DIGITAL SIGNAL PROCESSING

Covers Signals and Systems

Includes MATLAB Programs

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Tata McGraw-Hill

21st reprint 2007
DZLCRRXYRCYZR

This edition can be exported from India by the publishers, Tata McGraw-Hill Publishing Company Limited

ISBN 0-07-463996-X

Published by Tata McGraw-Hill Publishing Company Limited, 7 West Patel Nagar, New Delhi 110 008, typeset at Anvi Composers, A1/33 Pashchim Vihar, New Delhi 110 063 and printed at A P Offset Pvt. Ltd., Naveen Shahdara, Delhi 110 032

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Chapter 1

Classification of Signals and Systems

1.1 INTRODUCTION

Signals play a major role in our life. In general, a signal can be a function of time, distance, position, temperature, pressure, etc., and it represents some variable of interest associated with a system. For example, in an electrical system the associated signals are electric current and voltage. In a mechanical system, the associated signals may be force, speed, torque, etc. In addition to these, some examples of signals that we encounter in our daily life are speech, music, picture and video signals. A signal can be represented in a number of ways. Most of the signals that we come across are generated naturally. However, there are some signals that are generated synthetically. In general, a signal carries information, and the objective of signal processing is to extract this information.

Signal processing is a method of extracting information from the signal which in turn depends on the type of signal and the nature of information it carries. Thus signal processing is concerned with representing signals in mathematical terms and extracting the information by carrying out algorithmic operations on the signal. Mathematically, a signal can be represented in terms of basic functions in the domain of the original independent variable or it can be represented in terms of basic functions in a transformed domain. Similarly, the information contained in the signal can also be extracted either in the original domain or in the transformed domain.

A system may be defined as an integrated unit composed of diverse, interacting structures to perform a desired task. The task may vary such as filtering of noise in a communication receiver, detection of range of a target in a radar system, or monitoring steam pressure in a boiler. The function of a system is to process a given input sequence to generate an output sequence.

It is said that digital signal processing techniques origin in the seventeenth century when finite difference methods, numerical integration methods, and numerical interpolation methods were developed to solve physical problems involving continuous variables and functions. There has been a tremendous growth since then and today digital signal processing techniques are applied in almost every field. The main reasons for such wide applications are due to the numerous advantages of digital signal processing techniques. Some of these advantages are discussed subsequently.

Digital circuits do not depend on precise values of digital signals for their operation. Digital circuits are less sensitive to changes in component values. They are also less sensitive to variations in temperature, ageing and other external parameters.

In a digital processor, the signals and system coefficients are represented as binary words. This enables one to choose any accuracy by increasing or decreasing the number of bits in the binary word.

Digital processing of a signal facilitates the sharing of a single processor among a number of signals by time-sharing. This reduces the processing cost per signal.

Digital implementation of a system allows easy adjustment of the processor characteristics during processing. Adjustments in the processor characteristics can be easily done by periodically changing the coefficients of the algorithm representing the processor characteristics. Such adjustments are often needed in adaptive filters.

Digital processing of signals also has a major advantage which is not possible with the analog techniques. With digital filters, linear phase characteristics can be achieved. Also multirate processing is possible only in the digital domain. Digital circuits can be connected in cascade without any loading problems, whereas this cannot be easily done with analog circuits.

Storage of digital data is very easy. Signals can be stored on various storage media such as magnetic tapes, disks and optical disks without any loss. On the other hand, stored analog signals deteriorate rapidly as time progresses and cannot be recovered in their original form.

For processing very low frequency signals like seismic signals, analog circuits require inductors and capacitors of a very large size whereas, digital processing is more suited for such applications.

Though the advantages are many, there are some drawbacks associated with processing a signal in the digital domain. Digital processing needs 'pre' and 'post' processing devices like analog-to-digital and digital-to-analog converters and associated reconstruction filters. This increases the complexity of the digital system. Also, digital techniques suffer from frequency limitations. For reconstructing a signal from its sample, the sampling frequency must be atleast twice the highest frequency component present in that signal. The available frequency range of operation of a digital signal processor is primarily determined by the sample-and-hold circuit and the analog-to-digital converter, and as a result is limited by the technology available at that time. The highest sampling frequency is presently around 1GHz reported by K.Poulton, etal., in 1987. However, such high sampling frequencies are not used since the resolution of the A/D converter decreases with an increase in the speed of the converter. But the advantages of digital processing techniques outweigh the disadvantages in many applications. Also, the cost of DSP hardware is decreasing continuously. Consequently, the applications of digital signal processing are increasing rapidly.

1.2 CLASSIFICATION OF SIGNALS

Signals can be classified based on their nature and characteristics in the time domain. They are broadly classified as (i) *continuous-time signals* and (ii) *discrete-time signals*. A continuous-time signal is a mathematically continuous function and the function is defined continuously in the time domain. On the other hand, a discrete-time signal is specified only at certain time instants. The amplitude of the discrete-time signal between two time instants is just not defined. Figure 1.1 shows typical continuous-time and discrete-time signals.

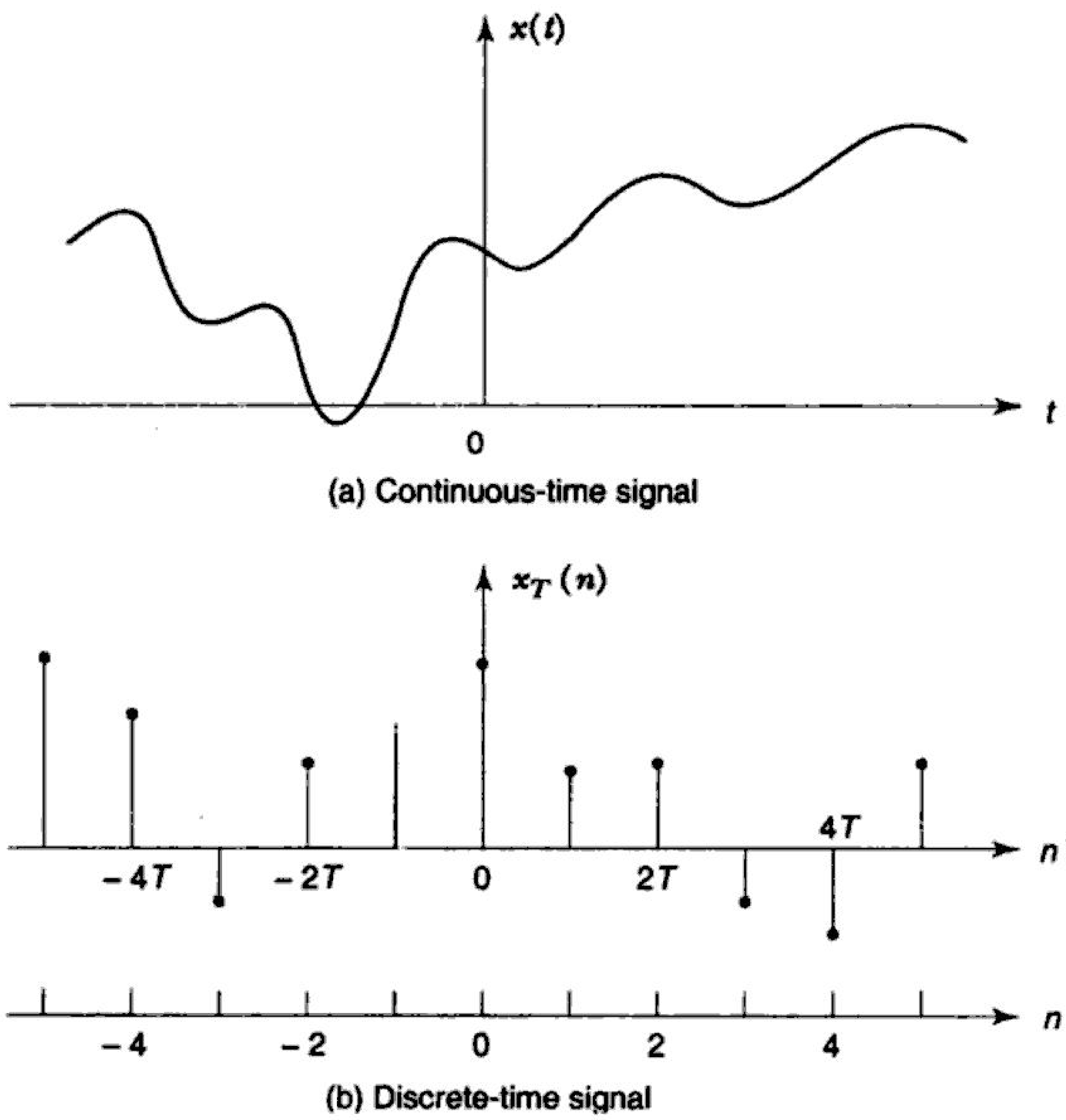


Fig. 1.1 *Continuous-Time and Discrete-Time Signals*

Both continuous-time and discrete-time signals are further classified as

(i) Deterministic and non-deterministic signals
(ii) Periodic and aperiodic signals
(iii) Even and odd signals, and
(iv) Energy and power signals.

1.2.1 Deterministic and Non-deterministic Signals

Deterministic signals are functions that are completely specified in time. The nature and amplitude of such a signal at any time can be predicted. The pattern of the signal is regular and can be characterised mathematically. Examples of deterministic signals are

(i) $x(t) = \alpha t$ This is a ramp whose amplitude increases linearly with time and slope is α.
(ii) $x(t) = A \sin \omega t$. The amplitude of this signal varies sinusoidally with time and its maximum amplitude is A.
(iii) $x(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$ This is a discrete-time signal whose amplitude is 1 for the sampling instants $n \geq 0$ and for all other samples, the amplitude is zero.

For all the signals given above, the amplitude at any time instant can be predicted in advance. Contrary to this, a non-deterministic signal is one whose occurrence is random in nature and its pattern is quite irregular. A typical example of a non-deterministic signal is **thermal noise** in an electrical circuit. The behaviour of such a signal is probabilistic in nature and can be analysed only stochastically. Another example which can be easily understood is the number of accidents in an year. One cannot exactly predict what would be the figure in a particular year and this varies randomly. Non-deterministic signals are also called **random signals**.

1.2.2 Periodic and Aperiodic Signals

A continuous-time signal is said to be periodic if it exhibits periodicity, i.e.

$$x(t + T) = x(t), \quad -\infty < t < \infty \tag{1.1}$$

where T is the period of the signal. The smallest value of T that satisfies Eq. 1.1 is called the fundamental period, T_o, of the signal. A periodic signal has a definite pattern that repeats over and over, with a repetition period of T_o. For a discrete-time signal, the condition for periodicity can be written as,

$$x(n + N_o) = x(n), \quad -\infty < n < \infty \tag{1.2}$$

where N_o is the sampling period measured in units of number of sample spacings. Periodic signals can be in general, expressed as

(i) Continuous-Time Periodic Signals

$$x_p(t) = \sum_{i=-\infty}^{\infty} X(t - iT_o) \tag{1.3}$$

where

$$X(t) = \begin{Bmatrix} x(t), & t_1 \le t < t_1 + T_o \\ 0, & \text{elsewhere} \end{Bmatrix} \tag{1.4}$$

(ii) Discrete-Time Periodic Signals

$$x_p(n) = \left[\sum_{i=-\infty}^{\infty} X(n - iN_o), T \right] \tag{1.5}$$

where

$$X(n) = \begin{Bmatrix} x(n), & n_1 \le n < (n_1 + N_o) \\ 0, & \text{elsewhere} \end{Bmatrix} \tag{1.6}$$

and T is the sampling period in seconds.

A signal which does not satisfy either Eq. 1.1 or 1.2 is called an **aperiodic signal**. Some examples of periodic signals are shown in Fig. 1.2. Some periodic signals can be simply modelled using a single equation. For example,

$$x(t) = A \sin\left(\frac{2\pi t}{T_o} - \varphi\right) \tag{1.7}$$

is a continuous-time sinusoidal signal which is valid for all t. The constant A represents the maximum amplitude and φ represents the phase shift of the sinusoidal signal. Similarly, a periodic discrete-time sinusoidal signal is represented as

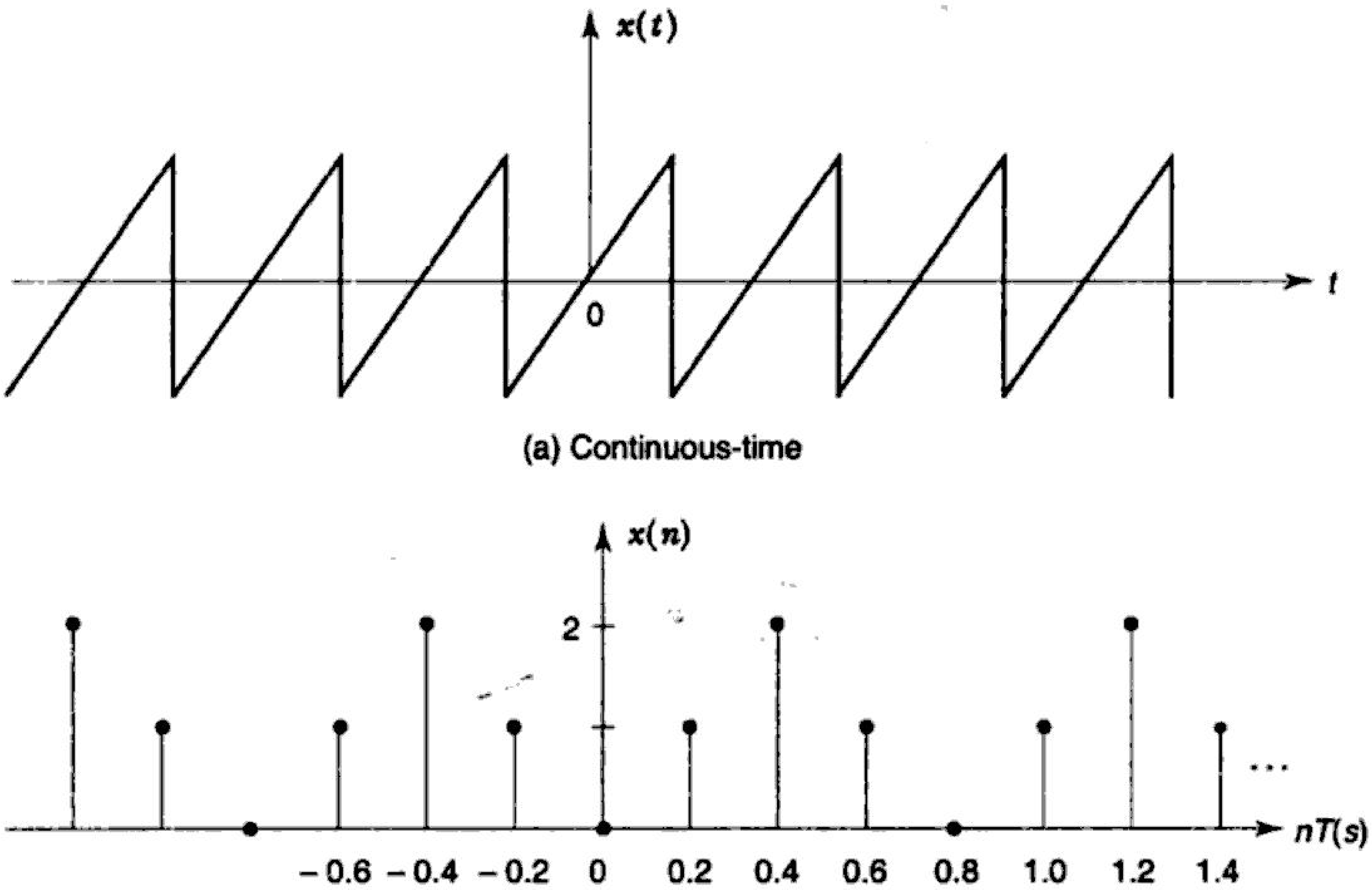
$$x(n) = \left[A \sin\left(\frac{2\pi n}{N_0} - \beta\right), T \right] \tag{1.8}$$

The term β represents the delay and T represents the sampling period.

The sum of two or more periodic continuous-time signals need not be periodic. They will be periodic if and only if the ratio of their fundamental periods is rational. In order to determine whether the sum of two or more periodic signals is periodic or not, the following steps may be used.

(i) Determine the fundamental period of the individual signals in the sum signal.
(ii) Find the ratio of the fundamental period of the first signal with the fundamental periods of every other signal.
(iii) If all these ratios are rational, then the sum signal is also periodic.

In the case of discrete-time signals, the sum of a number of periodic signals is always periodic because the ratio of individual periods is always the ratio of integers, which is rational.



(a) Continuous-time

$$x_p = \left[\sum_{i=-\infty}^{\infty} X[n - iN_0], T \right]$$

Where

$$x[n] = \begin{cases} -n, & -2 \le n < 0 \\ n, & 0 \le n < 2 \\ 0, & \text{otherwise} \end{cases}$$

$N_0 = 4$ and $T = 0.25$

(b) *Discrete-time*

Fig. 1.2 *Some Examples of Periodic Signals*

Example 1.1 Determine which of the following signals are periodic.

(a) $x_1(t) = \sin 15\,\pi t$ (b) $x_2(t) = \sin 20\,\pi t$ (c) $x_3(t) = \sin \sqrt{2}\,\pi t$
(d) $x_4(t) = \sin 5\pi t$ (e) $x_5(t) = x_1(t) + x_2(t)$
(f) $x_6(t) = x_2(t) + x_4(t)$

Solution

(a) $x_1(t) = \sin 15\,\pi t$ is periodic.

The fundamental period is $T_o = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{15\pi}$

$= 0.1333333333\ldots$ seconds

(b) $x_2(t) = \sin 20\,\pi t$ is periodic.

The fundamental period is $T_o = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{20\pi} = 0.1$ seconds

(c) $x_3(t) = \sin \sqrt{2}\pi t$ is periodic.

The fundamental period is $T_o = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{2}\pi}$

$= 1.41421356...$ seconds

(d) $x_4(t) = \sin 5\pi t$ is periodic.

The fundamental period is $T_o = \frac{2\pi}{\omega} = \frac{2\pi}{5\pi} = 0.4$ seconds

(e) $x_5(t) = x_1(t) + x_2(t)$

The fundamental period of $x_1(t) = T_{o1} = 0.13333333...$ seconds and the fundamental period of $x_2(t) = T_{o2} = 0.1$ seconds. The ratio of fundamental frequencies, $\frac{T_{o1}}{T_{o2}} = \frac{0.13333333333\ldots}{0.1}$, cannot be expressed as a ratio of integers. Hence, $x_5(t)$ is not periodic.

(f) $x_6(t) = x_2(t) + x_4(t)$

The fundamental period of $x_2(t) = T_{o2} = 0.1$ seconds and the fundamental period of $x_4(t) = T_{o4} = 0.4$ seconds. The ratio of fundamental frequencies, $\frac{T_{o2}}{T_{o4}} = \frac{0.1}{0.4} = \frac{1}{4}$, can be expressed as a ratio of integers. Hence, $x_5(t)$ is periodic.

1.2.3 Even and Odd Signals

If a signal exhibits symmetry in the time domain, it is called an *even signal*. The signal must be identical to its reflection about the origin. Mathematically, an even signal satisfies the following relation.

For a continuous-time signal, $x(t) = x(-t)$ (1.9a)

For a discrete-time signal, $x(n) = x(-n)$ (1.9b)

An *odd signal* exhibits anti-symmetry. The signal is not identical to its reflection about the origin, but to its negative. An odd signal satisfies the following relation.

For a continuous-time signal, $x(t) = -x(-t)$ (1.10a)

For a discrete-time signal, $x(n) = -x(-n)$ (1.10b)

$x_1(t) = \sin \omega t$ and $x_2(t) = \cos \omega t$ are good examples of odd and even signals, respectively. Figure 1.3 shows the typical odd and even signals. An even signal which often occurs in the analysis of signals is the sinc function. The sinc function may be expressed in the following two ways according to our convenience: (i) sinc $(x) = \frac{\sin x}{x}$ and (ii) sinc $(x) = \frac{\sin(\pi x)}{\pi x}$. In chapter 2, the first expression is used.

The area under the sinc function is unity. The sinc function is shown in Fig. 1.3c. The positive portions of the sinc function have angles of

$\pm n\pi$ where n is an even integer, and the negative portions of the sinc function have angles of $\pm m\pi$ where m is odd. It can be seen from Fig. 1.3c that the sinc function exhibits symmetry about $x = 0$.

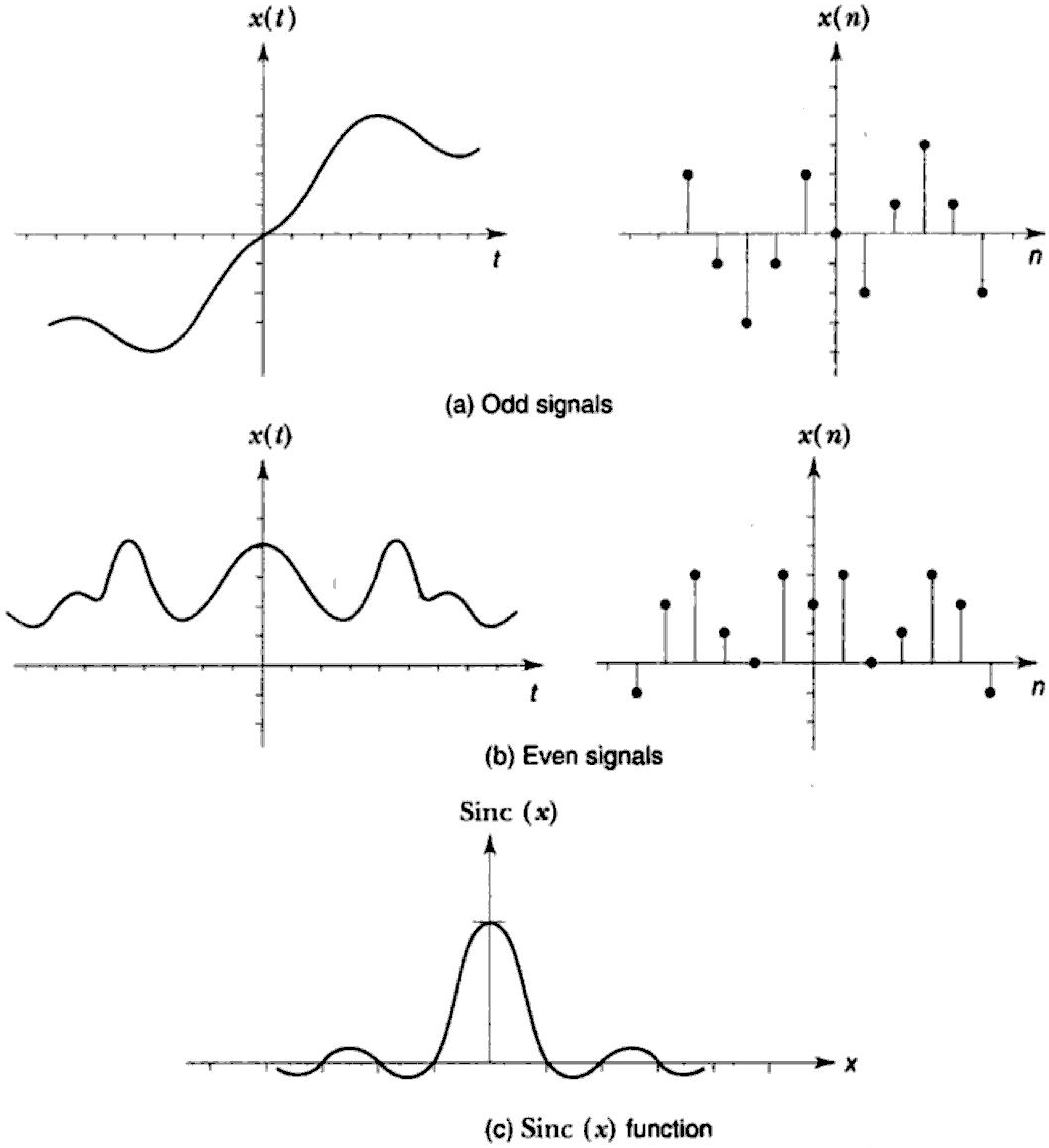


Fig. 1.3 *Typical Examples for (a) Odd Signal and (b) Even Signal (c) the Sinc (x) Function*

A signal can be expressed as a sum of two components, namely, the even component of the signal and the odd component of the signal. The even and odd components can be obtained from the signal itself, as given below.

$$x(t) = x_{\text{even}}(t) + x_{\text{odd}}(t) \tag{1.11}$$

where

$$x_{\text{even}}(t) = \frac{1}{2}[x(t) + x(-t)] \text{ and } x_{\text{odd}}(t) = \frac{1}{2}[x(t) - x(-t)]$$

1.2.4 Energy and Power Signals

Signals can also be classified as those having finite energy or finite average power. However, there are some signals which can neither be classified as energy signals nor power signals. Consider a voltage source $v(t)$, across a unit resistance R, conducting a current $i(t)$. The instantaneous power dissipated by the resistor is

$$p(t) = v(t)\, i(t) = \frac{v^2(t)}{R} = i^2(t)R$$

Since $R = 1$ ohm, we have

$$p(t) = v^2(t) = i^2(t) \tag{1.12}$$

The total energy and the average power are defined as the limits

$$E = \lim_{T \to \infty} \int_{-T}^{T} i^2(t)\, dt, \quad \text{joules} \tag{1.13}$$

and

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} i^2(t)\, dt, \text{ watts} \tag{1.14}$$

The total energy and the average power normalised to unit resistance of any arbitrary signal $x(t)$ can be defined as

$$E = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2\, dt, \text{ Joules} \tag{1.15}$$

and

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt, \text{ watts} \tag{1.16}$$

The ***energy signal*** is one which has finite energy and zero average power, i.e. $x(t)$ is an energy signal if $0 < E < \infty$, and $P = 0$. The ***power signal*** is one which has finite average power and infinite energy, i.e. $0 < P < \infty$, and $E = \infty$. If the signal does not satisfy any of these two conditions, then it is neither an energy nor a power signal.

1.3 SINGULARITY FUNCTIONS

Singularity functions are an important classification of non-periodic signals. They can be used to represent more complicated signals. The unit-impulse function, sometimes referred to as delta function, is the basic singularity function and all other singularity functions can be derived by repeated integration or differentiation of the delta function. The other commonly used singularity functions are the unit-step and unit-ramp functions.

1.3.1 Unit-Impulse Function

The unit-impulse function is defined as

$$\delta(t) = 0, t \neq 0 \tag{1.17}$$

and

$$\int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = 1 \tag{1.18}$$

The Equations 1.17 and 1.18 indicate that the area of the impulse function is unity and this area is confined to an infinitesimal interval on the t-axis and concentrated at $t = 0$. The unit impulse function is very useful in continuous-time system analysis. It is used to generate the system response providing fundamental information about the system characteristics. In discrete-time domain, the unit-impulse signal is called a unit-sample signal. It is defined as

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \tag{1.19}$$

1.3.2 Unit-step Function

The integral of the impulse function $\delta(t)$ gives,

$$\int_{-\infty}^{t} \delta(t)\,\mathrm{d}t = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \tag{1.20}$$

Since, the area of the impulse function is all concentrated at $t = 0$, for any value of $t < 0$ the integral becomes zero and for $t > 0$, from Eq.1.18, the value of the integral is unity. The integral of the impulse function is also a singularity function and called the unit-step function and is represented as

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \tag{1.21}$$

The value at $t = 0$ is taken to be finite and in most cases it is unspecified. The discrete-time unit-step signal is defined as

$$u(n) = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases} \tag{1.22}$$

1.3.3 Unit-ramp Function

The unit-ramp function, $r(t)$ can be obtained by integrating the unit-impulse function twice or integrating the unit-step function once, i.e.

$$r(t) = \int_{-\infty}^{t} \int_{-\infty}^{\alpha} \delta(\tau)\,\mathrm{d}\tau\,\mathrm{d}\alpha \tag{1.23}$$

$$= \int_{-\infty}^{t} u(\alpha)\,\mathrm{d}\alpha$$

$$= \int_{0^+}^{t} 1 \cdot d\alpha$$

That is,

$$r(t) = \begin{cases} 0, & t < 0 \\ t, & t > 0 \end{cases} \quad (1.24)$$

A ramp signal starts at $t = 0$ and increases linearly with time, t. In discrete-time domain, the unit-ramp signal is defined as

$$r(n) = \begin{cases} 0, & n < 0 \\ n, & n \geq 0 \end{cases} \quad (1.25)$$

1.3.4 Unit-pulse Function

An unit-pulse function, $\Pi(t)$, is obtained from unit-step signals as shown below.

$$\Pi(t) = u\left(t + \frac{1}{2}\right) - u\left(t - \frac{1}{2}\right) \quad (1.26)$$

The signals $u\left(t + \frac{1}{2}\right)$ and $u\left(t - \frac{1}{2}\right)$ are the unit-step signals shifted by $\frac{1}{2}$ units in the time axis towards the left and right, respectively.

Figure 1.4 shows some of the singularity functions. The advantage of the singularity function is that any arbitrary signal that is made up of straight line segments can be represented in terms of step and ramp functions.

1.3.5 Properties of δ (t)

1. $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

2. $\int_{-\infty}^{\infty} x(t)\, \delta(t)\, dt = x(0)$

3. $\int_{-\infty}^{\infty} x(t)\, \delta(t - t_0)\, dt = x(t_0)$

4. $\int_{-\infty}^{\infty} x(\lambda)\, \delta(t - \lambda)\, d\lambda = x(t)$

5. $\delta(at) = \frac{1}{|a|}\, \delta(t)$

6. $x(t)\, \delta(t - t_0) = x(t_0)$

7. $x(t_0)\, \delta(t - t_0) = x(t_0)$

8. $\int_{t_1}^{t_2} x(t)\, \delta^n (t - t_0)\, dt = (-1)^n\, x^n(t_0)$

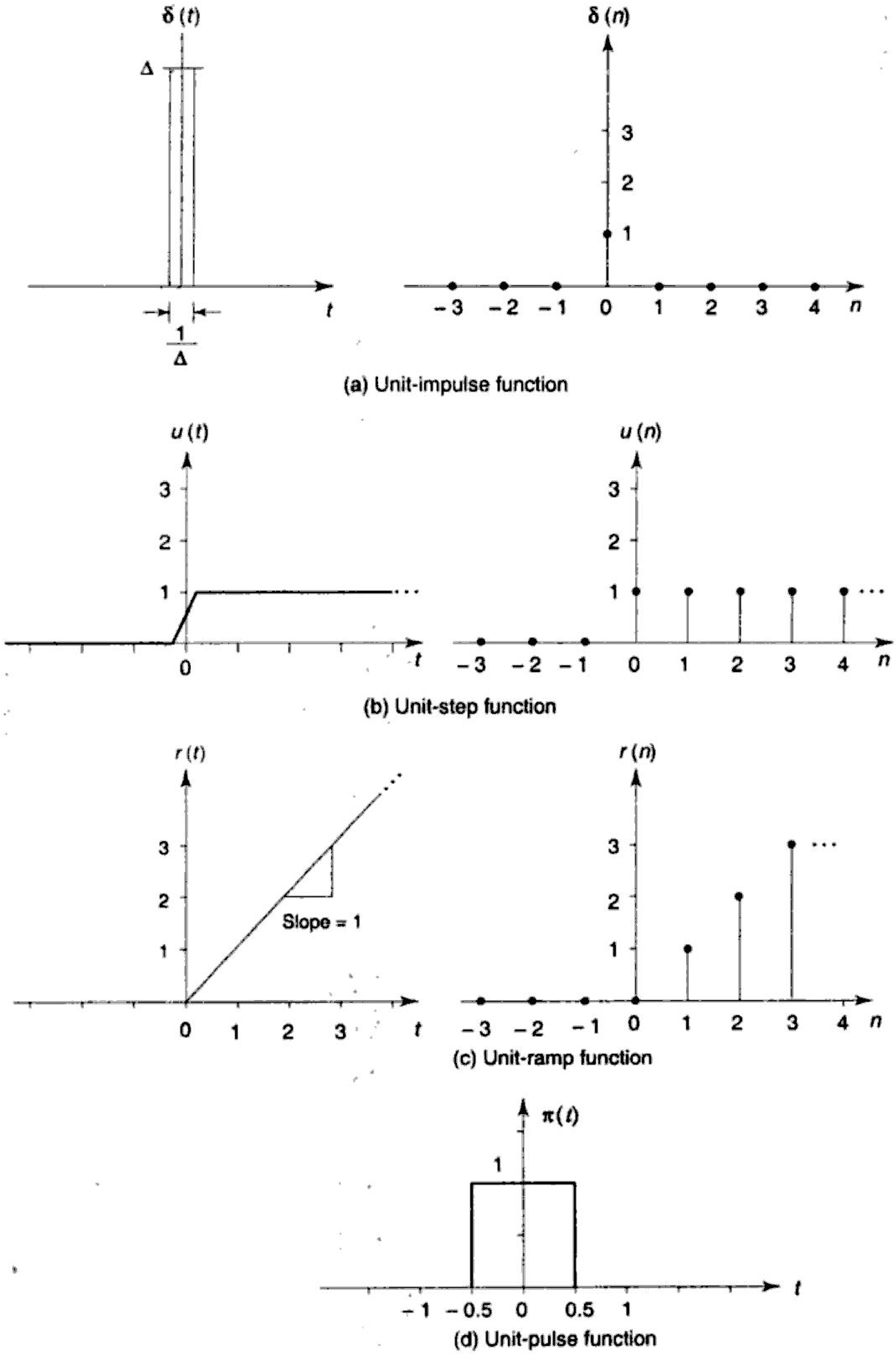


Fig. 1.4 *Singularity Functions (a) Unit-Impulse Function (b) Unit-Step Function (c) Unit-Ramp Function (d) Unit-Pulse Function*

Proof

$$\frac{d}{dt}[x(t)\,\delta(t-t_0)] = x(t)\,\dot{\delta}(t-t_0) + \dot{x}(t)\,\delta(t-t_0)$$

$$= x(t)\,\dot{\delta}(t-t_0) + \dot{x}(t_0)\,\delta(t-t_0),\ t_1 < t_0 < t_2$$

Integrating, we get

$$\int_{t_1}^{t_2} \frac{d}{dt}[x(t)\,\delta(t-t_0)]\,dt = \int_{t_1}^{t_2} [x(t)\,\dot{\delta}(t-t_0)]\,dt + \int_{t_1}^{t_2} [\dot{x}(t_0)\,\delta(t-t_0)]\,dt$$

$$[x(t)\,\delta(t-t_0)]_{t_1}^{t_2} = \int_{t_1}^{t_2} x(t)\,\dot{\delta}(t-t_0)\,dt + \dot{x}(t_0)$$

LHS = 0.

Therefore, $\int_{t_1}^{t_2} x(t)\,\dot{\delta}(t-t_0)\,dt + \dot{x}(t_0) = 0$

i.e. $\int_{t_1}^{t_2} x(t)\,\dot{\delta}(t-t_0)\,dt = -\dot{x}(t_0)$

Similarly,

$$\int_{t_1}^{t_2} x(t)\,\ddot{\delta}(t-t_0)\,dt = \ddot{x}(t_0)$$

Hence, $\int_{t_1}^{t_2} x(t)\,\delta^n(t-t_0)\,dt = (-1)^n\,x^n(t_0)$

1.3.6 Representation of Signals

In the signal given by $x(at+b)$, i.e., $x(a(t+b/a))$, a is a scaling factor and b/a is a pure shift version in the time domain.

If b/a is positive, then the signal $x(t)$ is shifted to left.

If b/a is negative, then the signal $x(t)$ is shifted to right.

If a is positive, then the signal $x(t)$ will have positive slope.

If a is negative, then the signal $x(t)$ will have negative slope.

If a is less than 0, then the signal $x(t)$ is reflected or reversed through the origin.

If $|a| < 1$, $x(t)$ is expanded, and if $|a| > 1$, $x(t)$ is compressed.

Example 1.2 Sketch the following signals

(a) $x(t) = \Pi(2t+3)$ (b) $x(t) = 2\Pi(t-1/4)$

(c) $x(t) = \cos(20\pi t - 5\pi)$ and (d) $x(t) = r(-0.5t+2)$

Solution

(a) $\Pi(2t+3) = \Pi(2(t+3/2))$

Here the signal shown in Fig. E1.2(a) is shifted to left, with centre at $-3/2$. Since $a = 2$, i.e. $|a| > 1$, the signal is compressed. The signal width becomes 1/2 with unity amplitude.

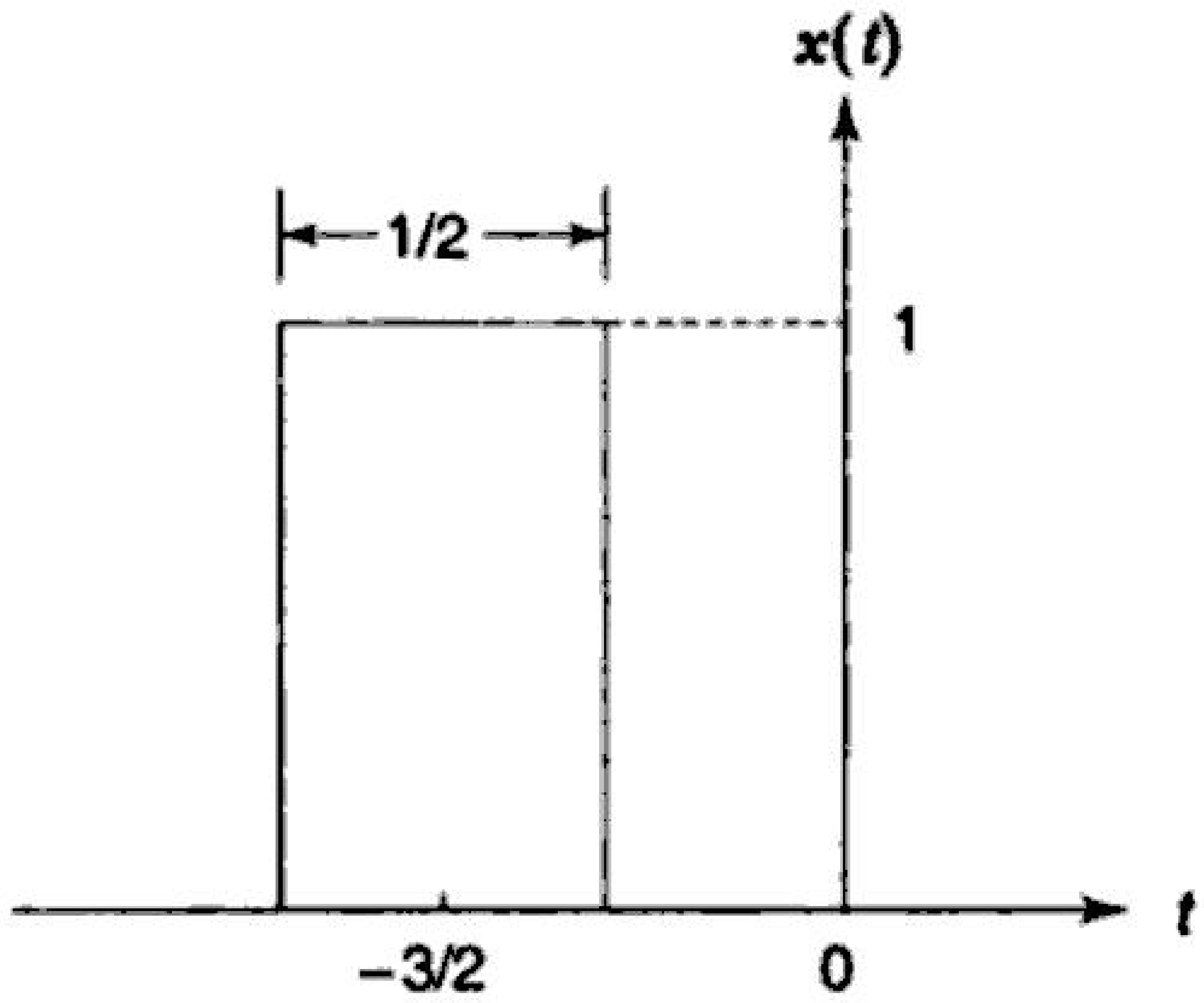


Fig. E1.2 (a)

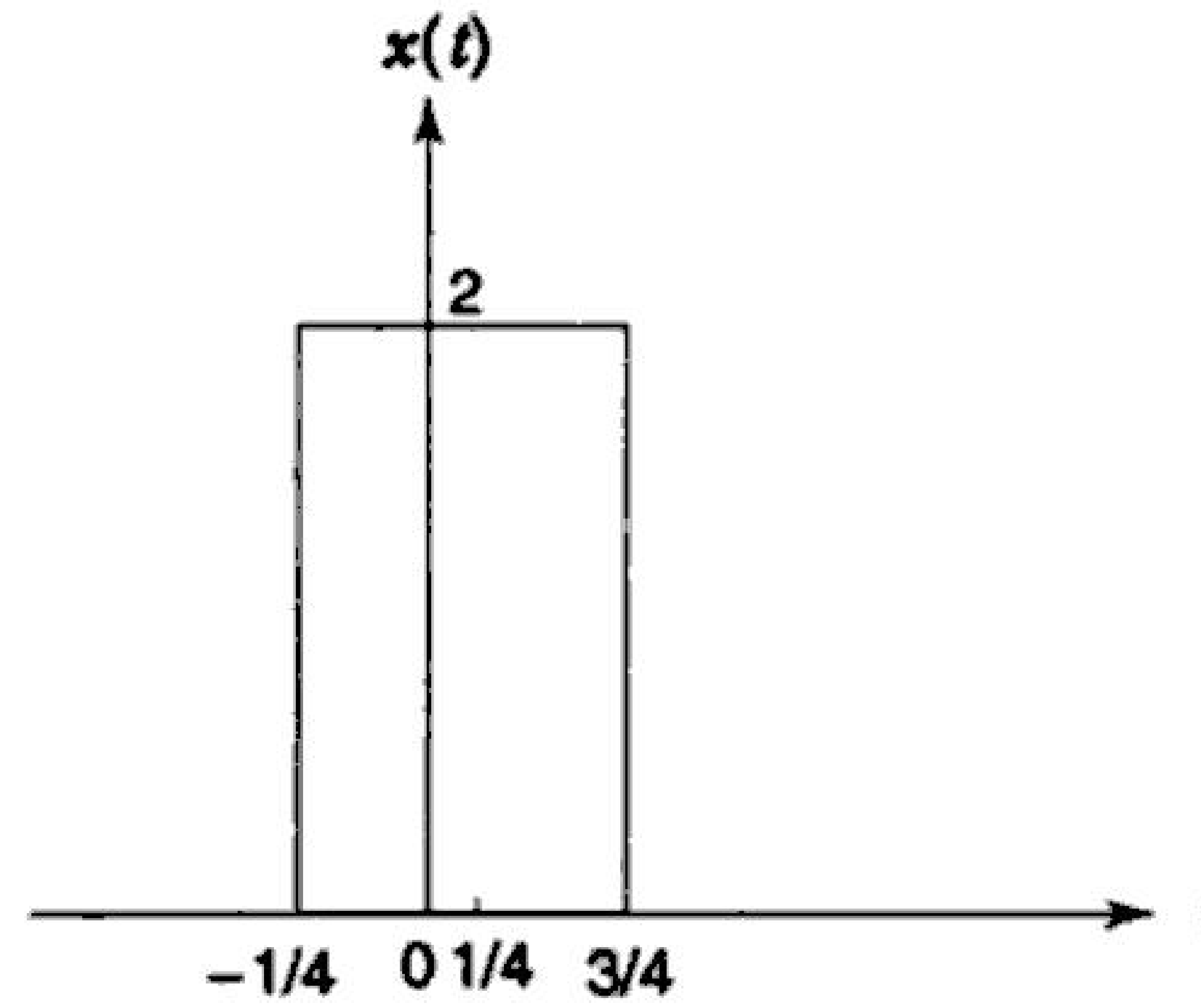


Fig. E1.2 (b)

(b) $x(t) = 2\Pi(t - 1/4)$

Here the signal shown in Fig. E1.2(b) is shifted to the right, with centre at 1 /4. Since $a = 1$, the signal width is 1 and amplitude is 2.

(c) $x(t) = \cos(20\,\pi t - 5\pi)$

$$= \cos\left(20\,\pi\left(t - \frac{5\,\pi}{20\,\pi}\right)\right)$$

$$= \cos\left(20\pi\left(t - \frac{1}{4}\right)\right)$$

Here the signal $x(t)$ shown in Fig. E1.2(c) is shifted by quarter cycle to the right.

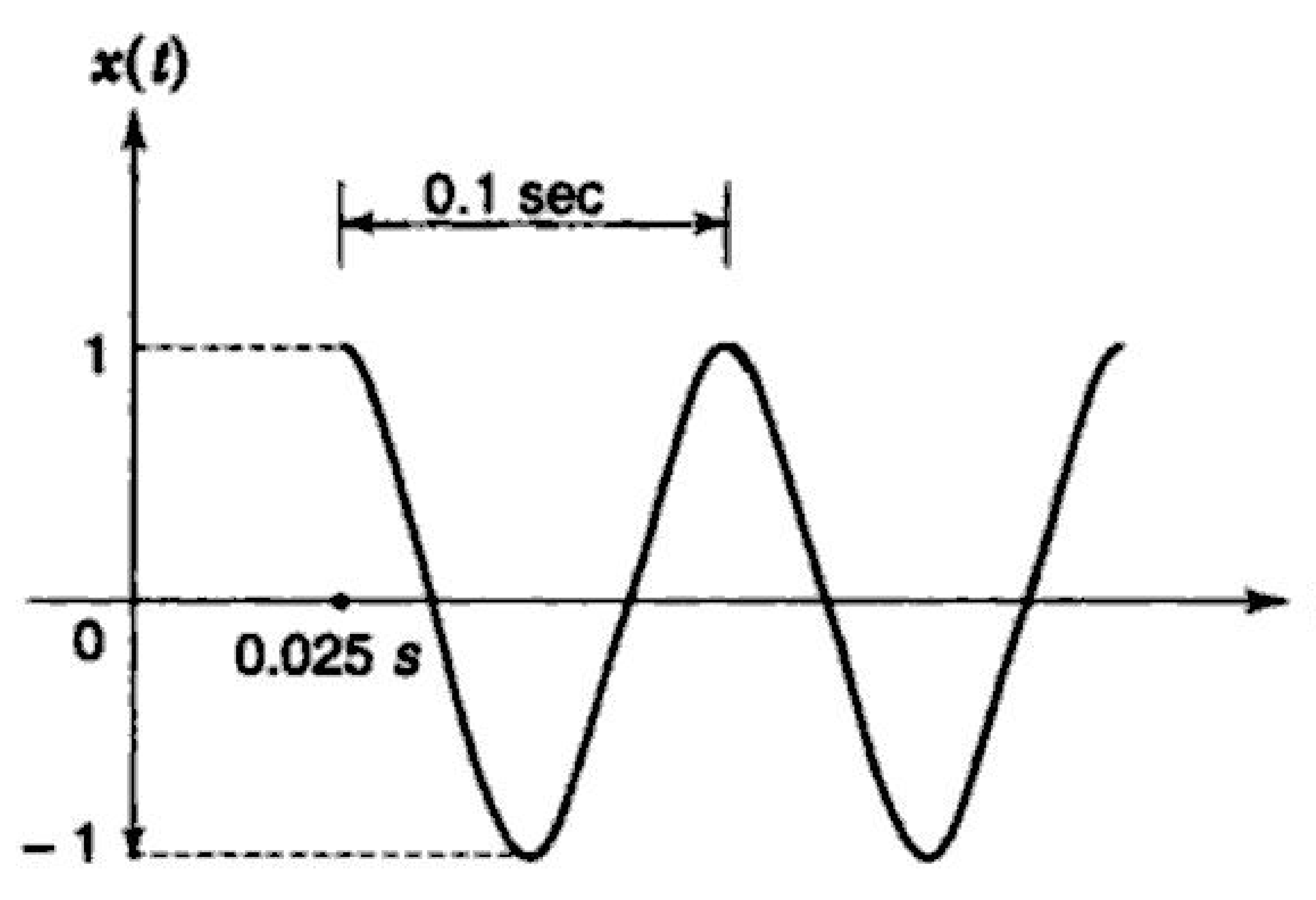


Fig. E1.2(c)

(d) $x(t) = r(-0.5t + 2)$

$$= r\left(-0.5\left(t - \frac{2}{0.5}\right)\right)$$

$$= r(-0.5\,(t - 4))$$

The given ramp signal is reflected through the origin and shifted to right at $t = 4$.

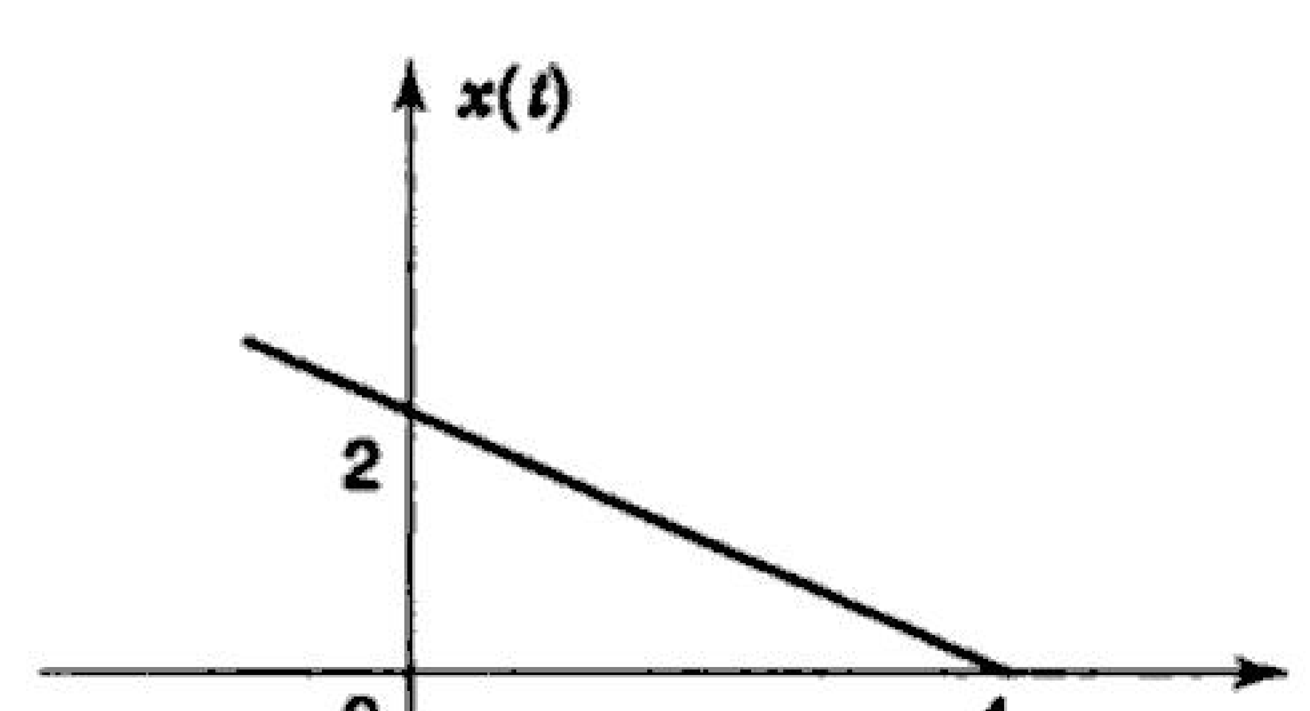


Fig. E1.2 (d)

The signal is expanded by $\dfrac{1}{0.5} = 2$. When $t = 0$, the magnitude of the signal $x(t) = 2$, shown in Fig. E1.2(d).

Example 1.3 Write down the corresponding equation for the given signal.

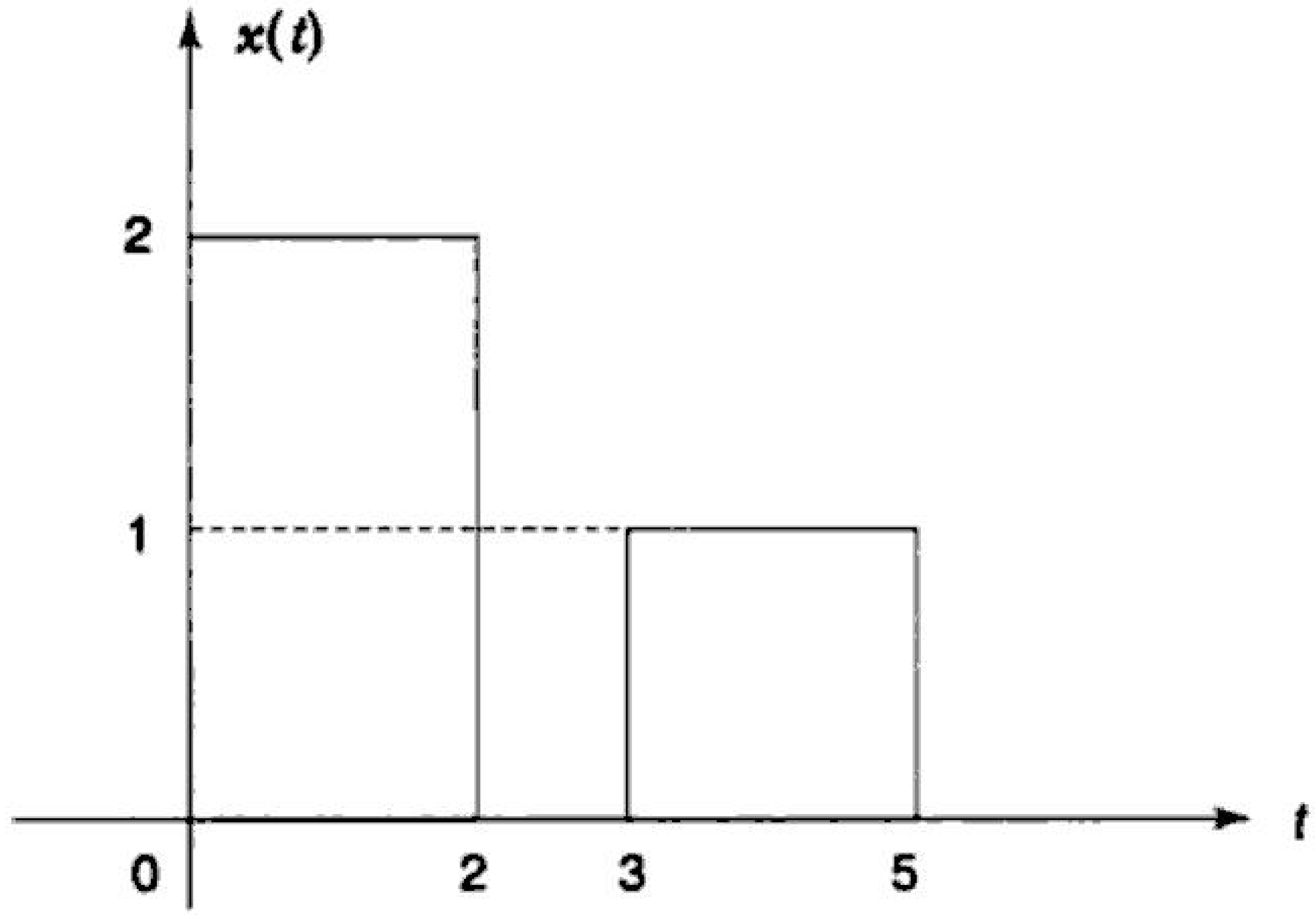


Fig. E1.3

Solution

Representation through addition of two unit step functions

The signal $x(t)$ can be obtained by adding both the pulses, i.e.

$$x(t) = 2[u(t) - u(t-2)] + [u(t-3) - u(t-5)]$$

Representation through multiplication of two unit step functions

$$x(t) = 2[u(t)\,u(-t+2)] + [u(t-3)\,u(-t+5)]$$
$$= 2(u(t)\,u(2-t) + u(t-3)\,u(5-t))$$

1.4 AMPLITUDE AND PHASE SPECTRA

Let us consider a cosine signal of peak amplitude A, frequency f and phase shift Φ, in order to introduce the concept of amplitude and phase spectra, i.e.,

$$x(t) = A\cos(2\pi ft + \Phi) \tag{1.27}$$

The amplitude and phase of this signal can be plotted as a function of frequency. The amplitude of the signal as a function of frequency is referred to as *amplitude spectrum* and the phase of the signal as a function of frequency is referred to as *phase spectrum* of the signal. The amplitude and phase spectra together is called the *frequency spectrum* of the signal. The units of the amplitude spectrum depends on the signal. For example, the unit of the amplitude spectrum of a voltage signal is measured in volts, and the unit of the amplitude spectrum of a current signal is measured in amperes. The unit of the phase spectrum is usually radians. The frequency spectrum drawn only for positive values of frequencies alone is called a *single-sided spectrum*.

The cosine signal can also be expressed in phasor form as the sum of the two counter rotating phasors with complex-conjugate magnitudes, i.e.

$$x(t) = A\left[\frac{e^{j(2\pi ft+\Phi)} + e^{-j(2\pi ft+\Phi)}}{2}\right]$$

From this the amplitude spectrum for the signal $x(t)$ consists of two components of amplitude, viz. $A/2$ at frequency 'f' and $A/2$ at frequency '$-f$'. Similarly, the phase spectrum also consists of two phase components one at 'f' and the other at '$-f$'. The frequency spectrum of the signal, in this case, is called a *double-sided spectrum*. The following example illustrates the single-sided and double-sided frequency spectra of a signal.

Example 1.4 Sketch the single-sided and double-sided amplitude and phase spectra of the signal

$$x(t) = 8\sin\left(20\pi t - \frac{\pi}{6}\right), -\infty < t < \infty$$

Solution: The single-sided spectra is plotted by expressing $x(t)$ as the real part of the rotating phasor. Using the trigonometric identity, $\cos\left(u - \frac{\pi}{2}\right) = \sin u$, the given signal is converted into a form as in Eq.1.27, i.e.

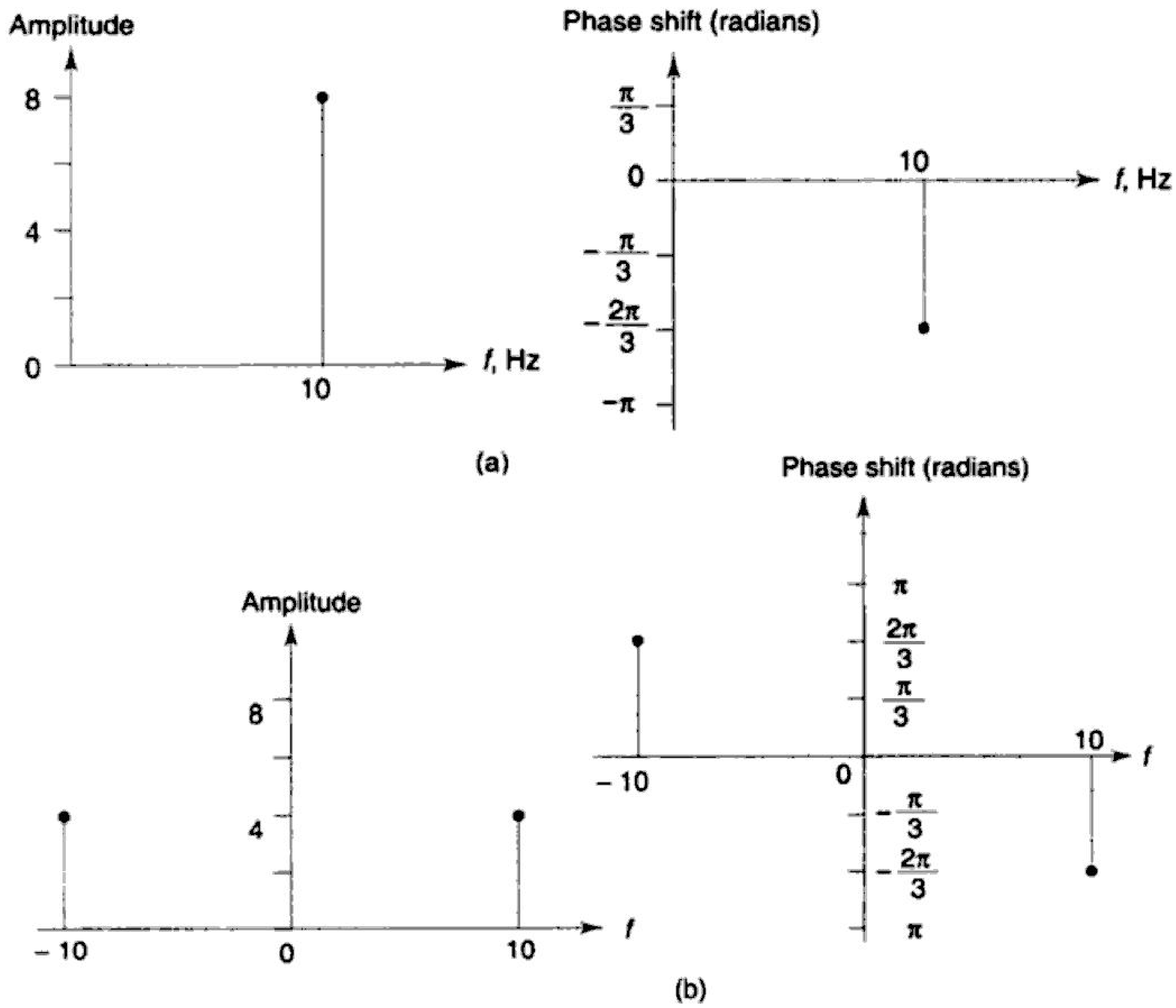


Fig. E.1.4 *Amplitude and Phase Spectra (a) Single-Sided and (b) Double-Sided*

$$x(t) = 8\sin\left(20\pi t - \frac{\pi}{6}\right) = 8\cos\left(20\pi t - \frac{\pi}{6} - \frac{\pi}{2}\right)$$
$$= 8\cos\left(20\pi t - \frac{2\pi}{3}\right)$$

The single-sided amplitude and phase spectra are shown in Fig. E.1.4a. The signal has an amplitude of 8 units at $f = 10$ Hz and a phase angle of $-\frac{2\pi}{3}$ radians at $f = 10$ Hz. To plot the double-sided spectrum, the signal is converted into the form as in Eq.1.28. Therefore,

$$x(t) = 4e^{j\left(20\pi t - \frac{2\pi}{3}\right)} + 4e^{-j\left(20\pi t - \frac{2\pi}{3}\right)}$$

The double-sided amplitude and phase spectra are shown in Fig. E.1.4b. The signal has two components at $f = 10$ Hz and $f = -10$ Hz. The amplitude of these components are 4 units each and the phase of these components are $-\frac{2\pi}{3}$ and $\frac{2\pi}{3}$ radians, respectively.

1.5 CLASSIFICATION OF SYSTEMS

As with signals, systems are also broadly classified into continuous-time and discrete-time systems. In a continuous-time system, the associated signals are also continuous, i.e. the input and output of the system are both continuous-time signals. On the other hand, a discrete-time system handles discrete-time signals. Here, both the input and output signals are discrete-time signals.

Both continuous and discrete-time systems are further classified into the following types.

(i) Static and dynamic systems
(ii) Linear and non-linear systems
(iii) Time-variant and time-invariant systems
(iv) Causal and non-causal systems, and
(v) Stable and unstable systems.

1.5.1 Static and Dynamic Systems

The output of a static system at any specific time depends on the input at that particular time. It does not depend on past or future values of the input. Hence, a static system can be considered as a system with no memory or energy storage elements. A simple resistive network is an example of a static system. The input/output relation of such systems does not involve integrals or derivatives.

The output of a dynamic system, on the other hand at any specified time depends on the inputs at that specific time and at other times. Such systems have memory or energy storage elements. The equation characterising a dynamic system will always be a differential equation for continuous-time system or a difference equation for a discrete-time system. Any electrical circuit consisting of a capacitor or an inductor is an example of a dynamic system. The following equations characterise the dynamic systems.

(i) $\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + x(t)$

(ii) $y(n-1) + 2y(n) = 4x(n) - x(n-1)$

1.5.2 Linear and Non-linear Systems

A linear system is one in which the principle of superposition holds. See Fig. 1.5. For a system with two inputs $x_1(t)$ and $x_2(t)$, the superposition is defined as follows.

$$H[a_1x_1(t) + a_2x_2(t)] = a_1H[x_1(t)] + a_2H[x_2(t)] \tag{1.28}$$

where, a_1 and a_2 are the weights added to the inputs, and $H[x(t)] = y(t)$ is the response of the continuous-time system to the input $x(t)$. Thus, a *linear system* is defined as one whose response to the sum of the weighted inputs is same as the sum of the weighted responses. If a system does not satisfy Eq.1.28, then the system is *non-linear*. For a discrete-time system, the condition for linearity is given by Eq.1.29.

$$H[a_1x_1(n) + a_2x_2(n)] = a_1H[x_1(n)] + a_2H[x_2(n)] \tag{1.29}$$

where $H[x(n)] = y(n)$ is the response of the discrete-time system to the input $x(n)$.

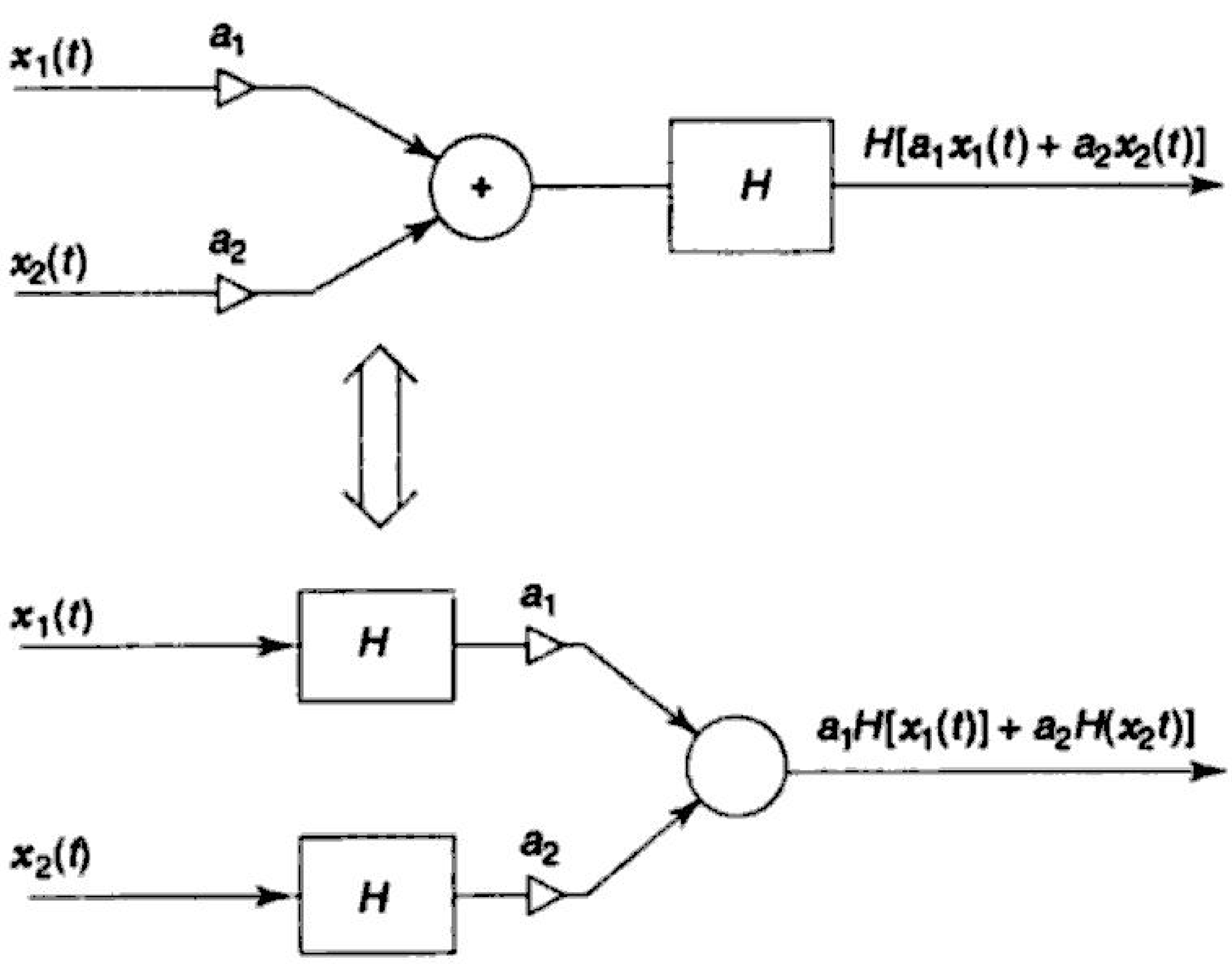


Fig. 1.5 *Illustration of the Superposition Principle*

Example 1.5 Determine whether the system described by the differential equation $\frac{dy(t)}{dt} + 2y(t) = x(t)$ is linear.

Solution Let the response of the system to $x_1(t)$ be $y_1(t)$ and the response of the system to $x_2(t)$ be $y_2(t)$. Thus, for the input $x_1(t)$, the describing equation is

$$\frac{dy_1(t)}{dt} + 2y_1(t) = x_1(t)$$

and for the input $x_2(t)$,

$$\frac{dy_2(t)}{dt} + 2y_2(t) = x_2(t)$$

Multiplying these equations by a_1 and a_2, respectively, and adding yields,

$$a_1\frac{dy_1(t)}{dt} + a_2\frac{dy_2(t)}{dt} + 2a_1y_1(t) + 2a_2y_2(t) = a_1x_1(t) + a_2x_2(t)$$

i.e.

$$\frac{d}{dt}(a_1y_1(t) + a_2y_2(t)) + 2(a_1y_1(t) + a_2y_2(t)) = a_1x_1(t) + a_2x_2(t)$$

The response of the system to the input $a_1x_1(t) + a_2x_2(t)$ is $a_1y_1(t) + a_2y_2(t)$. Thus, the superposition condition is satisfied and hence the system is linear.

Example 1.6 Determine whether the system described by the differential equation $\frac{dy(t)}{dt} + y(t) + 4 = x(t)$ is linear.

Solution Let the response of the system to $x_1(t)$ be $y_1(t)$ and the response of the system to $x_2(t)$ be $y_2(t)$. Thus, for input $x_1(t)$, the describing equation is

$$\frac{dy_1(t)}{dt} + y_1(t) + 4 = x_1(t)$$

and for input $x_2(t)$,

$$\frac{dy_2(t)}{dt} + y_2(t) + 4 = x_2(t)$$

Multiplying these equations by a_1 and a_2, respectively, and adding yields,

$$\frac{d}{dt}(a_1y_1(t) + a_2y_2(t)) + (a_1y_1(t) + a_2y_2(t)) + 4(a_1 + a_2)$$
$$= a_1x_1(t) + a_2x_2(t)$$

This equation cannot be put into the same form as the original differential equation describing the system. Hence, the system is non-linear.

1.5.3 Time-variant and Time-invariant Systems

A time-invariant system is one whose input-output relationship does not vary with time. A time-invariant system is also called a *fixed system*. The condition for a system to be fixed is

$$H\,[x(t-\tau)] = y\,(t-\tau) \tag{1.30}$$

A time-invariant system satisfies Eq.1.30 for any $x\,(t)$ and any value of τ. Equation 1.30 states that **if $y\,(t)$ is the response of the system to any input $x(t)$, then the response of the system to the time-shifted input is the response of the system to $x\,(t)$ time shifted by the same amount.** In discrete time, this property is also referred to as shift-invariance. For a discrete-time system, the condition for shift-invariance is given by

$$H\,[x\,(n-k)] = y\,(n-k) \tag{1.31}$$

where k is an integer. A system not satisfying either Eq.1.30 or 1.31 is said to be time-variant. The systems satisfying both linearity and time-invariant conditions are called ***linear***, ***time-invariant*** systems, or simply ***LTI systems***.

1.5.4 Causal and Non-causal Systems

A causal system is non-anticipatory. The response of the causal system to an input does not depend on future values of that input, but depends only on the present and/or past values of the input. If the response of the system to an input depends on the future values of that input, then the system is non-causal or anticipatory. Non-causal systems are unrealisable. The following difference equations describe causal systems.

(i) $y(n) = 0.5\,x\,(n) - x\,(n-2)$

(ii) $y(n) = x\,(n)$

(iii) $y(n-2) + y\,(n) = x(n) + 0.98\,x\,(n-1)$

The following equations describe non-causal systems.

(i) $y(n-1) = x(n)$

(ii) $y(n) = 0.11x(n\!-\!1) + x(n) - 0.8\,x(n+1)$

1.5.5 Stable and Unstable Systems

A system is said to be bounded-input, bounded-output (BIBO) stable, if every bounded input produces a bounded output. A bounded signal has an amplitude that remains finite. Thus, a BIBO stable system will have a bounded output for any bounded input so that its output does not grow unreasonably large. The conditions for a system to be BIBO stable are given below.

(i) If the system transfer function is a rational function, the degree of the numerator must be no larger than the degree of the denominator.

(ii) The poles of the system must lie in the left half of the s-plane or within the unit circle in the z-plane.

(iii) If a pole lies on the imaginary axis, it must be a single-order one, i.e. no repeated poles must lie on the imaginary axis.

The systems not satisfying the above conditions are unstable.

1.6 SIMPLE MANIPULATIONS OF DISCRETE-TIME SIGNALS

When a signal is processed, the signal undergoes many manipulations involving the independent variable and the dependent variable. Some of these manipulations include (i) shifting the signal in the time domain, (ii) folding the signal and (iii) scaling in the time-domain. A brief introduction of these manipulations here, will help the reader in the following chapters.

1.6.1 Transformation of the Independent Variable

Shifting

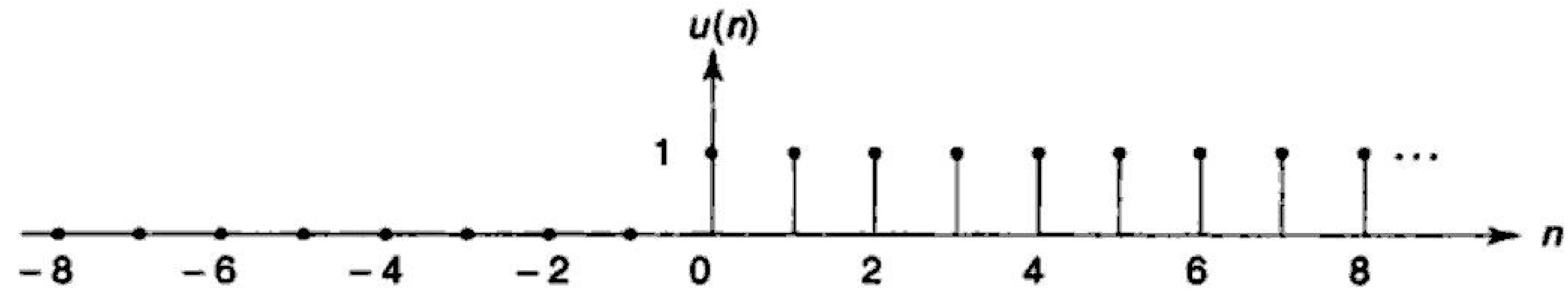
In the case of discrete-time signals, the independent variable is the time, n. A signal $x(n)$ may be shifted in time, i.e. the signal can be either advanced in the time axis or delayed in the time axis. The shifted signal is represented by $x(n - k)$, where k is an integer. If 'k' is positive, the signal is delayed by k units of time and if k is negative, the time shift results in an advance of signal by k units of time. However, advancing the signal in the time axis is not possible always. If the signal is available in a magnetic disk or other storage units, then the signal can be delayed or advanced as one wishes. But in real time, advancing a signal is not possible since such an operation involves samples that have not been generated. As a result, in real-time signal processing applications, the operation of advancing the time base of the signal is physically unrealizable.

Folding

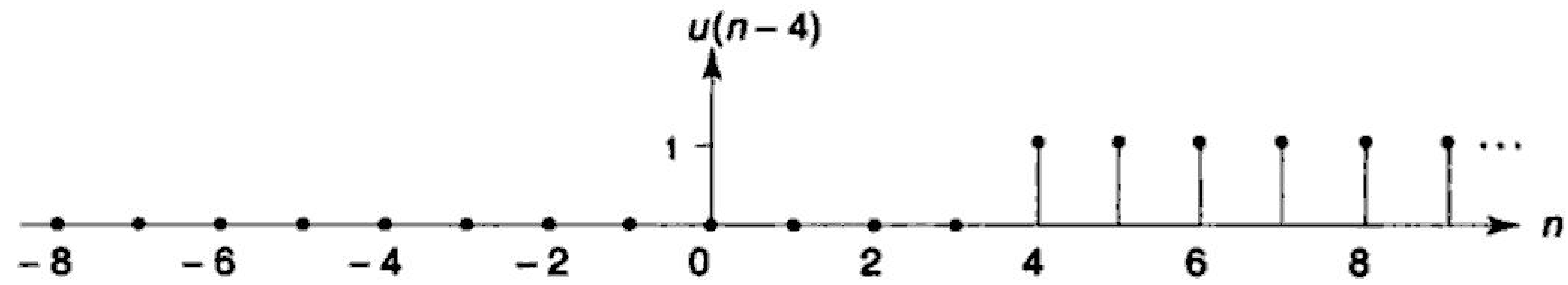
This operation is done by replacing the independent variable n by $-n$. This results in folding of the signal about the origin, i.e. $n = 0$. Folding is also known as the reflection of the signal about the time origin $n = 0$. Folding of a signal is done while convoluting the signal with another.

