{y} \le y_1,\ \mathbf{z} \le z_1\}$$

$$- P\{\mathbf{x} \le x_1,\ \mathbf{y} \le y_2,\ \mathbf{z} \le z_2\} + P\{\mathbf{x} \le x_1,\ \mathbf{y} \le y_2,\ \mathbf{z} \le z_1\}$$

$$+ P\{\mathbf{x} \le x_1,\ \mathbf{y} \le y_1,\ \mathbf{z} \le z_2\} - P\{\mathbf{x} \le x_1,\ \mathbf{y} \le y_1,\ \mathbf{z} \le z_1\}$$

7-2 $P\{\mathbf{x}_A = 1,\ \mathbf{x}_B = 1,\ \mathbf{x}_C = 1\} = P(ABC) = 1/4$

$P\{\mathbf{x}_A = 1\} = P(A) = 1/2$ $\qquad$ $P\{\mathbf{x}_B = 1\} = P(B) = 1/2$

$P\{\mathbf{x}_C = 1\} = P(C) = 1/2$ hence

$$P\{\mathbf{x}_A = 1,\ \mathbf{x}_B = 1,\ \mathbf{x}_C = 1\} \ne P\{\mathbf{x}_A = 1\}P\{\mathbf{x}_B = 1\}P\{\mathbf{x}_C = 1\}$$

hence $\mathbf{x}_A$, $\mathbf{x}_B$, $\mathbf{x}_C$ are not independent. But

$P\{\mathbf{x}_A = 1,\ \mathbf{x}_B = 1\} = P(AB) = 1/4 = P\{\mathbf{x}_A = 1\}P\{\mathbf{x}_B = 1\}$

Similarly for any other combination, e.g.,
since $P(A) = P(AB) + P(A\bar{B})$, we conclude that
$P(A\bar{B}) = 1/2 - 1/4 = 1/4$ $\qquad$ $P(\bar{B}) = 1 - P(B) = 1/2$
$P\{\mathbf{x}_A = 1,\ \mathbf{x}_B = 0\} = P(A\bar{B}) = 1/4$
$P\{\mathbf{x}_B = 0\} = P(\bar{B}) = 1/2$ hence
$P\{\mathbf{x}_A = 1,\ \mathbf{x}_B = 0\} = P\{\mathbf{x}_A = 1\}P\{\mathbf{x}_B = 0\}$

7-3 If $\underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z}$ are independent in pairs, then

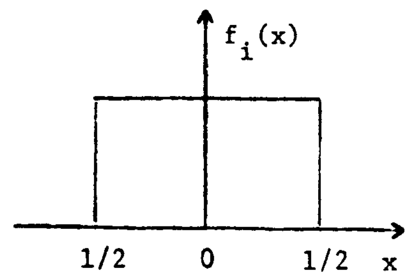
$$r_{xy} = r_{xz} = r_{yz} = 0$$

and (7-60) yields (we assume $\eta_x = \eta_y = \eta_z = 0$)

$$\Phi(\omega_1, \omega_2, \omega_3) = \exp\left\{-\frac{1}{2}\left(\sigma_1^2\omega_1^2 + \sigma_2^2\omega_2^2 + \sigma_3^2\omega_3^2\right)\right\}$$

$$f(x_1, x_2, x_3) = f(x_1)f(x_2)f(x_3)$$

7-4 $\underset{\sim}{x} = \underset{\sim}{x}_1 + \underset{\sim}{x}_2 + \underset{\sim}{x}_3$. To determine $E\{\underset{\sim}{x}^4\}$ we shall use char. functions



$$\Phi_i(\omega) = \int_{-1/2}^{1/2} e^{j\omega x}\,dx = \frac{2\sin(\omega/2)}{\omega}$$

$$\Phi(\omega) = \left[\frac{2\sin(\omega/2)}{\omega}\right]^3 = \left(1 - \frac{\omega^2}{24} + \frac{\omega^4}{1920} - \cdots\right)^3$$

The coefficient of ω^4 in this expansion equals

$$\frac{13}{1920} \text{ hence } \frac{1}{4!}\,\frac{d^4\Phi(0)}{d\omega^4} = \frac{13}{1920}$$

and [see (5-103)]

$$E\{\underset{\sim}{x}^4\} = m_4 = \frac{13 \times 4!}{1920} = \frac{13}{80}$$

7-5 (a) The joint density f(x,y) has circular symmetry because

$$f(x,y) = \int_{-\infty}^{\infty} f(\sqrt{x^2 + y^2 + z^2})\, dz$$

depends only on $x^2 + y^2$. The same holds for f(x,z) and f(y,z). And since the RVs $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ are independent, they must be normal [see (6-29)].

(b) From (a) it follows that the RVs $\mathbf{v}_x, \mathbf{v}_y, \mathbf{v}_z$ are $N(0; \sqrt{kT/m})$. With $\sigma^2 = kT/m$ and n = 3 it follows from (7-62)-(7-63) and (5-25) that

$$f_v(v) = \sqrt{\frac{2m^3}{\pi k^3 T^3}}\; v^2\, e^{-mv^2/2kT}\, U(v)$$

$$E\{\mathbf{v}\} = 2\sqrt{\frac{2kT}{\pi m}} \qquad E\{\mathbf{v}^{2n}\} = 1 \times 3 \cdots (2n+1)\left(\frac{kT}{m}\right)^n$$

7-6 From Prob. 6-52: $\mathbf{y} = a\mathbf{x} + b$, $\mathbf{z} = c\mathbf{y} + d$, hence,

$$\mathbf{z} = A\mathbf{x} + B \qquad \eta_z = A\eta_x + B \qquad \sigma_z = A\sigma_x$$

$$E\{(\mathbf{z} - \eta_z)(\mathbf{x} - \eta_x\} = E\{A(\mathbf{x} - \eta_x)(\mathbf{x} - \eta_x)\} = A\sigma_x^2 = \sigma_x \sigma_z$$

7-7 It follows from (6-241) with $g_1(x) = x$, $g_2(y) = y$ if we replace all densities with conditional densities assuming $\mathbf{x}_3$.

7-8 Reasoning as in (7-82), we conclude that

$E\{[\mathbf{y}-(a_1\mathbf{x}_1+a_2\mathbf{x}_1)]^2\}$ is minimum if

$$E\{[\mathbf{y}-(a_1\mathbf{x}_1+a_2\mathbf{x}_2)]\mathbf{x}_i\} = 0 \qquad i = 1,2$$

With $R_{0i} = E\{\mathbf{y}\,\mathbf{x}_i\}$, $R_{ij} = E\{\mathbf{x}_i\mathbf{x}_j\}$, the above yields

$$R_{01} = a_1R_{11} + a_2R_{12} \qquad\qquad R_{02} = a_1R_{12} + a_2R_{22}$$

But $\quad \hat{E}\{\mathbf{y}|\mathbf{x}_1\} = A\,\mathbf{x}_1 \qquad\qquad A = R_{01}/R_{11} = a_1 + a_2R_{12}/R_{11}$

$$\hat{E}\{\hat{E}\{\mathbf{y}|\mathbf{x}_1,\mathbf{x}_2\}|\mathbf{x}_1\} = \hat{E}\{a_1\mathbf{x}_1 + a_2\mathbf{x}_2|\mathbf{x}_1\}$$

$$= a_1\mathbf{x}_1 + a_2\hat{E}\{\mathbf{x}_2|\mathbf{x}_1\} = \left(a_1 + a_2\frac{R_{12}}{R_{11}}\right)\mathbf{x}_1 = A\,\mathbf{x}_1$$

7-9 As in Probl. 6-51

$$E^2\{\mathbf{x}_i\mathbf{x}_j\} \le E^2\{\mathbf{x}_i\}E^2\{\mathbf{x}_j\} = M^2 \qquad\qquad |E\{\mathbf{x}_i\mathbf{x}_j\}| \le M$$

$$E\{\mathbf{s}^2|\mathbf{n} = n\} = E\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{x}_i\mathbf{x}_j\right\} \le Mn^2$$

Hence [see (6-240)]

$$E\{\mathbf{s}^2\} = E\{E\{\mathbf{s}^2|\mathbf{n}\}\} < E\{M\mathbf{n}^2\}$$

7-10 As we know,

$$1 + x + \cdots + x^n + \cdots = \frac{1}{1-x} \qquad |x| < 1$$

Differentiating, we obtain

$$1 + 2x + \cdots + n\,x^{n-1} + \cdots = \sum_{k=1}^{\infty} k\,x^{k-1} = \frac{1}{(1-x)^2} \qquad \text{(i)}$$

The RV $\underset{\sim}{x}_1$ equals the number of tosses until heads shows for the first time, Hence, $\underset{\sim}{x}_1$ takes the values 1,2,... with $P\{\underset{\sim}{x}_1 = k\} = pq^{k-1}$. Hence, [see (3-12) and (i)]

$$E\{\underset{\sim}{x}_1\} = \sum_{k=1}^{\infty} k\,P\{\underset{\sim}{x}_1 = k\} = \sum_{k=1}^{\infty} k\,p\,q^{k-1} = \frac{p}{(1-q)^2} = \frac{1}{p}$$

Starting the count after the first head shows, we conclude that the RV $\underset{\sim}{x}_2 - \underset{\sim}{x}_1$ has the same statistics as the RV $\underset{\sim}{x}_1$. Hence,

$$E\{\underset{\sim}{x}_2 - \underset{\sim}{x}_1\} = E\{\underset{\sim}{x}_1\} \qquad E\{\underset{\sim}{x}_2\} = 2E\{\underset{\sim}{x}_1\} = \frac{2}{p}$$

Reasoning similarly, we conclude that

$E\{\underset{\sim}{x}_n - \underset{\sim}{x}_{n-1}\} = E\{\underset{\sim}{x}_1\}$. Hence (induction)

$$E\{\underset{\sim}{x}_n\} = E\{\underset{\sim}{x}_{n-1}\} + E\{\underset{\sim}{x}_1\} = \frac{n-1}{p} + \frac{1}{p} = \frac{n}{p}$$

7-11 If n accidents occur in a day, the probability that m of them will be fatal equals $\binom{n}{m} p^m q^{n-m}$ for $m \le n$ and zero for $m > n$. Hence,

$$P\{\underset{\sim}{m} = m \mid \underset{\sim}{n} = n\} = \begin{cases} 0 & m > n \\ \binom{n}{m} p^m q^{n-m} & m \le n \end{cases}$$

This yields

$$E\{e^{j\omega\underset{\sim}{m}} \mid \underset{\sim}{n} = n\} = \sum_{m=0}^{n} e^{j\omega m} \binom{n}{m} p^m q^{n-m} = (p\,e^{j\omega} + q)^n$$

But

$$P\{\underset{\sim}{n} = n\} = e^{-a}\frac{a^n}{n!} \qquad n = 0,1,\ldots$$

Hence,

$$E\{e^{j\omega \underset{\sim}{m}}\} = E\{E\{e^{j\omega \underset{\sim}{m}} \mid \underset{\sim}{n}\}\} = E\{(p\,e^{j\omega} + q)^{\underset{\sim}{n}}\}$$

$$\sum_{n=0}^{\infty} (p\,e^{j\omega} + q)^n\, e^{-a} \frac{a^n}{n!} = e^{a(p\,e^{j\omega} + q)} e^{-a} = e^{a\,p\,(e^{j\omega} - 1)}$$

This shows that the RV $\underset{\sim}{m}$ is Poisson distributed with parameter a p [see (5-119)] .

7-12 We shall determine first the conditional distribution

$$F_s(s \mid \underset{\sim}{n} = n) = \frac{P\{\underset{\sim}{s} \le s,\ \underset{\sim}{n} = n\}}{P\{\underset{\sim}{n} = n\}}$$

The event $\{\underset{\sim}{s} < s,\ \underset{\sim}{n} = n\}$ consists of all outcomes such that $\underset{\sim}{n} = n$ and $\sum_{k=1}^{n} \underset{\sim}{x}_k \le s$. Since the RV $\underset{\sim}{n}$ is independent of the RVs $\underset{\sim}{x}_k$, this yields

$$F_s(s \mid \underset{\sim}{n} = n) = P\{\sum_{k=1}^{n} \underset{\sim}{x}_k \le s\}P\{\underset{\sim}{n} = n\}/P\{\underset{\sim}{n} = n\}$$

From the above and the independence of the RVs $\underset{\sim}{x}_k$ it follows that [see (7-51)]

$$f_s(s \mid \underset{\sim}{n} = n) = f_1(s) * f_2(s) * \cdots * f_n(s)$$

Setting $A_k = \{\underset{\sim}{n} = k\}$ in (4-74), we obtain

$$f_s(s) = \sum_k p_k\, [f_1(s) * \cdots * f_k(s)]$$

7-13 From the independence of the RVs n and x_i it follows that

$$E\{e^{sy} \mid n = k\} = E\{e^{s(x_1 + \cdots + x_k)}\}$$

$$= E\{e^{sx_1}\} \cdots E\{e^{sx_k}\} = \Phi_x^k(s)$$

Hence,

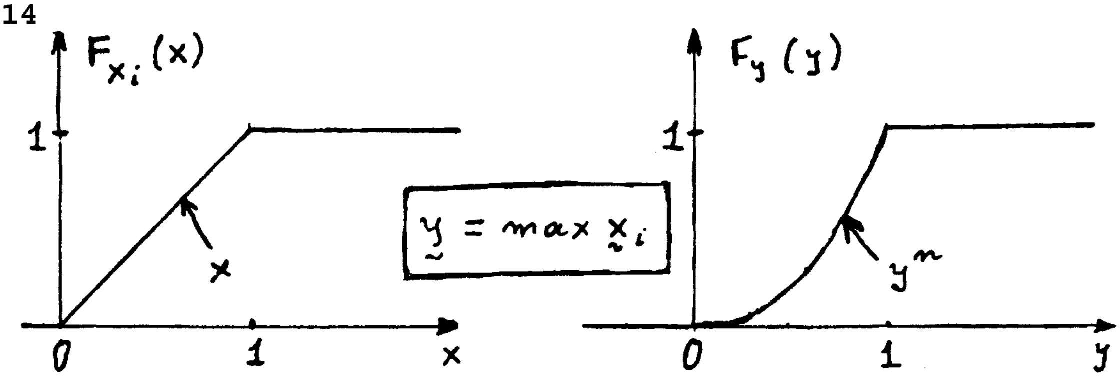
$$\Phi_y(s) = E\{e^{sy}\} = E\{E\{e^{sy} \mid n\}\} = E\{\Phi_x^n(s)\}$$

$$= \Gamma_n[\Phi_x(s)] \text{ because } E\{z^n\} = \Gamma_n(z)$$

Special case. If n is Poisson with parameter a, then [see (5-119)]

$$\Gamma_n(z) = e^{az - a} \qquad \Phi_y(s) = e^{a\Phi_x(s) - a}$$

7-14



$$\{y \le y\} = \{x_1 \le y,\ x_2 \le y,\ \ldots,\ x_n \le y\}$$

From the independence of x_i and the abcve it follows that

$$F_y(y) = P\{y \le y\} = P\{x_1 \le y\} \cdots P\{x_n \le y\}$$

$$= F_1(y) \cdots F_n(y)$$

where $F_i(y) = y$ for $0 \le y \le 1$.

7-15 The RV $\underset{\sim}{x}$ is defined in the space S. The set

$$C = \{z < \underset{\sim}{z} \le z + dz,\ w < \underset{\sim}{w} \le w + dz\} \qquad z > w$$

is an event in the space S_n of repeated trials and its probability equals

$$P(C) = f_{zw}(z,w)dzdw$$

We introduce the events

$$D_1 = \{\underset{\sim}{x} \le w\} \qquad D_2 = \{w < \underset{\sim}{x} \le w + dw\} \qquad D_3 = \{w + dw < \underset{\sim}{x} \le z\}$$

$$D_4 = \{z < \underset{\sim}{x} \le z + dz\} \qquad D_5 = \{z + dz < \underset{\sim}{x}\}$$

These events form a partition of S and their probabilities $p_i = P(D_i)$ equal

$$F_x(w) \qquad f_x(w)dw \qquad F_x(z)-F_x(w+dw) \qquad f_z(z)dz \qquad 1-F_x(z+dz)$$

respectively. The event C occurs iff the smallest of the RVs $\underset{\sim}{x}_i$ is in the interval (w, w+dw), the largest is in the interval (z, z+dz), and, consequently, all others are between w+dw and z. This is the case iff D_1 does not occur at all, D_2 occurs once, D_3 occurs n-2 times, D_4 occurs once, and D_5 does not occur at all. With

$$k_1=0 \qquad k_2=1 \qquad k_3=n-2 \qquad k_4=1 \qquad k_5=0$$

it follows from (4-102) that

$$P(C) = \frac{n!}{(n-2)!} p_2\, p_3^{n-2}\, p_4 = n(n-1)f_x(w)dw\, [f_x(z)-F_x(w+dw)]^{n-1} f_x(z)dz$$

for z > w, and 0 otherwise.

7-16 If $\underset{\sim}{z}$ is $N(\eta,1)$ then

$$E\{e^{s\underset{\sim}{z}^2}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{sz} e^{-(z-\eta)^2/2} dz$$

$$sz^2 - \frac{(z-\eta)^2}{2} = \left(s - \frac{1}{2}\right)\left(z - \frac{\eta}{1-2s}\right)^2 + \frac{\eta^2 s}{1-1s}$$

Since

$$\frac{1}{\sqrt{2\pi}}\int_{-\eta}^{\infty} e^{-a(z-b)^2} dz = \frac{1}{\sqrt{2a}}$$

the above yields

$$E\{e^{s\underset{\sim}{z}^2}\} = \frac{1}{\sqrt{2(1/2-S)}} \exp\left\{\frac{\eta^2 S}{1-2S}\right\}$$

$$\Phi_w(s) = \frac{1}{\sqrt{1-2s}} \exp\left\{\frac{\eta_1 s}{1-2s}\right\} \cdots \frac{1}{\sqrt{1-2s}} \exp\left\{\frac{\eta_n s}{1-2s}\right\}$$

7-17 We wish to show that the RVs

$$\bar{\underset{\sim}{x}} = \frac{1}{n}\sum_{i=1}^{n} \underset{\sim}{x}_i \qquad \underset{\sim}{s}^2 = \frac{1}{n-1}\sum_{i=1}^{n} (\underset{\sim}{x}_i - \bar{x})^2$$

are independent. Since $\underset{\sim}{s}^2$ is a function of the n RVs $\underset{\sim}{x}_i - \bar{\underset{\sim}{x}}$, it suffices to show that each of these RVs is independent of $\bar{\underset{\sim}{x}}$. We assume for simplicity that $E\{\underset{\sim}{x}_i\}=0$. Clearly,

$$E\{\underset{\sim}{x}_i \bar{x}\} = \frac{1}{n} E\{\underset{\sim}{x}_i^2\} = \frac{\sigma^2}{n} \qquad E\{\bar{\underset{\sim}{x}}\bar{\underset{\sim}{x}}\} = \frac{1}{n^2}\sum_{i=1}^{n} \underset{\sim}{x}_i^2 = \frac{\sigma^2}{n}$$

because $E\{\underset{\sim}{x}_i \underset{\sim}{x}_j\}=0$ for $i \neq i$. Hence,

$$E\{(\underset{\sim}{x}_i - \bar{\underset{\sim}{x}})\bar{\underset{\sim}{x}}\} = 0$$

Thus, the RVs $\underset{\sim}{x}_i - \bar{\underset{\sim}{x}}$ and $\bar{\underset{\sim}{x}}$ are orthogonal; and since they are jointly normal, they are independent.

7-18 Since $\eta_s = \alpha_0 + \alpha_1\eta_1 + \alpha_2\eta_2$ [see (7-87)], the mean of the error

$$\underset{\sim}{\varepsilon} = \underset{\sim}{s} - (\alpha_0 + \alpha_1\underset{\sim}{x}_1 + \alpha_2\underset{\sim}{x}_2) = (\underset{\sim}{s} - \eta_s) - [\alpha_1(x_1 - \eta_1) + \alpha_2(\underset{\sim}{x}_2 - \eta_2)]$$

is zero. Furthermore, $\underset{\sim}{\varepsilon}$ is orthogonal to $\underset{\sim}{x}_i$, hence, it is also orthogonal to $\underset{\sim}{x}_i - \eta_i$.

7-19 From the orthogonality principle:

$$\hat{E}\{\underset{\sim}{y}|\underset{\sim}{x}_1,\underset{\sim}{x}_2\} = a_1\underset{\sim}{x}_1 + a_2\underset{\sim}{x}_2 \qquad \underset{\sim}{y} - \{a_1x_1 + a_2\underset{\sim}{x}_2\} \perp \underset{\sim}{x}_1, x_2$$

$$\hat{E}\{y|x_1\} = A\,\underset{\sim}{x}_1 \qquad y - A\,\underset{\sim}{x}_1 \perp \underset{\sim}{x}_1$$

Hence

$$\underset{\sim}{y} - (a_1\underset{\sim}{x}_1 + a_2\underset{\sim}{x}_2) - (\underset{\sim}{y} - A\,x_1) = a_1\underset{\sim}{x}_1 + a_2\underset{\sim}{x}_2 - A\,\underset{\sim}{x}_1 \perp \underset{\sim}{x}_1$$

From this it follows that

$$\hat{E}\{a_1x_1 + a_2\underset{\sim}{x}_2|\underset{\sim}{x}_1\} = A\,\underset{\sim}{x}_1$$

$$\hat{E}\{\hat{E}\{\underset{\sim}{y}|\underset{\sim}{x}_1,\underset{\sim}{x}_2\}|\underset{\sim}{x}_1\} = \hat{E}\{\underset{\sim}{y}|\underset{\sim}{x}_1\}$$

7-20 The event $\{\underset{\sim}{x} \le x\}$ occurs if there is at least one point in the interval $(0,x)$; the event $\{\underset{\sim}{y} \le y\}$ occurs if all the points are in the interval $(0,y)$:

$$A_x = \{\text{at least one point in } (0,x)\} = \{\underset{\sim}{x} \le x\}$$

$$B_y = \{\text{no points in } (y,1)\}$$

$$= \{\text{all points in } (0,y)\} = \{\underset{\sim}{y} \le y\}$$

Hence, for $0 \le x \le 1$, $0 \le y \le 1$

$$F_x(x) = P(A_x) = 1 - P(\bar{A}_x) = 1 - (1 - x)^n$$

$$F_y(y) = P(B_y) = y^n$$

Furthermore,

$$\{\underset{\sim}{x} \le x,\ \underset{\sim}{y} \le y\} = A_x B_y \qquad A_x B_y + \bar{A}_x B_y = B_y$$

If $x \le y$ then

$$\bar{A}_x B_y = \{\text{all points in } (x,y)\}$$

$$P(\bar{A}_x B_y) = (y - x)^n$$

If $x > y$, then $\bar{A}_x B_y = \{\emptyset\}$. Hence

$$F_{xy}(x,y) = P(A_x B_y) = \begin{cases} y^n - (y-x)^n & x \le y \\ y^n & x > y \end{cases}$$

7-21 Suppose that $E\{\underset{\sim}{x}_i\} = 0$, $E\{\underset{\sim}{x}_i^2\} = \sigma^2$, $E\{\underset{\sim}{x}_i^4\} = \mu_4$

If $\underset{\sim}{A} = \sum_{i=1}^{n} \underset{\sim}{x}_i^2$, then $E\{\underset{\sim}{A}\} = n\sigma^2$

$$E\{\underset{\sim}{A}^2\} = \sum_{i,j=1}^{n} E\{\underset{\sim}{x}_i^2 \underset{\sim}{x}_j^2\} = n\mu_4 + (n^2 - n)\sigma^4$$

because

$$E\{\underset{\sim}{x}_i^2 \underset{\sim}{x}_j^2\} = \begin{cases} \mu_4 & i = j \\ \sigma^4 & i \neq j \end{cases}$$

Furthermore

$$E\{\bar{\underset{\sim}{x}}^2 \underset{\sim}{x}_j^2\} = \frac{1}{n^2} E\left\{\left(\sum_{i=1}^{n} \underset{\sim}{x}_i\right)^2 \underset{\sim}{x}_j^2\right\} = \frac{1}{n^2} [\mu_4 + (n - 1)\sigma^4]$$

$$E\{\bar{\underset{\sim}{x}}^2 \underset{\sim}{A}\} = \frac{1}{n} [\mu_4 + (n - 1)\sigma^4]$$

$$E\{\bar{\underset{\sim}{x}}^4\} = \frac{1}{n^4} E\left\{\left(\sum_{i=1}^{n} \underset{\sim}{x}_i\right)^4\right\} = \frac{1}{n^4} [n\mu_4 + 3n(n - 1)\sigma^4]$$

because

$$E\{\underset{\sim}{x}_i \underset{\sim}{x}_j \underset{\sim}{x}_k \underset{\sim}{x}_r\} = \begin{cases} \mu_4 & i = j = k = r & \text{[n such terms]} \\ \sigma^4 & i = j \neq k = r & \text{[3n(n-1) such terms]} \\ 0 & \text{otherwise} \end{cases}$$

Clearly, $(n - 1)\,\bar{\underset{\sim}{V}} = \sum_{i=1}^{n} (\underset{\sim}{x}_i - \bar{\underset{\sim}{x}})^2 = \underset{\sim}{A} - n\bar{\underset{\sim}{x}}^2$, $E\{\bar{\underset{\sim}{V}}\} = \sigma^2$. Hence

$$(n - 1)^2 E\{\underset{\sim}{V}^2\} = E\{\underset{\sim}{A}^2\} - 2nE\{\bar{\underset{\sim}{x}}^2 \underset{\sim}{A}\} + n^2 E\{\bar{\underset{\sim}{x}}^4\}$$

$$= n\mu_4 + (n^2 - n)\sigma^4 - 2[\mu_4 + (n - 1)\sigma^4] + \frac{1}{n}[\mu_4 + 3(n - 1)\sigma^4]$$

This yields

$$E\{\bar{\underset{\sim}{V}}^2\} = \frac{\mu_4}{n} + \frac{n^2 - 2n + 3}{n(n - 1)}\sigma^4 = \sigma^4 + \sigma_{\bar{V}}^2$$

<u>Note</u> If the RVs $\underset{\sim}{x}_i$ are $N(0,\sigma^2)$, then $\mu_4 = 3\sigma^4$

$$\sigma_{\bar{V}}^2 = \frac{1}{n}\left(3\sigma^4 - \frac{n - 3}{n - 1}\sigma^4\right) = \frac{2}{n - 1}\sigma^4$$

7-22 From Prob. 6-49:

$$E\{|\underset{\sim}{x}_{2i} - \underset{\sim}{x}_{2i-1}|\} = \frac{2\sigma}{\sqrt{\pi}} \qquad E\{|\underset{\sim}{x}_{2i} - \underset{\sim}{x}_{2i-1}|^2\} = 2\sigma^2$$

Hence,

$$E\{|\underset{\sim}{x}_{2i} - \underset{\sim}{x}_{2i-1}||\underset{\sim}{x}_{2j} - \underset{\sim}{x}_{2j-1}|\} = \begin{cases} 2\sigma^2 & i = j \\ 4\sigma^2/\pi & i \neq j \end{cases}$$

$$E\{\underset{\sim}{z}\} = \frac{\sqrt{\pi}}{2n} \frac{2\sigma n}{\sqrt{\pi}} = \sigma$$

$$E\{\underset{\sim}{z}^2\} = \frac{\pi}{4n^2}\left[2n\sigma^2 + \frac{4\sigma^2}{\pi}(n^2 - n)\right]$$

$$\sigma_z^2 = \frac{\pi}{2n}\sigma^2 + \left(1 - \frac{1}{n}\right)\sigma^2 - \sigma^2 = \frac{\pi - 2}{2n}\sigma^2$$

7-23 If $R^{-1} = \begin{bmatrix} a_{11} \cdots a_{1n} \\ \\ a_{n1} \cdots a_{nn} \end{bmatrix}$ then $\sum_j a_{ij}R_{ji} = 1$

Hence,

$$E\{\underset{\sim}{X}R^{-1}\underset{\sim}{X}^t\} = E\{\sum_{i=1}^{n}\sum_{j=1}^{n} \underset{\sim}{x}_i a_{ij} \underset{\sim}{x}_j\}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}R_{j1} = \sum_{i=1}^{n} 1 = n$$

7-24 The density $f_z(z)$ of the sum $\underset{\sim}{z} = \underset{\sim}{x}_1 + \cdots + \underset{\sim}{x}_n$ tends to a normal curve with variance $\sigma_1^2 + \cdots + \sigma_n^2 \to \infty$ as $n \to \infty$ (we assume $\sigma_i > c > 0$). Hence, $f_z(z)$ tends to a constant in any interval of length 2π. The result follows as in (5-37) and Prob. 5-20.

7-25 Since $a_n - a \to 0$, we conclude that

$$E\{(\mathbf{x}_n - a)^2\} = E\{[(\mathbf{x}_n - a_n) + (a_n - a)]^2\}$$

$$= E\{(\mathbf{x}_n - a_n)^2\} + 2(a_n - a)E\{\mathbf{x}_n - a_n\} + (a_n - a)^2 \to 0$$

as $n \to \infty$.

7-26 If $E\{\mathbf{x}_n \mathbf{x}_m\} \to a$ as $n,m \to \infty$, then, given $\varepsilon > 0$, we can find a number n_o such that

$$E\{\mathbf{x}_n \mathbf{x}_n\} = a + \theta(n,m) \qquad |\theta| < \varepsilon \qquad \text{if } n,m > 0$$

Hence,

$$E\{(\mathbf{x}_n - \mathbf{x}_m)^2\} = E\{\mathbf{x}_n^2\} + E\{\mathbf{x}_m^2\} - 2E\{\mathbf{x}_n \mathbf{x}_m\}$$

$$= a + \theta_1 + a + \theta_2 - 2(a + \theta_3) = \theta_1 + \theta - 2\theta_3$$

and since $|\theta_1 + \theta_2 - 2\theta_3| < 4\,\varepsilon$ for any ε, it follows that

$E\{(\mathbf{x}_n - \mathbf{x}_m)^2\} \to 0$, hence (Cauchy) $\mathbf{x}_n$ tends to a limit.

<u>Conversely</u> If $\mathbf{x}_n \to \mathbf{x}$ in the MS sense, then

$E\{(\mathbf{x}_n - \mathbf{x})^2\} \to 0$. Furthermore,

$$E\{\mathbf{x}_n^2\} \to E\{\mathbf{x}^2\} \qquad E\{\mathbf{x}\,\mathbf{x}_n\} \to E\{\mathbf{x}^2\}$$

because (see Prob. 6-51)

$$E^2\{\mathbf{x}_n^2 - \mathbf{x}^2\} = E^2\{(\mathbf{x}_n - \mathbf{x})(\mathbf{x}_n + \mathbf{x})\}$$

$$\le E\{(\mathbf{x}_n - \mathbf{x})^2\}E\{(\mathbf{x}_n + \mathbf{x})^2\} \to 0$$

$$E^2\{\mathbf{x}(\mathbf{x}_n - \mathbf{x})\} \le E\{\mathbf{x}^2\}E\{(\mathbf{x}_n - \mathbf{x})^2\} \to 0$$

Similarly, $E\{(\mathbf{x}_n - \mathbf{x})(\mathbf{x}_m - \mathbf{x})\} \to 0$. Hence,

$$E\{\mathbf{x}_n\mathbf{x}_m\} + E\{\mathbf{x}^2\} - E\{\mathbf{x}\,\mathbf{x}_n\} - E\{\mathbf{x}\,\mathbf{x}_m\} \to 0$$

Combining, we conclude that $E\{\mathbf{x}_n\mathbf{x}_m\} \to E\{\mathbf{x}^2\}$.

7-27 $$E\{\mathbf{x}_k\} = 0 \qquad\qquad E\{\mathbf{x}_k^2\} = \sigma_k^2$$

$$E\left\{\left(\sum_{k=n_1}^{n_2} \mathbf{x}_k\right)^2\right\} = \sum_{k=n_1}^{n_2} E\{\mathbf{x}_k^2\}$$

If $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$, then given $\varepsilon > 0$, we can find n_o such that $\sum_{k=n+1}^{n+m} \sigma_k^2 < \varepsilon$

for any m and $n > n_o$. Thus

$$E\{(\mathbf{y}_{n+m} - \mathbf{y}_n)^2\} = E\left\{\left(\sum_{k=n+1}^{n+m} \mathbf{x}_k\right)^2\right\} = \sum_{k=n+1}^{n+m} \sigma_k^2 < \varepsilon$$

This shows that (Cauchy), $\mathbf{y}_k$ converges in the MS sense. The proof of the converse is similar.

7-28 If $f_i(x) = c\,e^{-cx}U(x)$ then $\Phi_i(s) = \dfrac{c}{c-s}$

$$\Phi(s) = \Phi_1(s) \cdots \Phi_n(s) = \frac{c^n}{(c-s)^n}$$

Hence (see Example 5-29) $f(x) = \dfrac{c^n x^{n-1}}{(n-1)!} e^{-cx}\, U(x)$

7-29 From Prob. 7-28 it follows that f(x) is the density of the sum $\mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_n$. Furthermore,

$$E\{\mathbf{x}\} = \frac{n}{c} \qquad\qquad \sigma_x^2 = \frac{n}{c^2}$$

From the central limit theorem it follows, therefore, that for large n, the Erlang density is nearly equal to a normal curve with mean n/c and variance n/c^2.

7-30 $\quad E\{\underset{\sim}{r}_i\} = 500 \qquad \sigma_i^2 = 50^2/3$

$\underset{\sim}{r} = \underset{\sim}{r}_1 + \underset{\sim}{r}_2 + \underset{\sim}{r}_3 + \underset{\sim}{r}_4 \qquad E\{\underset{\sim}{r}\} = 2{,}000 \qquad \sigma_r^2 = 10^4/3$

Thus, $\underset{\sim}{r}$ is approximately $N(2000;10^2/\sqrt{3})$

$P\{1900 \le \underset{\sim}{r} \le 2100\} = 2\ G\ (\frac{100\sqrt{3}}{100}) - 1 = 0.9169.$

7-31 The RVs $\underset{\sim}{x}_i$ are independent with (see Prob. 5-37)

$$f_i(x) = \frac{c_i}{\pi(c_i^2 + x^2)} \qquad \Phi_i(\omega) = e^{-c_i|\omega|}$$

In that case, (7-104) does not hold because

$$\int_{-\infty}^{\infty} x^\alpha f(x)dx = \frac{c_i}{\pi}\int_{-\infty}^{\infty}\frac{x^\alpha}{c_i^2 + x^2}\ dx = \infty \qquad \alpha > 2$$

In fact, the density of $\underset{\sim}{x} = \underset{\sim}{x}_1 + \cdots + \underset{\sim}{x}_n$ is Cauchy with parameter $c = c_1 + \cdots + c_n$ because

$$\Phi(\omega) = e^{-c_1|\omega|} \cdots e^{-c_n|\omega|} = e^{-(c_1 + \cdots + c_n)|\omega|}$$

7-32 In this problem, $\sigma_z^2 = E\{|\underset{\sim}{z}|^2\} = E\{\underset{\sim}{x}^2 + \underset{\sim}{y}^2\} = 2\sigma^2$

$$f_z(x) = f_x(x)f_y(y) = \frac{1}{2\pi\sigma^2}e^{-(x^2+y^2)/2\sigma^2} = \frac{1}{2\pi\sigma_z^2}e^{-|z|^2/\sigma_z^2}$$

$$\Phi_z(\Omega) = \Phi_x(u)\Phi_y(v) = \exp\left\{-\frac{1}{2}\sigma^2(u^2+v^2)\right\} = \exp\left\{-\frac{1}{4}\sigma_z^2|\Omega|^2\right\}$$

CHAPTER 8

8-1 (a) From (8-11) with γ=.95, u=.975, $z_{.975} \simeq 2$, σ=0.1, and n=9 we obtain

$$c = \frac{z_u\sigma}{\sqrt{n}} = 0.066$$

(b) From (8-11) with c=91.01-91=0.05mm:

$$z_u = \frac{c\sqrt{n}}{\sigma} = 1.5 \qquad u = .933 \qquad \gamma = .866$$

8-2 (a) From (8-11) with σ=1 and n=4: $\bar{x} \pm \sigma z_u/\sqrt{n} \simeq 203 \pm 1$mm

(b) From (8-12) with δ=.05: $c = \sigma/\sqrt{n\delta} = 2.236$mm

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8-3 From (8-4)with γ=.9, u=.95: $\bar{x} \pm z_u\sigma/\sqrt{n} = 25{,}000 \pm 1{,}028$ miles

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8-4 We wish to find n such that $P\{|\bar{x}-a|<0.2\} = 0.95$ where $a=E\{\bar{x}\}$.From (8-4) it follows with u=.975 and σ=0.1mm that

$$\frac{z_u\sigma}{\sqrt{n}} \le 0.2, \text{ hence, } n=1$$

8-5 In this problem, x is uniform with $E\{x\}=\theta$ and $\sigma^2=4/3$.We can use, however, the normal approximation for $\bar{x}$ because n=100. With γ=.95, (8-11) yields the interval

$$\bar{x} \pm z_{.975}\ \sigma\sqrt{n} = 30 \pm 0.227$$

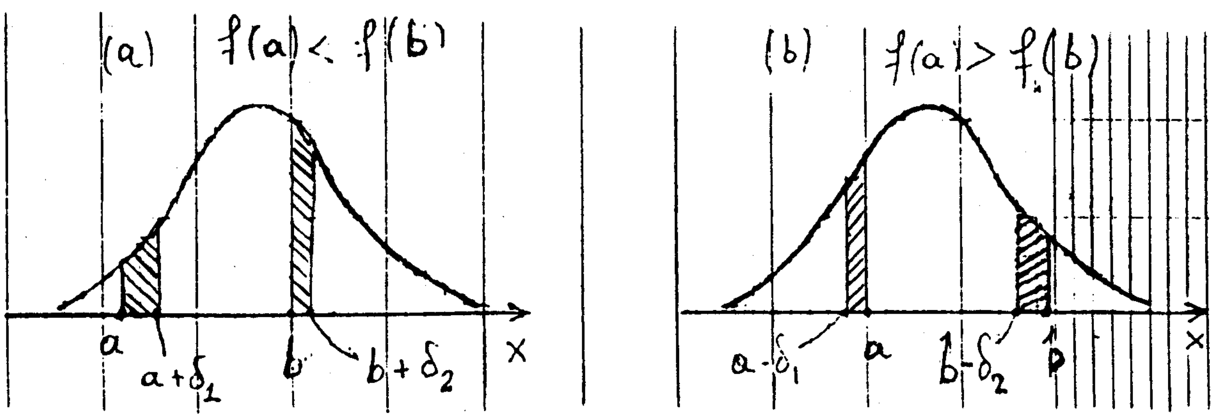
8-6 We shall show that if $f(x)$ is a density with a single maximum and $P\{a<x<b\}=\gamma$, then $b-a$ is minimum if $f(a)=f(b)$. The density $xe^{-x}U(x)$ is a special case. It suffices to show that $b-a$ is not minimum if $f(a)< f(b)$ or $f(a)>f(b)$.

Suppose first that $f(a)< f(b)$ as in figure (a). Clearly. $f'(a)> 0$ and $f'(b)< 0$, hence, we can find two constants $\delta_1>0$ and $\delta_2>0$ such that $P\{a+\delta_1 < x < b+\delta_2\} = \gamma$ and

$$f(a) < f(a+\delta_1) < f(b+\delta_2) < f(b)$$

From this it follows that $\delta_1 > \delta_2$, hence, the length of thenew interval $(a+\delta_1, b+\delta_2)$ is smaller than $b-a$.

If $f(a) > f(b)$, we form the interval $(a-\delta_1, b-\delta_2)$ (Fig. 8-6b) and proceed similarly.



Special case. If $f(x)=xe^{-x}$ then (see Problem 4-9) $F(x)=1-e^{-x}-xe^{-x}$, hence,

$$P\{a < x < b\} = e^{-a} + ae^{-a} - e^{-b} - be^{-b} = .95$$

And since $f(a)=f(b)$, the system

$$ae^{-a} = be^{-b} \qquad e^{-a} - e^{-b} = .95$$

results. Solving, we obtain $a \simeq 0.04 \quad b \simeq 5.75$.

A numerically simpler solution results if we set

$$0.025 = P\{x \le a\} = F(a) \qquad 0.025 = P\{x > b\} = 1 - F(b)$$

as in (9-5). This yields the system

$$0.025 = 1 - e^{-a} - ae^{-a} \qquad 0.025 = e^{-b} + be^{-b}$$

Solving, we obtain $a=0.242$, $b=5.572$. However, the length $5.572-0.242=5.33$ of the resulting interval is larger than the length $4.75-0.04=4.71$ of the optimum interval.

8-7 We start with the general problem: We observe the n samples x_i of an $N(\eta,10)$ RV x and we wish to predict the value x of x at a future trial in terms of the average $\bar{x}$ of the observations. If η is known, we have an ordinary prediction problem. If it is unknown, we must first estimate it. To do so, we form the RV $w=x-\bar{x}$. This RV is

$N(0,\sigma_w)$ where $\sigma_w^2=\sigma_x^2+\sigma_{\bar{x}}^2=\sigma^2+\sigma^2/n$. With $c=z_{.975}\sigma_w$$ it follows that $P\{|w| < c\}=.95$. Hence

$$P\{\bar{x} - c < x < \bar{x} + c\} = 0.95$$

For $n=20$ and $\sigma=10$ the above yields $\sigma_w=10.25$ and $c \simeq 20.5$. Thus, we can expect with .95 confidence coefficient that our bulb will last at least $80-20.5=59.5$ and at most $80+20=100.5$ hours.

8-8 The time of arrival of the 40th patient is the sum $x_1 + \cdots + x_n$ of $n=39$ RVs with exponential distribution. We shall estimate the mean $\eta=1/\theta$ of x in terms of its sample mean $\bar{x}=240/39=6.15$ minutes using two methods. The first is approximate (large n) and is based on (8-11).

Normal approximation. With $\lambda=\eta$ and$z_{.975}/\sqrt{39}=0.315$:

$$P\left\{\frac{\bar{x}}{1.315} < \eta < \frac{\bar{x}}{0.685}\right\} = .95 \qquad 4.68 < \eta < 8.98 \text{ minutes}$$

Exact solution. The RVs $\underset{\sim}{x}_i$ are i.i.d. with exponential distribution. From this and (7-52) it follows that their sum

$\underset{\sim}{y}=\underset{\sim}{x}_1 + \cdots + \underset{\sim}{x}_n = n\underset{\sim}{\bar{x}}$ has an Erlang distribution:

$$\Phi_y(s) = \frac{\theta^n}{(\theta-s)^n} \qquad f_y(y) = \frac{\theta^n}{(n-1)!} y^{n-1} e^{-\theta y} U(y)$$

and the RV $\underset{\sim}{z}=2\theta\underset{\sim}{y} = 2n\theta\underset{\sim}{\bar{x}}$ has a $\chi^2(2n)$ distribution:

$$f_z(z) = \frac{1}{2\theta} f_y\left(\frac{z}{2\theta}\right) U(z) = \frac{z^{n-1}}{2^n(n-1)!} e^{-z/2} U(z)$$

Hence,

$$P\left\{ \chi^2_{\delta/2}(2n) < \frac{2n\bar{x}}{\eta} < \chi^2_{1-\delta/2}(2n) \right\} = \gamma = 1-\delta$$

Since $\chi^2_{.025}(78) = 54.6$, $\chi^2_{.975}(78) = 104.4$, and $2n\bar{x}=480$, this yields the interval

$$4.60 < \eta < 8.79 \text{ minutes}$$

8-9 From (8-19) with $\bar{x}=2{,}550/200=12.75$ $n=200$ and $z_u \simeq 2$

$$\lambda^2 - 25.52\,\lambda + 12.75^2 = 0 \qquad \lambda_1 = 12.255 < \lambda < 13.265 = \lambda_2$$

8-10 From (8-21) with $\bar{x}=2{,}080/4000=0.52$, $n=4{,}000$ and $z_u \simeq 2.326$.

$$p_{1,2} \approx \bar{x} \pm z_u \sqrt{\frac{\bar{x}(1-\bar{x})}{n}} = .52 \pm .018$$

Hence, $.502 < p < .538$.