Time scaling

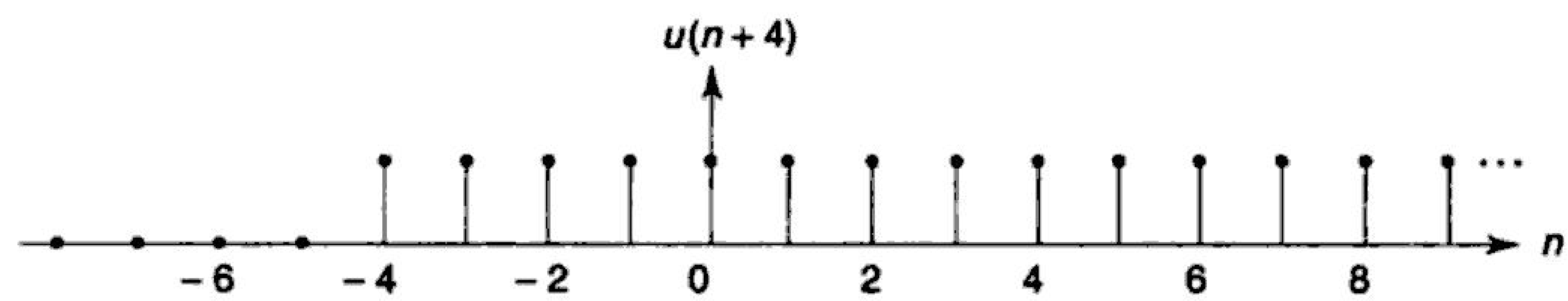
This involves replacing the independent variable n by kn, where k is an integer. This process is also called as *down sampling*. If $x(n)$ is the discrete-time signal obtained by sampling the analog signal, $x(t)$, then $x(n) = x(nT)$, where T is the sampling period. If time-scaling is done, then the time-scaled signal, $y[n] = x(kn) = x(knT)$. This implies that the sampling rate is changed from $1/T$ to $1/kT$. This decreases the sampling rate by a factor of k. Down-sampling operations are discussed in detail in Chapter 11 of this book. The folding and time scaling operations are shown in Fig. 1.7(a) and (b).



(a) Unit-step signal

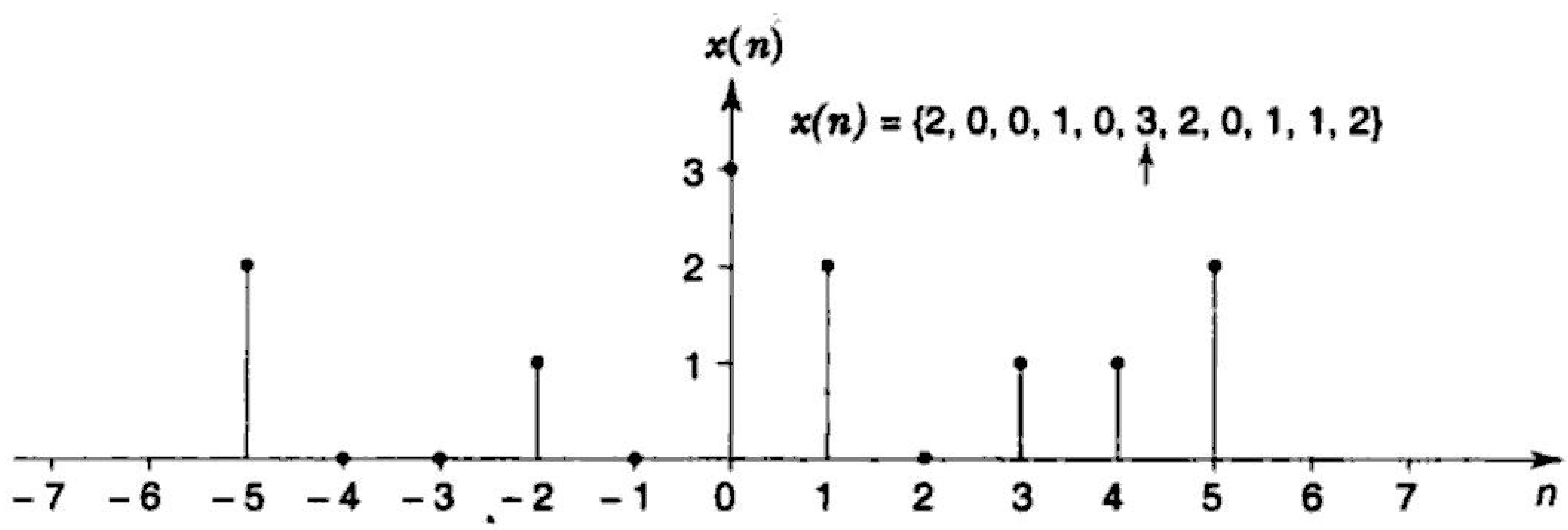


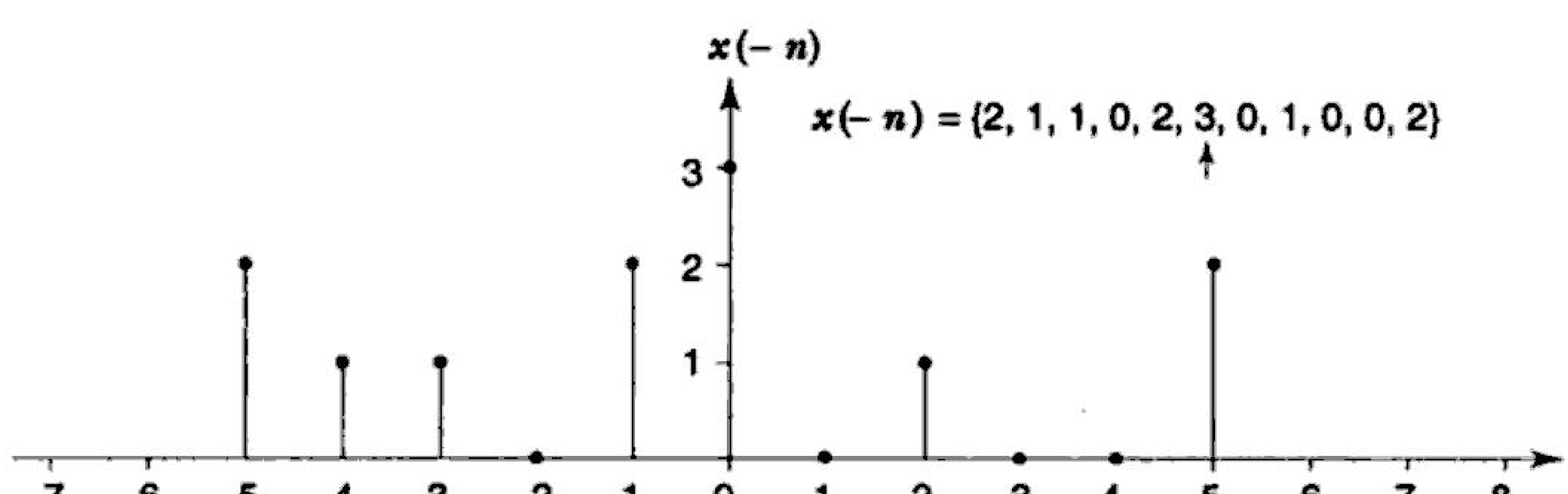
(b) Delaying the unit-step signal by 4 units



(c) Advancing the unit-step signal by 4 units

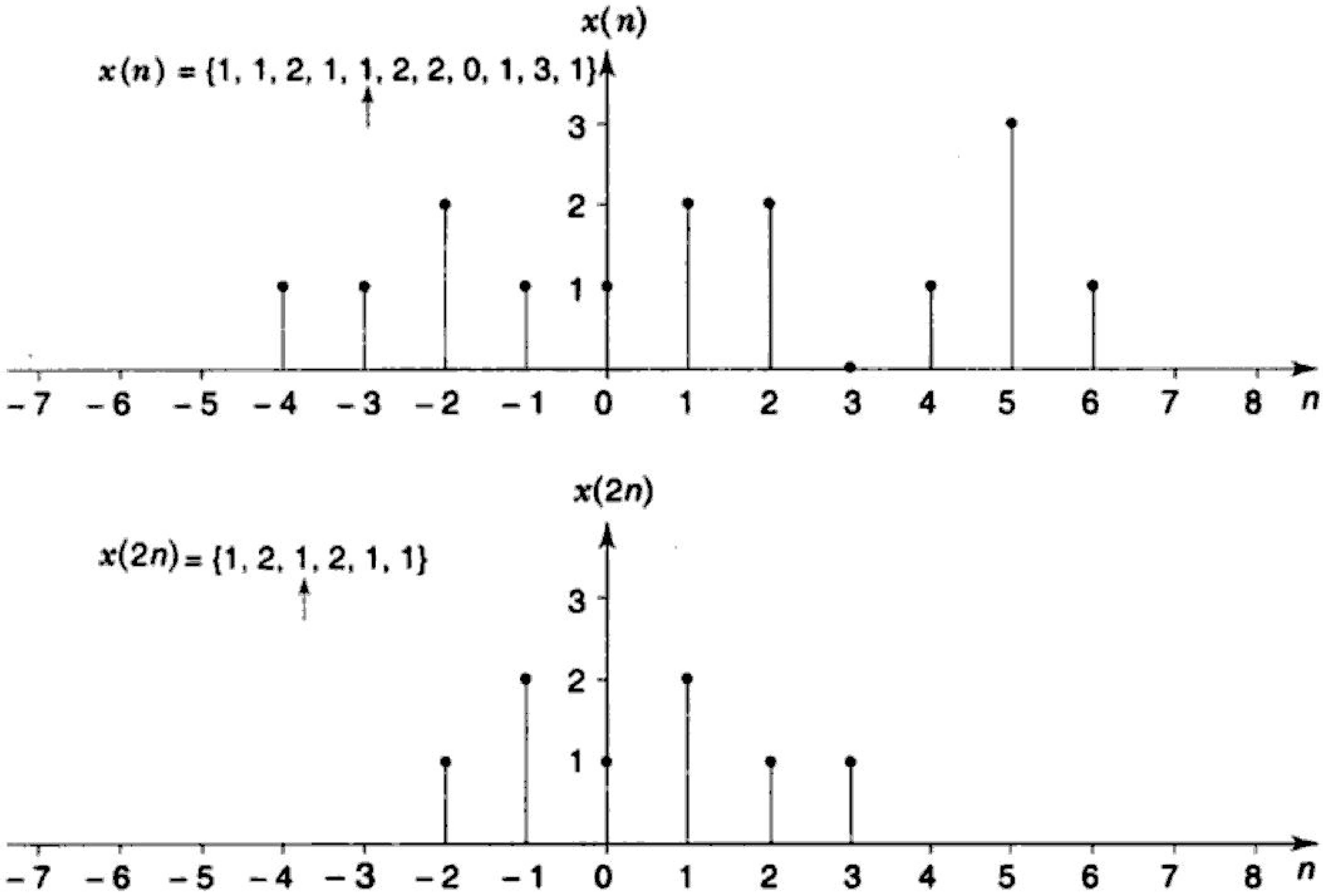
Fig. 1.6 *Graphical Representation of a Signal, and Its Delayed and Advanced Versions*





(a) Folding

Fig. 1.7 *(Contd.)*



(b) Time scaling

Fig. 1.7 *Illustration of Folding and Time Scaling Operations*

1.7 REPRESENTATIONS OF SYSTEMS

The representation of a system helps in visualising the system with its components and their interconnections. A system can be represented using a diagram featuring various components of the system. These components are represented by symbols. An electrical system is thus represented by a diagram consisting of different symbols representing resistors, capacitors or any other device. A mechanical system can be represented using symbols for different elements like damper, acceleration, zero friction, etc. Thus, a system can be visualised when represented in the form of a diagram.

A more convenient form of representing a system is the block-diagram representation. In this form of representation, each box is an operator on the input signal and the operation is shown on the box itself. Some typical operations include integration, differentiation, scalar multiplication, delay, etc. The lines connecting the individual boxes are directional and these lines show the direction of the signal flow. A number of lines may terminate at a node. The node may be either an accumulator or a multiplier. These nodes are represented by circles with the symbols '+' or '×' marked on them. Figure 1.8 shows the block diagram representation of continuous-time and discrete-time systems.

A system can also be represented using mathematical models. The analysis of system characteristics and performance can be carried out using the mathematical model of a system. The mathematical model of a

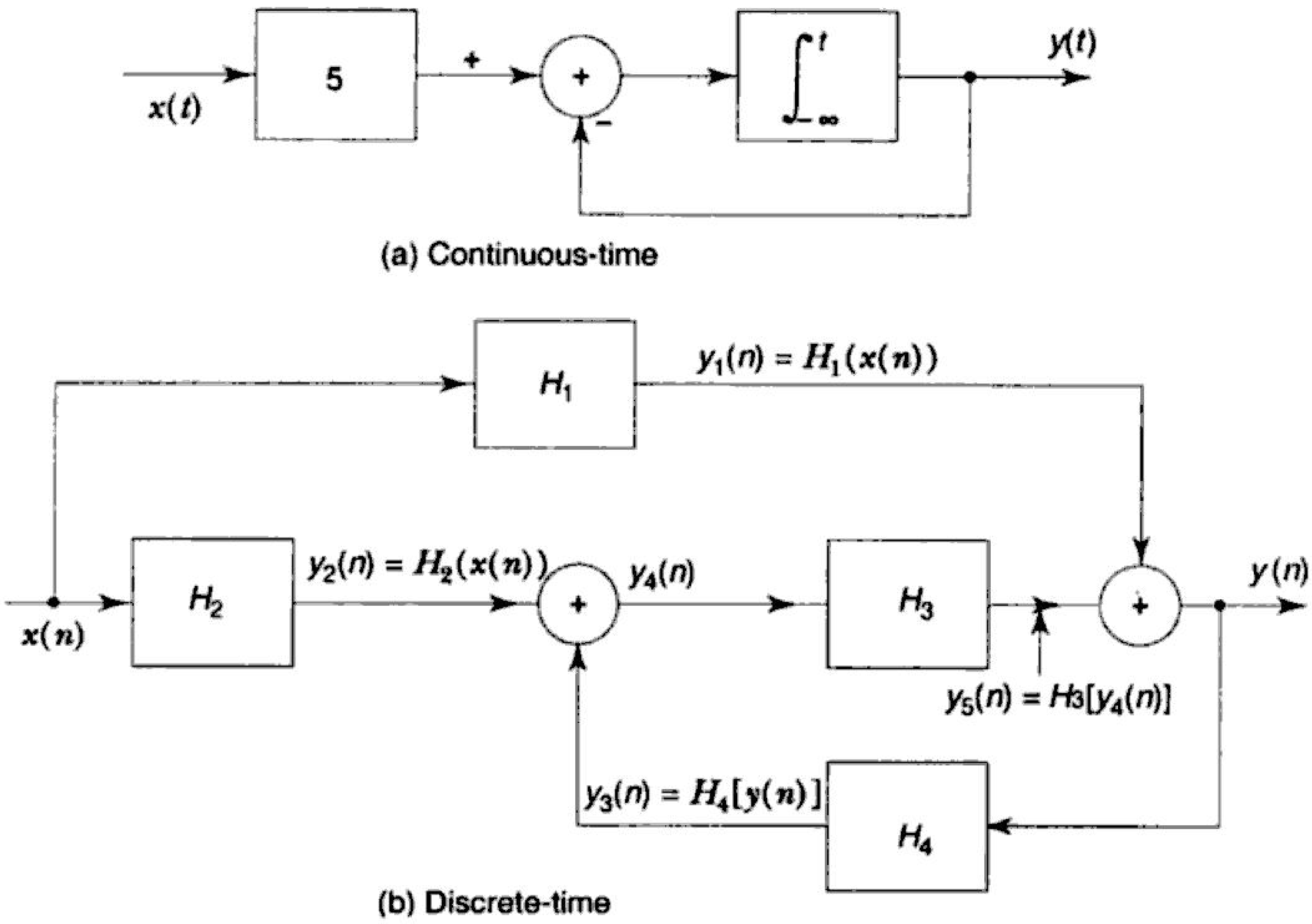


Fig. 1.8 *Block Diagram Representation of (a) Continuous-Time System (b) Discrete-Time System*

system consists of equations relating to the signals of interest. Both discrete-time and continuous-time signals can be modelled using the following three methods.

(i) A linear difference/differential equation
(ii) The impulse-response sequence
(iii) A state-variable or matrix description.

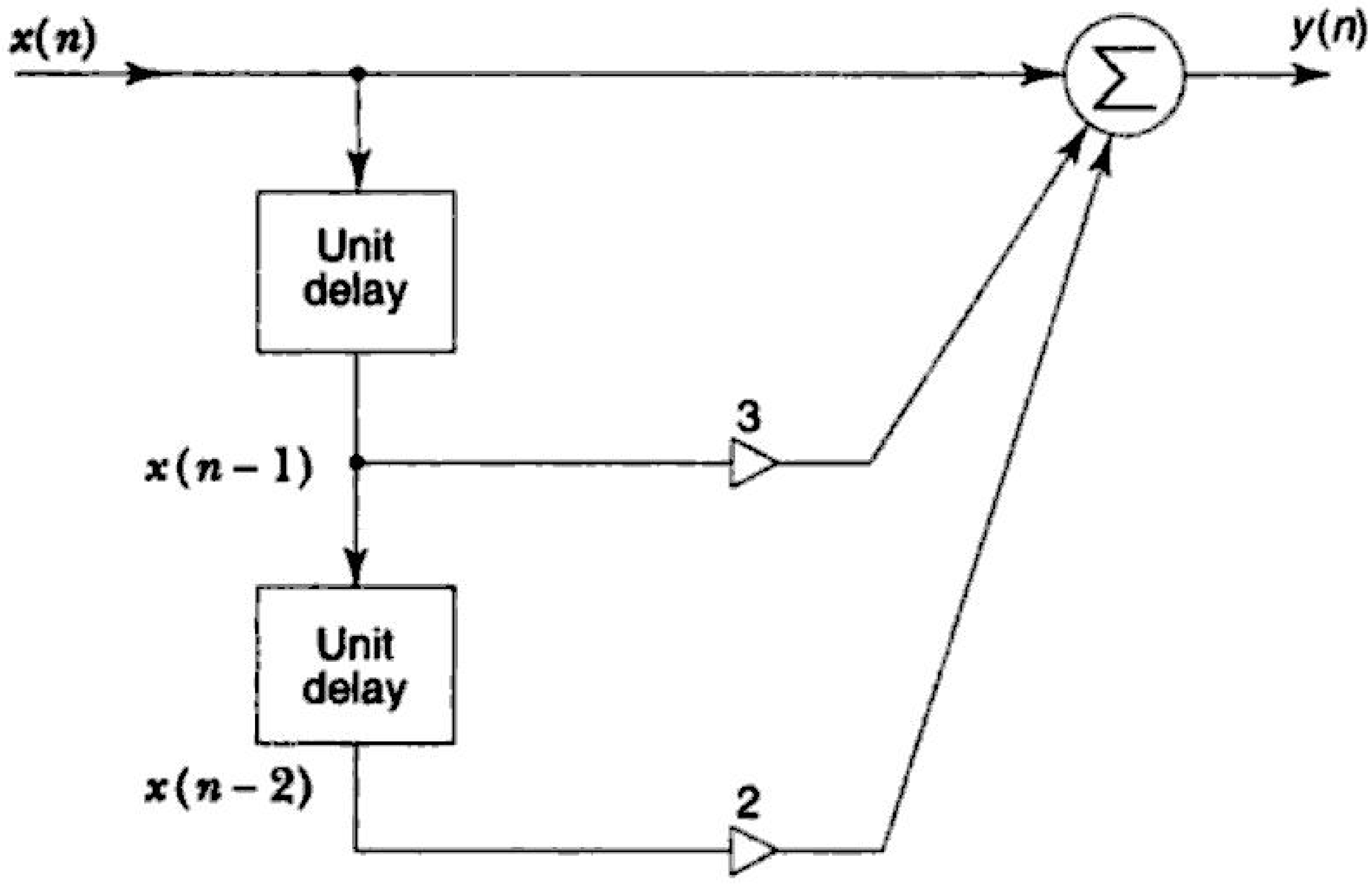
All the above three methods help in determining the output of the system from the knowledge of the input to the system. A system can be interpreted in different ways depending on the model since each model emphasise certain aspects of the system. Together, these models provide a very good understanding of the system and how the system works. In the following sections, these models are discussed further.

1.7.1 Linear Difference/Differential Equations

A discrete-time system is modelled by a difference equation, whereas a continuous-time system is modelled by a differential equation. In a linear discrete-time system, the input sequence $\{x_n\}$ is transformed into an output sequence $\{y_n\}$ according to some difference equation. For example,

$$y(n) = x(n) + 3x(n-1) + 2x(n-2) \tag{1.32}$$

is a linear difference equation which tells that the nth member of the output sequence $y(n)$ is obtained by accumulating (adding) the input at the present moment, $x(n)$, with thrice the previous input, $x(n-1)$ and twice the input delayed twice, $x(n-2)$. Let the input sequence be $x(n) = \{0, 1, 1, 2, 0, 0, 0, \ldots\}$. The output sequence for the system as described by Eq. 1.32 is $y(n) = \{0, 1, 4, 7, 8, 4, 0, 0, \ldots\}$. The block diagram representation of the system described by Eq.1.32 is shown in Fig. 1.9.



$y[n] = x[n] + 3x[n-1] + 2x[n-2]$

Fig. 1.9 *Discrete-Time System Corresponding to Eq. 1.32*

In digital signal processing applications, our prime concern is of linear, time-invariant discrete-time systems. Such systems are modelled using linear difference equations with constant coefficients. The block diagram representation of these systems contain only unit delays, constant multipliers and adders.

A continuous-time system is modelled by a linear differential equation. An ordinary linear differential equation with constant coefficients characterises linear, constant parameter systems. For example, an nth order system is represented by

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + y(t) = x(t) \qquad (1.33)$$

The general solution of the above equation consists of two components, namely, the homogeneous solution and the particular solution. The homogeneous solution is the sourcefree, natural solution of the system, whereas the particular solution is the component due to the source $x(t)$.

1.7.2 Impulse Response of a System

The impulse response of a system is another method for modelling a system. The impulse response of a linear, time-invariant system is the response of the system when the input signal is an unit-impulse function. The system is assumed to be initially relaxed, i.e. the system has zero initial conditions. The impulse response of a system is represented by the notation $h(t)$ (continuous-time) or $h(n)$ (discrete-time). If $y(t)$ is the system response for an input $x(t)$, then the response of the system when $x(t) = \delta(t)$ is $y(t) = h(t)$.

The impulse response of a system can be directly obtained from the solution of the differential or difference equation characterising the system. The impulse response is also determined by finding out the output of the system to the rectangular pulse input $x(t) = \frac{1}{\varepsilon}\Pi\left(\frac{t}{\varepsilon}\right)$ and then taking the limit of the resulting system response, $y(t)$ as $\varepsilon \to 0$. The unit-impulse function is nothing but the derivative of the unit-step signal. Therefore, the impulse response of the system can also be obtained by computing the derivative of the step response of the system.

1.7.3 State-Variable Technique

The state-variable technique provides a convenient formulation procedure for modelling a multi-input, multi-output system. This technique also facilitates the determination of the internal behaviour of the system very easily. The *state* of a system at time t_0 is the minimum information necessary to completely specify the condition of the system at time t_0 and it allows determination of the system outputs at any time $t > t_0$, when inputs upto time t are specified. The state of a system at time t_0 is a set of values, at time t_0, of a set of variables. These variables are called the *state variables*. The number of state variables is equal to the order of the system. The state variables are chosen such that they correspond to physically measurable quantities. It is also convenient to consider an n-dimensional space in which each coordinate is defined by one of the state variables $x_1, x_2, \ldots, x_n$, where n is the order of the system. This n-dimensional space is called the *state space*. The ***state vector*** is an n-vector x whose elements are the state variables. The state vector defines a point in the state space at any time t. As the time changes, the system state changes and a set of points, which is nothing but the locus of the tip of the state vector as time progresses, is called a *trajectory* of the system.

A linear system of order n with m inputs and k outputs can be represented by n first-order differential equations and k output equations as shown below.

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \ldots + b_{1m}u_m \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \ldots + b_{2m}u_m \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \ldots + b_{nm}u_m
\end{aligned} \tag{1.34}$$

and

$$
\begin{aligned}
y_1 &= c_{11}x_1 + c_{12}x_2 + \ldots + c_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \ldots + d_{1m}u_m \\
y_2 &= c_{21}x_1 + c_{22}x_2 + \ldots + c_{2n}x_n + d_{21}u_1 + d_{22}u_2 + \ldots + d_{2m}u_m \\
&\quad \vdots \\
y_k &= c_{k1}x_1 + c_{k2}x_2 + \ldots + c_{kn}x_n + d_{k1}u_1 + d_{k2}u_2 + \ldots + d_{km}u_m
\end{aligned} \tag{1.35}
$$

where u_i, $i = 1, 2, \ldots, m$ are the system inputs, x_i, $i = 1, 2, 3, \ldots, n$ are called the state variables and y_i, $i = 1, 2, 3, \ldots, k$ are the system outputs. Equations 1.34 are called the *state equations*, and Eqs 1.35 are the *output equations*. Equations 1.34 and 1.35 together constitute the state-equation model of the system. Generally, the a's, b's, c's and d's may be functions of time. The solution of such a set of time-varying state equations is very difficult. If the system is assumed to be time-invariant, then the solution of the state equations can be obtained without much difficulty.

The state variable representation of a system offers a number of advantages. The most obvious advantage of this representation is that multiple-input, multiple-output systems can be easily represented and analysed. The model is in the time-domain, and one can obtain the simulation diagram for the equations directly. This is of much use when computer simulation methods are used to analyse the system. Also, a compact matrix notation can be used for the state model and using the laws of linear algebra the state equations can be very easily manipulated. For example, Eqs 1.34 and 1.35 expressed in a compact matrix form is shown below. Let us define vectors

$$
x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix} \tag{1.36}
$$

and matrices

$$
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix} \tag{1.37}
$$

$$
C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ c_{k1} & c_{k2} & \cdots & c_{kn} \end{bmatrix}, \quad D = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ d_{n1} & d_{k2} & \cdots & d_{km} \end{bmatrix}
$$

Now, Eqs 1.34 and 1.35 can be compactly written as

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu} \tag{1.38a}$$

$$y = \boldsymbol{Cx} + \boldsymbol{Du} \tag{1.38b}$$

where $\dot{x} = dx/dt$. Equations 1.38 may be illustrated schematically as shown in Fig.1.10. The double lines indicate a multiple-variable signal flow path. The blocks represent matrix multiplication of the vectors and matrices. The integrator block consists of n integrators with appropriate connections specified by the **A** and **B** matrices.

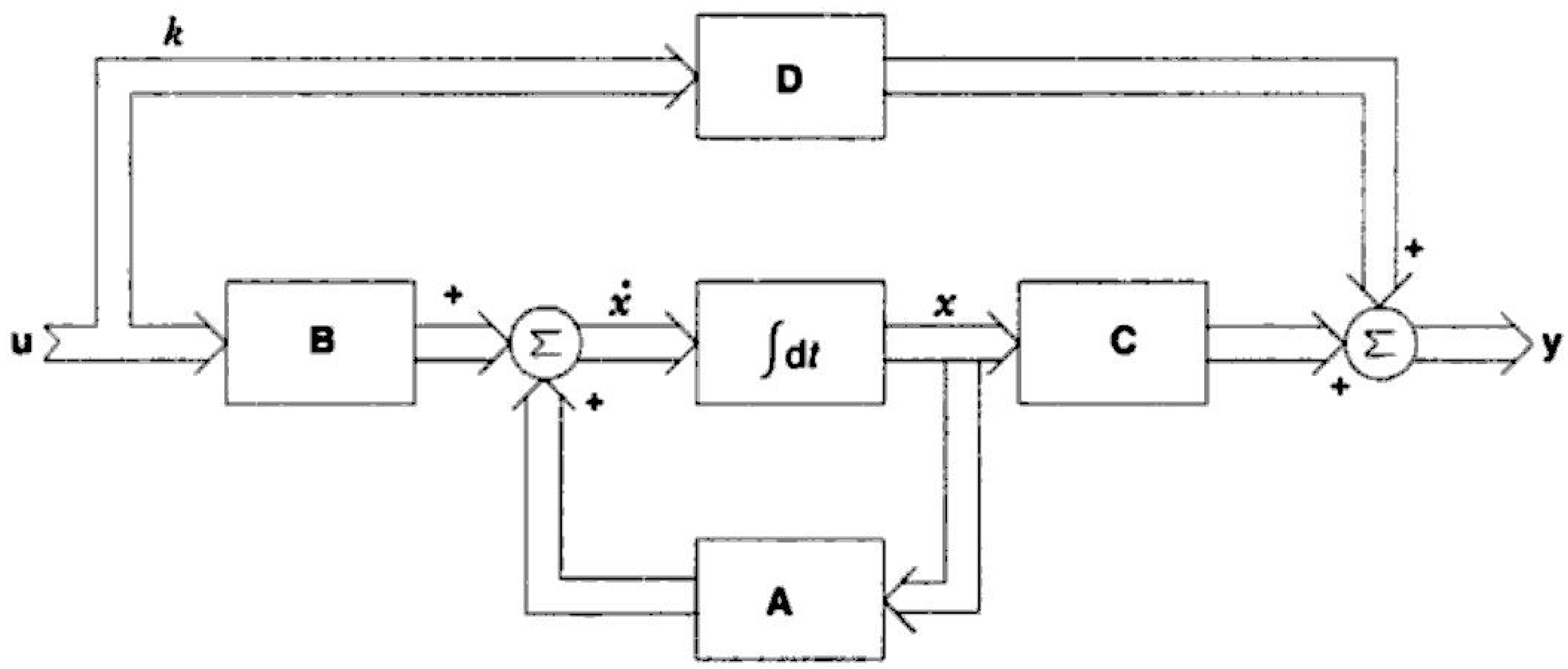


Fig. 1.10 *Block Diagram of the State-Variable Model of Eq. 1.38*

State Equations for Discrete-time Systems

For a discrete-time system, the state equations form a set of first-order difference equations constituting a recursion relation. This recursion relation allows determination of the state of a system at the sampling time kT from the state of the system and the input at the sampling time $(k - 1)T$, where k is an integer. The state-equations for a discrete-time system can be modelled as shown below.

$$\boldsymbol{x}_{k+1} = \boldsymbol{Fx} + \boldsymbol{Gu}_k \tag{1.39a}$$

$$\boldsymbol{y}_k = \boldsymbol{Hx}_k + \boldsymbol{Ju}_k \tag{1.39b}$$

The dependence of these parameters on T is suppressed for simplicity. For a single input, single output system $\boldsymbol{u}_k$ and $\boldsymbol{y}_k$ are scalars and $\boldsymbol{G}$ and $\boldsymbol{H}$ become vectors $\boldsymbol{g}$ and $\boldsymbol{h}$, and $\boldsymbol{J}$ is a null in most cases. The state-variable modelling of a discrete-time system finds application in the digital simulation of a continuous time systems.

1.8 ANALOG-TO-DIGITAL CONVERSION OF SIGNALS

A discrete-time signal is defined by specifying its value only at discrete times, called *sampling instants*. When the sampled values are quantised and encoded, a digital signal is obtained. A digital signal can be obtained from the analog signal by using an analog-to-digital converter. In the following sections the process of analog-to-digital conversion is discussed in some detail and this enables one to understand the relationship between the digital signals and discrete-time signals.

Figure 1.11 shows the block diagram of an analog-to-digital converter. The *sampler* extracts the sample values of the input signal at the sampling instants. The output of the sampler is the discrete-time signal with continuous amplitude. This signal is applied to a *quantiser* which converts this continuous amplitude into a finite number of sample values. Each sample value can be represented by a digital word of finite word length. The final stage of analog-to-digital conversion is encoding. The *encoder* assigns a digital word to each quantised sample. Sampling, quantizing and encoding are discussed in the following sections.

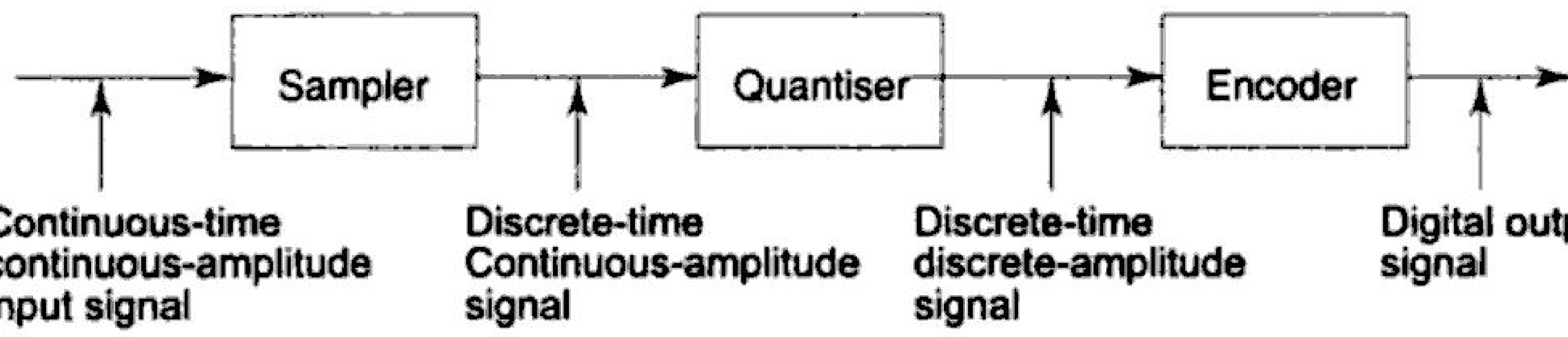


Fig. 1.11 *Analog-to-Digital Converter*

1.8.1 Sampling of Continuous-time Signals

Sampling is a process by which a continuous-time signal is converted into a discrete-time signal. This can be accomplished by representing the continuous-time signal $x(t)$, at a discrete number of points. These discrete number of points are determined by the sampling period, T, i.e. the samples of $x(t)$ can be obtained at discrete points $t = nT$, where n is an integer. The process of sampling is illustrated in Fig.1.12. The sampling unit can be thought of as a switch, where, to one of its inputs the continuous-time signal is applied. The signal is available at the output only during the instants the switch is closed. Thus, the signal at the output end is not a continuous function of time but only discrete samples. In order to extract samples of $x(t)$, the switch closes briefly every T seconds. Thus, the output signal has the same amplitude as $x(t)$ when the switch is closed and a value of zero when the switch is open. The switch can be any high speed switching device.

The continuous-time signal $x(t)$ must be sampled in such a way that the original signal can be reconstructed from these samples. Otherwise, the sampling process is useless. Let us obtain the condition necessary to faithfully reconstruct the original signal from the samples of that signal. The condition can be easily obtained if the signals are analysed in the frequency domain. Let the sampled signal be represented by $x_s(t)$. Then,

$$x_s(t) = x(t)\, g(t) \tag{1.40}$$

where $g(t)$ is the *sampling function*. The sampling function is a continuous train of pulses with a period of T seconds between the pulses, and it models the action of the sampling switch. The sampling function is shown in Fig. 1.12(c) and (d). The frequency spectrum of the sampled

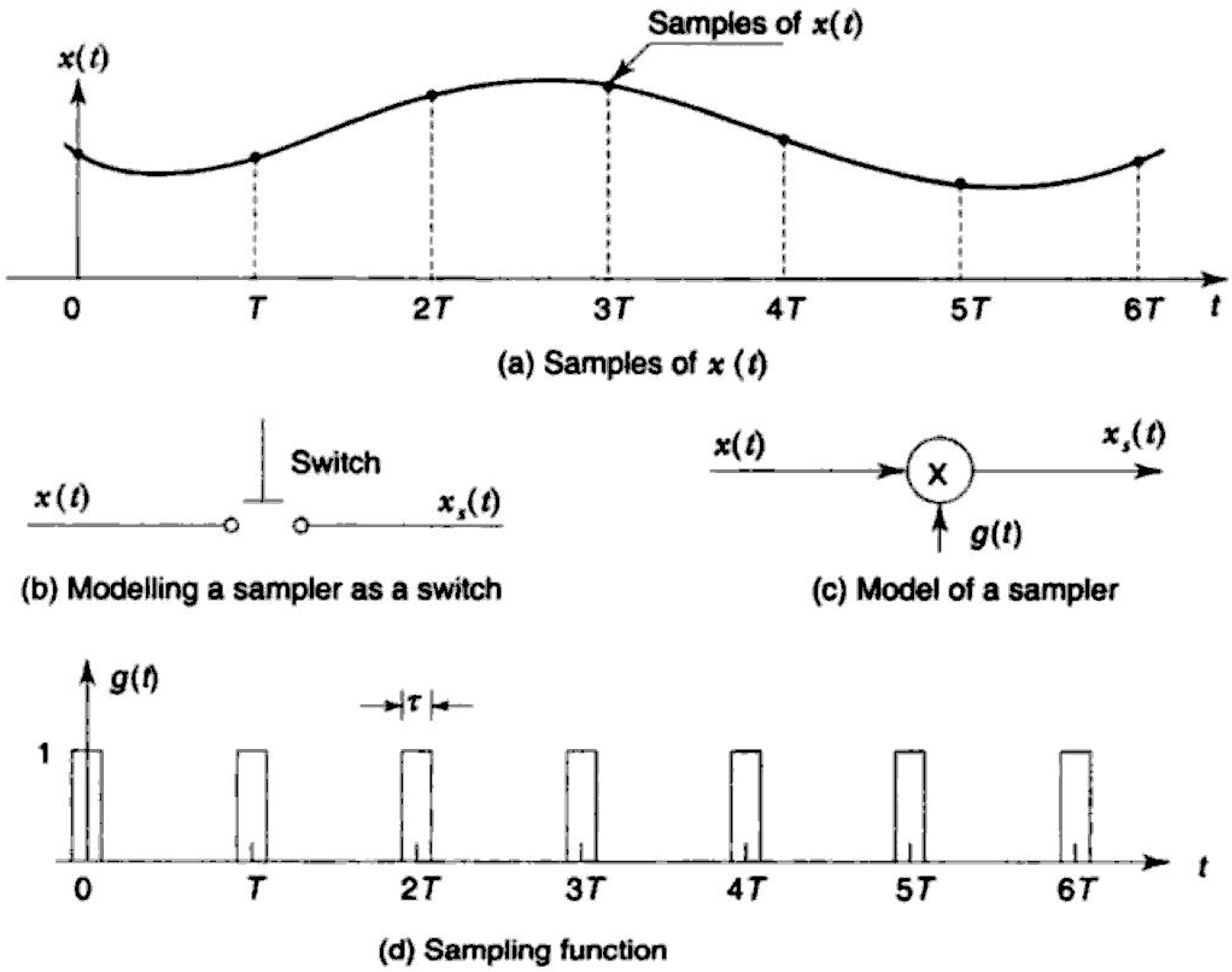


Fig. 1.12 *The Sampling Process*

signal $x_s(t)$ helps in determining the appropriate values of T for reconstructing the original signal. The sampling function $g(t)$ is periodic and can be represented by a Fourier series (Fourier Series and transforms are discussed in Chapter six), i.e.

$$g(t) = \sum_{n=-\infty}^{\infty} C_n \, e^{jn2\pi f_s t} \tag{1.41}$$

where

$$C_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} g(t) e^{-jn2\pi f_s t} \, \mathrm{d}t \tag{1.42}$$

is the nth Fourier coefficient of $g(t)$, and $f_s = \dfrac{1}{T}$ is the fundamental frequency of $g(t)$. The fundamental frequency, f_s is also called the *sampling frequency*. From Eqs.1.40, and 1.41, we have

$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} C_n e^{jn2\pi f_s t} = \sum_{n=-\infty}^{\infty} C_n x(t) e^{jn2\pi f_s t} \tag{1.43}$$

The spectrum of $x_s(t)$, denoted by $X_s(f)$, can be determined by taking the Fourier transform of Eq. 1.43, i.e.

$$X_s(f) = \int_{-\infty}^{\infty} x_s(t)\, e^{-j2\pi f t}\, dt \tag{1.44}$$

Using Eq.1.43 in the above equation,

$$X_s(f) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_n x(t)\, e^{jn2\pi f_s t}\, e^{-j2\pi f t} dt \tag{1.45}$$

Interchanging the order of integration and summation,

$$X_s(f) = \sum_{n=-\infty}^{\infty} C_n \int_{-\infty}^{\infty} x(t) e^{-j2\pi(f - nf_s)t}\, dt \tag{1.46}$$

But from the definition of the Fourier transform

$$\int_{-\infty}^{\infty} x(t) e^{-j2\pi(f - nf_s)t}\, dt = X(f - nf_s)$$

Thus,

$$X_s(f) = \sum_{n=-\infty}^{\infty} C_n X(f - nf_s) \tag{1.47}$$

From Eq. 1.47, it is understood that the spectrum of the sampled continuous-time signal is composed of the spectrum of $x(t)$ plus the spectrum of $x(t)$ translated to each harmonic of the sampling frequency. The spectrum of the sampled signal is shown in Fig. 1.13. Each frequency translated spectrum is multiplied by a constant. To reconstruct the original signal, it is enough to just pass the spectrum of $x(t)$ and suppress the spectra of other translated frequencies. The amplitude response of such a filter is also shown in Fig. 1.13. As this filter is used to reconstruct the original signal, it is often referred to as a *reconstruction filter*. The output of the reconstruction filter will be $C_0X(f)$ in the frequency domain and $x(t)$ in the time-domain.

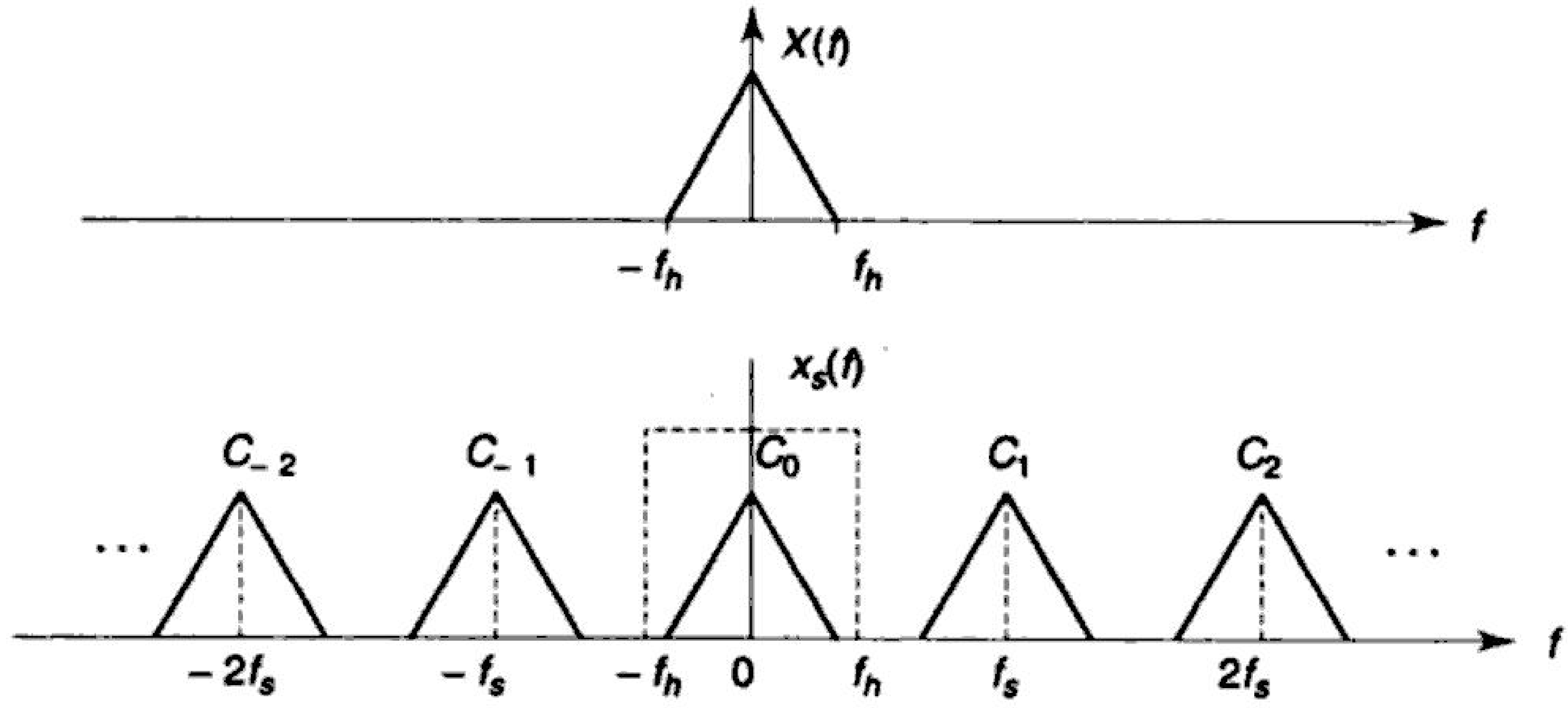


Fig. 1.13 *Spectrum of Sampled Signal*

The signal $x(t)$, in this case, is assumed to have no frequency components above f_h, i.e. in the frequency domain, $X(f)$ is zero for $|f| \geq f_h$. Such a signal is said to be ***bandlimited***. From Fig.1.13, it is clear that in order to recover $X(f)$ from $X_s(f)$, we must have

$$f_s - f_h \geq f_h$$

or equivalently,

$$f_s \geq 2f_h \text{, hertz} \tag{1.48}$$

That is, in order to recover the original signal from the samples, the sampling frequency must be greater than or equal to twice the maximum frequency in $x(t)$. *The sampling theorem is thus derived, which states that a bandlimited signal $x(t)$ having no frequency components above f_h hertz, is completely specified by samples that are taken at a uniform rate greater than $2f_h$ hertz.* The frequency equal to twice the highest frequency in $x(t)$, i.e. $2f_h$, is called the *Nyquist rate.*

Sampling by Impulse Function

The sampling function $g(t)$, discussed above, was periodic. The pulse width of the sampling function must be very small compared to the period, T. The samples in digital systems are in the form of a number, and the magnitude of these numbers represent the value of the signal $x(t)$ at the sampling instants. In this case, the pulse width of the sampling function is infinitely small and an infinite train of impulse functions of period T can be considered for the sampling function. That is,

$$g(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{1.49}$$

The sampling function as given in Eq.1.49 is shown in Fig.1.14. When this sampling function is used, the weight of the impulse carries the sample value.

The sampling function $g(t)$ is periodic and can be represented by a Fourier series as in Eq.1.41, which is repeated here.

$$g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn2\pi f_s t}$$

where

$$C_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)\, e^{-jn2\pi f_s t} dt \tag{1.50}$$

Since $\delta(t)$ has its maximum energy concentrated at $t = 0$, a more formal mathematical definition of the unit-impulse function may be defined as a functional

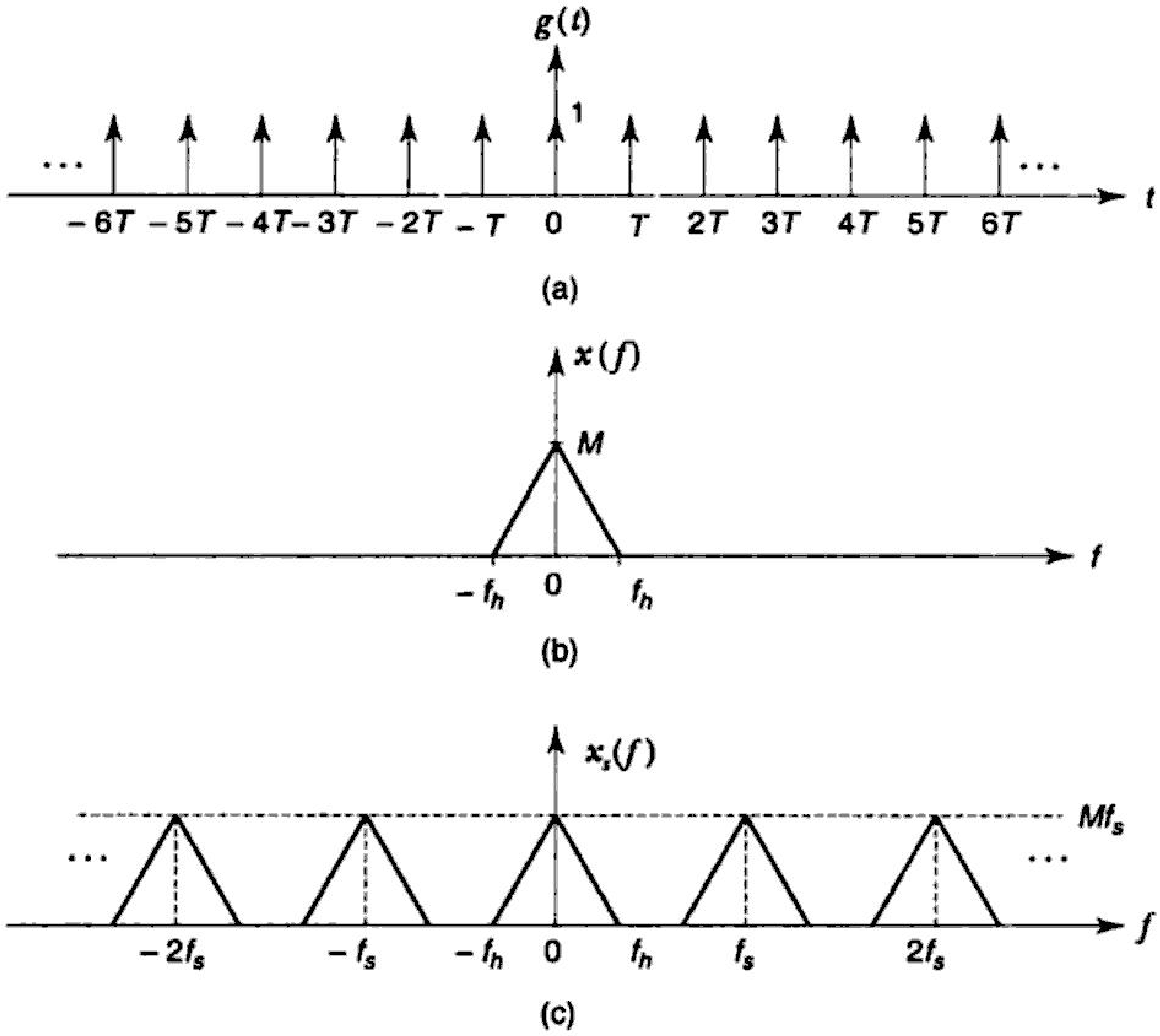


Fig. 1.14 *(a) Impulse Sampling Function (b) Spectrum of the Signal x(t) (c) Spectrum of Impulse Sampled Signal*

$$\int_{-\infty}^{\infty} x(t)\,\delta(t)\,dt = x(0) \tag{1.51}$$

where $x(t)$ is continuous at $t = 0$. Using Eq. 1.51 in Eq. 1.50, we have

$$C_n = \frac{1}{T}e^0 = \frac{1}{T} = f_s \tag{1.52}$$

Thus C_n is same as the sampling frequency f_s, for all n. The spectrum of the impulse sampled signal, $x_s(t)$ is given by

$$X_s(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \tag{1.53}$$

The spectra of the signal $x(t)$ and the impulse sampled signal X_s (t) are shown in Figs 1.14 (b) and (c). The effect of impulse sampling is same as sampling with a train of pulses. However, all the frequency translated spectra have the same amplitude. The original signal $X(f)$ can be reconstructed from $X_s(f)$ using a low-pass filter. Figure 1.15 shows the effect of sampling at a rate lower than the Nyquist rate.

Consider a bandlimited signal $x(t)$, with f_h as its highest frequency content, being sampled at a rate lower than the Nyquist rate, i.e., sampling frequency $f_s < 2f_h$. This results in overlapping of adjacent

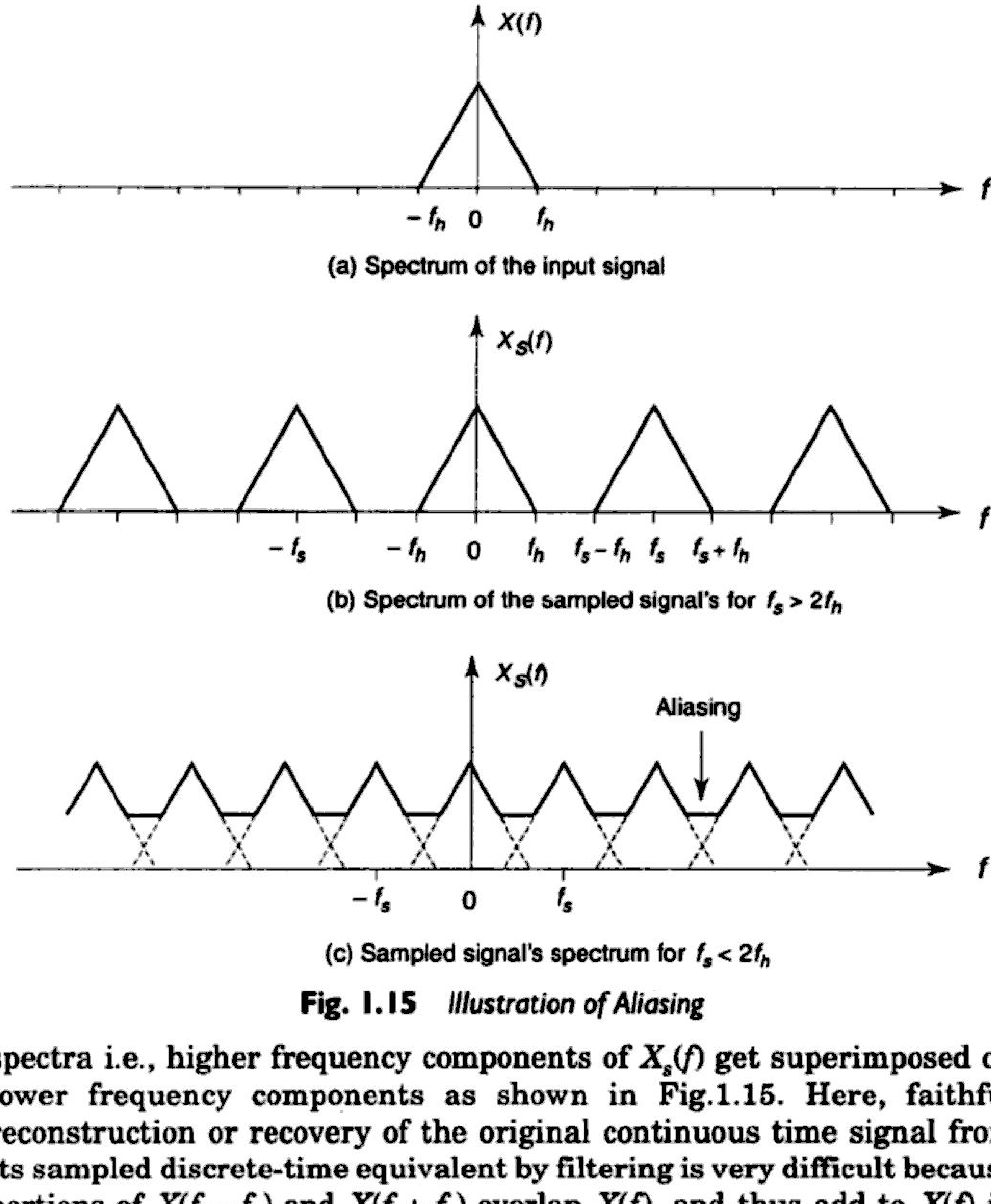


Fig. 1.15 *Illustration of Aliasing*

spectra i.e., higher frequency components of $X_s(f)$ get superimposed on lower frequency components as shown in Fig.1.15. Here, faithful reconstruction or recovery of the original continuous time signal from its sampled discrete-time equivalent by filtering is very difficult because portions of $X(f - f_s)$ and $X(f + f_s)$ overlap $X(f)$, and thus add to $X(f)$ in producing $X_s(f)$. The original shape of the signal is lost due to undersampling, i.e. down-sampling. This overlap is known as *aliasing* or *overlapping* or *fold over*. Aliasing, as the name implies, means that a signal can be impersonated by another signal. In practice, no signal is strictly bandlimited but there will be some frequency beyond which the energy is very small and negligible. This frequency is generally taken as the highest frequency content of the signal.

To prevent aliasing, the sampling frequency f_s should be greater than two times the frequency f_h of the sinusoidal signal being sampled. The condition to be satisfied by the sampling frequency to prevent aliasing is called the *sampling theorem*. In some applications, an analog anti-aliasing filter is placed before sample/hold circuit in order to prevent the aliasing effect.

A useful application of aliasing due to undersampling arises in the *sampling oscilloscope*, which is meant for observing very high frequency waveforms.

1.8.2 Signal Reconstruction

Any signal $x(t)$ can be faithfully reconstructed from its samples if these samples are taken at a rate greater than or equal to the Nyquist rate. It can be seen from the spectrum of the sampled signal, $X_s(t)$ that it consists of the spectra of the signal and its frequency translated harmonics. Thus, if the spectrum of the signal alone can be separated from that of the harmonics then the original signal can be obtained. This can be achieved by filtering the sampled signal using a low-pass filter with a bandwidth greater than f_h and less than $f_s - f_h$. hertz.

If the sampling function is an impulse sequence, we note from Eq.1.53 that the spectrum of the sampled signal has an amplitude equal to $f_s = 1/T$. Therefore, in order to remove this scaling constant, the low-pass filter must have an amplitude response of $1/f_s = T$. Assuming that sampling has been done at the Nyquist rate, i.e. $f_s = 2f_h$, the bandwidth of the low-pass filter will be $f_h = \frac{f_s}{2}$. Therefore, the unit impulse response of an ideal filter for this bandwidth is

$$h(t) = T \int_{-f_s/2}^{f_s/2} e^{j2\pi ft}\, df \tag{1.54}$$

That is

$$h(t) = \frac{T}{j2\pi t}\left(e^{j\pi f_s t} - e^{-j\pi f_s t}\right)$$

The above expression can be alternatively written as

$$h(t) = Tf_s \frac{\sin \pi f_s t}{\pi f_s t} = \text{sinc } f_s\, t \tag{1.55}$$

The ideal reconstruction filter is shown in Fig.1.16a. The input to this filter is the sampled signal $x(nT)$ and the output of the filter is the reconstructed signal $x(t)$. The output signal $x(t)$ is given by

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\, h(t - nT)$$

Using Eq.1.55, we get

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT) \,\text{sinc } f_s(t - nT) \tag{1.56}$$

The above expression is a ***convolution*** expression and the signal $x(t)$ is reconstructed by convoluting its samples with the unit-impulse response of the filter. Eq. 1.56 can also be interpreted as follows. The original signal can be reconstructed by weighting each sample by a sinc function and adding them all. This process is shown in Fig. 1.16b.

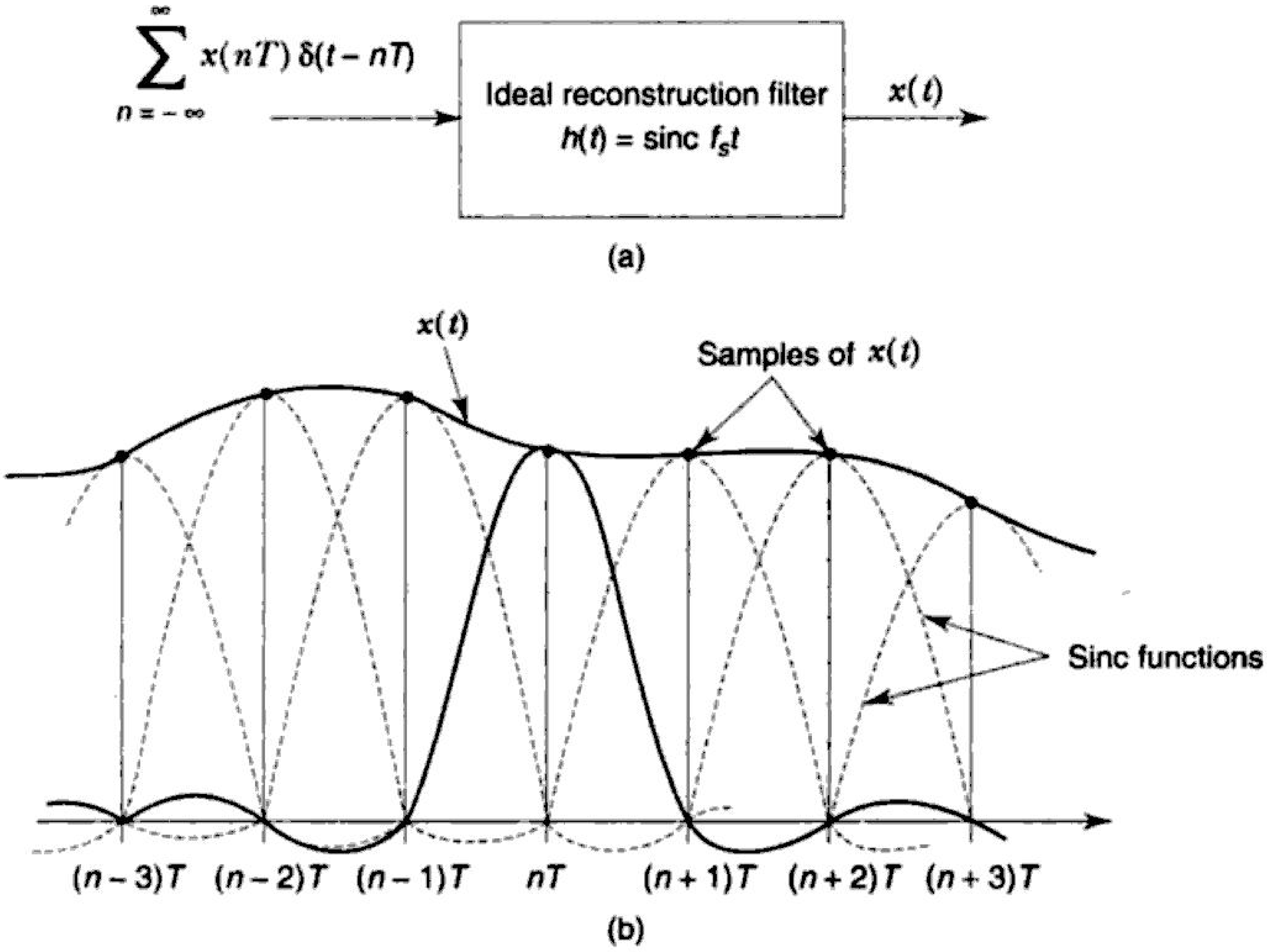


Fig. 1.16 *Signal Reconstruction (a) Reconstruction Filter (b) Time Domain Representation*

1.8.3 Signal Quantisation and Encoding

A discrete-time signal with continuous-valued amplitudes is called a *sampled data signal*, whereas a continuous-time signal with discrete-valued amplitudes is referred to as a quantised boxcar signal. Quantisation is a process by which the amplitude of each sample of a signal is rounded off to the nearest permissible level. That is, quantisation is conversion of a discrete-time continuous-amplitude signal into a discrete-time, discrete-valued signal. Then encoding is done by representing each of these permissible levels by a digital word of fixed wordlength.

The process of quantisation introduces an error called *quantisation error* and it is simply the difference between the value of the analog input and the analog equivalent of the digital representation. This error will be small if there are more permissible levels and the width of these quantization levels is very small. In the analog-to-digital conversion process, the only source of error is the quantiser. Even if there are more quantisation levels, error can occur if the signal is at its maximum or minimum value for significant time intervals. Figure 1.17 shows how a continuous-time signal is quantised in a quantiser that has 16 quantising levels.

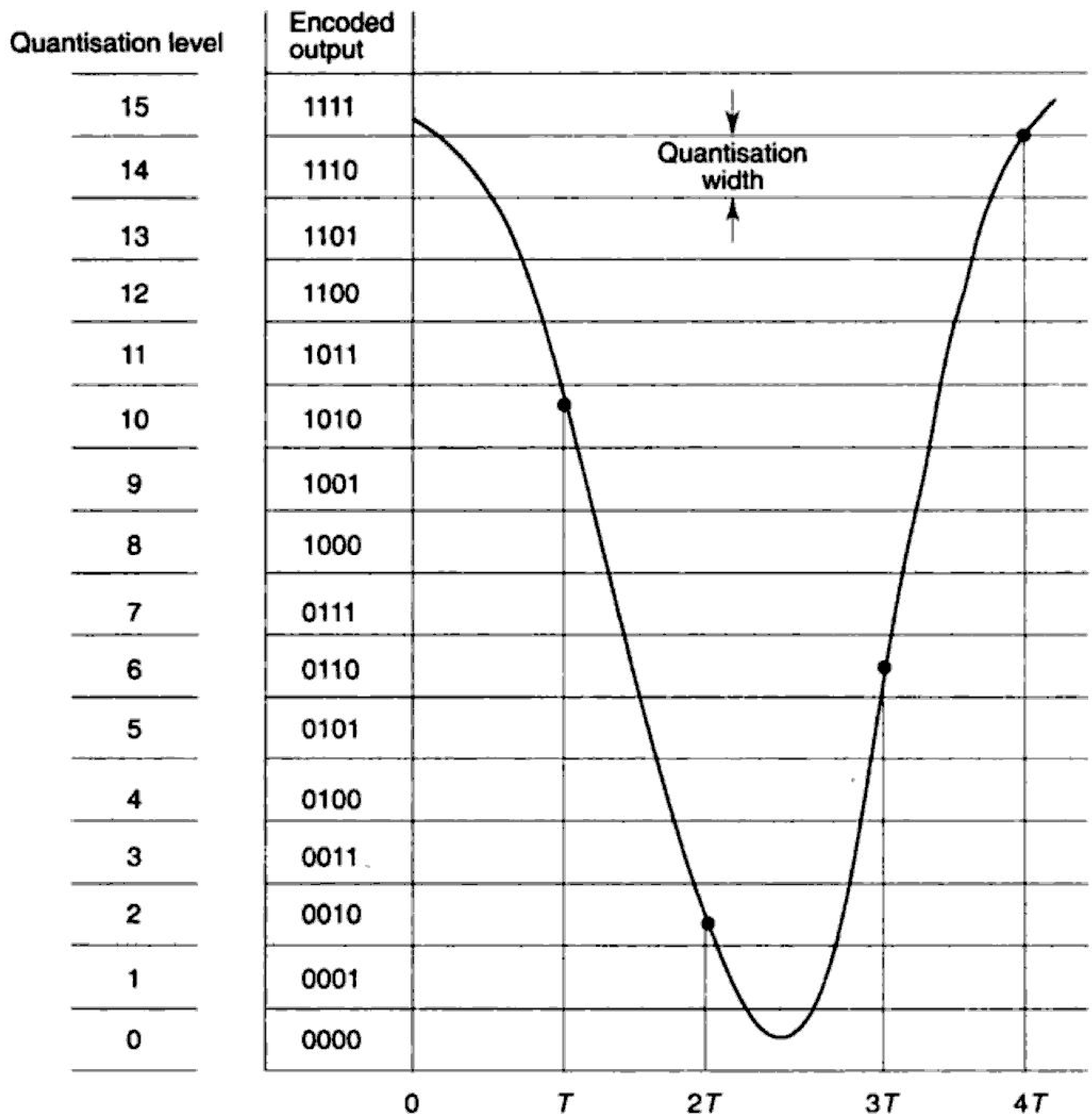


Fig. 1.17 *Quantizing and Encoding*

REVIEW QUESTIONS

1.1 What are the major classifications of signals?

1.2 With suitable examples distinguish a deterministic signal from a random signal.

1.3 What are periodic signals? Give examples.

1.4 Describe the procedure used to determine whether the sum of two periodic signals is periodic or not.

1.5 Determine which of the following signals are periodic and determine the fundamental period.

(a) $x_1(t) = 10 \sin 25\pi t$ (b) $x_2(t) = 10 \sin \sqrt{5}\, \pi t$

(c) $x_3(t) = \cos 10\pi t$ (d) $x_4(t) = x_1(t) + x_2(t)$

(e) $x_5(t) = x_1(t) + x_3(t)$ (f) $x_4(t) = x_2(t) + x_3(t)$

1.6 What are even signals? Give examples.

1.7 What are odd signals? Give examples.

1.8 What is energy signal?

1.9 What is power signal?

1.10 What are singularity functions?

1.11 Define unit-impulse function?

1.12 What is unit-step function? How it can be obtained from an unit-impulse function?

1.13 What is unit-ramp function? How it can be obtained from an unit-impulse function?

1.14 What is pulse function?

1.15 Evaluate

(a) $\int_{-\infty}^{\infty} e^{-at^2}\,\delta(t-10)\,dt$ (b) $\int_{-\infty}^{\infty} e^{-2t^2}\,\delta(t+5)\,dt$

(c) $\int_{-\infty}^{\infty} 40e^{-2t^2}\,\delta(t-10)\,dt$ and (d) $\int_{-\infty}^{\infty} e^{-2t^2}\,\delta(t-10)\,dt$

Ans (a) $e^{-100\alpha}$ (b) 0 (c) $40\,e^{-200}$ (d) e^{-200}

1.16 Explain the terms single-sided spectrum and double-sided spectrum with respect to a signal.

1.17 Sketch the single-sided and double-sided frequency spectra of the signals

(a) $x_1(t) = 10\sin\left(10\pi t - \frac{2\pi}{3}\right), -\infty < t < \infty$

(b) $x_2(t) = 25\cos\left(5\pi t - \frac{\pi}{2}\right), -\infty < t < \infty$

(c) $x_3(t) = 100\sin\left(10\pi t - \frac{2\pi}{3}\right) + 50\cos\left(25\pi t - \frac{\pi}{3}\right), -\infty < t < \infty$

1.18 How are systems classified?

1.19 Distinguish static systems from dynamic systems.

1.20 What is linear system ?

1.21 Determine whether the following systems are linear

(a) $\frac{dy(t)}{dt} + 5y(t) + 2 = x(t)$ (b) $5\frac{dy(t)}{dt} + y(t) = 5x(t)$

(c) $\frac{dy(t)}{dt} + y(t) + 5 = 10x(t)$

1.22 What is LTI system?

1.23 What is a causal system? Why are non-causal systems unrealisable?

1.24 What is BIBO stability?

1.25 What are the conditions for BIBO stability?
1.26 With illustrations, explain shifting, folding and time scaling operations on discrete-time signals.
1.27 What are the different ways of representing a system?
1.28 Explain how difference/differential equations are used to model a system.
1.29 Explain how impulse response can model a system.
1.30 Discuss the state-variable modelling of a system.
1.31 Explain the terms. (i) state variable (ii) state space (iii) state vector (iv) trajectory (v) state equations and (vi) output equations.
1.32 With a block diagram explain the process of analog-to-digital conversion.
1.33 What is meant by sampling? State the sampling theorem.
1.34 Explain how sampling can be done with an impulse function.
1.35 Draw the spectrum of a sampled signal and explain aliasing.
1.36 Explain the process of reconstruction of the signal from its samples. Obtain the impulse response of an ideal reconstruction filter.
1.37 What is meant by quantization and encoding?
1.38 What is a quantized boxcar signal?

Chapter 2

Fourier Analysis of Periodic and Aperiodic Continuous-Time Signals and Systems

2.1 INTRODUCTION

A signal which is repetitive is a periodic function of time. Any periodic function of time $f(t)$ can be represented by an infinite series called the ***Fourier Series***. A function of time $f(t)$ is said to be periodic of period T if $f(t) = f(t + T)$ for all t. For example, the periodic waveforms of sinusoidal and exponential forms are shown in Fig. 2.1.

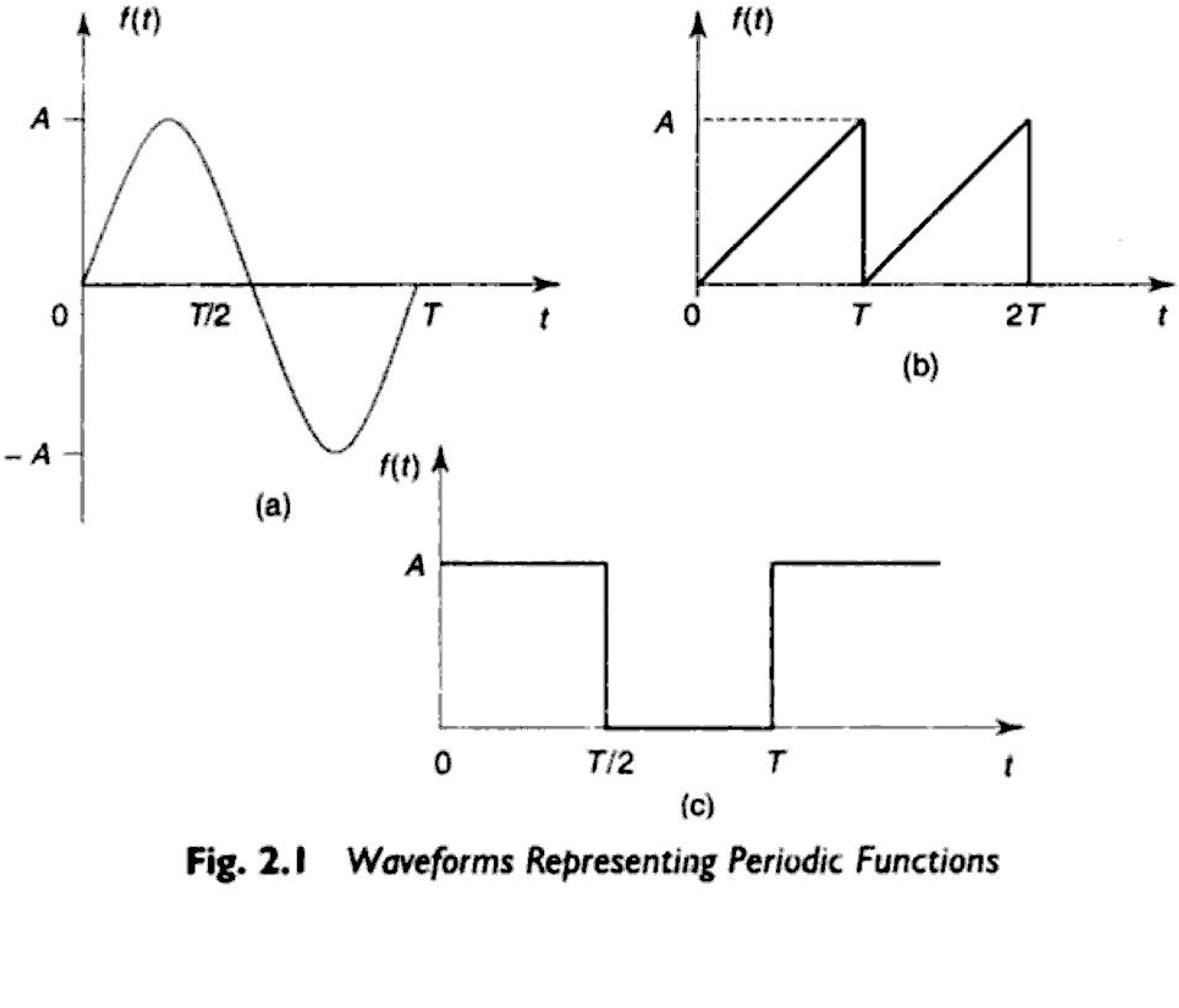


Fig. 2.1 *Waveforms Representing Periodic Functions*

Examples of periodic processes are the vibration of a tuning fork, oscillations of a pendulum, conduction of heat, alternating current passing through a circuit, propagation of sound in a medium, etc. Fourier series may be used to represent either functions of time or functions of space co-ordinates. In a similar manner, functions of two and three variables may be represented as double and triple Fourier series respectively. Periodic waveforms may be expressed in the form of Fourier series. Non-periodic waveforms may be expressed by Fourier transforms.