8-11 (a) In this problem, $\bar{x}=0.40$, $n=900$ and $z_u \simeq 2$. From (8-21) : Margin of error

$$\pm\ 100\ z_u\sqrt{\frac{\bar{x}(1-\bar{x})}{n}} = \pm\ 3.27\%$$

(b) We wish to find z_u. From (9-21) and Table 1a:

$$100 z_u\sqrt{\frac{\bar{x}(1-\bar{x})}{n}} = 2 \qquad z_u = 1.225 \qquad u = .89$$

This yields the confidence coefficient $\gamma = 2u - 1 = .78$

8-12 From (8-21) with $\bar{x}=0.29$ and $z_u=2$:

$$z_u\sqrt{\frac{\bar{x}(1-\bar{x})}{n}} = 0.04 \qquad\qquad n > \frac{\bar{x}(1-\bar{x})}{.04^2}\, z^2_u = 515$$

8-13 The problem is to find n such that [see (8-20)] $z_u\sqrt{\frac{p(1-p)}{n}} \le .02$ *for every p.* Since $z_u \simeq 2$ and $p(1-p) \le 1/4$, this is the case if

$$z_u\sqrt{1/4n} \le .02 \qquad n \ge 2,500$$

8-14 From (8-36) with $k=1$

$$f(p) = \begin{cases} 5 & .4 < p < .6 \\ 0 & \text{otherwise} \end{cases} \qquad P\{k=1\} = 5\int_{.4}^{.6} p\,dp = .5 = \frac{1}{\gamma}$$

$$f_p(p|1) = \begin{cases} 10p & .4 < p < .6 \\ 0 & \text{otherwise} \end{cases} \qquad \hat{p} = 10\int_{.4}^{.6} p^2 dp = .5067$$

8-15 From Prob. 8-8: $f_{\bar{x}}(\bar{x}|\theta) = \frac{(\theta n)^n}{(n-1)!} \bar{x}^{n-1} e^{-n\theta\bar{x}}$

From (8-32):

$$f_\theta(\theta\,|\bar{x}) = \frac{(c+n\bar{x})^{n+1}}{n!} \theta^n e^{-(c+n\bar{x})\theta}$$

From (8-31):

$$\hat{\theta} = \frac{(c+n\bar{x})^{n+1}}{n!} \int_0^\infty \theta^{n+1} e^{-(c+n\bar{x})\theta} d\theta = \frac{n+1}{c+n\bar{x}}$$

8-16 The sum $n\bar{x}$ is a Poisson RV with mean $n\theta$ (see Prob. 8-8). In the context of Bayesian estimation, this means that

$$f_{\bar{x}}(\bar{x}|\theta) = e^{-n\theta} \frac{(n\theta)^k}{k!} \qquad k = n\bar{x} = 0,1,\ldots$$

Inserting into (8-32), we obtain [see (4-76)]

$$f_\theta(\theta|\bar{x}) = \frac{(n+c)^{n\bar{x}+b+1}}{\Gamma(n\bar{x}+b+a)} \theta^{n\bar{x}+b} e^{-(n+c)\theta}$$

and (8-31) yields

$$\hat{\theta} = \frac{(n+c)^{n\bar{x}+b+1}}{\Gamma(n\bar{x}+b+1)} \frac{\Gamma(n\bar{x}+b+2)}{(n+c)^{n\bar{x}+b+2}} = \frac{n\bar{x}+b+1}{n+c} \xrightarrow[n\to\infty]{} \bar{x}$$

8-17 From (8-17) with $t_{.95}(9)=2.26$

$$\bar{x} \pm \frac{t_u s}{\sqrt{n}} = 90 \pm 3.57 \qquad 86.43 < \eta \cdot 93.57$$

From (8-24) with $\chi^2_{.975}(9)=19.02$, $\chi^2_{.025}(9)=2.70$.

$$\frac{9 \times 5^2}{19.02} = 11.83 < \sigma^2 < \frac{9 \times 5^2}{2.70} = 83.33 \qquad 3.44 < \sigma < 9.13$$

8-18 The RVs x_i/σ are N(0,1), hence, the sum $z=(x_1^2 + \cdots + x^2_{10})/\sigma^2$ has a $\chi^2(10)$ distribution. This yields

$$P\{\chi^2_{.025}(10) < z < \chi^2_{.975}(10)\} = .95$$

$$\chi^2_{.025}(10) = 3.25 < \frac{4}{\sigma^2} < \chi^2_{.975}(10) = 20.48$$

$$0.442 < \sigma < 1.109$$

8-19 From (8-23) with $n=4, \chi^2_{.025}(4)=0.48,\ \chi^2_{.975}(4)=11.14$

$$n\hat{v} = .1^2 + .15^2 + .05^2 + .04^2 = .0366$$

$$\frac{.0366}{.048} > \sigma^2 > \frac{.0366}{11.14} \qquad 0.276 > \sigma > 0.057$$

8-20 In this problem $n=3,\ x_1+x_2+x_3=9.8$

$$f(x,c) \sim c^4x^3e^{-cx} \qquad f(X,c) = c^{4n}(x_1 \ldots x_n)^{3n}e^{-cn\bar{x}}$$

$$\frac{\partial f(X,c)}{\partial c} = \left(\frac{4n}{c} - n\bar{x}\right)f(X,\theta) = 0 \qquad \hat{c} = \frac{4}{\bar{x}} = 1.224$$

8-21 The joint density

$$f(X,c) = c^n e^{-cn(\bar{x}-x_0)} \qquad x_i > x_0$$

has an interior maximum if

$$\frac{\partial f(X,c)}{\partial c} = 0 \qquad \hat{c} = \frac{1}{\bar{x}-x_0}$$

8-22 The probability

$$p = 1 - F_x(200) = e^{-200c}$$

of the event $\{x > 200\}$ is a monoton decreasing function of c. To find the ML estimate $\hat{c}$ of c it suffices to find the ML estimate $\hat{p}$ of p. From Example 8-28 it follows with k=62 and n=80 that

$$\hat{p} = \frac{62}{80} = .775 \text{ hence}$$

$$\hat{c} = -\frac{1}{200} \ln \hat{p} = 0.0013$$

8-23 The samples of x are the integers x_i and the joint density of the RVs x_i equals

$$f(X,\theta) = e^{-n\theta} \prod \frac{\theta^{x_i}}{x_i!} = e^{-n\theta} \frac{\theta^{n\bar{x}}}{\Pi x_i!}$$

Hence, $f(X,\theta)$ is maximum if $-n + n\bar{x}/\theta = 0$. This yields $\hat{\theta} = \bar{x}$

8-24 If $L = \ln f(x,\theta)$ then

$$\frac{\partial L}{\partial \theta} = \frac{1}{f}\frac{\partial f}{\partial \theta} \qquad \frac{\partial^2 L}{\partial \theta^2} = \frac{1}{f}\frac{\partial^2 f}{\partial \theta^2} - \frac{1}{f^2}\left(\frac{\partial f}{\partial \theta}\right)^2 \qquad \frac{\partial^2 L}{\partial \theta^2} + \left(\frac{\partial L}{\partial \theta}\right)^2 = \frac{1}{f}\frac{\partial^2 f}{\partial \theta^2}$$

But

$$E\left\{\frac{1}{f}\frac{\partial^2 f}{\partial \theta^2}\right\} = \int_R \frac{1}{f}\frac{\partial^2 f}{\partial \theta^2} f dX = 0 \quad \text{hence} \quad E\left\{\frac{\partial^2 L}{\partial \theta^2} + \left(\frac{\partial L}{\partial \theta}\right)^2\right\} = 0$$

8-25 (a) From (8-307): Critical region

$$\bar{x} > c = \eta_0 + z_{1-\alpha}\frac{\sigma}{\sqrt{n}} = 8+2.326 \times \frac{2}{8} = 8.58$$

If η=8.7, then $\eta_q = \frac{8.7-8}{218} = 2.8$

$$\beta(\eta) = G(2.36 - 2.8) = .32$$

(b) We assume that α=.01, β(8.7)=.05 and wish to find n and c.

$$G(z_{1-\alpha}-\eta_q) = \beta \qquad z_{1-\alpha}-\eta_q = z_\beta$$

$$\eta_q = z_{.99} - z_{.05} = 4.97 = \frac{8.7-8}{2/\sqrt{n}}$$

$$n = 129 \qquad c = 8 + \frac{2}{\sqrt{129}}z_{.99} = 8.41$$

8-26 Our objective is to test the composite null hypothesis $\eta>\eta_0=28$ against the hypothesis $\eta<\eta_0$. Consider first the simple null hypothesis $\eta=\eta_0=28$. In this case, we can use (8-301) with

$$q = \frac{\bar{x}-\eta_0}{s/\sqrt{n}} \qquad \bar{x} = \frac{1}{17}\sum x_i = 27.67 \qquad s^2 = \frac{1}{16}\sum(x_i-\bar{x}) = 17.6$$

This yields s=4.2 and q=-0.33. Since

$$q_u = t_u(n-1) = t_{0.05}(16) = -1.95 < -0.33$$

we conclude that the evidence does not support the rejection of the hypothesis η=28. The resulting OC function $\beta_0(\eta)$ is determined from (9-60c).

If η_0>28, then the corresponding value of q is larger than -0.33. From this it follows that the evidence does not support the hypothesis η_0 for any $\eta_0 > 28$. We note, however, that the corresponding OC function $\beta(\eta)$ is smaller than the function $\beta_0(\eta)$ obtained from (8-301) with $\eta_0=28$.

8-27 From (8-297) with $q_u = t_u(n-1)$: Critical region $|\bar{x}-\eta_0| > t_{1-\alpha/2}(n-1)s/\sqrt{n}$

1. $\underline{\alpha = .1}$ $\quad t_{.95}(63) = 1.67 \quad |\bar{x}-8| > 1.67 \times 1.5/8 = 0.313$

Since $\bar{x}=7.7$ is in the interval 8 ± 0.317, we accept H_0

2. $\underline{\alpha = .01}$ $\quad t_{.995}(63) = 2.62 \quad |\bar{x}-8| > 2.62 \times 1.5/8 = 0.49$

Since $\bar{x}=7.7$ is outside the interval 8 ± 0.49, we reject H_0.

8-28 We assume that the RVs $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are normal and independent. We form the difference $\underset{\sim}{w}=\underset{\sim}{\bar{x}}-\underset{\sim}{\bar{y}}$ of their sample means

$$\underset{\sim}{\bar{x}} = \frac{1}{16}\sum_{i=1}^{16} \underset{\sim}{x}_1 \qquad \underset{\sim}{\bar{y}} = \frac{1}{26}\sum_{i=1}^{26} \underset{\sim}{y}_i$$

and use as test statistic the ratio

$$\underset{\sim}{q} = \frac{\underset{\sim}{w}}{\sigma_w} \qquad \sigma_w^2 = \frac{\sigma_x^2}{16} + \frac{\sigma_y^2}{26}$$

The RV $\underset{\sim}{q}$ is normal with $\sigma_q=1$ and under hypothesis H_0, $E\{\underset{\sim}{q}\}=0$. We can, therefore, use (8-307) because $q_u = z_u$. To find q, we must determine σ_w. Since σ_x and σ_y are not specified, we shall use the approximations $\sigma_x \simeq s_x = 1.1$ and $\sigma_y \simeq s_y = 0.9$. This yields

$$\sigma_w^2 \simeq \frac{1.1^2}{16} + \frac{0.9^2}{26} = 0.107 \qquad q = \frac{\bar{x}-\bar{y}}{\sigma_w} = \frac{0.4}{0.327} = 1.223$$

Since $z_{0.95} = 1.645 > 1.223$, we accept H_0.

8-29 (a) In this problem, $n=64$, $k=22$, $p_0=q_0=0.5$

$$q = \frac{k-np_0}{\sqrt{np_0q_0}} = 2.5 \qquad z_{\alpha/2} = -z_{1-\alpha/2} \simeq -2$$

Since 2.5 is outside the interval (2, -2), we reject the fair coin hypothesis [see (8-313)].

(b) From (8-313) with $n=16$, $p_0=q_0=0.5$:

$$\frac{k_1-np_0}{\sqrt{np_0q_0}} = z_{\alpha/2} \qquad \frac{k_2-np_0}{\sqrt{np_0q_0}} = -z_{\alpha/2}$$

This yields $k_1 = 8-2\times2 = 4$, $k_2 = 8+2\times2 = 12$

8-30 We shall use as test statistic the sum

$$\mathbf{q} = \mathbf{x}_1 + \cdots + \mathbf{x}_m \qquad n = 22$$

The critical region of the test is $q<q_\alpha$ where $q=x_1+\cdots+x_n=90$ [see (8-301)].

The RV $\underset{\sim}{q}$ is Poisson distributed with parameter $n\lambda$. Under hypothesis H_0, $\lambda=\lambda_0=5$; hence, $\eta_q=n\lambda_0=110=\sigma_q^2$. To find q_α we shall use the normal approximation. With $\alpha=0.05$ this yields

$$q_\alpha = n\lambda_0 + z_\alpha \sqrt{n\lambda_0} = 90-17.25 = 72.75$$

Since 90 > 72.75, we accapt the hypothesis that $\lambda=5$.

8-31 From (9-75) with $n=102$ and $p_{0i}=1/6$

$$q = \sum_{i=1}^{6} \frac{(k_i-17)^2}{17} = 2 \qquad \chi^2_{.95}(5) \simeq 11$$

Since 2<11, we accept the fair die hypothesis.

8-32 Uniformly distributed integers from 0 to 9 means that they have the same probability of appearing. With $m=10$, $p_{01}=.1$, and $n=1{,}000$, it follows from (8-325) that

$$q = \sum_{j=0}^{9} \frac{(n_j-100)^2}{100} = 17.76 \qquad \chi^2_{.95}(9) = 16.92$$

Since 17.76 > 16.92, we reject the uniformity hypothesis.

8-33 In this problem

$$f(x,\theta) = e^{-\theta}\frac{\theta^x}{x!} \qquad f(X,\theta) = \frac{e^{-n\theta}\,\theta^{n\bar{x}}}{x_1!\cdots x_n!}$$

$f(X,\theta)$ is maximum for $\theta=\theta_m=\bar{x}$. And since $\theta_{m0}=\theta_0$ we conclude that

$$\lambda(X) = \frac{e^{-n\theta_0}\theta_0^{n\bar{x}}}{e^{-n\bar{x}}\bar{x}^{n\bar{x}}} \qquad w = -2\ ln\ \lambda = 2n(\theta_0-\bar{x}) + \bar{x}\ ln(\bar{x}/\theta_0)$$

With $n=50$, $\theta_0=20$, $\bar{x}=1{,}058/50=21.16$, this yields $w=3$. Since $m_0=1$, $m=1$, and $\chi^2_{.95}(1)=3.84>3$, we accept H_0.

8-34 We form the RVs

$$\underset{\sim}{z} = \sum_{i=1}^{m}\left(\frac{x_i-\eta_x}{\sigma_x}\right)^2 \qquad \underset{\sim}{w} = \sum_{i=1}^{n}\left(\frac{y_i-\eta_y}{\sigma_y}\right)^2$$

These RVs are $\chi^2(m)$ and $\chi^2(n$ respectively. If $\sigma_x=\sigma_y$, then

$$\underset{\sim}{q} = \frac{z/m}{\underset{\sim}{w}/n}$$

Hence (see Prob. 6-23), $\underset{\sim}{q}$ has a Snedecor distribution. To test the hypothesis $\sigma_x=\sigma_y$, we use (8-297) where $q_u=F_u(m,n)$ is the tabulated u percentile of the Snedecor distribution. This yields the following test:

Accept H_o iff $\qquad F_{\alpha/2}(m,n) < q < F_{1-\alpha/2}(m,n)$.

8-35 If $\underset{\sim}{x}$ has a student-t distribution, then $f(-x)=f(x)$, hence (see Prob. 6-75)

$$E\{\underset{\sim}{x}\} = 0 \qquad \sigma_x^2 = E\{\underset{\sim}{x}^2\} = \frac{n}{n-2}$$

8-36 (a) Suppose that the probability $P(A)$ that player A wins a set equals $p=1-q$. He wins the match in five sets if he wins two of the first four sets and the fifth set. Hence, the probability $p_5(A)$ that he wins in five equals $6p^3q^2$. Similarly, the probability $p_5(B)$ that player B wins in five equals $6p^2q^3$. Hence,

$$p_5 = p_5(A) + p_5(B) = 6p^3q^2 + 6p^2q^3 = 6p^2q^2$$

is the probability that the match lasts five sets. If $p=q=1/2$, then $p_5=3/8$.

(b) Suppose now that $P(A) = \underset{\sim}{p}$ is an RV with density $f(p)$. In this case,

$$\underset{\sim}{p}_5 = 6\underset{\sim}{p}^2(1-\underset{\sim}{p}^2)$$

is an RV. We wish to find its best bayesian estimate. Using the MS criterion, we obtain

$$\hat{p}_5 = E\{\underset{\sim}{p}_5\} = \int_0^1 6p^2(1-p^2)f(p)dp$$

If $f(p)=1$, then $\hat{p}_5 = 1/5$.

8-37 Given

$$f_v(v) \sim e^{-v^2/2\sigma^2} \qquad f_\theta(\theta) \sim e^{-(\theta-\theta_o)^2/2\sigma_o^2}$$

To show that

$$f_\theta(\theta|X) \sim e^{-(\theta-\theta_1)^2/2\sigma_1^2}$$

where

$$\frac{1}{\sigma_1^2} \equiv \frac{1}{\sigma_o^2} + \frac{n}{\sigma^2} \qquad \theta_1 \equiv \frac{\sigma_1^2}{\sigma_o^2}\theta_o + \frac{n\sigma_1^2}{\sigma^2}\bar{x}$$

<u>Proof</u>

$$f_x(x|\theta) = f_v(x-\theta) \sim \exp\left\{-\frac{(x-\theta)^2}{2\sigma^2}\right\}$$

$$f(X)|\theta \sim \exp\left\{-\frac{1}{2\sigma^2}\sum(x_i-\theta)^2\right\}$$

Since $\sum(x_i-\theta)^2 = \sum(x_i-\bar{x})^2 + n(\bar{x}-\theta)^2$, we conclude from (8-32) omitting factors that do not depend on θ that

$$f(\theta|X) \sim \exp\left\{-\frac{1}{2}\left[\frac{(\theta-\theta_o)^2}{\sigma_o^2} + \frac{n(\bar{x}-\theta)^2}{\sigma^2}\right]\right\}$$

The above bracket equals

$$\left(\frac{1}{\sigma_o^2} + \frac{n}{\sigma^2}\right)\theta^2 - 2\left(\frac{\theta_o}{\sigma_o^2} + \frac{n\bar{x}}{\sigma^2}\right)\theta + \cdots = \frac{1}{\sigma_1^2}(\theta^2 - 2\theta\theta_1) + \cdots$$

and (i) follows.

8-38 The likelihood function of X equals

$$f(X,\theta) = \frac{1}{(\sqrt{2\pi\theta})^n} \exp\left\{-\frac{1}{2\theta}\sum (x_i-\eta)^2\right\}$$

where $\theta = \sigma^2$ is the unknown parameter. Hence

$$L(X,\theta) = -\frac{n}{2}\ln(2\pi\theta) - \frac{1}{2\theta}\sum (x_i-\eta)^2$$

$$\frac{\partial L(X,\theta)}{\partial\theta} = -\frac{n}{2\theta} + \frac{1}{2\theta^2}\sum (x_i-\theta)^2 = 0 \qquad \hat{\theta} = \frac{1}{n}\sum (x_i-\eta)^2$$

8-39 The estimators $\hat{\underset{\sim}{\theta}}_1$ and $\hat{\underset{\sim}{\theta}}_2$ have the same variance because otherwise one or the other would not be best. Thus

$$E\{\hat{\underset{\sim}{\theta}}_1\} = E\{\hat{\underset{\sim}{\theta}}_2\} = \theta \qquad \text{var}\,\hat{\underset{\sim}{\theta}}_1 = \text{var}\,\hat{\underset{\sim}{\theta}}_2 = \sigma^2$$

If $\hat{\underset{\sim}{\theta}} = \frac{1}{2}(\hat{\underset{\sim}{\theta}}_1 + \hat{\underset{\sim}{\theta}}_2)$, then

$$E\{\hat{\underset{\sim}{\theta}}\} = \theta \qquad \sigma_{\hat{\theta}}^2 = \frac{1}{2}(\sigma^2 + \sigma^2 + 2r\sigma^2) = \frac{1}{2}(1+r)\sigma^2$$

where σ is the correlation coefficient of $\hat{\underset{\sim}{\theta}}_1$ and $\hat{\underset{\sim}{\theta}}_2$. If $r<1$ then $\sigma_{\hat{\theta}}<\sigma$ which is impossible. Hence, $r=1$ and $\hat{\underset{\sim}{\theta}}_1 = \hat{\underset{\sim}{\theta}}_2$ (see Prob. 6-53).

8-40 $k_1+k_2-np_1-np_2 = n-n(p_1+p_2) = 0$; Hence, $|k_1-np_1| = |k_2-np_2|$

$$\frac{(k_1-np_1)^2}{np_1} + \frac{(k_2-np_2)^2}{np_2} = (k_1-np_1)^2\left(\frac{1}{np_1} + \frac{1}{np_2}\right) = \frac{(k_1-np_1)^2}{np_1p_2}$$

8.41 It is given that

$$E\{T(X)\} = \int_{-\infty}^{\infty} T(X)\, f(X;\theta)\, dx = \psi(\theta),$$

so that after differentiating and making use of (8-81) we get

$$\int_{-\infty}^{\infty} T(X)\, \frac{\partial f(X;\theta)}{\partial \theta}\, dx = \psi^{'}(\theta) \qquad (8.41-1)$$

Also using (8-80)

$$\int_{-\infty}^{\infty} \psi(\theta)\, \frac{\partial f(X;\theta)}{\partial \theta}\, dx = 0, \qquad (8.41-2)$$

and the above two expressions give

$$\int_{-\infty}^{\infty} [T(X) - \psi(\theta)]\, \frac{\partial f(X;\theta)}{\partial \theta}\, dx = \psi^{'}(\theta) \qquad (8.41-3)$$

But

$$\frac{\partial f(X;\theta)}{\partial \theta} = \frac{1}{f(X;\theta)}\, \frac{\partial \log f(X;\theta)}{\partial \theta}$$

so that (8.41-3) simplifies to

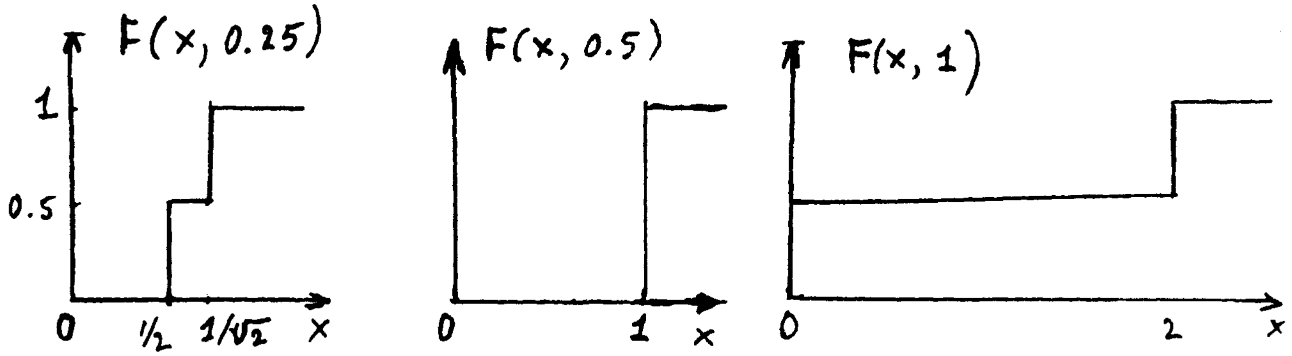
$$\int_{-\infty}^{\infty} \left[\{T(X) - \psi(\theta)\}\sqrt{f(X;\theta)}\right] \left[\sqrt{f(X;\theta)}\ \frac{\partial \log f(X;\theta)}{\partial \theta}\right] dx = \psi^{'}(\theta)$$

and application of Cauchy-Schwarz inequality as in (8-89)-(8-92), Text gives

$$E\left[\{T(X) - \psi(\theta)\}^2\right] \geq \frac{[\psi^{'}(\theta)]^2}{E\left\{\left(\frac{\partial \log f(X;\theta)}{\partial \theta}\right)^2\right\}}$$

9-1 (a) $E\{\mathbf{x}(t)\} = t + 0.5\sin\pi t$

$$\mathbf{x}(t,\text{ heads}) = \sin\pi t = \begin{cases} 1/\sqrt{2} & t = 0.25 \\ 1 & t = 0.5 \\ 0 & t = 1 \end{cases} \qquad \mathbf{x}(t,\text{ tails}) = 2t = \begin{cases} 0.5 \\ 1 \\ 2 \end{cases}$$



9-2

$$\mathbf{x}(t) = e^{\mathbf{a}t}$$

$$\eta(t) = \int_{-\infty}^{\infty} e^{at} f_a(a)\,da \qquad R(t_1,t_2) = \int_{-\infty}^{\infty} e^{at_1} e^{at_2} f_a(a)\,da$$

From (5-16) with $x = g(a) = e^{ta}$ $\quad g'(a) = t\,e^{ta} = tx$

$$f(x,t) = \frac{1}{x|t|} f_a\left(\frac{1}{t}\ln x\right)U(x)$$

9-3 As we know, $E\{\underset{\sim}{x}(t)\} = \lambda t$ and var $\underset{\sim}{x}(t) = \lambda^2t^2$ [see (9-18)]. But $E\{\underset{\sim}{x}(9) = 6\}$ by assumption, hence, $\lambda = 2/3$

(a) $E\{\underset{\sim}{x}(8)\} = 24$ var $\underset{\sim}{x}^2(t) = 24^2$

(b) The RV $\underset{\sim}{x}(2)$ is Poisson distributed with parameter $2\lambda = 6$. Hence,

$$P\{\underset{\sim}{x}(2) \le 3\} = e^{-2\lambda} \sum_{k=0}^{3} \frac{(2\lambda)^k}{k!}$$

(c) The RVs $\underset{\sim}{z} = \underset{\sim}{x}(2)$ and $\underset{\sim}{w} = \underset{\sim}{x}(4) - \underset{\sim}{x}(2)$ are independent and Poisson distributed with parameter 2λ. Hence,

$$P\{\underset{\sim}{z}=k\} = e^{-2\lambda}\frac{(2\lambda)^k}{k!} \qquad P\{\underset{\sim}{z} = k, \underset{\sim}{w} = m\} = e^{-4\lambda}\frac{(2\lambda)^k}{k!}\frac{(2\lambda)^m}{m!}$$

$$P\{\underset{\sim}{x}(4) \le 5 \mid \underset{\sim}{x}(2) \le 3\} = \frac{P\{z\le 3,\ w\le 5-z\}}{P\{\underset{\sim}{z}\le 3\}} \qquad P\{\underset{\sim}{z} \le 3\} = \sum_{k=0}^{3} p\{\underset{\sim}{z}=k\}$$

$$P\{\underset{\sim}{z} \le 3, \underset{\sim}{w} \le 5 - \underset{\sim}{z}\} = \sum_{k=0}^{3}\ \sum_{m=0}^{5-k} P\{\underset{\sim}{z} = k, \underset{\sim}{w} = m\}$$

9-4
$$\underset{\sim}{x}(t) = U(t - \underset{\sim}{c}) \qquad \underset{\sim}{y}(t) = \delta(t - \underset{\sim}{c}) = \underset{\sim}{x}'(t)$$

For t_1 or $t_2 < 0$, $R(t_1,t_2) = 0$; for t_1 and $t_2 > T$, $R(t_1,t_2) = 1$. Otherwise,

$$R_x(t_1,t_2) = \frac{1}{T}\min(t_1,t_2) \qquad \frac{\partial R_x}{\partial t_1} = \frac{1}{T}U(t_1 - t_2) \qquad -\frac{\partial^2 R_x}{\partial t_1 \partial t_2} = \frac{1}{T}\delta(t_1 - t_2)$$

From this and (9-105) it follows that $T R_y(t_1 - t_2) = \delta(t_1 - t_2)$ for $0 < t_1, t_2 < T$ and 0 otherwise.