2.2 TRIGONOMETRIC FOURIER SERIES

A periodic function $f(t)$ can be expressed in the form of trigonometric series as

$$\begin{aligned} f(t) &= \frac{1}{2}a_0 + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + a_3 \cos 3\omega_0 t + \ldots \\ &\quad + b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t + b_3 \sin 3\omega_0 t + \ldots \end{aligned} \tag{2.1}$$

where $\omega_0 = 2\pi f = \frac{2\pi}{T}$, f is the frequency and a's and b's are the coefficients. The Fourier series exists only when the function $f(t)$ satisfies the following three conditions called ***Dirichlet's conditions***.

(i) $f(t)$ is well defined and single-valued, except possibly at a finite number of points, i.e.
$f(t)$ has a finite average value over the period T.

(ii) $f(t)$ must posses only a finite number of discontinuities in the period T.

(iii) $f(t)$ must have a finite number of positive and negative maxima in the period T.

Equation 2.1 may be expressed by the Fourier series

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \tag{2.2}$$

where a_n and b_n are the coefficients to be evaluated.

Integrating Eq. 2.2 for a full period, we get

$$\int_{-T/2}^{T/2} f(t)\,dt = \frac{1}{2}a_0 \int_{-T/2}^{T/2} dt + \int_{-T/2}^{T/2} \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)\,dt$$

Integration of cosine or sine function for a complete period is zero.

Therefore, $$\int_{-T/2}^{T/2} f(t)\,dt = \frac{1}{2}a_0 T$$

Hence, $$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt \tag{2.3}$$

or, equivalently $a_0 = \frac{2}{T}\int_0^T f(t)\,dt$

Multiplying both sides of Eq. 2.2 by $\cos m\omega_0 t$ and integrating, we have

$$\int_{-T/2}^{T/2} f(t)\cos m\omega_0 t\,dt = \frac{1}{2}\int_{-T/2}^{T/2} a_0\cos m\,\omega_0 t\,dt +$$

$$\int_{-T/2}^{T/2}\sum_{n=1}^{\infty} a_n \cos n\,\omega_0\,t\cos m\,\omega_0 t\,dt + \int_{-T/2}^{T/2}\sum_{n=1}^{\infty} b_n \sin n\,\omega_0\,t\cos m\,\omega_0 t\,dt$$

Here, $\frac{1}{2}\int_{-T/2}^{T/2} a_0\cos m\,\omega_0 t\,dt = 0$

$$\int_{-T/2}^{T/2} a_n\cos n\,\omega_0 t\cos m\,\omega_0\,dt = \frac{a_n}{2}\int_{-T/2}^{T/2}[\cos(m+n)\,\omega_0\,t + \cos(m-n)\,\omega_0 t]\,dt$$

$$= \begin{cases} 0, & \text{for } m \neq n \\ \frac{T}{2}a_n, & \text{for } m = n \end{cases}$$

$$\int_{-T/2}^{T/2} b_n \sin n\,\omega_0 t\cos m\,\omega_0\,t\,dt = \frac{bn}{2}\int_{-T/2}^{T/2}[\sin(m+n)\,\omega_0 t - \sin(m-n)\omega_0 t]\,dt$$

$$= 0$$

Therefore, $\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t\,dt = \frac{T a_n}{2}$, for $m = n$

Hence, $$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\,\omega_0\,t\,dt \tag{2.4}$$

or, equivalently $a_n = \frac{2}{T}\int_0^T f(t)\cos n\,\omega_0\,t\,dt$

Similarly, multiplying both sides of Eq. 2.2 by $\sin\ m\ \omega_0\ t$ and integrating, we get

$$\int_{-T/2}^{T/2} f(t)\sin m\,\omega_0\,t\,dt = \frac{1}{2}\int_{-T/2}^{T/2} a_0\sin m\,\omega_0\,t\,dt$$

$$+ \int_{-T/2}^{T/2}\sum_{n=1}^{\infty} a_n \cos n\,\omega_0 t\sin m\,\omega_0\,t\,dt + \int_{-T/2}^{T/2}\sum_{n=1}^{\infty} b_n \sin n\,\omega_0 t\sin m\,\omega_0\,t\,dt$$

Here, $\frac{1}{2}\int_{-T/2}^{T/2} a_0 \sin m\omega_0 t \, dt = 0$

$$\int_{-T/2}^{T/2} a_n \cos n\omega_0 t \sin m\omega_0 t \, dt = 0$$

$$\int_{-T/2}^{T/2} b_n \sin n\omega_0 t \sin m\omega_0 t \, dt = \begin{cases} 0, & \text{for } m \neq n \\ \frac{T}{2} b_n, & \text{for } m = n \end{cases}$$

Therefore, $\int_{-T/2}^{T/2} f(t) \sin m\omega_0 t \, dt = \frac{T}{2} b_n$, for $m = n$

Hence, $$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t) \sin n\omega_0 t \, dt \tag{2.5}$$

or, equivalently, $b_n = \frac{2}{T}\int_{0}^{T} f(t) \sin n\omega_0 t \, dt$

The number $n = 1, 2, 3, \ldots$ gives the values of the harmonic frequencies.

Symmetry Conditions

(i) If the function $f(t)$ is even, then $f(-t) = f(t)$. For example, $\cos t$, t^2, $t \sin t$, are all even. The cosine is an even function, since it may be expressed as the power series

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots$$

The waveforms representing the even functions of t are shown in Fig. 2.2. Geometrically, the graph of an even function will be symmetrical with respect to the y-axis and only cosine terms are present (d.c. term optional). When $f(t)$ is even,

$$\int_{-a}^{a} f(t) \, dt = 2\int_{0}^{a} f(t) \, dt$$

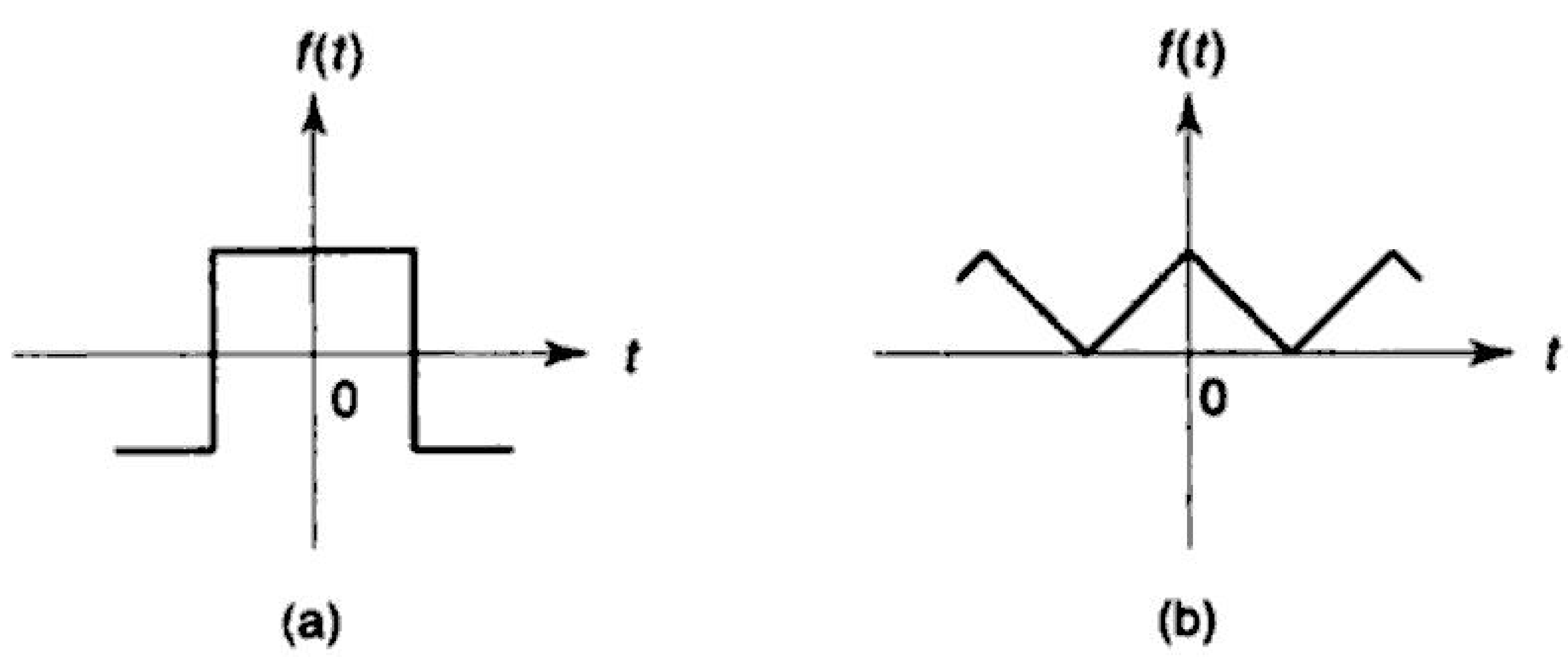


Fig. 2.2 *(Contd.)*

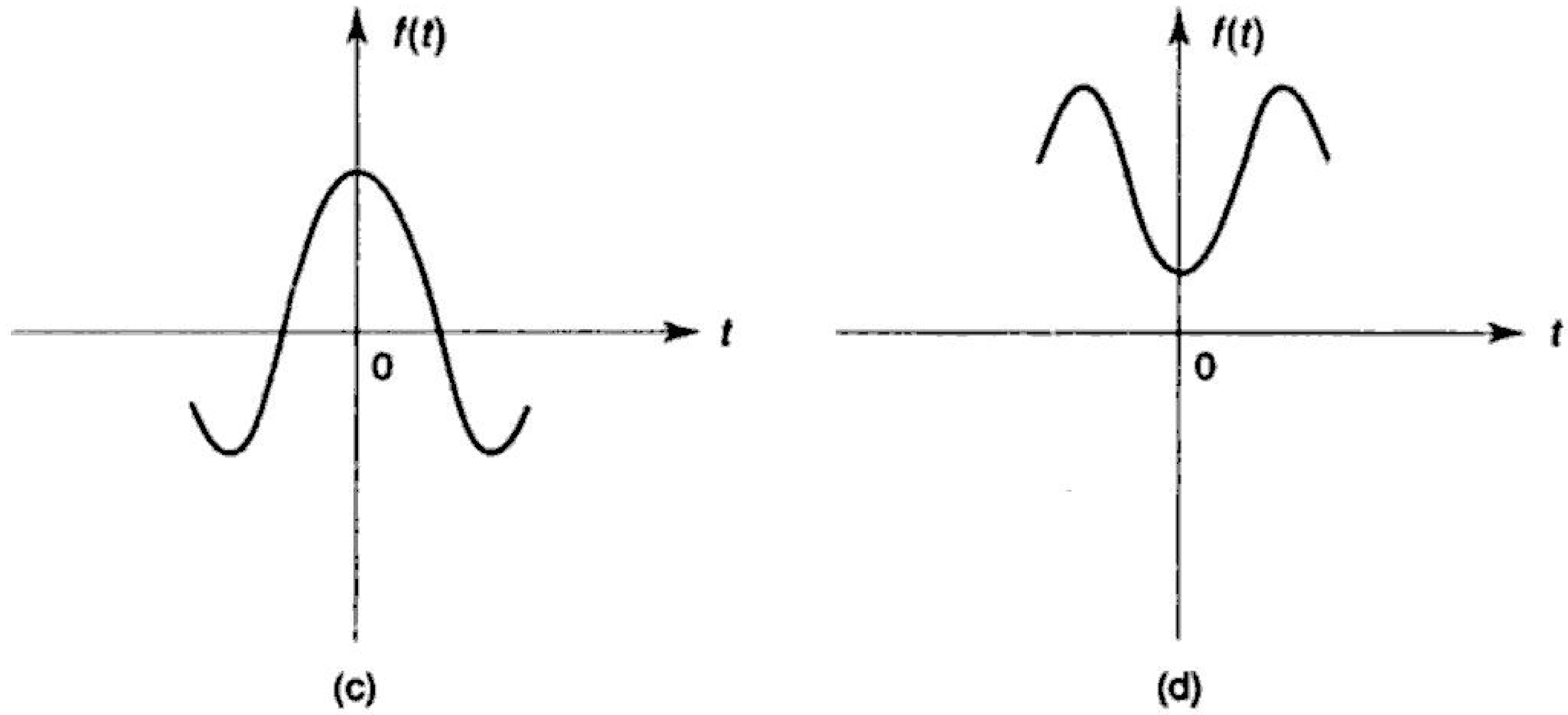


Fig. 2.2 *Waveforms Representing Even Functions*

The sum or product of two or more even functions is an even function.

(ii) If the function $f(t)$ is odd, then $f(-t) = -f(t)$ and only sine terms are present (d.c. term optional). For example, $\sin t$, t^3, $t \cos t$ are all odd. The waveforms shown in Fig. 2.3 represent odd functions of t. The graph of an odd function is symmetrical about the origin. If $f(t)$ is odd, $\int_{-a}^{a} f(t)\, dt = 0$. The sum of two or more odd functions is an odd function and the product of two odd functions is an even function.

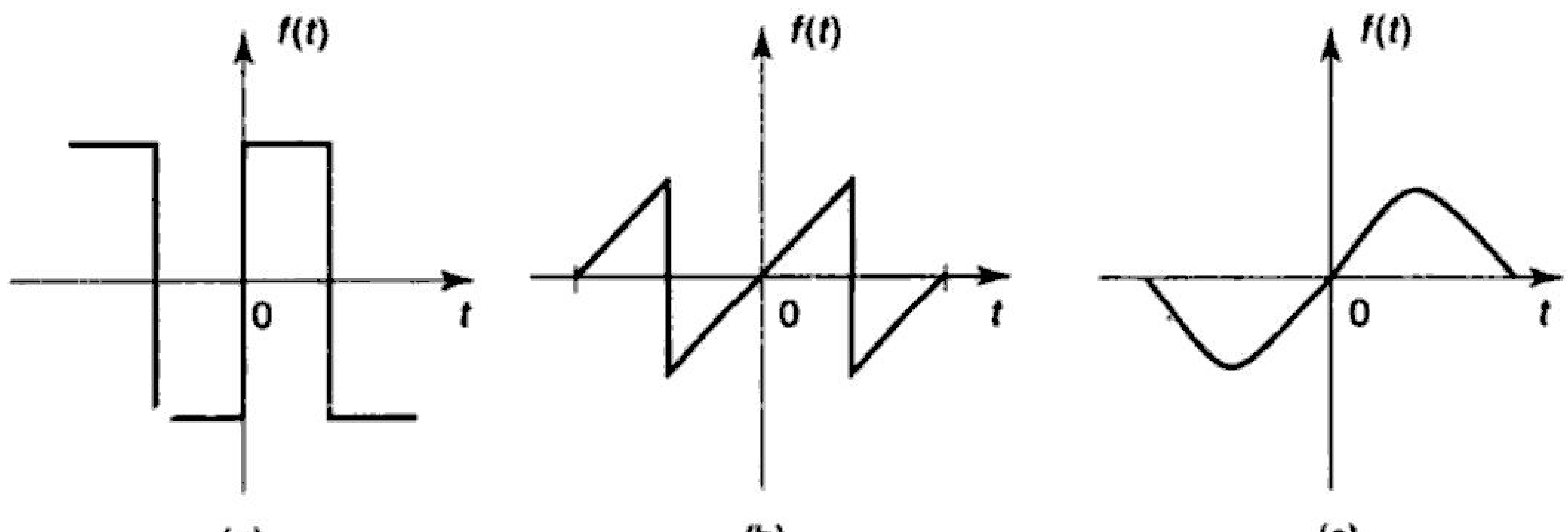


Fig. 2.3 *Waveforms Representing Odd Functions*

(iii) If $f(t + T/2) = f(t)$, only even harmonics are present.

(iv) If $f(t + T/2) = -f(t)$, only odd harmonics are present and hence the waveform has half-wave symmetry.

Example 2.1 **Obtain the Fourier components of the periodic square wave signal which is symmetrical with respect to the vertical axis at time $t = 0$, as shown in Fig. E2.1.**

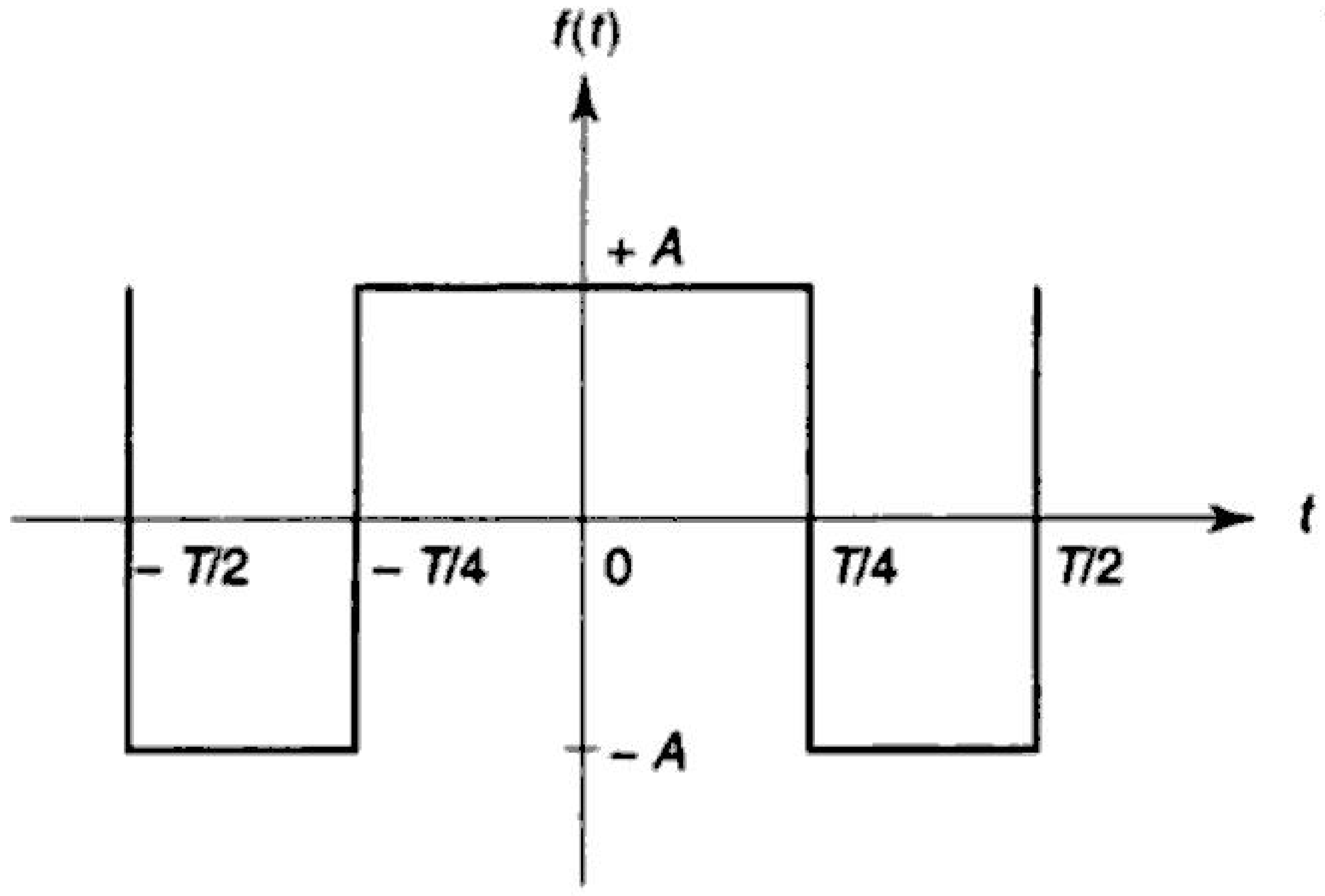


Fig. E2.1

Solution Since the given waveform is symmetrical about the horizontal axis, the average area is zero and hence the d.c. term $a_0 = 0$. In addition, $f(t) = f(-t)$ and so only cosine terms are present, i.e., $b_n = 0$.

Now, $$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\,\omega_0 t\,dt$$

where $$f(t) = \begin{cases} -A, & \text{from } -T/2 < t < -T/4 \\ +A, & \text{from } -T/4 < t < +T/4 \\ -A, & \text{from } +T/4 < t < +T/2 \end{cases}$$

Therefore ,

$$a_n = \frac{2A}{T}\left[\int_{-T/2}^{-T/4}(-\cos n\omega_0 t)dt + \int_{-T/4}^{T/4}\cos n\omega_0 t\,dt + \int_{T/4}^{T/2}(-\cos n\,\omega_0\,t)\,dt\right]$$

$$= \frac{2A}{T}\left[\left(\frac{-\sin n\omega_0 t}{n\,\omega_0}\right)_{-T/2}^{-T/4} + \left(\frac{\sin n\,\omega_0 t}{n\,\omega_0}\right)_{-T/4}^{T/4} + \left(\frac{-\sin n\,\omega_0 t}{n\,\omega_0}\right)_{T/4}^{T/2}\right]$$

$$= \frac{2A}{n\,\omega_0 T}\left[-\sin\left(\frac{-n\,\omega_0 T}{4}\right) + \sin\left(\frac{-n\,\omega_0 T}{2}\right) + \sin\left(\frac{n\,\omega_0 T}{4}\right)\right.$$

$$\left. -\sin\left(\frac{-n\,\omega_0 T}{4}\right) - \sin\left(\frac{n\,\omega_0 T}{2}\right) + \sin\left(\frac{n\,\omega_0 T}{4}\right)\right]$$

$$= \frac{8A}{n\,\omega_0 T}\sin\left(\frac{n\,\omega_0 T}{4}\right) - \frac{4A}{n\,\omega_0 T}\sin\left(\frac{n\,\omega_0 T}{2}\right)$$

When $\omega_0\,T = 2\pi$, the second term is zero for all integer values of n. Hence,

$$a_n = \frac{8A}{2n\pi}\sin\left(\frac{n\pi}{2}\right) = \frac{4A}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$a_0 = 0 \text{ (d.c. term)}$$

$$a_1 = \frac{4A}{\pi}\sin\left(\frac{\pi}{2}\right) = \frac{4A}{\pi}$$

$$a_2 = \frac{4A}{\pi}\sin(\pi) = 0$$

$$a_3 = \frac{4A}{3\pi}\sin\left(\frac{3\pi}{2}\right) = -\frac{4A}{3\pi}$$

. . . .

. . . .

Substituting the values of the coefficients in Eq. 2.2, we get

$$f(t) = \frac{4A}{\pi}\left[\cos(\omega_0 t) - \frac{1}{3}\cos(3\omega_0 t) + \frac{1}{5}\cos(5\omega_0 t) - \cdots\right]$$

Example 2.2 Obtain the Fourier Components of the periodic rectangular waveform shown in Fig. E2.2.

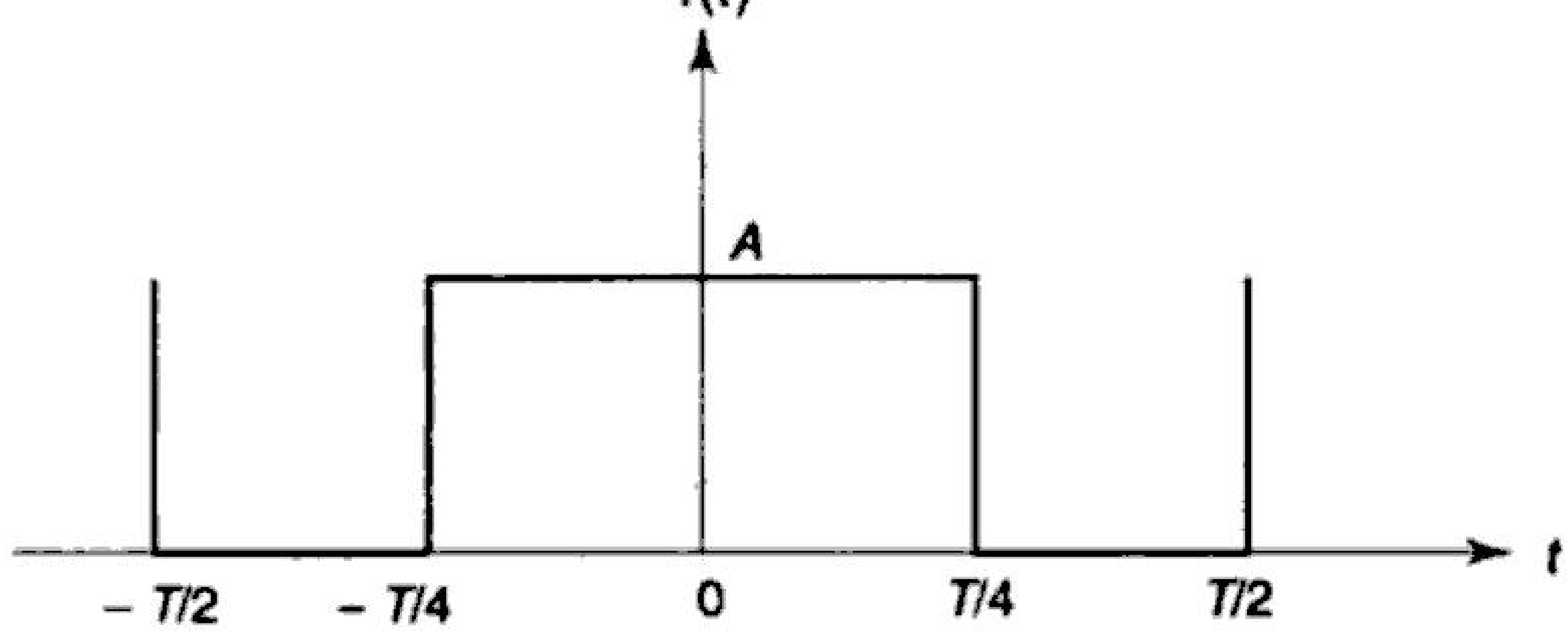


Fig. E2.2

Solution The given waveform for one period can be written as

$$f(t) = \begin{cases} 0, & \text{for } -T/2 < t < -T/4 \\ A, & \text{for } -T/4 < t < T/4 \\ 0, & \text{for } T/4 < t < T/2 \end{cases}$$

For the given waveform, $f(-t) = f(t)$ and hence it is an even function and has $b_n = 0$.
The value of the d.c.term is

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt$$

$$= \frac{2}{T}\int_{-T/4}^{T/4} A\,dt = \frac{2A}{T} \times \frac{T}{2} = A$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t\,dt$$

$$= \frac{2}{T}\int_{-T/4}^{T/4} A\cos n\omega_0 t\, dt$$

$$= \frac{2A}{T}\left[\frac{\sin n\omega_0 t}{n\omega_0}\right]_{-T/4}^{T/4}$$

$$= \frac{4A}{n\omega_0 T}\sin(n\omega_0\, T/4)$$

When $\omega_0 T = 2\pi$, we have

$$a_n = \frac{2A}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$= 0, \quad \text{for } n = 2, 4, 6, \ldots$$

$$= \frac{2A}{n\pi}, \quad \text{for } n = 1, 5, 9, 13, \ldots$$

$$= -\frac{2A}{n\pi}, \quad \text{for } n = 3, 7, 11, 15, \ldots$$

Substituting the values of the coefficients in Eq. 2.2, we obtain

$$f(t) = \frac{A}{2} + \frac{2A}{\pi}\left(\cos\omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t - \cdots\right)$$

Example 2.3 Obtain the trigonometric Fourier series for the half-wave rectified sine wave shown in Fig. E2.3.

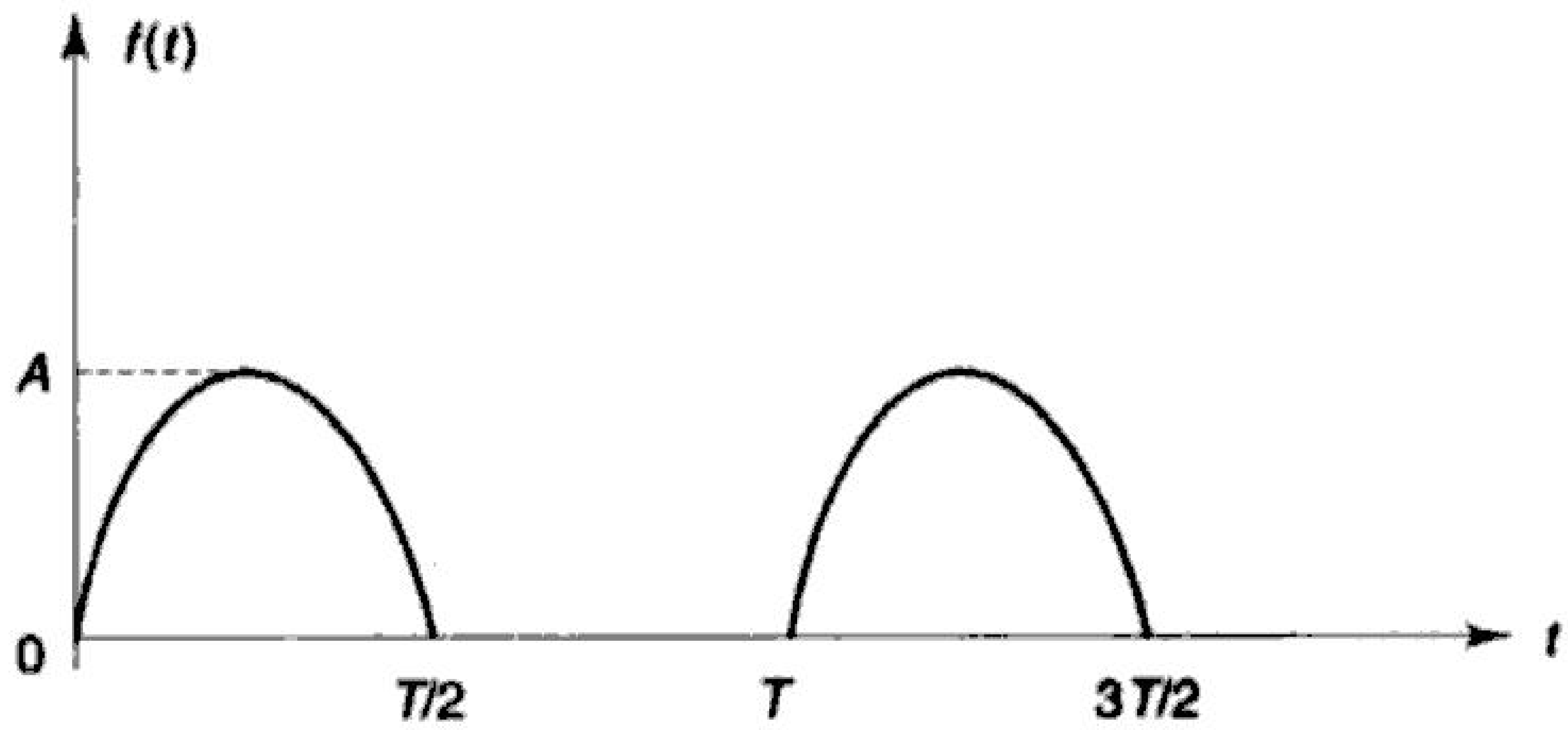


Fig. E2.3

Solution As the waveform shows no symmetry, the series may contain both sine and cosine terms. Here, $f(t) = A\sin\omega_0 t$

To evaluate a_0:

$$a_0 = \frac{2}{T}\int_0^T A\sin\omega_0 t\, dt$$

$$= \frac{2}{T}\int_0^{T/2} A\sin\omega_0 t\, dt$$

$$= \frac{2A}{\omega_0 T}\left[-\cos\omega_0 t\right]_0^{T/2} = \frac{2A}{\omega_0 T}\left[-\cos(\omega_0 T/2) + 1\right]$$

Substituting $\omega_0 T = 2\pi$, we have $a_0 = \dfrac{2A}{\pi}$.

To evaluate a_n:

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\,dt$$

$$= \frac{2}{T}\int_0^{T/2} A\sin\omega_0 t\cos n\omega_0 t\,dt$$

$$= \frac{2A}{\omega_0 T}\left[\frac{-n\sin\omega_0 t\sin n\omega_0 t - \cos n\omega_0 t\cos\omega_0 t}{-n^2+1}\right]_0^{T/2}$$

Substituting $\omega_0 T = 2\pi$, we have

$$a_n = \frac{A}{\pi(1-n^2)}\left[\cos n\pi + 1\right]$$

Hence, $\quad a_n = \dfrac{2A}{\pi(1-n^2)}$, for n even

$= 0$, for n odd

For $n = 1$, this expression is infinite and hence we have to integrate separately to evaluate a_1.

Therefore, $\quad a_1 = \dfrac{2}{T}\displaystyle\int_0^{T/2} A\sin\omega_0 t\cos\omega_0 t\,dt$

$$= \frac{A}{T}\int_0^{T/2}\sin 2\omega_0 t\,dt$$

$$= \frac{A}{2\omega_0 T}\left[-\cos 2\omega_0 t\right]_0^{T/2}$$

When $\omega_0 T = 2\pi$, we have $a_1 = 0$.
To find b_n:

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\,dt$$

$$= \frac{2}{T}\int_0^{T/2} A\sin\omega_0 t\sin n\omega_0 t\,dt$$

$$= \frac{2A}{\omega_0 T}\left[\frac{n\sin\omega_0 t\cos n\omega_0 t - \sin n\omega_0 t\cos\omega_0 t}{-n^2+1}\right]_0^{T/2}$$

When $\omega_0 T = 2\pi$, we have $b_n = 0$.
For $n = 1$, the expression is infinite and hence b_1 has to be calculated separately.

$$b_1 = \frac{2A}{T}\int_0^{T/2} \sin^2 \omega_0 t \, dt$$

$$= \frac{2A}{\omega_0 T}\left[\frac{\omega_0 t}{2} - \frac{\sin 2\omega_0 t}{4}\right]_0^{T/2}$$

When $\omega_0 T = 2\pi$, we have $b_1 = \dfrac{A}{2}$.

Substituting the values of the coefficients in Eq. 2.2, we get

$$f(t) = \frac{A}{\pi}\left\{1 + \frac{\pi}{2}\sin\omega_0 t - \frac{2}{3}\cos 2\omega_0 t - \frac{2}{15}\cos 4\omega_0 t - \cdots\right\}$$

Example 2.4 Obtain the trigonometric Fourier series of the triangular waveform shown in Fig. E2.4.

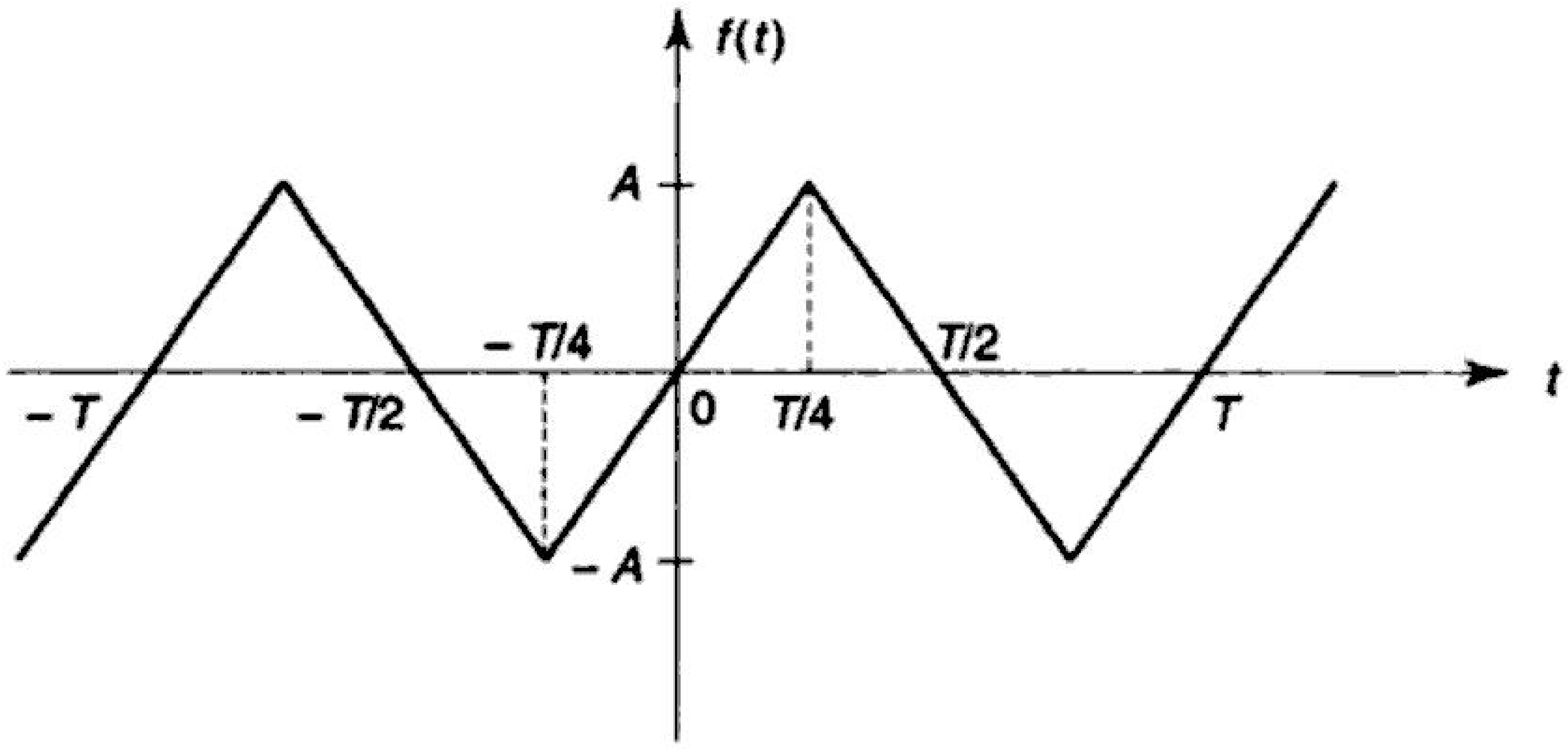


Fig. E2.4

Solution

(i) As the waveform has equal positive and negative area in one cycle, the average value of $a_0 = 0$.

(ii) As $f(t) = -f(t)$, it is an odd function and hence $a_n = 0$

$$\text{and } b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t \, dt$$

(iii) Here, $f(t \pm T/2) = -f(t)$. Hence it has half-wave odd symmetry and $a_n = b_n = 0$ for n even.

(iv) To find $f(t)$ for the given waveform

The equation of a straight line is $\dfrac{y - y_1}{x - x_1} = \dfrac{y_1 - y_2}{x_1 - x_2}$

For the region $0 < t < \dfrac{T}{4}$

$$\frac{f(t)-0}{t-0} = \frac{0-A}{0-T/4}$$

Therefore, $f(t) = \dfrac{4A}{T}t$

For the region $\dfrac{T}{4} < t < \dfrac{T}{2}$,

$$\frac{f(t)-A}{t-\dfrac{T}{4}} = \frac{A-0}{\dfrac{T}{4}-\dfrac{T}{2}}$$

$$f(t)-A = -\frac{4A}{T}\left(t-\frac{T}{4}\right) = -\frac{4A}{T}t + A$$

Therefore, $f(t) = -\dfrac{4A}{T}t + 2A$.

$$\text{Now, } b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t\, dt$$

$$= \frac{4}{T}\int_0^{T/4}\left(\frac{4A}{T}\right)t\sin n\omega_0 t\, dt + \frac{4}{T}\int_{T/4}^{T/2}\left(-\frac{4A}{T}t+2A\right)\sin n\omega_0 t\, dt$$

$$= \frac{16A}{T^2}\int_0^{T/4} t\sin n\omega_0 t\, dt - \frac{16A}{T^2}\int_{T/4}^{T/2} t\sin n\omega_0 t\, dt + \frac{8A}{T}\int_{T/4}^{T/2}\sin n\omega_0 t\, dt$$

$$= \frac{16A}{T^2}\left[\left\{t\frac{\cos n\omega_0 t}{-n\omega_0}\right\}_0^{T/4} - \int_0^{T/4}\frac{\cos n\omega_0 t}{-n\omega_0}dt\right]$$

$$- \frac{16A}{T^2}\left[\left\{t\frac{\cos n\omega_0 t}{-n\omega_0}\right\}_{T/4}^{T/2} - \int_{T/4}^{T/2}\frac{\cos n\omega_0 t}{-n\omega_0}dt\right]$$

$$+ \frac{8A}{T}\left\{\frac{\cos n\omega_0 t}{-n\omega_0}\right\}_{T/4}^{T/2}$$

$$= \frac{16A}{T^2}\left[\frac{T}{4}\frac{\cos n\omega_0 T/4}{(-n\omega_0)} + \left\{\frac{\sin n\omega_0 t}{n^2\omega_0^2}\right\}_0^{T/4}\right]$$

$$- \frac{16A}{T^2}\left[\frac{T}{2}\frac{\cos n\omega_0 T/2}{(-n\omega_0)} + \frac{T}{4}\frac{\cos n\omega_0 T/4}{n\omega_0} + \left\{\frac{\sin n\omega_0 t}{n^2\omega_0^2}\right\}_{T/4}^{T/2}\right]$$

$$+ \frac{8A}{T}\left[\frac{\cos n\omega_0 T/2}{(-n\omega_0)} + \frac{\cos n\omega_0 T/4}{n\omega_0}\right]$$

Substituting $\omega_0 = \dfrac{2\pi}{T}$, we have

$$b_n = \frac{16A}{T^2}\frac{2\sin n\pi/2}{n^2 4\pi^2/T^2} + \frac{16A}{T^2}\cdot\frac{T}{2}\frac{\cos n\pi}{n.2\pi/T} - \frac{8A}{T}\frac{\cos n\pi}{n2\pi/T}$$

Simplifying, we get

$$b_n = \frac{8A}{n^2\pi^2}\sin(n\pi/2)$$

$$b_1 = \frac{8A}{\pi^2}\sin(\pi/2) = \frac{8A}{\pi^2}$$

$$b_3 = \frac{8A}{3^2\pi^2}\sin\frac{3\pi}{2} = -\frac{8A}{3^2\pi^2}$$

$$b_5 = \frac{8A}{5^2\pi^2}\sin\frac{5\pi}{2} = \frac{8A}{5^2\pi^2}$$

....

....

Substituting the values of the coefficients in Eq. 2.2, we get

$$f(t) = \frac{8A}{\pi^2}\left[\sin\omega_0 t - \frac{1}{3^2}\sin 3\omega_0 t + \frac{1}{5^2}\sin 5\omega_0 t + \cdots\right]$$

Example 2.5 Deduce the Fourier series for the waveform of a positive going rectangular pulse train shown in Fig. E2.5.

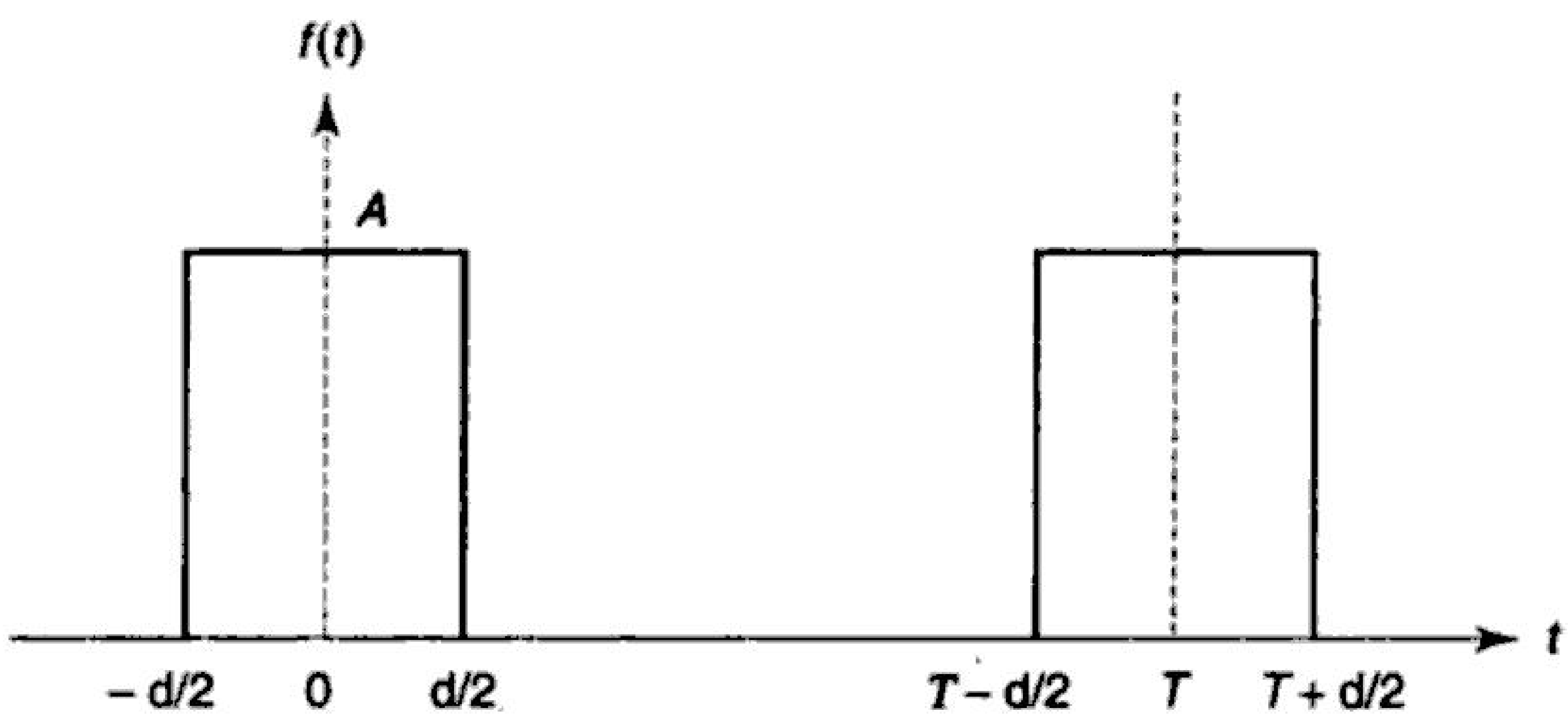


Fig. E2.5

Solution The periodic function of the Fourier series for the given pulse train is expressed by

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

where $\quad a_0 = \dfrac{2}{T}\displaystyle\int_{-T/2}^{T/2} f(t)\,dt$

$$= \frac{2}{T} \int_{-d/2}^{d/2} A dt = \frac{2A}{T} [t]_{-d/2}^{d/2} = \frac{2Ad}{T}$$

Here, since the choice of $t = 0$ is at the centre of a pulse, the b_n coefficients are zero.

Therefore, $a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos n\omega_0 t \, dt = \frac{2}{T} \int_{-d/2}^{d/2} \cos n\omega_0 t \, dt$

$$= \frac{2A}{T} \left[\frac{\sin n\omega_0 t}{n\omega_0} \right]_{-d/2}^{d/2} = \frac{2A}{n\omega_0 T} \left[\sin\left(\frac{n\omega_0 d}{2}\right) - \sin\left(\frac{-n\omega_0 d}{2}\right) \right]$$

$$= \frac{4A}{n\omega_0 T} \sin \frac{n\omega_0 d}{2}$$

Hence, $f(t) = \frac{Ad}{T} + \frac{2Ad}{T} \sum_{n=1}^{\infty} \frac{\sin(n\omega_0 d/2)}{n\omega_0 d/2} \cos n\omega_0 t$

2.3 COMPLEX OR EXPONENTIAL FORM OF FOURIER SERIES

From Eq. 2.2, the trigonometric form of the Fourier series is

$$f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

An alternative but convenient way of writing the periodic function $f(t)$ is in exponential form with complex quantities. Since

$$\cos n\omega_0 t = \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}$$

$$\sin n\omega_0 t = \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}$$

Substituting these quantities in the expression for the Fourier series gives

$$f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \left(\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} \right) + \sum_{n=1}^{\infty} b_n \left(\frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j} \right)$$

$$= \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left(\frac{(a_n - jb_n) e^{jn\omega_0 t}}{2} \right) + \left(\frac{(a_n + jb_n) e^{-jn\omega_0 t}}{-jb} \right)$$

Here, taking $c_n = \frac{1}{2} (a_n - jb\,n)$

$$c_{-n} = \frac{1}{2}(a_n + jb_n) \tag{2.6}$$

$$c_0 = a_0$$

Where c_{-n} is the complex conjugate of c_n. Substituting expressions for the coefficients a_n and b_n from Eqs 2.4 and 2.5 gives

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)[\cos n\,\omega_0\,t - j\sin n\,\omega_0\,t]\,\mathrm{d}t$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,\mathrm{d}t \tag{2.7}$$

and $$c_{-n} = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\;[\cos n\;\omega_0\;t + j\sin n\;\omega_0\;t]\;\mathrm{d}t$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{jn\omega_0 t}\,\mathrm{d}t \tag{2.8}$$

with $$f(t) = c_0 + \sum_{n=1}^{\infty} c_n\,e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1} c_n\,e^{jn\omega_0 t} \tag{2.9}$$

where the values of n are negative in the last term and are included under the Σ sign. Also, c_0 may be included under the Σ sign by using the value of $n = 0$. Therefore,

$$f(t) = \sum_{n=-\infty}^{\infty} c_n\,e^{jn\omega_0 t} \tag{2.10}$$

It is clear from the result given in Eq. 2.10 that the periodic function $f(t)$ may be expressed mathematically by an infinite set of positive and negative frequency components. The negative frequencies have not only mathematical significance, but also physical significance, since a positive frequency may be associated with an anti-clockwise rotation and a negative frequency with a clockwise rotation.

The complex Fourier series furnishes a method of decomposing a signal in terms of a sum of elementary signals of the form $\left\{e^{jn\omega_0 t}\right\}$. This representation may be used for signals $f(t)$ that are

(i) Periodic, $f(t) = f(t + T)$, in which case the representation is valid on $(-\infty, \infty)$

(ii) Aperiodic, in which case the representation is valid on a finite interval (t_1, t_2). The periodic extension of $f(t)$ is obtained outside of (t_1, t_2).

Note that similar to the evaluation of integrals a_n and b_n, the limits of integration in Eq. 2.7 may be the end points of any convenient full period and not essentially 0 to T or 0 to 2π. For $f(t)$ to be real, $C_{-n} = C_n^*$, so that only positive value of n are considered in Eq. 2.7. Also, we have

$$a_n = 2\text{Re}\,[c_n] \quad \text{and} \quad b_n = -\,2\,\text{Im}\,[c_n] \tag{2.11}$$

For an even waveform, the trigonometric Fourier series has only cosine terms and hence, by Eq. 2.6, the exponential Fourier series coefficients will be pure real numbers. Similarly, for an odd waveform, the trigonometric Fourier series contains only sine terms and hence the exponential Fourier series coefficients will be pure imaginary.

Example 2.6

(a) Find the trigonometric Fourier series of the waveform shown in Fig. E2.6 and
(b) Determine the exponential Fourier series and hence find a_n and b_n of the trigonometric series and compare the results.

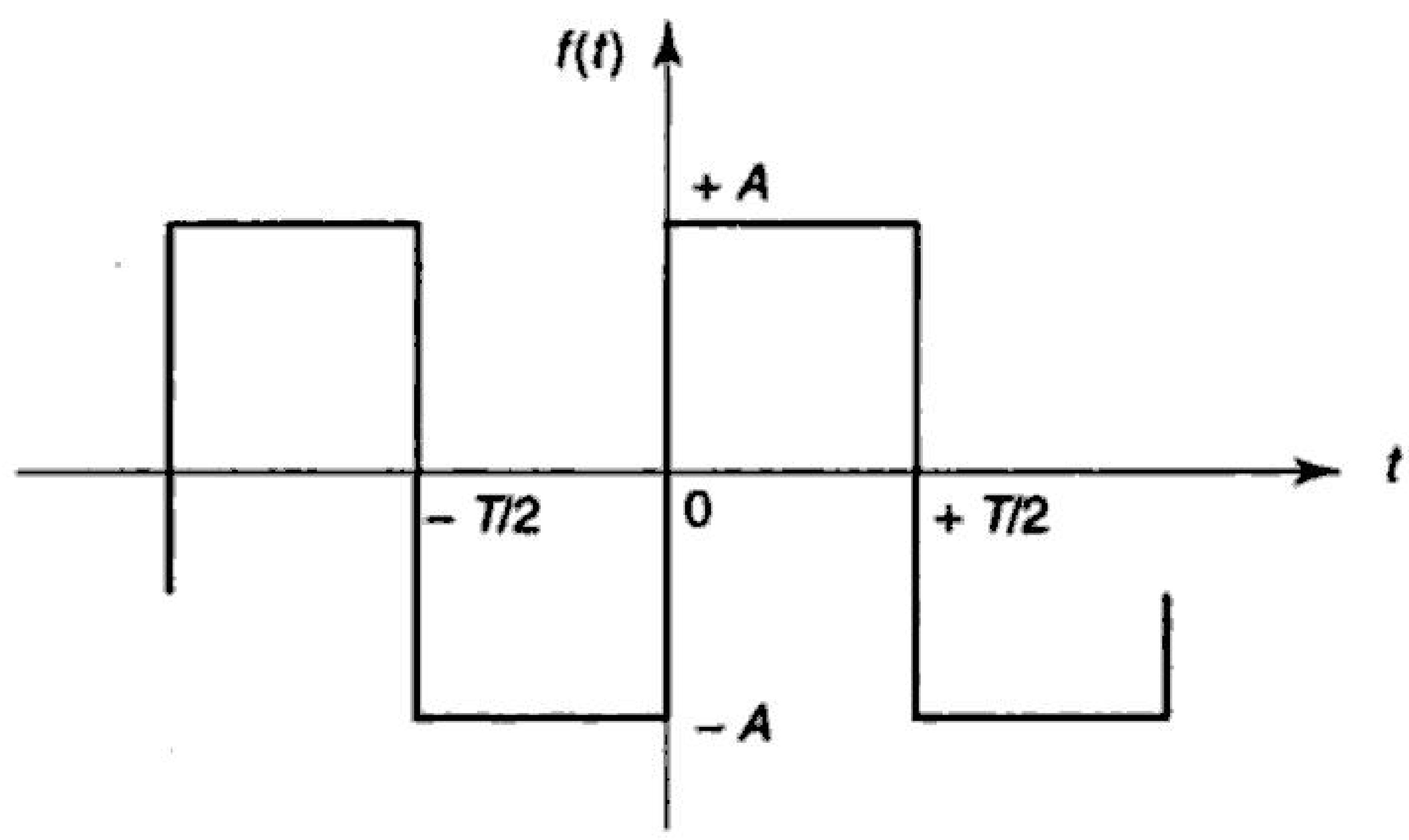


Fig. E2.6

Solution The function of the given waveform for one period can be written as

$$f(t) = \begin{cases} -A, & \text{for } -T/2 < t < 0 \\ +A, & \text{for } 0 < t < T/2 \end{cases}$$

As the waveform is symmetrical about the origin, the function of the waveform is odd and hence $a_0 = a_n = 0$, and

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\,\omega_0\,t\,dt$$

$$= \frac{2}{T}\left[\int_{-T/2}^{0}(-A\sin n\,\omega_0\,t)\,dt + \int_{0}^{T/2} A\sin n\,\omega_0\,t\,dt\right]$$

$$= \frac{2A}{T}\left\{\left[\frac{\cos n\,\omega_0\,t}{n\,\omega_0}\right]_{-T/2}^{0} + \left[\frac{-\cos n\,\omega_0\,t}{n\,\omega_0}\right]_{0}^{T/2}\right\}$$

$$= \frac{2A}{n\,\omega_0\,T}\{[1-\cos(n\,\omega_0\,T/2)] + [1-\cos(n\,\omega_0\,T/2)]\}$$

$$= \frac{4A}{n\omega_0 T}[1 - \cos(n\omega_0 T/2)]$$

When $\omega_0 = \dfrac{2\pi}{T}$, we have

$$b_n = \frac{4A}{n \cdot 2\pi}\left[1 - \cos n\left(\frac{2\pi}{T}\cdot T/2\right)\right]$$

$$= \frac{2A}{n\pi}[1 - \cos n\pi]$$

$$b_n = \begin{cases} 0 \quad , & \text{if } n \text{ is even} \\ \dfrac{4A}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

Substituting the values of the coefficients in Eq. 2.2, we obtain

$$f(t) = \frac{4A}{\pi}\left[\sin\omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \ldots\right], \text{ where } \omega_0 = \frac{2\pi}{T}$$

(b) To determine exponential Fourier series

Here $c_0 = \left|\frac{1}{2}a_0\right| = 0$

To evaluate c_n

Since the wave is odd, c_n consists of pure imaginary coefficients. From Eq. 2.7, we have

$$c_n = \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T}\left[\int_{-T/2}^{0} (-A)\, e^{-jn\omega_0 t}\, dt + \int_0^{T/2} A\, e^{-jn\omega_0 t}\, dt\right]$$

$$= \frac{A}{T}\left\{\left[(-1)\frac{1}{(-jn\omega_0)}e^{-jn\omega_0 t}\right]_{-T/2}^{0} + \left[\frac{1}{(-jn\omega_0)}e^{-jn\omega_0 t}\right]_0^{T/2}\right\}$$

$$= \frac{A}{T}\cdot\frac{1}{(-jn\omega_0)}\left\{-e^0 + e^{jn\omega_0(T/2)} + e^{-jn\omega_0(T/2)} - e^0\right\}$$

When $\omega_0 = \dfrac{2\pi}{T}$, we get

$$c_n = \frac{A}{T}\cdot\frac{T}{-jn2\pi}\left\{-e^0 + e^{jn(2\pi/T)(T/2)} + e^{-jn(2\pi/2)(T/2)} - e^0\right\}$$

$$= \frac{A}{(-j2\pi n)}\left\{-e^0 + e^{jn\pi} + e^{-jn\pi} - e^0\right\} = j\frac{A}{n\pi}(e^{jn\pi} - 1)$$

Here, $e^{jn\pi} = +1$ for even n and $e^{jn\pi} = -1$ for odd n

Therefore, $c_n = -j\left(\dfrac{2A}{n\pi}\right)$ for odd n only.

Hence, the exponential Fourier series is

$$f(t) = \ldots + j\frac{2A}{3\pi}e^{-j3\omega_0 t} + j\frac{2A}{\pi}e^{-j\omega_0 t} - j\frac{2A}{\pi}e^{j\omega_0 t} - j\frac{2A}{3\pi}e^{j3\omega_0 t}$$

By using Eq. 2.11, the trigonometric Fourier series coefficients a_n and b_n can be evaluated as

$$a_n = 2\,\text{Re}[c_n] = 2\,|c_n| = 0 \text{ and } b_n = -2\,\text{Im}\,[c_n] = \frac{4A}{n\pi} \text{ for odd } n \text{ only.}$$

These coefficients are the same as the coefficients obtained in the trigonometric Fourier series.

Example 2.7

(a) Find the trigonometric Fourier series of the waveform shown in Fig. E2.7 and

(b) Determine the exponential Fourier series and hence find a_n and b_n of the trigonometric series and compare the results.

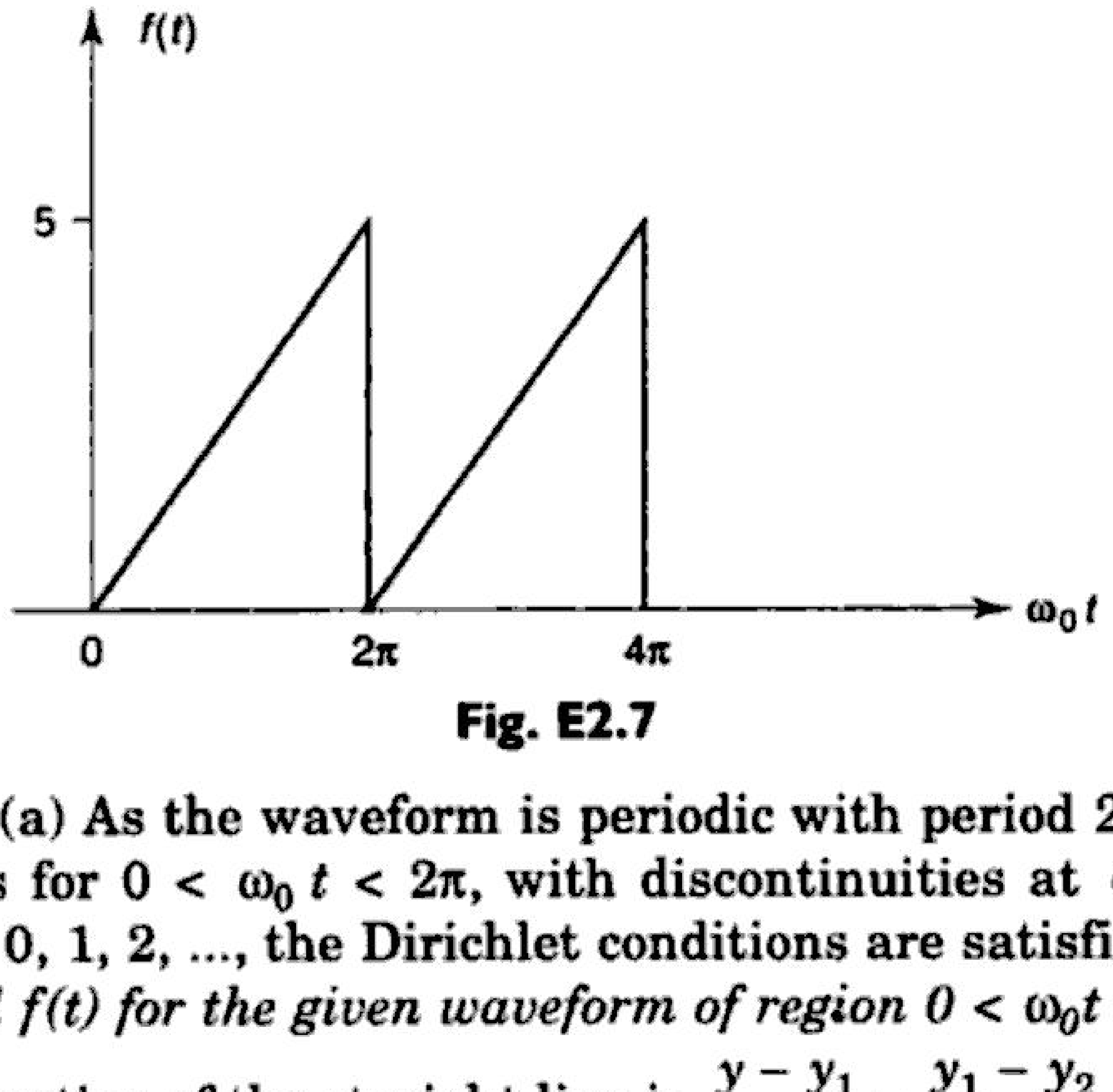


Fig. E2.7

Solution (a) As the waveform is periodic with period 2π in $\omega_0 t$ and continuous for $0 < \omega_0 t < 2\pi$, with discontinuities at $\omega_0 t = n\,(2\pi)$, where $n = 0, 1, 2, \ldots$, the Dirichlet conditions are satisfied.

To find f(t) for the given waveform of region $0 < \omega_0 t < 2\pi$:

The equation of the straight line is $\dfrac{y - y_1}{x - x_1} = \dfrac{y_1 - y_2}{x_1 - x_2}$

Substituting $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (2\pi, 5)$, we get

$$\frac{f(t) - 0}{\omega_0 t - 0} = \frac{0 - 5}{0 - 2\pi}$$

Therefore, $f(t) = \left(\dfrac{5}{2\pi}\right)\omega_0 t$

To find Fourier coefficients

Using Eq. 2.3, we obtain the average term,

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,\mathrm{d}t$$

$$= \frac{2}{2\pi}\int_0^{2\pi} \frac{5}{2\pi}\,\omega_0 t\,\mathrm{d}(\omega_0 t)$$

$$= \frac{10}{(2\pi)^2}\left[\frac{(\omega_0 t)^2}{2}\right]_0^{2\pi}$$

$$= \frac{10}{(2\pi)^2}\,\frac{(2\pi)^2}{2} = 5$$

Using Eq. 2.4, we obtain $a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\,\omega_0 t\,\mathrm{d}t$

$$= \frac{2}{2\pi}\int_0^{2\pi}\left(\frac{5}{2\pi}\right)\omega_0 t\cos n\,\omega_0 t\,\mathrm{d}(\omega_0 t)$$

$$= \frac{5}{2\pi^2}\left[\frac{\omega_0 t}{n}\sin n\,\omega_0 t + \frac{1}{n^2}\cos n\,\omega_0 t\right]_0^{2\pi}$$

$$= \frac{5}{2\pi^2 n^2}(\cos n\,2\pi - \cos 0) = 0$$

Hence, the series contains no cosine terms.

Using Eq. 2.5, we obtain $b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\,\omega_0 t\,\mathrm{d}t$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{5}{2\pi}\right)\omega_0 t\sin n\,\omega_0 t\,\mathrm{d}(\omega_0 t)$$

$$= \frac{5}{2\pi^2}\left[-\frac{\omega_0 t}{n}\cos n\,\omega_0 t + \frac{1}{n^2}\sin n\,\omega_0 t\right]_0^{2\pi}$$

$$= -\frac{5}{n\pi}$$

Combining the average term and the sine-term coefficients, the series becomes

$$f(t) = \frac{1}{2}a_0 + b_1\sin\omega_0 t + b_2\sin 2\omega_0 t + \ldots$$

$$= \frac{5}{2} - \frac{5}{\pi}\sin\omega_0 t - \frac{5}{2\pi}\sin 2\omega_0 t - \frac{5}{3\pi}\sin 3\omega_0 t - \ldots$$

$$= \frac{5}{2} - \frac{5}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\,\omega_0\, t}{n}$$

(b) *To determine exponential Fourier series*

Here, $c_0 = \left|\frac{1}{2}a_0\right| = \frac{5}{2}$

To evaluate c_n:

From Eq. 2.7, we have

$$c_n = \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega_0 t}\, dt$$

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{5}{2\pi}\right) \omega_0\, t\, e^{-jn\omega_0 t}\, d(\omega_0\, t)$$

$$= \frac{5}{(2\pi)^2}\left[\frac{e^{-jn\omega_0 t}}{(-jn)^2}(-jn\omega_0 t - 1)\right]_0^{2\pi} = j\frac{5}{2\pi n}$$

Substituting the coefficients c_n in Eq. 2.9, the exponential Fourier series is

$$f(t) = \ldots - j\frac{5}{4\pi}e^{-j2\omega_0 t} - j\frac{5}{2\pi}e^{-j\omega_0 t} + \frac{5}{2} + j\frac{5}{2\pi}e^{j\omega_0 t}$$
$$+ j\frac{5}{4\pi}e^{-j2\omega_0 t} + \ldots$$

By using Eq. 2.11, the trigonometric Fourier series coefficients a_n and b_n can be evaluated as

$$a_n = 2\mathrm{Re}[c_n] = 2|c_n| = 0 \quad \text{and} \quad b_n = -2\,\mathrm{Im}\,[c_n] = -\frac{5}{n\pi}$$

Hence, $f(t) = \frac{5}{2} - \frac{5}{\pi}\sin\omega_0 t - \frac{5}{2\pi}\sin 2\,\omega_0 t - \frac{5}{3}\pi\sin 3\omega_0 t - \ldots$

This result is the same as that of the trigonometric Fourier series method.

2.4 PARSEVAL'S IDENTITY FOR FOURIER SERIES

A periodic function $f(t)$ with a period T is expressed by the Fourier series as

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\,\omega_0\, t + b_n \sin n\,\omega_0\, t)$$

Now, $\quad [f(t)]^2 = \frac{1}{2}a_0\, f(t) + \sum_{n=1}^{\infty}[a_n\, f(t)\cos n\,\omega_0\, t + b_n\, f(t)\sin n\,\omega_0\, t]$

Therefore, $\frac{1}{T}\int_{-T/2}^{T/2}[f(t)]^2\, dt = \frac{(a_0/2)}{T}\int_{-T/2}^{T/2}[f(t)]\, dt$

$$+\frac{1}{T}\sum_{n=1}^{\infty}\left[a_n\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t\,dt + b_n\int_{-T/2}^{T/2} f(t)\sin n\omega_0 t\,dt\right]$$

From Eqns 2.2, 2.3 and 2.4, we have

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t\,dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega_0 t\,dt$$

Therefore, substituting all these values, we get

$$\frac{1}{T}\int_{-T/2}^{T/2}[f(t)]^2\,dt = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) \tag{2.12}$$

This is the ***Parseval's identity.***

2.5 POWER SPECTRUM OF A PERIODIC FUNCTION

The power of a periodic signal spectrum $f(t)$ in the time domain is defined as

$$P = \frac{1}{T}\int_{-T/2}^{T/2}[f(t)]^2\,dt$$

The Fourier series for the signal $f(t)$ is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{jn\omega_0 t}$$

According to Parseval's relation, we have

$$P_{av} = \frac{1}{T}\int_{-T/2}^{T/2}[f(t)]^2\,dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} f(t)\sum_{n=-\infty}^{\infty} c_n\, e^{jn\omega_0 t}\,dt$$

$$= \sum_{n=-\infty}^{\infty} c_n \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{jn\omega_0 t}\,dt$$

$$= \sum_{n=-\infty}^{\infty} c_n\, c_{-n}$$

$$= \sum_{n=-\infty}^{\infty} |c_n|^2, \text{ watts}$$

From Eq. 2.12, the above equation becomes

$$\left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \sum_{n=0}^{\infty}|c_n|^2$$

Here, $c_0 = \left|\frac{a_0}{2}\right|$ and $c_n = \sqrt{a_n^2 + b_n^2}$, $(n \geq 1)$ (2.13)

Thus the power in $f(t)$ is

$$P = \ldots + |c_{-n}|^2 + \ldots + |c_{-1}|^2 + |c_0|^2 + |c_1|^2 + \ldots + |c_n|^2 + \ldots \quad (2.14)$$

$$P = |c_0|^2 + 2|c_1|^2 + 2|c_2|^2 + \ldots + |c_n|^2 + \ldots$$

Hence, for a periodic function, the power in a time waveform $f(t)$ can be evaluated by adding together the powers contained in each harmonic, i.e. frequency component of the signal $f(t)$.

The power for the n^{th} harmonic component at $n\,\omega_0$ radians per sec is $|c_n|^2$ and that of $-n\,\omega_0$ is $|c_{-n}|^2$. For the single real harmonic, we have to consider both the frequency components $\pm\, n\,\omega_0$.

Here, $c_n = c_{-n}^*$ and hence $|c_n|^2 = |c_{-n}|^2$. The power for the n^{th} real harmonic $f(t)$ is

$$P_n = |c_n|^2 + |c_{-n}|^2 = 2|c_n|^2$$

The effective or RMS value of $f(t)$

Using Eqns 2.12, 2.13 and 2.14, the RMS value of the function $f(t)$ expressed by Eq. 2.1 is

$$F_{\text{rms}} = \sqrt{\left(\frac{a_0}{2}\right)^2 + \frac{1}{2}a_1^2 + \frac{1}{2}a_2^2 + \ldots + \frac{1}{2}b_1^2 + \frac{1}{2}b_2^2 + \ldots}$$

$$= \sqrt{c_0^2 + \frac{1}{2}c_1^2 + \frac{1}{2}c_2^2 + \ldots} \quad (2.15)$$

Example 2.8 The complex exponential Fourier representation of a signal $f(t)$ over the interval $(0, T)$ is

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{3}{4 + (n\pi)^2}\, e^{jn\pi t}$$

(a) What is the numerical value of T?
(b) One of the components $f(t)$ is $A \cos 3\pi t$. Determine the value of A.
(c) Determine the minimum number of terms which must be retained in the representation of $f(t)$ in order to include 99.9% of the energy in the interval.

Note: $\sum_{n=-\infty}^{\infty} \left| \frac{3}{4+(n\pi)^2} \right| \approx 0.669$

Solution The complex exponential Fourier transform representation of a signal $f(t)$ is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{jn\omega_0 t} \quad \text{where } \omega_0 = \frac{2\pi}{T}$$

The given signal $f(t)$ over the interval (0, T) is

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{3}{4+(n\pi)^2} \, e^{jn\pi t}$$

(a) Comparing the above two equations, we get

$$c_n = \frac{3}{4+(n\pi)^2} \text{ and}$$

$$e^{jn\frac{2\pi}{T}t} = e^{jn\pi t}$$

Hence, $\frac{2\pi}{T} = \pi$, i.e. $T = 2$

(b) When $n = 3$, the component of $f(t)$ will be

$$c_3 = \frac{3}{4+(3\pi)^2} e^{j3\pi t} = \frac{3}{4+(3\pi)^2} [\cos 3\pi t + j \sin 3\pi t]$$

Similarly, when $n = -3$, the component will be

$$c_{-3} = \frac{3}{4+(-3\pi)^2} e^{-j3\pi t} = \frac{3}{4+(3\pi)^2} [\cos 3\pi t - j \sin 3\pi t]$$

Therefore, $c_3 + c_{-3} = \frac{6}{4+(3\pi)^2} \cos 3\pi t$

Hence, when one of the components of $f(t)$ is $A \cos 3\pi t$, the value of A is

$$A = \frac{6}{4+(3\pi)^2}$$

(c) Total (maximum) power $P_t = \sum_{n=-\infty}^{\infty} \left| \frac{3}{4+(n\pi)^2} \right| \approx 0.669$

The power in $f(t)$ is

$$P = |c_0|^2 + 2\left[|c_1|^2 + |c_2|^2 + |c_3|^2 + |c_4|^2\right]$$

$$= \left|\frac{3}{4}\right|^2 + 2\left[\left|\frac{3}{4+(\pi)^2}\right|^2 + \left|\frac{3}{4+(2\pi)^2}\right|^2 + \left|\frac{3}{4+(3\pi)^2}\right|^2 + \left|\frac{3}{4+(4\pi)^2}\right|^2\right]$$

$= 0.5625 + 0.0935 + 9.52 \times 10^{-3} + 2.088 \times 10^{-3} + 6.866 \times 10^{-4}$

$= 0.66836$

Therefore, energy contained in the four terms is

$$\frac{P}{P_t} \times 100 = \frac{0.66836}{0.669} \times 100 = 99.9\%$$

Hence, the first four terms include 99.9% of the total energy.

2.6 FOURIER TRANSFORM

The plot of amplitudes at different frequency components for a periodic wave is known as discrete (line) frequency spectrum because amplitude values have significance only at discrete values of $n\,\omega_0$ where $\omega_0 = 2\pi/T$ is the separation between two adjacent (consecutive) harmonic components. If the repetition period T increases, ω_0 decreases .

Hence, when the repetition period T becomes infinity, i.e. $T \to \infty$, the wave $f(t)$ will become non-periodic, the separation between two adjacent harmonic components will be zero, i.e. $\omega_0 = 0$. Therefore, the discrete spectrum will become a continuous spectrum. When $T \to \infty$, the adjacent pulses virtually never occur and the pulse train reduces to a single isolated pulse. The exponential form of the Fourier series given in Eq. 2.10 can be extended to aperiodic waveforms such as single pulses or single transients by making a few changes.

Assuming $f(t)$ is initially periodic, from Eq. 2.10, we have,

$$f(t) = \sum_{-\infty}^{\infty} c_n\, e^{jn\omega_0 t}$$

where $$c_n = 1/T \int_{-T/2}^{T/2} f(t)\, e^{-jn\omega_0 t}\, dt$$

In the limit, for a single pulse, we have

$T \to \infty$, $\omega_0 = 2\pi/T \to d\omega$ (a small quantity)

or $1/T = \omega_0/2\pi \to d\omega/2\pi$

Furthermore, the n^{th} harmonic in the Fourier series is $n\,\omega_0 \to n\,d\omega$. Here n must tend to infinity as ω_0 approaches zero, so that the product is finite, i.e. $n\,\omega_0 \to \omega$.

In the limit, the Σ sign leads to an integral and we have

$$c_n = \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

and, $$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \right] e^{j\omega t}$$

When evaluated, the quantity in bracket is a function of frequency only and is denoted as $F(j\,\omega)$ where

$$F(j\,\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, \mathrm{d}t \tag{2.16}$$

It is called the *Fourier transform* of $f(t)$.
Substituting for $f(t)$ above, We obtain

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\,\omega)\, e^{j\omega t}\, \mathrm{d}\omega$$

or, equivalently,

$$f(t) = \int_{-\infty}^{\infty} F(j\,\omega)\, e^{j\omega t}\, \mathrm{d}f \tag{2.17}$$

which is called the *inverse* Fourier *transform*. Now the time function $f(t)$ represents the expression for a single pulse or transient only. Equations 2.16 and 2.17 constitute a Fourier *transform pair*.

From Eqns 2.16 and 2.17, it is apparent that the Fourier transform and inverse Fourier transform are similar, except for a sign change on the exponential component.