9-5 $\underset{\sim}{a} - \underset{\sim}{b}t = 0$ iff $t = \underset{\sim}{t}_1 = \underset{\sim}{a}/\underset{\sim}{b}$. Setting $\sigma_1 = \sigma_2 = \sigma$ and $r = 0$ in (6-63), we obtain

$$P\{0 < \underset{\sim}{t}_1 < T\} = \frac{1}{2} + \frac{1}{\pi}\text{arc tan}\, T - \left(\frac{1}{2} + \frac{1}{\pi}\text{arc tan } 0\right)$$

9-6 The equations

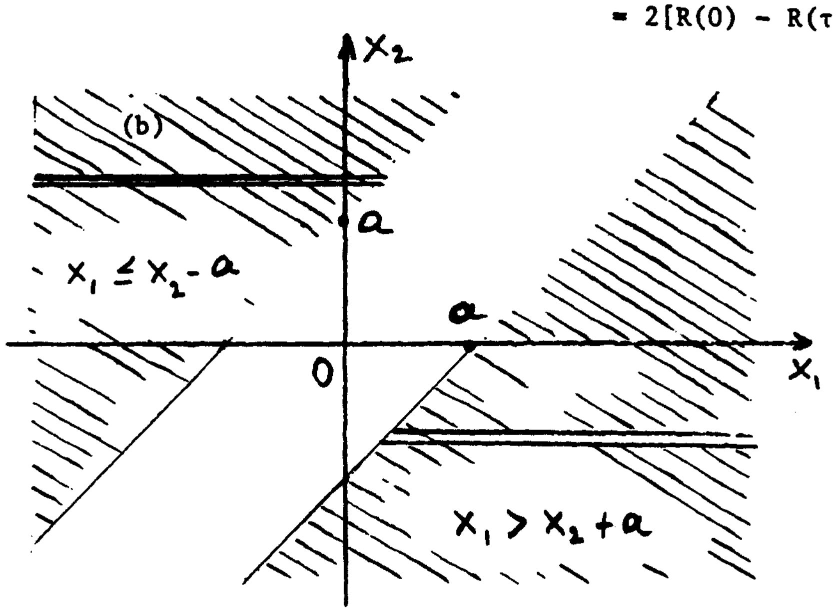
$$\underset{\sim}{w}''(t) = \underset{\sim}{\nu}(t)U(t) \qquad \underset{\sim}{w}(0) = \underset{\sim}{w}'(0) = 0$$

specify a system with input $\underset{\sim}{\nu}(t)U(t)$ and impulse response $h(t) = t\,U(t)$. Hence [see (9-100)]

$$E\{\underset{\sim}{w}^2(t)\} = q(t)U(t) * t^2U(t) = \int_0^t (t-\tau)^2 q(\tau)\,d\tau$$

9-7 (a) From (5-88) with $\underset{\sim}{x} = \underset{\sim}{x}(t+\tau) - \underset{\sim}{x}(t)$, and (8-101):

$$P\{|\underset{\sim}{x}(t+\tau) - \underset{\sim}{x}(t)| \geq a\} \leq \frac{E\{[\underset{\sim}{x}(t+\tau) - \underset{\sim}{x}(t)]^2\}}{a^2}$$
$$= 2[R(0) - R(\tau)]/a^2$$



The above probability equals the mass in the regions (shaded) $x_2 - x_1 > a$ and $x_2 - x_1 < -a$. Hence,

$$P\{|\underset{\sim}{x}(t+\tau) - \underset{\sim}{x}(t)| \geq a\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{x_2-a} f(x_1,x_2;\tau)\,dx_1dx_2 + \int_{-\infty}^{\infty}\int_{x_2+a}^{\infty} f(x_1,x_2;\tau)\,dx_1dx_2$$

9-8 (a) The RV $\underset{\sim}{x}(t)$ is normal with zero mean and variance $E\{\underset{\sim}{x}^2(t)\} = R(0)=4$, hence it is $N(0,2)$ and $P\{\underset{\sim}{x}(t)\le 3\} = F(3) = G(1.5) = 0.933$

(b) $E\{[\underset{\sim}{x}(t+1) - \underset{\sim}{x}(t-1)]\} = 2[R(0)-R(2)] = 8(1-e^{-4})$

9-9 If $\underset{\sim}{x}(t) = \underset{\sim}{c}e^{j(\omega t+\theta)}$ and $\eta_c = 0$ then

$$\eta_x(t) = \eta_c e^{j(\omega t+\theta)} = 0 \qquad R_{xx}(t+\tau,t) = \sigma_c^2 e^{j\omega\tau}$$

hence, $\underset{\sim}{x}(t)$ is WSS. We shall prove the converse:

If the process $\underset{\sim}{x}(t) = \underset{\sim}{c}\,w(t)$ is WSS, then $\eta_c=0$ and $w(t) = e^{j(\omega t+\theta)}$ within a constant factor.

<u>Proof</u> $\eta_x(t) = \eta_c w(t)$ is independent of t; hence, $\eta_c=0$. The function $R_{xx}(t_1,t_2) = \sigma_c^2 w(t_1)w^*(t_2)$ depends only on $\tau=t_1-t_2$; hence, $w(t+\tau)w^*(t)=g(\tau)$. With $\tau=0$ this yields

$$|w(t)|^2 = g(0) = \text{constant} \qquad w(t) = ae^{j\phi(t)}$$

$$w(t+\tau)w^*(t) = a^2 e^{j[\phi(t+\tau)-\phi(t)]}$$

Hence the difference $\phi(t+\tau)-\phi(t)$ depends only on τ:

$$\phi(t+\tau)-\phi(t) = f(\tau) \tag{i}$$

From this it follows that, if $\phi(t)$ is continuous then, $\phi(t)$ is a linear function of t. To simplify the proof, we shall assume that $\phi(t)$ is differentiable. Differentiating with respect to t, we obtain $\phi'(t+\tau) = \phi'(t)$ for every τ. With t=0 this yields

$$\phi'(\tau) = \phi'(0) = \text{constant} \qquad \phi(\tau) = a\tau+b$$

9-10 We shall show that if $\underset{\sim}{x}(t)$ is a normal process with zero mean and $\underset{\sim}{z}(t) = \underset{\sim}{x}^2(t)$, then $C_{zz}(\tau) = 2C_{xx}^2(\tau)$.

From (7-61): If the RVs $\underset{\sim}{x}_k$ are normal and $E(\underset{\sim}{x}_k)=0$, then

$$E\{\underset{\sim}{x}_1\underset{\sim}{x}_2\underset{\sim}{x}_3\underset{\sim}{x}_4\} = E\{\underset{\sim}{x}_1\underset{\sim}{x}_2\}\, E\{\underset{\sim}{x}_3\underset{\sim}{x}_4\} + E\{\underset{\sim}{x}_1\underset{\sim}{x}_3\}\, E\{\underset{\sim}{x}_2\underset{\sim}{x}_4\} + E\{\underset{\sim}{x}_1\underset{\sim}{x}_4\}\, E\{\underset{\sim}{x}_2\underset{\sim}{x}_3\}$$

With $\underset{\sim}{x}_1=\underset{\sim}{x}_2=\underset{\sim}{x}(t+\tau)$ and $\underset{\sim}{x}_3 = \underset{\sim}{x}_4 = \underset{\sim}{x}(t)$, we conclude that the autocorrelation of $\underset{\sim}{z}(t)$ equals

$$E\{\underset{\sim}{x}^2(t+\tau)\underset{\sim}{x}^2(t)\} = E^2\{\underset{\sim}{x}^2(t+\tau)\} + 2E^2\,\{\underset{\sim}{x}(t+\tau)\underset{\sim}{x}(t)\} = R_{xx}{}^2(0) + 2R_{xx}{}^2(\tau)$$

And since $R_{xx}(\tau)=C_{xx}(\tau)$, and $E\{\underset{\sim}{z}(t)\} = R_{xx}(0)$, the above yields

$$C_{zz}(\tau) = R_{zz}(\tau) - E^2\{\underset{\sim}{z}(t)\} = 2C_{xx}{}^2(\tau)$$

9-11 $\underset{\sim}{y}''(t) + 4\underset{\sim}{y}'(t) + 13\underset{\sim}{y}(t) = \underset{\sim}{x}(t)$ all t

The process $\underset{\sim}{y}(t)$ is the response of a system with input $\underset{\sim}{x}(t) = 26 + \underset{\sim}{\nu}(t)$ and

$$H(s) = \frac{1}{s^2+45+13} \qquad h(t) = \frac{1}{3}\, e^{-2t}\sin 3tU(t)$$

Since $\eta_x = 26$, this yields $\eta_y = \eta_x H(0) = 2$. The centered process $\underset{\sim}{\tilde{y}}(t) = \underset{\sim}{y}(t)-\eta_y$ is the response due to $\underset{\sim}{\nu}(t)$. Hence [see (9-100)]

$$E\{\underset{\sim}{\tilde{y}}^2(t)\} = q\int_0^\infty h^2(t)dt = \frac{10}{104}$$

With b=4 and c=13 it follows that (see Example 9-276)

$$R_{\underset{\sim}{yy}}(\tau) = \frac{10}{104}e^{-2|\tau|}\left(\cos 3\tau - \frac{2}{3}\sin 3|\tau|\right) + 4$$

If $\underset{\sim}{\nu}$ is normal, then $\underset{\sim}{y}(t)$ is normal with mean 2 and variance $R_{yy}(0) - 4 = 10/104$; hence,

$$P\{\underset{\sim}{y}(t) \le 3\} = G\left(\frac{3-2}{0.31}\right) = G(3.24)$$

9-12 $E\{\underset{\sim}{y}(t)\} = 0 \qquad R_{yy}(t_1,t_2) = \dfrac{R_{xx}(t_1,t_2)}{f(t_1)f(t_2)} = w(t_1-t_2)$

$$E\{\underset{\sim}{z}(t)\} = 0 \qquad R_{zz}(t_1,t_2) = \frac{R_{xx}(t_1,t_2)}{\sqrt{q(t_1)}\,\sqrt{q(t_2)}} = \delta(t_1-t_2)$$

because $q(t_1)\delta(t_1-t_2) = \sqrt{q(t_1)}\,\sqrt{q(t_2)}\,\delta(t_1-t_2)$.

9-13 From (9-181) and the identity $4ab \le (a+b)^2$ it follows that

$$|R_{xy}(\tau)|^2 \le R_{xx}(0)R_{yy}(0) \le \frac{1}{4}[R_{xx}(0)+R_{yy}(0)]^2$$

9-14 Clearly (stationarity assumption)

$$E\{|\underset{\sim}{x}^*(t) - \underset{\sim}{y}^*(t)|^2\} = E\{|\underset{\sim}{x}(0) - \underset{\sim}{y}(0)|^2\} = 0$$

Furthermore,

$$E\{\underset{\sim}{x}(t+\tau)[\underset{\sim}{x}^*(t) - \underset{\sim}{y}^*(t)]\} = R_{xx}(\tau) - R_{xy}(\tau)$$

and [see (9-177)]

$$|E\{\underset{\sim}{x}(t+\tau)[\underset{\sim}{x}^*(t) - \underset{\sim}{y}^*(t)]\}|^2 \le E\{|\underset{\sim}{x}(t+\tau)|^2\}E\{|\underset{\sim}{x}^*(t) - \underset{\sim}{y}^*(t)|^2\} = 0$$

Hence, $R_{xx}(\tau) - R_{xy}(\tau) = 0$; similarly, $R_{yy}(\tau) = R_{xy}(\tau)$

9-15 $$E\{|\underset{\sim}{x}(t+\tau) - \underset{\sim}{x}(t)|^2\} = E\{[\underset{\sim}{x}(t+\tau) - \underset{\sim}{x}(t)][\underset{\sim}{x}^*(t+\tau) - \underset{\sim}{x}^*(t)]\}$$

$$= R(0) - R(\tau) - R^*(\tau) + R(0) = 2R(0) - 2\,\underline{\text{Re}}\,R(\tau)$$

9-16 From $\Phi(1) = \Phi(2) = 0$ it follows that

$$E\{\cos\underset{\sim}{\phi}\} = E\{\sin\underset{\sim}{\phi}\} = E\{\cos 2\underset{\sim}{\phi}\} = E\{\sin 2\underset{\sim}{\phi}\} = 0$$

Hence, $E\{\underset{\sim}{x}(t)\} = \cos\omega t\, E\{\cos\underset{\sim}{\phi}\} - \sin\omega t\, E\{\sin\underset{\sim}{\phi}\} = 0$

and as in Example 9-14

$$2\cos[\omega(t+\tau)+\underset{\sim}{\phi}]\cos(\omega t+\underset{\sim}{\phi}) = \cos\omega\tau + \cos(2\omega t + \omega\tau + 2\underset{\sim}{\phi})$$

$$2R_x(\tau) = \cos\omega\tau$$

If $\underset{\sim}{\phi}$ is uniform in $(-\pi,\pi)$, then

$$\Phi(\lambda) = \frac{\sin\pi\omega}{\pi\omega} \qquad \Phi(1) = \Phi(2) = 0$$

9-17 (a) $\underset{\sim}{x}(t_1)\underset{\sim}{x}(t_2) = [\underset{\sim}{x}(t_1) - \underset{\sim}{x}(0)][\underset{\sim}{x}(t_2) - \underset{\sim}{x}(t_1) + \underset{\sim}{x}(t_1) - x(0)]$

$$R(t_1,t_2) = E\{[\underset{\sim}{x}(t_1) - \underset{\sim}{x}(0)]^2\} = E\{\underset{\sim}{x}^2(t_1)\} = R(t_1,t_1)$$

(b) If $t_1 + \varepsilon < t_2$, then $R_y(t_1,t_2) = 0$; if

$t_1 < t_2 < t_1 + \varepsilon$ then

$$E\{[\underset{\sim}{x}(t_1+\varepsilon) - \underset{\sim}{x}(t_1)][\underset{\sim}{x}(t_2+\varepsilon) - \underset{\sim}{x}(t_2)]\} = q(t_1 + \varepsilon - t_2)$$

Hence, $\varepsilon^2 R_y(\tau) = q(\varepsilon - |\tau|)$ for $|\tau| = |t_2 - t_1| \leq \varepsilon$

9-18

$$E\{\underset{\sim}{x}(t)\underset{\sim}{y}(t)\} = \int_{-\infty}^{\infty} E\{\underset{\sim}{x}(t)\underset{\sim}{x}(t-\tau)\}h(\tau)d\tau$$

$$= \int_{-\infty}^{\infty} R_{xx}(t,t-\tau)h(\tau)d\tau = \int_{-\infty}^{\infty} q(t)\delta(\tau)h(\tau)d\tau = h(0)q(t)$$

9-19 As in Prob. 5-14, $g(x) = 6 + 3\,F_x(x)$. In this case,

$E\{\underset{\sim}{x}^2(t)\} = 4$, hence, $\underset{\sim}{x}(t)$ is $N(0,2)$ and $F_x(x) = G(x/2)$

9-20 $\underset{\sim}{x}(t)$ is SSS, hence, $P\{x(t) \leq y\} = F_x(y)$ does not depend on t. The RVs $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{x}(t)$ are independent, hence, [see (6-238)]

$$F_y(y) = P\{\underset{\sim}{x}(t-\underset{\sim}{\varepsilon}) \leq y \mid \underset{\sim}{\varepsilon} = \varepsilon\} = P\{\underset{\sim}{x}(t-\varepsilon) \leq y \mid \underset{\sim}{\varepsilon} = \varepsilon\}$$

$$= P\{\underset{\sim}{x}(t-\varepsilon) < y\} = F_x(y)$$

is independent of t. Similarly for higher order distributions.

9-21 $E\{x(t)\} = \eta =$ constant, hence, [see (9-102)] $E\{x'(t)\} = 0$

Furthermore, $R_{xx}(-\tau) = R_{xx}(\tau)$. hence, $R'_{xx}(0) = 0$ and (10-97) yields

$$E\{x(t)x'(t)\} = R_{xx'}(0) = 0$$

9-22 (a) $E\{z\,w\} = R_x(2) = 4e^{-4}$ $\qquad E\{z^2\} = E\{w^2\} = R_x(0) = 4$

$$E\{(z+w)^2\} = R_x(0) + R_x(0) + 2R_x(2) = 8(1+e^{-4})$$

(b) z is $N(0,2)$ $\qquad P\{z < 1\} = F_z(1) = G(1/2)$

$r_{zw} = e^{-4}$, $\quad f_{zw}(z,w)$: $N(0,0;2,2;e^{-4})$

9-23 The RV $x'(t)$ is normal with zero mean and variance

$$E\{|x'(t)|^2\} = R_{x'x'}(0) = -R''(0)$$

Hence, $P\{x'(t) \le a\} = F_{x'}(a) = G\left[a/\sqrt{|R''(0)|}\right]$

9-24 The function arc sin x is odd, hence, it can be expanded into a sine series in the interval (-1,1):

$$\alpha(x) \equiv \arcsin x = \sum_{n=1}^{\infty} b_n \sin n\pi x \qquad |x| \le 1$$

$$b_n = \int_{-1}^{1} \alpha(x)\sin n\pi x\,dx = -\frac{1}{n\pi}\int_{-1}^{1} \alpha(x)\,d\cos n\pi x$$

$$= -\frac{\alpha(x)\cos n\pi x}{n\pi}\Bigg|_{-1}^{1} + \frac{1}{n\pi}\int_{-1}^{1} \cos n\pi x\,d\alpha(x)$$

$$= -\frac{\cos n\pi}{n} + \frac{1}{n\pi}\int_{-\pi/2}^{\pi/2} \cos(n\pi \sin x)\,dx$$

and the result follows because [see (9-81)]

$$R_y(\tau) = \frac{2}{\pi}\arcsin\frac{R_x(\tau)}{R_x(0)} \qquad J_o(z) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos(z\sin x)\,dx$$

9-25 As we know [see (5-100) and (6-193)]

$$E\{e^{j\omega \underset{\sim}{x}(t)}\} = \exp\{-\frac{R(0)}{2}\omega^2\}$$

$$E\{e^{j[\omega_1 \underset{\sim}{x}(t+\tau)+\omega_2 \underset{\sim}{x}(t)]}\} = \exp\{-\frac{1}{2}[R(0)\omega_1^2 + 2R(\tau)\omega_1\omega_2 + R(0)\omega_2^2]\}$$

Hence, with $j\omega = a$

$$E\{I\, e^{a\underset{\sim}{x}(t)}\} = \exp\{\frac{a^2}{2} R_x(0)\}\, I$$

$$E\{I\, e^{a\underset{\sim}{x}(t+\tau)} I\, e^{a\underset{\sim}{x}(t)}\} = I^2 \exp\{a\,[R_x(0) + R_x(\tau)]\}$$

9-26 (a) $R_y(\tau) = a^2 E\{\underset{\sim}{x}[c(t+\tau)]\underset{\sim}{x}(ct)\} = a^2 R(c\tau)$

(b) If $\underset{\sim}{z}_\varepsilon(t) = \sqrt{\varepsilon}\,\underset{\sim}{x}(\varepsilon t)$ then $R_{z_\varepsilon}(\tau) = \varepsilon R_x(\varepsilon\tau)$ [as in (a)].

If $\delta > 0$ is sufficiently small and $\phi(t)$ is continuous at the origin, then

$$\int_{-\delta}^{\delta} R_{z_\varepsilon}(\tau)\phi(\tau)d\tau \simeq \phi(0)\int_{-\delta}^{\delta} \varepsilon R_x(\varepsilon\tau)\, d\tau$$

$$= \phi(0)\int_{-\varepsilon\delta}^{\varepsilon\delta} R(\tau)d\tau \xrightarrow[\varepsilon\to\infty]{} \phi(0)\int_{-\infty}^{\infty} R(\tau)d\tau = q\,\phi(0)$$

Hence, $R_{z_\varepsilon}(\tau) \longrightarrow q\,\delta(\tau)$ as $\varepsilon \to \infty$.

9-27

$$\underset{\sim}{y}(t) = \int_{t-T}^{t} \underset{\sim}{x}(\tau)h(t-\tau)d\tau$$

Hence, $\underset{\sim}{y}(t_1)$ and $\underset{\sim}{y}(t_2)$ depend linearly on the values of $\underset{\sim}{x}(t)$ in the intervals $(t_1 - T,\ t_1)$ and $(t_2 - T,\ t_2)$ respectively. If $|t_1 - t_2| > T$ then these intervals do not overlap and since $E\{\underset{\sim}{x}(\tau_1)\underset{\sim}{x}(\tau_2)\} = 0$ for $\tau_1 \neq \tau_2$, it follows that $E\{\underset{\sim}{y}(t_1)\underset{\sim}{y}(t_2)\} = 0$.

9-28 (a)

$$I(t) = E\left\{\int_0^t \int_0^t h(t,\alpha)\underset{\sim}{x}(\alpha)h(t,\beta)\underset{\sim}{x}(\beta)\ d\alpha d\beta\right\}$$

$$= \int_0^t \int_0^t h(t,\alpha)h(t,\alpha)q(\alpha)\delta(\alpha-\beta)d\alpha d\beta = \int_0^t h^2(t,\alpha)q(\alpha)d\alpha$$

(b) If $\underset{\sim}{y}'(t) + c(t)\underset{\sim}{y}(t) = \underset{\sim}{x}(t)$, then $\underset{\sim}{y}(t)$ is the output of a linear time-varying system as in (a) with impulse response $h(t,\alpha)$ such that

$$\frac{\partial h(t,\alpha)}{\partial t} + c(t)h(t,\alpha) = \delta(t-\alpha) \qquad h(\alpha^-,\alpha) = 0$$

or equivalently

$$\frac{\partial h(t,\alpha)}{\partial t} + c(t)h(t,\alpha) = 0 \qquad t > 0 \quad h(\alpha^+,\alpha) = 1$$

This yields

$$h(t,\alpha) = e^{-\int_\alpha^t c(\tau)d\tau}$$

Hence, if

$$I(t) = \int_0^t h^2(t,\alpha)q(\alpha)d\alpha \quad \text{then} \quad I'(t) + 2c(t)I(t) = q(t)$$

because the impulse response of this equation equals

$$e^{-2\int_\alpha^t c(\tau)d\tau} = h^2(t,\alpha)$$

9-29 (a) If $\underset{\sim}{y}'(t) + 2\underset{\sim}{y}(t) = \underset{\sim}{x}(t)$, then $\underset{\sim}{y}(t) = \underset{\sim}{x}(t)*h(t)$

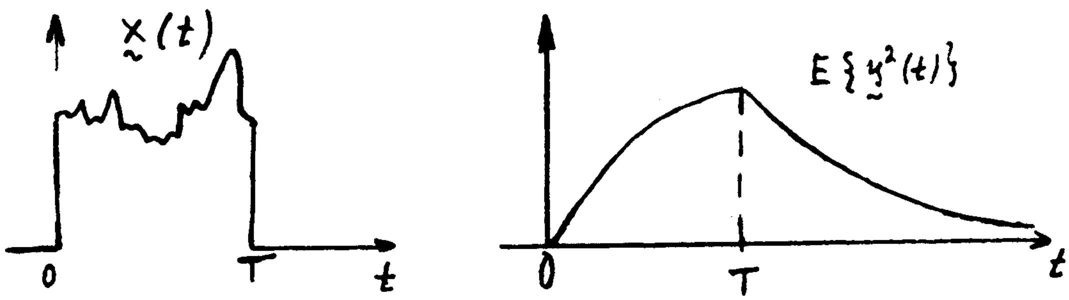
where $h(t) = e^{-2t}U(t)$ and with $q(t) = 5$, (10-90) yields

$$E\{\underset{\sim}{y}^2(t)\} = 5*e^{-4t}U(t) = 5\int_0^{\infty} e^{-4\tau}d\tau = \frac{5}{4}$$

(b) As in (a) with $q(t) = 5U(t)$. Hence, for $t > 0$

$$E\{\underset{\sim}{y}^2(t)\} = 5U(t)*e^{-4t}U(t) = 5\int_0^{t} e^{-4\tau}d\tau = \frac{5}{4}(1 - e^{-4t})$$

9-30



From (9-90) with $q(t) = N[U(t) - U(t-T)]$

$$E\{\underset{\sim}{y}^2(t)\} = \begin{cases} AN\int_0^{t} e^{-2\alpha(t-\tau)}d\tau = \frac{AN}{2\alpha}(1-e^{-2\alpha t}) & 0 \le t < T \\ \\ AN\int_0^{T} e^{-2\alpha(t-\tau)}d\tau = \frac{AN}{2\alpha}(e^{2\alpha T} - 1)e^{-2\alpha t} & t > T \end{cases}$$

9-31 Since $\underset{\sim}{x}(t)$ is WSS, the moments of S equal the moments of

$$\underset{\sim}{z} = \int_{-5}^{5} \underset{\sim}{x}(t)dt$$

Hence, (see Fig. 9-5)

$$E\{\underset{\sim}{s}^2\} = \int_{-5}^{5}\int_{-5}^{5} R_x(t_1-t_2)dt_1dt_2 = \int_{-10}^{10}(10-|\tau|)R_x(\tau)d\tau$$

$$E\{\underset{\sim}{s}\} = 80 \qquad \sigma_s^2 = 2\int_0^{10}(10-\tau)10e^{-2\tau}d\tau$$

9-32 $\underset{\sim}{y}(t) = \underset{\sim}{x}(t)*h(t) \qquad h(t) = e^{-2t}U(t)$

(a) $E\{\underset{\sim}{y}^2(t)\} = 5*e^{-4t}\ U(t) = 5/4$

$$R_{xy}(t_1,t_2) = 5\ \delta(t_1-t_2)*e^{-2t_2}\ U(t_2) = 5e^{-2(t_2-t_1)}U(t_2-t_1)$$

$$R_{yy}(t_1,t_2) = 5e^{-2(t_2-t_1)}U(t_2-t_1)*e^{-2t_1}\ U(t_1)$$

$$= \frac{5}{4}\ e^{-2|t_1-t_2|}$$

The first equation follows from (9-100) with $q(t) = 5$; the second from (9-94) with $R_{xx}(t_1,t_2) = 5\delta(t_1-t_2)$, and the third from (9-96).

(b) With $R_{xx}(t_1,t_2) = 5\delta(t_1-t_2)U(t_1)U(t_2)$, (9-94) and (9-96) yield the following: For t_1 or $t_2 < 0$, $R_{xy}(t_1,t_2) = R_{yy}(t_1,t_2) = 0$. For $0 < t_1 < t_2$

$$R_{xy}(t_1,t_2) = 5\delta(t_1-t_2)*e^{-2t_2} = 5\ e^{-2t_2}$$

$$R_{yy}(t_1,t_2) = \int_0^{t_1} 5\ e^{-2(t_1-\tau)}\ e^{-2(t_1-\tau)}\ d\tau = \frac{5}{4}\ e^{-2(t_2-t_1)}(1-e^{-4t_1})$$

9-33 $$\int_{-\infty}^{\infty} e^{-\alpha\tau^2} e^{-s\tau} d\tau = e^{s^2/4\alpha} \int_{-\infty}^{\infty} e^{-\alpha(\tau + s/2\alpha)^2} d\tau = \sqrt{\frac{\pi}{\alpha}}\, e^{s^2/4\alpha}$$

This yields

$$e^{-\alpha\tau^2} \longleftrightarrow \sqrt{\frac{\pi}{\alpha}}\, e^{-\omega^2/4\alpha}$$

$$e^{-\alpha\tau^2} \cos \omega_o \tau \longleftrightarrow \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \left[e^{-(\omega-\omega_o)^2/4\alpha} + e^{-(\omega+\omega_o)^2/4\alpha} \right]$$

9-34 $$G(x_1,x_2;\omega) = \int_{-\infty}^{\infty} f(x_1,x_2;\tau) e^{-j\omega\tau} d\tau$$

$$R(\tau) = E\{\underset{\sim}{x}(t+\tau)\underset{\sim}{x}(t)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2\, f(x_1,x_2;\tau) dx_1 dx_2$$

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} e^{-j\omega\tau} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2\, f(x_1,x_2;\tau) dx_1\, dx_2 d\tau$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 \int_{-\infty}^{\infty} e^{-j\omega\tau}\, f(x_1,x_2;\tau) d\tau dx_1 dx_2$$

9-35 The process $\underset{\sim}{y}(t) = \underset{\sim}{x}(t+a) - \underset{\sim}{x}(t-a)$ is the output of a system with input $\underset{\sim}{x}(t)$ and system function

$$H(\omega) = e^{ja\omega} - e^{-ja\omega} = 2j \sin a\omega$$

Hence [see (9-150)]

$$S_y(\omega) = 4 \sin^2 a\omega\, S_x(\omega) = (2 - e^{j2a\omega} - e^{-j2a\omega}) S_x(\omega)$$

$$R_y(\tau) = 2 R_x(\tau) - R_x(\tau + 2a) - R_x(\tau - 2a)$$

9-36 Since $S(\omega) \geq 0$, we conclude with (9-136) that

$$R(0) - R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)(1 - \cos \omega\tau)\, d\omega$$

$$\geq \frac{1}{8\pi} \int_{-\infty}^{\infty} S(\omega)(1 - \cos 2\omega\tau)\, d\omega = \frac{1}{4}\,[R(0) - R(2\tau)]$$

and the result follows for $n = 1$. Repeating the above, we obtain the general result.

9-37 **From** (6-197)

$$E\{x^2(t+\tau)x^2(t)\} = E\{x^2(t+\tau)\}E\{x^2(t)\} + 2E^2\{x^2(t+\tau)x^2(t)\}$$

Hence,

$$R_y(\tau) = R_x^2(0) + 2\,R_x^2(\tau) = I^2(1 + e^{-2\alpha|\tau|} + e^{-2\alpha|\tau|}\cos 2\beta\tau)$$

$$S_y(\omega) = \left[2\pi\delta(\omega) + \frac{4\alpha}{4\alpha^2 + \omega^2} + \frac{2\alpha}{4\alpha^2 + (\omega - 2\beta)^2} + \frac{2\alpha}{4\alpha^2 + (\omega + 2\beta)^2}\right]$$

Furthermore,

$$\eta_y = E\{x^2(t)\} = R_x(0) \qquad C_y(\tau) = 2R_x^2(\tau)$$

9-38

$$\int_{-\infty}^{\infty} S(\omega)\left|\sum_i a_i\, e^{j\omega\tau_i}\right|^2 d\omega = \int_{-\infty}^{\infty} S(\omega) \sum_{i,k} a_i a_k^* \, e^{j\omega(\tau_i - \tau_k)}\, d\omega$$

$$= \sum_{i,k} a_i a_k^* \, R(\tau_i - \tau_k) \geq 0$$

9-39 (a) $S(s) = \frac{1}{1+s^4} = \frac{1}{(s^2+\sqrt{2}s+1)(s^2-\sqrt{2}s+1)}$

A special case of example 9-27b with $b = \sqrt{2}$, $c = 1$. Hence,

$$R(\tau) = \frac{1}{2\sqrt{2}} e^{-|\tau|/\sqrt{2}} \left(\cos\frac{\tau}{\sqrt{2}} + \sin\frac{|\tau|}{\sqrt{2}}\right)$$

(b) From the pair $e^{-2|\tau|} \leftrightarrow 4/(4+\omega^2)$ and the convolution theorem it follows that

$$e^{-2|\tau|} * e^{-2|\tau|} \leftrightarrow \frac{16}{(4+\omega^2)^2}$$

Hence, for $\tau > 0$

$$16\,R(\tau) = \int_{\infty}^{\infty} e^{-2|x|}\, e^{-2|\tau-x|}dx = \int_{-\infty}^{0} e^{2x}\, e^{-2(\tau-x)}dx$$

$$+ \int_0^{\tau} e^{-2x}\, e^{-2(\tau-x)}\,dx + \int_{\tau}^{\infty} e^{-2x}\, e^{2(\tau-x)}\,dx = \frac{1}{2} e^{-2\tau}(1+2\tau)$$

And since $R(-\tau) = R(\tau)$, the above yields

$$e^{-2|\tau|}\,\frac{1+2|\tau|}{32} \longleftrightarrow \frac{1}{(4+\omega^2)^2}$$

9-40 $H^*(-s^*)\Big|_{s=j\omega} = H^*(j\omega) \qquad H^*(1/z^*)\Big|_{z=e^{j\omega T}} = H^*(e^{j\omega T})$

Hence

$$H(s)H^*(-s^*)\Big|_{s=j\omega} = |H(j\omega)|^2 \qquad H(z)H^*(1/z^*)\Big|_{z=j\omega T} = |H(e^{j\omega T})|^2$$

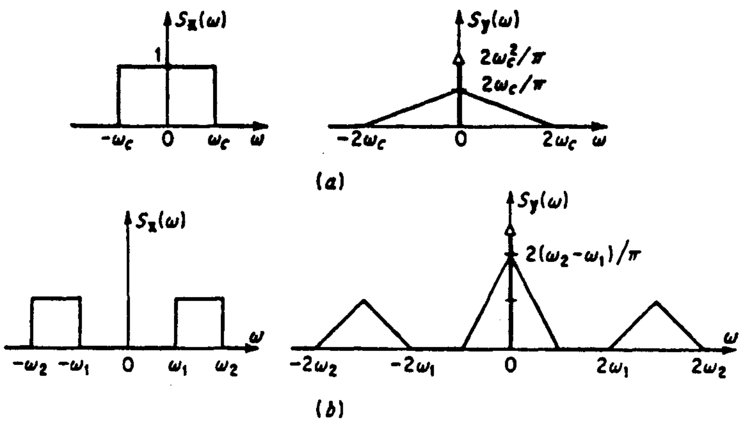
9-41 From (6-197)

$$R_y(\tau) = E\{\underset{\sim}{x}^2(t+\tau)\underset{\sim}{x}^2(t)\}$$

$$= E\{\underset{\sim}{x}^2(t+\tau)\}E\{\underset{\sim}{x}^2(t)\} + 2\,E^2\{\underset{\sim}{x}(t+\tau)\underset{\sim}{x}(t)\} = R_x^2(0) + 2\,R_x^2(\tau)$$

From the above and the frequency convolution theorem it follows that

$$S_y(\omega) = 2\pi R_x^2(0)\delta(\omega) + \frac{1}{\pi}\,S_x(\omega) * S_x(\omega)$$



9-42 $\underset{\sim}{y}(t) = 2\underset{\sim}{x}(t) + 3\underset{\sim}{x}'(t)$ $\qquad \eta_x = 5 \qquad C_{xx}(\tau) = 4e^{-2|\sigma|}$

The process $\underset{\sim}{y}(t)$ is the output of the system H(s) = 2+3s with input $\underset{\sim}{x}(t)$. Hence, η_y=5H(0)=10

$$S_{yy}{}^c(\omega) = S_{xx}{}^c(\omega)|2+3j\omega|^2 = \frac{16}{4+\omega^2}(4+9\omega^2) = 144 - \frac{512}{4+\omega^2} = S_{yy}(\omega) - 2\pi\eta_y{}^2\delta(\omega)$$

9-43 (a) $y'(t) + 3y(t) = x(t)$, $R_{xx}(\tau) = 5\delta(\tau)$. The process $y(t)$ is the output of the system

$$H(s) = \frac{1}{s+3} \qquad h(t) = e^{-3t}U(t)$$

Hence, [see (9-100) and (9-150)]

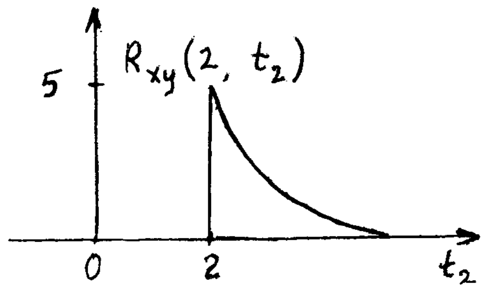
$$E\{y^2(t)\} = 5\int_0^{\infty} e^{-6t}dt = \frac{5}{6}$$

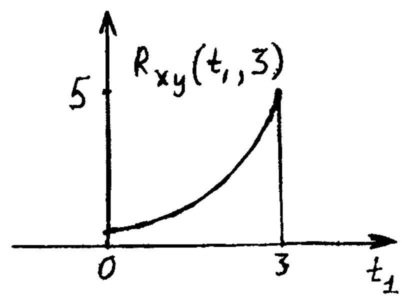
$$S_{yy}(\omega) = \frac{5}{\omega^2+9} \qquad R_{yy}(\tau) = \frac{5}{6}\,e^{-3|\tau|}$$

(b) As in Example 9-18:

$$E\{y^2(t)\} = 5\int_0^{t} e^{-6\alpha}d\alpha = \frac{5}{6}\,(1-e^{-6t}) \qquad t > 0$$

$$R_{xy}(t_1,t_2) = 5e^{-2|t_2-t_1|}U(t_1)U(t_2)U(t_2-t_1)$$





9-44 We shall show that: If $x(t)$ is a complex process with autocorrelation $R(\tau)$ and $|R(\tau_1)|=R(0)$ for some τ_1, then $R(\tau)=e^{j\omega_o\tau}w(\tau)$ where $w(\tau)$ is a periodic function with period τ_1. Furthermore, the process $y(t) = e^{-j\omega_o t}x(t)$ is MS periodic.

<u>Proof</u> Clearly, $R(\tau_1) = R(0)e^{j\phi}$. With $\omega_o = \phi/\tau_1$,

$$R_{yy}(\tau) = E\{x(t+\tau)e^{-j\omega_o(t+\tau)}x^*(t)e^{j\omega_o t}\} = R(\tau)e^{-j\omega\tau}$$

Hence, $R_{yy}(\tau_1)=e^{-j\omega_o\tau_1}R(\tau_1) = R(0) = R_{yy}(0)$. From this and (10-168) it follows that the function $w(\tau) = R_{yy}(\tau)$ is periodic.

9-45 (a) The cross spectrum $S_{\check{x}x}(\omega) = -j\,\mathrm{sgn}\,\omega\, S_{xx}(\omega)$ is an odd function. Hence,

$$E\{\underset{\sim}{x}(t)\check{\underset{\sim}{x}}'(t)\} = \frac{-j}{2\pi}\int_{\infty}^{\infty} \mathrm{sgn}\,\omega S_{xx}(\omega)\,d\omega = 0$$

(b) The process $\check{\check{\underset{\sim}{x}}}(t)$ is the output of the system

$$(-j\,\mathrm{sgn}\,\omega)(-j\,\mathrm{sgn}\,\omega) = -1$$

with input $\underset{\sim}{x}(t)$. Hence, $\check{\check{\underset{\sim}{x}}}(t) = -\underset{\sim}{x}(t)$.

9-46 In general

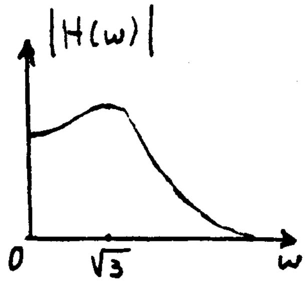
$$E\{\underset{\sim}{y}^2(t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\,|H(\omega)|^2 d\omega$$

$$\leq |H(\omega_m)|^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\,d\omega = E\{\underset{\sim}{x}^2(t)\}\,|H(\omega_m)|^2$$

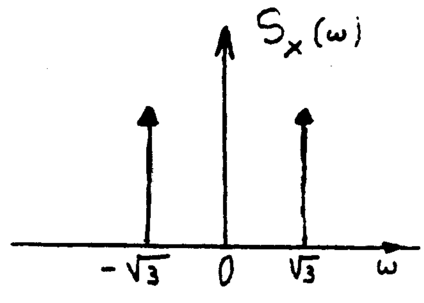
where $|H(\omega_m)|$ is the maximum of $|H(\omega)|$. In our case,

$$|H(\omega)|^2 = \frac{1}{(5-\omega)^2 + 4\omega^2} \quad \text{is maximum for } \omega = \sqrt{3}$$

and $|H(\omega_m)|^2 = 1/16$. Hence $E\{\underset{\sim}{y}^2(t)\} \leq 10/16$ with equality if $R_x(10) = 10\cos\sqrt{3}\,\tau$ (Fig. b).



(a)



(b)

9-47 If $R_x(\tau) = e^{j\omega_o\tau}$, then $S_x(\omega) = 2\pi\delta(\omega-\omega_o)$, hence, the integral of $S_x(\omega)$ equals zero in any interval not including the point $\omega = \omega_o$. From (9-182) it follows that the same is true for the integral of $S_{xy}(\omega)$. This shows that $S_{xy}(\omega)$ is a line at $\omega = \omega_o$ for any $\underset{\sim}{y}(t)$.

9-48 (a) As in (9-147) and (9-149)

$$R_{yx}(\tau) = R_{xx}(\tau) * h(\tau) = \int_{-\infty}^{\infty} e^{j\alpha(\tau-\gamma)}h(\gamma)\, d\gamma = e^{j\alpha\tau}H(\alpha)$$

$$R_{yy}(\tau) = R_{xx}(\tau) * \rho(\tau) = \int_{-\infty}^{\infty} e^{j\alpha(\tau-\gamma)}\rho(\gamma)d\gamma = e^{j\alpha\tau}|H(\alpha)|^2$$

(b) As in (9-94) and (9-95)

$$R_{yx}(t_1,t_2) = e^{-j\beta t_2}\int_{-\infty}^{\infty} e^{j\alpha(t_1-\gamma)} h(\gamma)d\gamma = e^{j(\alpha t_1-\beta t_2)}H(\alpha)$$

$$R_{yy}(t_1,t_2) = e^{-j\alpha t_1} H(\alpha)\int_{-\infty}^{\infty} e^{-j\beta(t_2-\gamma)} h(\gamma)d\gamma = e^{j(\alpha t_1-\beta t_2)}H(\alpha)H^*(\beta)$$

because $h(t)$ is real and $H(-\beta) = H^*(\beta)$.

9-49 If $S_{xx}(\omega)S_{yy}(\omega) \equiv 0$ then $S_{xx}(\omega) = 0$ or $S_{yy}(\omega) = 0$ in any interval (a,b). From this and (10-168) it follows that the integral of $S_{xy}(\omega)$ in any interval equals zero, hence, $S_{xy}(\omega) \equiv 0$.

9-50 This is the discrete-time version of theorem (9-162). From (9-163)

$$E^2\{(\underset{\sim}{x}[n+m+1] - \underset{\sim}{x}[n+m])\underset{\sim}{x}[n]\} \le E\{|\underset{\sim}{x}[n+m+1] - \underset{\sim}{x}[n+m]|^2\}E\{|\underset{\sim}{x}[n]|^2\}$$

$$(R[m+1] - R[m])^2 \le 2(R[0] - R[1])R[0] = 0$$

Hence, $R[m+1] = R[m]$ for any m.

9-51 We shall show that

$$2\,\frac{R^2[1]}{R[0]} - R[0] \le R[2] \le R[0] \qquad \text{(i)}$$

The covariance matrix of the RVs $\underset{\sim}{x}[n]$, $\underset{\sim}{x}[n+1]$, and $\underset{\sim}{x}[n+2]$ is non-negative [see (7-29)]:

$$\begin{vmatrix} R[0] & R[1] & R[2] \\ R[1] & R[0] & R[1] \\ R[2] & R[1] & R[0] \end{vmatrix} \ge 0$$

This yields

$$R[0]R^2[2] - 2\,R^2[1]R[2] - R^3[0] + 2R[0]R^2[1] \le 0$$

The above is a quadratic in R[2] with roots

$$R[0] \text{ and } -R[0] + 2\,R^2[1]/R[0]$$

Since it is nonpositive, R[2] must be between the roots as in (i)

9-52 If $\underset{\sim}{x}[n] = Ae^{jn\underset{\sim}{\omega}T}$ then

$$R_x[m] = A^2E\{e^{j(m+n)\underset{\sim}{\omega}T}e^{-jn\underset{\sim}{\omega}T}\} = A^2\int_{-\sigma}^{\sigma} e^{jm\omega T}\, f(\omega)d\omega$$

But [see (9-194)]

$$R[m] = \frac{1}{2\sigma}\int_{\sigma}^{\sigma} S_x(\omega)e^{jm\omega T}d\omega$$

hence, $A^2f(\omega) = S_x(\omega)/2\sigma$

9-53 (a) If $\underset{\sim}{y}(0) = \underset{\sim}{y}'(0) = 0$, then $\underset{\sim}{y}(t)$ is the output of a system with input $\underset{\sim}{x}(t)U(t)$ and impulse response $h(t)$ such that

$$h''(t) + 7h'(t) + 10h(t) = \delta(t) \qquad h(0^-) = h'(0^-) = 0$$

$$h(t) = \frac{1}{3}(e^{-2t} - e^{-5t})U(t)$$

and with $q(t) = 5\,U(t)$, (9-100) yields

$$E\{\underset{\sim}{y}^2(t)\} = \frac{5}{9}\int_0^t (e^{-2\tau} - e^{-5\tau})^2 d\tau$$

(b) If $\underset{\sim}{y}[-1] = \underset{\sim}{y}[-2] = 0$, then $\underset{\sim}{y}[n]$ is the output of a system with input $\underset{\sim}{x}[n]U[n]$ and delta response $h[n]$ such that

$$8h[n] - 6h[n-1] + h[n-2] = \delta[n] \qquad h[-1] = h[-2] = 0$$

$$h[n] = \left(\frac{1}{2^{n+2}} - \frac{1}{2^{2n+3}}\right)U[n]$$

and with $q[n] = 5\,U[n]$, (10-176) yields

$$E\{\underset{\sim}{y}^2[n]\} = 5\sum_{k=0}^{n}\left(\frac{1}{2^{k+2}} - \frac{1}{2^{2k+3}}\right)^2$$

9-54 $$\underset{\sim}{y}[n] = \underset{\sim}{x}[n]*h[n] \qquad h[n] = 2^{-n}U[n]$$

(a) $$E\{\underset{\sim}{y}^2[n]\} = 5*2^{-2n}U[n] = 0$$

$$R_{xy}[m_1,m_2] = 5\delta[m_1-m_2]*2^{-m_2}U[m_2] = 5\,2^{-(m_2-m_1)}U[m_2-m_1]$$

$$R_{yy}[m_1,m_2] = 5\times 2^{-(m_2-m_1)}U[m_2-m_1]*2^{-m_1}U[m_1]$$

$$= \frac{20}{3}\times 2^{-|m_1-m_2|}$$

The first equation follows from (9-190) with $q[n] = 5$; the second and third from (9-191) with $R_{xx}[m_1,m_2] = 5\,\delta[m_1-m_2]$.