2.6.1 Energy Spectrum for a Non-Periodic Function

For a non-periodic energy signal, such as a single pulse, the total energy in $(-\infty, \infty)$ is finite, whereas the average power, i.e. energy per unit time, is zero because $\frac{1}{T}$ tends to zero as T tends to infinity. Hence, the total energy associated with $f(t)$ is given by

$$E = \int_{-\infty}^{\infty} f^2(t)\, \mathrm{d}t$$

Since, $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\,\omega)\, e^{j\omega t}\, \mathrm{d}\omega$, we obtain

$$E = \int_{-\infty}^{\infty} f(t) \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\,\omega)\, e^{j\omega t}\, \mathrm{d}\omega\, \mathrm{d}t$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\,\omega) \left[\int_{-\infty}^{\infty} f(t)\, e^{j\omega t}\, \mathrm{d}t \right] \mathrm{d}\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\,\omega)\, F(-j\,\omega)\, \mathrm{d}\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\,\omega)\, F^{*}(j\,\omega)\, \mathrm{d}\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 \, d\omega$$

$$= \int_{-\infty}^{\infty} |F(f)|^2 \, df, \text{ joules}$$

$$E = \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \int_{-\infty}^{\infty} |F(f)|^2 \, df \tag{2.18}$$

This result is called ***Rayleigh's energy theorem or Parseval's theorem*** for Fourier transform. The quantity $|F(f)|^2$ is referred to as the *energy spectral density, S(f)*, which is equal to the energy per unit frequency.

The integration in Eq. 2.18 is carried out over positive and negative frequencies. If $f(t)$ is real, then $|F(j\omega)| = |F(-j\omega)|$, then the Eq. 2.18 becomes,

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 \, d\omega = \frac{1}{\pi} \int_{0}^{\infty} |F(j\omega)|^2 \, d\omega = \int_{0}^{\infty} S(\omega)\, d\omega$$

Here the integration is carried out over only positive frequencies. The quantity $S(\omega) = |F(j\omega)|^2/\pi$ is called the energy spectral density.

2.7 PROPERTIES OF FOURIER TRANSFORM

Table 2.1 presents important properties of the Fourier transform.

Table 2.1 *Important properties of the Fourier transform*

Operation	*f(t)*	*F(j ω)*
Transform	$f(t)$	$\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$
Inverse transform	$\frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega)\, e^{j\omega t}\, d\omega$	$F(j\omega)$
Linearity	$af_1(t) + bf_2(t)$	$aF_1(j\omega) + bF_2(j\omega)$
Time-reversal	$f(-t)$	$F(-j\omega) = F^*(j\omega)$, $f(t)$ real
Time-shifting (Delay)	$f(t - t_0)$	$F(j\omega)\, e^{-j\omega t_0}$
Time-Scaling	$f(at)$	$\frac{1}{\lvert a \rvert} F\left(\frac{j\omega}{a}\right)$
Time-differentiation	$\frac{d^n}{dt^n} f(t)$	$(j\omega)^n F(j\omega)$
Frequency-differentiation	$(-jt)\, f(t)$	$\frac{dF(j\omega)}{d\omega}$

(Contd.)

Operation	*f(t)*	*F(j ω)*
Time-integration	$\int_{-\infty}^{t} f(\tau)\, d\tau$	$\frac{1}{j\omega} F(j\omega) + \pi F(0)\, \delta(\omega)$
Frequency-integration	$\frac{1}{(-jt)} f(t)$	$\int^{\omega} f(j\omega')\, d\omega'$
Time convolution	$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau$	$F_1(j\omega)\, F_2(j\omega)$
Frequency convolution (Multiplication)	$f_1(t) \cdot f_2(t)$	$\frac{1}{2\pi} [F_1(j\omega) * F_2(j\omega)]$
Frequency shifting (Modulation)	$f(t)\, e^{j\omega_0 t}$	$F(j\omega - j\omega_0)$
Symmetry	$F(jt)$	$2\pi f(-\omega)$
Real-time function	$f(t)$	$F(j\omega) = F^*(-j\omega)$ $\mathrm{Re}[F(j\omega)] = \mathrm{Re}[F(-j\omega)]$ $\mathrm{Im}[F(j\omega)] = -\mathrm{Im}[F(-j\omega)]$ $\lvert F(j\omega)\rvert = \lvert F(-j\omega)\rvert$ $\Phi f(j\omega) = -\Phi f(j\omega)$
Parseval's theorem	$E = \int_{-\infty}^{\infty} \lvert f(t)\rvert^2\, dt$	$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert F(j\omega)\rvert^2\, d\omega$
Duality	If $f(t) \Leftrightarrow g(j\omega)$, then $g(t) \Leftrightarrow 2\pi f(-j\omega)$	

2.7.1 Linearity

The Fourier transform is a linear operation. Therefore, if

$$f_1(t) \Leftrightarrow F_1(j\,\omega)$$
$$f_2(t) \Leftrightarrow F_2(j\,\omega)$$

then, $af_1(t) + bf_2(t) \Leftrightarrow aF_1(j\,\omega) + bF_2(j\,\omega)$

where a and b are arbitrary constants.

2.7.2 Symmetry

If $\quad f(t) \Leftrightarrow F(j\,\omega)$

then, $F(jt) \Leftrightarrow 2\pi f(-\omega)$

Proof

Since $\quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega)\, e^{j\omega t}\, d\omega$

$$2\pi f(-t) = \int_{-\infty}^{\infty} F(j\omega')\, e^{-j\omega' t}\, d\omega'$$

where the dummy variable ω is replaced by ω′.

Now if t is replaced by ω, we have

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(j\omega')\, e^{-j\omega'\omega}\, d\omega'$$

Finally, ω' is replaced by t to obtain a more recognisable form and we have

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(jt)\, e^{-j\omega t}\, dt = \mathcal{F}\,[F(jt)]$$

Therefore, $F(jt) \Leftrightarrow 2\pi f(-\omega)$
If $f(t)$ is an even function, $f(t) = f(-t)$.
Hence, $\mathcal{F}[F(jt)] = 2\pi f(\omega)$

2.7.3 Scaling

If $\quad f(t) \Leftrightarrow F(j\omega)$,

then, $\quad f(at) \Leftrightarrow \dfrac{1}{|a|} F\left(\dfrac{j\omega}{a}\right)$

Proof

If $a > 0$, then the transform of $f(at)$ is

$$\mathcal{F}\,[f(at)] = \int_{-\infty}^{\infty} f(at)\, e^{-j\omega t}\, dt$$

Putting $x = at$, we have $dx = a dt$. Substituting in the above equation, we get

$$\mathcal{F}[f(at)] = \mathcal{F}\,[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{\frac{-j\omega x}{a}} \frac{dx}{a} = \frac{1}{a} F\left(\frac{j\omega}{a}\right)$$

If $a < 0$, then $\mathcal{F}\,[f(at)] = -\dfrac{1}{a} F\left(\dfrac{j\omega}{a}\right)$

Combining these two results, we get

$$f(at) \Leftrightarrow \frac{1}{|a|} F\left(\frac{j\omega}{a}\right)$$

We conclude that larger the duration of the time function, smaller is the bandwidth of its spectrum by the same scaling factor. Conversely, smaller the duration of the time function, larger is the bandwidth of its spectrum. This scaling property provides an *inverse relationship* between time-duration and bandwidth of a signal i.e. the time-bandwidth product of an energy signal is a constant.

2.7.4 Convolution

Convolution is a powerful way of characterising the input-output relationship of time-invariant linear systems. There are two convolution theorems, one for the time domain and another for the frequency domain.

Time Convolution

If $x(t) \Leftrightarrow X(j\omega)$ and $h(t) \Leftrightarrow H(j\omega)$,

then $y(t) = x(t) * h(t)$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \Leftrightarrow Y(j\omega) = X(j\omega)\, H(j\omega)$$

Proof

$$\mathcal{F}\,[y(t)] = Y(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t}\left[\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau\right] dt$$

$$= \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)\, e^{-j\omega t}\, dt\right] d\tau$$

Putting $a = t - \tau$, then $t = a + \tau$ and $da = dt$

Therefore, $$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} h(a)\, e^{-j\omega(a+\tau)}\, da\right] d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau \int_{-\infty}^{\infty} h(a)\, e^{-j\omega a}\, da$$

$$= X(j\omega)\, H(j\omega)$$

Hence, the convolution of the signals in the time domain is equal to the multiplication of their individual Fourier transforms in the frequency domain.

Example 2.9 In the system shown in Fig. E2.9 determine the output response of the low-pass RC network for an input signal $x(t) = e^{-\frac{t}{RC}}$

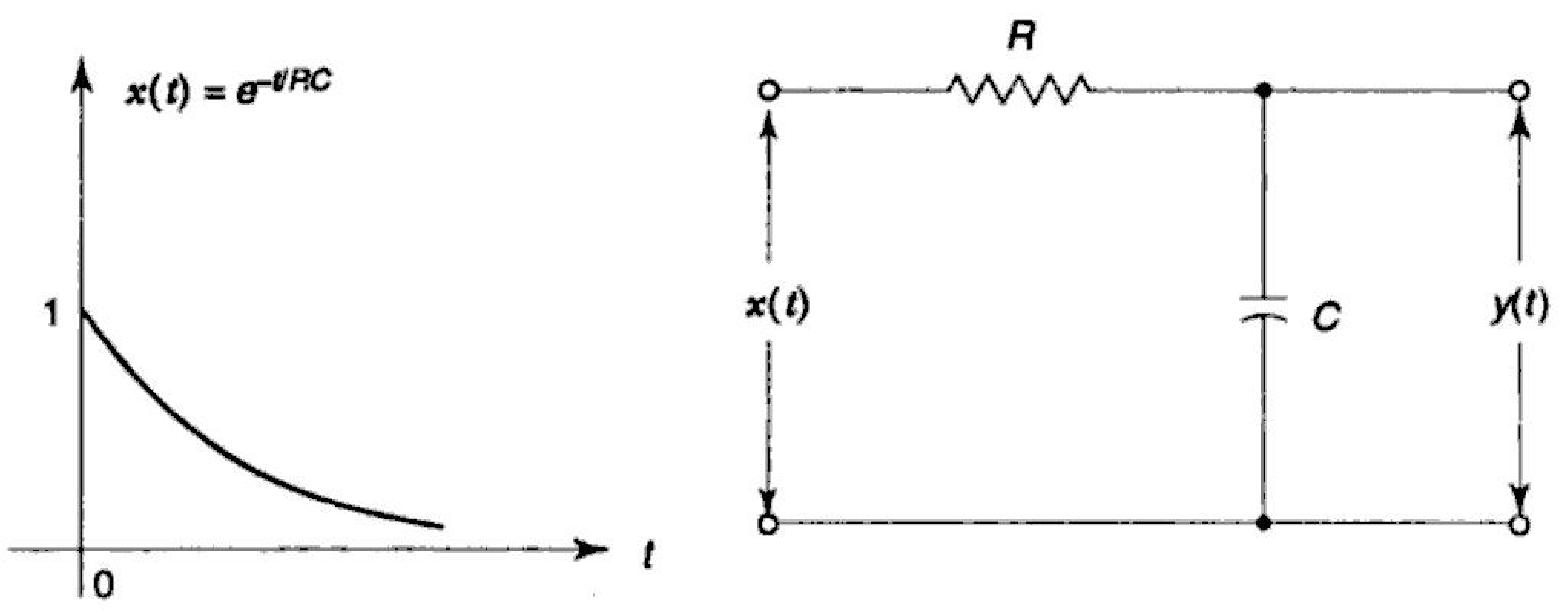


Fig. E2.9

Solution The input signal $x(t) = e^{-\frac{t}{RC}}$
Using the convolution theorem, we can find $y(t)$

$$y(t) = x(t) * h(t) = \mathcal{F}^{-1}[X(j\omega)\, H(j\omega)]$$

$$X(j\omega) = \mathcal{F}[x(t)] = \int_0^\infty e^{-\frac{t}{RC}}\, e^{-j\omega t}\, dt$$

$$= \int_0^\infty e^{-\left(\frac{1}{RC} + j\omega\right)t}\, dt$$

$$= \frac{1}{j\omega + \frac{1}{RC}}$$

Similarly, the transfer function of the network is

$$H(j\omega) = \frac{1/j\omega C}{\left(R + \frac{1}{j\omega C}\right)} = \frac{1}{(j\omega RC + 1)}$$

$$= \frac{1}{RC} \cdot \frac{1}{\left(j\omega + \frac{1}{RC}\right)}$$

Hence, $Y(j\omega) = X(j\omega)\, H(j\omega) = \dfrac{1}{RC} \cdot \dfrac{1}{\left(j\omega + \frac{1}{RC}\right)^2}$

$$y(t) = \mathcal{F}^{-1}[Y(j\omega)] = \frac{1}{RC}\, t e^{-\frac{t}{RC}}\, u(t)$$

Example 2.10 Determine the output response of the low-pass RC network due to an input $x(t) = t\,e^{-t/RC}$ by convolution.

Solution The transfer function of the network is

$$H(j\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{(1 + j\omega RC)}$$

The given input time function is $x(t) = te^{-t/RC}$

$$A = Y(j\omega)(a + j\omega)\big|_{j\omega = -a} = \frac{1}{(b-a)}$$

$$B = Y(j\omega)(b + j\omega)\big|_{j\omega = -b} = \frac{1}{(a-b)}$$

Hence, $Y(j\omega) = \dfrac{1}{(b-a)}\left[\dfrac{1}{(a+j\omega)} - \dfrac{1}{(b+j\omega)}\right]$

Taking inverse Fourier transform, we get

$$y(t) = \frac{1}{(b-a)}\left[e^{-at}\,u(t) - e^{-bt}\,u(t)\right]$$

When $b = a$, the partial fraction expansion is invalid. Hence,

$$Y(j\omega) = \frac{1}{(a+j\omega)^2}$$

$$= j\frac{\mathrm{d}}{\mathrm{d}\omega}\left[\frac{1}{a+j\omega}\right]$$

Using dual of the differentiation property,

$$e^{-at}\,u(t) \Leftrightarrow \frac{1}{a+j\omega}$$

$$te^{-at}\,u(t) \Leftrightarrow j\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{1}{a+j\omega}\right) = \frac{1}{(a+j\omega)^2}$$

Therefore, $y(t) = te^{-at}\,u(t)$

2.7.5 Frequency Convolution

If $f(t) \Leftrightarrow F(j\omega)$ and $g(t) \Leftrightarrow G(j\omega)$,

then $f(t)\,g(t) \Leftrightarrow \dfrac{1}{2\pi} F(j\omega) * G(j\omega)$

Proof

The inverse transform of $[F(j\omega) * G(j\omega)]/2\pi$ is

$$\mathcal{F}^{-1}\left[\frac{F(j\omega) * G(j\omega)}{2\pi}\right] = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} e^{j\omega t} \int_{-\infty}^{\infty} F(ju)\,G(j\omega - ju)\,\mathrm{d}u\,\mathrm{d}\omega$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} F(ju) \int_{-\infty}^{\infty} G(j\omega - ju)\,e^{j\omega t}\,\mathrm{d}\omega\,\mathrm{d}u$$

Putting $x = \omega - u$, then $\omega = x + u$ and $\mathrm{d}x = \mathrm{d}\omega$

Therefore

$$\mathcal{F}^{-1}\left[\frac{F(j\omega) * G(j\omega)}{2\pi}\right] = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} F(ju) \int_{-\infty}^{\infty} G(jx)\,e^{j(x+u)t}\,\mathrm{d}x\,\mathrm{d}u$$

2.7.12 Time Reversal

If $f(t) \Leftrightarrow F(j\omega)$, then $f(-t) \Leftrightarrow F(-j\omega)$

It is clear that the time-reversal theorem is similar to the scaling theorem with $a = -1$. When the signal is real, time-reversal affects only the phase spectrum because the amplitude spectrum is an even function of frequency.

2.7.13 Complex Conjugation

If $f(t) \Leftrightarrow F(j\omega)$, then $f^*(t) \Leftrightarrow F^*(-j\omega)$

2.7.14 Duality

If $f(t) \Leftrightarrow g(j\omega)$, then $g(t) \Leftrightarrow 2\pi f(-j\omega)$

2.7.15 Area Under *f*(t)

If $f(t) \Leftrightarrow F(f)$, then $\int_{-\infty}^{\infty} f(t)\,\mathrm{d}t = F(0)$

Thus, the area under a function $f(t)$ is equal to the value of its Fourier transform $F(f)$ at $f = 0$.

The result can be obtained by substituting $f = 0$ in the formula defining the Fourier transform of the function $f(t)$.

2.7.16 Area Under *F*(*f*)

If $f(t) \Leftrightarrow F(f)$, then $\int_{-\infty}^{\infty} F(f)\,\mathrm{d}f = f(0)$

Thus, the value of a function $f(t)$ at $t = 0$ is equal to the area under its Fourier transform $F(f)$.The result can be obtained by substituting $t = 0$ in the formula defining the inverse Fourier transform of $F(f)$.

Example 2.14 A certain function of time $f(t)$ has the following Fourier transform

$$F(j\omega) = \frac{1}{\omega^2 + 1} e^{-2\omega^2/(\omega^2+1)}$$

Using the properties of the Fourier transform, write the Fourier transforms of

(a) $f(2t)$, (b) $f(t-2)e^{jt}$, (c) $4\frac{\mathrm{d}}{\mathrm{d}t} f(t)$ and (d) $\int_{-\infty}^{t} f(\tau)\,\mathrm{d}\tau$

In each case state clearly the properties you will use.

Solution

$$F(j\omega) = \frac{1}{\omega^2 + 1} e^{-2\omega^2/(\omega^2+1)}$$

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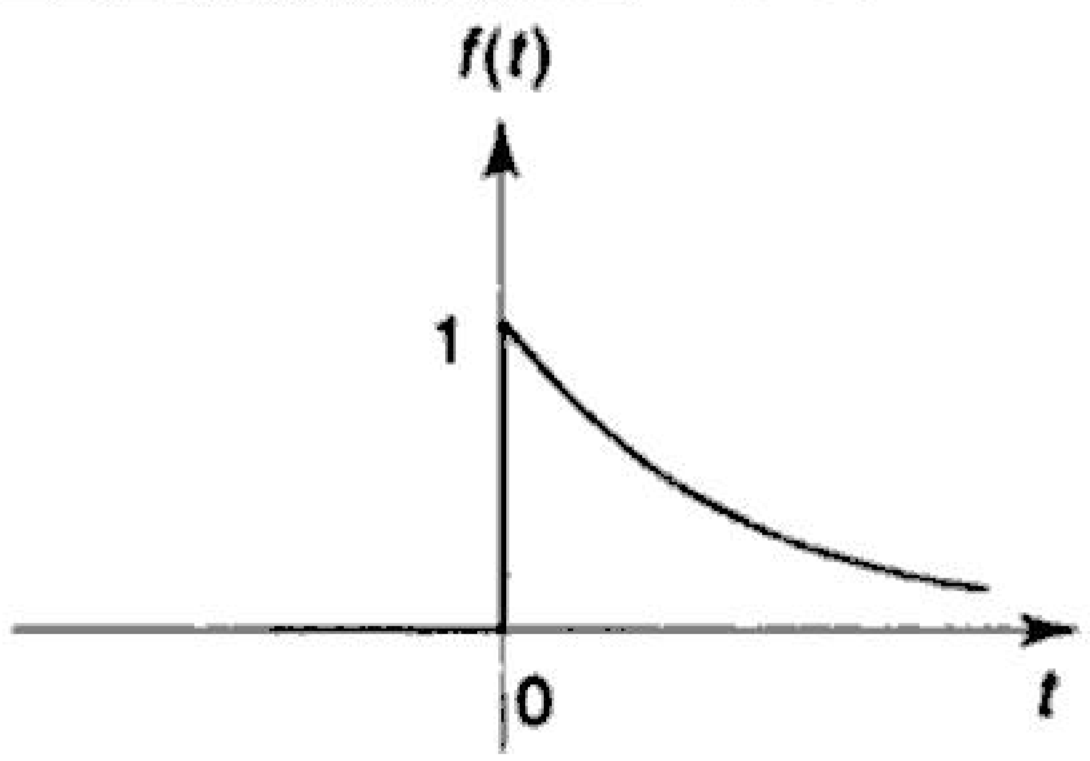
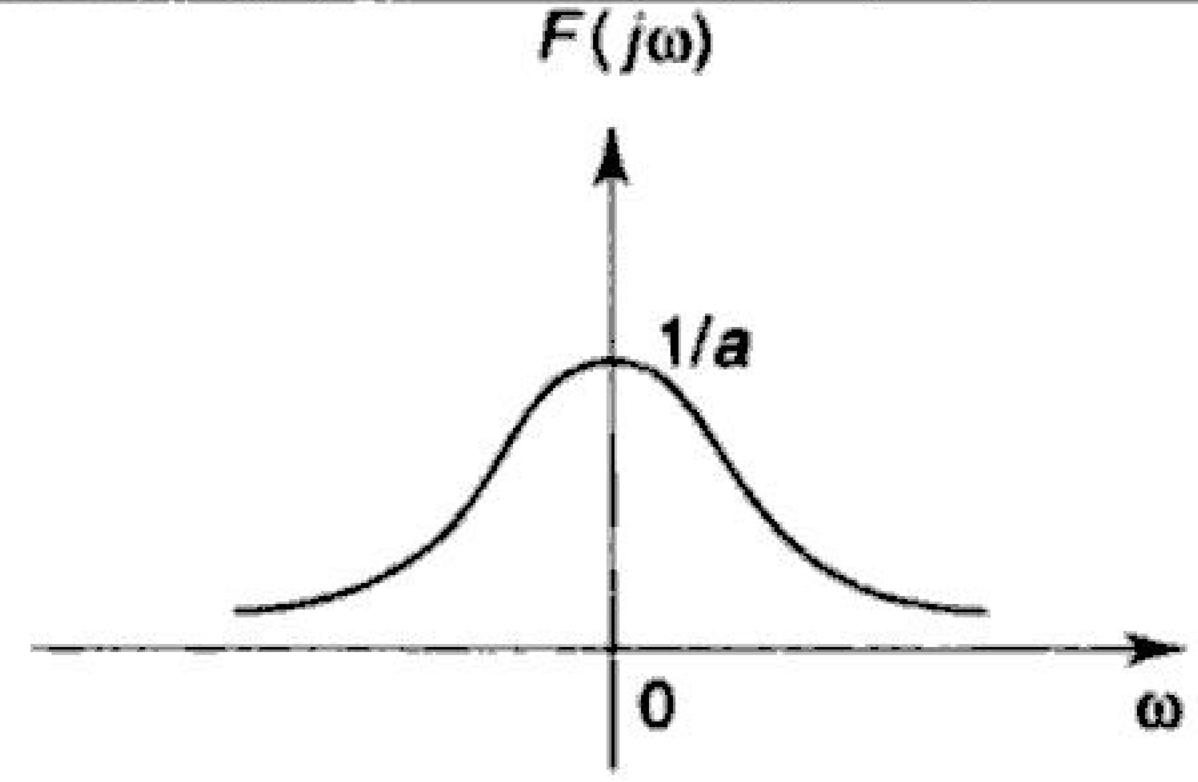
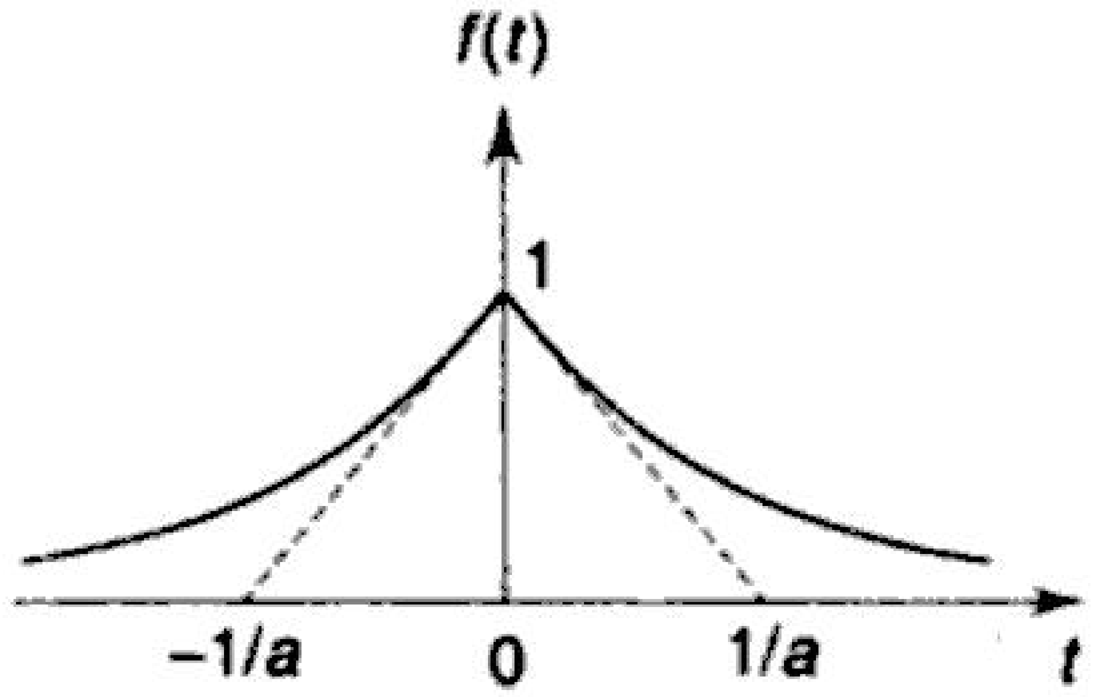
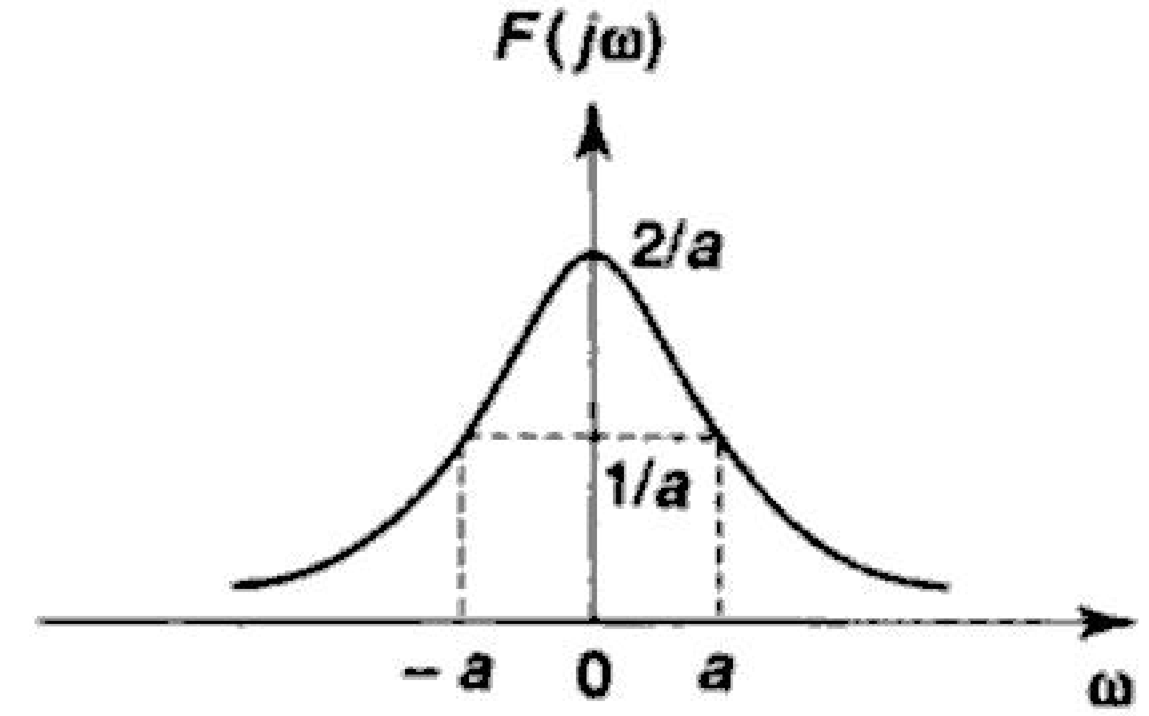
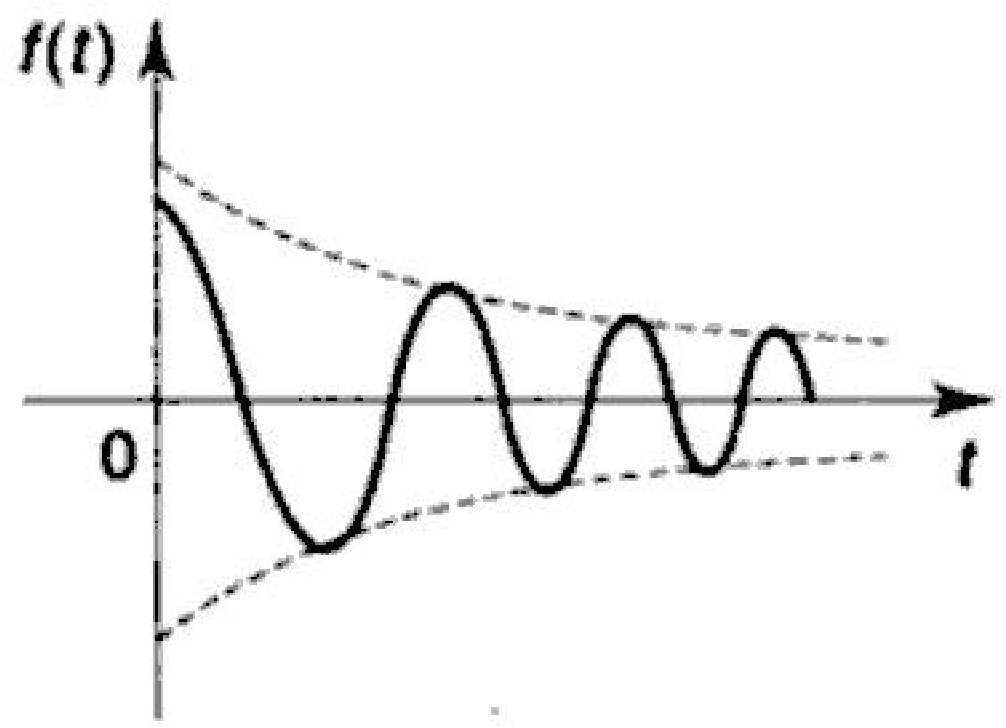
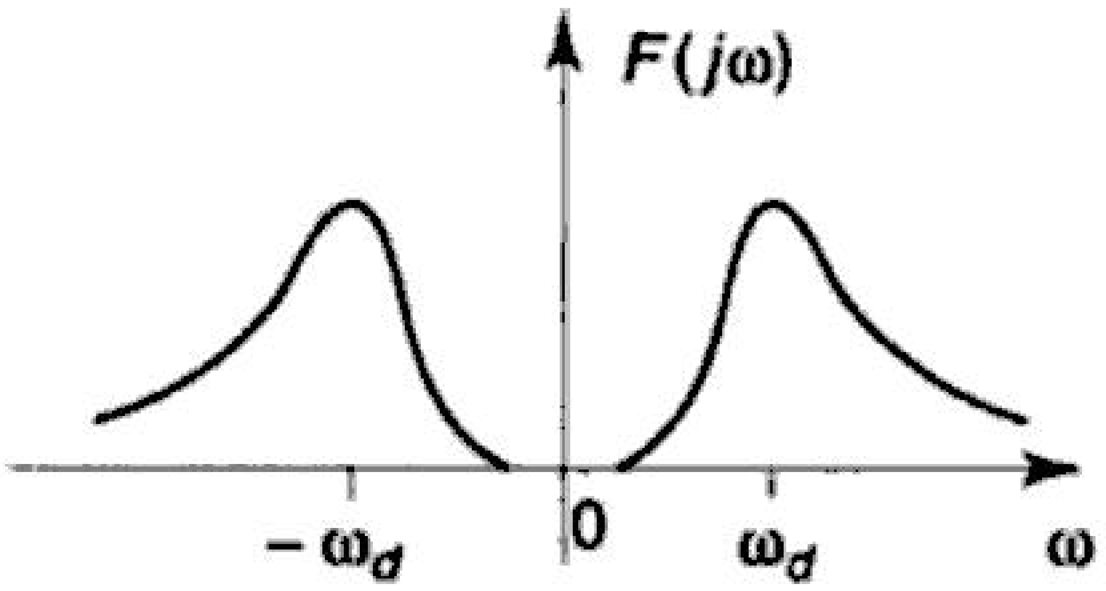
Sl. No.	Time domain $f(t)$		Frequency domain $F(j\omega)$	
8.		$e^{-at}\,u(t)$	$\dfrac{1}{a+j\omega}$	
9.	 Double exponential	$e^{-a\|t\|}$	$\dfrac{2a}{a^2+\omega^2}$	
10.		$e^{-at}\cos\omega_0 t\cdot u(t)$	$\dfrac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	

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2.8.1 Gate Function

Let us consider the single gate function (rectangular pulse) shown in Fig. 2.4. It has the analytic expression given by

$$f(t) = \begin{cases} 1, & \text{for } -T/2 < t < T/2 \\ 0, & \text{otherwise} \end{cases}$$

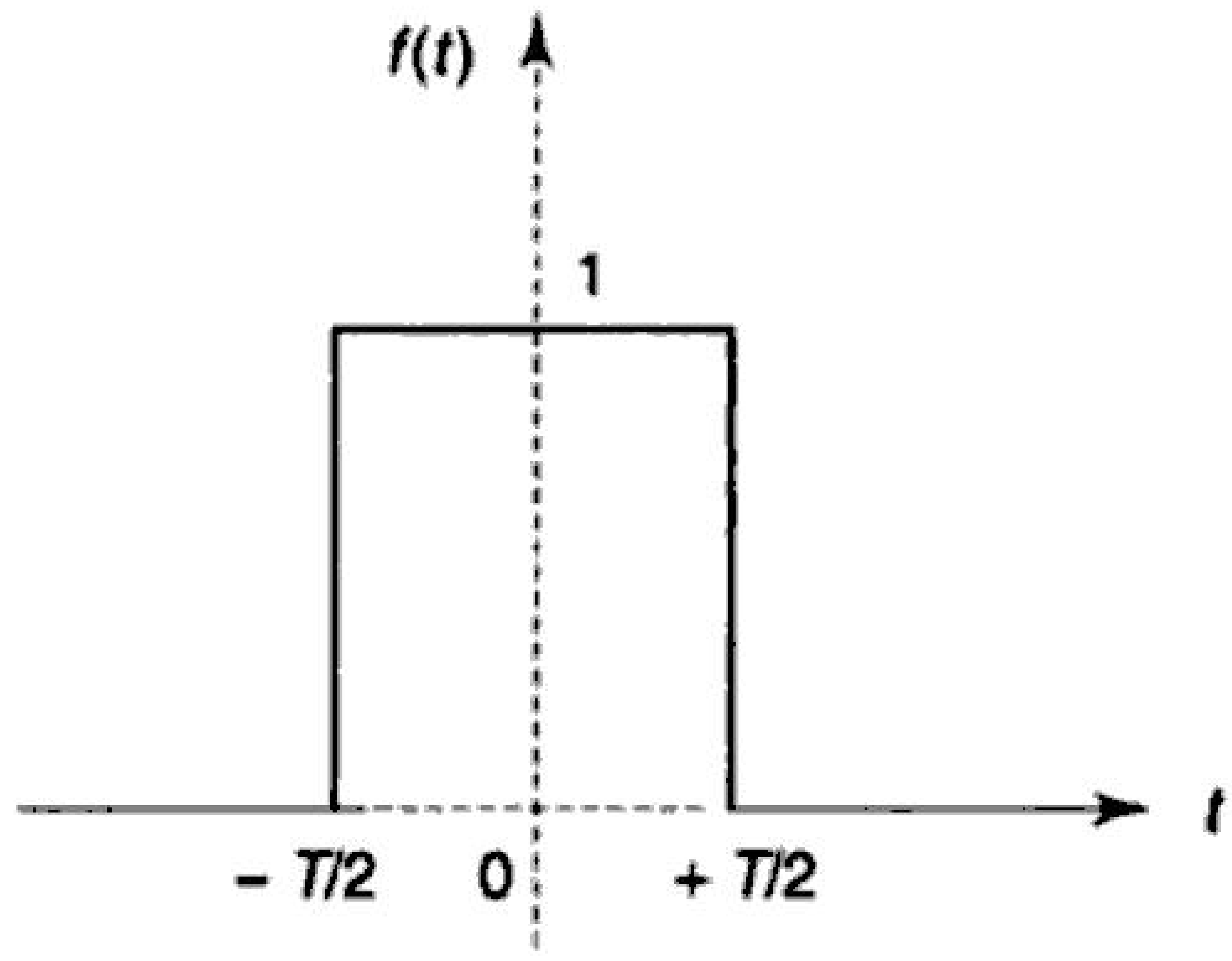

Fig. 2.4 *Single Gate Function*

The Fourier transform of $f(t)$ is

$$F(j\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T/2}^{T/2} 1 \cdot e^{-j\omega t}\, dt$$

$$= \frac{1}{-j\omega}\left[e^{-j\omega t}\right]_{-T/2}^{T/2}$$

$$= \frac{1}{-j\omega}\left[e^{-j\omega T/2} - e^{j\omega T/2}\right]$$

$$= T \cdot \frac{\sin\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)} = T \operatorname{sinc}\left(\frac{\omega T}{2}\right)$$

Hence, the amplitude spectrum is

$$|F(j\omega)| = T\left|\operatorname{sinc}\left(\frac{\omega T}{2}\right)\right|$$

and the phase spectrum is $\angle F(\omega) = \begin{cases} 0, & \operatorname{sinc}\left(\frac{\omega T}{2}\right) > 0 \\ \pi & \operatorname{sinc}\left(\frac{\omega T}{2}\right) < 0 \end{cases}$

The amplitude and phase spectra are shown in Fig. 2.5.

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Comments

(1) The phase of the amplitude spectrum is exactly the same as that in the previous case given in Fig. 2.7.

(2) There is an additional uniform phase shift factor $e^{-j\omega T/2}$ which changes the phase spectrum of the previous case.

(3) By using the time-shift theorem, $\mathcal{F}\left[f\left(t-\frac{T}{2}\right)\right] = F(j\omega)\, e^{-j\omega T/2}$ where $F(j\omega) = T \operatorname{sinc}\left(\frac{\omega T}{2}\right)$, the above result can readily be obtained.

Example 2.16 Find the Fourier transform of a rectangular pulse 2 seconds long with a magnitude of 10 volts as shown in Fig. E2.16.

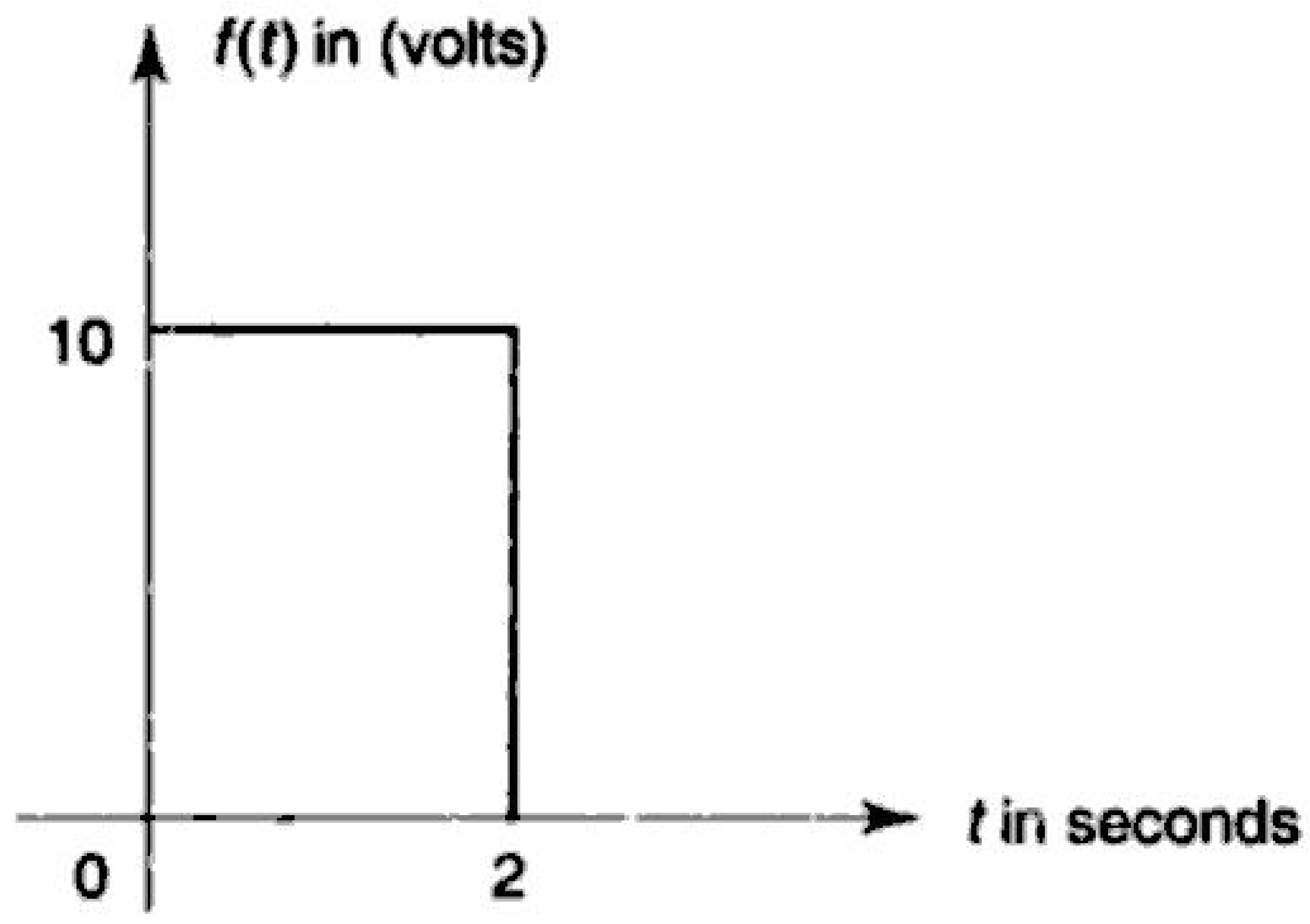

Fig. E2.16

Solution Fourier transform $F(j\omega)$ of the given pulse is given by

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_0^2 10\, e^{-j\omega t}\, dt = 10\left[\frac{e^{-j\omega t}}{-j\omega}\right]_0^2$$

$$= 10\left(\frac{e^{-j2\omega} - 1}{-j\omega}\right)$$

$$= 10\,\frac{e^{-j\omega}}{-j\omega}\left[e^{-j\omega} - e^{j\omega}\right]$$

$$= 20\,\frac{e^{-j\omega}}{\omega}\left[\frac{e^{j\omega} - e^{-j\omega}}{2j}\right]$$

$$= 20\, e^{-j\omega}\,\frac{\sin\omega}{\omega}$$

$$= 20\, e^{-j\omega} \operatorname{sinc}\omega$$

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2.8.4 Triangular Pulse

Consider the triangular pulse shown in Fig. 2.10.

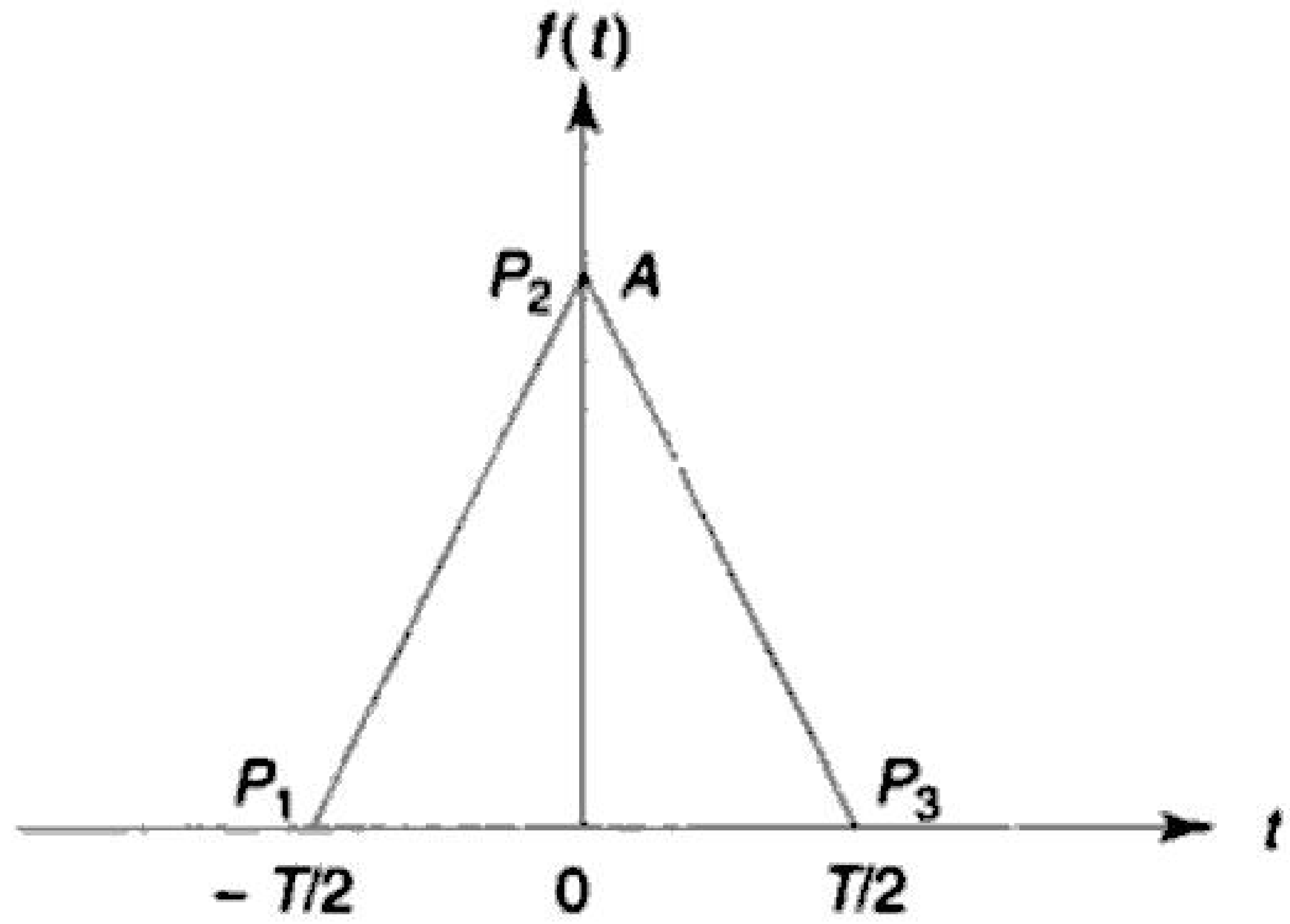

Fig. 2.10 *Triangular Pulse*

Equation of line P_1P_2 is

$$f(t)_{P_1P_2} = \frac{A}{T/2}t + A = A\left(1+\frac{2}{T}t\right)$$

Equation of line P_2P_3 is

$$f(t)_{P_2P_3} = -\frac{A}{T/2}t + A = A\left(1-\frac{2}{T}t\right)$$

Therefore $f(t) = A\left(1+\frac{2}{T}t\right)$ for $-\frac{T}{2} < t \le 0$

$$= A\left(1-\frac{2}{T}t\right) \quad \text{for} \quad 0 \le t \le \frac{T}{2}$$

Now,

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt = \int_{-T/2}^{T/2} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T/2}^{0} f(t)\, e^{-j\omega t}\, dt + \int_{0}^{T/2} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T/2}^{0} A\left(1+\frac{2}{T}t\right)e^{-j\omega t}\, dt + \int_{0}^{T/2} A\left(1-\frac{2}{T}t\right)e^{-j\omega t}\, dt$$

$$= A\int_{-T/2}^{0} e^{-j\omega t}\, dt + A\int_{0}^{T/2} e^{-j\omega t}\, dt$$

$$+ \frac{2A}{T}\int_{-T/2}^{0} t\, e^{-j\omega t}\, dt - \frac{2A}{T}\int_{0}^{T/2} t\, e^{-j\omega t}\, dt$$

Therefore,

$$\mathcal{F}\,[e^{-a^2t^2}] = \frac{\sqrt{\pi}}{a}\,e^{-\left(\frac{\pi f}{a}\right)^2}$$

2.8.6 Impulse Function (Unit Impulse)

The impulse function is called the **Dirac delta function** $\delta(t)$ which has an infinite amplitude and is infinitely narrow. This is defined as

(i) $\delta(t) = 0$ for all values except at $t = 0$.

(ii) $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$, i.e. the area within the pulse is unity.

The Fourier transform of the impulse function $\delta(t)$ is obtained as

$$F(j\omega) = \mathcal{F}\,[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)\,e^{-j\omega t}\,dt = 1$$

Hence, we have the pair $\delta(t) \Leftrightarrow 1$.

The frequency spectrum of the impulse function $\delta(t)$ shown in Fig. 2.13 (a) has a constant amplitude and extends over positive and negative frequencies.

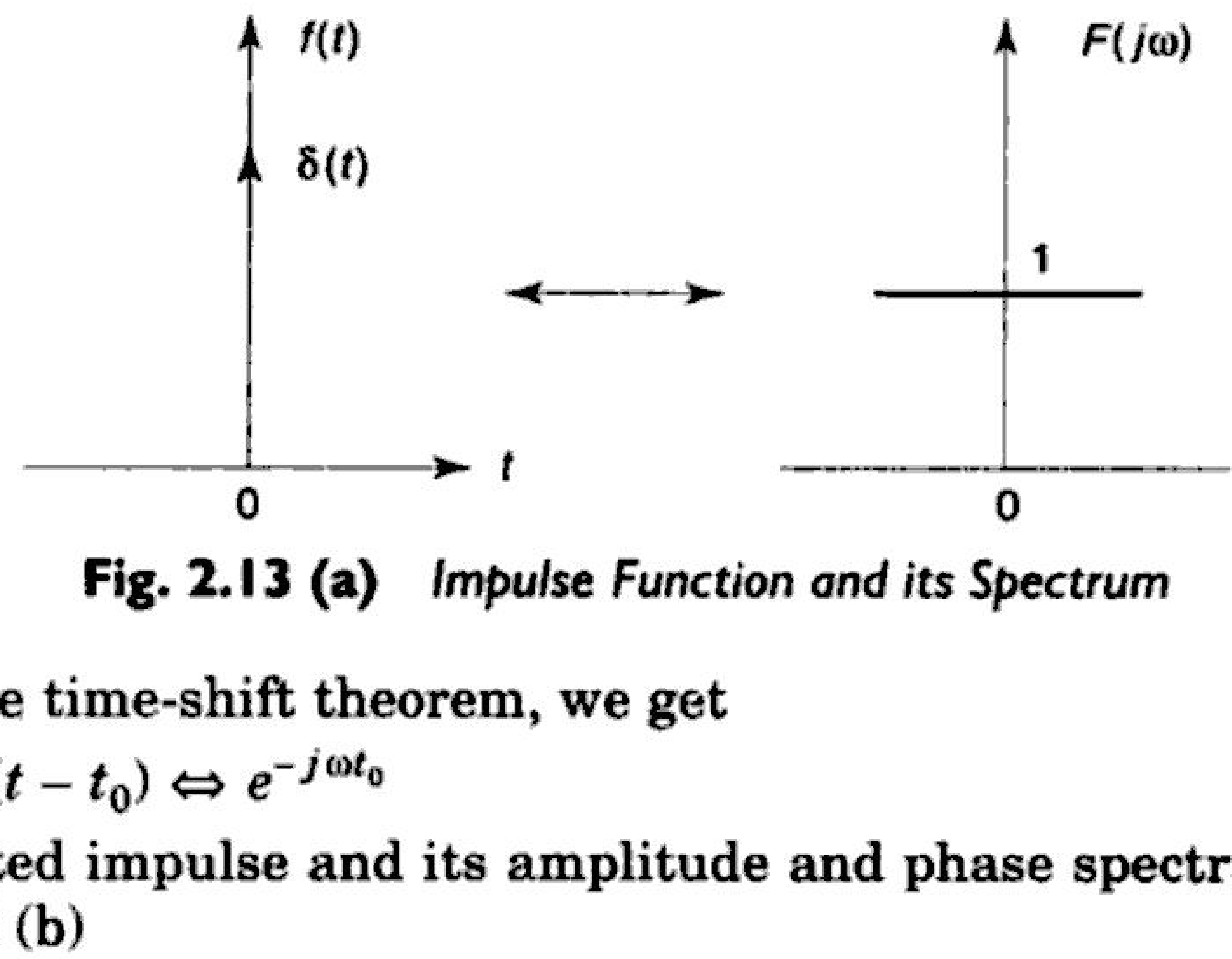

Fig. 2.13 (a) *Impulse Function and its Spectrum*

Using the time-shift theorem, we get

$$\delta(t - t_0) \Leftrightarrow e^{-j\omega t_0}$$

The shifted impulse and its amplitude and phase spectra are shown in Fig. 2.13 (b)

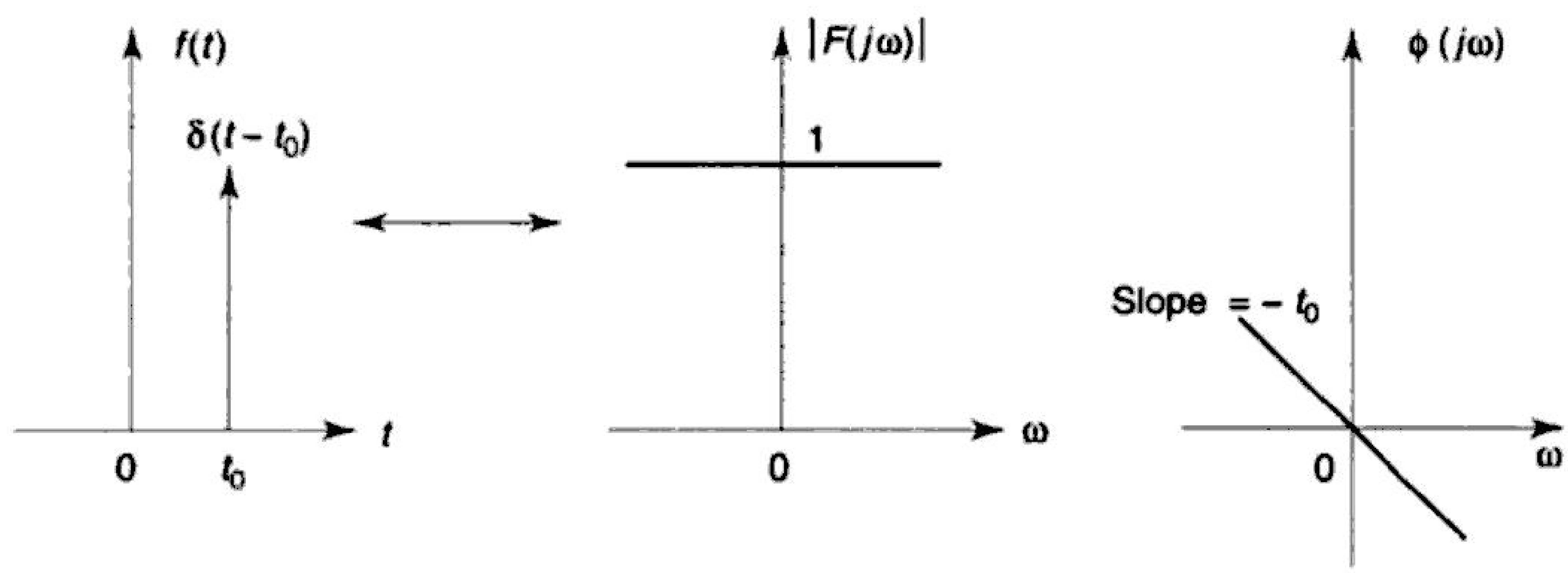

Fig. 2.13 (b) *Shifted Impulse and its Spectrum*

$$F(j\omega) = \mathcal{F}[\text{sgn}(t)] = \int_{-\infty}^{\infty} \text{sgn}(t)\, e^{j\omega t}\, dt$$

$$= \int_{-\infty}^{0} (-1)\, e^{-j\omega t}\, dt + \int_{0}^{\infty} (1)\, e^{-j\omega t}\, dt$$

$$= -\left[\frac{e - j\omega t}{-j\omega}\right]_{-\infty}^{0} + \left[\frac{e - j\omega t}{-j\omega}\right]_{0}^{\infty}$$

$$= \frac{1}{j\omega} + \frac{1}{j\omega} = \frac{2}{j\omega}$$

Therefore, $\mathcal{F}[\text{sgn}(t)] = \dfrac{2}{j\omega}$

Hence, we have the transform pair $\text{sgn}(t) \Leftrightarrow \dfrac{2}{j\omega}$

The amplitude and phase spectra of the signum function are shown in Figs 2.18 (a) and (b) respectively.

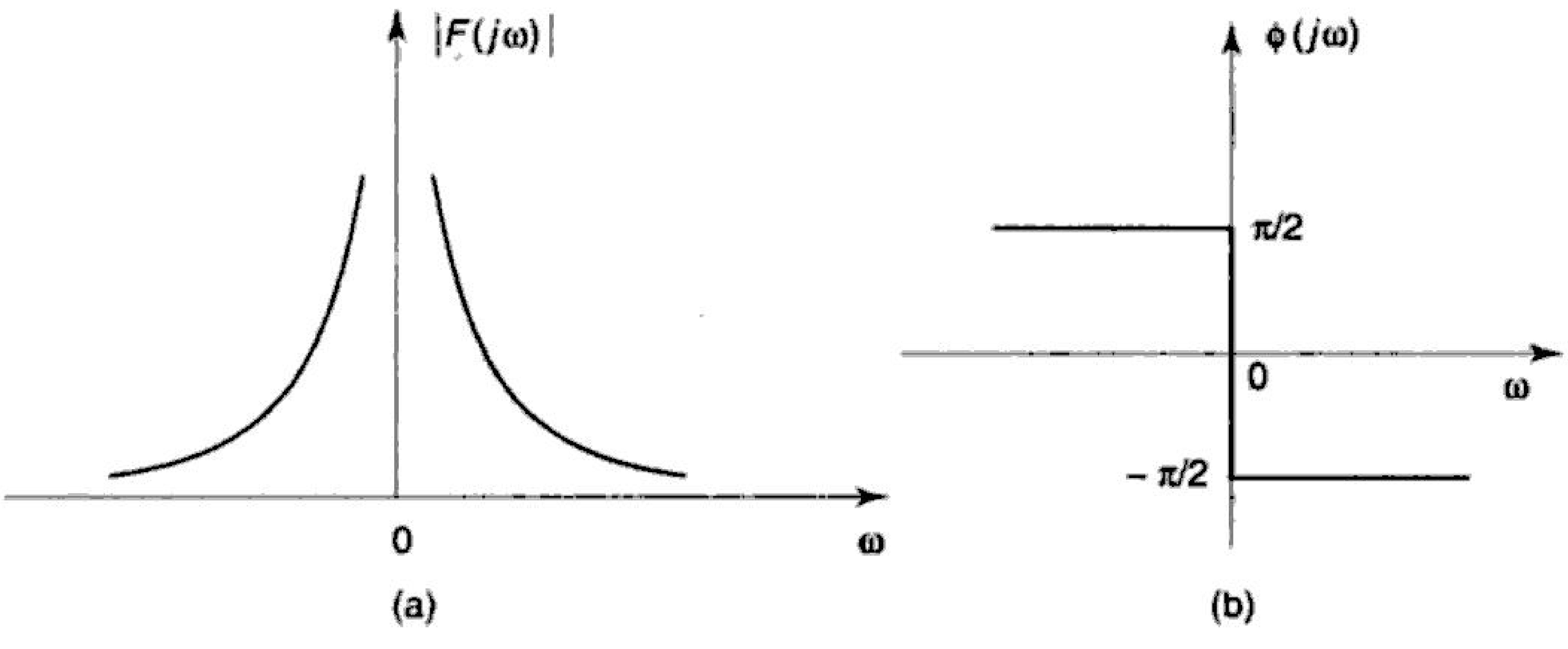

Fig. 2.18 *(a) Amplitude and (b) Phase Spectra of the Signum Function*

Example 2.19 Determine the Fourier transform of $f(t) = e^{-a|t|}$ sgn(t).

Solution $f(t) = e^{-a|t|}\, \text{sgn}(t)$

$$= \begin{cases} -e^{-at}, & \text{for } t < 0 \\ e^{-at}, & \text{for } t > 0 \end{cases}$$

Therefore, $\mathcal{F}[f(t)] = \displaystyle\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$

$$= -\int_{-\infty}^{0} e^{at}\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt$$

Comparing the powers P and P_{fzc}, we see that $(0.181/0.2) \times 100 = 90.5\%$ of the total power in $f(t)$ is contained within the first zero crossing of the spectrum for $f(t)$.

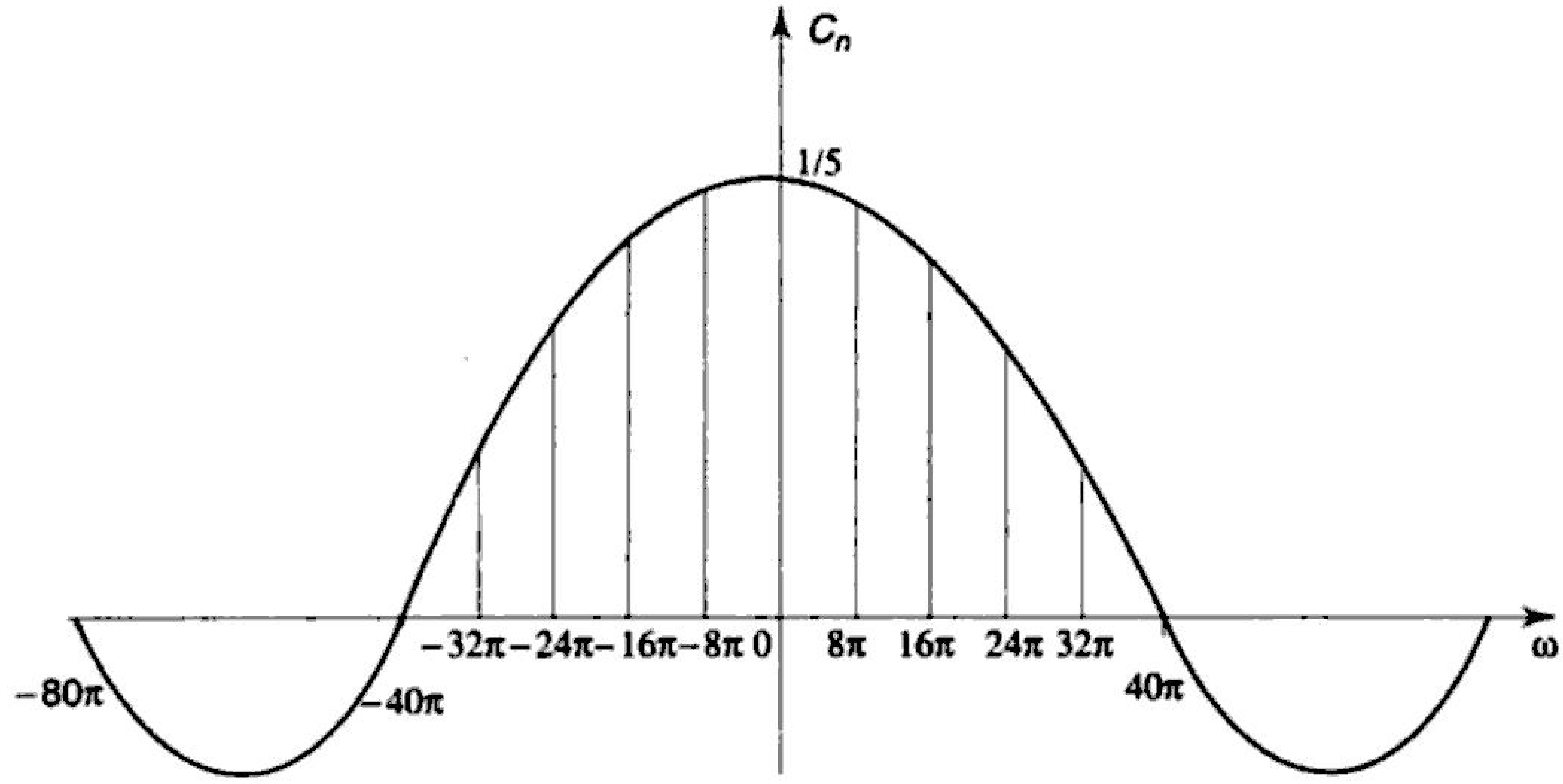

Fig. E2.20(b) *Spectrum of the Function*

2.8.11 Unit Impulse Train

Since $f(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$ is a periodic function, as shown in Fig. 2.20, the Fourier series representation of this unit impulse train is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

where,
$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jn\omega_0 t}\, dt = \frac{1}{T}$$

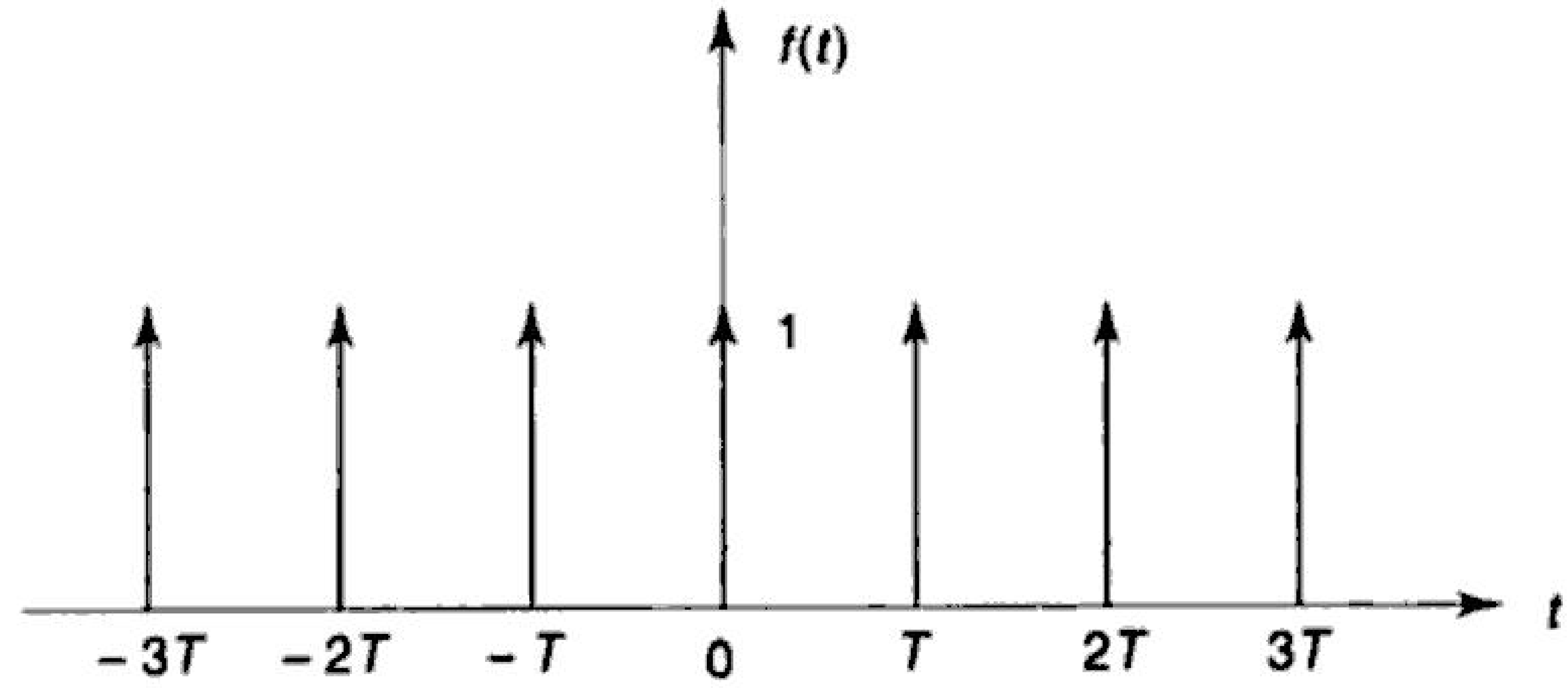

Fig. 2.20 *Unit Impulse Train*

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Solution Here, the signal period is $T = \frac{1}{\alpha}$. Since $x(t)$ is a periodic signal, it cannot be an energy signal. Therefore, the power signal is evaluated as

$$P_x = \frac{1}{T}\int_{t_1}^{t_1+T} |x(t)|^2 \, dt$$

$$P_x = \alpha \int_{t_1}^{t_1+\left(\frac{1}{\alpha}\right)} \left|A e^{2\pi\alpha t}\right|^2 dt$$

$$= \alpha \int_{t_1}^{t_1+\left(\frac{1}{\alpha}\right)} A^2 \, dt$$

$$= \alpha \left[A^2 t\right]_{t_1}^{t_1+\frac{1}{\alpha}} = A^2$$

Since the signal has finite power, it is a power signal and $E_x = \infty$.

Example 2.24 Determine the magnitude and phase spectrum of the pulse shown in Fig. E 2.24(a).

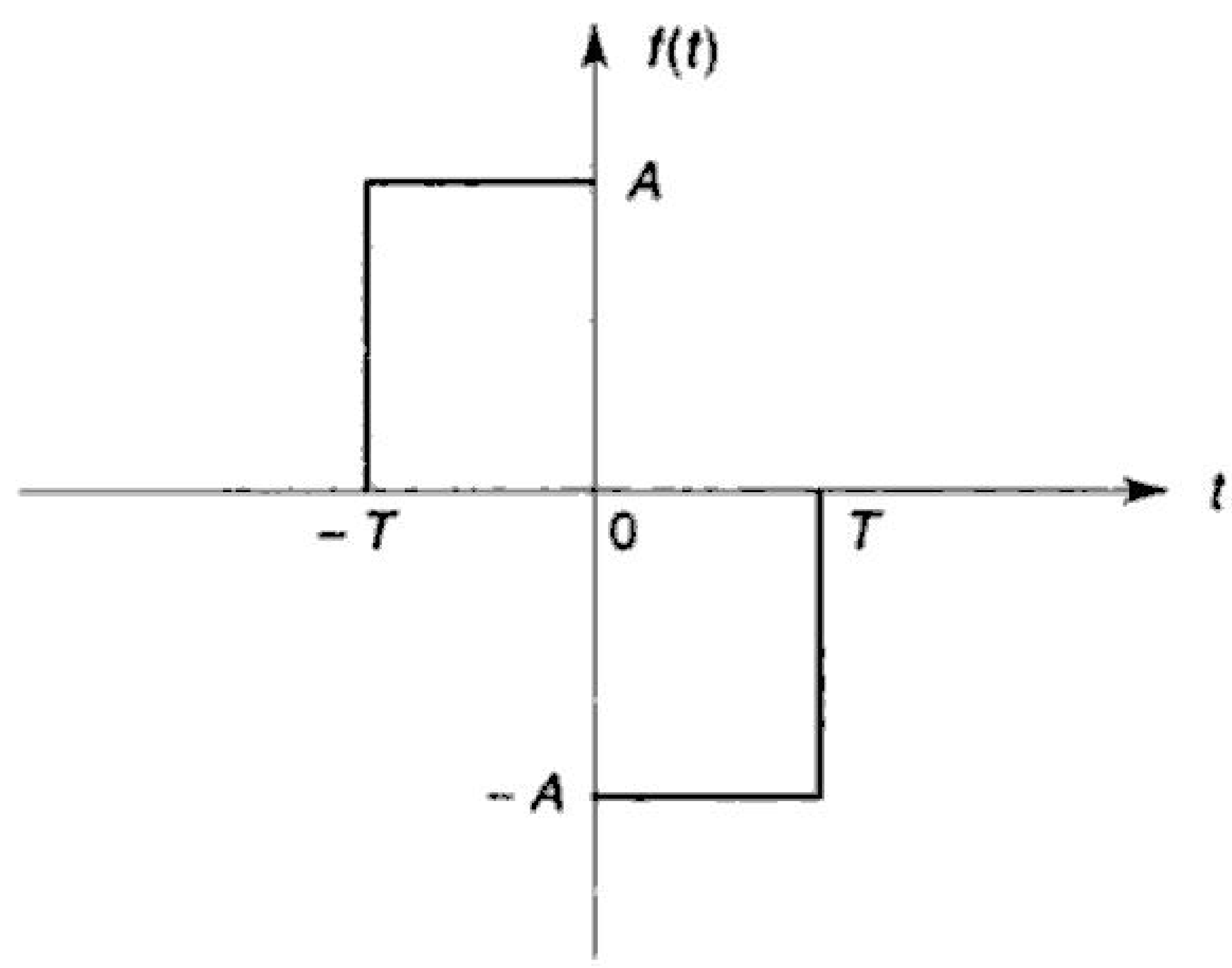

Fig. E 2.24(a)

Solution

Here $\quad f(t) = A, \quad$ for $-T \le t \le 0$

$\quad = -A,$ for $0 \le t \le T$

$\quad = 0, \quad$ otherwise

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{0} f(t)\, e^{-j\omega t}\, dt + \int_{0}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T}^{0} f(t)\, e^{-j\omega t}\, dt + \int_{0}^{T} f(t)\, e^{-j\omega t}\, dt$$

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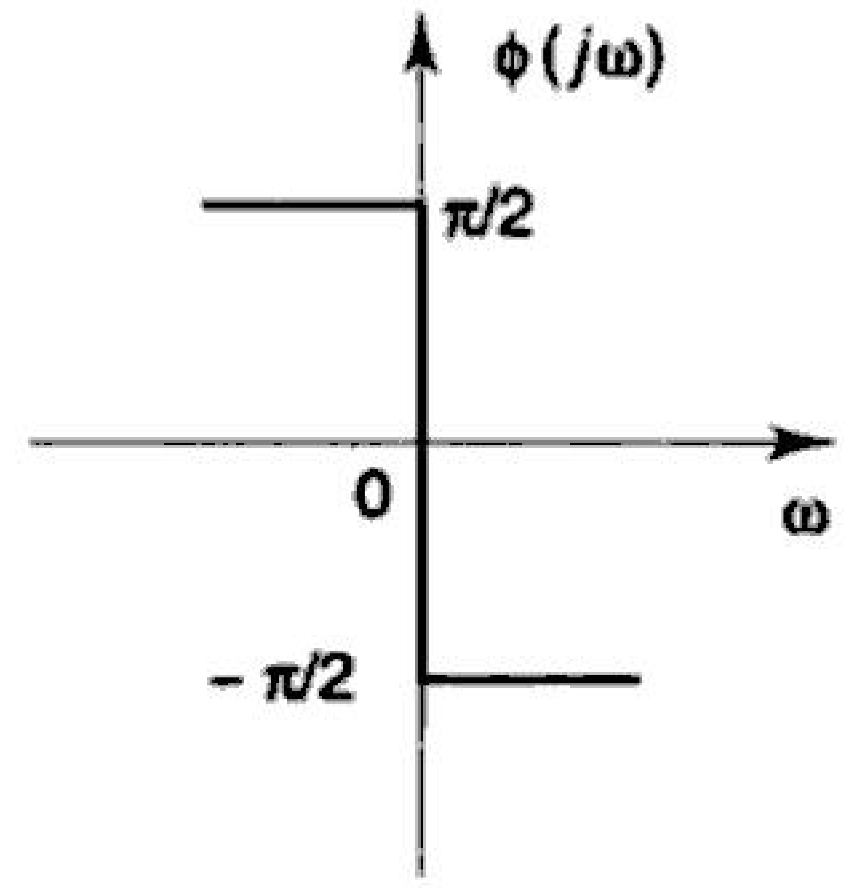

Fig. E2.26 (b) *Phase Spectrum*

Solution The magnitude $|F(j\omega)| = \pi$, for $-T \le \omega \le T$

$$\Phi(\omega) = \angle F(j\omega) = \pi/2, \text{ for } \omega < 0$$
$$= -\pi/2, \text{ for } \omega > 0$$
$$= 0, \quad \text{for } \omega = 0$$

For the limits $-T \le \omega \le 0$, $F(j\omega) = \pi e^{j\pi/2}$

For the limits $0 \le \omega \le T$, $F(j\omega) = \pi e^{-j\pi/2}$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-T}^{0} \pi e^{j\pi/2}\, e^{j\omega t}\, d\omega + \int_{0}^{T} \pi e^{-j\pi/2}\, e^{j\omega t}\, d\omega\right]$$

$$= \frac{1}{2} e^{j\pi/2}\left[\frac{e^{j\omega t}}{jt}\right]_{-T}^{0} + \frac{1}{2} e^{-j\pi/2}\left[\frac{e^{j\omega t}}{jt}\right]_{0}^{T}$$

$$= \frac{1}{2}\frac{e^{j\pi/2}}{jt}\left[1 - e^{-jTt}\right] + \frac{1}{2}\frac{e^{-j\pi/2}}{jt}\left[e^{jTt} - 1\right]$$

$$= \frac{1}{j2t}\left[e^{j\pi/2} - e^{j\pi/2}\, e^{-jTt} + e^{-j\pi/2}\, e^{jTt} - e^{-j\pi/2}\right]$$

$$= \frac{1}{t}\left[\frac{e^{j\pi/2} - e^{-j\pi/2}}{j2}\right] - \frac{1}{t}\left[\frac{e^{j(\pi/2 - Tt)} - e^{-j(\pi/2 - Tt)}}{j2}\right]$$

$$= \frac{1}{t}\left[\sin(\pi/2) - \sin(\pi/2 - Tt)\right]$$

$$= \frac{1}{t}\left[1 - \cos Tt\right] = \frac{2\sin^2\left(\frac{Tt}{2}\right)}{t} = \frac{T^2 t}{2}\operatorname{sinc}^2\left(\frac{Tt}{2}\right)$$

Example 2.27 Obtain the Fourier transform of the trapezoidal pulse shown in Fig. E2.27.

$$= \frac{1}{2}\int_0^\infty \left[e^{-(a-jb+j\omega)t} + e^{-(a+jb+j\omega)t}\right]$$

$$= \frac{1}{2}\left[\frac{e^{-(a+j\omega-jb)t}}{-(a+j\omega-jb)} + \frac{e^{-(a+j\omega+jb)t}}{-(a+j\omega+jb)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{1}{(a+j\omega-jb)} + \frac{1}{(a+j\omega+jb)}\right]$$

$$= \frac{1}{2}\left[\frac{a+j\omega+jb+a+j\omega-jb}{(a+j\omega)^2+b^2}\right]$$

$$= \frac{a+j\omega}{(a+j\omega)^2+b^2}$$

Example 2.30 Find the Fourier transform of $f(t) = t\cos at$

Solution

$$\mathcal{F}[t\cos at] = \mathcal{F}\left[t\left\{\frac{e^{jat}+e^{-jat}}{2}\right\}\right]$$

$$= \int_0^\infty t\left[\frac{e^{jat}+e^{-jat}}{2}\right]e^{-j\omega t}\,dt$$

$$= \frac{1}{2}\int_0^\infty \left(te^{-j(-a+\omega)t} + te^{-j(a+\omega)t}\right)dt$$

$$= \frac{1}{2}\left[t\left\{\frac{e^{-j(-a+\omega)t}}{-j(-a+\omega)}\right\} - \frac{e^{-j(-a+\omega)t}}{[-j(-a+\omega)]^2}\right]_0^\infty$$

$$+ \frac{1}{2}\left[t\left\{\frac{e^{-j(a+\omega)t}}{-j(a+\omega)}\right\} - \frac{e^{-j(a+\omega)t}}{[-j(a+\omega)]^2}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{1}{(-j)^2(-a+\omega)^2} + \frac{1}{(-j)^2(a+\omega)^2}\right]$$

$$= -\frac{1}{2}\left[\frac{1}{(-a+\omega)^2} + \frac{1}{(a+\omega)^2}\right]$$

$$= -\frac{1}{2}\frac{a^2+\omega^2+2a\omega+a^2+\omega^2-2a\omega}{\left(\omega^2-a^2\right)^2}$$

$$= -\frac{\omega^2+a^2}{\left(\omega^2-a^2\right)^2} \quad \text{or} \quad = -\frac{a^2+\omega^2}{\left(a^2-\omega^2\right)^2}$$

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Example 2.33 Determine the Fourier transform of the sinusoidal pulse shown in Fig. E2.33.