(b) With $R_{xx}[m_1,m_2] = 5\,\delta[m_1-m_2]U[m_1]U[m_2]$, Prob. 9-25a yields the following: For m_1 or $m_2 < 0$, $R_{xy}[m_1,m_2] = R_{yy}[m_1,m_2] = 0$. For $0 < m_1 < m_2$

$$R_{xy}[m_1,m_2] = 5\,\delta[m_1-m_2]*2^{-m_2} = 5\times 2^{-m_2}$$

$$R_{yy}[m_1,m_2] = \sum_{k=0}^{m_1} 5\times 2^{-(m_2-k)}\,2^{-(m_1-k)} = \frac{5}{3}\,2^{-(m_2-m_1)}(4 - 2^{-2m_1})$$

9-55

(a) $R_x[m_1,m_2] = q[m_1]\delta[m_1-m_2]$

$$E\{\underset{\sim}{s}^2\} = \sum_{n=0}^{N}\sum_{k=0}^{N} a_n a_k \; E\{\underset{\sim}{x}[n]\underset{\sim}{x}[k]\}$$

$$= \sum_{n=0}^{N}\sum_{k=0}^{N} a_n a_k \; q[n]\delta[n-k] = \sum_{n=0}^{N} a_n^2 \; q[n]$$

(b) $R_x(t_1,t_2) = q(t_1)\delta(t_1-t_2)$

$$E\{s^2\} = \int_0^T\int_0^T a(t)a(\tau) \; E\{x(t)x(\tau)\}d\tau dt$$

$$= \int_0^T\int_0^T a(t)a(\tau)q(t)\delta(t-\tau)d\tau dt = \int_0^T a^2(t)q(t)dt$$

CHAPTER 10

10-1

(a) If $\underset{\sim}{x}(t)$ is a Poisson process as in Fig. 9-3a, then for a fixed t, $\underset{\sim}{x}(t)$ is a Poisson RV with parameter λt. Hence [see (5-119)] its characteristic function equals $\exp\{\lambda t(e^{j\omega}-1)\}$.

(b) If $\underset{\sim}{x}(t)$ is a Wiener process then f(x,t) is $N(0,\sqrt{\alpha t})$. Hence [see (5-100)] its first order characteristic function equals $\exp\{-\alpha t\omega^2/2\}$.

10-2 For large $\underset{\sim}{t}$, $\underset{\sim}{x}(t)$ and $\underset{\sim}{y}(t)$ can be approximated by two independent Wiener processes as in (10-52):

$$f_x(x,t) = \frac{1}{\sqrt{2\pi\alpha t}}\, e^{-x^2/2\alpha t} \qquad f_y(y,t) = \frac{1}{\sqrt{2\pi\alpha t}}\, e^{-y^2/2\alpha t}$$

Hence, $\underset{\sim}{z}(t)$ has a Rayleigh density [see (6-70)]. [Note. Exactly, $\underset{\sim}{z}(t)$ is a discrete-type RV taking the values $s\sqrt{m^2+n^2}$ where m and n are integers]. The product $f_z(z,t)dz$ equals approximately the probability that $\underset{\sim}{z}(t)$ is between z and z + dz provided that $dz \gg T$.

10-3 The voltage $\underset{\sim}{v}(t)$ is the output of a system with input $\underset{\sim}{n}_e(t)$ and system function

$$H_1(s) = \frac{1}{LCs^2 + RCs + 1}$$

Hence,

$$S_v(\omega) = S_{n_e}(\omega)\,|H_1(j\omega)|^2 = \frac{2kTR}{(1-\omega^2 LC)^2 + R^2C^2\omega^2}$$

Furthermore,

$$Z_{ab}(s) = \frac{R + Ls}{LCs^2 + RCs + 1} \qquad \underline{Re}\; Z_{ab}(j\omega) = \frac{R}{(1-\omega^2 LC)^2 + R^2C^2\omega^2}$$

in agreement with (10-75).

The current $\underset{\sim}{i}(t)$ is the output of a system with input $\underset{\sim}{n}_e(t)$ and system function

$$H_2(s) = \frac{1}{R + Ls}$$

Hence,

$$S_i(\omega) = S_{n_e}(\omega)\,|H_2\;(j\omega)|^2 = \frac{2kTR}{R^2 + \omega^2 L^2}$$

Furthermore (short circuit admittance)

$$Y_{ab}(s) = \frac{1}{R + LS} \qquad \underline{Re}Y_{ab}(j\omega) = \frac{2kTR}{R^2 + L^2\omega^2}$$

in agreement with (10-78).

10-4 The equation $m\underset{\sim}{x}''(t) + f\underset{\sim}{x}'(t) = \underset{\sim}{F}(t)$ specifies a system with

$$H(s) = \frac{1}{ms^2 + fs} \qquad h(t) = \frac{1}{f}(1 - e^{-ft/m})U(t)$$

and (9-100) yields

$$E\{\underset{\sim}{x}^2(t)\} = \frac{2kTf}{f^2}\int_0^t (1 - e^{-2\alpha\tau})^2\, d\tau \qquad \alpha = \frac{f}{2m}$$

10-5 As in Example 12-2, a and b are such that

$$\underset{\sim}{x}(\tau) - a\,\underset{\sim}{x}(0) - b\underset{\sim}{v}(0) \perp \underset{\sim}{x}(0),\ \underset{\sim}{v}(0)$$

This yields

$$R_{xx}(\tau) = aR_{xx}(0) + b\,R_{xv}(0) \qquad\qquad \text{(i)}$$

$$R_{xv}(\tau) = aR_{xv}(0) + b\,R_{vv}(0)$$

where [see (10-163)]

$$R_{xx}(\tau) = A\,e^{-\alpha\tau}(\cos\beta\tau + \frac{\alpha}{\beta}\sin\beta\tau) \qquad \tau > 0$$

$$R_{xv}(\tau) = -R'_{xx}(\tau) = A\ e^{-\alpha\tau}(\sin\beta\tau)\,\frac{\alpha^2+\beta^2}{\beta}$$

$$R_{vv}(\tau) = R'_{xv}(\tau) = A\ e^{-\alpha\tau}(\cos\beta\tau - \frac{\alpha}{\beta}\sin\beta\tau)\,\frac{\alpha^2+\beta^2}{\beta^2}$$

Inserting into (i) and solving, we obtain

$$a = e^{-\alpha\tau}(\cos\beta\tau + \frac{\alpha}{\beta}\sin\beta\tau)$$

$$b = \frac{1}{\beta}\,e^{-\alpha\tau}\ \sin\beta\tau$$

Finally,

$$P = E\{[\underset{\sim}{x}(t) - a\,\underset{\sim}{x}(0) - b\,\underset{\sim}{v}(0)]\underset{\sim}{x}(t)\} = R_{xx}(0) - a\,R_{xx}(t) - b\,R_{xv}(t)$$

$$= \frac{2kTf}{m^2}\left[1 - e^{-2\alpha t}(1 + \frac{2\alpha^2}{\beta}\sin^2\beta t + \frac{\alpha}{\beta}\sin 2\beta t)\right]$$

10-6 If $\underset{\sim}{x}(t) = \underset{\sim}{w}(t^2)$ then [see (10-70)]

$$R_x(t_1,t_2) = E\{\underset{\sim}{w}(t_1^2)\underset{\sim}{w}(t_2^2)\} = \alpha t_1^2$$

If $\underset{\sim}{y}(t) = \underset{\sim}{w}^2(t)$ then [see (6-197)]

$$R_y(t_1,t_2) = E\{\underset{\sim}{w}^2(t_1)\underset{\sim}{w}^2(t_2)\}$$

$$= E\underset{\sim}{w}^2(t_1)E\{\underset{\sim}{w}^2(t_2)\} + 2\,E^2\{\underset{\sim}{w}(t_1)\underset{\sim}{w}(t_2)\} = \alpha^2 t_1 t_2 + 2\alpha^2 t_1^2$$

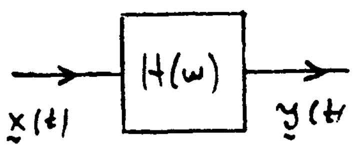
10-7 From (10-112):

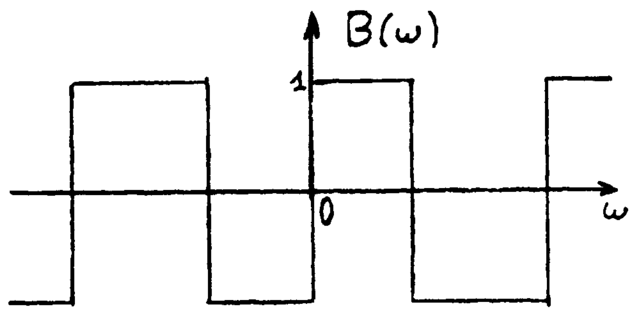
$$\eta_s = 3\int_0^{10} 2\,dt = 60 \qquad \sigma_s^2 = 3\int_0^{10} 4dt = 120 \qquad E\{\underset{\sim}{s}^2\} = 3720$$

$\underset{\sim}{s}(7) = 0$ if there are no points in the interval (7-10, 7). The number of points in this interval is a Poission RV with parameter $10\lambda = 30$. Hence, $P\{\underset{\sim}{s}(7) = 0\} = e^{-30}$.

10-8

$H(\omega) = j\,B(\omega)$





From the assumption: $S_{xx}(\omega) = S_{yy}(\omega) \qquad S_{xy}(-\omega) = -S_{xy}(\omega)$

From (9-148): $S_{yy}(\omega) = S_{xx}(\omega)|H(\omega)|^2 \qquad S_{xy}(\omega) = S_{xx}(\omega)H^*(\omega)$

Combining, we obtain

$$|H(\omega)|^2 = 1 \qquad H(-\omega) = -H(\omega)$$

Since h(t) is real, the second equation yields $H(\omega) = jB(\omega)$ and from the first it follows that

$$|B(\omega)| = 1$$

as in the figure.

10-9 With $\underset{\sim}{i}(t) = \underset{\sim}{a}(t)$, $\underset{\sim}{q}(t) = \underset{\sim}{b}(t)$, (11-63) yields

$$S_i(\omega) = S_q(\omega) \qquad S_{iq}(\omega) = -S_{qi}(\omega) = S_{qi}(-\omega)$$

Hence [see (11-75) and (11-82)],

$$S_w(\omega) = 2\, S_i(\omega) + 2j\, S_{qi}(\omega)$$

$$S_w(-\omega) = 2\, S_i(\omega) - 2j\, S_{qi}(\omega)$$

Adding and subtracting, we obtain

$$4\, S_i(\omega) = S_w(\omega) + S_w(-\omega) \qquad 4j\, S_{iq}(\omega) = S_w(-\omega) - S_w(\omega)$$

10-10 From (10-133)

$$\underset{\sim}{x}(t) = \underset{\sim}{\mathrm{Re}}\, [\underset{\sim}{w}(t)e^{j\omega_0 t}]$$

$$\underset{\sim}{x}(t-\tau) = \underset{\sim}{\mathrm{Re}}[\underset{\sim}{w}_\tau(t)e^{j\omega_0 t}] = \underset{\sim}{\mathrm{Re}}[\underset{\sim}{w}(t-\tau)e^{j\omega_0(t-\tau)}]$$

$$\underset{\sim}{w}_\tau(t) = \underset{\sim}{w}(t-\tau)e^{-j\omega_0\tau}$$

10-11 $$R''_x(\tau) \longleftrightarrow -\omega^2 S_x(\omega)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2 S_x(\omega)d\omega = -R''_x(0)$$

and with ω_0 the optimum carrier frequency, (10-150) yields

$$E\{|\underset{\sim}{w}'(t)|^2\} = \frac{M}{2\pi} = -2R''_x(0) - 2\omega_0^2 R_x(0)$$

10-12 From the stationarity of the process $\underset{\sim}{x}(t)\cos\omega t + \underset{\sim}{y}(t)\sin\omega t$ it follows that [see (10-130)]

$$C_{xx}(\tau) = C_{yy}(\tau) \qquad C_{xy} = -C_{yx}(\tau) \tag{i}$$

Using these identities, we shall express the joint density f(X,Y) of the 2n RVs

$$\underset{\sim}{X} = [\underset{\sim}{x}(t_1), \ldots, \underset{\sim}{x}(t_n)] \qquad \underset{\sim}{Y} = [\underset{\sim}{y}(t_1), \ldots, \underset{\sim}{y}(t_n)]$$

in terms of the covariance matrix C_{ZZ} of the complex vector $\underset{\sim}{Z} = \underset{\sim}{X} + j\underset{\sim}{Y}$. From (i) it follows that

$$E\{\underset{\sim}{x}(t_i)\underset{\sim}{x}(t_j)\} = E\{\underset{\sim}{y}(t_i)\underset{\sim}{y}(t_j)\} \qquad E\{\underset{\sim}{x}(t_i)\underset{\sim}{y}(t_j)\} = -E\{\underset{\sim}{y}(t_i)\underset{\sim}{x}(t_j)\}$$

This yields

$C_{XX} = C_{YY}$, and $C_{XY} = -C_{YX}$; hence, f(X,Y) is given by (8-62).

10-13 The signal $\underset{\sim}{c}(t) = f(t)$ is an extreme case of a cyclostationary process as in (10-178) with

$$h(t) = \begin{cases} f(t) & 0 \le t < T \\ 0 & \text{otherwise} \end{cases} \qquad \longleftrightarrow \qquad H(\omega) = \int_0^T f(t)e^{-j\omega t}dt$$

and $\underset{\sim}{c}_m = 1$, $R[m] = 1$. Hence [see (10A-2)]

$$\sum_{m=-\infty}^{\infty} R_m e^{-jm\omega T} = \sum_{m=-\infty}^{\infty} e^{-jm\omega T} = T\sum_{m=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{T}m)$$

From the above and (10-180) it follows that the process $\underset{\sim}{x}(t) = f(t - \underset{\sim}{\theta})$ is stationary with power spectrum

$$S(\omega) = \left| \int_0^T f(t)e^{-j\omega t}dt \right|^2 \sum_{m=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{T}m)$$

10-14 The process

$$\underline{y}_N(t) = \underline{x}(t+\tau) - \sum_{n=-N}^{N} \underline{x}(t+nT)\,\frac{\sin\sigma(\tau-nT)}{\sigma(\tau-nT)}$$

is the output of a system with input $\underline{x}(t)$ and system function

$$H_N(\omega) = e^{j\omega\tau} - \sum_{n=-N}^{N} \frac{\sin\sigma(\tau-nT)}{\sigma(\tau-nT)}\, e^{jnT\omega}$$

Furthermore, $\underline{\varepsilon}_N(\tau) = \underline{y}_N(0)$, hence [see (9-153)]

$$E\{\underline{\varepsilon}_N^2(\tau)\} = E\{\underline{y}^2(0)\} = \frac{1}{2\pi}\int_{\infty}^{\infty} S(\omega)\,|H_N(\omega)|^2 d\omega \qquad \text{(i)}$$

The function $H_N(\omega)$ is the truncation error in the Fourier series expansion of $e^{j\omega\tau}$ in the interval $(-\sigma,\sigma)$. Hence, for $N > N_0$

$$|H_N(\omega)| < \varepsilon \qquad\qquad |\omega| < \sigma$$

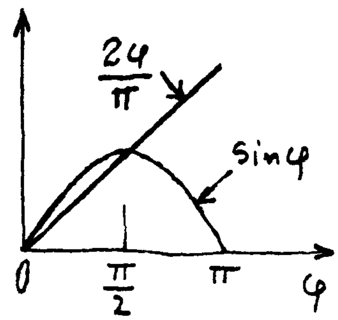
From this and (i) it follows that, if $S(\omega) = 0$ for $|\omega| < \sigma$, then

$$E\{\underline{\varepsilon}_N^2(\tau)\} = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S(\omega)\,|H_N(\omega)|^2 d\omega < \varepsilon\, R(0) \qquad N > N_0$$

10-15 [see after (10-195)]

$$R(0) - R(\tau) = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S(\omega)(1-\cos\omega\tau)\,d\omega$$

$$\leq \frac{\tau^2}{4\pi}\int_{-\sigma}^{\sigma} \omega^2 S(\omega)\,d\omega = \frac{-\tau^2}{2} R''(0)$$



Furthermore, since

$$\sin\phi \geq \frac{2\phi}{\pi} \qquad 0 \leq \phi \leq \frac{\pi}{2}$$

we obtain

$$R(0) - R(\tau) = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S(\omega)\, 2\sin^2\frac{\omega\tau}{2}\, d\omega$$

$$\geq \frac{2\tau^2}{\pi^2}\,\frac{1}{2\pi}\int_{-\sigma}^{\sigma} \omega^2 S(\omega)\,d\omega = \frac{-2\tau^2}{\pi^2} R''(0)$$

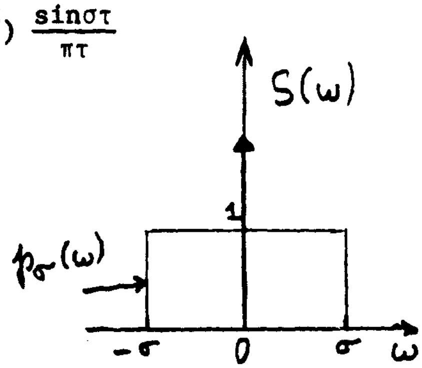
10-16 With $T = \pi/\sigma$

$$R(mT) = E\{\underset{\sim}{x}(nT+mT)\underset{\sim}{x}(nT)\} = \begin{cases} I & m = 0 \\ \eta^2 & m \neq 0 \end{cases}$$

Hence [see (10-196)]

$$R(\tau) = \sum_{m=-\infty}^{\infty} R(mT)\,\frac{\sin\sigma(\tau-mT)}{\sigma(\tau-mT)} = \eta^2 + (I-\eta^2)\,\frac{\sin\sigma\tau}{\pi\tau}$$

$$S(\omega) = 2\pi\eta^2\delta(\omega) + 2\pi(I-\eta^2)p_\sigma(\omega)$$

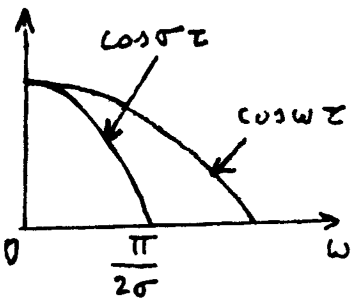


10-17 Given $E\{\underset{\sim}{x}(n+m)\underset{\sim}{x}(n)\} = N\delta[m]$

This is a special case of Prob. 10-16 with $\eta = 0$, $I = N$.

10-18 If $|\tau| < \pi/2\sigma$, then

$$\cos \omega\tau \geq \cos \sigma\tau \qquad |\omega| \leq \sigma$$



$$R(\tau) = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S(\omega)\cos\omega\tau\, d\omega$$

$$\geq \frac{\cos\sigma\tau}{2\pi}\int_{-\sigma}^{\sigma} S(\omega)\, d\omega = R(0)\cos\sigma\tau$$

10-19 From (10-133) with $c = \sigma$

$$P_1(\omega,\tau) + j\omega\, P_2(\omega,\tau) = 1$$

$$P_1(\omega,\tau) + j(\omega+\tau)P_2(\omega,\tau) = e^{j\sigma\tau}$$

Hence,

$$P_1(\omega,\tau) = 1 - \frac{\omega}{\sigma}(e^{j\sigma\tau} - 1) \qquad P_2(\omega,\tau) = \frac{1}{j\sigma}(e^{j\sigma\tau} - 1)$$

Inserting into (11-141), we obtain

$$p_1(\tau) = \frac{4\sin^2(\sigma\tau/2)}{\sigma^2\tau^2} \qquad p_2(\tau) = \frac{4\sin^2(\sigma\tau/2)}{\sigma^2\tau}$$

and with $t = 0$, the desired result follows from (10-206) because $\bar{T} = 2T$ and

$$\sin^2 \frac{\sigma(\tau-2nT)}{2} = \sin^2\left(\frac{\sigma\tau}{2} - n\pi\right) = \sin^2 \frac{\sigma\tau}{2}$$

10-20 As in (10-213)

$$\underset{\sim}{P}(\omega) = \frac{1}{\lambda}\int_{-a}^{a} \cos\omega t \ \underset{\sim}{z}(t) \cos\omega_c t \, dt$$

$$E\{\underset{\sim}{P}(\omega)\} = \int_{-a}^{a} \cos\omega t \cos\omega_c t \, dt$$

$$\sigma^2_{P(\omega)} = \frac{1}{\lambda}\int_{-a}^{a} \cos^2\omega_c t_2 \cos^2 \omega t_2 \, dt_2$$

10-21 We shall show that if

$$\underset{\sim}{X}_c(\omega) = \frac{1}{\lambda} \sum_{|t_i|<c} \underset{\sim}{x}(t_i)e^{-j\omega t_i} = \frac{1}{\lambda}\int_{-a}^{a} \underset{\sim}{x}(t)\underset{\sim}{z}(t)e^{-j\omega t}dt$$

where $\underset{\sim}{z}(t) = \sum \delta(t-\underset{\sim}{t}_i)$ is a Poisson impulse train, then

$$E\{|\underset{\sim}{X}_c(\omega)|^2\} \simeq 2cS_x(\omega) + \frac{2c}{\lambda} R_x(0)$$

Proof

Since $R_z(\tau) = \lambda^2 + \lambda\delta(\tau)$, it follows that

$$E\left\{|\underset{\sim}{X}_c(\omega)|^2\right\} = \frac{1}{\lambda^2}\int_{-c}^{c}\int_{-c}^{c} R_x(t_1-t_2)e^{-j\omega(t_1-t_2)}dt_1dt_2$$

$$= \int_{-c}^{c} e^{j\omega t_2}\int_{-c}^{c} R_x(t_1-t_2)e^{-j\omega t_1}dt_1dt_2 + \frac{1}{\lambda}\int_{-c}^{c} R_x(0)dt_2$$

If $\int_{-\infty}^{\infty} |R_x(\tau)| < \infty$ then for sufficient large c, the inner integral on the right is nearly equal to $S_x(\omega)^{-j\omega t_2}$ and (i) follows.

10-22 $\quad E\{\underset{\sim}{z}(t)\} = g(t) \qquad E\{\underset{\sim}{w}(t)\} = g(t) - g(T)t/T = g(t)$

$$\underset{\sim}{w}(t) = (1 - \frac{t}{T}) \int_0^t \underset{\sim}{x}(\alpha)d\alpha - \frac{t}{T}\int_t^T \underset{\sim}{x}(\alpha)d\alpha$$

The above two integrals are uncorrelated because $\underset{\sim}{n}(t)$ is white noise. Hence, as in Example 9-5

$$\sigma_w^2 = (1 - \frac{t}{T})^2\, Nt + \frac{t^2}{T^2}\, N(T - t) = Nt(1 - \frac{t}{T})$$

Note The above shows that the information that $g(T) = 0$ can be used to improve the estimate of $g(t)$. Indeed, if we use $\underset{\sim}{w}(t)$ instead of $\underset{\sim}{z}(t)$ for the estimate of $g(t)$ in terms of the data $\underset{\sim}{x}(t)$, the variance is reduced from Nt to $Nt(1-t/T)$.

10-23 (a) Since $|\sum_i a_i b_i| \le \sum |a_i||b_i|$, it suffices to assume that the numbers a_i and b_i are real. The quadratic

$$I(z) = \sum_i (a_i - z\, b_i)^2 = z^2 \sum_i b_i^2 - 2z \sum_i a_i b_i + \sum_i a_i^2$$

is nonnegative for every real z, hence, its discriminant cannot be positive. This yields (i).

(b) With $f[n]$ and $R_\nu[m] = S_o\delta[m]$ as in Prob. 10-24a (white noise)

$$y_f[n_0] = \sum h[n] f[n_0-n] \qquad \underset{\sim}{y}_\nu[n] = \sum h[n]\underset{\sim}{\nu}[n]$$

$$E\{\underset{\sim}{y}_\nu^2[n]\} = S_0\, \rho[0] = S_o \sum |h[n]|^2$$

[see (9-213)] And (i) yields

$$\frac{y_f^2[n_0]}{E\{\underset{\sim}{y}_\nu^2[n]\}} = \frac{|\sum h[n] f[n_0-n]|^2}{S_0 \sum h^2[n]} \le \frac{1}{S_0} \sum |h[n]|^2$$

with equality iff $h[n] = kf^*[n_0-n]$.

10-24 (a) Given $F(z)$ and $S_\nu(\omega) = S_0 \equiv$ constant. The z transform of $y_f[n]$ equals $F(z)H(z)$. Hence, [see (9-109)]

$$y_f[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{j\omega T})H(e^{j\omega T})e^{jn\omega T}d\omega$$

$$\frac{y_f^2[n]}{E\{\underset{\sim}{y}_\nu^2[n]\}} = \frac{\left|\int_{-\pi}^{\pi} F(e^{j\omega T})H(e^{j\omega T})d\omega\right|^2}{S_0 \int_{-\pi}^{\pi}\left|H(e^{j\omega T})\right|^2 d\omega}$$

$$\le \frac{1}{S_0}\int_{\pi}^{\pi}\left|F(e^{j\omega T})\right|^2 d\omega$$

The last inequality follows from Schwarz's inequality with equality iff

$$H(e^{j\omega T}) = kF^*(e^{j\omega T}) = kF(e^{-j\omega T}),\ \text{i.e., iff } H(z) = kF(z^{-1})$$

(b) Given arbitrary $R_\nu[m]$, $F(z)$, and the form of $H(z)$ (FIR); to find the coefficients a_m of $H(z)$. In this case

$$y_f[n] = a_0 f[n] + a_1 f[n-1] + \cdots + a_N f[n-N]$$

$$\underset{\sim}{y}_\nu[n] = a_0\underset{\sim}{\nu}[n] + a_1\underset{\sim}{\nu}[n-1] + \cdots + a_N\underset{\sim}{\nu}[n-N]$$

To maximize the signal-to-noise ratio it suffices to minimize

$$E\{\underset{\sim}{y}_\nu^2[n]\} = \sum_{k,r=0}^{N} a_k a_r R_\nu[k-r]$$

subject to the constraint that the sum

$$y_f[0] = a_0 f[0] + a_1 f[-1] + \cdots + a_N f[-N]$$

is constant. With λ a constant (Lagrange multiplier), we minimize the sum

$$I = \sum_{k,r=0}^{N} a_k a_r R\ [k-r] - \lambda \left[\sum_{k=0}^{N} a_k f[-k] - y_f[0] \right]$$

this yields the system

$$\frac{\partial I}{\partial a_k} = 0 = \sum_{r=0}^{N} \left[a_r R_\nu [k-r] - \lambda f[-k] \right] \qquad k = 0,.., N$$

whose solution yields a_k.

10-25 $$B = A|H(\omega_0)| = \frac{A}{\sqrt{\alpha^2+\omega_0^2}} \qquad S_{y_n}(\omega) = \frac{N}{\alpha^2+\omega^2}$$

$$R_{y_n}(\tau) = \frac{N}{2\alpha}\ e^{-\alpha|\tau|} \qquad E\{\underset{\sim}{y}_n^2(t)\} = R_{y_n}(0) = \frac{N}{2\alpha}$$

$$\frac{B^2}{E\{\underset{\sim}{y}_n^2(t)\}} = \frac{2A^2}{N} \frac{\alpha}{\alpha^2+\omega_0^2}$$ Max. if $\alpha = \omega_0$

10-26 Since $H(\omega)$ is determined within a constant factor, we can sssume that the response $y_f(t_o)$ of the optimum $H(\omega)$ due to $f(t)$ is constant:

$$y_f(t_o) = \sum_{i=0}^{m} a_i\, f(t_o - iT) = c \tag{i}$$

Our problem is to minimize the variance

$$V = E\{\underset{\sim}{y}_\nu^2(t)\} = \sum_{n=0}^{m} a_n \sum_{i=0}^{m} a_i\, R(nT-iT) \tag{ii}$$

of $\underset{\sim}{y}_\nu(t)$ subject to the constraint (i). This yields the system

$$\frac{\partial V}{\partial a_n} = \sum_{i=0}^{m} a_i\, R(nT-iT) - kf(t_o - nT) = 0$$

where k is a constant (lagrange multiplier). With a_n so determined, we conclude from (ii) that

$$V = \sum_{n=0}^{m} ka_n\, f(t_o - nT) = ky_f(t_o) \qquad r^2 = \frac{y_f^2(t_o)}{ky_f(t_o)}$$

10-27 $R_{yyy}(\mu,\nu) = E\{\underset{\sim}{x}(t+\mu)+c[\,[\underset{\sim}{x}(t+\nu)+c]\,[\underset{\sim}{x}(t)+c]\} = R(\mu,\nu) + cR(\mu) + cR(\nu) + cR(\mu-\nu) + c^3$

because $E\{\underset{\sim}{x}(t)\} = 0$. Furthermore,

$$R(\mu) \leftrightarrow 2\pi S(u)\delta(v) \qquad R(\nu) = 2\pi\delta(u)S(v) \qquad c^3 \leftrightarrow 4\pi^2\delta(u)\delta(v)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R(\mu-\nu)e^{-j(u\mu+v\nu)}d\mu d\nu = \int_{-\infty}^{\infty} R(\tau)e^{-ju\tau}d\tau \int_{-\infty}^{\infty} e^{-j(u+v)\nu}d\nu = 2\pi S(u)\delta(u+v)$$

10-28 We shall use the equations $E\{\underset{\sim}{\bar{x}}(t)\} = 0$, $E\{\underset{\sim}{\bar{x}}^2(t)\} = \lambda t$. Suppose that $t_1 < t_2 < t_3$. Clearly,

$$\underset{\sim}{\bar{x}}(t_2) = \underset{\sim}{\bar{x}}(t_1) + [\underset{\sim}{\bar{x}}(t_2)-\underset{\sim}{\bar{x}}(t_1)]$$

$$\underset{\sim}{\bar{x}}(t_3) = \underset{\sim}{\bar{x}}(t_1) + [\underset{\sim}{\bar{x}}(t_2) - \underset{\sim}{\bar{x}}(t_1)] + [\underset{\sim}{\bar{x}}(t_3)-\underset{\sim}{\bar{x}}(t_2)] \qquad \text{(i)}$$

Inserting into the product $\underset{\sim}{\bar{x}}(t_1)\underset{\sim}{\bar{x}}(t_2)\underset{\sim}{\bar{x}}(t_3)$ and using the identity $E\{\underset{\sim}{\bar{x}}(t_i)-\underset{\sim}{\bar{x}}(t_j)\} = 0$ and the independence of the three terms on the right of (i), we obtain

$$E\{\underset{\sim}{x}(t_1)\underset{\sim}{x}(t_2)\underset{\sim}{x}(t_3)\} = E\{\underset{\sim}{x}^3(t_1)\} = \lambda t_1 = \lambda \min(t_1, t_2, t_3)$$

Since $\underset{\sim}{\bar{z}}(t) = \underset{\sim}{\bar{x}}'(t)$, we conclude from (9-120)-(9-122) that

$$R_{\bar{z}\bar{z}\bar{z}}(t_1,t_2,t_3) = \frac{\partial^3 R_{xxx}(t_1,t_2,t_3)}{\partial t_1 \partial t_2 \partial t_3} = \lambda \frac{\partial^3 \min(t_1,t_2,t_3)}{\partial t_1 \partial t_2 \partial t_3}$$

It suffices therefore to show that the right side equals $\lambda\delta(t_1-t_2)\delta(t_1-t_2)$. This is a consequence of the following:

$$\frac{\partial \min(t_1,t_2,t_3)}{\partial t_3} = t_1 U(t_2-t_1)\delta(t_3-t_1) + t_2\, U(t_1-t_2)\,\delta\,(t_3-t_2)$$

$$+ U(t_1-t_3)U(t_2-t_3) - t_3\delta(t_1-t_3)U(t_2-t_3) - t_3 U(t_1-t_3)\delta(t_2-t_3)$$

$$= U(t_1-t_3)U(t_2-t_3)$$

because $t_i\delta(t_i-t_j) = t_j\delta(t_j-t_i)$. Hence,

$$\frac{\partial^2 \min(t_1,t_2,t_3)}{\partial t_2 \partial t_3} = U(t_1-t_3)\delta(t_2-t_3) \qquad \frac{\partial^2 \min(t_1,t_2,t_3)}{\partial t_1 \delta t_2 \partial t_3} = \delta(t_1-t_2)\delta(t_1-t_3)$$

10-29 See outline given in text.

CHAPTER 11

11-1 $$S_x(z) = \frac{5-2(z+1/z)}{10-3(z+1/z)} = \frac{2}{3} + \frac{5/9}{10/3 - (z+1/z)}$$

$$R[m] = \frac{2}{3} + \frac{5}{18}\,3^{-|m|} \qquad \Gamma(z) = \frac{3z-1}{2z-1}$$

11-2 $$S_x(s) = \frac{s^4+64}{s^4-10s^2+9} = \frac{s^2+4s+8}{s^2+4s+3}\;\frac{s^2-4s+8}{s^2-4s+3}$$

$$L(s) = \frac{s^2+4s+8}{s^2+4s+3}$$

11-3 First proof

$$\underset{\sim}{s}[n] = \sum_{k=0}^{\infty} \ell[n]\underset{\sim}{i}[n-k] \qquad E\{\underset{\sim}{x}^2[n]\} = \sum_{k=0}^{\infty} \ell^2[k]$$

Second proof

$$S(z) = L(z)L(1/z) \qquad R[m] = \ell[m] * \ell[-m] = \sum_{k=0}^{\infty} \ell[k]\ell[k-m]$$

$$R[0] = \sum_{k=0}^{\infty} \ell^2[k]$$

11-4 (a) This is a special case of (11-22) and (11-23).

(b) From (a) it follows that

$$R''_{yx}(\tau) + 3R'_{yx}(\tau) + 2R_{yx}(\tau) = q\delta(\tau)$$

Since $R_{xx}(\tau) = 0$ for $\tau < 0$, the above shows that

$$R_{yx}(\tau) = 0 \text{ for } \tau \le 0^- \qquad R'_{yx}(0^-) = 0$$

Furthermore,

$$S_{yx}(s) = \frac{q}{s^2 + 3s + 2}$$

hence (initial value theorem)

$$R_{yx}(0^+) = \lim_{s \to \infty} s\, S_{yx}(s) = 0 \qquad R'_{yx}(0^+) = \lim_{s \to \infty} s^2\, S_{yx}(s) = q$$

Similarly,

$$R''_{yy}(\tau) + 3R'_{yy}(\tau) + 2R_{yy}(\tau) = R_{xy}(\tau) = R_{yx}(-\tau) = 0 \quad \text{for } \tau > 0$$

$$S_{yy}(s) = \frac{q}{(s^2 + 3s + 2)(s^2 - 3s + 2)} = \frac{qs/12 + q/4}{s^2 + 3s + 2} + \frac{-qs/12 + q/4}{s^2 - 3s + 2}$$

$$S^+_{yy}(s) = \frac{qs/12 + q/4}{s^2 + 3s + 2}$$

$$R^+_{yy}(0^+) = R_{yy}(0) = \lim_{s \to \infty} s^2\, S^+_{yy}(s) = \frac{q}{12}$$

$$R'_{yy}(0) = \lim_{s \to \infty} s\left[s\, S^+_{yy}(s) - \frac{q}{12}\right] = 0$$

11-5

$$S_x(z) = S_s(z) + S_\nu(z) = \frac{1}{D(z)} + q = \frac{1 + qD(z)}{D(z)}$$

If $R_s[m] = 2^{-|m|}$ and $S_\nu(z) = 5$, then (see Example 9-31)

$$S_s(z) = \frac{1.5}{2.5 - (z^{-1} + z)} \qquad S_x(z) = \frac{5 - 14z^{-1} + 5z^{-2}}{1 - 2.5z^{-1} + z^{-2}}$$

11-6 The process

$$\underset{\sim}{y}[n] = \frac{1}{n}\sum_{k=1}^{n} \underset{\sim}{x}(nT+kT)$$

is the output of a system with input $\underset{\sim}{x}[n]$ and system function

$$H(z) = \frac{1}{n}\sum_{k=1}^{n} z^k$$

Furthermore, $\underset{\sim}{s} = \underset{\sim}{y}[0]$ and

$$n^2\left|H(e^{j\omega T})\right|^2 = \left|\sum_{k=1}^{n} e^{jk\omega T}\right|^2$$

$$= \left|\frac{e^{j\omega T} - e^{j(n+1)\omega T}}{1 - e^{j\omega T}}\right|^2 = \frac{\sin^2 n\omega T/2}{\sin^2 \omega T/2}$$

Hence [see (9-51)]

$$E\{\underset{\sim}{s}^2\} = R_y[0] = \frac{1}{2\pi n^2}\int_{-\infty}^{\infty} S_x(\omega)\,\frac{\sin^2 n\omega T/2}{\sin^2 \omega T/2}\,d\omega$$

11-7 Since $R(t_1,t_2) = e^{-c|t_1-t_2|}$, (12-58) yields

$$\int_{-a}^{t_1} e^{-c(t_1-t_2)}\phi(t_2)dt_2 + \int_{t_1}^{a} e^{c(t_1-t_2)}\phi(t_2)dt_2 = \lambda\phi(t_1) \qquad \text{(i)}$$

Differentiating twice and using (i) we obtain (omitting details)

$$\lambda\phi''(t) + (2c - \lambda c^2)\phi(t) = 0$$

Hence,

$$\phi(t) = \beta\cos\omega t \quad \text{and} \quad \phi(t) = \beta'\cos\omega' t$$

To determine ω, we insert into (i). This yields

$$\frac{2c}{c^2+\omega^2} + \frac{\omega\sin a\omega - c\cos a\omega}{c^2+\omega^2}\, e^{-ac}\,(e^{ct} + e^{-ct}) = 2c\,\lambda\cos\omega t$$

This yields

$$\omega_n \sin a\omega_n - c \cos a\omega_n = 0 \qquad \lambda_n = \frac{2c}{c^2+\omega_n^2}$$

The constants β_n are determined from (normalization)

$$1 = \int_{-a}^{a} \beta_n^2 \cos^2\omega_n t\, dt \qquad \beta_n^2 = \frac{1}{a+c\lambda_n}$$

Similarly for $\beta_n' \sin\omega_n' t$.

11-8 As in (9-60)

$$E\{|\underline{X}_T(\omega)|^2\} = \int_{-T/2}^{T/2} R(t_1-t_2)e^{-j\omega(t_1-t_2)}dt_1dt_2$$

$$= \int_{-T}^{T} (T-|\tau|)R(\tau)\; e^{-j\omega\tau}d\tau$$

Differentiating with respect to T and using the fact that if

$$\phi(t) = \int_{-t}^{t} f(x;t)dx$$

then

$$\frac{d\phi(t)}{dt} = f(t;t) - f(-t,t) + \int_{-t}^{t} \frac{\partial f}{\partial t}(x,t)dx$$

we obtain

$$\frac{\partial E\{|\underline{X}_T(\omega)|^2\}}{\partial T} = \int_{-T}^{T} R(\tau)e^{-j\omega\tau}d\tau = E\{\frac{\partial}{\partial T}|\underline{X}_T(\omega)|^2\}$$

The above approaches $S(\omega)$ as $T \to \infty$.

11-9 $$E\{\underset{\sim}{X}(\omega)\} = \int_{-a}^{a} 5\cos 3t\, e^{-j\omega t}dt = \frac{5\sin a(\omega-3)}{\omega-3} + \frac{5\sin a(\omega+3)}{\omega+3}$$

Var. $\underset{\sim}{X}(\omega) = 2\,q\,a = 4a.$

11-10 $$E\{\underset{\sim}{X}(u)\underset{\sim}{X}(v)\} = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \sigma_n^2\delta[n-k]\; e^{-j(nu-kv)T}$$

$$= \sum_{n=-\infty}^{\infty} \sigma_n^2\; e^{-jn(u-v)T}$$

11-11 Shifting the origin, we set

$$\underset{\sim}{c}_n = \frac{1}{T}\int_{-T/2}^{T/2} \underset{\sim}{x}(t)e^{-jn\omega_o t}dt \qquad \beta_n(\alpha) = \frac{1}{T}\int_{-T/2}^{T/2} R(\tau-\alpha)e^{-jn\omega_o\tau}d\tau$$

(a) We shall show that if

$$\hat{\underset{\sim}{x}}(t) = \sum_{n=-\infty}^{\infty} \underset{\sim}{c}_n e^{jn\omega_o t} \text{ then } E\{|\underset{\sim}{x}(t)-\hat{\underset{\sim}{x}}(t)|^2\} = 0 \text{ for } |t| < T/2 \qquad \text{(i)}$$

<u>Proof</u> $$E\{\underset{\sim}{c}_n\underset{\sim}{x}^*(\alpha)\} = \frac{1}{T}\int_{-T/2}^{T/2} E\{\underset{\sim}{x}(t)\underset{\sim}{x}^*(\alpha)\}e^{-jn\omega_o t}dt = \beta_n(\alpha)$$

The functions $\beta_n(\alpha)$ are the coefficients of the Fourier expansion of $R(\tau-\alpha)$:

$$R(\tau-\alpha) = \sum_{n=-\infty}^{\infty} \beta_n(\alpha)e^{jn\omega_o\tau} \qquad |\tau| < T/2 \qquad \text{(ii)}$$

Hence

$$E\{\hat{\underset{\sim}{x}}(t)\underset{\sim}{x}^*(t)\} = \sum_{n=-\infty}^{\infty} E\{\underset{\sim}{c}_n\underset{\sim}{x}^*(t)\}^{jn\omega_o t} = \sum_{n=-\infty}^{\infty} \beta_n(t)e^{jn\omega_o t}$$

From (ii) it follows with $\tau = \alpha = t$ that the last sum equals R(0). Similarly, $E\{\hat{\underset{\sim}{x}}^*(t)\underset{\sim}{x}(t)\} =$ R(0) and (i) results.

(b) $E\{\underset{\sim}{c}_n\underset{\sim}{c}_m^*\} = \frac{1}{T}\int_{-T/2}^{T/2} E\{c_n\underset{\sim}{x}^*(t)\}e^{jn\omega_o t}dt = \frac{1}{T}\int_{-T/2}^{T/2} \beta_n(t)e^{jn\omega_o t}dt$

(c) If T is sufficiently large, then

$$T\beta_n(\alpha) = \int_{-T/2}^{T/2} R(\tau-\alpha)e^{-jn\omega_o\tau}d\tau \simeq S(n\omega_o)e^{-jn\omega_o\alpha}$$

$$E\{c_n c_m^*\} = \frac{S(n\omega_o)}{T^2}\int_{-T/2}^{T/2} e^{j(m-n)\omega_o\alpha}d\alpha = \begin{cases} S(n\omega_o)/T & m=n \\ 0 & m\neq n \end{cases}$$

Thus, for large T, the coefficients $\underset{\sim}{c}_n$ of an arbitrary WSS process are nearly orthogonal.

11-12 $$E\{\underset{\sim}{x}(t_1)\underset{\sim}{x}^*(t_2)\} = \frac{1}{4\pi^2} E\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\{\underset{\sim}{X}(u)\underset{\sim}{X}^*(v)\}e^{j(ut_1-vt_2)}dudv\right.$$

$$= \frac{1}{4\pi^2} E\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q(u)\delta(u-v)e^{j(ut_1-vt_2)}dudv = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} Q(u)e^{ju(t_1-t_2)}du\right.$$

This depends only on $\tau = t_1 - t_2$:

$$R_{xx}(\tau) = \frac{1}{x_2}\int_{-\infty}^{\infty} Q(u)e^{ju\tau}du \qquad S_{xx}(\omega) = \frac{Q(\omega)}{2\pi}$$

11-13 Equations (11-79) can be written in the following form:

$$E\{\underset{\sim}{A}(u)\underset{\sim}{A}(v)\} = Q(u)\delta(u-v) = E\{\underset{\sim}{B}(u)\underset{\sim}{B}(v)\} \qquad E\{\underset{\sim}{A}(u)\underset{\sim}{B}(v)\} = 0$$

for $u \geq 0, v \geq 0$. We shall show that if the above is true and $E\{\underset{\sim}{A}(\omega)\} = E\{\underset{\sim}{B}(\omega)\} = 0$, then the process

$$\underset{\sim}{x}(t) = \frac{1}{\pi}\int_0^{\infty}\left[\underset{\sim}{A}(\omega)\cos\omega t - \underset{\sim}{B}(\omega)\sin\omega t\right] d\omega$$

is WSS.

<u>Proof</u> Clearly, $E\{\underset{\sim}{x}(t)\} = 0$ and

$E\{\underset{\sim}{x}(t+\tau)\underset{\sim}{x}(t)\}$

$$= \frac{1}{\pi^2}\int_0^\infty\int_0^\infty E\{\underset{\sim}{A}(u)\cos u(t+\tau) - \underset{\sim}{B}(u)\sin u(t+\tau)\}\,[\underset{\sim}{A}(v)\cos vt - B(v)\sin vt]\,du dv$$

$$= \frac{1}{\pi}\int_0^\infty\int_0^\infty Q(u)\delta(u-v)\,[\cos u(t+\tau)\cos vt + \sin u(t+\tau)\sin v(t)\,du dv\,]\,dv du$$

$$= \frac{1}{\pi^2}\int_0^\infty Q(u)\,[\cos u(t+\tau)\cos u\,t + \sin u(t+\tau)\sin ut]\,du$$

$$= \frac{1}{\pi^2}\int_0^\infty Q(u)\cos u\tau du$$

From this and (9-136) it follows that $\underset{\sim}{x}(t)$ is WSS with $S_{xx}(\omega) = Q(\omega)/\pi$.