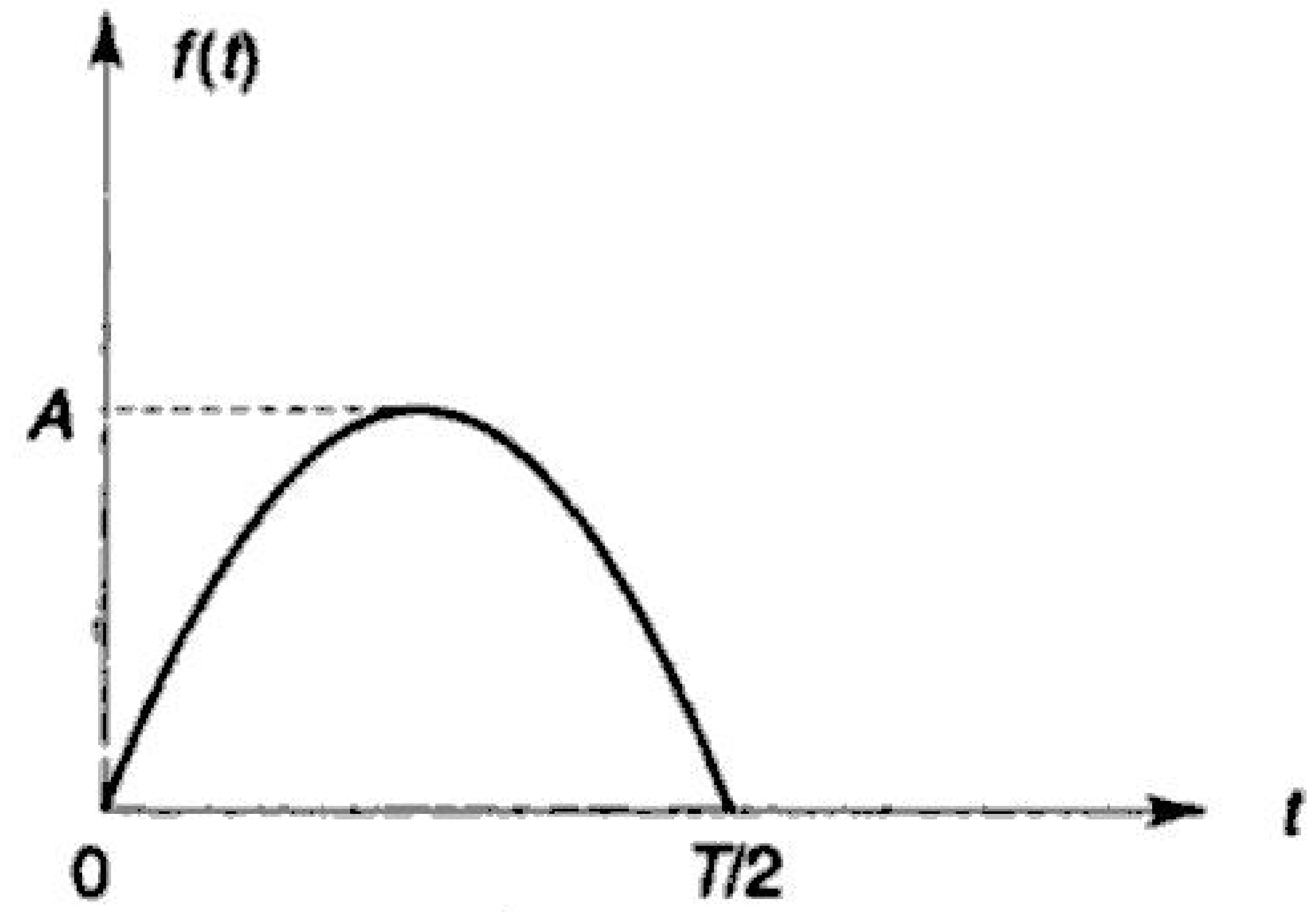

Fig. E2.33

Solution

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

Here $f(t) = A \sin \omega_0 t$ for $0 < t \leq = T/2$ where $\omega_0 = 2\pi/T$.
$= 0$, otherwise

$$F(j\omega) = \int_0^{T/2} A \sin \omega_0 t\, e^{-j\omega t}\, dt$$

$$= A \int_0^{T/2} \left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \right) e^{-j\omega t}\, dt$$

$$= \frac{A}{2j} \int_0^{T/2} \left[e^{j(\omega_0 - \omega)t} - e^{-j(\omega_0 + \omega)t} \right] dt$$

$$= \frac{A}{2j} \left[\frac{e^{j(\omega_0 - \omega)t}}{j(\omega_0 - \omega)} - \frac{e^{-j(\omega_0 + \omega)t}}{-j(\omega_0 + \omega)} \right]_0^{T/2}$$

$$= -\frac{A}{2} \left[\frac{e^{j(\omega_0 - \omega)T/2} - 1}{(\omega_0 - \omega)} + \frac{e^{-j(\omega_0 + \omega)T/2} - 1}{(\omega_0 + \omega)} \right]$$

$$= -\frac{A}{2} \left[\frac{\left(e^{j(\omega_0 - \omega)T/2} - 1\right)(\omega_0 + \omega) + \left(e^{-j(\omega_0 + \omega)T/2} - 1\right)(\omega_0 - \omega)}{(\omega_0^2 - \omega^2)} \right]$$

$$= \frac{-A}{2(\omega_0^2 - \omega^2)} \left[\omega_0 \left(e^{j(\omega_0 - \omega)T/2} + e^{-j(\omega_0 + \omega)T/2} \right) \right.$$

$$\left. + \omega \left(e^{j(\omega_0 - \omega)T/2} - e^{-j(\omega_0 + \omega)T/2} \right) - 2\omega_0) \right]$$

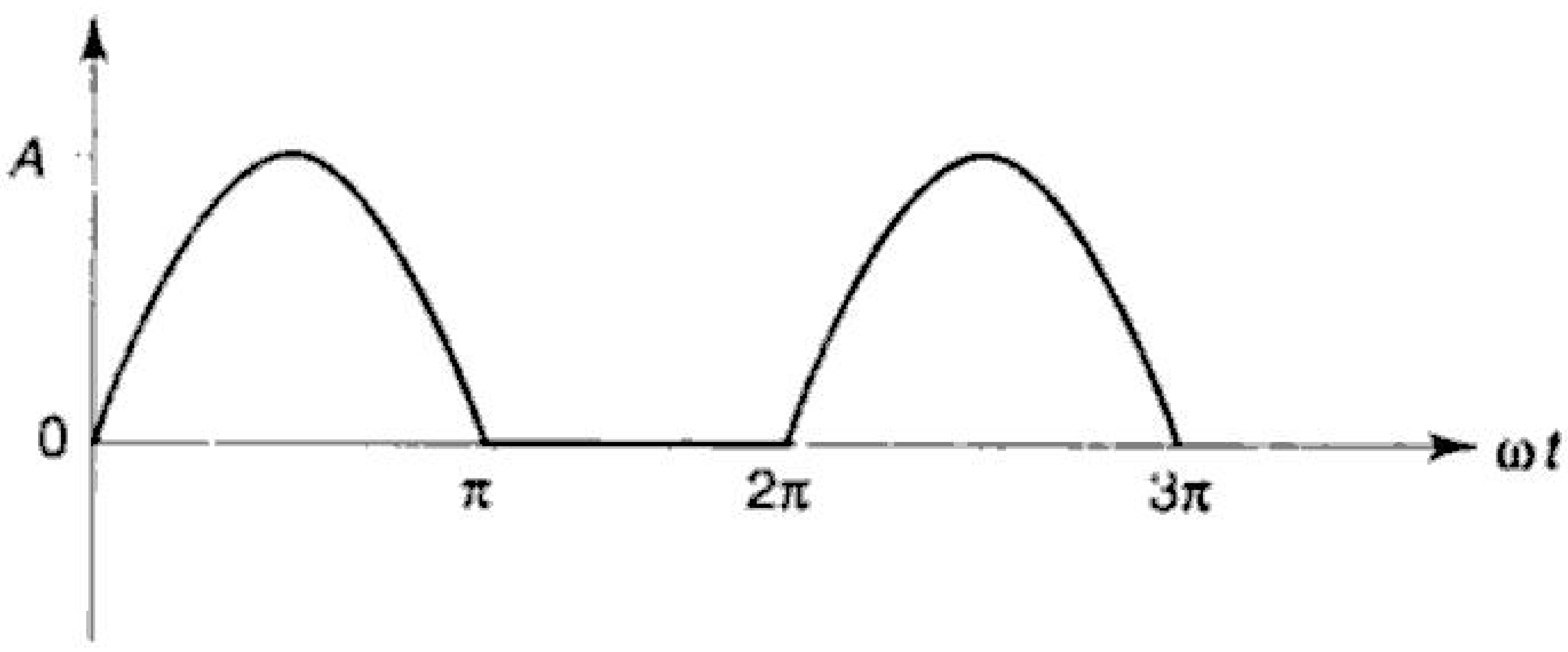

Fig. Q2.12

2.13 *Obtain the Fourier series for the full wave rectified sine waves shown in Figs Q2.13 (a) and (b).*

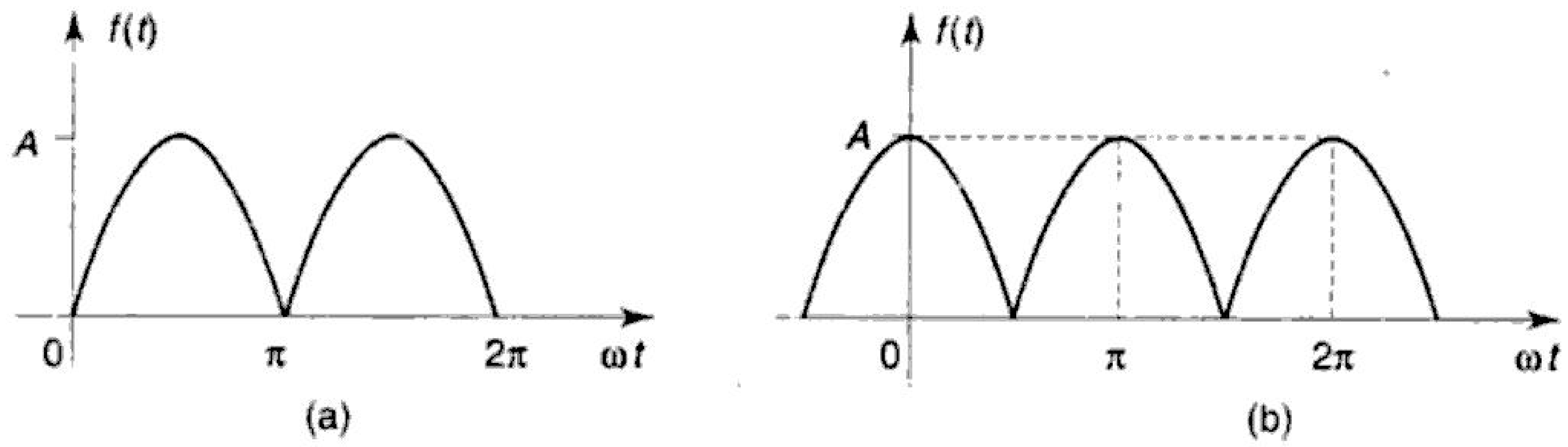

Fig. Q2.13

$$Ans : (a)\ f(t) = \frac{2A}{\pi}\left(1 - \frac{2}{3}\cos 2\,\omega t - \frac{2}{15}\cos 4\,\omega t - \frac{2}{35}\cos 6\,\omega t - \cdots\right)$$

$$(b)\ f(t) = \frac{2A}{\pi}\left(1 + \frac{2}{3}\cos 2\,\omega t - \frac{2}{15}\cos 4\,\omega t + \frac{2}{35}\cos 6\,\omega t - \cdots\right)$$

2.14 *Obtain the Trigonometric Fourier series expansion of the periodic signal*

$$x(t) = \begin{cases} A & for \quad kT < t \le (k+1)T \\ -A & for \quad (k+1)T < t \le -kT \end{cases}$$

with k taking the values 0, 2, 4, 6, . . .

2.15 *A periodic triangular waveform starts at the origin with zero value and increases linearly with respect to time. After a time T, it becomes zero. Obtain its Fourier series.*

2.16 *With regard to Fourier series representation, justify the following statement :*

(i) *Odd functions have only sine terms*
(ii) *Even functions have no sine terms*
(iii) *Functions with half-wave symmetry have only odd harmonics.*

2.17 *Obtain the exponential Fourier series for the waveforms shown in Figs Q17 (a) and (b).*

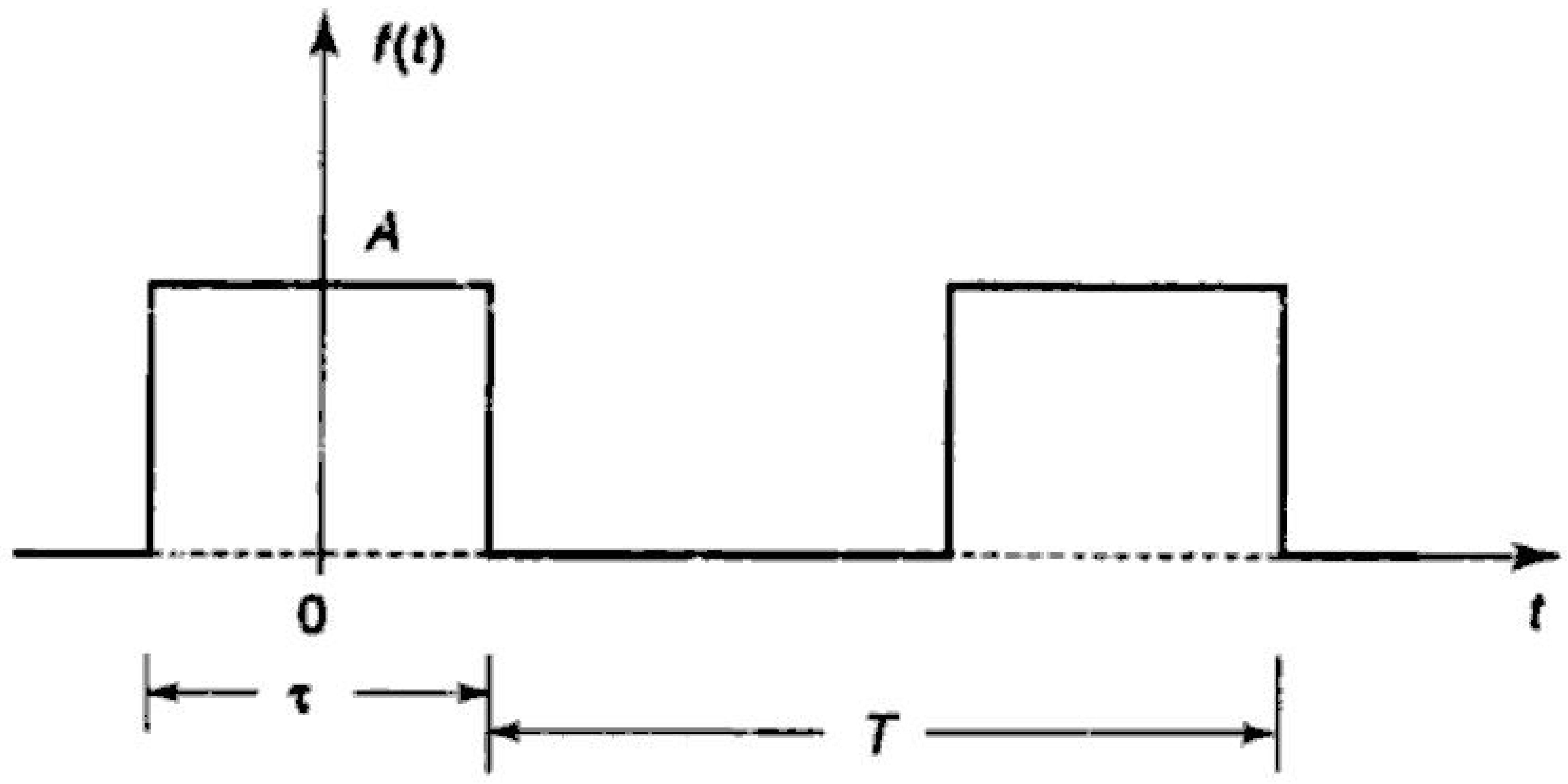

Fig. Q2.43

2.44 Defining x(t) and y(t) as

$$x(t) = \begin{cases} e^{-t} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \quad \text{and}$$

$$y(t) = \begin{cases} \alpha e^{-at} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

Find $f(t) = x(t) * y(t)$.

2.45 With suitable waveform, explain the convolution of x(t), the input and h(t), the system transfer function to get y(t), the output.

2.46 Evaluate the Fourier transform of a single unit pulse of 1 volt.

2.47 Obtain the Fourier transform of a single symmetrical triangular pulse.

2.48 Show that a time shift in the time domain is equal to a phase shift in the frequency domain.

2.49 Find the Fourier transform of $x(t) = Ae^{-t/T}u(t)$ and sketch its magnitude and phase as functions of frequency.

2.50 Determine the Fourier transform of a two-sided exponential pulse $x(t) = e^{-|t|}$.

2.51 Find the Fourier transform of $x(t) = A\cos(\omega_c t + \theta)$.

2.52 Find the Fourier transform of $f(t) = e^{-at} \sin bt$

2.53 Find Fourier transform of the following functions

(a) $5 \sin^2 3t$ (b) $\cos(8t + 0.1\pi)$

Ans: (a) $2.5\pi[2\,\delta(\omega) + \delta(\omega + 6) + \delta(\omega - 6)$

(b) $18.85\angle 18°\ \delta(\omega - 8) + 18.85\ \angle -18°\ \delta(\omega + 8)$

2.54 Find the Fourier transform of the single triangular pulse with period T = 8 sec and amplitude A = 10 V.

Ans: $40\ \text{sinc}^2\, 2\omega$

2.55 Determine the Fourier transform of a one-cycle sine wave.

$$= -\frac{A}{a+s}\left[e^{-(a+s)t}\right]_0^{\infty} = \frac{A}{(s+a)}$$

Hence, $\mathcal{L}\{Ae^{-at}\} = \dfrac{A}{(s+a)}$ (3.4)

3. Sine Function

$$f(t) = \sin \omega_0 t$$

Using Euler's identity, we have

$$\sin \omega_0 t = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$$

Hence, $\mathcal{L}\{\sin \omega_0 t\} = \dfrac{1}{2j}\left[\mathcal{L}(e^{j\omega_0 t}) - \mathcal{L}(e^{-j\omega_0 t})\right]$

$$= \frac{1}{2j}\left[\frac{1}{s-j\omega_0} - \frac{1}{s+j\omega_0}\right] = \frac{\omega_0}{s^2+\omega_0^2}$$

Hence, $\mathcal{L}\{\sin \omega_0 t\} = \dfrac{\omega_0}{s^2+\omega_0^2}$ (3.5)

4. Cosine Function

$$f(t) = \cos \omega_0 t$$

We know that $\cos \omega_0 t = \dfrac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$

$$\mathcal{L}\{\cos \omega_0 t\} = \frac{1}{2}\left[\mathcal{L}(e^{j\omega_0 t}) + \mathcal{L}(e^{-j\omega_0 t})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0}\right] = \frac{s}{s^2+\omega_0^2}$$

Hence, $\mathcal{L}\{\cos \omega_0 t\} = \dfrac{s}{s^2+\omega_0^2}$ (3.6)

5. Hyperbolic Sine and Cosine Functions

$$\sinh \omega_0 t = \frac{1}{2}\left[e^{\omega_0 t} - e^{-\omega_0 t}\right]$$

$$\cosh \omega_0 t = \frac{1}{2}\left[e^{\omega_0 t} + e^{-\omega_0 t}\right]$$

$$\mathcal{L}\{\sinh \omega_0 t\} = \frac{1}{2}\left[\mathcal{L}(e^{\omega_0 t}) - \mathcal{L}(e^{-\omega_0 t})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-\omega_0} - \frac{1}{s+\omega_0}\right] = \frac{\omega_0}{s^2-\omega_0^2}$$

$\mathcal{L}\{\sinh \omega_0 t\} = \dfrac{\omega_0}{s^2-\omega_0^2}$ (3.7)

Similarly, $\mathcal{L}\{\cosh \omega_0 t\} = \dfrac{1}{2}\left[\mathcal{L}(e^{\omega_0 t}) + \mathcal{L}(e^{-\omega_0 t})\right]$

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$$\underset{t \to 0^+}{\text{Lt}} f(t) = \underset{s \to \infty}{\text{Lt}} sF(s)$$

Proof We know that

$$\mathcal{L}\{f'(t)\} = s[\mathcal{L}(f(t))] - f(0)$$

By taking the limit $s \to \infty$ on both sides

$$\underset{s \to \infty}{\text{Lt}} \mathcal{L}\{f'(t)\} = \underset{s \to \infty}{\text{Lt}} [sF(s) - f(0)]$$

$$\underset{s \to \infty}{\text{Lt}} \int_0^\infty f'(t)\, e^{-st}\, dt = \underset{s \to \infty}{\text{Lt}} [sF(s) - f(0)]$$

As $s \to \infty$, the integration of LHS becomes zero

i.e. $$\int_0^\infty \underset{s \to \infty}{\text{Lt}} [f'(t)\, e^{-st}] dt = 0$$

$$\underset{s \to \infty}{\text{Lt}} sF(s) - f(0) = 0$$

Therefore, $$\underset{s \to \infty}{\text{Lt}} sF(s) = f(0) = \underset{t \to 0+}{\text{Lt}} f(t)$$

3.5.2 Final Value Theorem

If $f(t)$ and $f'(t)$ are Laplace transformable, then

$$\underset{t \to \infty}{\text{Lt}} f(t) = \underset{s \to 0}{\text{Lt}} sF(s) \tag{3.15}$$

Proof We know that

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

Taking the limit $s \to 0$ on both sides, we get

$$\underset{s \to 0}{\text{Lt}} \mathcal{L}\{f'(t)\} = \underset{s \to 0}{\text{Lt}} [sF(s) - f(0)]$$

$$\underset{s \to 0}{\text{Lt}} \int_0^\infty f'(t)\, e^{-st}\, dt = \underset{s \to 0}{\text{Lt}} sF(s) - f(0)]$$

Therefore, $$\int_0^\infty f'(t) dt = \underset{s \to 0}{\text{Lt}} [sF(s) - f(0)]$$

$$[f(t)]_0^\infty = \underset{t \to \infty}{\text{Lt}} f(t) - \underset{t \to 0}{\text{Lt}} f(t) = \underset{s \to 0}{\text{Lt}} sF(s) - f(0)$$

Since $f(0)$ is not a function of s, it gets cancelled from both sides of the above equation.

Therefore, $$\underset{t \to \infty}{\text{Lt}} f(t) = \underset{s \to 0}{\text{Lt}} sF(s)$$

3.6 CONVOLUTION INTEGRAL

If $X(s)$ and $H(s)$ are the Laplace transforms of $x(t)$ and $h(t)$, then the product of $X(s)H(s) = Y(s)$, where $Y(s)$ is the Laplace transform of $y(t)$ given by

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$$= \frac{1}{RC} \cdot \frac{1}{\left(s + \frac{1}{RC}\right)^3}$$

Taking inverse Laplace transform, we obtain

$$y(t) = \frac{1}{RC} \frac{t^2 e^{-t/RC}}{2} u(t)$$

3.7 TABLE OF LAPLACE TRANSFORMS

Table 3.1 presents some functions and their corresponding Laplace transforms. Table 3.2 lists the properties of the Laplace transform. Table 3.3 gives the elements needed to develop the s-domain image of a given time domain circuit.

Table 3.1 *Laplace transforms pairs*

S.No.	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$\delta(t-a)$	e^{-as}
3.	$u(t)$	$\frac{1}{s}$
4.	$u(t-a)$	$\frac{e^{-as}}{s}$
5.	$\frac{t^n}{n!} u(t)$, n positive interger	$(-1)^n \frac{1}{s^{n+1}}$
6.	$e^{-at} u(t)$	$\frac{1}{s+a}$
7.	$\frac{t^n e^{-at}}{n!} u(t)$	$\frac{1}{(s+a)^{n+1}}$
8.	$\sin(\omega_0 t)\, u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$
9.	$\cos(\omega_0 t)\, u(t)$	$\frac{s}{s^2+\omega_0^2}$
10.	$t\cos(\omega_0 t)\, u(t)$	$\frac{s^2-\omega_0^2}{(s^2+\omega_0^2)^2}$
11.	$t\sin(\omega_0 t)\, u(t)$	$\frac{2\omega_0 s}{(s^2+\omega_0^2)^2}$
12.	$e^{-at}\sin(\omega_0 t)\, u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$

(Contd.)

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For evaluating the constants $A_0, A_1, A_2, \ldots A_{n-1}$, we have to multiply both sides of the above equation by $(s - p_i)^n$.

Hence, $(s - p_i)^n F(s) = F_1(s) = A_0 + A_1(s - p_i) + A_2(s - p_i)^2 + \ldots$

$$+ A_{n-1}(s - p_i)^{n-1} + \frac{N_1(s)}{D_1(s)}(s - p_i)^n$$

Substituting $s = p_i$, we get

$$A_0 = (s - p_i)^n F(s) \,|_{s = p_i}$$

Differentiating $F_1(s)$ with respect to s, we get

$$\frac{d}{ds} F_1(s) = A_1 + 2A_2(s - p_i) + \ldots + A_{n-1}(n-1)(s - p_i)^{n-2}$$

$$+ \frac{d}{ds}\left(\frac{N_1(s)}{D_1(s)}(s - p_i)^n\right)$$

Substituting $s = p_i$ in the above equation, we get

$$A_1 = \frac{d}{ds} F_1(s)\,|_{s = p_i}$$

Similarly, $$A_2 = \frac{1}{2!}\frac{d^2}{ds^2} F_1(s)\,|_{s = p_i}$$

Generally, $$A_n = \frac{1}{n!}\frac{d^n}{ds^n} F_1(s)\,|_{s = p_i} \quad \text{where } n = 0, 1, 2, \ldots n - 1.$$

3.9 NETWORK TRANSFER FUNCTION

The transfer function $H(s)$ of the LTI system, as shown in Fig. 3.2, is equal to the ratio of the Laplace transform $Y(s)$ of the output signal to the Laplace transform $X(s)$ of the input signal when initial conditions are zero. Thus

$$H(s) = \frac{Y(s)}{X(s)} = \left.\frac{\text{Laplace transform of output}}{\text{Laplace transform of input}}\right|_{\textit{All initial conditions are zero}} \tag{3.18}$$

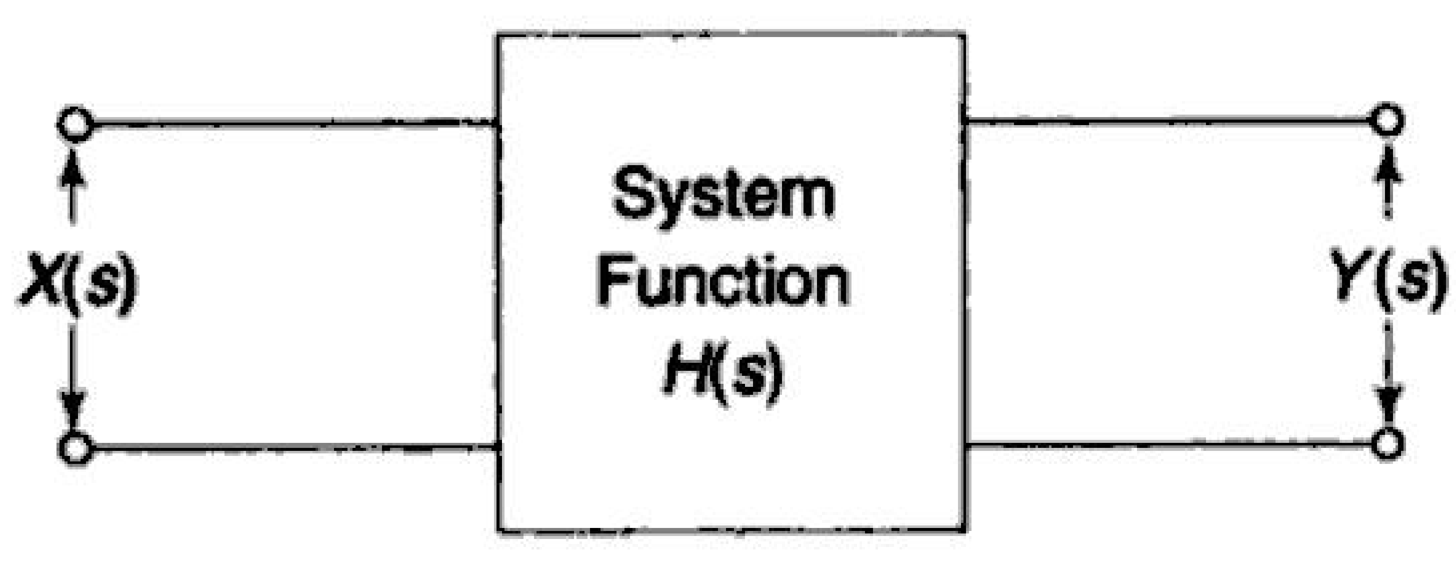

Fig. 3.2 *Transfer Function of a System*

The transfer function of a system $H(s)$ is the Laplace transform of the impulse response $h(t)$. The transfer function $H(s)$ is strictly analogous to

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The magnitude response of the pure delay is $|H(j\omega)| = 1$ and the phase response of the pure delay is $\Phi(j\omega) = -\omega\tau$.

Therefore, the time delay τ is equal to the minus the derivative of the phase response, i.e.

$$\tau = -\frac{d\,\Phi(j\omega)}{d\,\omega}$$

Example 3.10 Draw the poles and zero for the current $I(s)$ in a network given by

$$I(s) = \frac{3s}{(s+2)(s+4)}$$

and hence obtain $i(t)$.

Solution The zero occurs at $s = 0$ and the poles at $s = -2$ and $s = -4$ as shown in Fig. E3.10.

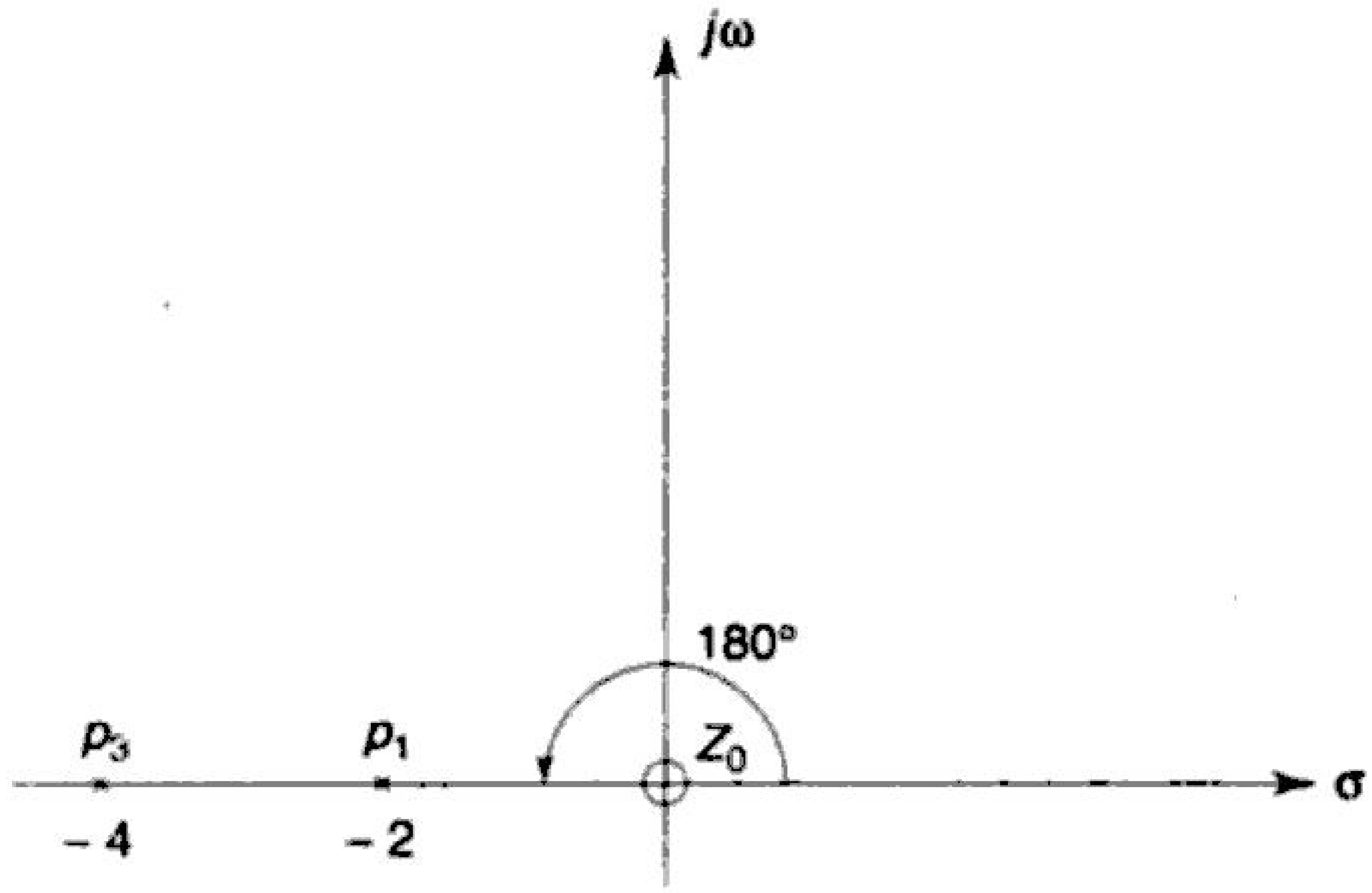

Fig. E3.10 *Pole Zero Plot of I(s)*.

The given function $I(s)$ can be expanded by partial fraction as

$$I(s) = \frac{A_1}{(s+2)} + \frac{A_2}{(s+4)}$$

The coefficients A_1 and A_2 may be evaluated from the pole-zero diagram.

$$A_1 = k\,\frac{\text{Magnitude and phase angle of phasor from zero at } z_0 \text{ to pole at } p_1}{\text{Magnitude and phase angle of phasor from pole at } p_2 \text{ to pole at } p_1}$$

$$= 3\cdot\frac{2\angle 180^\circ}{2\angle 0^\circ} = 3\angle 180^\circ = 3\cdot(\cos 180^\circ + j\sin 180^\circ) = 3\times -1 = -3$$

Similarly,

$$A_2 = 3\cdot\frac{4\angle 180^\circ}{2\angle 180^\circ} = 6$$

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$$\text{and Phase} \quad \Phi(j2) = \frac{\angle 90^\circ}{\angle \tan^{-1}(1) \angle \tan^{-1}(3)} = \frac{\angle 90^\circ}{\angle 45^\circ \angle 71.8^\circ}$$

$$= 90^\circ - 45^\circ - 71.8^\circ = -26.8^\circ$$

3.11 LAPLACE TRANSFORM OF PERIODIC FUNCTIONS

The time-shift theorem is useful in determining the transform of periodic time functions. Let function $f(t)$ be a causal periodic waveform which satisfies the condition $f(t) = f(t + nT)$ for all $t > 0$ where T is the period of the function and $n = 0, 1, 2, \ldots$

$$F(s) = \int_0^\infty f(t) e^{-st}\, dt$$

$$= \int_0^T f(t) e^{-st}\, dt + \int_T^{2T} f(t) e^{-st}\, dt + \cdots + \int_{nT}^{(n+1)T} f(t) e^{-st}\, dt + \cdots$$

As $f(t)$ is periodic, the above equation becomes

$$= \int_0^T f(t) e^{-st}\, dt + e^{-sT} \int_0^T f(t) e^{-st}\, dt + \cdots + e^{-nsT} \int_0^T f(t) e^{-st}\, dt + \cdots$$

$$= \left[1 + e^{-sT} + e^{-2sT} + \cdots + e^{-nsT} + \cdots\right] \int_0^T f(t)\, e^{-st}\, dt$$

$$= \left[1 + e^{-sT} + \left(e^{-sT}\right)^2 + \cdots + \left(e^{-sT}\right)^n + \cdots\right] F_1(s)$$

where $\quad F_1(s) = \int_0^T f(t)\, e^{-st}\, dt$

Here, $F_1(s) = \mathcal{L}\{[u(t) - u(t - T)]\, f(t)\}$, which is the transform of the first period of the time function, and $\{[u(t) - u(t - T)]\, f(t)\}$ has non-zero only in the first period of $f(t)$.

When we apply the binomial theorem to the bracketed expression, it becomes $1/(1 - e^{-sT})$

$$F(s) = \frac{1}{1 - e^{-sT}} \int_0^T f(t)\, e^{-st}\, dt = \frac{F_1(s)}{1 - e^{-sT}}$$

Example 3.13 Find the Laplace transform of the periodic rectangular waveform shown in Fig. E3.13.

Solution Here the period is $2T$

Therefore, $\mathcal{L}\{f(t)\} = \dfrac{1}{1 - e^{-2sT}} \left[\displaystyle\int_0^{2T} f(t)\, e^{-st}\, dt\right]$

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Therefore, $I(s)[Ls + R] = \dfrac{V_0}{s}$

Hence, $$I(s) = \frac{V_0}{L} \cdot \frac{1}{s\left(s + \frac{R}{L}\right)}$$

$$= \frac{V_0}{L} \cdot \frac{L}{R}\left[\frac{1}{s} - \frac{1}{s + R/L}\right] = \frac{V_0}{R}\left[\frac{1}{s} - \frac{1}{s + R/L}\right]$$

Taking inverse Laplace transform, we get

$$i(t) = \frac{V_0}{R} \cdot \left[1 - e^{-\frac{R}{L}t}\right] \tag{3.27}$$

Impulse Response

For the impulse response, the input excitation is $x(t) = \delta(t)$. Hence, the differential equation becomes

$$L\,\frac{di(t)}{dt} + Ri(t) = \delta(t)$$

$$L\,\{sI(s) - i\,(o^+)\} + RI\,(s) = 1$$

Since $i\,(o^+) = 0$,

$$I(s) = \frac{1}{R + Ls} = \frac{1}{L} \cdot \frac{1}{s + R/L}$$

Taking inverse Laplace transform, we get

$$i(t) = \frac{1}{L} \cdot e^{-(R/L)t}\, u(t)$$

3.12.2 Step and Impulse Responses of Series R-C Circuit

Step Response

For the step response, the input excitation is $x(t) = V_0 \,.\, u(t)$. In the series *RC* circuit shown in Fig. 3.4, the integro-differential equation is

$$\frac{1}{C}\int_{-\infty}^{t} i(t)\,dt + Ri(t) = V_0\, u(t) \tag{3.28}$$

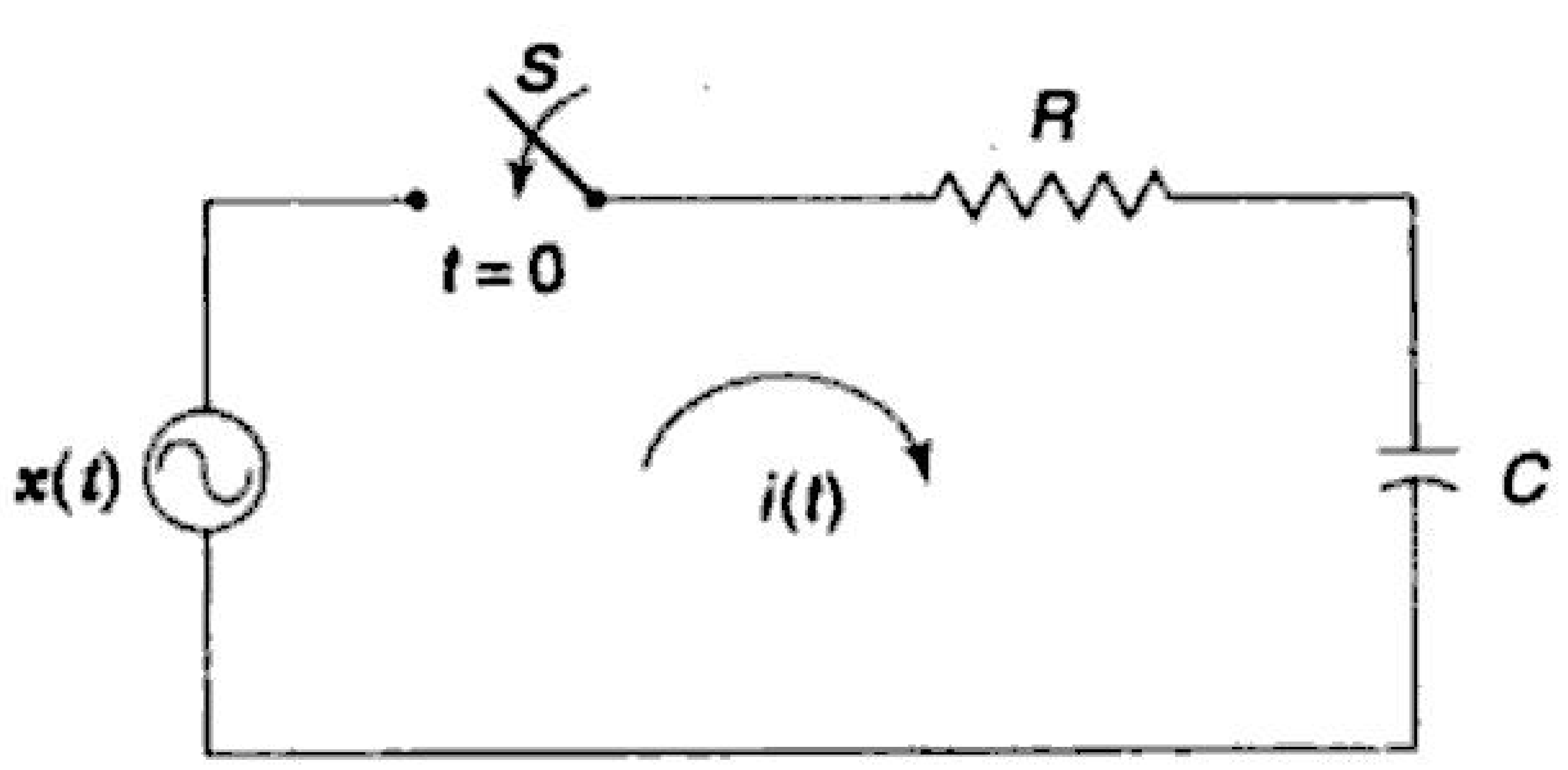

Fig. 3.4 *Series R-C circuit*

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Solution Applying Kirchoff's voltage law, the loop equation can be written as

$$2i(t) + 1\,\frac{di(t)}{dt} = 10\sin 25t$$

Taking Laplace transform, we get

$$2I(s) + [sI(s) - i(0)] = 10 \times \frac{25}{s^2 + (25)^2}$$

where $i(0)$ is the initial current passing through the circuit. As the inductor does not allow sudden changes in currents, $i(0) = 0$.

Therefore, $$sI(s) + 2I(s) = \frac{10 \times 25}{s^2 + (25)^2}$$

$$I(s) = \frac{250}{\left(s^2 + 625\right)(s + 2)}$$

Using partial fractions, the above equation can be expanded as

$$I(s) = \frac{250}{(s + 2)(s + j25)(s - j25)}$$

$$I(s) = \left[\frac{A_1}{s + 2} + \frac{A_2}{s + j25} + \frac{A_3}{s - j25}\right]$$

where $$A_1 = (s + 2)\,I(s)\big|_{s=-2}$$

$$= \left.\frac{250}{\left[s^2 + (25)^2\right]}\right|_{s=-2}$$

$$= \frac{250}{629}$$

$$A_2 = (s + j25)\,I(s)\big|_{s=-j25}$$

$$= \left.\frac{250}{(s + 2)(s - j25)}\right|_{s=-j25}$$

$$= \frac{250}{(2 - j25)(-j50)} = \frac{-5}{(25 + j2)}$$

$$A_3 = (s - j25)\,I(s)\big|_{s=j25}$$

$$= \left.\frac{250}{(s + 2)(s + j25)}\right|_{s=j25}$$

$$= \frac{250}{(2 + j25)(j50)} = \frac{-5}{(25 - j2)}$$

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$$A_0 = F(s)s^2|_{s=0} = \left.\frac{1}{s+1}\right|_{s=0} = 1$$

$$A_1 = \left.\frac{1}{ds}\frac{1}{s+1}\right|_{s=0} = -\frac{1}{(s+1)^2} = -1$$

$$A_2 = (s+1)\,F(s)|_{s=-1} = \left.\frac{1}{s^2}\right|_{s=-1} = 1$$

Therefore, $$\frac{1}{s^2(s+1)} = \frac{1}{s^2} - \frac{1}{s} + \frac{1}{(s+1)}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)}\right\} = t - 1 + e^{-t}$$

Therefore, $$i(t) = \mathcal{L}^{-1}[I(s)] = \mathcal{L}^{-1}\left\{\frac{1-2e^{-s}+e^{-2s}}{2s^2(s+1)}\right\}$$

$$= \frac{1}{2}\left[t-1+e^{-t}\right][u(t) - 2u(t-1) + u(t-2)]$$

Example 3.22 For the circuit shown in Fig. E3.22, determine the resultant current $i(t)$ when the switch is moved from position 1 to position 2 at $t = 0$. Initially the switch has been at position 1 for a long time to get the steady state values.

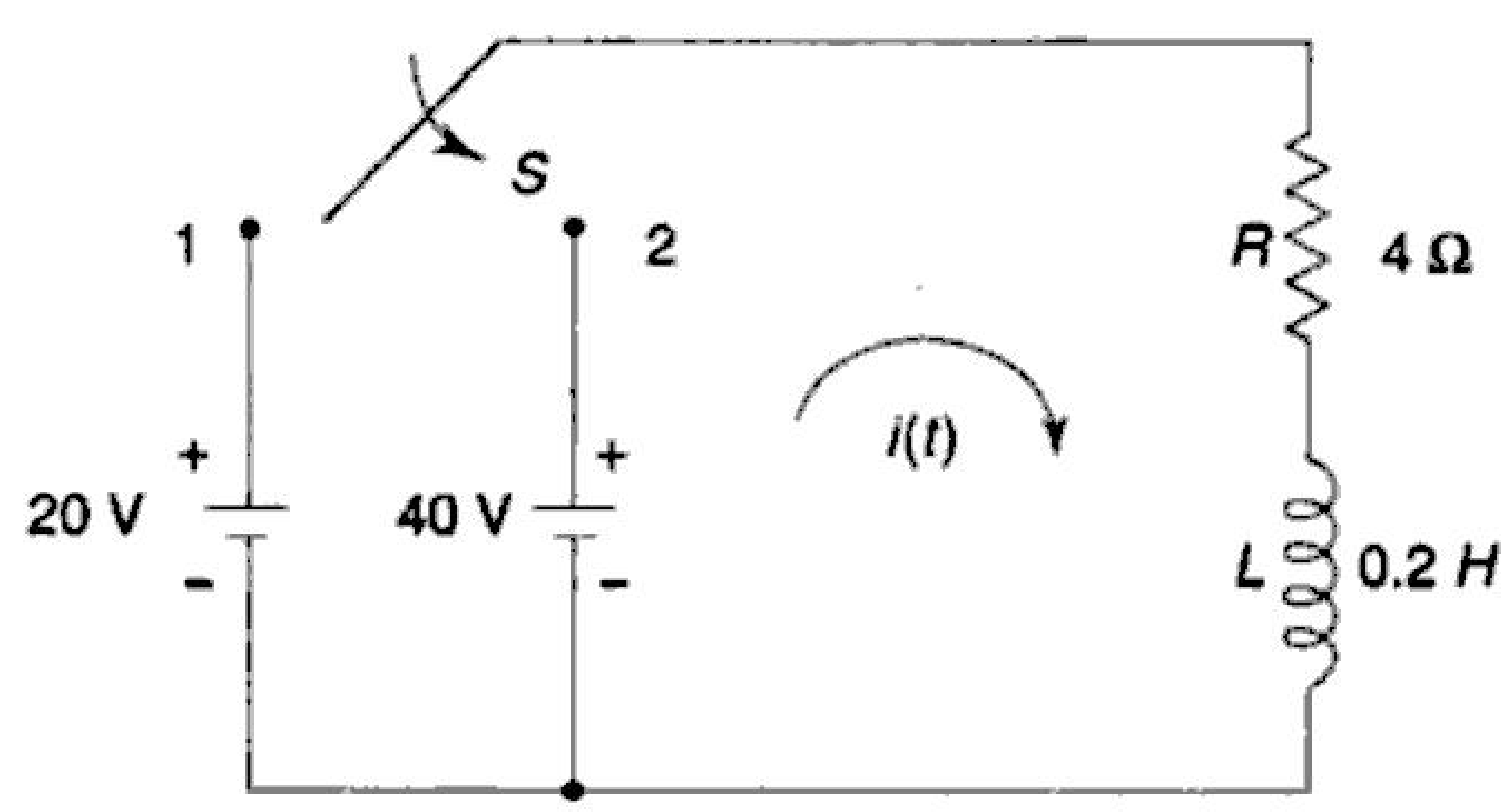

Fig. E3.22

Solution Let us consider the switch be at position 2. By applying Kirchhoff's law, we have

$$0.2\frac{di(t)}{dt} + 4i(t) = 40$$

Taking Laplace transform on both sides, we get

$$0.2[sI(s) - i(0)] + 4I(s) = \frac{40}{s}$$

$i(0)$ is the initial current passing through the circuit just after the switch is at position 2. Since the inductor does not allow sudden

i e., $i(t) = \dfrac{\omega}{1+\omega^2 R^2 C^2}[C\cos\omega t + \omega C^2 R\sin\omega t - Ce^{-t/RC}]$

Example 3.28 A periodic waveform shown in Fig. E3.28(a) is applied to the RC network of Fig. E3.28(b). Find the transient current and periodic or steady-state current.

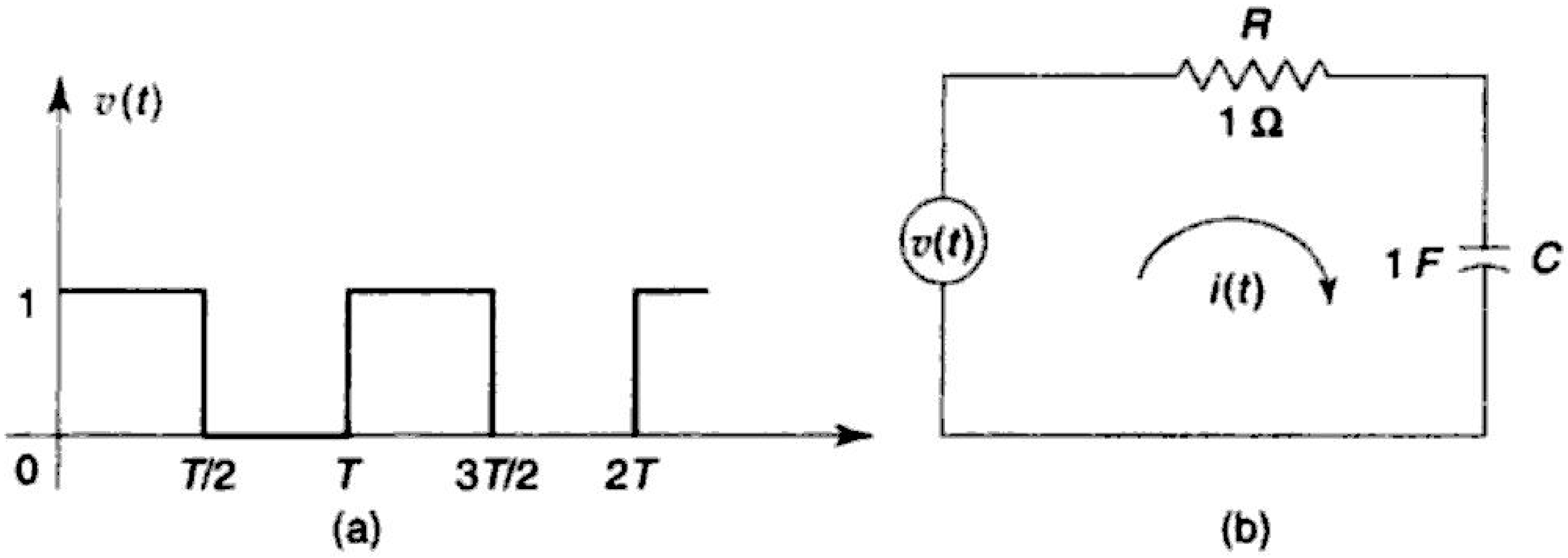

Fig. E3.28

Solution The function for the first period of the given waveform is

$$v(t) = 1 \quad \text{for} \quad 0 < t \le T/2$$
$$= 0 \quad \text{for } T/2 < t \le T$$

$$V(s) = \frac{1}{1-e^{-sT}}\int_0^T f(t)\,dt$$

$$= \frac{1}{1-e^{-sT}}\left[\int_0^{T/2} 1\cdot e^{-st}\,dt + \int_{T/2}^{T} 0\cdot e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-sT}}\left[\frac{e^{-st}}{-s}\right]_0^{T/2}$$

$$= \frac{1}{1-e^{-sT}}\left[\frac{1-e^{-sT/2}}{s}\right]$$

Alternate method to find Laplace transform of the given periodic waveform

The input periodic pulse train can be represented as

$$v(t)= u(t)-u(t-T/2) + u(t-T)-u(t-3T/2) + u(t-2T)-u(t-5T/2) + \ldots$$

Its Laplace transform is

$$V(s) = \frac{1}{s}[1-e^{-sT/2} + e^{-sT} - e^{-3sT/2} + e^{-2sT} - e^{-5sT/2} + \ldots]$$

$$= \frac{1}{s}[1-e^{-sT/2} + e^{-sT}(1-e^{-sT/2}) + e^{-2sT}(1-e^{-sT/2})]$$

Comparing the coefficients of s, we get

$$A_1 + A_2 + A_3 = 0$$
$$1 - 1 + A_3 = 0$$
$$A_3 = 0$$

Therefore, $H(s) = \dfrac{1}{(s+1)\left(s^2+s+1\right)} = \dfrac{1}{s+1} - \dfrac{s}{s^2+s+1}$

We know that $Y(s) = H(s)\,X(s)$

$$Y(s) = \left(\frac{1}{s+1} - \frac{s}{s^2+s+1}\right)\left(\frac{1}{s} + \frac{1}{s+3} - \frac{1}{s+1}\right)$$

$$= \frac{1}{s(s+1)} + \frac{1}{(s+1)(s+3)} - \frac{1}{(s+1)^2}$$

$$- \frac{1}{\left(s^2+s+1\right)} - \frac{s}{(s+3)\left(s^2+s+1\right)} + \frac{s}{(s+1)\left(s^2+s+1\right)}$$

By using partial fraction expansions, the above functions can be expanded as :

(i) $$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

(ii) $$\frac{1}{(s+1)(s+3)} = \frac{1}{2(s+1)} - \frac{1}{2(s+3)}$$

(iii) $$\frac{s}{(s+3)\left(s^2+s+1\right)} = \frac{-3/7}{s+3} + \frac{3/7s+1/7}{s^2+s+1}$$

(iv) $$\frac{s}{(s+1)\left(s^2+s+1\right)} = -\frac{1}{s+1} + \frac{s+1}{s^2+s+1}$$

Therefore, $Y(s) = \dfrac{1}{s} - \dfrac{1}{s+1} + \dfrac{1}{2(s+1)} - \dfrac{1}{2(s+3)}$

$$- \frac{1}{(s+1)^2} - \frac{1}{s^2+s+1} + \frac{3/7}{s+3} - \frac{3/7s+1/7}{s^2+s+1}$$

$$- \frac{1}{s+1} + \frac{s+1}{s^2+s+1}$$

where $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+s+1}\right\} = \mathcal{L}^{-1}\left\{\dfrac{s+\frac{1}{2}-\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\} - \frac{1}{\sqrt{3}}\,\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\}$$

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3.44 *A sinusoidal voltage $25\sin t$ is applied at the instant $t = 0$ to an RL circuit with $R = 5\,\Omega$ and $L = 1H$. Determine $i(t)$ by using Laplace transform method.*

3.45 *In the circuit shown in Fig. Q3.45, the steady state condition exists with the switch in position 1. The switch is moved to position 2 at $t = 0$. Calculate the current through the coil at the switching instant and current for all values $t > 0$.*

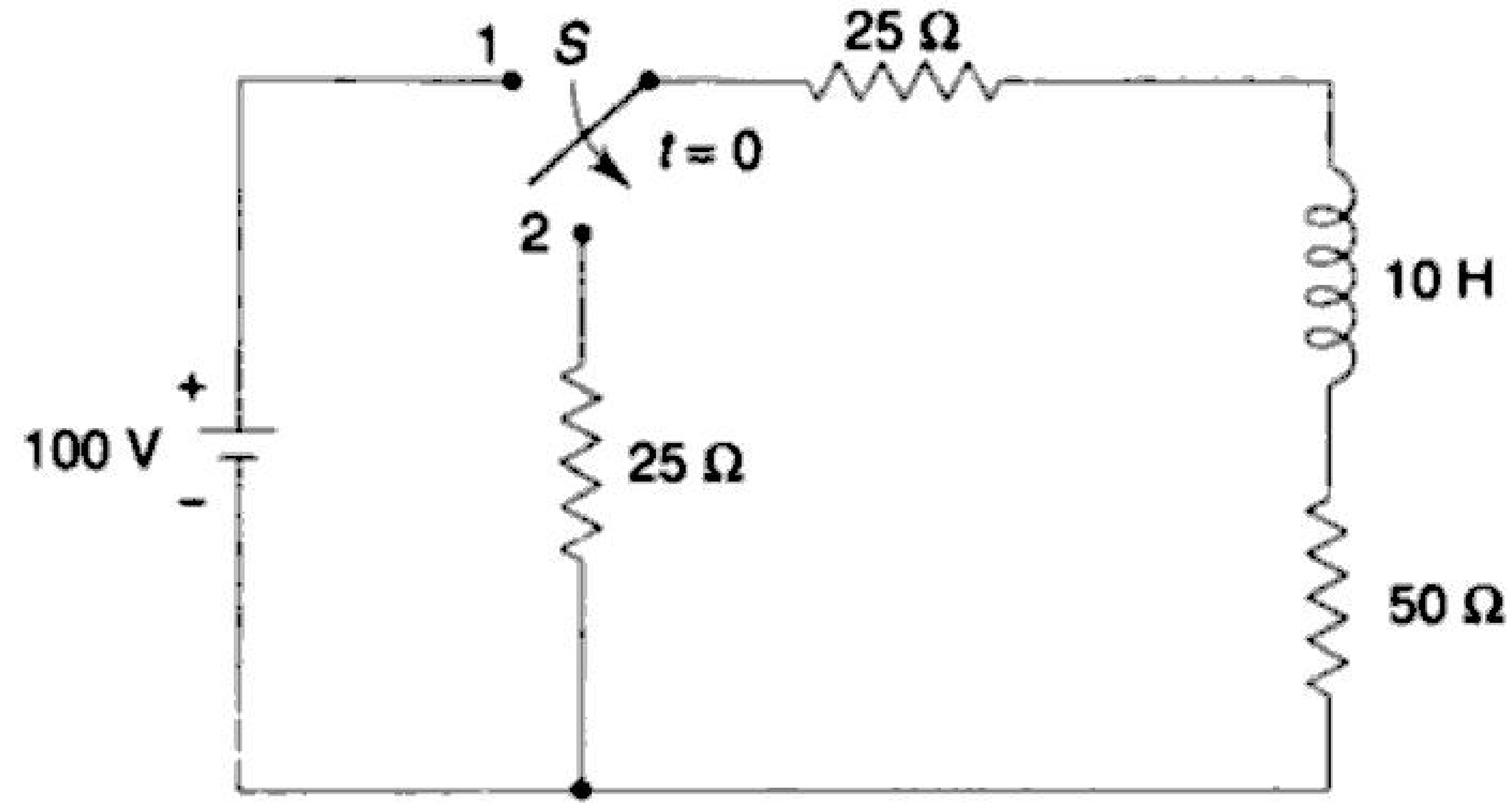


Fig. Q3.45

3.46 *In the circuit of Fig. Q 3.46, the switch S is closed and steady-state conditions have been reached. At $t = 0$, the switch S is opened. Obtain the expression for the current through the inductor.*

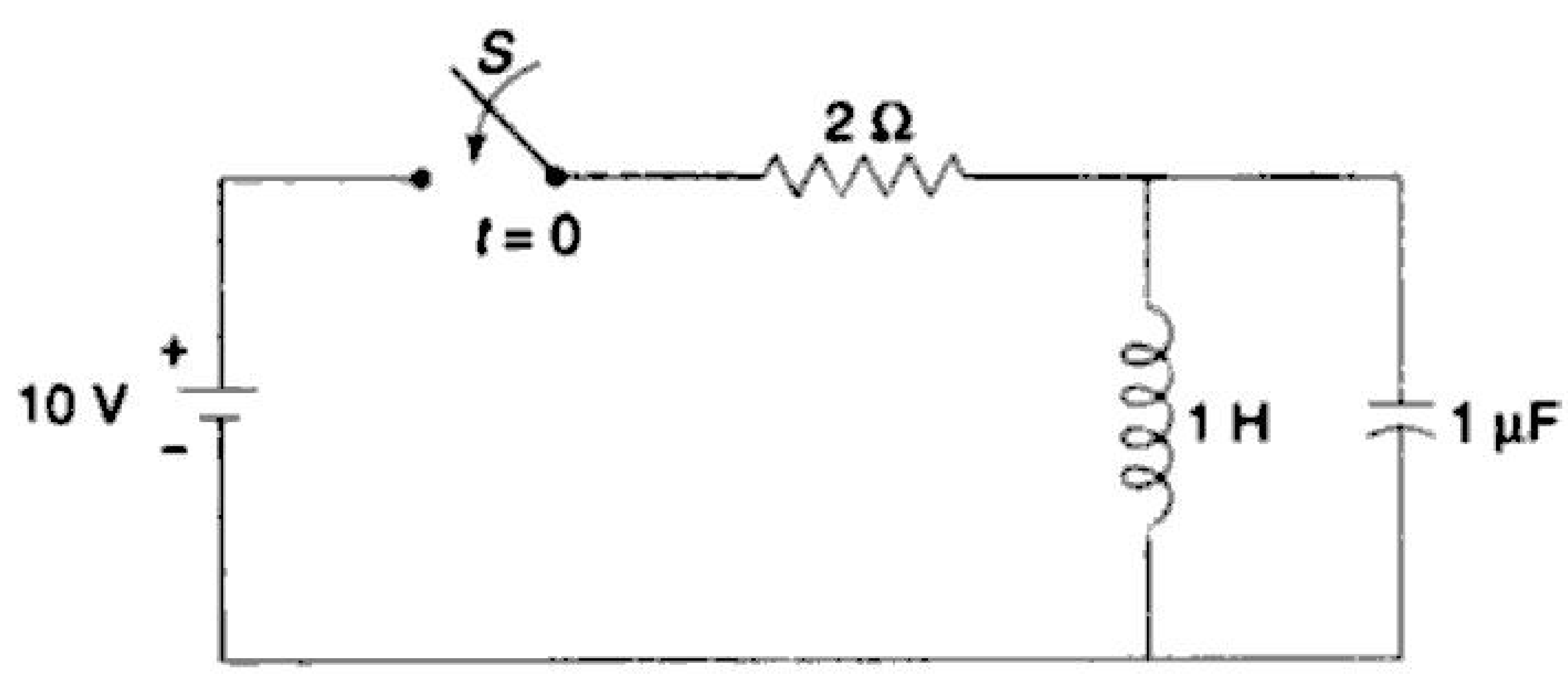


Fig. Q3.46

Ans : $5\cos 1000t$

3.47 *In the circuit of Fig. Q3.47, the switch S is closed at $t = 0$ after the switch is kept open for a long time. Determine the voltage across the capacitor.*

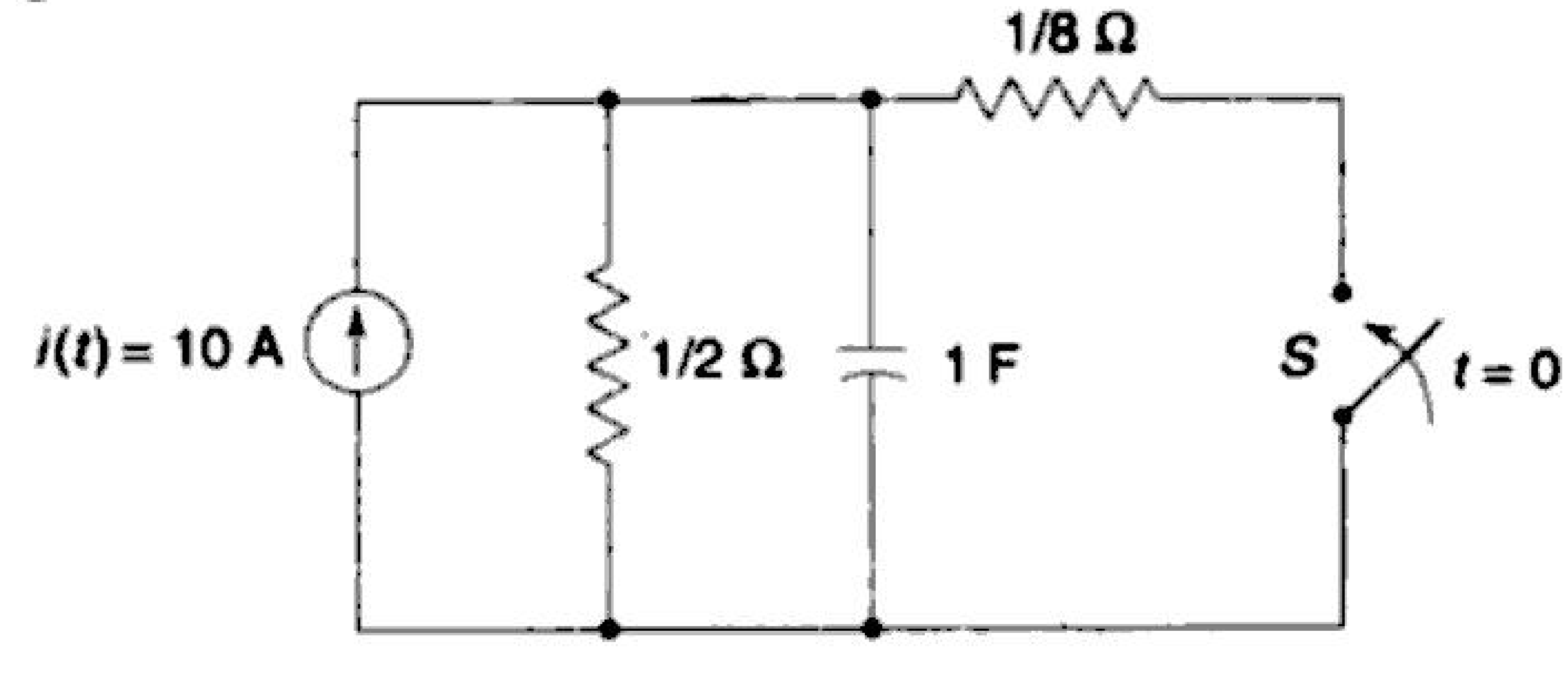


Fig. Q3.47

The analysis of any sampled signal or sampled data system in the frequency domain is extremely difficult using s-plane representation because the signal or system equations will contain infinite long polynomials due to the characteristic infinite number of poles and zeros. Fortunately, this problem may be overcome by using the z-transform, which reduces the poles and the zeros to a finite number in the z-plane.

The purpose of the z-transform is to map (transform) any point $s = \pm\sigma \pm j\omega$ in the s-plane to a corresponding point $z(r \angle \theta)$ in the z-plane by the relationship

$$z = e^{sT}, \quad \text{where } T \text{ is the sampling period (seconds)}$$

Table 4.1

$\sigma = 0,\ \omega_s = \dfrac{2\pi}{T}$									
$j\omega$	0	$\omega_s/8$	$\omega_s/4$	$3\omega_s/8$	$\omega_s/2$	$5\omega_s/8$	$3\omega_s/4$	$7\omega_s/8$	ω_s
$Z = 1\angle\omega T$	$1\angle 0°$	$1\angle 45°$	$1\angle 90°$	$1\angle 135°$	$1\angle 180°$	$1\angle 225°$	$1\angle 270°$	$1\angle 315°$	$1\angle 360°$

Under this mapping, the imaginary axis, $\sigma = 0$ maps on to the unit circle $|z| = 1$ in the z-plane. Also, the left hand half-plane $\sigma < 0$ corresponds to the interior of the unit circle $|z| = 1$ in the z-plane. This correspondence is shown in Fig. 4.1.

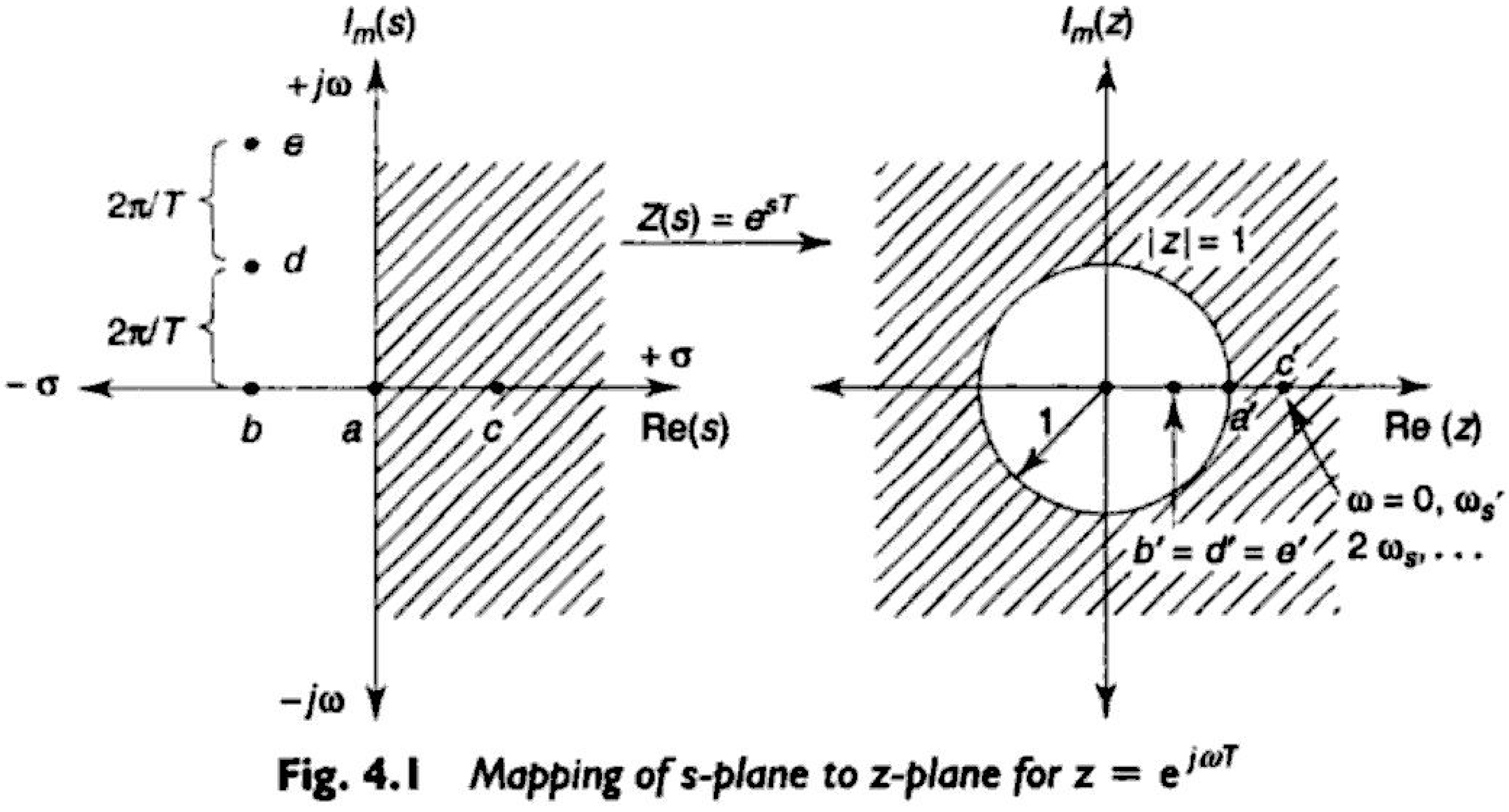


Fig. 4.1 *Mapping of s-plane to z-plane for $z = e^{j\omega T}$*

Considering that the real part of x is zero, i.e. $\sigma = 0$, we have $z = e^{j\omega T} = 1 \angle \pm j\omega T$, which gives the values of z (in polar form) shown as in Table 4.1.

We know that the Laplace transform gives

$$\mathcal{L}[x^*(t)] = X(s) = \sum_{n=0}^{\infty} x(nT)e^{-nsT}$$

(vi) If $x(n)$ is two-sided, and if the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z| = r_0$. That is, the ROC includes the intersection of the ROC's of the components.

(vii) If $X(z)$ is rational, then the ROC extends to infinity, i.e. the ROC is bounded by poles.

(viii) If $x(n)$ is causal, then the ROC includes $z = \infty$.

(ix) If $x(n)$ is anti-causal, then the ROC includes $z = 0$.

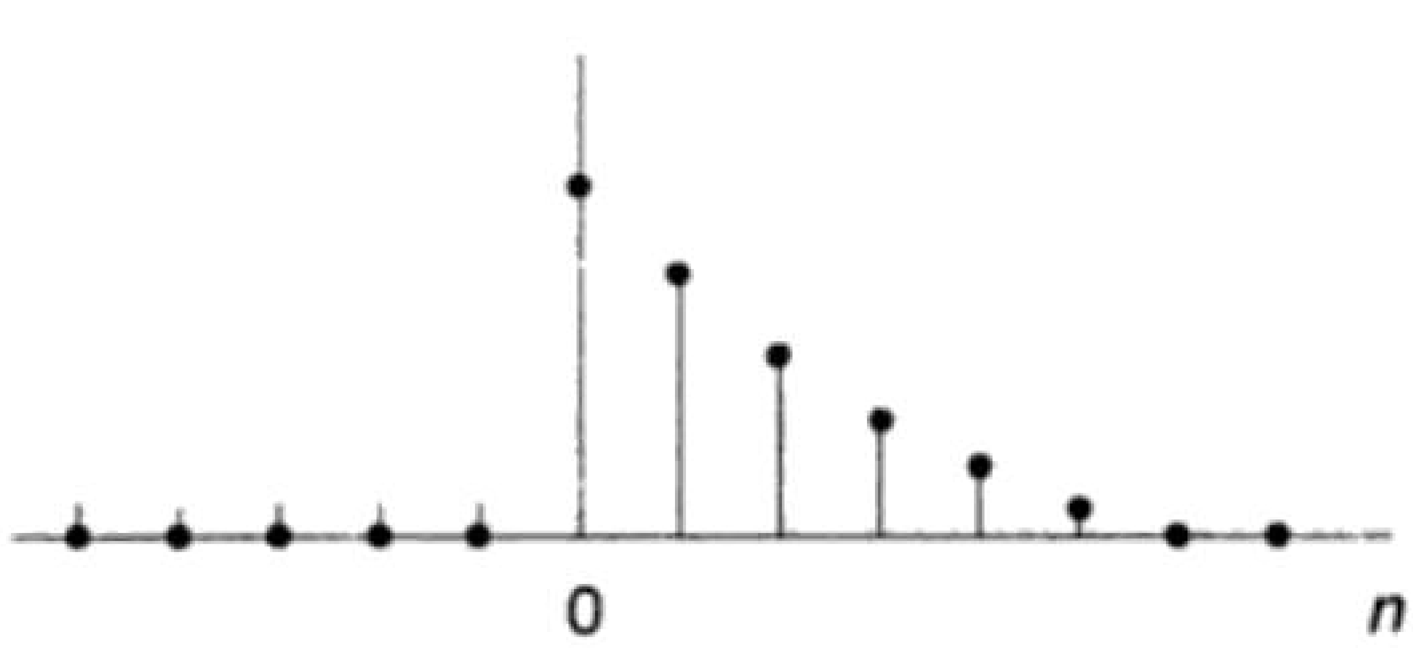
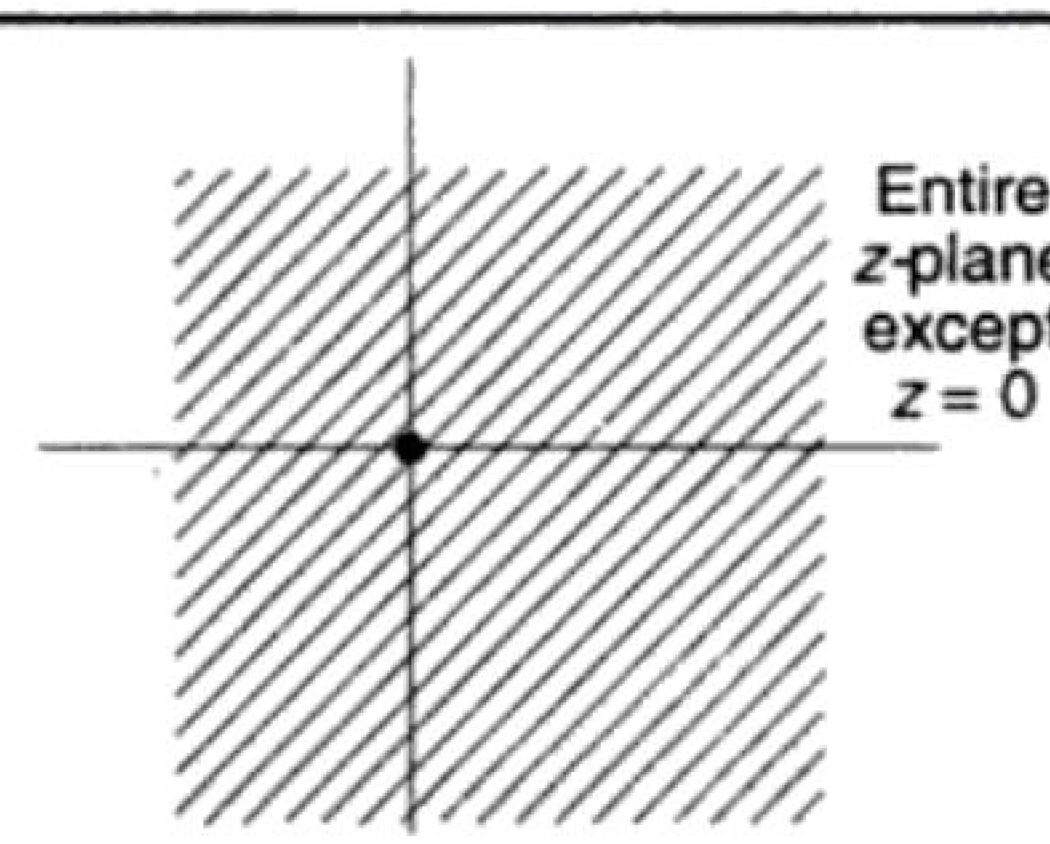
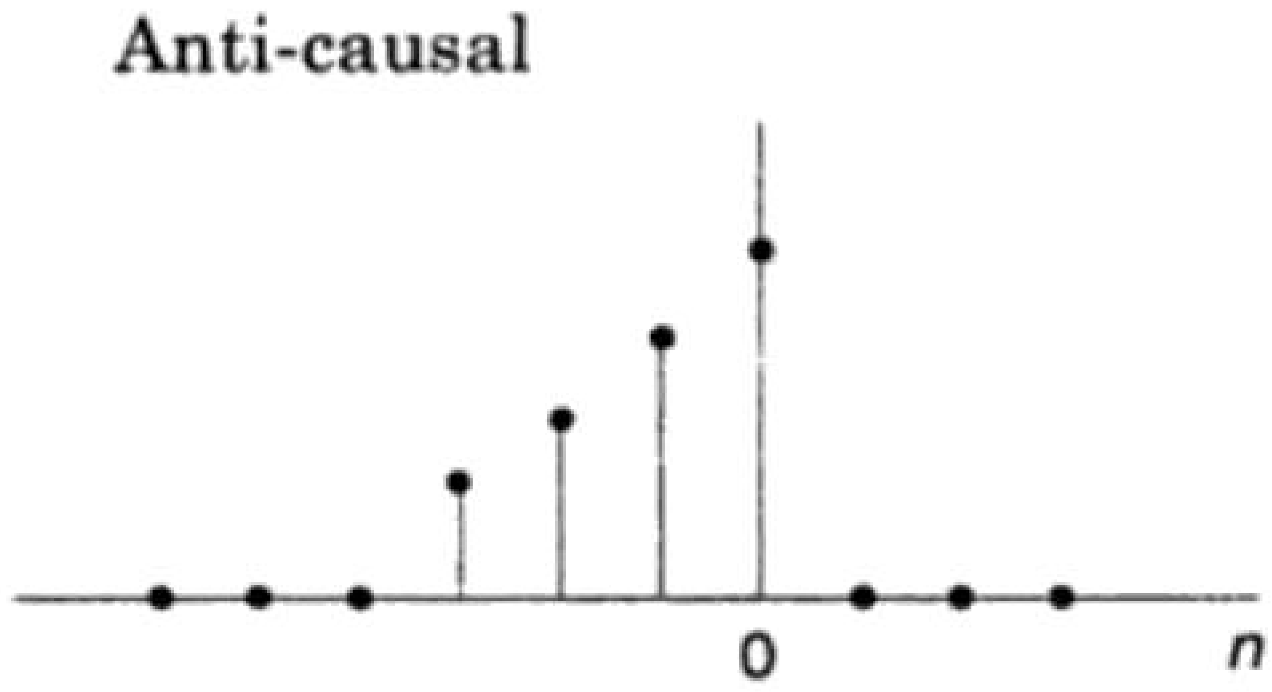
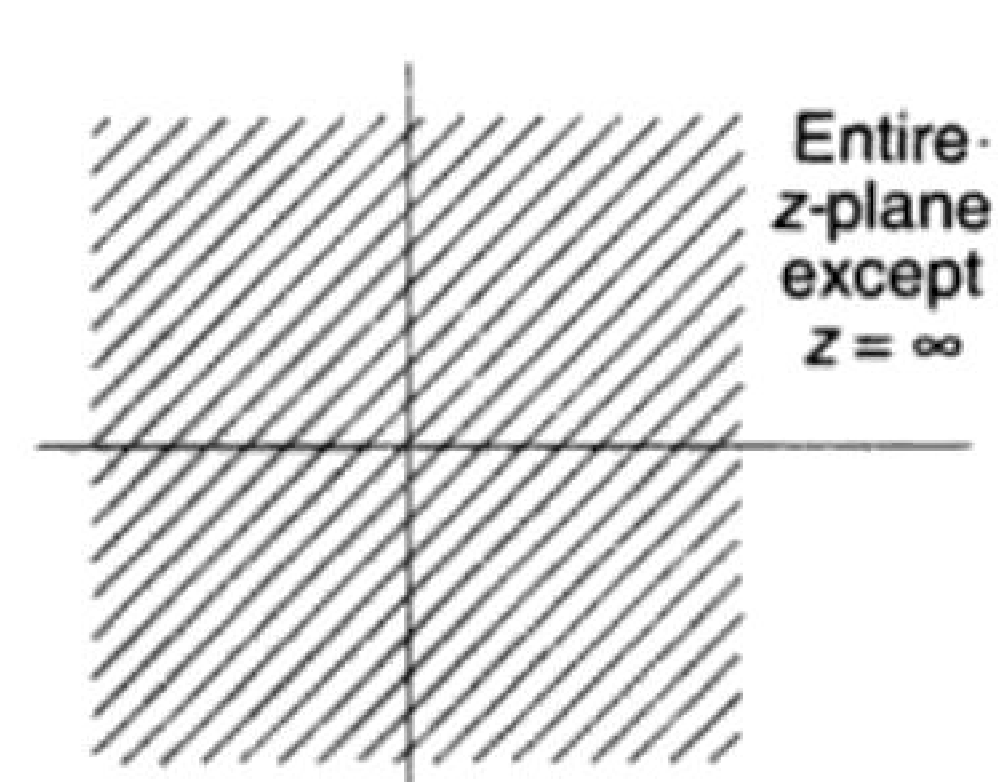
To determine the ROC for the series expressed by the Eq. 4.2, which is called a two-sided signal z-transform, this equation can be written as

$$\sum_{n=-\infty}^{\infty} x(n)\, r^{-n} = \sum_{n=-\infty}^{-1} x(n)\, z^{-n} + \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=1}^{\infty} x(-n)\, z^{+n} + \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

The first series, a non-causal sequence, converges for $|z| < r_2$, and the second series, a causal sequence, converges for $|z| > r_1$, resulting in an annular region of convergence. Then the Eq. 4.2 converges for $r_1 < |z| < r_2$. provided $r_1 < r_2$. The causal, anti-causal and two-sided signals with their corresponding ROCs are shown in Table 4.2. Some important commonly used z-transform pairs are given in Table 4.3.

Table 4.2 *The Causal, anti-causal and two-sided signals and their ROCs*

Signals	*ROCs*
(a) Finite duration signals	
Causal 	
Anti-causal 	

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(d) $x(n) = \{0, 0, 1, 2, 5, 4, 0, 1\}$

Taking z-transform, we get

$$X(z) = z^{-2} + 2z^{-3} + 5z^{-4} + 4z^{-5} + z^{-7}.$$

ROC: Entire z-plane except $z = 0$.

(e) $x(n) = \delta(n)$, hence $X(z) = 1$, ROC: Entire z-plane.

(f) $x(n) = \delta(n - k)$, $k > 0$, hence $X(z) = z^{-k}$, ROC: Entire z-plane except $z = 0$

(g) $x(n) = \delta(n + k)$, $k > 0$, hence $X(z) = z^{k}$, ROC: Entire z-plane except $z = \infty$.

Example 4.3 Determine the z-transform including the region of convergence of

$$x(n) = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Solution The z-transform for the given $x(n)$ is

$$X(z) = Z[a^n] = \sum_{n=-\infty}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

We know that $\sum_{0}^{\infty} a^n = \dfrac{1}{1-a}, |a| < 1$

Hence, $X(z) = \dfrac{1}{1 - az^{-1}} = \dfrac{z}{z - a}$

This converges when $|az^{-1}| < 1$ or $|z| > |a|$. Values of z for which $X(z) = 0$ are called zeros of $X(z)$, and values of z for which $X(z) \to \infty$ are called poles of $X(z)$.

Here the poles are at $z = a$ and zeros at $z = 0$. The region of convergence is shown in Fig. E 4.3.

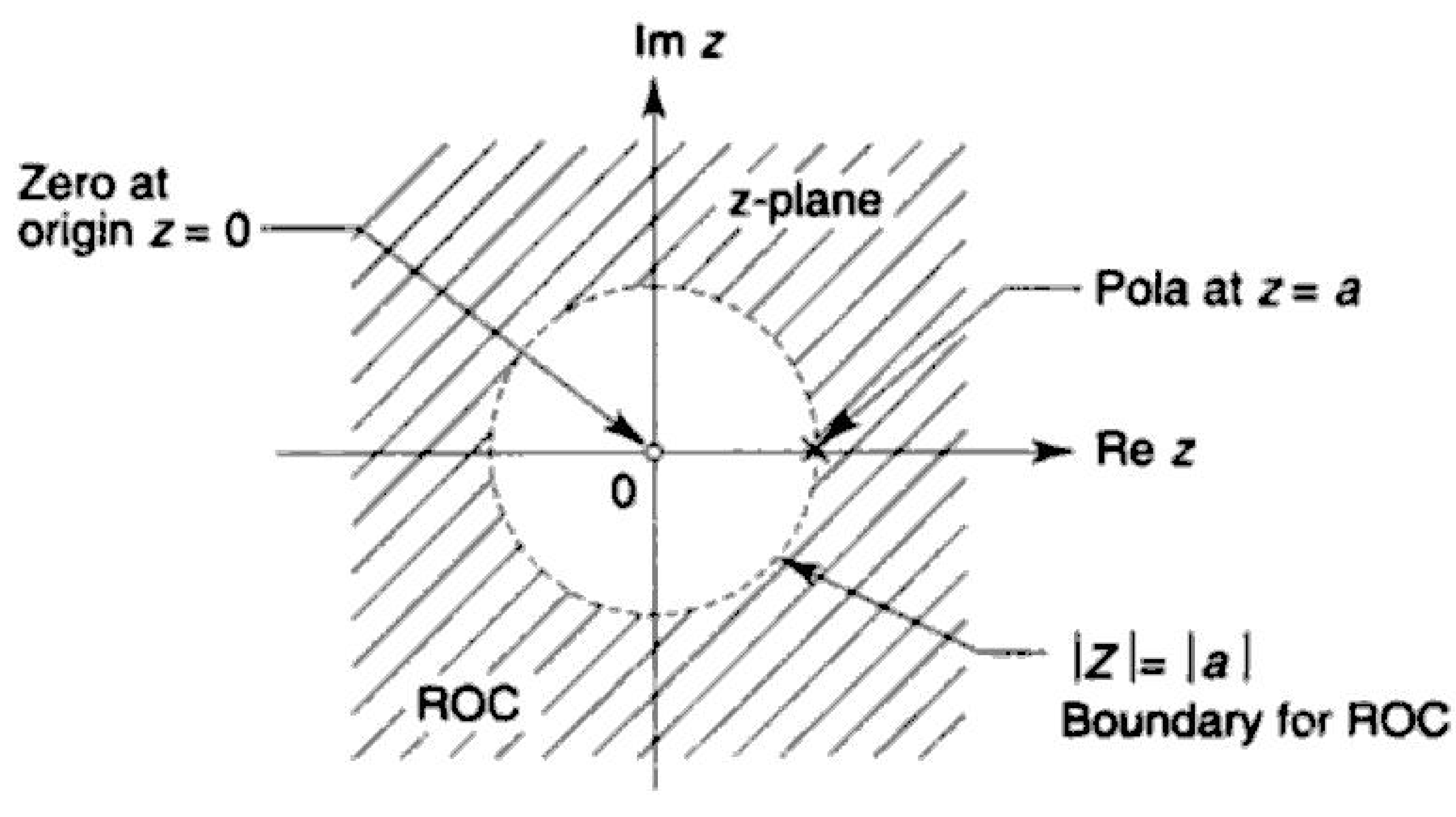

Fig. E 4.3 *ROC for the z-transform of x (n) = a^n.*

The ROC of $z^{-k} X(z)$ is the same as that of $X(z)$ except for $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$.

Example 4.8 By applying the time shifting property, determine the z-transform of the signal

$$X(z) = \frac{z^{-1}}{1 - 3z^{-1}}$$

Solution

$$X(z) = \frac{z^{-1}}{1 - 3z^{-1}} = z^{-1} X_1(z)$$

where $X_1(z) = \dfrac{1}{1 - 3z^{-1}}$

Here, from the time shifting property, we have $k = 1$ and $x(n) = (3)^n u(n)$

Hence $x(n) = (3)^{n-1} u(n-1)$

Example 4.9 Find $x(n)$

if $$X(z) = \frac{1 + \frac{1}{2}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

Solution Given $$X(z) = \frac{1 + \frac{1}{2}z^{-1}}{1 - \frac{1}{2}z^{-1}} = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{\frac{1}{2}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

Therefore, $$x(n) = Z^{-1}\left[\frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{2}\frac{z^{-1}}{1 - \frac{1}{2}z^{-1}}\right]$$

$$= \left(\frac{1}{2}\right)^n u(n) + \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} u(n-1)$$

$$= \left(\frac{1}{2}\right)^n [u(n) + u(n-1)]$$

$$= \left(\frac{1}{2}\right)^n [u(n) - u(n-1) + 2u(n-1)]$$

$$= \left(\frac{1}{2}\right)^n [\delta(n) + 2u(n-1)]$$

Taking inverse z-transform, we get

$$x_1(n) = z^{-1}\left[\frac{1}{\left(1-\frac{1}{2}z^{-1}\right)}\right] = \left(\frac{1}{2}\right)^n u(n)$$

$$x_2(n) = z^{-1}\left[\frac{1}{\left(1+\frac{1}{4}z^{-1}\right)}\right] = \left(-\frac{1}{4}\right)^n u(n)$$

Therefore, $x(n) = x_1(n)*x_2(n) = \sum_{k=0}^{n} x_1(n-k)x_2(k)$

$$= \sum_{k=0}^{n}\left(\frac{1}{2}\right)^{n-k}\left(\frac{-1}{4}\right)^k$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n}\left[\frac{(-1/4)}{(1/2)}\right]^k$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n}\left(-\frac{1}{2}\right)^k = \left(\frac{1}{2}\right)^n \frac{1-\left(-\frac{1}{2}\right)^{n+1}}{1-\left(-\frac{1}{2}\right)}$$

$$= \left(\frac{1}{2}\right)^n \cdot \frac{2}{3}\left[1-\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)^n\right] = \left[\frac{2}{3}\left(\frac{1}{2}\right)^n + \frac{1}{3}\left(\frac{-1}{4}\right)^n\right]u(n)$$

4.3.7 Correlation

If $x_1(n) \xleftrightarrow{z} X_1(z)$ and $x_2(n) \xleftrightarrow{z} X_2(z)$, then,

$$r_{x_1x_2}(l) = \sum_{n=-\infty}^{\infty} x_1(n)x_2(n-l) \xleftrightarrow{z} R_{x_1x_2}(z) = X_1(z)X_2(z^{-1}) \qquad (4.11)$$

Example 4.16 Determine the cross-correlation sequence $r_{x_1x_2}(l)$ of the sequences:

$x_1(n) = (1, 2, 3, 4)$

$x_2(n) = (4, 3, 2, 1)$

Solution Cross-correlation sequence can be obtained using the correlation property of z-transform, given in Eq. 4.11. Hence, for the given $x_1(n)$ and $x_2(n)$,

$X_1(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$

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with the coefficients representing the sequence values in the time domain, namely

$$X(z) = \frac{N(z)}{D(z)} = \sum_{n=0}^{\infty} a_n z^{-n} = a z^0 + a_1 z^{-1} + a_2 z^{-2} + \cdots$$

where the coefficients a_n are the values of $x(n)$.

From the above equation it is clear that expansion does not result in a closed form solution. Hence, if $X(z)$ can be expanded in a power series, the coefficients represent the inverse sequence values. Thus, the coefficient of z^{-K} is the K^{th} term in the sequence. The region of convergence will determine whether the series has positive or negative exponents. For right hand sequences, called causal sequences will have primarily negative exponents, while left hand sequences the anti-causal sequences will have positive exponents. For annular regions of convergence, a Laurent expansion will give both the positive and negative exponents.

This method is only useful for having a quick look at the first few samples of the corresponding signals.

Example 4.19 A system has an impulse response $h(n) = \{1, 2, 3\}$ and output response

$y(n) = \{1, 1, 2, -1, 3\}$. Determine the input sequence $x(n)$.

Solution Performing the z-transform of $h(n)$ and $y(n)$, we have

$$H(z) = Z[h(n)] = Z[1, 2, 3] = 1 + 2z^{-1} + 3z^{-2}$$

$$Y(z) = Z[y(n)] = Z[1, 1, 2, -1, 3] = 1 + z^{-1} + 2z^{-2} - z^{-3} + 3z^{-4}$$

We know that $H(z) = \dfrac{Y(z)}{X(z)}$

Therefore, $X(z) = \dfrac{Y(z)}{H(z)} = \dfrac{1 + z^{-1} + 2z^{-2} - z^{-3} + 3z^{-4}}{1 + 2z^{-1} + 3z^{-2}}$

$$\begin{array}{r|l}
 & 1 - z^{-1} + z^{-2} \\
\hline
1 + 2z^{-1} + 3z^{-2} & 1 + z^{-1} + 2z^{-2} - z^{-3} + 3z^{-4} \\
 & 1 + 2z^{-1} + 3z^{-2} \\
\hline
 & -z^{-1} - z^{-2} - z^{-3} \\
 & -z^{-1} - 2z^{-2} - 3z^{-3} \\
\hline
 & z^{-2} + 2z^{-3} + 3z^{-4} \\
 & z^{-2} + 2z^{-3} + 3z^{-4} \\
\hline
 & 0
\end{array}$$

Therefore, $X(z) = 1 - z^{-1} + z^{-2}$

Taking inverse z-transform, we get

$$x(n) = \left(\underset{\uparrow}{1}, -1, 1\right)$$

Example 4.22 Determine the causal signal $x(n)$ having the z-transform

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2}$$

Solution Expanding the given $X(z)$ in terms of the positive powers of z.

$$X(z) = \frac{z^3}{(z+1)(z-1)^2}$$

Hence $F(z) = \dfrac{X(z)}{z} = \dfrac{z^2}{(z+1)(z-1)^2} = \dfrac{A_1}{(z+1)} + \dfrac{A_2}{(z-1)} + \dfrac{A_3}{(z-1)^2}$

Here, $A_1 = (z+1)F(z)|_{z=-1} = \dfrac{z^2}{(z-1)^2}\Big|_{z=-1} = \dfrac{1}{4}$

$$A_3 = (z-1)^2 F(z)|_{z=1} = \frac{z^2}{(z+1)}\Big|_{z=1} = \frac{1}{2}$$

$$A_2 = \frac{d}{dz}\left[\frac{z^2}{(z+1)}\right]_{z=1} = \left.\frac{(z+1)2z - z^2}{(z+1)^2}\right|_{z=1} = \frac{3}{4}$$

Therefore, $F(z) = \dfrac{1}{4}\dfrac{1}{(z+1)} + \dfrac{3}{4}\dfrac{1}{(z-1)} + \dfrac{1}{2}\dfrac{1}{(z-1)^2}$

Therefore, $X(z) = \dfrac{1}{4}\dfrac{z}{(z+1)} + \dfrac{3}{4}\dfrac{z}{(z-1)} + \dfrac{1}{2}\dfrac{z}{(z-1)^2}$

Taking inverse z-transform of $X(z)$, we obtain

$$x(n) = \frac{1}{4}(-1)^n u(n) + \frac{3}{4}u(n) + \frac{1}{2}nu(n)$$

$$= \left[\frac{1}{4}(-1)^n + \frac{3}{4} + \frac{1}{2}n\right]u(n)$$

Alternate Method

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})}$$

$$= \frac{A_1}{1+z^{-1}} + \frac{A_2}{1-z^{-1}} + \frac{A_3}{(1-z^{-1})^2}$$

Equating the numerators, we get

$$1 = A_1(1-z^{-1})^2 + A_2(1+z^{-1})(1-z^{-1}) + A_3(1+z^{-1})$$

$$= A_1(1-2z^{-1}+z^{-2}) + A_2(1-z^{-2}) + A_3(1+z^{-1})$$

Here, $A_1 + A_2 + A_3 = 1$

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$$X(z) = \frac{2+3z^{-1}}{(1+z^{-1})\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

Also verify the results in each case for $0 \le n \le 3$.

Solution

(i) Long Division Method

$$X(z) = \frac{2+3z^{-1}}{1+\frac{5}{4}z^{-1}+\frac{1}{8}z^{-2}-\frac{1}{8}z^{-3}}$$

$$\begin{array}{r|l}
 & 2+\frac{1}{2}z^{-1}-\frac{7}{8}z^{-2}+\frac{41}{32}z^{-3} \\
\hline
1+\frac{5}{4}z^{-1}+\frac{1}{8}z^{-2}-\frac{1}{8}z^{-3} & 2+3z^{-1} \\
 & 2+\frac{5}{4}z^{-1}+\frac{1}{4}z^{-2}-\frac{1}{4}z^{-3} \\
\hline
 & \frac{1}{2}z^{-1}-\frac{1}{4}z^{-2}+\frac{1}{4}z^{-3} \\
 & \frac{1}{2}z^{-1}+\frac{5}{8}z^{-2}+\frac{1}{16}z^{-3}-\frac{1}{16}z^{-4} \\
\hline
 & -\frac{7}{8}z^{-2}+\frac{3}{16}z^{-3}+\frac{1}{16}z^{-4} \\
 & -\frac{7}{8}z^{-2}+\frac{35}{32}z^{-3}-\frac{7}{64}z^{-4}+\frac{7}{64}z^{-5} \\
\hline
 & \frac{41}{32}z^{-3}+\cdots
\end{array}$$

Therefore, $X(z) = 2+\frac{1}{2}z^{-1}-\frac{7}{8}z^{-2}+\frac{41}{32}z^{-3}$

Taking inverse z-transform, we get $x(n) = \left[\underset{\uparrow}{2}, \frac{1}{2}, -\frac{7}{8}, \frac{41}{32}, \cdots\right]$

(ii) Partial Fraction Expansion Method

$$X(z) = \frac{2+3z^{-1}}{(1+z^{-1})\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

$$= \frac{A_1}{1+z^{-1}} + \frac{A_2}{1+\frac{1}{2}z^{-1}} + \frac{A_3}{1-\frac{1}{4}z^{-1}}$$

$$A_1 = \left.\frac{2+3z^{-1}}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}\right|_{z^{-1}=-1} = -\frac{8}{5}$$

$$= \sum_{n=1}^{\infty} -\frac{1}{n}(-1)^n a^n z^{-n}, \; |z| > |a|$$

Taking inverse z-transform, we get

$$x(n) = \frac{(-1)^{n+1} a^n}{n} u(n-1)$$

Example 4.29 For a low-pass *RC* network (R = 1 MΩ and C = 1μF) shown in Fig. E4.29, determine the equivalent discrete time expressions for the circuit output response $y(n)$, when the input is $x(t) = e^{-2t}$ and the sampling frequency is f_s = 50 Hz.

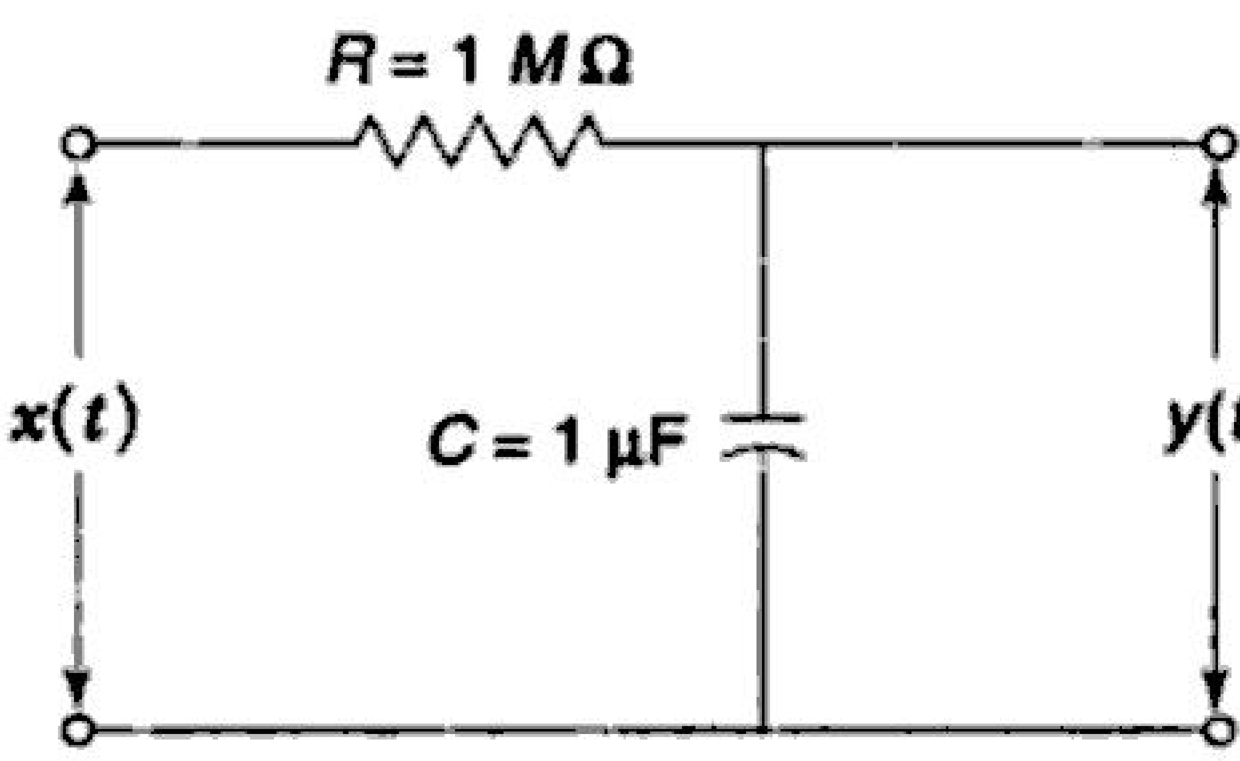

Fig. E4.29

Solution The transfer function of the given circuit in the s-domain can be expressed as

$$H(s) = \frac{1/RC}{s + 1/RC} = \frac{1}{s+1}$$

Taking inverse Laplace transform, we get

$$h(t) = e^{-t}$$

z-domain approach:
Using Table 4.3, the above transfer function may be expressed in z-plane as

$$H(z) = \frac{z}{z - e^{-T}}$$

Also, the given input function $x(t) = e^{-2t}$ may be expressed in the z-plane as

$$X(z) = \frac{z}{z - e^{-2T}}$$

We know that the output function $Y(z) = H(z)\,X(z)$.
Therefore,

$$Y(z) = \frac{z}{(z - e^{-T})} \cdot \frac{z}{(z - e^{-2T})}$$

$$= 2 \quad \text{for} \quad 6 \le n \le 10$$
$$= 3 \quad \text{for} \quad n > 10$$

Ans: $X(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + 2(z^{-6} + z^{-7} + z^{-8} + \dots + z^{-10}) + 3(z^{-11} + z^{-12} + \dots)$

4.24 *Determine the z-transform of the following sequences*

(a) $u(n-4)$ Ans: $\dfrac{z^{-4}}{1-z^{-1}}, |z| > 1$

(b) $e^{jn\pi/4}u(n)$ Ans: $\dfrac{1}{1-e^{jn\pi/4}z^{-1}}$

(c) $\delta(n-5)$ Ans: z^{-5}

(d) $\left(\dfrac{1}{3}\right)^n u(-n)$ Ans: $\dfrac{1}{1-3z}, |z| < \dfrac{1}{3}$

(e) $3^n u(n-2)$ Ans: $\dfrac{9z^{-2}}{1-3z^{-1}}$

4.25 *Find the z-transform of the sequence* $x(n) = na^n u(n)$

Ans: $X(z) = \dfrac{az^{-1}}{(1-az^{-1})^2}, |z| > |a|$

4.26 *Find the two-sided z-transform of*

$$x(n) = (1/3)^n \quad n \ge 0$$
$$= (-2)^n \quad n \le -1$$

Ans: $X(z) = \dfrac{z}{z-\dfrac{1}{3}} - \dfrac{z}{z+2}$

4.27 *Use convolution to find* $x(n)$ *if* $X(z)$ *is given by*

$$X(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}$$

Ans: $x(n) = \dfrac{2}{3}\left(\dfrac{1}{2}\right)^n u(n) + \dfrac{1}{3}\left(-\dfrac{1}{4}\right)^n u(n)$

4.28 *Find* $y(n)$ *using the convolution property of z-transform when*

$$x(n) = \{1, 2, 3, 1, -1, 1\} \text{ and } h(n) = \{1, 1, 1\}$$

Ans: $y(n) = \left\{\begin{matrix} 1, 3, 6, 6, 3, 1, 0, 1 \\ \uparrow \qquad\qquad\qquad\quad \end{matrix}\right\}$

4.29 *Convolve the sequences* $x(n)$ *and* $h(n)$ *where*

$$x(n) = 0, \ n < 0$$
$$= a^n, \ n \ge 0$$

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4.45 *Find the causal signal x(n) for* $X(z) = \dfrac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}}$

Ans: $x(n) = \sqrt{10}\left(\dfrac{1}{\sqrt{2}}\right)^n \cos\left(\dfrac{\pi n}{4} - 71.565°\right)u(n)$

4.46 *Find the inverse z-transform of*

$$X(z) = \frac{z}{3z^2 - 4z + 1}$$

where the ROC is (i) $|z| > 1$ *and (ii)* $|z| < \dfrac{1}{3}$ *using the long division method.*

Ans: (i) $x(n) = \left\{\underset{\uparrow}{0}, \dfrac{1}{3}, \dfrac{4}{9}, \dfrac{13}{27}, \dfrac{40}{81}, \cdots\right\}$

(ii) $x(n) = \left\{\cdots 121, 40, 13, 4, 1, \underset{\uparrow}{0}\right\}$

4.47 *Using long division, determine the inverse z-transform of*

$$X(z) = \frac{1+2z^{-1}}{1-2z^{-1}+z^{-2}}$$

if (a) x(n) is causal and (b) x(n) is anti-causal

Ans: (a) $x(n) = \left\{\underset{\uparrow}{1}, 4, 7, 10, 13, \cdots\right\}$

(b) $x(n) = \left\{\cdots 14, 11, 8, 5, 2, \underset{\uparrow}{0}\right\}$

4.48 *Determine the causal signal x(n) having the z-transform*

$X(z) = \dfrac{z^2+z}{\left(z-\dfrac{1}{2}\right)^3\left(z-\dfrac{1}{4}\right)}$ *for the region of convergence* $|z| > \dfrac{1}{2}$

Ans: $x(n) = \left\{80\left(\dfrac{1}{2}\right)^n - 20n\left(\dfrac{1}{2}\right)^{n-1} + 6[n(n-1)/2]\left(\dfrac{1}{2}\right)^{n-2} - 80\left(\dfrac{1}{4}\right)^n\right\}u(n)$

4.49 *Using (i) the long division method, (ii) partial fraction method and (iii) residue method, find x(n) and verify the results in each case for n in the range* $0 \le n \le 3$.

(a) $X(z) = \dfrac{z+3}{z-0.25}$

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This expression gives the output response $y(n)$ of the LTI system as a function of the input signal $x(n)$ and the unit impulse (sample) response $h(n)$ and is referred to as a **convolution sum**.

5.1.2 Unit Step Response [$u(n)$]

The unit step sequence $u(n)$ is defined by

$$u(n)=\begin{cases}0, & n<0\\ 1, & n\geq 0\end{cases} \tag{5.3}$$

The shifted unit step sequence $u(n-k)$ is given by

$$u(n-k)=\begin{cases}0, & n<k\\ 1, & n\geq k\end{cases}$$

The graphical representations of $u(n)$ and $u(n-2)$ are shown in Fig. 5.5.

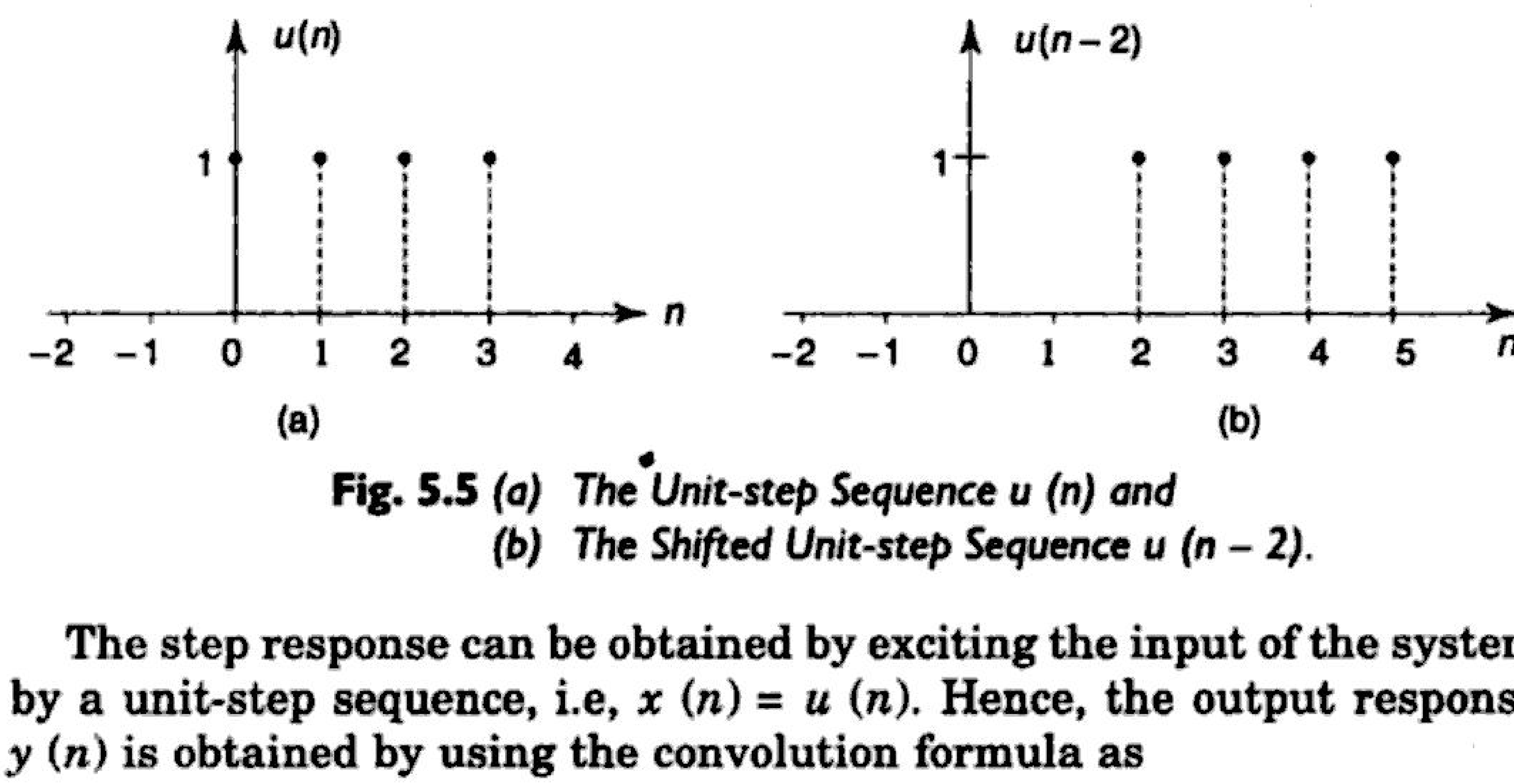

Fig. 5.5 *(a) The Unit-step Sequence u (n) and*
(b) The Shifted Unit-step Sequence u (n – 2).

The step response can be obtained by exciting the input of the system by a unit-step sequence, i.e, $x(n) = u(n)$. Hence, the output response $y(n)$ is obtained by using the convolution formula as

$$y(n)=\sum_{k=-\infty}^{\infty} h(k)\,u(n-k)$$

Relation Between the Unit Sample and the Unit-step Sequences

The unit sample sequence $\delta(n)$ and the unit-step sequence $u(n)$ are related as

$$u(n)=\sum_{m=0}^{n}\delta(m),\quad \delta(\mathrm{n})=u(n)-u(n-1) \tag{5.4}$$

5.2 PROPERTIES OF A DSP SYSTEM

The properties of linearity, time invariance, causality and stability of the difference equations are required for the DSP system to be practically realisable.

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Therefore,

$$aF[x_1(t)] + bF[x_2(t)] = a\frac{d^2y_1(t)}{dt^2} + b\frac{d^2y_2(t)}{dt^2} + ay_1(t)\frac{dy_1(t)}{dt} + by_2(t)\frac{dy_2(t)}{dt} + ay_1(t) + by_2(t)$$

$$F[a\,x_1(t) + b\,x_2(t)] = \frac{d^2}{dt^2}[a\,y_1(t) + b\,y_2(t)] + [a\,y_1(t) + b\,y_2(t)]\left[\frac{d}{dt}(a\,y_1(t) + b\,y_2(t)) + 1\right]$$

Here $aF[x_1(t)] + bF[x_2(t)] \neq F[ax_1(t) + bx_2(t)]$ and hence the system is non-linear.

5.2.2 Time- Invariance

A DSP system is said to be time-invariant if the relationship between the input and output does not change with time. It is mathematically defined as

$$\text{if } y(n) = F[x(n)], \text{ then } y(n-k) = F[x(n-k)] = z^{-k}F[x(n)] \quad (5.6)$$

for all values of k. This is true for all possible excitations. The operator z^{-k} represents a signal delay of k samples.

Example 5.8 Determine whether the DSP systems described by the following equations are time invariant.

(a) $y(n) = F[x(n)] = a\,n\,x(n)$.

(b) $y(n) = F[x(n)] = a\,x(n-1) + b\,x(n-2)$

Solution

(a) The response to a delayed excitation is

$$F[x(n-k)] = an[x(n-k)]$$

The delayed response is $y(n-k) = a(n-k)[x(n-k)]$

Here $F[x(n-k)] \neq y(n-k)$ and hence the system is not time invariant, i.e. the system is time dependent .

(b) Here, $F[x(n-k)] = ax[(n-k)-1] + bx[(n-k)-2]$

$$= y(n-k)$$

Hence the system is time invariant.

Example 5.9 Check whether the following systems are linear and time invariant.

(a) $F[x(n)] = n[x(n)]^2$

(b) $F[x(n)] = a[x(n)]^2 + b\,x(n)$

Solution

(a) (i) $F[x(n)] = n[x(n)]^2$

Here, $F[x_1(n)] = n[x_1(n)]^2$ and

$F[x_2(n)] = n[x_2(n)]^2$

Therefore, $F[x_1(n)] + F[x_2(n)] = n[\{x_1(n)\}^2 + \{x_2(n)\}^2]$

$$= 2\,[(\infty - \tan^{-1}\infty) - (0 - \tan^{-1} 0)]$$
$$= 2\,[\infty - \pi/2] = \infty$$

Since the Paley-Wiener criterion is not satisfied, the amplitude function is not a suitable amplitude response for a causal LTI system.

Example 5.13 Using Paley-Wiener criterion, determine whether the magnitude function $|H(j\,\omega)| = \dfrac{1}{\sqrt{1+\omega^2}}$ is realisable.

Solution According to Paley-Wiener criterion, the magnitude function $|H(j\,\omega)|$ is realisable only when the following condition is satisfied.

$$\int_{-\infty}^{\infty} \frac{|ln\,|H(j\omega)||}{1+\omega^2}\,d\omega < \infty$$

Here, the given magnitude function is $|H(j\,\omega)| = \dfrac{1}{\sqrt{1+\omega^2}}$

$$ln\left(\frac{1}{\sqrt{1+\omega^2}}\right) = ln\,1 - \log(1+\omega^2)^{1/2}$$
$$= -\frac{1}{2}\log(1+\omega^2)$$
$$\left|ln\left(\frac{1}{\sqrt{1+\omega^2}}\right)\right| = \frac{1}{2}\log(1+\omega^2)$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{\left|ln\left(\dfrac{1}{\sqrt{1+\omega^2}}\right)\right|}{1+\omega^2}\,d\omega = \int_{-\infty}^{\infty} \frac{\frac{1}{2}\log(1+\omega^2)}{1+\omega^2}\,d\omega$$
$$= 2\cdot\frac{1}{2}\int_0^{\infty} \frac{\log(1+\omega^2)}{1+\omega^2}\,d\omega$$
$$= \int_0^{\infty} \frac{\log(1+\omega^2)}{1+\omega^2}\,d\omega$$

Substituting $\omega = \tan\theta$, we get

$$1+\omega^2 = 1+\tan^2\theta = \sec^2\theta,$$
$$d\,\omega = \sec^2\theta\, d\theta \quad \text{and} \quad \theta \text{ becomes 0 to } \pi/2$$
$$= \int_0^{\pi/2} \frac{\log \sec^2\theta}{\sec^2\theta}\,.\,\sec^2\theta\, d\theta$$

Example 5.16 Check the stability condition for the DSP systems described by the following equations .

(a) $y(n) = a^n u(n)$

(b) $y(n) = x(n) + e^a y(n-1)$

Solution

(a) $y(n) = a^n u(n)$

Taking z-transform, we have

$$Y(z) = \frac{1}{1 - a z^{-1}} = \frac{z}{z - a}$$

Here the pole is at $z = a$ and hence for the system to be stable, $|a| < 1$.

(b) $y(n) = x(n) + e^a y(n-1)$

Taking z-transform, we have

$$Y(z) = X(z) + e^a z^{-1} Y(z)$$

$$Y(z)[1 - e^a z^{-1}] = X(z)$$

Therefore, $$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - e^a z^{-1}} = \frac{z}{z - e^a}$$

Here the pole is at $z = e^a$ and hence $|e^a| < 1$, i.e. $a < 0$ for stability.

5.2.5 Bounded Input- Bounded Output (BIBO) Stability

Stability is of utmost importance in any system design. There are many definitions for stability. One of them is BIBO stability. A sequence $x(n)$ is bounded if there exists a finite M such that $|x(n)| < M$ for all n. Any system is said to be BIBO stable if and only if every bounded input gives a bounded output.

For any linear time invariant (LTI) system, the BIBO stability depends on the impulse response of that system. To obtain the necessary and sufficient condition for BIBO stability, consider the convolution property which relates the input and the output of a LTI system as

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)\, h(k)$$

It follows that

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} x(n-k)\, h(k) \right|$$

$$\leq \sum_{k=-\infty}^{\infty} |x(n-k)|\; |h(k)|$$

$$\leq M \sum_{k=-\infty}^{\infty} |h(k)| \;\; [\text{since } |x(n)| < M \text{ for all } n]$$

where M is a finite constant.

Therefore, the output is bounded if and only if

The necessary and sufficient condition for BIBO stability is

$$\sum_{k=0}^{\infty} |h(k)| < \infty$$

Here, $\sum_{k=0}^{\infty} |h(k)| = |b|$

Hence, the given system is BIBO stable if $|b| < \infty$.

(e) $y(n) = ax(n) \,.\, x(n-1)$

If $x(n) = \delta(n)$, then $y(n) = h(n)$.

The above equation can be changed into $h(n) = a\delta(n) \,.\, \delta(n-1)$

When $n = 0$, $h(0) = a\delta(0) \,.\, \delta(-1) = 0$

When $n = 1$, $h(1) = a\delta(1) \,.\, \delta(0) = 0$

When $n = 2$, $h(2) = a\delta(2) \,.\, \delta(1) = 0$

The necessary and sufficient condition for BIBO stability is

$$\sum_{k=0}^{\infty} |h(k)| < \infty$$

Here, $\sum_{k=0}^{\infty} |h(k)| = 0$.

So the given system is BIBO stable .

(f) $y(n) = \text{max. of } [x(n), x(n-1), x(n-2)]$.

If $x(n) = \delta(n)$, then $y(n) = h(n)$.

The above equation can be changed into

$h(n) = \text{max . of } [\delta(n), \delta(n-1), \delta(n-2)]$

When $n = 0$, $h(0) = \text{max . of } [\delta(0), \delta(-1), \delta(-2)] = 1$

When $n = 1$, $h(1) = \text{max . of } [\delta(1), \delta(0), \delta(-1)] = 1$

When $n = 2$, $h(2) = \text{max . of } [\delta(2), \delta(1), \delta(0)] = 1$

When $n = 3$, $h(3) = \text{max . of } [\delta(3), \delta(2), \delta(1)] = 0$

In general,

$$\begin{aligned} h(n) &= 1, \quad \text{for } n = 0, 1, 2 . \\ &= 0, \quad \text{otherwise } (i.e., n > 2) . \end{aligned}$$

The necessary and sufficient condition for BIBO stability is

$$\sum_{k=0}^{\infty} |h(k)| < \infty.$$

Here, $\sum_{k=0}^{\infty} |h(k)| = |h(0)| + |h(1)| + |h(2)| + \cdots + |h(k)| + \cdots$

$$= 1 + 1 + 1 + 0 + \cdots = 3.$$

So, the given system is BIBO stable.

(g) $y(n) = \text{average of } [x(n+1), x(n), x(n-1)]$.

If $x(n) = \delta(n)$, then $y(n) = h(n)$.

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$$h(n) = \begin{cases} b_k/a_0 & , \quad 0 \le n \le m \\ 0 & , \quad \text{otherwise} \end{cases} \tag{5.16}$$

As this is obviously of finite duration, it represents a FIR system.

Example 5.20 A DSP system is described by the linear difference equation

$$y(n) = 0.2\,x(n) - 0.5\,x(n-2) + 0.4\,x(n-3)$$

Given that the digital input sequence {–1, 1, 0, –1} is applied to this DSP system, determine the corresponding digital output sequence.

Solution Taking z-transform of the given linear difference equation, we get

$$Y(z) = 0.2\,X(z) - 0.5\,z^{-2}X(z) + 0.4\,z^{-3}X(z)$$

Therefore,

$$H(z) = \frac{Y(z)}{X(z)} = 0.2 - 0.5\,z^{-2} + 0.4\,z^{-3}$$

The given input sequence is $x(n) = \{-1, 1, 0, -1\}$ and its z-transform is

$$X(z) = -1 + z^{-1} - z^{-3}$$

Therefore, $Y(z) = H(z).X(z)$

$$= -0.2 + 0.2\,z^{-1} + 0.5\,z^{-2} - 1.1\,z^{-3} + 0.4\,z^{-4} + 0.5\,z^{-5} - 0.4\,z^{-6}$$

Taking inverse z-transform, we get the digital output sequence

$$y(n) = \{-0.2, 0.2, 0.5, -1.1, 0.4, 0.5, -0.4\}$$

Example 5.21 Determine $H(z)$ and its poles and zeros if

$$y(n) + \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) + x(n-1)$$

Solution Given

$$y(n) + \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) + x(n-1)$$

Taking z-transform,we get

$$Y(z) + \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = X(z) + z^{-1}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1+\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}} = \frac{z(z+1)}{z^2+\frac{3}{4}z+\frac{1}{8}}$$

$$= \frac{z(z+1)}{\left(z+\frac{1}{2}\right)\left(z+\frac{1}{4}\right)}$$

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Using the convolution property of the z-transform, we get

$$H(z) = H_1(z)\,H_2(z) \cdots H_L(z) \tag{5.18}$$

Hence, $H(e^{j\omega}) = H_1(e^{j\omega})\,H_2(e^{j\omega}) \cdots H_L(e^{j\omega})$

Here we observe that the cascade connection involves convolution of the impulse responses in the time domain and multiplication of the frequency responses in the frequency domain.

5.4.3 Energy Density Spectrum

In the digital system, the spectrum of the signal at the output of the system is

$$Y(z) = H(z)\,X(z)\Big|_{z=e^{j\omega}} \tag{5.19}$$

Hence, $Y(e^{j\omega}) = H(e^{j\omega})\,X(e^{j\omega})$. This is the desired input-output relation in the frequency domain, which means that the spectrum of the signal at the output of the system is equal to the frequency response of the system multiplied by the spectrum of the signal at the input.

$$|Y(e^{j\omega})|^2 = |H(e^{j\omega})|^2\,|X(e^{j\omega})|^2 \tag{5.20}$$

As the energy density spectra of $x(n)$ is $S_{xx}(e^{j\omega}) = |X(e^{j\omega})|^2$ and the energy density spectra of $y(n)$ is $S_{yy}(e^{j\omega}) = |Y(e^{j\omega})|^2$, we have

$$S_{yy}(e^{j\omega}) = |H(e^{j\omega})|^2\,S_{xx}(e^{j\omega}) \tag{5.21}$$

5.4.4 Magnitude and Phase Spectrum

The magnitude response is the absolute value of a filter's complex frequency response. The phase response is the angle component of a filter's frequency response. For a linear time invariant system with a real-valued impulse response, the magnitude and phase functions posses symmetry properties which are detailed below. From the definition of z-transform, $H(e^{j\omega})$, a complex function of the real variable ω can be expressed as

$$\begin{aligned}
H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h(n)\,e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} h(n)\cos\omega n - j\sum_{n=-\infty}^{\infty} h(n)\sin\omega n \\
&= H_R(e^{j\omega}) + jH_I(e^{j\omega}) \\
&= |H(e^{j\omega})|\,e^{j\Phi(\omega)} \\
&= \sqrt{H_R^2(e^{j\omega}) + H_I^2(e^{j\omega})}\;e^{j\tan^{-1}[H_I(e^{j\omega})/H_R(e^{j\omega})]}
\end{aligned}$$

where $H_R(e^{j\omega})$ and $H_I(e^{j\omega})$ denote the real and imaginary components of $H(e^{j\omega})$.

Therefore, $|H(e^{j\omega})| = \sqrt{H_R^2(e^{j\omega}) + H_I^2(e^{j\omega})}$

$$\Phi(\omega) = \tan^{-1}\left[\frac{H_I(e^{j\omega})}{H_R(e^{j\omega})}\right]$$

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Therefore, $H(z)|_{z=e^{j\omega}} = H(e^{j\omega}) = \frac{1}{2}\left[\frac{1}{1-\frac{1}{4}e^{-j\omega}} + \frac{1}{1-\frac{3}{4}e^{-j\omega}}\right]$

Magnitude and phase responses:

$$H(z) = \frac{z\left(z-\frac{1}{2}\right)}{\left(z-\frac{1}{4}\right)\left(z-\frac{3}{4}\right)}$$

$$H(e^{j\omega}) = \frac{e^{j\omega}\left(e^{j\omega}-\frac{1}{2}\right)}{\left(e^{j\omega}-\frac{1}{4}\right)\left(e^{j\omega}-\frac{3}{4}\right)}$$

Therefore $|H(e^{j\omega})| = \frac{\left|\left(e^{j\omega}-\frac{1}{2}\right)\right|}{\left|\left(e^{j\omega}-\frac{1}{4}\right)\right|\left|\left(e^{j\omega}-\frac{3}{4}\right)\right|}$

$$\Phi(\omega) = \omega + \arg\left(e^{j\omega}-\frac{1}{2}\right) - \arg\left(e^{j\omega}-\frac{1}{4}\right) - \arg\left(e^{j\omega}-\frac{3}{4}\right)$$

Example 5.27 Determine the frequency response, magnitude response, phase response and time delay of the system given by

$$y(n) + \frac{1}{2}y(n-1) = x(n) - x(n-1)$$

Solution

To find the frequency response $H(e^{j\omega})$:

Given, $y(n) + \frac{1}{2}y(n-1) = x(n) - x(n-1)$

Taking z-transform, we get

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) - z^{-1}X(z)$$

$$Y(z)\left[1+\frac{1}{2}z^{-1}\right] = X(z)\,[1-z^{-1}]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-z^{-1}}{1+\frac{1}{2}z^{-1}}$$

Therefore, the frequency response is, $H(e^{j\omega}) = \frac{1-e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega}}$

Points A and C in Fig. 5.8 correspond to frequencies 0 and $\omega_s/2$, the Nyquist frequency or folding frequency, which is equal to πf_s or π/T, where f_s is the sampling frequency and T is the sampling period. One complete revolution of the phase $e^{j\omega T}$ about the origin corresponds to a frequency increment of $\omega_s = \dfrac{2\pi}{T}$. Here, $H(e^{j\omega T})$ is a periodic function of frequency with a period ω_s . The frequency response has the property $H(e^{j\omega T}) = H(e^{-j\omega T})$. Therefore, the magnitude function $M(\omega)$ is an even function of ω, and the phase function $\Phi(\omega)$ is an odd function of ω . Vectors are drawn from each pole and zero to $e^{j\omega T}$ point on the unit circle.

The transfer function of a digital system may be expressed in terms of its poles and zeros as

$$H(e^{j\omega T}) = M(\omega)\, e^{j\Phi(\omega)} = \frac{H_0 \prod_{i=1}^{p} (e^{j\omega T} - z_i)}{\prod_{i=1}^{q} (e^{j\omega T} - p_i)} \tag{5.23}$$

By substituting, $\quad (e^{j\omega T} - z_i) = M_{zi}\, e^{j\Phi z_i}$

and $\quad (e^{j\omega T} - p_i) = M_{pi}\, e^{j\Phi p_i}$

We have, Magnitude as

$$M(\omega) = |H(e^{j\omega})|$$

$$= \frac{H_0 \prod_{i=1}^{p} M_{zi}}{\prod_{i=1}^{q} M_{pi}}$$

$$= \frac{H_0 \prod_{i=1}^{p} \{\text{Vector magnitude from the } i^{\text{th}} \text{ zero to the frequency point on the circumference of the unit circle}\}}{\prod_{i=1}^{q} \{\text{Vector magnitude from the } i^{\text{th}} \text{ pole to the frequency point on the circumference of the unit circle}\}}$$

That is, the magnitude of the system function $M(\omega)$ equals the product of all zero vector lengths, divided by the product of all pole vector lengths.

Phase Shift

$$\Phi(\omega) = \angle H(e^{j\omega})$$

$$= \sum_{i=1}^{p} \Phi_{z_i} - \sum_{i=1}^{q} \Phi_{p_i}$$

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(e) $h(n) = \begin{cases} \cos\left(\dfrac{n\pi}{8}\right), & -1 < n < 15 \\ 0, & \text{otherwise} \end{cases}$

Ans: (a), (c) and (e) are causal and stable.

5.26 For each of the following discrete-time signals, determine whether or not the system is linear, shift-invariant, causal and stable.

(a) $y(n) = x(n+7)$, (b) $y(n) = x^3(n)$, (c) $y(n) = n\,x(n)$

(d) $y(n) = \alpha + \sum_{k=0}^{4} x(n-k)$, α is a non-zero constant.

(e) $y(n) = \alpha + \sum_{k=-4}^{4} x(n-k)$, α is a non-zero constant.

Ans: (a) Linear, time-invariant, non-causal and stable.
(b) Non-linear, time-invariant, causal and BIBO stable.
(c) Linear, not time-invariant, causal and not BIBO stable.
(d) Non-linear, time-invariant, causal and BIBO stable.
(e) Non-linear, time-invariant, non-causal and BIBO stable.

5.27 Determine whether the following systems are linear or non-linear, causal or non-causal, shift invariant or shift-variant .

(i) $y(nT) = x(nT+T) + x(nT-T)$.

(ii) $y(nT) = x^2(nT+T)\,e^{-nT}\sin\omega nT$

(iii) $y(n) = a\,y(n-1) + x(n)$

5.28 Using the Paley-Wiener criterion, show that $|H(f)| = e^{-\beta f^2}$ is not a suitable amplitude response for a causal LTI system.

5.29 Discuss the stability of the system described by

$$H(z) = \frac{z^{-1}}{1 - z^{-1} - z^{-2}}$$

5.30 Find the stability region for the causal system

$$H(z) = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

by evaluating its poles and restricting them to be inside the unit circle.

5.31 A causal LTI system is described by the difference equation

$$y(n) = y(n-1) + y(n-2) + x(n-1)$$

where $x(n)$ is the input and $y(n)$ is the output.

(i) Find the system function $H(z) = \dfrac{Y(z)}{X(z)}$ for this system, plot the poles and zeros of $H(z)$ and indicate the region of convergence.

(ii) Find the unit sample response of the system.

(iii) Is the system stable or not?

(Contd.)

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In the first case the impulse response $h(n)$ is folded and shifted, and $x(n)$ is the excitation signal. In the second case the input signal $x(n)$ is folded and shifted. Here $h(n)$ acts as the excitation signal.

Commutative Law

Convolution satisfies commutative law, i.e. $x(n) * h(n) = h(n) * x(n)$

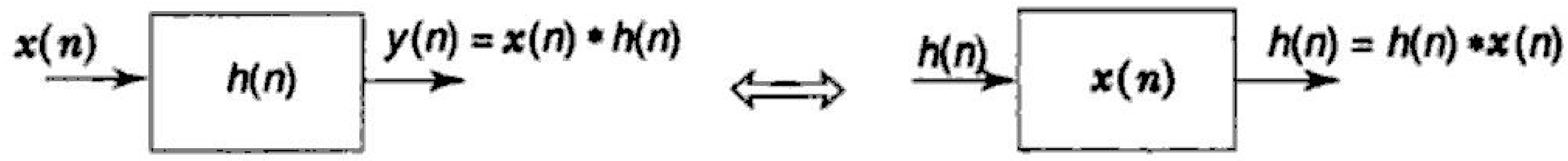

Fig. 6.3 *Commutative Property*

Associative Law

$$[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$$

Take LHS of the above equation.

Consider $x(n)$ to be the input signal to the LTI system with impulse response $h_1(n)$. The output $y_1(n)$ is given by

$$y_1(n) = x(n) * h_1(n)$$

This $y_1(n)$ signal now acts as the input signal to the second LTI system with impulse response $h_2(n)$.

Therefore, $y(n) = y_1(n) * h_2(n)$

$$= [x(n) * h_1(n)] * h_2(n)$$

Now consider the RHS of the equation, which indicates that the input $x(n)$ is applied to an equivalent system $h(n)$ and is given by

$$h(n) = h_1(n) * h_2(n)$$

and the output of the equivalent system to the input $x(n)$ is given by

$$y(n) = x(n) * h(n)$$
$$= x(n) * [h_1(n) * h_2(n)]$$

Since convolution satisfies commutative property, the cascading of two systems can be interchanged as shown in Fig. 6.4.

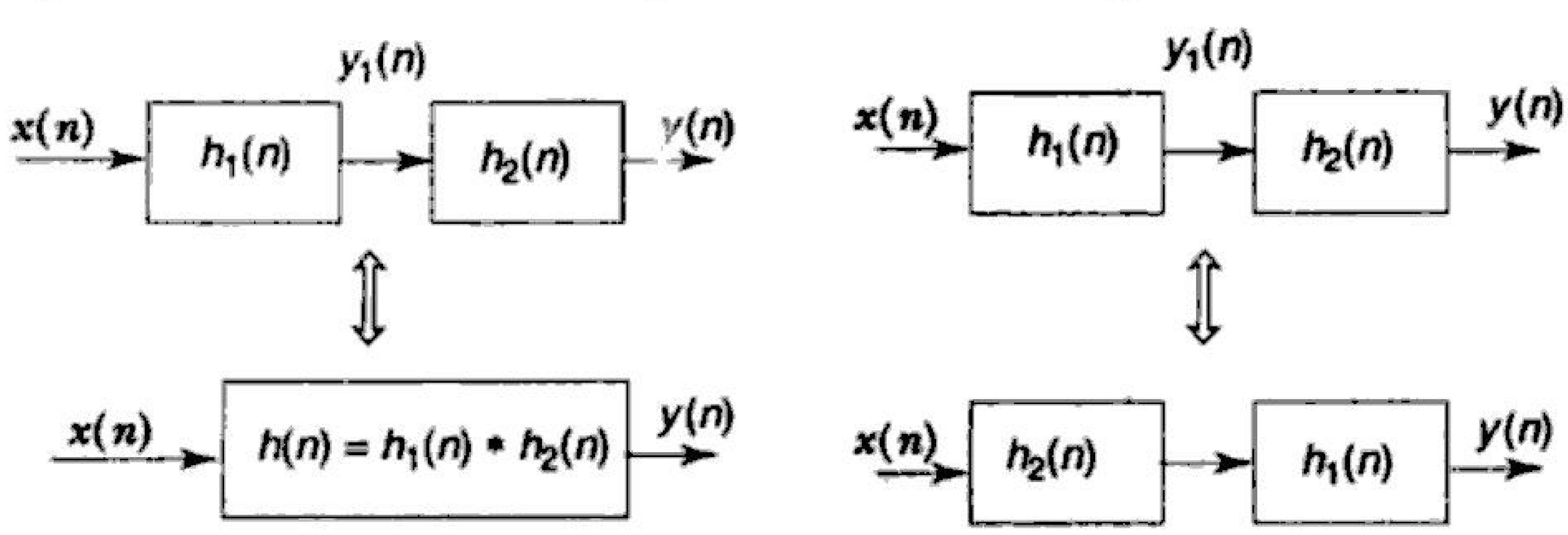

Fig. 6.4 *Associative Property*

If N linear time invariant systems are in cascade with impulse responses $h_1(n)$, $h_2(n)$,...$h_N(n)$, then the equivalent system impulse response is given by

$$h(n) = h_1(n) * h_2(n) * \ldots * h_N(n)$$

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$$= \frac{1}{4}[60 + 4 + 4] = 17$$

$$x_3(2) = \frac{1}{4}\left[60 - 4j\, e^{j\pi} + 4j\, e^{j3\pi}\right]$$

$$= \frac{1}{4}[60 - 4j\,(-1) + 4j\,(-1)] = 15$$

$$x_3(3) = \frac{1}{4}\left[60 + (-4j)\, e^{j3\pi/2} + 4j\, e^{j9\pi/2}\right]$$

$$= \frac{1}{4}[60 + (-4j)\,(-j) + 4j\,(j)]$$

$$= \frac{1}{4}[60 - 4 - 4] = 13$$

Therefore, $x_3(n) = [\ 15, 17, 15, 13\ ]$

Note: From the above results, we find that the resulting sequences obtained by both linear convolution and circular convolution have different values and length. Linear convolution results in an aperiodic sequence with a length of $(2N-1)$, i.e. seven in this case, whereas circular convolution results in a periodic sequence with a length of N, i.e. four in this case. Circular convolution will produce the same sequence values as those produced by linear convolution if three zeros are padded at the end of the two given sequences $x_1(n)$and $x_2(n)$.

Example 6.2 Find the response of an FIR filter with impulse response $h(n) = \{1, 2, 4\}$ to the input sequence $x(n) = \{1, 2\}$.

Solution

Linear Convolution

Given $h(n) = \{\ 1, 2, 4\ \}$ and $x(n) = \{\ 1, 2\ \}$
Here $N_1 = 3$ and $N_2 = 2$. Hence $N = N_1 + N_2 - 1 = 4$

We know that $y(n) = x(n) * h(n) = \sum\limits_{k=-\infty}^{\infty} x(k)h(n-k)$

Therefore,

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k)$$

$$= \ldots + x(0)\, h(0) + x(1)\, h(-1) + \ldots$$

$$= 0 + 1 + 0 \ldots = 1$$

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k)$$

When $\quad n = 1$

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)\, h\,(1-k)$$

$$= \ldots + x(-1)\, h(2) + x(0)\, h(1) + x(1)\, h(0) + \ldots$$

$$= 0 + (1)\,(1) + (1)\,(1) + 0 \ldots = 2$$

When $\quad n = 2$

$$y(2) = \sum_{k=-\infty}^{\infty} x\,(k)\, h(2-k)$$

$$= \ldots + x\,(-1)\, h(3) + x(0)\, h(2) + x(1)\, h(1) + \ldots$$

$$= 0 + (1)\,(1) + 0 \ldots = 1$$

When $\quad n = 3$

$$y(3) = \sum_{k=-\infty}^{\infty} x(k)\, h(3-k) = 0$$

When $\quad n = -1$

$$y(-1) = \sum_{k=-\infty}^{\infty} x(k)\, h(-1-k)$$

$$= \ldots + x(-1)\, h(1) + x(0)\, h(-1) + x(1)\, h(-2) + \ldots$$

$$= 0 + (1)\,(1) + (1)\,(1) + 0 \ldots = 2$$

When $\quad n = -2$

$$y(-2) = \sum_{k=-\infty}^{\infty} x(k)\, h(-2-k)$$

$$= \ldots + x(-1)\, h(1) + x(0)\, h(-2) + x(1)\, h(-3) + \ldots$$

$$= 0 + (1)\,(1) + 0 \ldots = 1$$

When $\quad n = -3$

$$y(-3) = \sum_{k=-\infty}^{\infty} x\,(k)\, h(-3-k) = 0$$

The convolution signal $y(n)$ is [See Fig. E6.3(b)]

$$y(n) = 0,\ n \le -3 \quad \text{and} \quad n \ge 3$$

$$y(n) = 1,\ n = \pm 2$$

$$y(n) = 2,\ n = \pm 1$$

$$y(n) = 3,\ n = 0$$

Note: For the convolved signal, the left extreme and the right extreme can be found using the left and right extremes of the two sequences to be convolved. That is,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

When $\quad n = 0$

$$y(0) = \sum_{k=-\infty}^{\infty} h(k)\, x(-k)$$

$$= \ldots + h(0)\, x(0) + h(1)\, x(-1) + \ldots$$

$$= 0 + 1 + 0 + \ldots = 1$$

When $\quad n = 1$

$$y(1) = \sum_{k=-\infty}^{\infty} h(k)\, x(1-k)$$

$$= \ldots + h(0)\, x(1) + h(1)\, x(0) + h(2)\, x(-1) + \ldots$$

$$= 0 + (1)(1) + (a)(1) + 0 \ldots = 1 + a$$

When $\quad n = 2$

$$y(2) = \sum_{k=-\infty}^{\infty} h(k)\, x(2-k)$$

$$= \ldots + h(0)\, x(2) + h(1)\, x(1) + h(2)\, x(0) + h(3)\, x(-1) + \ldots$$

$$= 0 + (1)(1) + (a)(1) + (a^2)(1) + 0 + \ldots = 1 + a + a^2$$

When $\quad n = 3$

$$y(3) = \sum_{k=-\infty}^{\infty} h(k)\, x(3-k)$$

$$= \ldots + h(0)\, x(3) + h(1)\, x(2) + h(2)\, x(1) + h(3)\, x(0) + h(4)\, x(-1) + \ldots$$

$$= 0 + (1)(1) + (a)(1) + (a2)(1) + (a3)(1) + 0 + \ldots$$

$$= 1 + a + a^2 + a^3$$

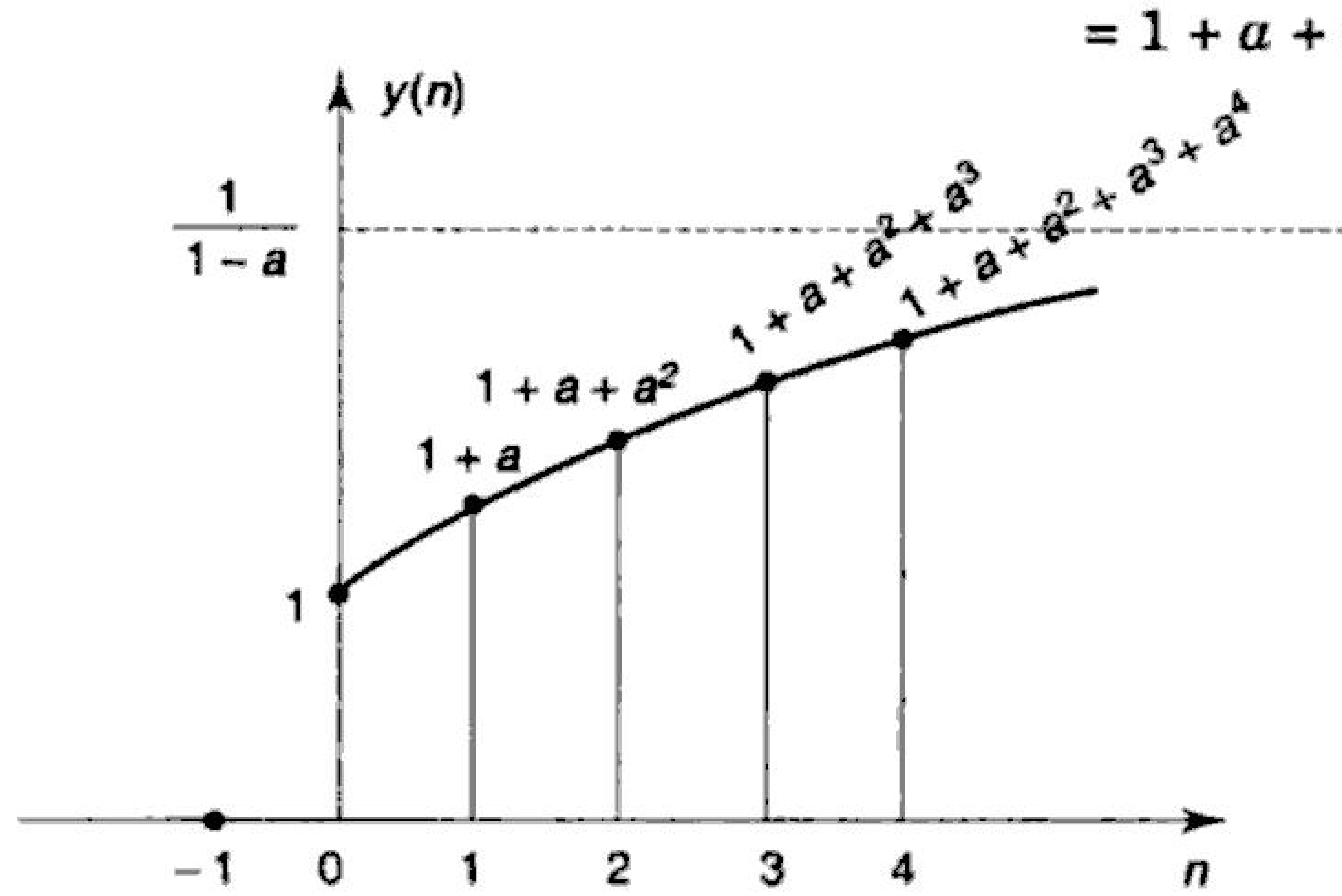

Fig. E6.4(b)

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When $\quad n = 3$

$$y(3) = \sum_{k=-\infty}^{\infty} x(k)\, h(3-k) = 0$$

$$= \ldots + x(0)\, h(3) + x(1)\, h(2) + x(2)\, h(1) + \ldots$$

$$= 0 + (1)(0) + (2)(-1) + 3(-2) + 0 \ldots = -8$$

When $\quad n = 4$

$$y(4) = \sum_{k=-\infty}^{\infty} x(k)\, h(4-k)$$

$$= \ldots + x(0)\, h(4) + x(1)\, h(3) + x(2)\, h(2) + \ldots$$

$$= 0 + (1)(0) + (2)(0) + 3(-1) + 0 \ldots. = -3$$

These sequence values are plotted in Fig. E 6.5(b).