11-14 $\qquad E\{\underset{\sim}{v}(t)\} = 0 \qquad E\{\underset{\sim}{X}_T(\omega)\} = \int_{-T}^{T} f(t)e^{-j\omega t}dt$

The above integral is the transform of the product $f(t)p_T(t)$, hence (frequency convolution theorem), it equals $F(\omega)*\sin T\omega/\pi\omega$.

$$\text{Var}\ \underset{\sim}{X}_T(\omega) = E\left\{\ \left|\ \int_{-T}^{T}\underset{\sim}{\nu}(t)e^{-j\omega t}dt\ \right|^2\right\}$$

The integral is the transform of the nonstationary white noise $\underset{\sim}{\nu}(t)p_T(t)$. The autocorrelation of this process equals $q(t_1)\delta(t_1-t_2)$ where $q(t) = qp_T(t)$. Hence, [see (11-69)]

$$\text{Var}\ \underset{\sim}{X}_T(\omega) = Q(0) = \int_{-T}^{T} q dt = 2qT$$

CHAPTER 14

14-1 It suffices to show that [see (14-41)]

$$H(A \cdot B|B_j) = H(A|B_j)$$

Since

$$A_iB_kB_j = \begin{cases} A_iB_j & k = j \\ \{\emptyset\} & k \neq j \end{cases} \qquad \text{and } P(A_iB_j|B_j) = P(A_i|B_j)$$

(14-40) yields

$$H(A \cdot B|B_j) = -\sum_{i,k} P(A_iB_k|B_j) \log P(A_iB_k|B_j)$$

$$= -\sum_i P(A_i|B_j) \log P(A_i|B_j) = H(A|B_j)$$

14-2 If $\alpha < \beta$, then $\phi'(\alpha) > \phi'(\beta)$ because

$$\phi'(\alpha) - \phi'(\beta) = \log(\beta/\alpha) > 0. \quad \text{Hence,}$$

$$\int_a^b \phi'(\alpha)d\alpha > \int_{a+c}^{b+c} \phi'(\alpha)d\alpha \qquad c > 0$$

This yields

$$\phi(p_1 + p_2) - \phi(p_1) = \int_{p_1}^{p_1+p_2} \phi'(\alpha)d\alpha < \int_0^{p_2} \phi'(\alpha)d\alpha = \phi(p_2)$$

Similarly

$$\phi(p_1 + \varepsilon) - \phi(p_1) - \phi(p_2) + \phi(p_2 - \varepsilon)$$

$$= \int_{p_1}^{p_1+\varepsilon} \phi'(\alpha)d\alpha - \int_{p_2-\varepsilon}^{p_2} \phi'(\alpha)d\alpha > 0$$

14-3 Applying the identity

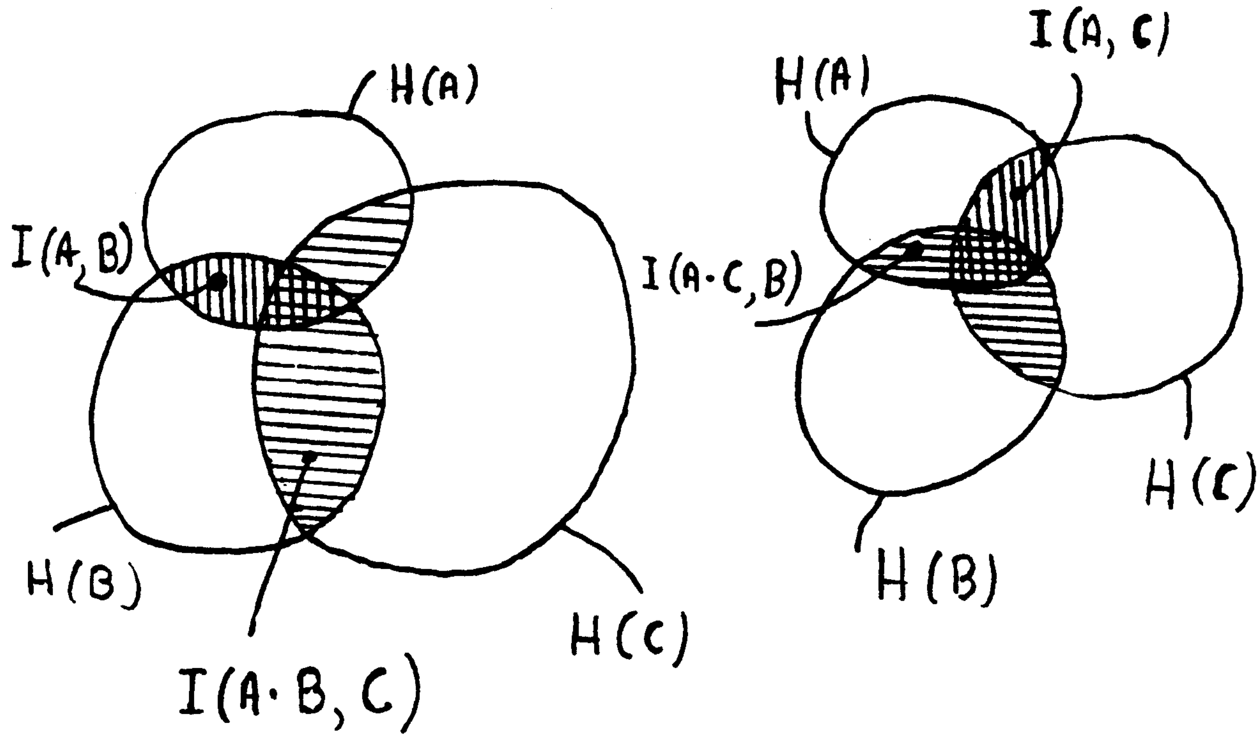
$$H(A_1 \cdot A_2) = H(A_1) + H(A_2|A_1) \tag{i}$$

to the partitions $A_1 = A$, $A_2 = B \cdot C$ and $A_1 = A \cdot B$, $A_2 = C$, we obtain the first line. The second line follows from the first [see (i)]. The third line is a consequence of the first two.

14-4 It follows if we apply the identity

$$I(A_1, A_2) = H(A_1) + H(A_2) - H(A_1 \cdot A_2)$$

to the partitions $A_1 = A \cdot B$, $A_2 = C$.



14-5 **(a)** From (14-53)

$$I(A,B \cdot C) = H(A) + H(B \cdot C) - H(A \cdot B \cdot C)$$

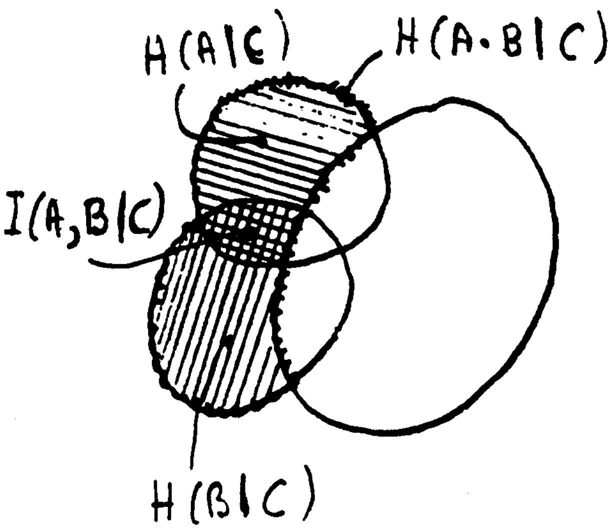
$$I(A,C) = H(A) + H(C) - H(A \cdot C)$$

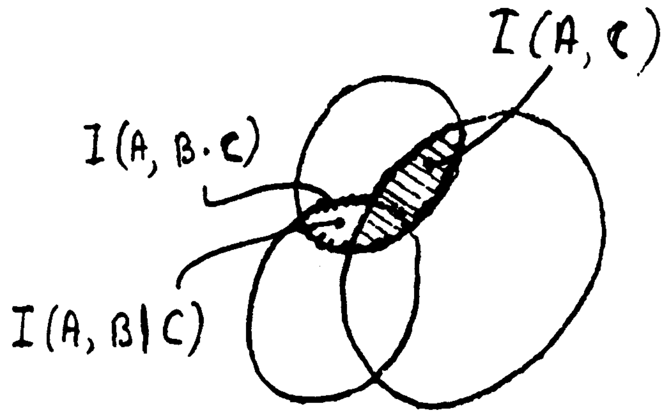
and since (see Prob. 14-4)

$$H(A \cdot B \cdot C) - H(A \cdot C) = H(A \cdot B|C) - H(A|C)$$

we conclude with (14-49) that

$$I(A,B \cdot C) - I(A \cdot C) = H(B|C) + H(A|C) - H(A \cdot B|C)$$





(b) If $B \cdot C$ is observed, then the resulting prediction in the uncertainty of A equals $I(A, B \cdot C)$. But, if $B \cdot C$ is observed, then C is observed, hence, the reduction in the uncertainty of A is at least $I(A,C)$. Hence

$$I(A,B \cdot C) \geq I(A,C)$$

with equality only if $I(A,B|C) = 0$, i.e., if in the subsequence of trials in which C occurred, knowledge of the occurrence of B gives no information about A.

14-6 The partition $H(A^3)$ has eight elements with respective probabilities

$$p^3, p^2q, p^2q, p^2q, pq^2, pq^2, pq^2, q^3$$

Hence

$$H(A^3) = -p^3\log p^3 - 3p^2q\log p^2q - 3pq^2\log pq^2 - q^3\log q^3$$

$$= -3p(p^2+2pq+q^2)\log p - 3q(p^2+2pq+q^2)\log q$$

$$= -3p\log p - 3q\log q = 3H(A)$$

14-7 The density of the RV $\underset{\sim}{w} = \underset{\sim}{x} + a$ equals $f_x(w-a)$. Hence,

$$H(\underset{\sim}{x}+a) = -\int_{-\infty}^{\infty} f_x(w-a)\log f_x(w-\alpha)\,dw$$

$$= -\int_{-\infty}^{\infty} f_x(x)\log f_x(x)\,dx = H(\underset{\sim}{x})$$

The joint density of the RVs $\underset{\sim}{x}$ and $\underset{\sim}{z} = \underset{\sim}{x}+\underset{\sim}{y}$ equals $f_{xy}(x,z-x)$. Hence [see (14-90)]

$$H(\underset{\sim}{z}|\underset{\sim}{x}) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,z-x)\log f_{xy}(x,z-x)/f_x(x)\,dx\,dz$$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y)\log f_{xy}(x,y)/f_x(x)\,dx\,dy = H(\underset{\sim}{y}|\underset{\sim}{x})$$

14-8 The RVs $\underset{\sim}{x}$ and $\underset{\sim}{y}$ take the values x_i and y_j respectively then $\underset{\sim}{z} = x_i + y_j$ iff $\underset{\sim}{x} = x_i$ and $\underset{\sim}{y} = y_j$ (assumption). Hence,

$$\{\underset{\sim}{z} = x_i + y_j\} = \{\underset{\sim}{x} = x_i\} \cap \{\underset{\sim}{y} = y_j\}$$

This shows that $A_z = A_x \cdot B_y$. Furthermore, since the RVs $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are independent, the events $\{\underset{\sim}{x} = x_i\}$ and $\{\underset{\sim}{y} = y_j\}$ are also independent. This shows that the partitions A_x and B_y are independent and [see (14-44) and Prob. 14-1]

$$H(A_z|A_x) = H(A_x \cdot A_y|A_x) = H(A_y|A_x) = H(A_y)$$

From this it follows that $H(\underset{\sim}{z}|\underset{\sim}{x}) = H(\underset{\sim}{y})$ because [see (14-88) and (14-41)]

$$H(\underset{\sim}{z}|\underset{\sim}{x}) = H(A_z|A_x)$$

14-9 As we see from (14-80)

$H(\underset{\sim}{x}) = \ln a$ where we assume that $a = N\delta$. The RV $\underset{\sim}{y}$ takes the values $0, \delta, \ldots, (N-1)\delta$ with probability $1/N$. The conditional density of $\underset{\sim}{x}$ assuming $\underset{\sim}{y} = k\delta$ is uniform in the interval $(k\delta, k\delta + \delta)$. Hence,

$$H(\underset{\sim}{x}|\underset{\sim}{y} = k\delta) = -\int_{k\delta}^{k\delta+\delta} f(x|\underset{\sim}{y} = k\delta) \ln f(x|\underset{\sim}{y} = k\delta)\, dx = \ln \delta$$

And as in (14-41)

$$H(\underset{\sim}{x}|\underset{\sim}{y}) = \sum_{k=0}^{N} H(\underset{\sim}{x}|\underset{\sim}{y} = k\delta) P\{\underset{\sim}{y} = k\delta\} = \ln \delta$$

Finally [see (14-95)]

$$I(\underset{\sim}{x}, \underset{\sim}{y}) = H(\underset{\sim}{x}) - H(\underset{\sim}{x}|\underset{\sim}{y}) = \ln a - \ln \delta$$

14-10 If $y_i = g(x_i)$, $y_j = g(x_j)$ and $y_i = y_j$ then $x_i = x_j$. Hence,

$$p_{ij} = \begin{cases} p_i & i = j \\ 0 & i \neq j \end{cases} \qquad p_i = P\{\underset{\sim}{x} = x_i\}$$

and

$$H(\underset{\sim}{x}, \underset{\sim}{y}) = -\sum_{i,j} p_{ij} \log p_{ij} = -\sum_i p_i \log p_i = H(\underset{\sim}{x})$$

14-11 From Prob. 10-10 it follows with $g(x) = x$ that $H(\underset{\sim}{x},\underset{\sim}{x}) = H(\underset{\sim}{x})$. And since [see (14-103)] $H(\underset{\sim}{x},\underset{\sim}{x}) = H(\underset{\sim}{x}|\underset{\sim}{x}) + H(\underset{\sim}{x})$ we conclude that $H(\underset{\sim}{x}|\underset{\sim}{x}) = 0$. From Prob. 14-3 it follows that

$$H(\underset{\sim}{y},\underset{\sim}{x}|\underset{\sim}{x}) = H(A_y \cdot A_x|A_x) = H(A_x \cdot A_x) + H(A_y|A_x \cdot A_x)$$

$$= H(A_y|A_x) = H(\underset{\sim}{y}|\underset{\sim}{x})$$

because $A_x \cdot A_x = A_x$ and $H(A_x \cdot A_x) = H(\underset{\sim}{x},\underset{\sim}{x}) = 0$.

14-12 $$E\{\underset{\sim}{x}_n\} = 0 \qquad E\{\underset{\sim}{x}_n^2\} = 5 \qquad E\{\underset{\sim}{y}_n\} = 0$$

$$E\{\underset{\sim}{y}_n^2\} = \sum_{k=0}^{\infty} 2^{-2k} E\{\underset{\sim}{x}_{n-k}^2\} = \frac{20}{3} \qquad E\{\underset{\sim}{x}_n\underset{\sim}{y}_n\} = E\{\underset{\sim}{x}_n^2\} = 5$$

(a) From (14-95), (14-84), and (15-86) with $\mu_{11} = 5$, $\mu_{22} = 20/3$, and $\mu_{12} = 5$

$$H(\underset{\sim}{x}) = \ln\sqrt{10\pi e} \qquad H(\underset{\sim}{y}) = \ln\sqrt{40\pi e/3} \qquad H(\underset{\sim}{x},\underset{\sim}{y}) = \ln 10\pi e/\sqrt{3}$$

$$I(\underset{\sim}{x},\underset{\sim}{y}) = \ln 2$$

(b) The process $\underset{\sim}{y}(t)$ is the output of the system

$$L(z) = \frac{1}{1 - 0.5\, z^{-1}} \qquad \ell_o = 1$$

with input $\underset{\sim}{x}_n$. Since $\bar{H}(\underset{\sim}{x}) = H(\underset{\sim}{x})$ and [see (12A-1)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln|L(e^{j\phi})|\,d\phi = \ln\ell_o = 0$$

(14-133) yields $\bar{H}(\underset{\sim}{y}) = \bar{H}(\underset{\sim}{x}) = H(\underset{\sim}{x}) = \ln\sqrt{10\pi e}$.

14-13
$$\bar{H}(\underset{\sim}{x}) = H(\underset{\sim}{x}) = -\frac{1}{2}\int_4^6 \ln\frac{1}{2}\,dx = \ln 2$$

And as in Prob. 14-12 with $\ell_o = 5$,

$$\bar{H}(\underset{\sim}{y}) = \bar{H}(x) + \ln 5 = \ln 10$$

14-14 Given that $f(x) = 0$ for $|x|>1$ and $E\{\underset{\sim}{x}\} = 0.3$, find $f(x)$. With $g(x) = x$, (14-143) yields $f(x) = Ae^{-\lambda x}$ where

$$A\int_{-1}^{1} e^{-\lambda x}\,dx = \frac{A}{\lambda}(e^{\lambda} - e^{-\lambda}) = 1$$

$$A\int_{-1}^{1} xe^{-\lambda x}\,dx = \frac{A}{\lambda^2}(e^{\lambda} - e^{-\lambda}) - \frac{A}{\lambda}(e^{\lambda} - e^{-\lambda}) = 0.31$$

Solving, we obtain $A \simeq 0.425$, $\lambda \simeq -1$

14-15 $f(x) = Ae^{-\lambda x}$ for $1<x<5$ and 0 otherwise,

$$A\int_1^5 e^{-\lambda x}\,dx = 0.31 \qquad A\int_1^5 xe^{-\lambda x}dx = 3\,\frac{37}{60}$$

Hence, $A \simeq 1.06$, $\lambda \simeq 0.5$

14-16 From (14-151) with $x_k=k$, $g_1(x_k) = g_1(k) = k$, $k=1, \ldots, 6$

$$g_2(x_k)=\begin{cases} 0 & k=1,3,5 \\ 1 & k=2,4,6 \end{cases} \quad p_k=\begin{cases} Ae^{-\lambda_1 k} & k=1,3,5 \\ Ae^{-\lambda_1 x-\lambda_2} & k=2,4,6 \end{cases}$$

Since $p_1 + p_3 + p_5 = 0.5$ and $E\{\underset{\sim}{x}\} = 4.44$, we conclude with $z = e^{-\lambda_2}$ and $w = e^{-\lambda_2}$ that

$$A(z+z^3+z^5) = Aw(z^2+z^4z^6)$$

$$A(\underset{\sim}{z}+3z^3+5z^5) + Aw(2z^2+4z^4+6z^6) = 4.44$$

This yields $A \simeq 0.0437$, $\underset{\sim}{z} = 1/w \simeq 1.468$

14-17 (a) The transformation $\underset{\sim}{y} = 3\underset{\sim}{x}$ is one-to-one, hence, $H(\underset{\sim}{y}) = H(\underset{\sim}{x})$

(b) From (14-113) with $g(x) = 3x$: $H(\underset{\sim}{y}) = H(\underset{\sim}{x}) + \ln 3$

14-18 (a) For fair dice, $P\{7\} = \frac{1}{6}$, $P(11) = \frac{1}{18}$, $P\{\text{neither 7 nor 11}\} = \frac{14}{18}$

$$H(A) = -\left(\frac{1}{6}\ln\frac{1}{6} + \frac{1}{18}\ln\frac{1}{18} + \frac{14}{18}\ln\frac{14}{18}\right) = 0.655$$

(b) From (14-10) with $n=100$ and $N=3$:

$$n_T \simeq e^{nH(A)} \simeq 2.79 \times 10^{28} \qquad n_a \simeq N^n \simeq 5.16 \times 10^{47}$$

14-19 The process $\underset{\sim}{x}_n$ is WSS with entropy rate $\overline{H}(x)$. Show that, if

$$\underset{\sim}{w}_n = \sum_{k=0}^{n} \underset{\sim}{x}_{n-k}\, \ell_k$$

then

$$\lim_{n\to\infty} \frac{1}{n+1} H(\underset{\sim}{w}_o,\ldots,\underset{\sim}{w}_n) = \overline{H}(x) + \ln|\ell_o| \qquad \text{(i)}$$

Proof. The RVs $\underset{\sim}{w}_o,\ldots,\underset{\sim}{w}_n$ are linear transformations of the RVs $\underset{\sim}{x}_o,\ldots,\underset{\sim}{x}_n$ and the transformation matrix equals

$$\begin{bmatrix} \ell_0 & 0 & \cdots & 0 \\ \ell_1 & \ell_0 & \cdots & 0 \\ \hline \ell_n & \ell_{n-1} & \cdots & 0 \end{bmatrix}$$

Since the determinant of this transformation equals $|\ell_o|^{n+1}$, (14-115) yields

$$H(\underset{\sim}{w}_o,\ldots,\underset{\sim}{w}_n) = H(\underset{\sim}{x}_o,\ldots\underset{\sim}{x}_n) + (n+1)\ln|\ell_o|$$

Dividing by (n+1) we obtain (i) as $n\to\infty$.

14-20 As in Example 14-19, $f(p) = A\, e^{-\lambda p}$. To find λ, we use the $\lambda-\eta$ curve of Fig. 14-16. This yields

$$\lambda = -1.23 \qquad f(p) = 0.51\, e^{1.23p}$$

14-21 As in Example 14-22, $p_k = A\, e^{-\lambda k}$. To find λ, we use the w-η curve of Fig. 14-17. This yields (see also Jaynes)

$$w \simeq 1.449 \qquad \lambda \simeq -0.371$$

p_1	p_2	p_3	p_4	p_5	p_6
0.054	0.079	0.114	0.165	0.240	0.348

14-22 The unknown density is normal as in (14-157) where

$$\Delta = \begin{vmatrix} 4 & 1 & 1 \\ 1 & 4 & m_{23} \\ 1 & m_{23} & 4 \end{vmatrix} = -4m_{23}^2 + 2m_{23} + 56$$

The moment $m_{23} = E\{\mathbf{x}_2\mathbf{x}_3\}$ must be such as to maximize Δ. This yields $m_{23} = 0.25$.

14-23

	p_i	0.3	0.2	0.15	0.15	0.1	0.06	0.04	
		$\frac{1}{4} \le p_i < \frac{1}{2}$	$\frac{1}{8} \le p_i < \frac{1}{4}$			$\frac{1}{16} \le p_i < \frac{1}{8}$	$\frac{1}{32} \le p_i < \frac{1}{6}$		$\sum_{i=1}^{7} \frac{1}{2^{n_i}}$
Shannon	n_i	2	3	3	3	4	5	5	0.75
		2	3	3	3	4	4	4	0.8125
		2	3	3	3	3	4	4	0.875
L = 2.7		2	3	3	3	3	3	4	0.9375
		2	3	3	3	3	3	3	1
	x_i	00	010	011	100	101	110	111	

Fano

$L = 2.6$

p_i	0.3	0.2	0.15	0.15	0.1	0.06	0.04
	A_0 0.5		A_1 0.5				
	A_{00} 0.3	A_{01} 0.2	A_{10} 0.3		A_{11} 0.2		
			A_{100} 0.15	A_{101} 0.15	A_{110} 0.1	A_{111} 0.1	
						A_{1110}	A_{1111}
x_i	00	01	100	101	110	1110	1111

Huffman

$L = 2.6$

	1	2	3	4	5	6	7
	1	2	3	4	5	6 / 0	7 / 1
	1	2	5 / 0	6 / 10	7 / 11	3	4
	1	3 / 0	4 / 1	2	5 / 0	6 / 10	7 / 11
	2 / 0	5 / 10	6 / 110	7 / 111	1	3 / 0	4 / 1
	1 / 0	3 / 10	4 / 11	2 / 0	5 / 10	6 / 110	7 / 111
	1 / 00	3 / 010	4 / 011	2 / 10	5 / 110	6 / 1110	7 / 1111
x_i	00	10	010	011	110	1110	1111

14-24 If $\underline{x}_n = 0$, then $\bar{x}_n = 000$ and $\underline{y}_n = 1$ iff $\bar{y}_n$ consists of one 0 or no zeros. The probability of one and only one zero equals $3\beta^2(1-\beta)$ [see (3-13)]; the probability of no zeros equals β^3. Hence,

$$P\{\underline{y}_n = 1 | \underline{x}_n = 0\} = 3\beta^2(1-\beta) + \beta^3$$

Thus, the redundantly coded channel of Example 14-29 is symmetrical as in (14-191) with probability of error $\beta_1 = \beta^2$.

14-25 If the received information is always wrong, then

$$P\{\underline{y}_n = 1 | \underline{x}_n = 0\} = \beta = 1, \text{ hence } C = 1 - r(\beta) = 1$$