Example 6.6 Compute the convolution $y(n) = x(n) * h(n)$ of the signals

$$x(n) = \left\{1, 1, \underset{\uparrow}{0}, 1, 1,\right\} \text{ and } h(n) = \left\{1, -2, -3, \underset{\uparrow}{4}\right\}$$

Solution The sequences of the given two signals are plotted in Fig. E6.6(a).

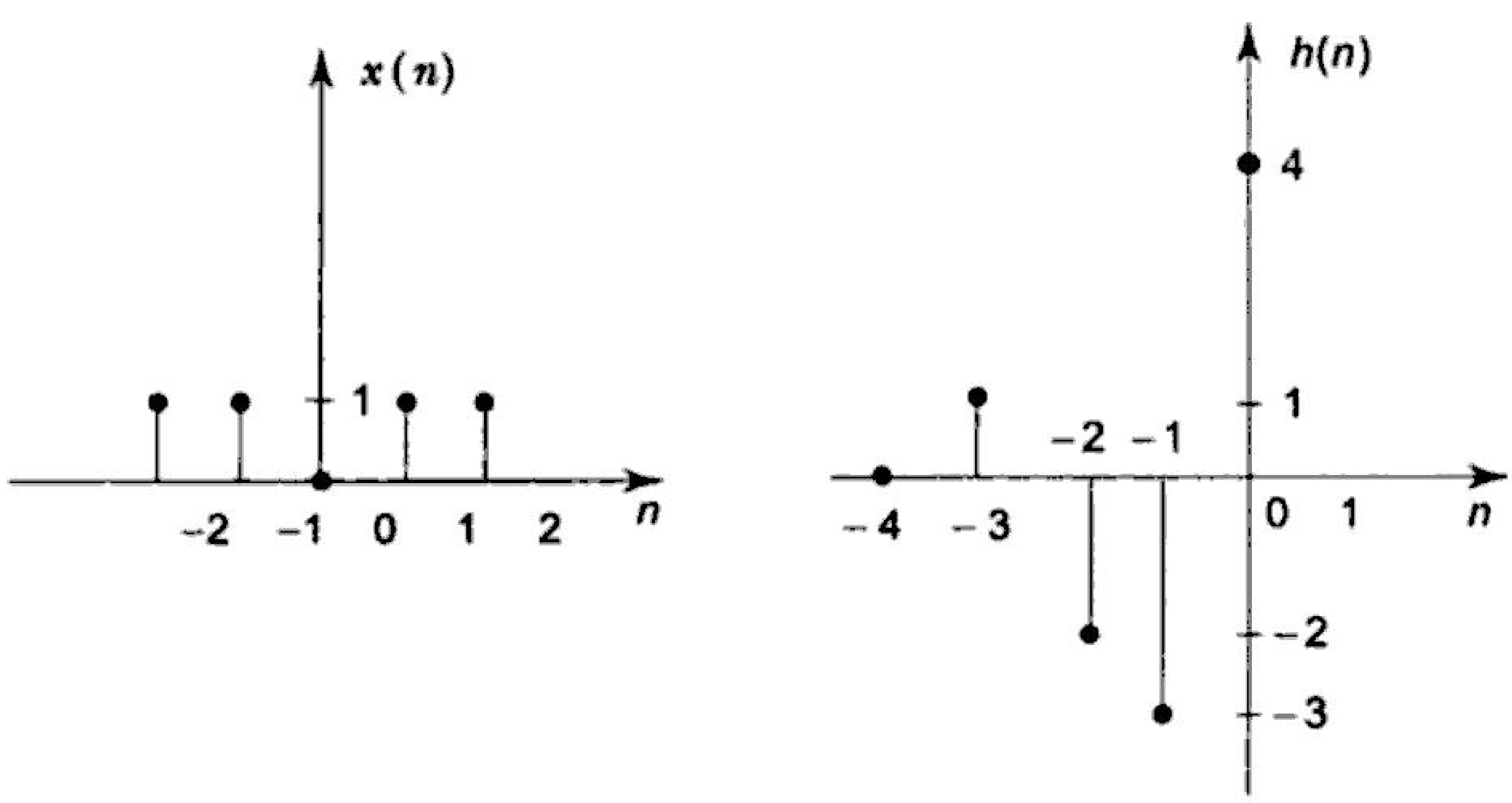

Fig. E6.6 (a)

From the graph,

$$x_l = -2, \quad x_r = 2, \; h_l = -3, \; h_r = 0$$

Hence the left and right extremes of the convoluted signal $y(n)$ are calculated as

$$y_l = x_l + h_l = -2 + (-3) = -5$$

$$y_r = x_r + h_r = 2 + 0 = 2$$

and the corresponding IDFT is

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j2\pi nk/N}, \quad n = 0, 1, \ldots, N-1.$$

6.3.1 Relationship between the DFT and Other Transforms

Relationship to the Fourier Series Coefficients of a Periodic Sequence

The Fourier Series of a periodic sequence $x_p(n)$ with fundamental period N is given by

$$x_p(n) = \sum_{k=0}^{N-1} c_k\, e^{j2\pi nk/N}, \quad -\infty < n < \infty$$

where the Fourier series coefficients are given by

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n) e^{-j2\pi nk/N}, \quad k = 0, 1, \ldots, N-1.$$

By comparing the above equations with that of DFT pair and defining a sequence $x(n)$ which is identical to $x_p(n)$ over a single period, we get

$$X(k) = Nc_k$$

If a periodic sequence $x_p(n)$ is formed by periodically repeating $x(n)$ every N samples, i.e.

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$$

The discrete frequency-domain representation is given by

$$X(k) = \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi nk/N} = NC_k, \quad k = 0, 1, \ldots, N-1.$$

and the IDFT is

$$x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j2\pi nk/N}, \quad -\infty < n < \infty$$

Relationship to the Spectrum of an Infinite Duration (Aperiodic) Signal

Let $x(n)$ be an aperiodic finite energy sequence. The Fourier transform is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

If $X(e^{j\omega})$ is sampled at N equally spaced frequencies,

Hence,

$$x^*(-n,(\text{mod } N)) = x^*(N-n) \xleftrightarrow[N]{DFT} X^*(k)$$

8. Circular Convolution

If $\qquad x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$ and $x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$, then

$$x_1(n) Ⓝ x_2(n) \xleftrightarrow[N]{DFT} X_1(k)\,X_2(k)$$

F where $x_1(n) Ⓝ x_2(n)$ denotes the circular convolution of the sequence $x_1(n)$ and $x_2(n)$ defined as

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m)\,x_2(n-m,(\text{mod}N))$$

$$= \sum_{m=0}^{N-1} x_2(m)\,x_1(n-m,(\text{mod}N))$$

9. Circular Correlation

For complex-valued sequences $x(n)$ and $y(n)$,

if $\qquad x(n) \xleftrightarrow[N]{DFT} X(k)$ and $y(n) \xleftrightarrow[N]{DFT} Y(k)$, then

$$r_{xy}(l) \xleftrightarrow[N]{DFT} R_{xy}(k) = X(k)Y^*(k)$$

where $r_{xy}(l)$ is the (unnormalised) circular cross-correlation sequence, given as

$$r_{xy}(l) = \sum_{n=0}^{N-1} x(n)\,y^*(n-l,(\text{mod}N))$$

10. Multiplication of Two Sequences

If $\qquad x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$ and $x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$, then

$$x_1(n)\,x_2(n) \xleftrightarrow[N]{DFT} \frac{1}{N} X_1(k) Ⓝ X_2(k)$$

11. Parseval's Theorem

For complex-valued sequences $x(n)$ and $y(n)$,

if $\qquad x(n) \xleftrightarrow[N]{DFT} X(k)$ and $y(n) \xleftrightarrow[N]{DFT} Y(k)$, then

$$\sum_{n=0}^{N-1} x(n)\,y^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\,Y^*(k)$$

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$$= 1 + 0.5 - j\,0.866 + 2(-0.5 - j\,0.866) + 2(-1) + 3(-0.5 + j\,0.866) + 3(0.5 + j\,0.866)$$

$$= -1.5 + j\,2.598$$

For $k = 2$

$$X(2) = \sum_{n=0}^{5} x(n)\, e^{-j2\pi(2)n/6}$$

$$= \sum_{n=0}^{5} x(n)\, e^{-2j\pi n/3}$$

$$= 1 + e^{-j2\pi/3} + 2e^{-j4\pi/3} + 2e^{-j2\pi} + 3e^{-j8\pi/3} + 3e^{-j10\pi/3}$$

$$= 1 + (-0.5) - j\,0.866 + 2(-0.5 + j\,0.866) + 2(1) + 3(-0.5 - j\,0.866) + 3(-0.5 + j\,0.866)$$

$$= -1.5 + j\,0.866$$

For $k = 3$

$$X(3) = \sum_{n=0}^{5} x(n)\, e^{-j2\pi(3)n/6}$$

$$= \sum_{n=0}^{5} x(n)\, e^{-j\pi n}$$

$$= 1 + e^{-j\pi} + 2e^{-j2\pi} + 2e^{-j3\pi} + 3e^{-j4\pi} + 3e^{-j5\pi}$$

$$= 1 - 1 + 2(1) + 2(-1) + 3(1) + 3(-1) = 0$$

For $k = 4$

$$X(4) = \sum_{n=0}^{5} x(n)\, e^{-j2\pi(4)n/6}$$

$$= \sum_{n=0}^{5} x(n)\, e^{-j4\pi n/3}$$

$$= 1 + e^{-j4\pi/3} + 2e^{-j8\pi/3} + 2e^{-j4\pi} + 3e^{-j16\pi/3} + 3e^{-j20\pi/3}$$

$$= 1 + (-0.5 + j\,0.866) + 2(-0.5 - j\,0.866) + 2(1) + 3(-0.5 + j\,0.866) + 3(-0.5 - j\,0.866)$$

$$= -1.5 - j\,0.866$$

For $k = 5$

$$X(5) = \sum_{n=0}^{5} x(n)\, e^{-j2\pi(5)n/6}$$

$$= \sum_{n=0}^{5} x(n)\, e^{-j5\pi n/3}$$

$$= 1 + e^{-j5\pi/3} + 2e^{-j10\pi/3} + 2e^{-j5\pi} + 3e^{-j20\pi/3} + 3e^{-j25\pi/3}$$

$$= \frac{1}{4}\left(1 + 2e^{j3\pi/2} + 3e^{j3\pi} + 4e^{j9\pi/2}\right)$$

$$= \frac{1}{4}(1 + 2(-j) + 3(-1) + 4j)$$

$$= \frac{1}{4}(-2 + 2j) = -\frac{1}{2} + j\frac{1}{2}$$

$$x(n) = \left\{\frac{5}{2}, -\frac{1}{2} - j\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} + j\frac{1}{2}\right\}$$

Example 6.14 Determine the IDFT of $X(k) = \{3, (2 + j), 1, (2 - j)\}$.

Solution The IDFT is defined as

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j2\pi nk/N}, 0 \le n \le N - 1.$$

Given $N = 4$, $x(n) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, e^{j\pi nk/2}, 0 \le n \le 3$

When $n = 0$

$$x(0) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, e^{0}$$

$$= \frac{1}{4}[3 + (2 + j) + 1 + (2 - j)] = 2$$

When $n = 1$

$$x(1) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, e^{j\pi k/2}$$

$$= \frac{1}{4}\left[3 + (2 + j)\, e^{j\pi/2} + e^{j\pi} + (2 - j)\, e^{j3\pi/2}\right]$$

$$= \frac{1}{4}[3 + (2 + j)j - 1 + (2 - j)(-j)] = 0$$

When $n = 2$

$$x(2) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, e^{j\pi k}$$

$$= \frac{1}{4}\left[3 + (2 + j)\, e^{j\pi} + e^{j2\pi} + (2 - j)\, e^{j3\pi}\right]$$

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Table 6.1

Index	*Binary representation*	*Bit reversed binary*	*Bit reversed index*
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

The basic computation in the DIT FFT algorithm is illustrated in Fig. 6.9, which is called a butterfly because the shape of its flow graph resembles a butterfly.

The symmetry and periodicity of W_N^r can be exploited to obtain further reductions in computation. The multiplications by $W_N^0 = 1$, $W_N^{N/2} = -1$, $W_N^{N/4} = j$ and $W_N^{3N/2} = -j$ can be avoided in the DFT computation process in order to save the computational complexity.

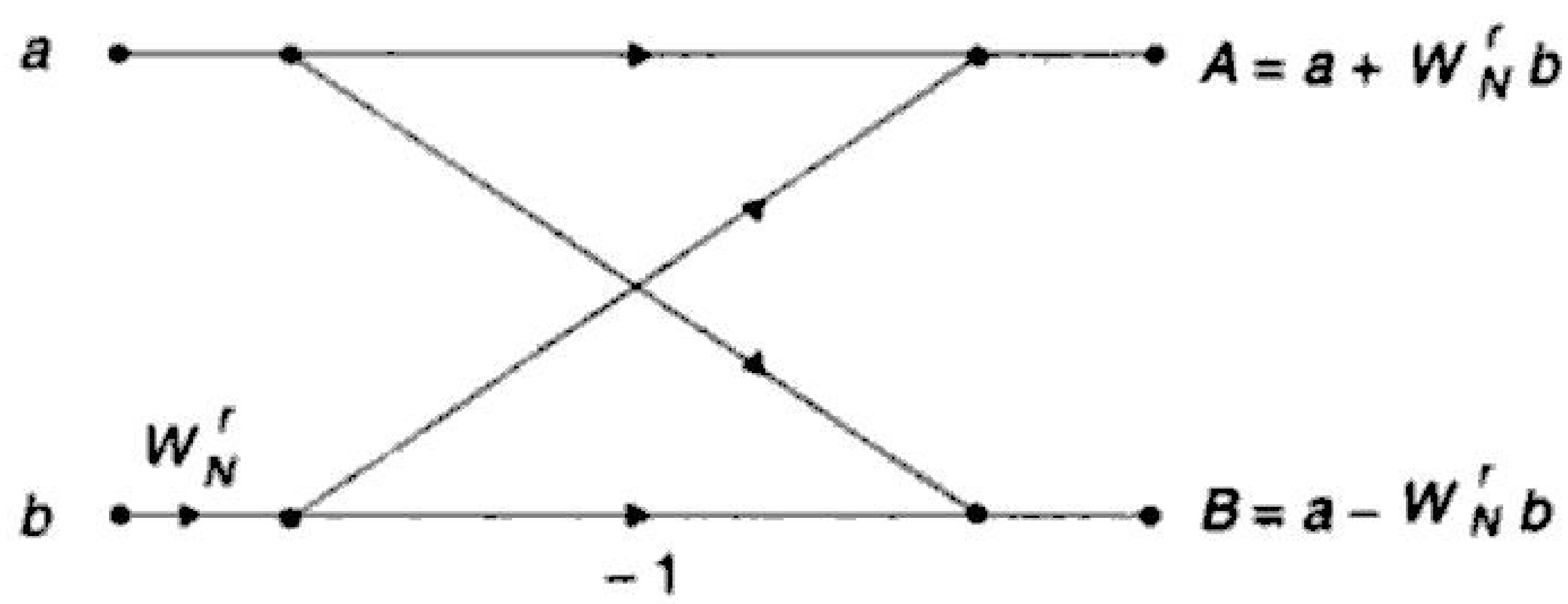

Fig. 6.9 *Basic Butterfly Flow Graph for the Computation in the DIT FFT Algorithm*

In the 8-point DIT FFT flow graph shown in Fig. 6.8, W_2^0, W_2^4 and W_2^8 are equal to 1, and hence these scale factors do not actually represent complex multiplications. Also, since W_2^0, W_2^4, and W_2^8 equal to −1, they do not represent a complex multiplication, where there is just a change in sign. Further, W_1^4, W_3^4, W_2^8 and W_8^6 are j or $-j$, they need only sign changes and interchanges of real and imaginary parts, even though they represent complex multiplications. When $N = 2^L$, the number of stages of computations is $L = \log_2 N$. Each stage has N complex multiplications and N complex additions. Therefore, the total number of complex multiplications and additions in computing all N-DFT samples is equal to $N \log_2 N$. Hence, the number of complex multiplications is reduced from N^2 to $N \log_2 N$.

In the reduced 8-point DIT FFT flow graph shown in Fig. 6.10, there are actually only four non-trivial complex multiplications corresponding to these scale factors. When the size of the transform is increased, the proportion of nontrivial complex multiplications is reduced and $N \log_2 N$ approximation becomes a little closer.

The reduced flow-graph for 16-point decimation-in-time FFT algorithm is shown in Fig. 6.11.

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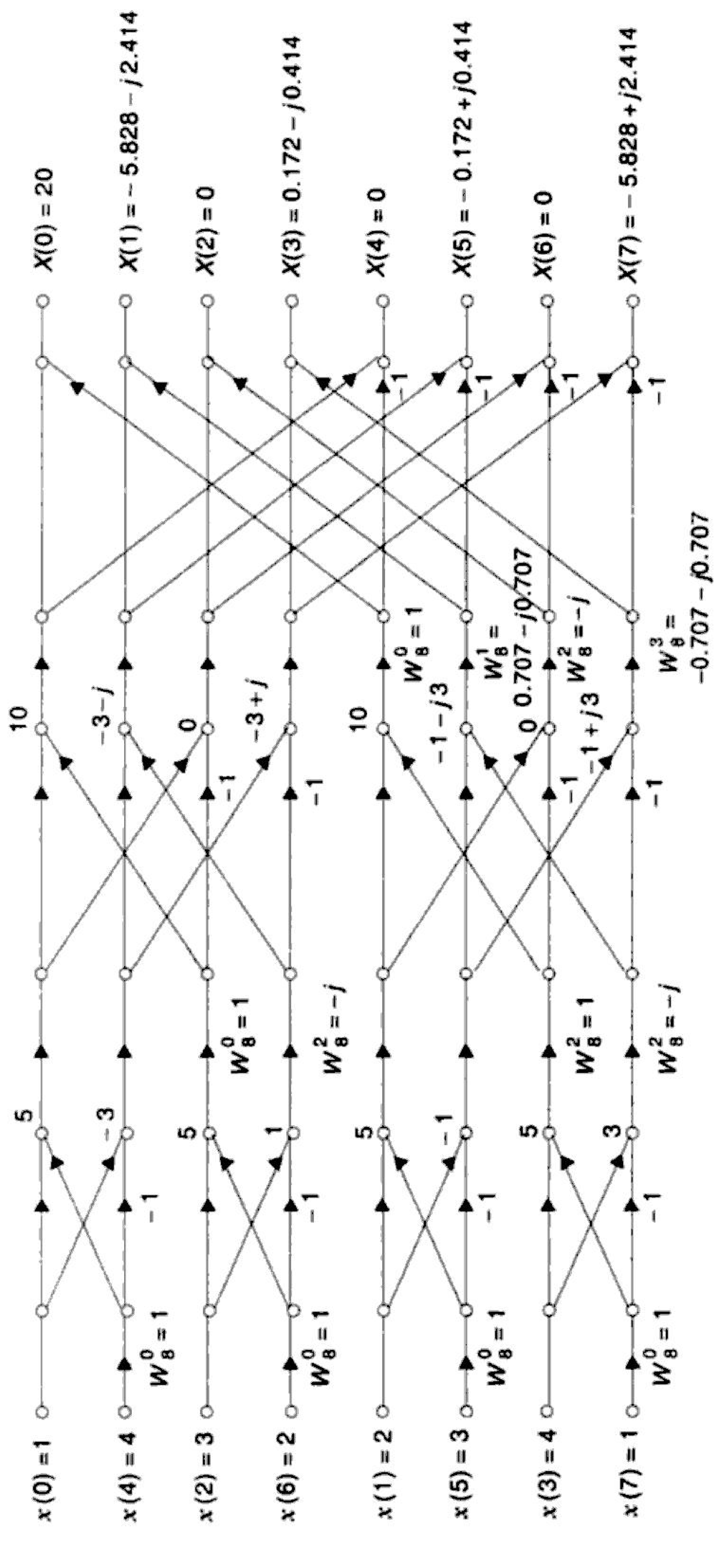
x(0) = 1
x(4) = 4
x(2) = 3
x(6) = 2
x(1) = 2
x(5) = 3
x(3) = 4
x(7) = 1
X(0) = 20
X(1) = − 5.828 − j2.414
X(2) = 0
X(3) = 0.172 − j0.414
X(4) = 0
X(5) = − 0.172 + j0.414
X(6) = 0
X(7) = − 5.828 + j2.414
W80 = 1
W82 = −j
W81 = 0.707 − j0.707
W83 = −0.707 − j0.707
−1
5
−3
1
3
10
−3 − j
0
−3 + j
−1 − j3
−1 + j3

Fig. E6.15

Using DIT FFT algorithm, we can find $X(k)$ from the given sequence $x(n)$ as shown in Fig. E 6.17.

Therefore, $X(k) = \{255, 48.63 + j166.05, -51 + j102, -78.63 + j46.05, -85, -78.63 - j46.05, -51 - j102, 48.63 - j166.05\}$

Example 6.18 Given $x(n) = \{0, 1, 2, 3\}$, find $X(k)$ using DIT FFT algorithm.

Solution Given $N = 4$

$$W_N^k = e^{-j\left(\frac{2\pi}{N}\right)k}$$

$$W_4^0 = 1 \text{ and } W_4^1 = e^{-j\pi/2} = -j$$

Using DIT FFT algorithm, we can find $X(k)$ from the given sequence $x(n)$ as shown in Fig. E 6.18.

Therefore, $X(k) = \{6, -2 + j2, -2, -2 - j2\}$

6.4.3 Decimation-in-Frequency (DIF) Algorithms

The decimation-in-time FFT algorithm decomposes the DFT by sequentially splitting input samples $x(n)$ in the time domain into sets of smaller and smaller subsequences and then forms a weighted combination of the DFTs of these subsequences. Another algorithm called decimation-in-frequency FFT decomposes the DFT by recursively splitting the sequence elements $X(k)$ in the frequency domain into sets of smaller and smaller subsequences. To derive the decimation-in-frequency FFT algorithm for N, a power of 2, the input sequence $x(n)$ is divided into the first half and the last half of the points as discussed below.

$$X(k) = \sum_{n=0}^{(N/2)-1} x(n)\, W_N^{nk} + \sum_{n=N/2}^{N-1} x(n)\, W_N^{nk}$$

$$= \sum_{n=0}^{(N/2)-1} x(n)\, W_N^{nk} + \sum_{n=0}^{(N/2)-1} x\left(n + \frac{N}{2}\right) W_N^{(n+N/2)k}$$

$$= \sum_{n=0}^{(N/2)-1} x(n)\, W_N^{nk} + W_N^{(N/2)k} + \sum_{n=0}^{(N/2)-1} x(n + N/2)\, W_N^{nk} \qquad (6.24)$$

Since, $W_N^{(N/2)k} = e^{-j\frac{2\pi}{N}\cdot\frac{N}{2}k} = \cos(\pi k) - j\sin \pi k = (-1)^k$, we obtain

$$X(k) = \sum_{n=0}^{(N/2)-1} x(n)\, W_N^{nk} + (-1)^k \sum_{n=0}^{(N/2)-1} x(n + N/2)\, W_N^{nk}$$

$$= \sum_{n=0}^{(N/2)-1} \left[x(n) + (-1)^k\, x\left(n + \frac{N}{2}\right)\right] W_N^{nk} \qquad (6.25)$$

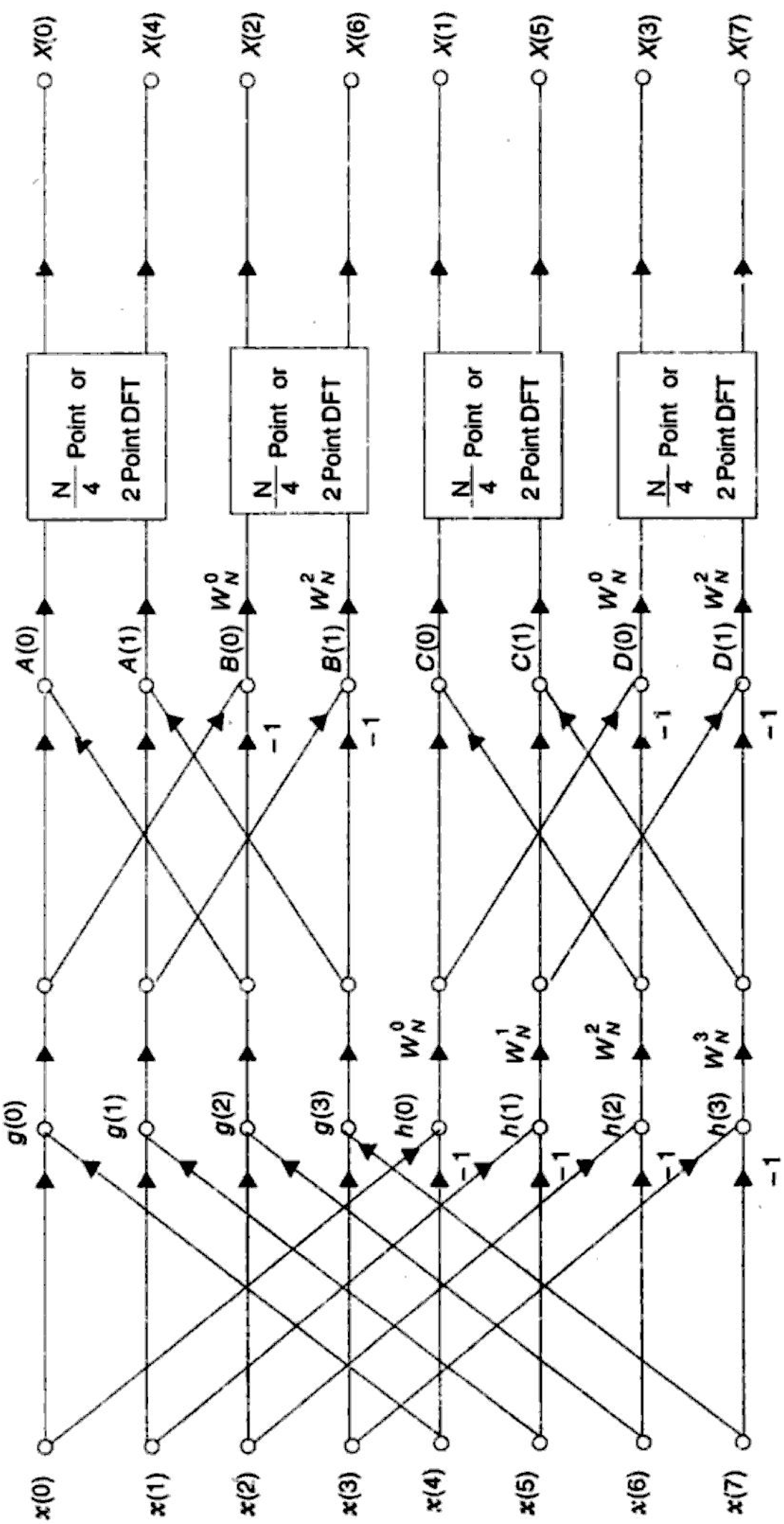

Fig. 6.13 *Flow Graph of the Second Stage of Decimation-In-Frequency FFT for $N = 8$*

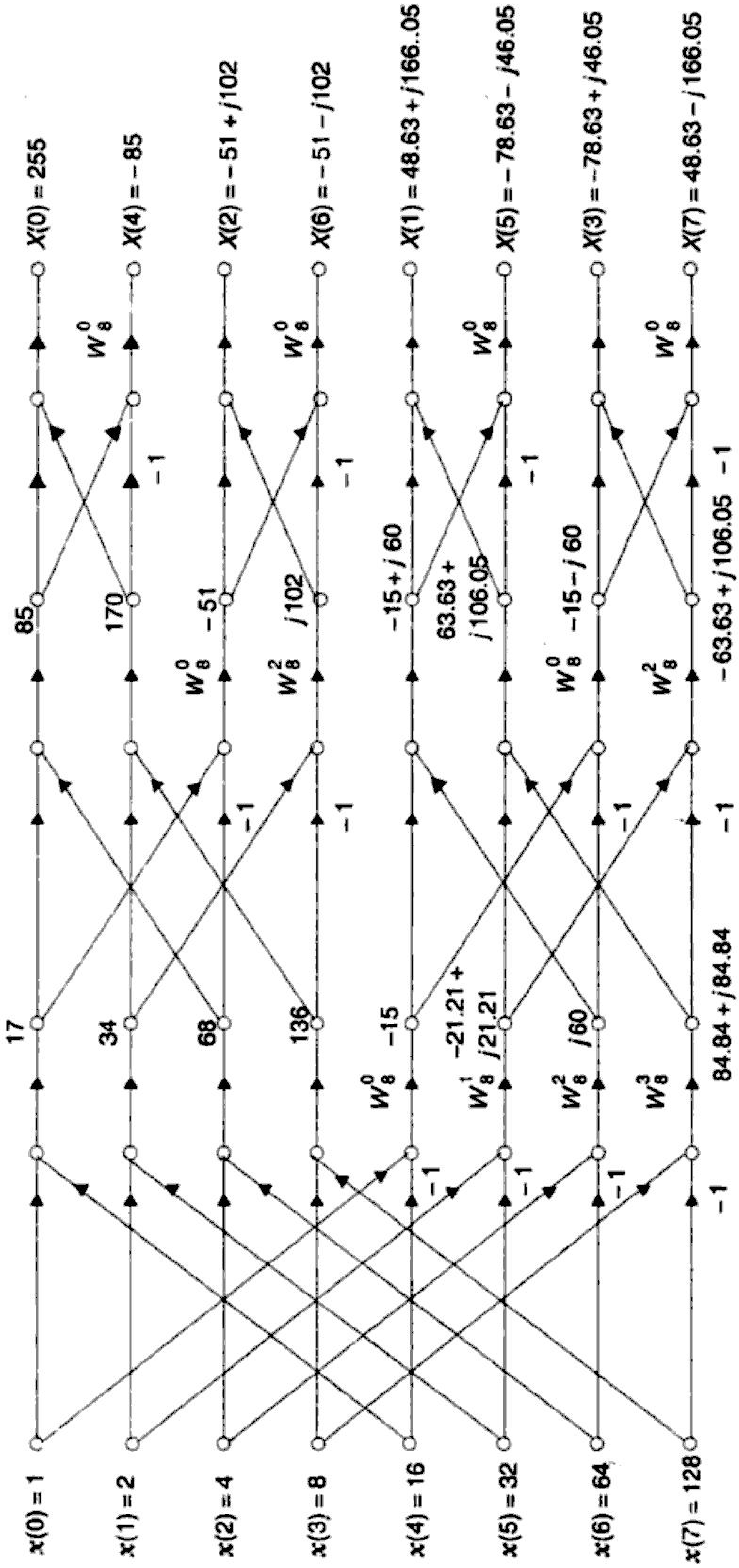
x(0) = 1
x(1) = 2
x(2) = 4
x(3) = 8
x(4) = 16
x(5) = 32
x(6) = 64
x(7) = 128
17
34
68
136
−15
−21.21 + j21.21
j60
84.84 + j84.84
85
170
−51
j102
−15 + j60
63.63 + j106.05
−15 − j60
−63.63 + j106.05
X(0) = 255
X(4) = −85
X(2) = −51 + j102
X(6) = −51 − j102
X(1) = 48.63 + j166..05
X(5) = −78.63 − j46.05
X(3) = −78.63 + j46.05
X(7) = 48.63 − j166.05

Fig. E6.20

Example 6.23 Use the 4-point inverse FFT and verify the DFT results {6, $-2+j2$, -2, $-2-j2$} obtained in Example 6.18 for the given input sequence {0, 1, 2, 3}.

Solution We know that $W_N^k = e^{-j\left(\frac{2\pi}{N}\right)k}$. Hence,

$$W_4^{-0} = 1 \text{ and } W_4^{-1} = e^{j\pi/2} = j$$

Using IFFT algorithm, we can find the input sequence $x(n)$ from the given DFT sequence $X(k)$ as shown in Fig. E6.23.

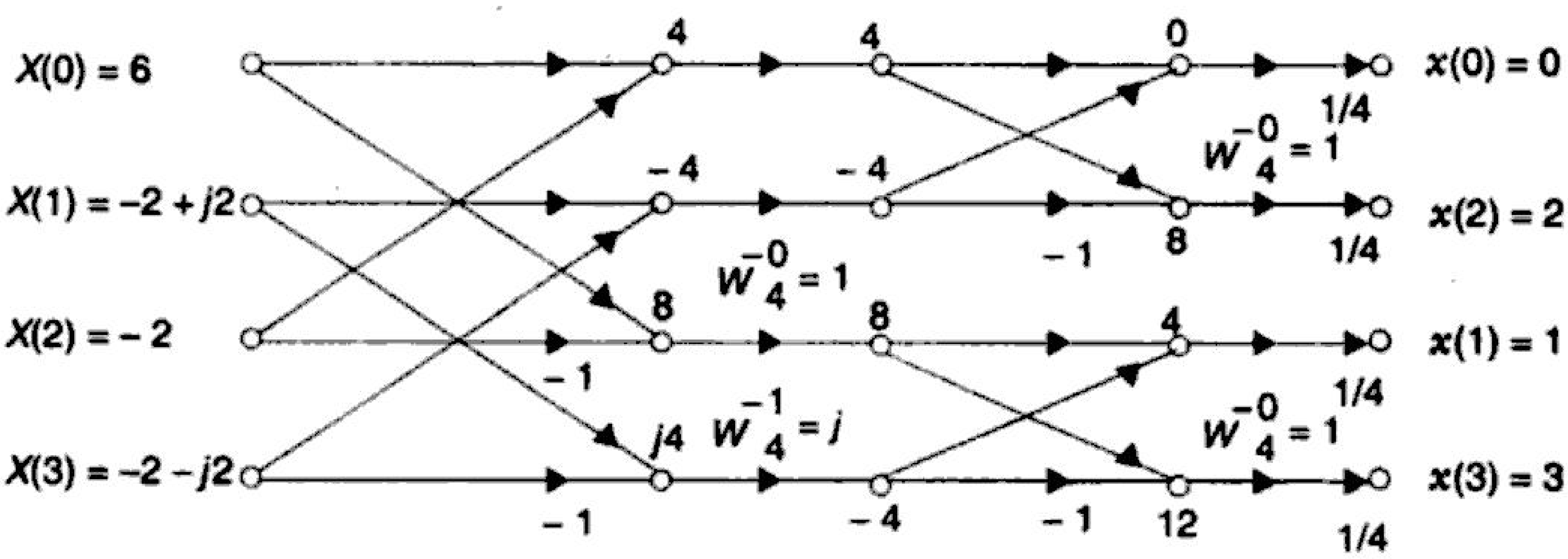

Fig. E6.23

Hence, $x(n) = \{0, 1, 2, 3\}$

Example 6.24 Given $X(k) = \{20, -5.828 - j\,2.414, 0, -0.172 - j\,0.414, 0, -0.172 + j\,0.414, 0, -5.828 + j\,2.414\}$, find $x(n)$.

Solution We know that $W_N^k = e^{-j\left(\frac{2\pi}{N}\right)k}$. Given $N = 8$. Hence,

$$W_8^{-0} = 1$$
$$W_8^{-1} = 0.707 + j\,0.707$$
$$W_8^{-2} = j$$
$$W_8^{-3} = -0.707 + j\,0.707$$

Using IFFT algorithm, we can find $x(n)$ from $X(k)$ as shown in Fig. E.6.24.

Hence, $x(n) = (1, 2, 3, 4, 4, 3, 2, 1)$

Example 6.25 Given $X(k) = \{255, 48.63 + j166.05, -51 + j102, -78.63 + j46.05, -85, -78.63 - j46.05, -51 - j102, 48.63 - j166.05\}$, find $x(n)$.

Solution We know that $W_N^k = e^{-j\left(\frac{2\pi}{N}\right)k}$. Given $N = 8$. Hence,

$$W_8^{-0} = 1$$

$$W_8^{-1} = 0.707 + j0.707$$

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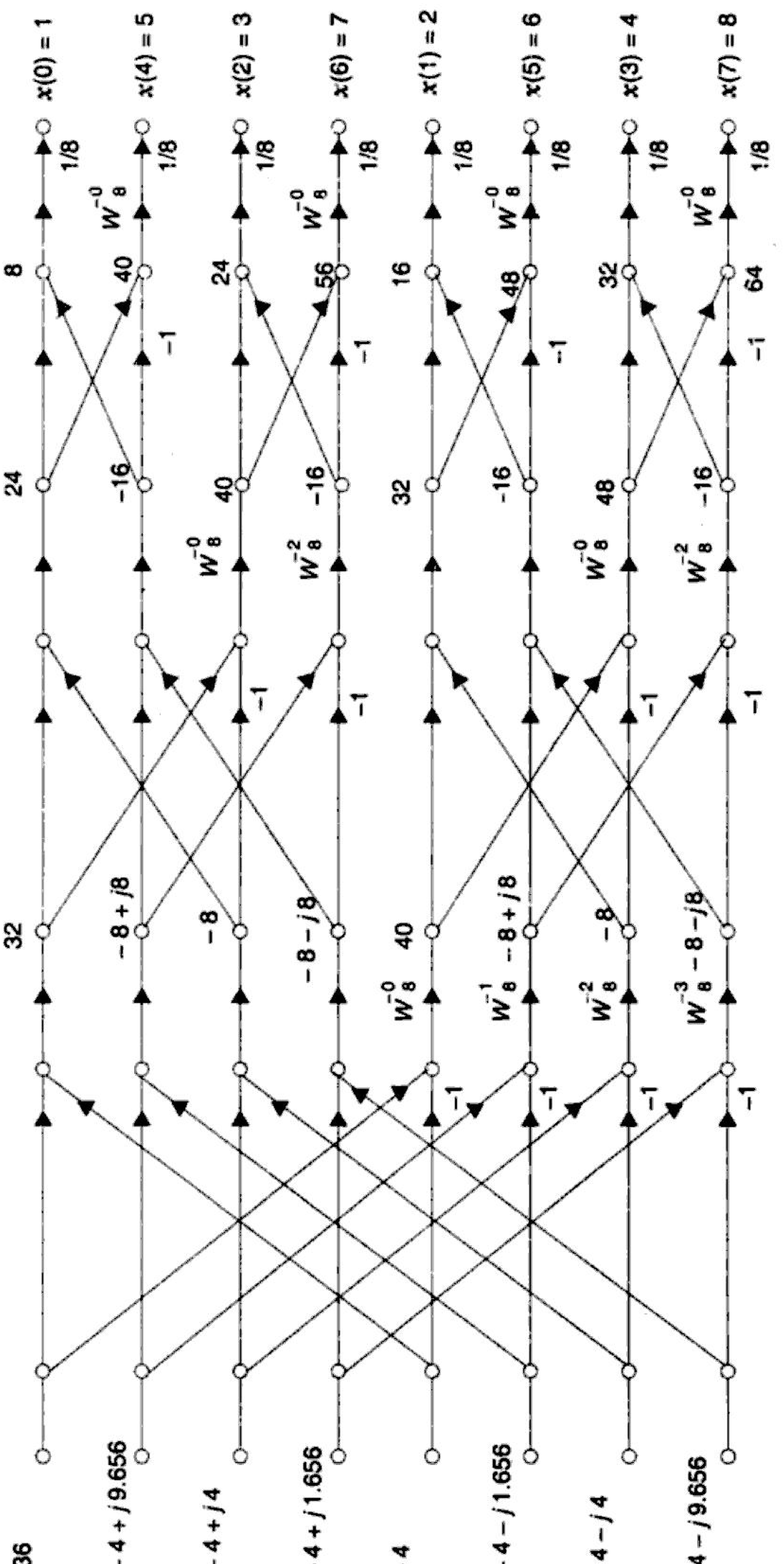
X(0) = 36
X(1) = −4 + j9.656
X(2) = −4 + j4
X(3) = −4 + j1.656
X(4) = −4
X (5) = −4 − j1.656
X(6) = −4 − j4
X(7) = −4 − j9.656
32
−8 + j8
−8
−8 − j8
40
−8 + j8
−8
−8 − j8
24
−16
40
−16
32
−16
48
−16
8
40
24
56
16
48
32
64
1/8
x(0) = 1
x(4) = 5
x(2) = 3
x(6) = 7
x(1) = 2
x(5) = 6
x(3) = 4
x(7) = 8

Fig. E6.26

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$$X(0) = X_1(0) + W_9^0 X_2(0) + W_9^0 X_3(0)$$

$$X(1) = X_1(1) + W_9^1 X_2(1) + W_9^2 X_3(1)$$

$$X(2) = X_1(2) + W_9^2 X_2(2) + W_9^4 X_3(2)$$

$$X(3) = X_1(0) + W_9^3 X_2(0) + W_9^6 X_3(0)$$

$$X(4) = X_1(1) + W_9^4 X_2(1) + W_9^8 X_3(1)$$

$$X(5) = X_1(2) + W_9^5 X_2(2) + W_9^{10} X_3(2)$$

$$X(6) = X_1(0) + W_9^6 X_2(0) + W_9^{12} X_3(0)$$

$$X(7) = X_1(1) + W_9^7 X_2(1) + W_9^{14} X_3(1)$$

$$X(8) = X_1(2) + W_9^8 X_2(2) + W_9^{16} X_3(2)$$

Figure E6.28 shows the radix-3 decimation-in-time FFT flow diagram for $N = 9$. Here, we have repeated the 3-point cat's cradle structure as we had repeated butterflies in the radix-2 case. The input sequence appears in digit-reversed order.

Example 6.29 Develop DIT FFT algorithms for decomposing the DFT for $N = 6$ and draw the flow diagrams for (a) $N = 2.3$ and (b) $N = 3.2$. (c) Also, by using the FFT algorithm developed in part (b), evaluate the DFT values for $x(n) = \{1, 2, 3, 4, 5, 6\}$.

Solution

(a) For $N = 6 = 2 \,.\, 3$, where $m_1 = 2$ and $N_1 = 3$, Eq. 6.40 becomes

$$X(k) = \sum_{n=0}^{2} x(2n)\, W_6^{2nk} + \sum_{n=0}^{2} x(2n+1)\, W_6^{(2n+1)k}$$

$$= \sum_{n=0}^{2} x(2n)\, W_6^{2nk} + W_6^k \sum_{n=0}^{2} x(2n+1)\, W_6^{2nk}$$

Also, $X_i(k+3) = X_i(k)$

$$X_1(k) = \sum_{n=0}^{2} x(2n)\, W_6^{2nk}$$

$$= x(0) + x(2) W_6^{2k} + x(4) W_6^{4k}$$

$$X_1(0) = x(0) + x(2) + x(4)$$

$$X_1(1) = x(0)\, W_6^0 + x(2) W_6^2 + x(4) W_6^4 = x(0) + x(2) W_6^2 + x(4) W_6^4$$

$$X_1(2) = x(0)\, W_6^0 + x(2) W_6^4 + x(4) W_6^8 = x(0) + x(2) W_6^4 + x(4) W_6^2$$

$$X(4) = X_1(0) + W_6^4 X_2(0) + W_6^2 X_3(0)$$
$$= 5 + (-0.5 + j0.866)(7) + (-0.5 - j0.866)(9)$$
$$= -3 - j1.732$$

$$X(5) = X_1(1) + W_6^5 X_2(1) + W_6^4 X_3(1)$$
$$= -3 + (0.5 + j0.866)(-3) + (-0.5 + j0.866)(-3)$$
$$= -3\,\{1 + 0.5 + j0.866 - 0.5 + j0.866\}$$
$$= -3\,\{1 + j1.732\}$$
$$= -3 - j5.196$$

The calculated values of DFT are also shown in Fig. E6.29(b).

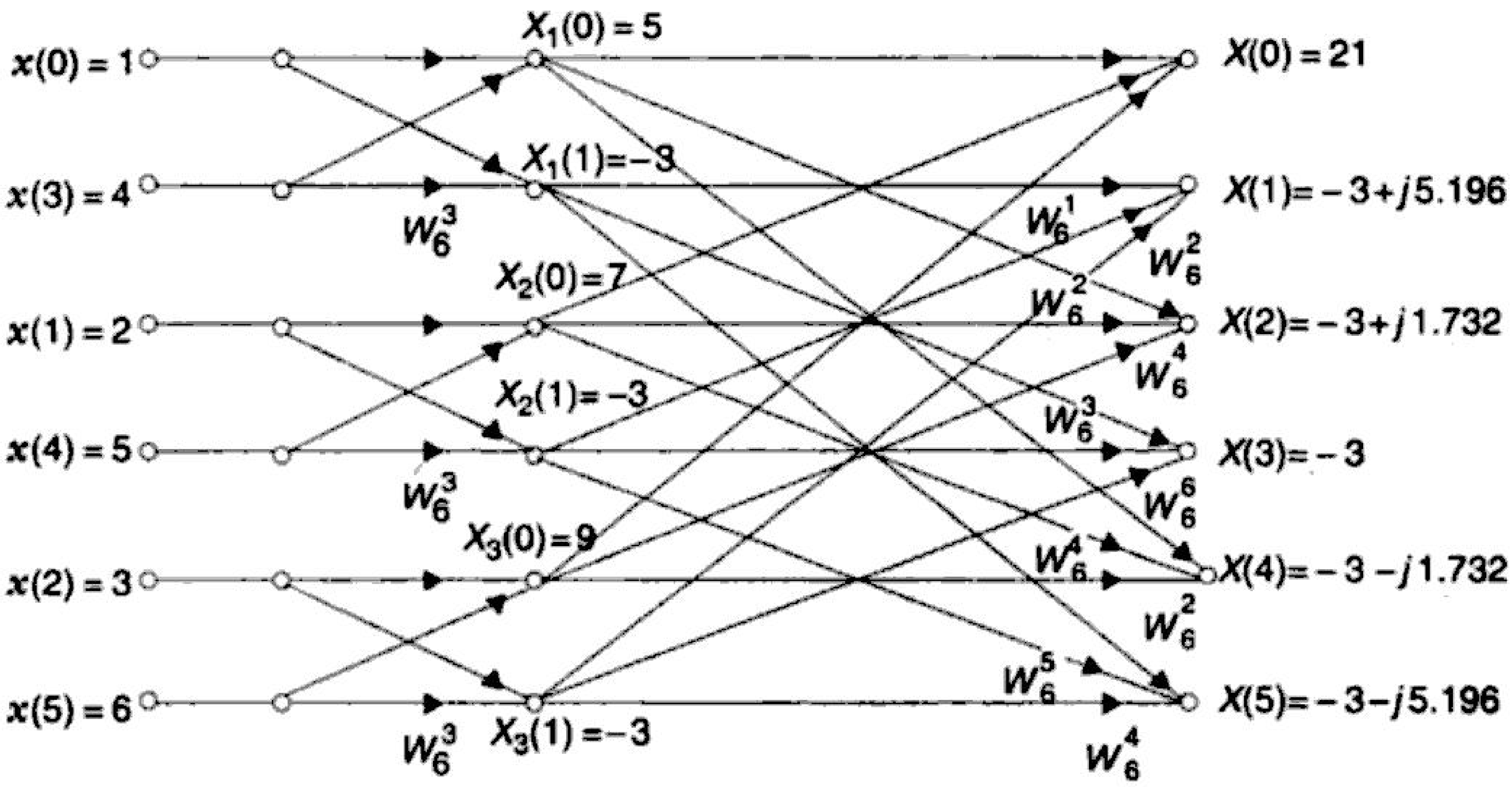

Fig. E6.29 (b) *DIT FFT Flow Diagram for N = 6 = 3 . 2*

Example 6.30 Develop the DIT FFT algorithm for decomposing the DFT for $N = 12$ and draw the flow diagram.

Solution For $N = 12 = 3 \,.\, 4$, where $m_1 = 3$ and $N_1 = 4$, Eq. 6.40 becomes

$$X(k) = \sum_{n=0}^{3} x(3n) W_{12}^{3nk} + \sum_{n=0}^{3} x(3n+1) W_{12}^{(3n+1)k} + \sum_{n=0}^{3} x(3n+2) W_{12}^{(3n+2)k}$$

$$= \sum_{n=0}^{3} x(3n) W_4^{nk} + W_{12}^{k} \sum_{n=0}^{3} x(3n+1) W_4^{nk} + W_{12}^{2k} \sum_{n=0}^{3} x(3n+2) W_4^{nk}$$

$$= X_1(k) + W_{12}^{k} X_2(k) + W_{12}^{2k} X_3(k)$$

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$$X(15) = X_1(3) + W_{16}^{15}X_2(3) + W_{16}^{30}X_3(3) + W_{16}^{45}X_4(3)$$

Figure E6.31 shows the radix-4 decimation-in-time FFT flow diagram for $N = 16$.

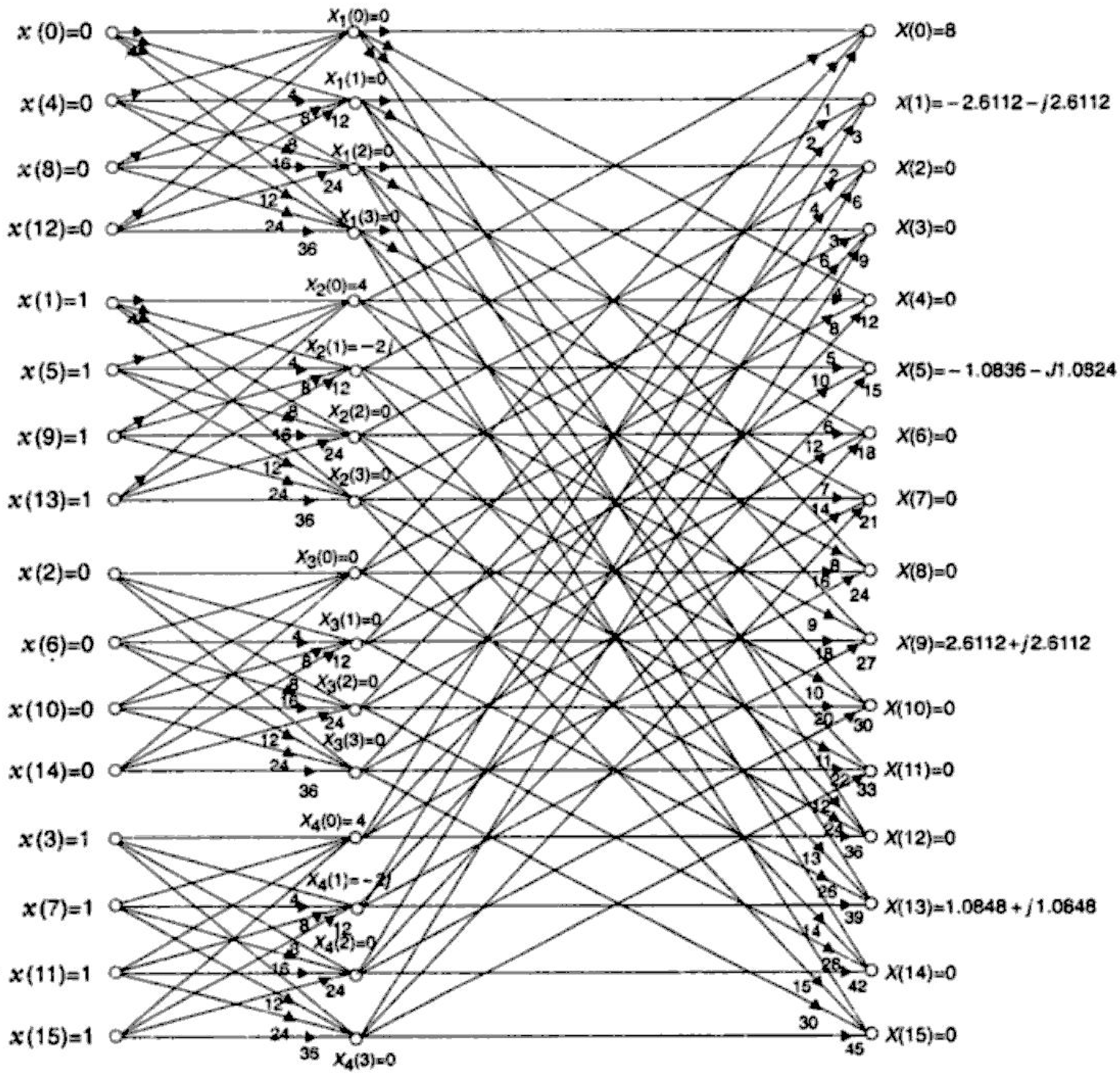

Fig. E6.31 *Radix-4 DIT FFT Flow Diagram for N = 16*

To determine the DFT of the given 16 – point sequence

$$x(n) = \{0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1\}$$

$$X_1(k) = x(0) + x(4)W_{16}^{4k} + x(8)W_{16}^{8k} + x(12)W_{16}^{12k}$$
$$= 0 + 0 + 0 + 0 = 0$$

$$X_2(k) = x(1) + x(5)W_{16}^{4k} + x(9)W_{16}^{8k} + x(13)W_{16}^{12k}$$
$$= 1 + 1(-j) + 1(-1) + 1(-j) = -2j$$

$$X_3(k) = x(2) + x(6)W_{16}^{4k} + x(10)W_{16}^{8k} + x(14)W_{16}^{12k}$$
$$= 0$$

$$X_1(0) = X_1(1) = X_1(2) = X_1(3) = 0$$

$$X_2(0) = 1 + 1 + 1 + 1 = 4$$

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(b) To develop DIF FFT algorithm for $N = 2.3$

$$X(k) = \sum_{n=0}^{5} x(n)\,W_6^{nk} = \sum_{n=0}^{1} x(n)\,W_6^{nk} + \sum_{n=2}^{3} x(n)\,W_6^{nk} + \sum_{n=4}^{5} x(n)\,W_6^{nk}$$

$$= \sum_{n=0}^{1}\left[x(n) + x(n+2)\,W_6^{2k} + x(n+4)\,W_6^{4k}\right] W_6^{nk}$$

$$X(3k) = \sum_{n=0}^{1}[x(n) + x(n+2) + x(n+4)]\,W_6^{3nk}$$

$$X(3k+1) = \sum_{n=0}^{1}\left[x(n) + x(n+2)\,W_6^{2} + x(n+4)\,W_6^{4}\right] W_6^{n}\,W_6^{3nk}$$

$$X(3k+2) = \sum_{n=0}^{1}\left[x(n) + x(n+2)\,W_6^{4} + x(n+4)\,W_6^{2}\right] W_6^{2n}\,W_6^{3nk}$$

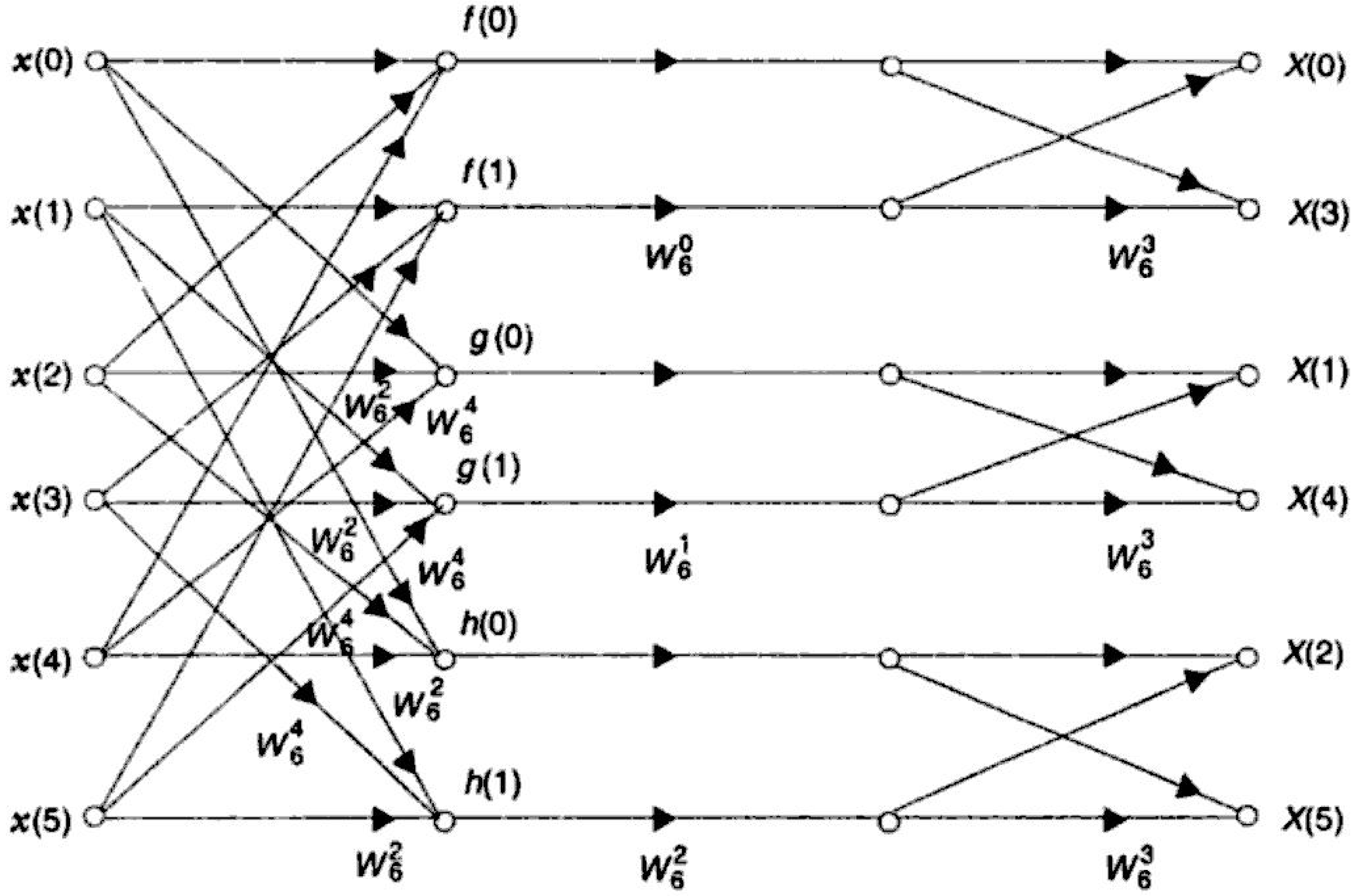

Fig. E6.32 (b) *DIF FFT Flow Diagram for Decomposing the DFT for N = 6 = 2.3*

$$f(n) = x(n) + x(n+2) + x(n+4)$$

$$g(n) = x(n) + x(n+2)W_6^2 + x(n+4)W_6^4$$

$$h(n) = x(n) + x(n+2)W_6^4 + x(n+4)W_6^2$$

$f(0) = x(0) + x(2) + x(4)$, $\qquad f(1) = x(1) + x(3) + x(5)$

$g(0) = x(0) + x(2)W_6^2 + x(4)W_6^4$, $\qquad g(1) = x(1) + x(3)W_6^2 + x(5)W_6^4$

$h(0) = x(0) + x(2)W_6^4 + x(4)W_6^2$, $\qquad h(1) = x(1) + x(3)W_6^4 + x(5)W_6^2$

(d) An L-point inverse FFT is performed on the product sequence obtained in step (c).

(e) The first $(M - 1)$ IFFT values obtained in step (d) is overlapped with the last $(M - 1)$ IFFT values for the previous block. Then addition is done to produce the final convolution output sequence $y(n)$.

(f) For the next data block, go to step (b).

Example 6.34 An FIR digital filter has the unit impulse response sequence, $h(n) = \{2, 2, 1\}$. Determine the output sequence in response to the input sequence $x(n) = \{3, 0, -2, 0, 2, 1, 0, -2, -1, 0\}$ using the overlap-add convolution method.

Solution The impulse response $h(n)$ has the length, $M = 3$. The length of the FFT/IFFT operation is selected as $L = 2^M = 2^3 = 8$. Then, $N = L - M + 1 = 8 - 3 + 1 = 6$, and the segmentation of the input sequence with the required zero padding is given in Fig. E6.34(a).

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	...
$x(n)$	3	0	−2	0	2	1	0	−2	−1	0	3	0	−2	0	...
$x_1(n)$	3	0	−2	0	2	1	0	0							
$x_2(n)$							0	−2	−1	0	3	0	0	0	
$x_3(n)$													−2	0	...

Fig. E6.34(a)

Steps (b), (c) and (d) are described below using the direct implementation of circular convolution.

Circular convolution of data blocks $x_1(n)$ and $x_2(n)$ with $h(n)$ padded with $(N - 1)$, i.e. five zeros is given in Fig. E6.34(b) and (c).

$x_1(n)$	0	−2	0	2	1	0	0	3	0	−2	0	2	1	0	0
$h(-(k-n))$	0	0	0	0	0	1	2	2							
$y_1(n)$								6	6	−1	−4	2	6	4	1

(b)

$x_2(n)$	−2	−1	0	3	0	0	0	0	−2	−1	0	3	0	0	0
$h(-(k-n))$	0	0	0	0	0	1	2	2							
$y_2(n)$								0	−4	−6	−4	5	6	3	0

(c)

Fig. E6.34 *(b) and (c)*

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$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n+l)\,y(n), \quad l = 0, \pm 1, \pm 2, \ldots \tag{6.42}$$

where index l is the (time) shift (or lag) parameter and the subscripts xy on the cross-correlation sequence $r_{xy}(l)$ show the sequences being correlated. The order of the subscripts, with x preceding y in Eq. 6.41, indicates that $x(n)$ is kept unshifted and $y(n)$ is shifted by l units in time, to the right for l positive and to the left for l negative. Similarly, in Eq. 6.42, $y(n)$ is kept unshifted and $x(n)$ is shifted by l units in time, to the left for l positive and to the right for l negative.

When the roles of $x(n)$ and $y(n)$ are reversed, the cross-correlation sequence becomes

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n)\,x(n-l) \tag{6.43}$$

or, equivalently,

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n+l)\,x(n) \tag{6.44}$$

Comparing Eq. 6.41 with Eq. 6.44 or Eq. 6.42 with Eq. 6.43, we find that

$$r_{xy}(l) = r_{yx}(-l) \tag{6.45}$$

This means that $r_{yx}(l)$ is the folded version of $r_{xy}(l)$, with respect to $l = 0$. Therefore, $r_{yx}(l)$ gives exactly the same information as $r_{xy}(l)$.

Autocorrelation Sequences

When $y(n) = x(n)$, the cross-correlation function becomes the autocorrelation function. As a result, $y(n)$ is replaced by $x(n)$ in Eq. 6.41 and Eq. 6.42 gives the autocorrelation function, $r_{xx}(l)$, which is defined as

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)\,x(n-l), \quad l = 0, \pm 1, \pm 2, \ldots \tag{6.46}$$

or, equivalently,

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n+l)\,x(n), \quad l = 0, \pm 1, \pm 2, \ldots \tag{6.47}$$

From the above equations, it is clear that the maximum autocorrelation value occurs at $l = 0$ because of an in-phase relationship between the two sequences. As l increases, the autocorrelation value increases.

Example 6.36 Determine the cross-correlation values of the two sequences $x(n) = \{1, 0, 0, 1\}$ and $h(n) = \{4, 3, 2, 1\}$.

6.38 *Compute the DFTs of the following sequences, where $N = 4$ using DIT algorithm.*

(a) $x(n) = 2^n$, (b) $x(n) = 2^{-n}$

(c) $x(n) = \sin\left(\frac{n\pi}{2}\right)$, (d) $x(n) = \cos\left(\frac{n\pi}{4}\right)$

Ans: (a) $X(k) = \{15, -3 + j6, -5, -3 - j6\}$

(b) $X(k) = \left(\frac{15}{8}, \frac{3}{4} - j\frac{3}{8}, \frac{5}{8}, \frac{3}{4} + j\frac{3}{8}\right)$

(c) $X(k) = (0, -j2, 0, j2)$

(d) $X(k) = (1, 1 - j\sqrt{2}, 1, 1 + j\sqrt{2})$

6.39 *Draw the butterfly line diagram for 8-point FFT calculation and briefly explain. Use decimation-in-time.*

6.40 *Find the DFT of the following sequence $x(n)$ using DIT FFT.*

$x(n) = (1, -1, -1, -1, 1, 1, 1, -1)$

Ans: $(0, -\sqrt{2} + j3.4142, 2 - j2, \sqrt{2} - j0.5858, 4, \sqrt{2} + j0.5858, 2 + j2, -\sqrt{2} - j3.4142)$

6.41 *Compute the 16-point DFT of the sequence*

$x(n) = \cos(\pi/2)$, $0 \le n \le 15$ *using DIT algorithm.*

6.42 *Find DFT (8-point) for a continuous time signal*

$x(t) = \sin(2\pi ft)$ *with* $f = 100$ *Hz*

6.43 *Compute the DFT of the sequence $x(n) = a^n$, where $N = 8$ and $a = 3$.*

6.44 *Compute the FFT for the sequence $x(n) = n^2 + 1$ where $N = 8$ using DIT algorithm.*

Ans: $X(k) = 100\ (1.48, -0.4686 + j0.7725, -0.24 + j0.32, -0.2731 + j0.1325, -0.28, -0.2731 - j0.1325, -0.24 - j0.32, -0.4686 - j0.7725)$

6.45 *Compute the FFT for the sequence $x(n) = n + 1$ where $N = 8$ using DIT algorithm.*

Ans: $X(k) = (36, -4 + j9.656, -4 + j4, -4 + j1.6568, -4, -4 - j1.6568, -4 - j4, -4 - j9.656)$

6.46 *Compute the DFT coefficients of a finite duration sequence (0, 1, 2, 3, 0, 0, 0, 0).*

Ans: $X(k) = (6, -\sqrt{2} - j4.8284, -2 + j2, \sqrt{2} - j0.8284, -2, \sqrt{2} + j0.8284, -2 - j2, -\sqrt{2} + j4.8284)$

6.47 *Draw the flow graph of an 8-point DIF FFT and explain.*

6.48 *Draw the butterfly line diagram for 8-point FFT calculation and briefly explain. Use decimation-in-frequency.*

6.49 *Repeat Q.No.38 using DIF algorithm.*

Ans: *Same as in Q.No. 38.*

6.50 *Draw the butterfly diagram for 16-point FFT calculation and briefly explain. Use decimation in frequency.*

The group delay is defined as the delayed response of the filter as a function of ω to a signal.

Linear phase filters are those filters in which the phase delay and group delay are constants, i.e. independent of frequency. Linear phase filters are also called constant time delay filters. Let us obtain the conditions FIR filters must satisfy in order to have constant phase and group delays and hence obtain the conditions for having a linear phase. For the phase response to be linear

$$\frac{\Phi(\omega)}{\omega} = -\tau, \quad -\pi \le \omega \le +\pi$$

Therefore,

$$\Phi(\omega) = -\omega\tau$$

where τ is a constant phase delay expressed in number of samples. Using Eq. 7.2,

$$\Phi(\omega) = \tan^{-1}\frac{\operatorname{Im} H(e^{j\omega})}{\operatorname{Re} H(e^{j\omega})} = -\omega\tau$$

or

$$\omega\tau = \tan^{-1}\frac{\sum_{n=0}^{M-1} h(n)\sin\omega n}{\sum_{n=0}^{M-1} h(n)\cos\omega n}$$

or

$$\tan\omega\tau = \frac{\sum_{n=0}^{M-1} h(n)\sin\omega n}{\sum_{n=0}^{M-1} h(n)\cos\omega n}$$

Simplifying, we get

$$\sum_{n=0}^{M-1} h(n)\sin(\omega\tau - \omega n) = 0 \tag{7.3}$$

and a solution to Eq.7.3 is given by

$$\tau = \frac{(M-1)}{2} \tag{7.4}$$

and

$$h(n) = h(M-1-n) \text{ for } 0 < n < M-1 \tag{7.5}$$

If Eqs 7.4 and 7.5 are satisfied, then the FIR filter will have constant phase and group delays and thus the phase of the filter will be linear. The phase and group delays of the linear phase FIR filter are equal and

where

$$H_r(f) = \sum_{n=-\infty}^{\infty} h(n) \cos 2\pi n f T \tag{7.11}$$

and

$$H_i(f) = \sum_{n=-\infty}^{\infty} h(n) \sin 2\pi n f T \tag{7.12}$$

From Eq. 7.11 and 7.12 we infer that $H_r(f)$ is an even function and $H_i(f)$ is an odd function of frequency. If $h(nT)$ is an even sequence, the imaginary part of the transfer function, $H_i(f)$, will be zero and if $h(nT)$ is an odd sequence, the real part of the transfer function, $H_r(f)$, will be zero. Thus an even unit impulse response yields a real transfer function and an odd unit impulse response yields an imaginary transfer function. A real transfer function has 0 or $\pm\,\pi$ radians phase shift, while an imaginary transfer function has $\pm\,\pi/2$ radians phase shift. Therefore, by making the unit impulse response either even or odd, one can generate a transfer function that is either real or imaginary.

In the design of digital filters two interesting situations are often sought after.

(i) For filtering applications, the main interest is in the amplitude response of the filter, where some portion of the input signal spectrum is to be attenuated and some portion is to be passed to the output with no attenuation. This should be accomplished without phase distortion. Thus the amplitude response is realised by using only a real transfer function. That is

$$H(e^{j\omega}) = H_r(f)$$

and

$$H_i(f) \equiv 0$$

(ii) For filtering plus quadrature phase shift. the applications include integrators, differentiators and Hilbert transform devices. For all these applications the desired transfer function is imaginary. Thus, the required amplitude response is realised by using only $H_i(f)$. That is

$$H(e^{j\omega}) = j\,H_i(f)$$

and

$$H_r(f) \equiv 0$$

Design Equations

The term $H(e^{j\omega})$ is periodic in the sampling frequency and hence both $H_r(f)$ and $H_i(f)$ are also periodic in the sampling frequency. Both $H_r(f)$ and $H_i(f)$ can be expanded in a Fourier series. Since the real part of the transfer function, $H_r(f)$, is an even function of frequency, its Fourier series will be of the form

$$H_r(f) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f T) \tag{7.13}$$

type-I design and type-II design. In the Type-I design, the set of frequency samples includes the sample at frequency $\omega = 0$. In some cases, it may be desirable to omit the sample at $\omega = 0$ and use some other set of samples. Such a design procedure is referred to as the Type-II design.

Type-I Design

The samples are taken at the frequency

$$\omega_k = \frac{2\pi k}{M}, \quad = 0, 1, \ldots, M-1 \tag{7.22}$$

The samples of the desired frequency response at these frequencies are given by

$$\begin{aligned}\tilde{H}(k) &= H_d\left(e^{j\omega}\right)\Big|_{\omega=\omega_k}, \quad k = 0, 1, \ldots, M-1 \\ &= H_d(e^{j2\pi k/M}), \quad k = 0, 1, \ldots, M-1\end{aligned} \tag{7.23}$$

This set of points can be considered as DFT samples, then the filter coefficients $h(n)$ can be computed using the IDFT,

$$h(n) = \frac{1}{M}\sum_{k=0}^{M-1} \tilde{H}(k)\, e^{j2\pi nk/M}, \; n = 0, 1, \ldots, M-1 \tag{7.24}$$

If these numbers are all real, then these can be considered as the impulse response coefficients of an FIR filter. This can happen when all the complex terms appear in complex conjugate pairs, and then all the terms can be matched by comparing the exponentials. The term $\tilde{H}(k)$ $e^{j2\pi nk/M}$ should be matched with the term that has the exponential $e^{-j2\pi nk/M}$ as a factor. The matching terms are then $\tilde{H}(k)\, e^{j2\pi nk/M}$ and $\tilde{H}(M-k)\, e^{j2\pi n(M-k)/M}$ since $2\pi n\,(M-k)/M = 2\pi n - (2\pi nk/M)$. These terms are complex conjugates if $\tilde{H}(0)$ is real and

(i) For M odd:

$$\tilde{H}(M-k) = \tilde{H}^*(k), \qquad k = 1, 2, \ldots, (M-1)/2 \tag{7.25}$$

(ii) For M even:

$$\begin{aligned}\tilde{H}(M-k) &= \tilde{H}^*(k), \qquad k = 1, 2, \ldots, M/2-1 \\ \tilde{H}(M/2) &= 0\end{aligned} \tag{7.26}$$

The desired frequency response $H_d(e^{j\omega})$ is chosen such that it satisfies the Eqs 7.25 and 7.26 for M odd or even, respectively. The filter coefficients can then be written as

$$h(n) = \frac{1}{M}\left[\tilde{H}(0) + 2\sum_{k=1}^{(M-1)/2} \mathrm{Re}\left[\tilde{H}(k)\, e^{j2\pi nk/M}\right]\right], \; M \text{ odd} \tag{7.27}$$

and $$h(n) = \frac{1}{M}\left[\tilde{H}(0) + 2\sum_{k=1}^{M/2-1} \mathrm{Re}\left[\tilde{H}(k)\, e^{j2\pi nk/M}\right]\right], \; M \text{ even} \tag{7.28}$$

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7.4.3.3 Hanning Window Function

The window function of a causal Hanning window is given by

$$w_{Hann}(n) = \begin{cases} 0.5 - 0.5\cos\dfrac{2\pi n}{M-1}, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases} \tag{7.44}$$

The window function of a non-causal Hanning window is expressed by

$$w_{Hann}(n) = \begin{cases} 0.5 + 0.5\cos\dfrac{2\pi n}{M-1}, & 0 < |n| < \dfrac{M-1}{2} \\ 0, & \text{otherwise} \end{cases}$$

The width of the main lobe is approximately $8\pi/M$ and the peak of the first side lobe is at –32dB.

7.4.3.4 Blackman Window Function

The window function of a causal Blackman window is expressed by

$$w_B(n) = \begin{cases} 0.42 - 0.5\cos\dfrac{2\pi n}{M-1} + 0.08\cos\dfrac{4\pi n}{M-1}, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

The window function of a non-causal Blackman window is given by

$$w_B(n) = \begin{cases} 0.42 + 0.5\cos\dfrac{2\pi n}{M-1} + 0.08\cos\dfrac{4\pi n}{M-1}, & \text{for } |n| < \dfrac{M-1}{2} \\ 0, & \text{otherwise} \end{cases} \tag{7.45}$$

The width of the main lobe is approximately $12\pi/M$ and the peak of the first side-lobe is at –58dB.

7.4.3.5 Bartlett Window Function

The window function of a non-causal Bartlett window is expressed by

$$w_{Bart}(n) = \begin{cases} 1+n, & -\dfrac{M-1}{2} < n < 1 \\ 1-n, & 1 < n < \dfrac{M-1}{2} \end{cases}$$

Table 7.1 gives the important frequency-domain characteristics of some window functions.

Table 7.1 *Frequency-Domain Characteristics of Some Window Functions*

Type of Window	*Approximate Transition Width of Main Lobe*	*Minimum Stopband Attenuation (dB)*	*Peak of first Sidelobe (dB)*
Rectangular	$4\pi / M$	–21	– 13
Bartlett	$8\pi / M$	–25	– 27
Hanning	$8\pi / M$	–44	– 32
Hamming	$8\pi / M$	–53	– 43
Blackman	$12\pi / M$	–74	–58

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Solution The filter coefficients are given by

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-3\pi/4}^{3\pi/4} e^{-j3\omega}\, e^{j\omega n}\, d\omega$$

$$h_d(n) = \frac{\sin 3\pi(n-3)/4}{\pi(n-3)}, \quad n \neq 3 \text{ and } h_d(3) = \frac{3}{4}$$

The filter coefficients are,

$h_d(0) = 0.0750$, $h_d(1) = -0.1592$, $h_d(2) = 0.2251$, $h_d(3) = 0.75$
$h_d(4) = 0.2251$, $h_d(5) = -0.1592$, $h_d(6) = 0.0750$

The Hamming window function is,

$$w(n) = \begin{cases} 0.54 - 0.46\cos\dfrac{2\pi n}{M-1}, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

Therefore, with $M = 7$,

$w(0) = 0.08$, $w(1) = 0.31$, $w(2) = 0.77$, $w(3) = 1$, $w(4) = 0.77$,

$w(5) = 0.31$, $w(6) = 0.08$.

The filter coefficients of the resultant filter are then,

$$h(n) = h_d(n).w(n) \quad n = 0, 1, 2, 3, 4, 5, 6.$$

Therefore,

$h(0) = 0.006$, $h(1) = -0.0494$, $h(2) = 0.1733$, $h(3) = 0.75$,
$h(4) = 0.1733$, $h(5) = -0.0494$ and $h(6) = 0.006$.

The frequency response is given by

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=0}^{6} h(n)\, e^{-j\omega n} \\ &= e^{-j3\omega}\,[\, h(3) + 2h(0)\cos 3\omega + 2h(1)\cos 2\omega + 2h(2)\cos\omega] \\ &= e^{-j3\omega}\,[\, 0.75 + 0.3466\cos\omega - 0.0988\cos 2\omega + 0.012\cos\omega] \end{aligned}$$

Example 7.9 Design an FIR digital filter to approximate an ideal low-pass filter with passband gain of unity, cut-off frequency of 850 Hz and working at a sampling frequency of $f_s = 5000$ Hz. The length of the impulse response should be 5. Use a rectangular window.

Solution The desired response of the ideal low-pass filter is given by

$$H_d(e^{j\omega}) = \begin{cases} 1, & 0 \le f \le 850 \text{ Hz} \\ 0, & f > 850 \text{ Hz} \end{cases}$$

The above response can be equivalently specified in terms of the normalised ω_c. The normalised $\omega_c = 2\pi f_c / f_s = 2\pi(850)/(5000) = 1.068$ rad/sec. Hence, the desired response is

High-Pass FIR Filter

$$h_d(n) = \begin{cases} -\left(\dfrac{2f_c}{F}\right)\dfrac{\sin 2\pi n f_c / F}{2\pi n f_c / F}, & \text{for } n > 0 \\ 1 - \dfrac{2f_c}{F}, & \text{for } n = 0 \end{cases} \tag{7.65}$$

where

$$f_c = 0.5\,(f_p + f_s) \quad \text{and} \quad \Delta F = f_p - f_s \tag{7.66}$$

Bandpass FIR Filter

$$h_d(n) = \begin{cases} \dfrac{1}{n\pi}\left[\sin\left(2\pi n f_{c2} / F\right) - \sin\left(2\pi n f_{c1} / F\right)\right], & \text{for } n > 0 \\ \dfrac{2}{F}\left(f_{c2} - f_{c1}\right), & \text{for } n = 0 \end{cases} \tag{7.67}$$

where

$$\begin{aligned} f_{c1} &= f_{p1} - \frac{\Delta F}{2} & f_{c2} &= f_{p2} + \frac{\Delta F}{2} \\ \Delta F_l &= f_{p1} - f_{s1} & \Delta F_h &= f_{s2} - f_{p2} \\ \Delta F &= \min\left[\Delta F_l, \Delta F_h\right] \end{aligned} \tag{7.68}$$

Bandstop FIR Filter

$$h_d(n) = \begin{cases} \dfrac{1}{n\pi}\left[\sin\left(2\pi n f_{c1} / F\right) - \sin\left(2\pi n f_{c2} / F\right)\right], & \text{for } n > 0 \\ \dfrac{2}{F}\left(f_{c1} - f_{c2}\right) + 1, & \text{for } n = 0 \end{cases} \tag{7.69}$$

where

$$\begin{aligned} f_{c1} &= f_{p1} + \frac{\Delta F}{2} & f_{c2} &= f_{p2} - \frac{\Delta F}{2} \\ \Delta F_l &= f_{s1} - f_{p1} & \Delta F_h &= f_{p2} - f_{s2} \\ \Delta F &= \min\left[\Delta F_l, \Delta F_h\right] \end{aligned} \tag{7.70}$$

Example 7.10 Design a low-pass digital FIR filter using Kaiser window satisfying the specifications given below.

Passband cut-off frequency, f_p = 150 Hz, stopband cut-off frequency, f_s = 250 Hz, passband ripple, A_p = 0.1 dB, stopband attenuation, A_s = 40 dB and sampling frequency, F = 1000 Hz.

Solution A computer program can be written for the design of Kaiser window digital filter using the functions given in Appendix. The computer output is given below.

From Eq. 7.53, δ = 0.005756.

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There are four different cases that result in a linear phase FIR filter, viz., (i) symmetric unit impulse response and the length of the filter, M odd, (ii) symmetric unit impulse response and M even, (iii) antisymmetric unit impulse response and M odd, and (iv) antisymmetric unit impulse response and M even. The first case is discussed below in detail and other cases are listed in Table 7.2.

In the symmetric unit impulse response case, $h(n) = H(M - 1 - n)$. The real-valued frequency response characteristics $|H(e^{j\omega})| = |H_r(e^{j\omega})|$, given in Eq. 7.14, is

$$|H(e^{j\omega})| = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{(M-3)}{2}} h(n)\cos\omega\left(\frac{M-1}{2} - n\right) \tag{7.74}$$

Let $k = (M - 1)/2 - n$. Then Eq. 7.74 can be written as

$$|H(e^{j\omega})| = \sum_{k=0}^{\frac{(M-1)}{2}} a(k)\cos\omega k \tag{7.75}$$

where

$$a(0) = h\left(\frac{M-1}{2}\right) \tag{7.76}$$

$$a(k) = 2h\left(\frac{M-1}{2} - k\right) \text{ for } 1 \le k \le \frac{M-1}{2}$$

The magnitude response for the other cases are similarly converted to a compact form as given in Table 7.2.

Table 7.2 *Magnitude Response Functions for Linear Phase FIR Filters*

Filter Type	$Q(\omega)$	$P(\omega)$
Case (i) - Symmetric and M odd $h(n) = h(M-1-n)$	1	$\sum_{k=0}^{(M-1)/2} a(k)\cos\omega k$
Case (ii) - Symmetric and M even $h(n) = h(M-1-n)$	$\cos\frac{\omega}{2}$	$\sum_{k=0}^{(M/2)-1} b(k)\cos\omega k$
Case (iii) - Antisymmetric and M odd $h(n) = -h(M-1-n)$	$\sin\omega$	$\sum_{k=0}^{(M-3)/2} c(k)\cos\omega k$
Case (iv) - Antisymmetric and M even $h(n) = -h(M-1-n)$	$\sin\frac{\omega}{2}$	$\sum_{k=0}^{(M/2)-1} d(k)\cos\omega k$

From Table 7.2, it can be seen that the magnitude response function can be written as given in Eq. 7.77, for the four different cases.

$$|H(e^{j\omega})| = Q(\omega)P(\omega) \tag{7.77}$$

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7.28 What is an FIR half-band digital filter ? Explain with a suitable illustration.

7.29 What is an optimal linear phase FIR filter ? What parameters are optimised in these filters ?

7.30 State and explain the alternation theorem.

7.31 What are extra ripple filters ?

7.32 What are maximal ripple filters ?

7.33. Explain the Remez exchange algorithm used in the design of optimal filters.

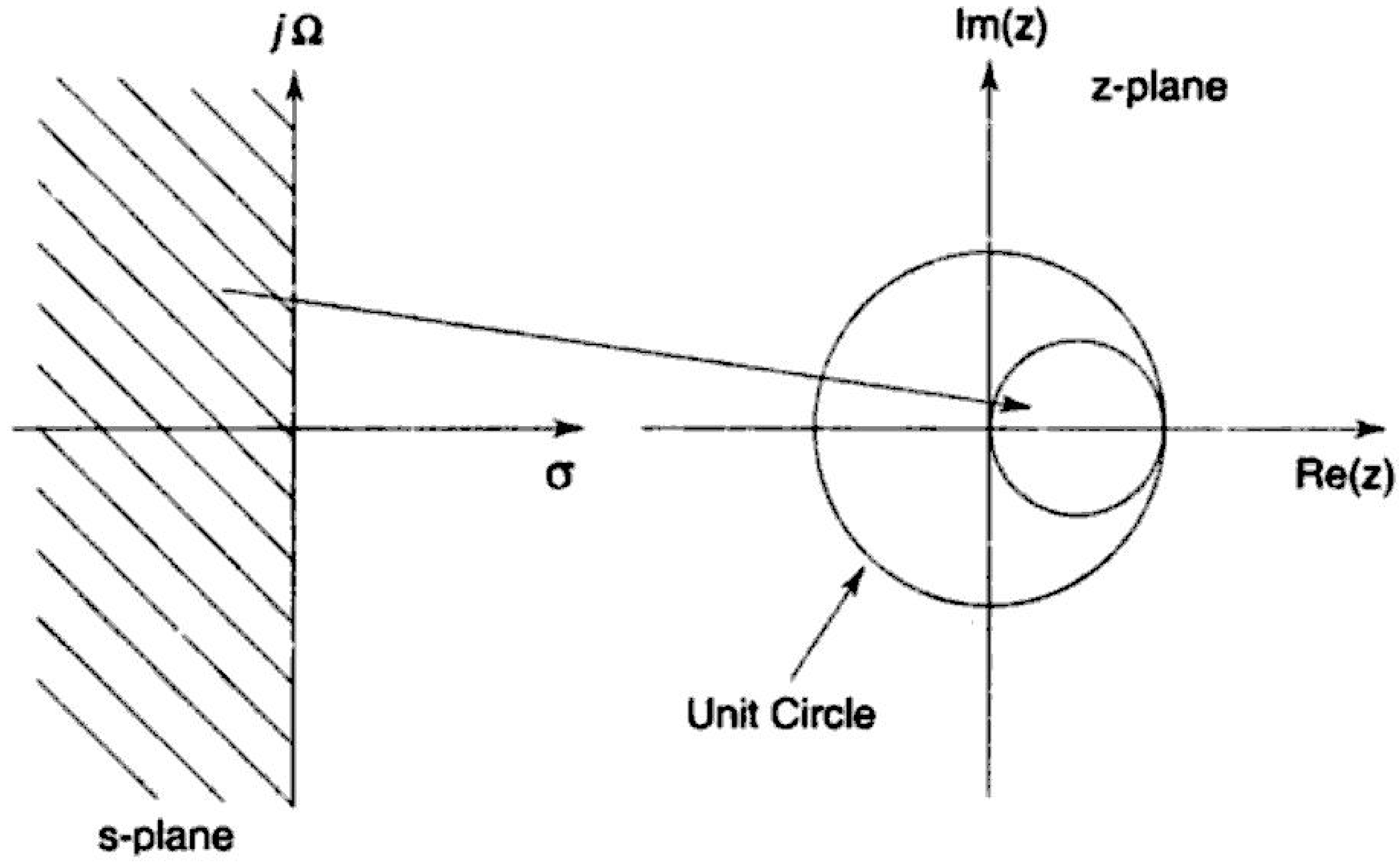

Fig. 8.2 *The Mapping of Eq. 8.5 into the z-plane*

It can be seen that the mapping of Eq. 8.5 takes the left-half plane of s-domain into the corresponding points inside the circle of radius 0.5 and centre at $z = 0.5$, and the right-half of the s-plane is mapped outside the unit circle. As a result, this mapping results in a stable analog filter transformed into a stable digital filter; however, as the locations of poles in the z-domain are confined to smaller frequencies, this design method can be used only for transforming analog low-pass filters and bandpass filters having smaller resonant frequencies. Neither a high-pass filter nor a band reject filter can be realised using this technique.

The forward difference can be substituted for the derivative instead of the backward difference. This gives,

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \frac{y(nT+T) - y(nT)}{T}$$

$$= \frac{y(n+1) - y(n)}{T} \tag{8.12}$$

The transformation formula will be

$$s = \frac{z-1}{T} \tag{8.13}$$

or,

$$z = 1 + sT \tag{8.14}$$

The mapping of Eq. 8.14 is shown in Fig. 8.3. This results in a worse situation than the backward difference substitution for the derivative. When $s = j\,\Omega$, the mapping of these points in the s-domain results in a straight line in the z-domain with coordinates $(z_{\text{real}}, z_{\text{imag}}) = (1, \Omega T)$. Consequently, stable analog filters do not always map into stable digital filters.

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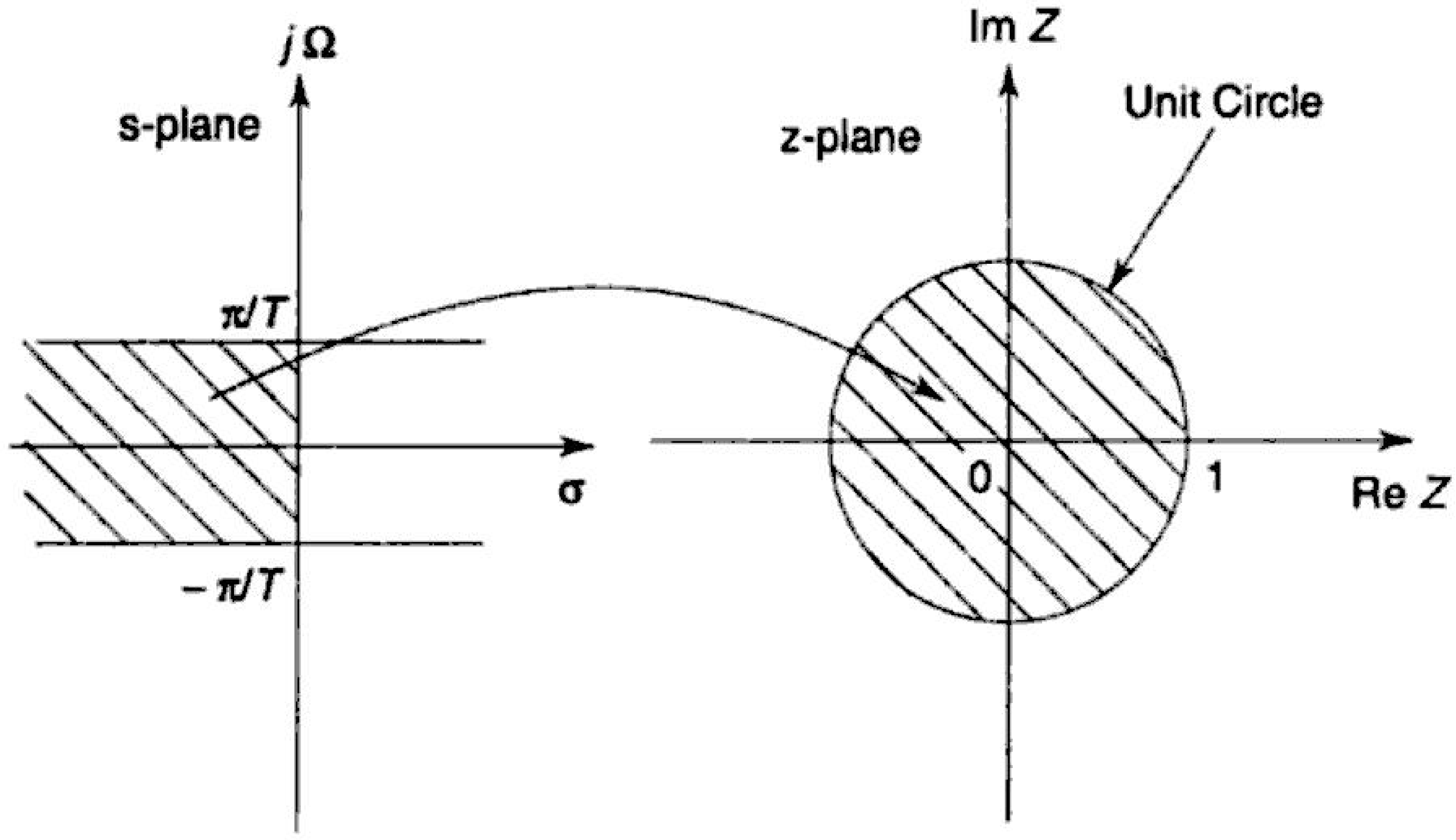

Fig. 8.4 *The Mapping of* $z = e^{sT}$

Example 8.4 Convert the analog filter into a digital filter whose system function is

$$H(s) = \frac{s + 0.2}{(s + 0.2)^2 + 9}$$

Use the impulse invariant technique. Assume $T = 1\text{s}$.

Solution The system response of the analog filter is of the standard form

$$H(s) = \frac{s + a}{(s + a)^2 + b^2}$$

where $a = 0.2$ and $b = 3$. The system response of the digital filter can be obtained using Eq. 8.27.

$$H(z) = \frac{1 - e^{-aT}(\cos bT)\, z^{-1}}{1 - 2e^{-aT}(\cos bT)\, z^{-1} + e^{-2aT} z^{-2}}$$

$$= \frac{1 - e^{-0.2T}(\cos 3T)\, z^{-1}}{1 - 2e^{-0.2T}(\cos 3T)\, z^{-1} + e^{-0.4T} z^{-2}}$$

Taking $T = 1\text{s}$,

$$H(z) = \frac{1 - (0.8187)(-0.99)\, z^{-1}}{1 - 2(0.8187)(-0.99)\, z^{-1} + 0.6703\, z^{-2}}$$

That is,

$$H(z) = \frac{1 + (0.8105)\, z^{-1}}{1 + 1.6210\, z^{-1} + 0.6703\, z^{-2}}$$

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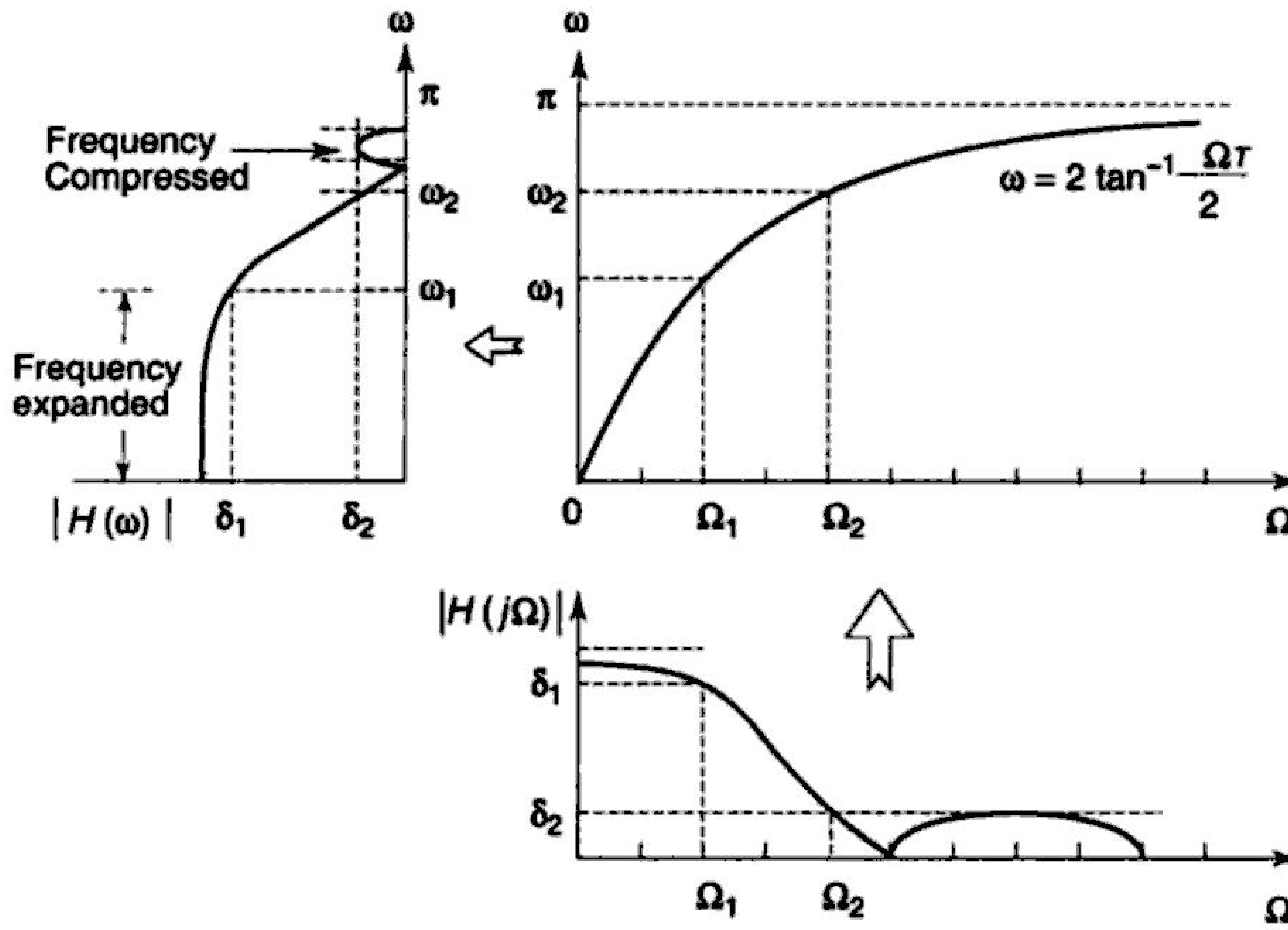

Fig. 8.5 *Relationship Between ω and Ω as Given in Eq. 8.40*

The sampling period is obtained from the above equation using

$$T = \frac{2}{\Omega_c} \tan \frac{\omega_r}{2} = \frac{2}{3} \tan \frac{\pi}{8} = 0.276 \text{ s}$$

Using bilinear transformation,

$$H(z) = H(s)\big|_{s = \frac{2}{T}\frac{(z-1)}{(z+1)}}$$

$$H(z) = \frac{\frac{2}{T}\frac{(z-1)}{(z+1)} + 0.1}{\left[\frac{2}{T}\frac{(z-1)}{(z+1)} + 0.1\right]^2 + 9}$$

$$= \frac{(2/T)(z-1)(z+1) + 0\cdot 1(z+1)^2}{[(2/T)(z-1) + 0.1(z+1)]^2 + 9(z+1)^2}$$

Substituting $T = 0.276$ s,

$$H(z) = \frac{1 + 0.027\, z^{-1} - 0.973\, z^{-2}}{8.572 - 11.84\, z^{-1} + 8.177\, z^{-2}}$$

Example 8.8 Apply bilinear transformation to

$$H(s) = \frac{2}{(s+1)(s+3)}$$

with $T = 0.1$s.

Step (ii) Determination of the order of the filter.
From Eq. 8.46,

$$N \geq \frac{1}{2}\frac{\log\left\{[(1/\delta_2^2 - 1)/(1/\delta_1^2 - 1)]\right\}}{\log(\Omega_2/\Omega_1)}$$

$$= \frac{1}{2}\frac{\log\{24/0.2346\}}{\log(2.414)} = 2.626$$

Let $N = 3$.
Step (iii) Determination of – 3 dB cut-off frequency.
From Eq. 8.47,

$$\Omega_c = \frac{\Omega_1}{\left[(1/\delta_1^2) - 1\right]^{1/2N}} = \frac{2}{\left[(1/0.9^2) - 1\right]^{1/6}} = 2.5467$$

Step (iv) Determination of $H_a(s)$.
From Eq. 8.49,

$$H(s) = \frac{B_0\,\Omega_c}{s + c_0\,\Omega_c}\prod_{k=1}^{(N-1)/2}\frac{B_k\,\Omega_c^2}{s^2 + b_k\,\Omega_c\,s + c_k\,\Omega_c^2}$$

$$= \left(\frac{B_0\,\Omega_c}{s + c_0\,\Omega_c}\right)\left(\frac{B_1\,\Omega_c^2}{s^2 + b_1\,\Omega_c\,s + c_1\,\Omega_c^2}\right)$$

From Eq. 8.50,

$$b_1 = 2\sin\frac{\pi}{6} = 1, \quad c_0 = 1 \quad c_1 = 1$$

$B_0\,B_1 = 1$. Therefore $B_0 = B_1 = 1$.

Therefore,

$$H(s) = \left(\frac{2.5467}{s + 2.5467}\right)\left(\frac{6.4857}{s^2 + 2.5467\,s + 6.4857}\right)$$

Step (v) Determination of $H(z)$.

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\frac{(z-1)}{(z+1)}}$$

That is,

$$H(z) = \left(\frac{2.5467}{2\dfrac{(z-1)}{(z+1)} + 2.5467}\right)\left(\frac{6.4857}{\left[2\,\dfrac{(z-1)}{(z+1)}\right]^2 + 2.5467\,s + 6.4857}\right)$$

Simplifying we get,

$$H(z) = \frac{16.5171\,(z+1)^3}{70.83\,z^3 + 31.1205\,z^2 + 27.2351\,z + 2.948}$$

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$$\frac{\varepsilon^2\, C_N^2\, (\Omega_2/\Omega)}{1+\varepsilon^2\, C_N^2\, (\Omega_2/\Omega)} \le \delta_2^2, \quad \Omega \ge \Omega_2 \tag{8.63b}$$

When $\Omega = \Omega_2$, Eq. 8.63b becomes

$$\delta_2^2 = \frac{\varepsilon^2}{1+\varepsilon^2}$$

Rearranging,

$$\varepsilon = \frac{\delta_2}{\left(1-\delta_2^2\right)^{0.5}} \tag{8.64}$$

When $\Omega = \Omega_c$, Eq. 8.63a becomes,

$$0.5 = \frac{\varepsilon^2\, C_N^2\, (\Omega_2/\Omega_c)}{1+\varepsilon^2\, C_N^2\, (\Omega_2/\Omega_c)}$$

$$0.5 + 0.5\,\varepsilon^2\, C_N^2\, (\Omega_2/\Omega_c) = \varepsilon^2\, C_N^2\, (\Omega_2/\Omega_c)$$

or, simplifying

$$C_N\, (\Omega_2/\Omega_c) = \frac{1}{\varepsilon} \tag{8.65}$$

Using Eq. 8.53,

$$\cosh\, [N \cosh^{-1}(\Omega_2/\Omega_c)] = \frac{1}{\varepsilon} \tag{8.66}$$

From Eqs. 8.66 and 8.64 we can get the order of the filter, N.

$$N = \frac{\cosh^{-1}(1/\varepsilon)}{\cosh^{-1}(\Omega_2/\Omega_c)} = \frac{\cosh^{-1}\left[\dfrac{1}{\delta_2^2} - 1\right]^{0.5}}{\cosh^{-1}(\Omega_2/\Omega_c)} \tag{8.67}$$

The value of N is chosen to be the nearest integer greater than the value given by Eq. 8.67.

8.8 ELLIPTIC FILTERS

The elliptic filter is sometimes called the Cauer filter. This filter has equiripple passband and stopband. Among the filter types discussed so far, for a given filter order, passband and stopband deviations, elliptic filters have the minimum transition bandwidth. The magnitude response of an odd ordered elliptic filter is shown in Fig. 8.9. The magnitude squared response is given by

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2\, U_N\, (\Omega/\Omega_c)} \tag{8.68}$$

where $U_N(x)$ is the Jacobian elliptic function of order N and ε is a constant related to the passband ripple.

8.12 *An analog filter has the following system function. Convert this filter into a digital filter using the impulse invariant technique.*

$$H(s) = \frac{1}{(s+0.1)^2 + 9}$$

8.13 *Convert the analog filter to a digital filter whose system function is*

$$H(s) = \frac{1}{(s+2)^3}$$

8.14 *Convert the analog filter to a digital filter whose system function is*

$$H(s) = \frac{36}{(s+0.1)^2 + 36}$$

The digital filter should have a resonant frequency of $\omega_r = 0.2\,\pi$. Use impulse invariant mapping.

8.15 *What is bilinear transformation ?*

8.16 *Compare bilinear transformation with other transformations based on their stability.*

8.17 *Obtain the transformation formula for the bilinear transformation.*

8.18 *An analog filter has the following system function. Convert this filter into a digital filter using bilinear transformation.*

$$H(s) = \frac{1}{(s+0.2)^2 + 16}$$

8.19 *Convert the analog filter to a digital filter whose system function is*

$$H(s) = \frac{1}{(s+2)^2\,(s+1)}$$

using bilinear transformation.

8.20 *Convert the analog filter to a digital filter whose system function is*

$$H(s) = \frac{36}{(s+0.1)^2 + 36}$$

The digital filter should have a resonant frequency of $\omega_r = 0.2\,\pi$. Use bilinear transformation.

8.21 *What is meant by frequency warping ? What is the cause of this effect ?*

8.22 *Describe Butterworth filters ?*

8.23 *Comment on the passband and stopband characteristics of Butterworth filters.*

8.24 *Describe Chebyshev filters ?*

8.25 *Describe inverse Chebyshev filters ?*

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For example, the difference equation of the first-order digital system may be written as

$$y(n) = a_1 y(n-1) + x(n) + b_1 x(n-1)$$

The basic realisation block diagram for this equation and the corresponding structure of the signal flow graph are shown in Figs 9.2 (a) and (b). Here, it is clear that there is direct correspondence between branches in the digital realisation structure and branches in the signal flow graph. But in the signal flow graph, the nodes represent both branch points and adders in the digital realisation block diagram.

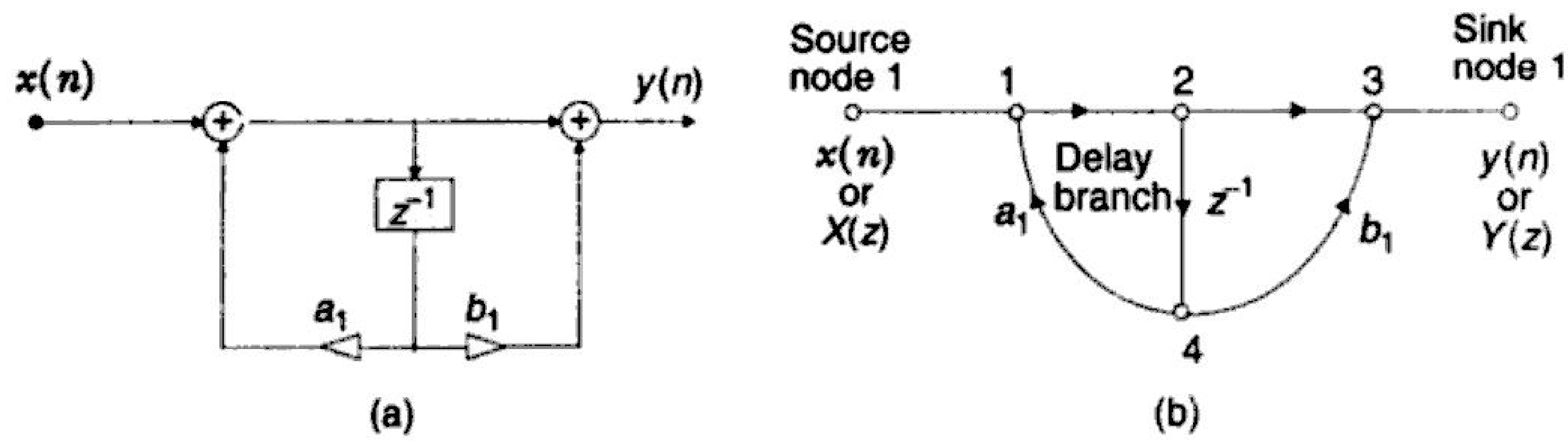

Fig. 9.2 *(a) Basic Realisation Block Diagram Representing a First-order Digital System and*
(b) Its Corresponding Signal Flow Graph.

Advantages of representing the digital system in block diagram form

(i) Just by inspection, the computation algorithm can be easily written
(ii) The hardware requirements can be easily determined
(iii) A variety of equivalent block diagram representations can be easily developed from the transfer function
(iv) The relationship between the output and the input can be determined.

9.2.1 Canonic and Non-Canonic Structures

If the number of delays in the realisation block diagram is equal to the order of the difference equation or the order of the transfer function of a digital filter, then the realisation structure is called **canonic**. Otherwise, it is a **non-canonic** structure.

9.3 BASIC STRUCTURES FOR IIR SYSTEMS

Causal IIR systems are characterised by the constant coefficient difference equation of Eq. 9.1 or equivalently, by the real rational transfer function of Eq. 9.2. From these equations, it can be seen that the realisation of infinite duration impulse response (IIR) systems involves a recursive computational algorithm. In this section, the most important filter structures namely direct Forms I and II, cascade and parallel realisations for IIR systems are discussed.

The direct Form II realisation requires only the larger of M or N storage elements. When compared to direct Form I realisation, the direct Form II uses the minimum number of storage elements and hence said to be a canonic structure. However, when the addition is performed sequentially, the direct Form II needs two adders instead of one adder required for the direct form I.

The Direct Form II realisation network structures are shown in Figs 9.6 and 9.7.

Though the direct Forms I and II are commonly employed, they have two drawbacks, viz. (i) they lack hardware flexibility and (ii) due to finite precision arithmetic, as to be discussed in Chapter ten, the sensitivity of the coefficients to quantisation effects increases with the order of the filter. This sensitivity may change the coefficient values and hence the frequency response, thereby causing the filter to become unstable. To overcome these effects, the cascade and parallel realisations can be implemented.

Example 9.2 Determine the direct Forms I and II realisations for a third-order IIR transfer function.

$$H(z)=\frac{0.28z^2+0.319z+0.04}{0.5z^3+0.3z^2+0.17z-0.2}$$

Solution Multiplying the transfer function numerator and denominator by $2z^{-3}$, we obtain the standard form of the transfer function.

$$H(z)=\frac{0.56z^{-1}+0.638z^{-2}+0.08z^{-3}}{1+0.6z^{-1}+0.34z^{-2}-0.4z^{-3}}$$

The direct Forms I and II realisations of the above transfer function are shown in Figs E 9.2(a) and (b) respectively.

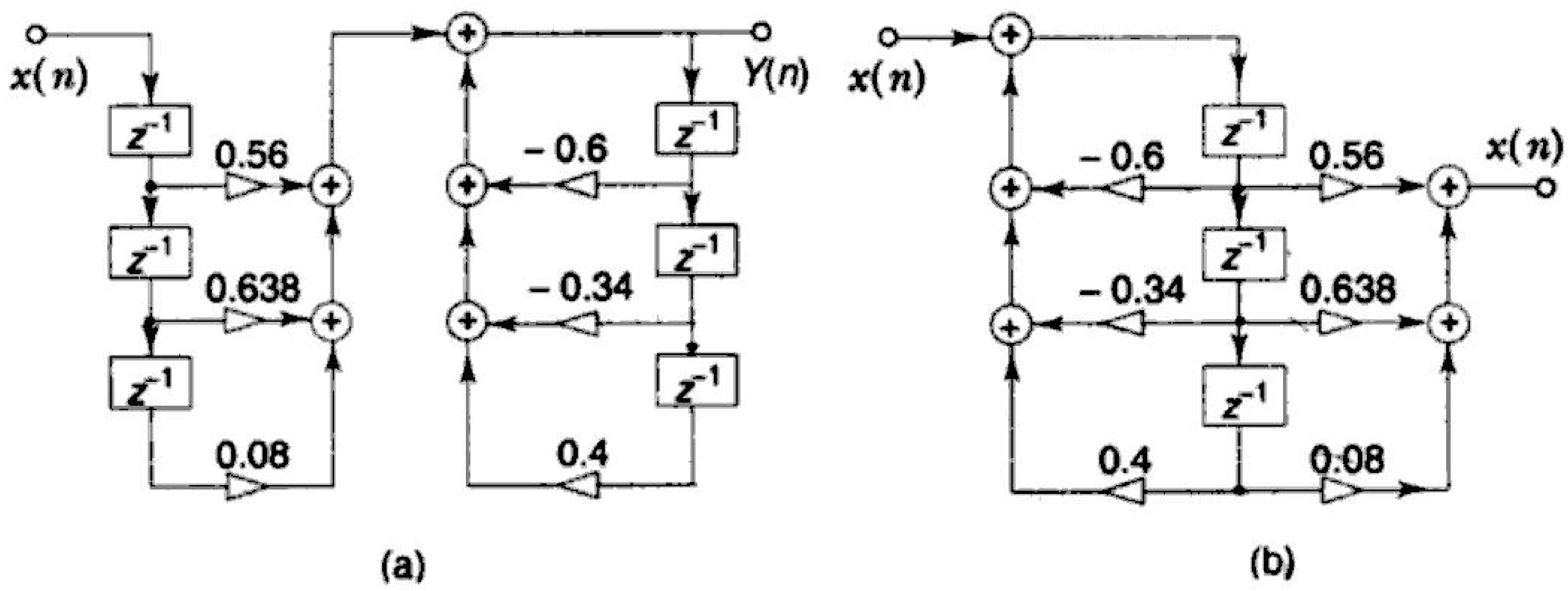

Fig. E9.2

The cascade realisation of the system transfer function is shown in Fig. E9.4.

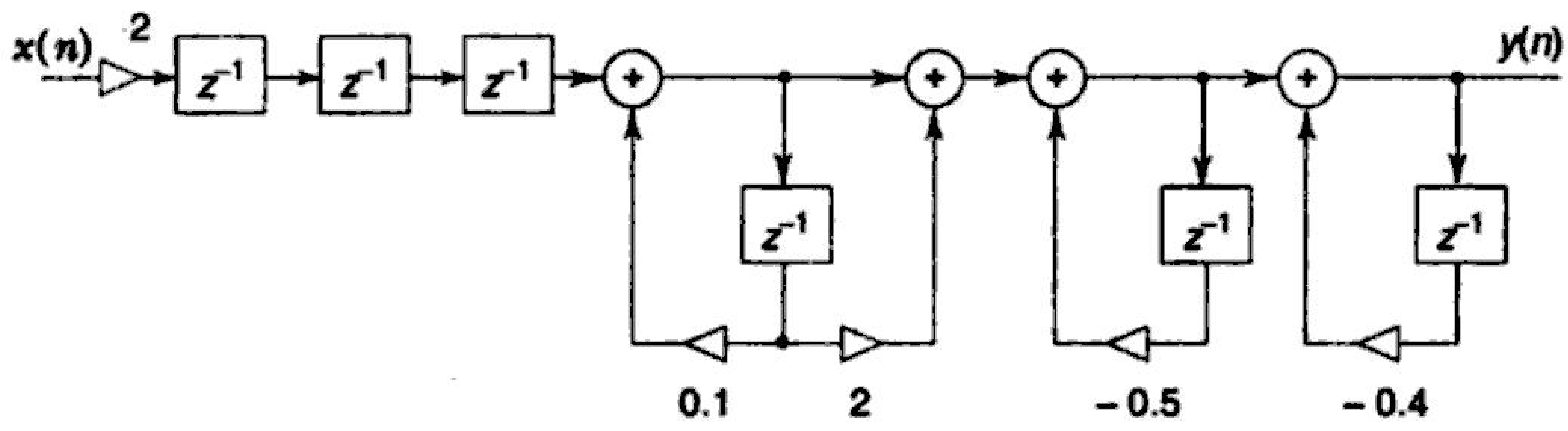

Fig. E9.4

9.3.3 Parallel Realisation of IIR Systems

By using the partial fraction expansion, the transfer function of an IIR system can be realised in a parallel form. A partial fraction expansion of the transfer function in the form given below will lead to the parallel form.

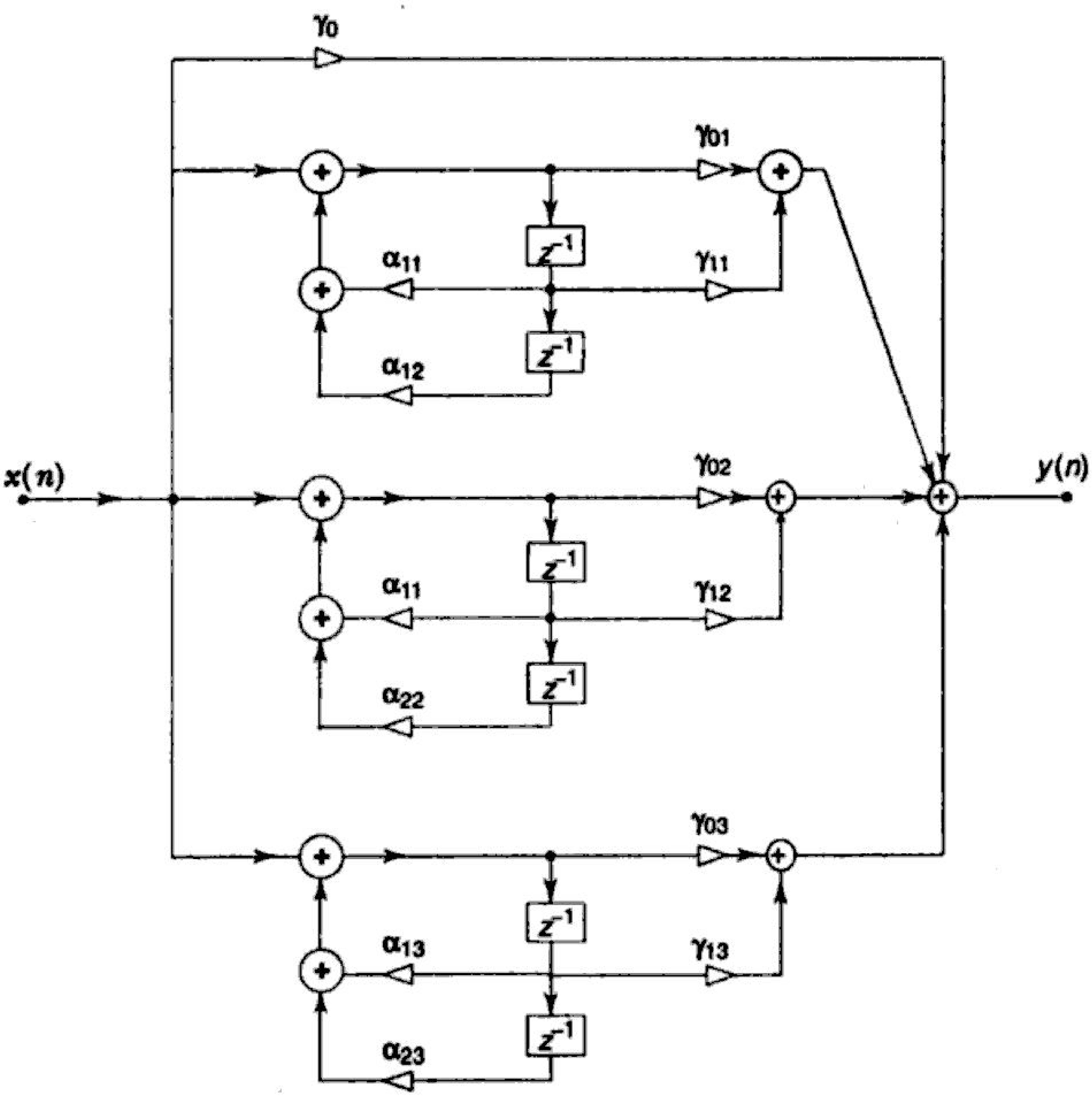

Fig. 9.9 *Parallel Form Realisation Structure With the Real and Complex Poles Grouped in Pairs*

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Example 9.8 Obtain the cascade and parallel realisations for the system function given by

$$H(z) = \frac{1+\frac{1}{4}z^{-1}}{\left(1+\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}\right)}$$

Solution

Cascade Realisation To obtain the cascade realisation, the transfer function is broken into a product of two functions as

$$H(z) = H_1(z)\, H_2(z)$$

where $H_1(z) = \dfrac{1+\frac{1}{4}z^{-1}}{1+\frac{1}{2}z^{-1}}$ and $H_2(z) = \dfrac{1}{1+\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}$

The cascade realisation structure for this system function is shown in Fig. E9.8(a).

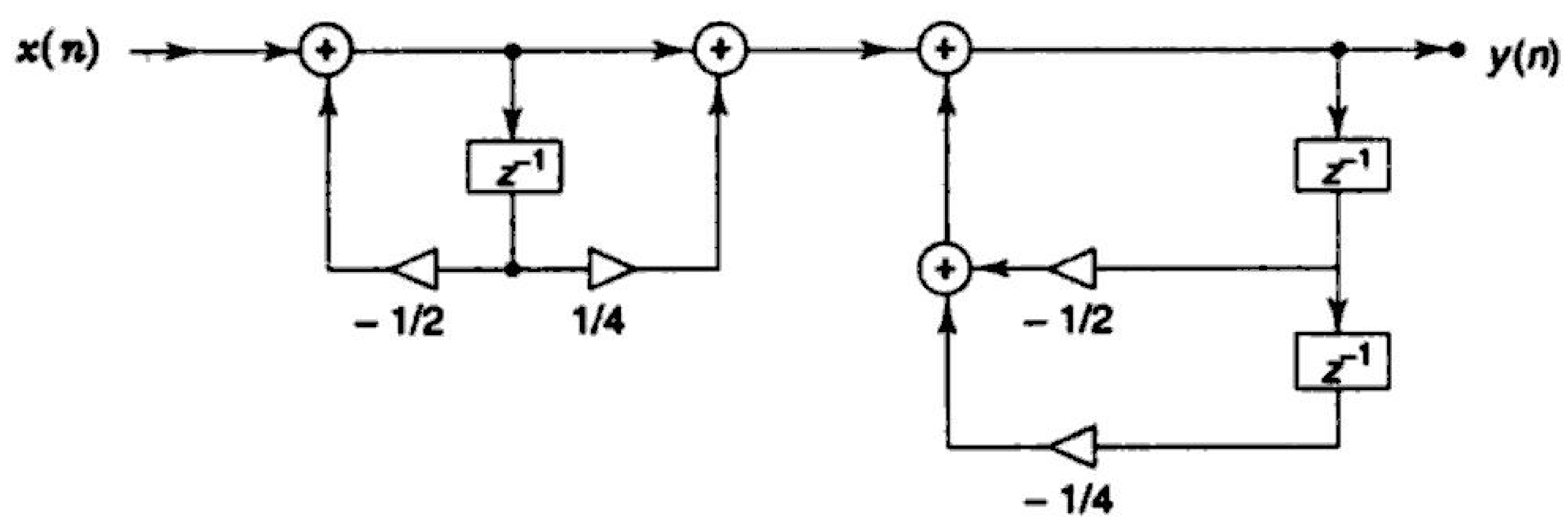

Fig. E9.8(a)

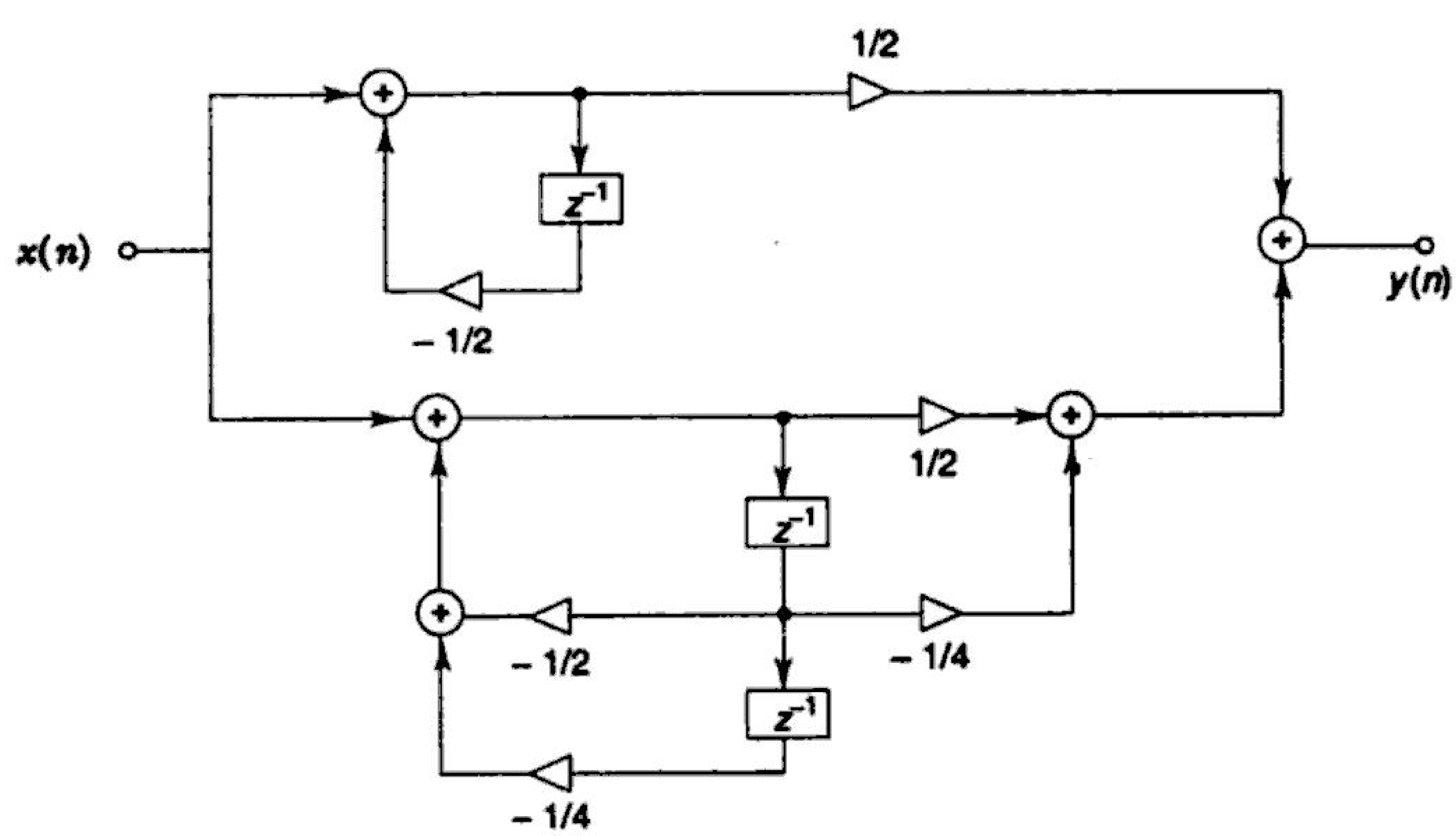

Fig. E9.8(b)

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The largest error occurs when all the discarded bits are one's. When the number x is negative, truncation results in reduction of the magnitude only. However, because of the negative sign, the resulting number will be greater than the original number. For example, let the number be $x = -0.375$. That is, in sign magnitude form it is represented as $x = 1\,011$ and after truncation of one bit, $Q(x) = 1\,01$. This is equivalent to -0.25 in decimal. But -0.25 is greater than -0.375. Therefore, the truncation error is positive and its range is

$$0 \le \varepsilon_T \le (2^{-B} - 2^{-L}) \tag{10.3}$$

The overall range of the truncation error for the sign magnitude representation is

$$-(2^{-B} - 2^{-L}) \le \varepsilon_T \le (2^{-B} - 2^{-L}) \tag{10.4}$$

(ii) *Truncation error for two's complement representation* When the input number is positive, truncation results in a smaller number, as in the case of sign magnitude numbers. Hence, the truncation error is negative and its range is same as that given in Eq. 10.2. If the number is negative, truncation of the number in two's complement form results in a smaller number and the error is negative. Thus the complete range of the truncation error for the two's complement representation is

$$-(2^{-B} - 2^{-L}) \le \varepsilon_T \le 0 \tag{10.5}$$

(iii) *Round-off error for sign magnitude and two's complement representation* The rounding of a binary number involves only the magnitude of the number and is independent of the type of fixed-point binary representation. The error due to rounding may be either positive or negative and the peak value is $\dfrac{(2^{-B} - 2^{-L})}{2}$. The round-off error is symmetric about zero and its range is

$$-\frac{(2^{-B} - 2^{-L})}{2} \le \varepsilon_R \le \frac{(2^{-B} - 2^{-L})}{2} \tag{10.6}$$

where ε_R is the round-off error.

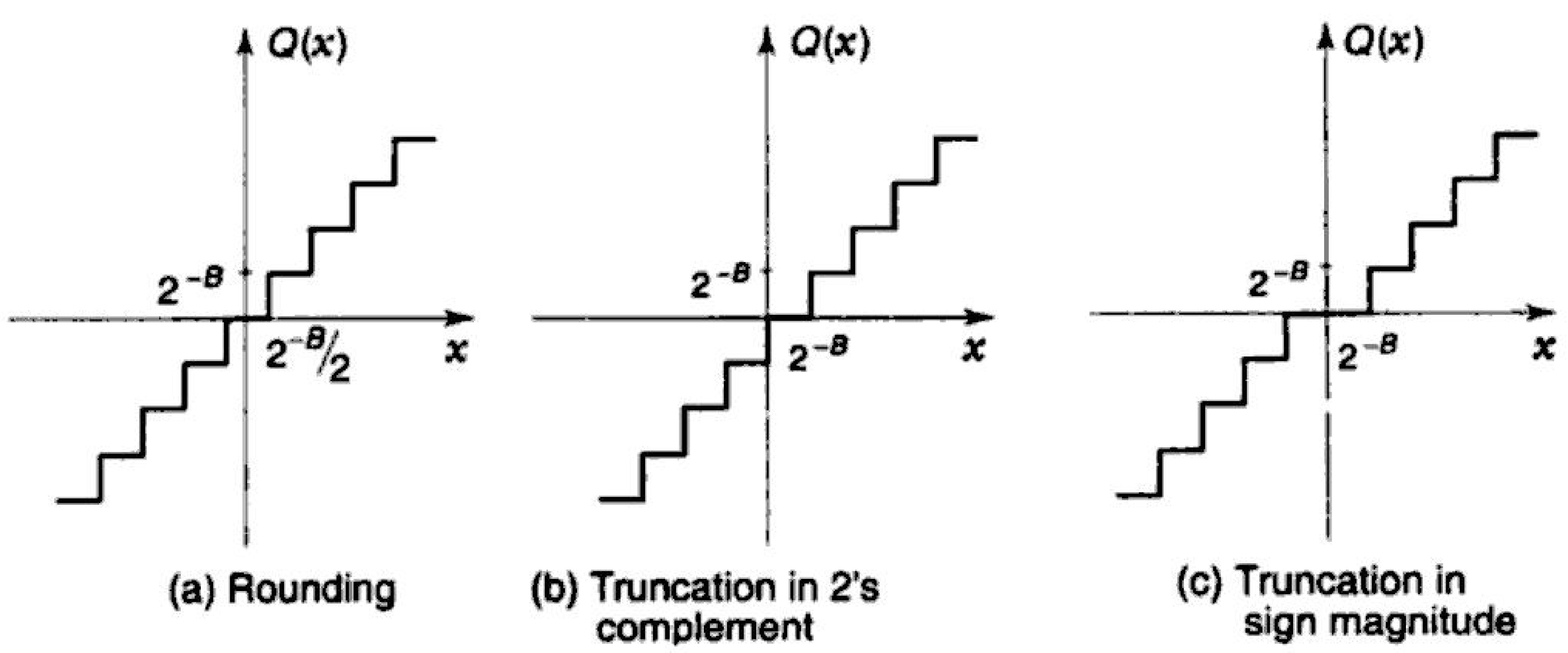

Fig. 10.1 *Quantization Error in Rounding and Truncation*

where E represents the statistical expectation. Using Eq. 10.18 in Eq. 10.19, we get

$$\gamma_{eo\,eo}(m) = E\left[\sum_{k=0}^{\infty} h(k)\, e^*(n-k) \cdot \sum_{k=0}^{\infty} h(k)\, e(n+m-k)\right]$$

$$= \sum_{k=0}^{\infty} h^2(k)\, E\,[e^*(n-k) \cdot e(n+m-k)]$$

$$\gamma_{eo\,eo}(m) = \sum_{k=0}^{\infty} h^2(k)\, \gamma_{ee}(m) \tag{10.20}$$

It has been assumed that the noise resulting from the quantisation process is a white noise. For this case, we have

$$\gamma_{eo\,eo}(m) = \sigma_{eo}^2 \quad \text{and} \quad \gamma_{ee}(m) = \sigma_e^2 \tag{10.21}$$

where σ_{eo}^2 is the output noise power (or power of the output error) and σ_e^2 is the input noise power. Using Eq. 10.21 in Eq. 10.20 and replacing the variable k with n,

$$\sigma_{eo}^2 = \sigma_e^2 \sum_{n=0}^{\infty} h^2(n) \tag{10.22}$$

Using Parseval's relation (see Example 10.1),

$$\sum_{n=0}^{\infty} h^2(n) = \frac{1}{2\pi j} \oint_C H(z)\, H(z^{-1})\, z^{-1}\, dz$$

in Eq. 10.22 we get,

$$\sigma_{eo}^2 = \frac{\sigma_e^2}{2\pi j} \oint_C H(z)\, H(z^{-1})\, z^{-1}\, dz \tag{10.23}$$

where the closed contour of integration is around the unit circle $|z| = 1$. This integration is evaluated using the method of residues, taking only the poles that lie inside the unit circle.

Example 10.1 Prove that

$$\sum_{n=0}^{\infty} x^2(n) = \frac{1}{2\pi j} \oint_C X(z)\, X(z^{-1})\, z^{-1}\, dz$$

Solution The z-transform of $x(n)$ is

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} \tag{E1}$$

Taking the z-transform of $x^2(n)$,

$$Z\,[x^2(n)] = \sum_{n=0}^{\infty} x(n)\, x(n)\, z^{-n} = \sum_{n=0}^{\infty} x^2(n)\, z^{-n} \tag{E2}$$

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10.6 COEFFICIENT QUANTISATION IN DIRECT FORM REALISATION OF FIR FILTERS

The statistical bounds on the error in the frequency response due to coefficient quantisation (rounding) is given here. The frequency response of a linear phase FIR filter is given by

$$H(e^{j\omega}) = e^{-j\omega(M-1)/2}\left\{h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{(M-3)/2} h(n)\cos\left[\left(\frac{M-1}{2}-n\right)\omega\right]\right\}$$

$$= e^{j\Phi(\omega)} M(\omega) \tag{10.33}$$

For a linear phase FIR filter, $h(n) = h(M-1-n)$. The term $e^{j\Phi(\omega)} = e^{-j\omega(M-1)/2}$ in the above expression represents the delay and is unaffected by quantisation. Hence, the quantisation effect is solely on the pseudomagnitude term $M(\omega)$. Let $\{h_q(n)\}$ be the sequence resulting from rounding $[h(n)\}$ to a quantisation step size of 2^{-B}. Therefore,

$$h_q(n) = h(n) + e(n) \tag{10.34}$$

and $h_q(n) = h_q(M-1-n)$ for $0 \le n \le (M-1)/2$. Let $e(n)$ be a random sequence and uniformly distributed over the range $-\frac{2^{-B}}{2}$ and $\frac{2^{-B}}{2}$. Let $H_q(z)$ be the z-transform of $\{h_q(n)\}$ and $M_q(\omega)$ be the pseudomagnitude of the quantised linear phase FIR filter. The error function is defined to be

$$E(e^{j\omega}) = M_q(\omega) - M(\omega) \tag{10.35}$$

or

$$E(e^{j\omega}) = e\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{(M-3)/2} e(n)\cos\left[\left(\frac{M-1}{2}-n\right)\omega\right]$$

where $E(e^{j\omega})$ is the frequency response of a linear phase FIR filter that has $\{e(n)\}$ as the impulse response for the first half and the second half can be obtained using $e(n) = e(M-1-n)$. Thus, the filter with its coefficients rounded-off can be considered as a parallel connection of the ideal filter (infinite precision) with a filter whose frequency response is $E(e^{j\omega})\, e^{-j\omega(M-1)/2}$. Since the error $e(n)$ due to rounding of the filter coefficients is always lesser than or equal to $\frac{2^{-B}}{2}$, a bound on $|E(e^{j\omega})|$ can be obtained as shown below.

$$|E(e^{j\omega})| \le \left|e\left(\frac{M-1}{2}\right)\right| + 2\sum_{n=0}^{(M-3)/2} |e(n)|\left|\cos\left[\left(\frac{M-1}{2}-n\right)\omega\right]\right| \tag{10.36}$$

Letting $\frac{M-1}{2} - n = k$ in the second term of the above expression,

$$|E(e^{j\omega})| \le \left|e\left(\frac{M-1}{2}\right)\right| + 2\sum_{k=1}^{(M-1)/2} \left|e\left(\frac{M-1}{2}-k\right)\right|\,|\cos k\omega|$$

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$$l \le \frac{0.5}{1-|b_2|} (0 < b_2 < 1) \tag{10.48}$$

OVERFLOW LIMIT CYCLES

Limit cycles can also occur due to overflow in digital filters implemented with finite precision arithmetic. The amplitudes of such overflow oscillations are much more serious in nature than zero input limit cycle oscillations. Consider a causal, all-pole second-order IIR digital filter implemented using two's complement arithmetic with a rounding of the sum of the products by a single quantiser. The difference equation describing the system is given by

$$\tilde{y}(n) = Q_R[-a_1 \tilde{y}(n-1) - a_2 \tilde{y}(n-2) + x(n)]$$

where Q_R [.] represents the rounding operation and $\tilde{y}(n)$ is the actual output of the filter. The filter coefficients are represented by signed 4-bit fractions. Let $a_1 = 1_\Delta 0\,0\,1 = -0.8751_d$ and $a_2 = 0_\Delta 111 = 0.875_d$ and the initial conditions be $\tilde{y}(-1) = 0_\Delta 1\,1\,0 = 0.75_d$ and $\tilde{y}(-2) = 1_\Delta 0\,1\,0 = -0.75_d$. For zero input, i.e. $x(n) = 0$ and for $n \ge 0$, we get the values for $\tilde{y}(n)$ as shown below.

Table 10.3

n	$\tilde{y}(n-1)$	$\tilde{y}(n-2)$	$-a_1 \tilde{y}(n-1) - a_2 \tilde{y}(n-2)$	$\tilde{y}(n) = Q_R[.]$	$\tilde{y}(n)$ in decimal
0	$\tilde{y}(-1) = 0_\Delta 110$	$\tilde{y}(-2) = 1_\Delta 010$	$1_\Delta 010100$	$1_\Delta 011$	-0.625
1	$\tilde{y}(0) = 1_\Delta 011$	$\tilde{y}(-1) = 0_\Delta 110$	$10_\Delta 110011$	$0_\Delta 110$	$+0.75$
2	$\tilde{y}(1) = 0_\Delta 110$	$\tilde{y}(0) = 1_\Delta 011$	$1_\Delta 001101$	$1_\Delta 010$	-0.75
3	$\tilde{y}(2) = 1_\Delta 010$	$\tilde{y}(1) = 0_\Delta 110$	$10_\Delta 101100$	$0_\Delta 110$	$+0.75$
4	$\tilde{y}(3) = 0_\Delta 110$	$\tilde{y}(2) = 1_\Delta 010$	$1_\Delta 010100$	$1_\Delta 011$	-0.625

For $n = 1$, the sum of two products has resulted in a carry bit to the left of the sign bit that is automatically lost, resulting in a positive number. The same thing happens for $n = 3$ and also for other values of n. It can be noted from the above table that the output swings between positive and negative values and the swing of oscillations is also large. Such limit cycles are referred to as overflow limit cycle oscillations.

The study of limit cycles is important for two reasons. In a communication environment, when no signal is transmitted, limit cycles can occur which are extremely undesirable. For example, in a telephone no one would like to hear unwanted noise when no signal is put in from the other end. Consequently, when digital filters are used in telephone exchanges, care must be taken regarding this problem. The second reason for studying limit cycles is that this effect can be effectively used in digital waveform generators. By producing desirable limit cycles in a reliable manner, these limit cycles can be used as a source in digital signal processing.

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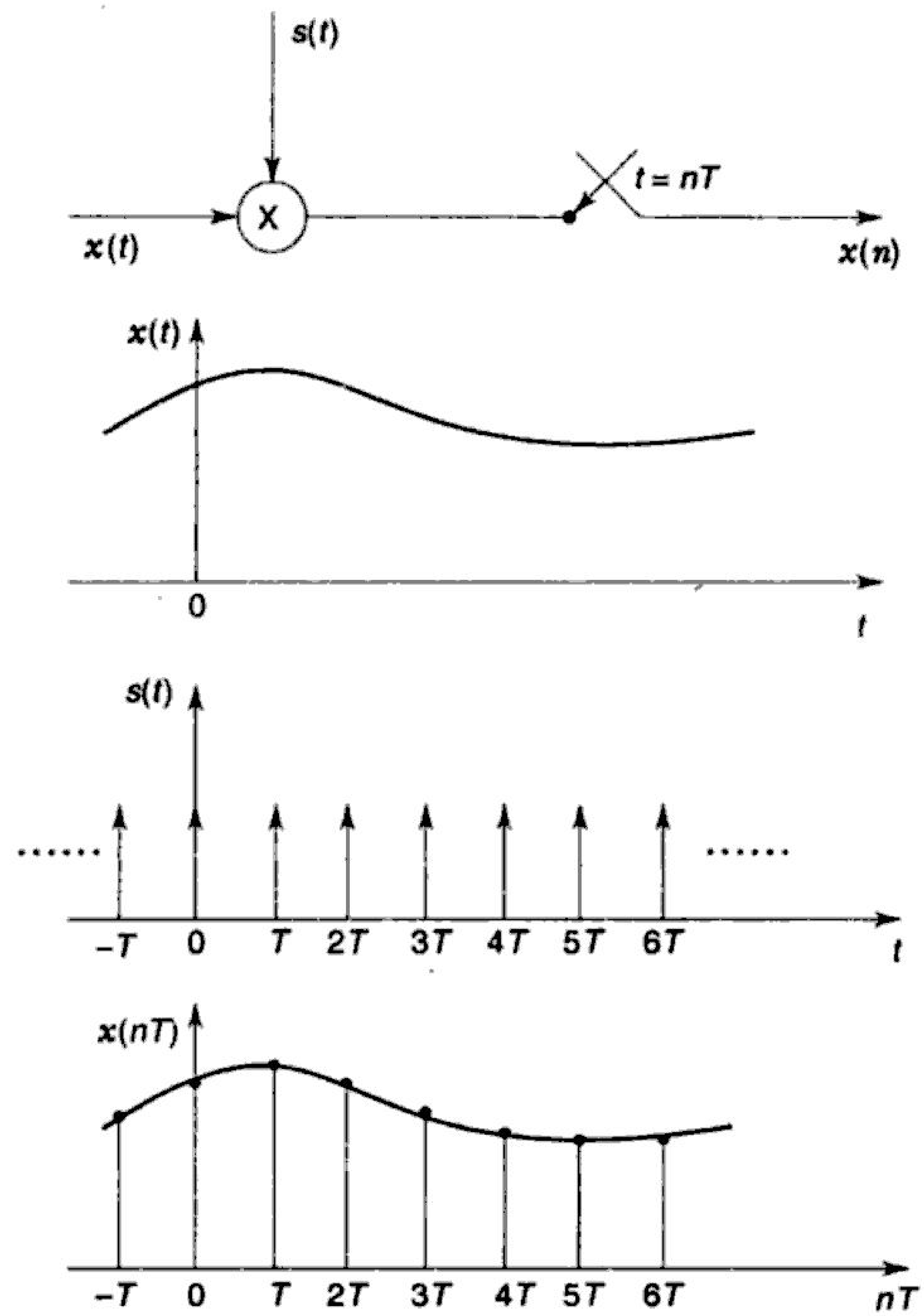


Fig. 11.2 *Periodic Sampling of $x(t)$*

11.3 SAMPLING RATE CONVERSION

Sampling rate conversion is the process of converting the sequence $x(n)$ which is got from sampling the continuous time signal $x(t)$ with a period T, to another sequence $y(k)$ obtained from sampling $x(t)$ with a period T'.

The new sequence $y(k)$ can be obtained by first reconstructing the original signal $x(t)$ from the sequence $x(n)$ and then sampling the reconstructed signal with a period T'.

Figure 11.3 shows the reconstruction of the original signal with a *D/A* converter, low-pass filter and resampler with sampling period T'.

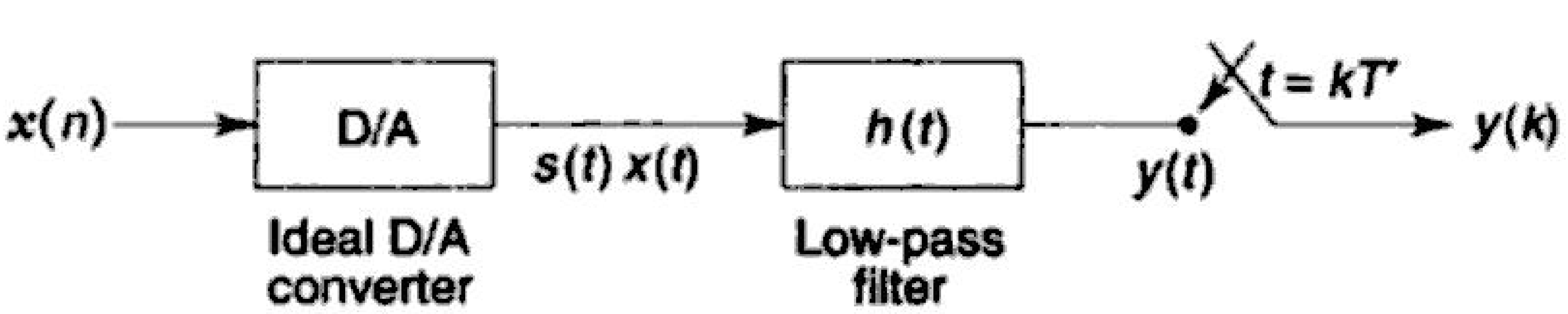


Fig. 11.3 *Conversion of a Sequence $x(n)$ to Another Sequence $y(k)$*

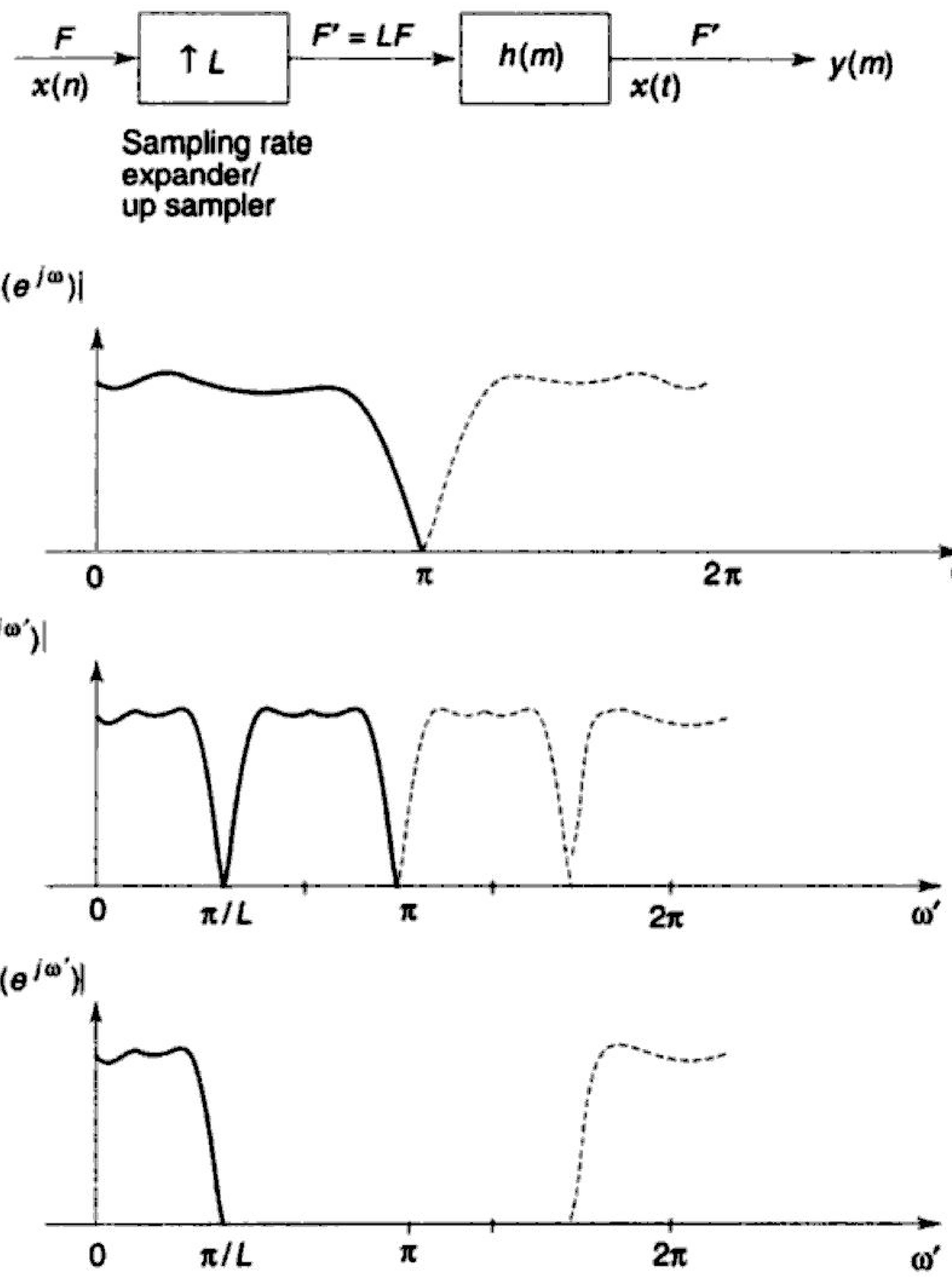


Fig. 11.7 *Interpolation of $x(n)$ by a Factor L*

$$y(n) = \begin{cases} x(n/M), & n = \text{multiples of } M \\ 0, & \text{otherwise} \end{cases}$$

where $M = 2$.

$$x(n) = 0, 1, 2, 3, 4, 5, \ldots$$
$$y(n) = 0, 0, 1, 0, 2, 0, 3, 0, 4, 0, 5, 0, \ldots$$

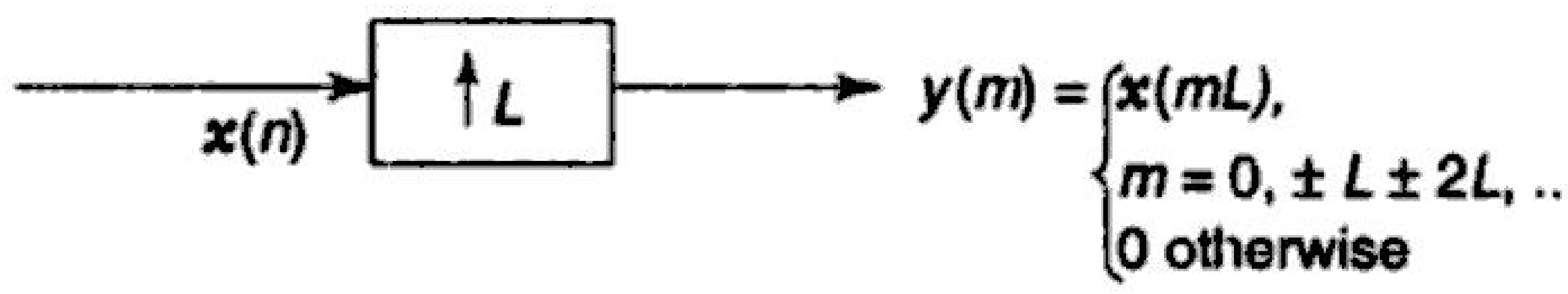
In general, to obtain the expanded signal $y(n)$ by a factor M, $(M-1)$ zeros are inserted between the samples of the original signal $x(n)$.

The z-transform of the expanded signal is

$$Y(z) = X(z^M), M = 2.$$

The input and output signals are shown in Fig. E11.2.

(f) Sampling rate expander / up sampler



(g) Modulator

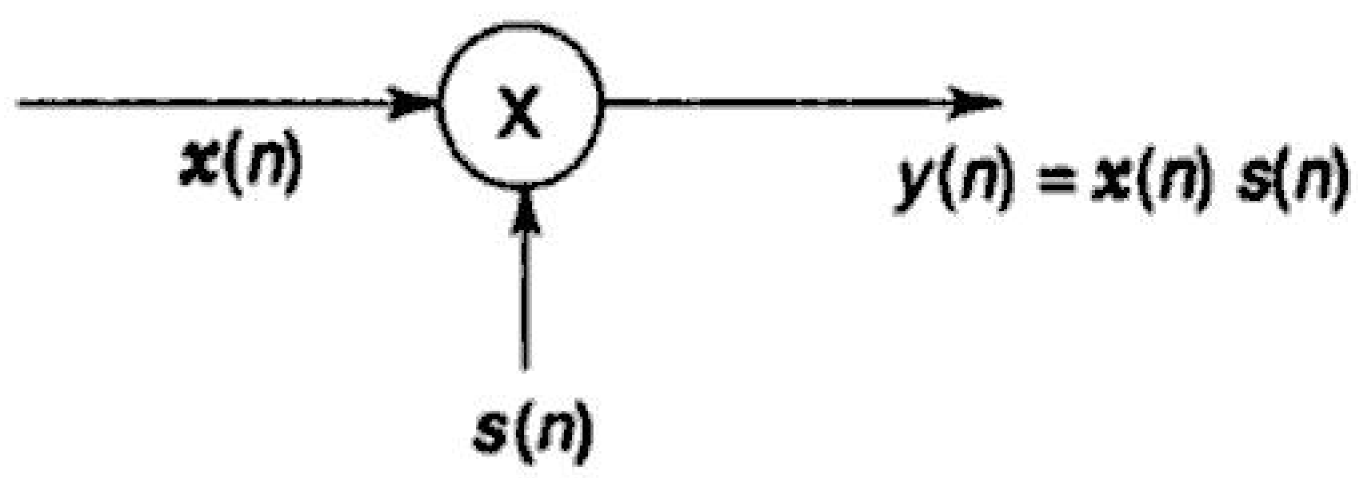


Fig. 11.9 *Branch Operations in Signal Flow Graphs*

node of the branch. External signals enter the input branches and signals at the output branches are terminal signals. The sum of the signals entering the node is equal to the sum of the signals leaving the node. Based on the signal flow graph of Fig. 11.10, the network equations can be written as follows.

At the input node,

$$r(n) = x(n) + a_1 r(n-1) + a_2 r(n-2) \tag{11.21}$$

At the output node,

$$y(n) = r(n) + b_1 r(n-1) + b_2 r(n-2) \tag{11.22}$$

Combining both the equations,

$$y(n) = x(n) + b_1 x(n-1) + b_2 x(n-2) + a_1 y(n-1) + a_2 y(n-2) \tag{11.23}$$

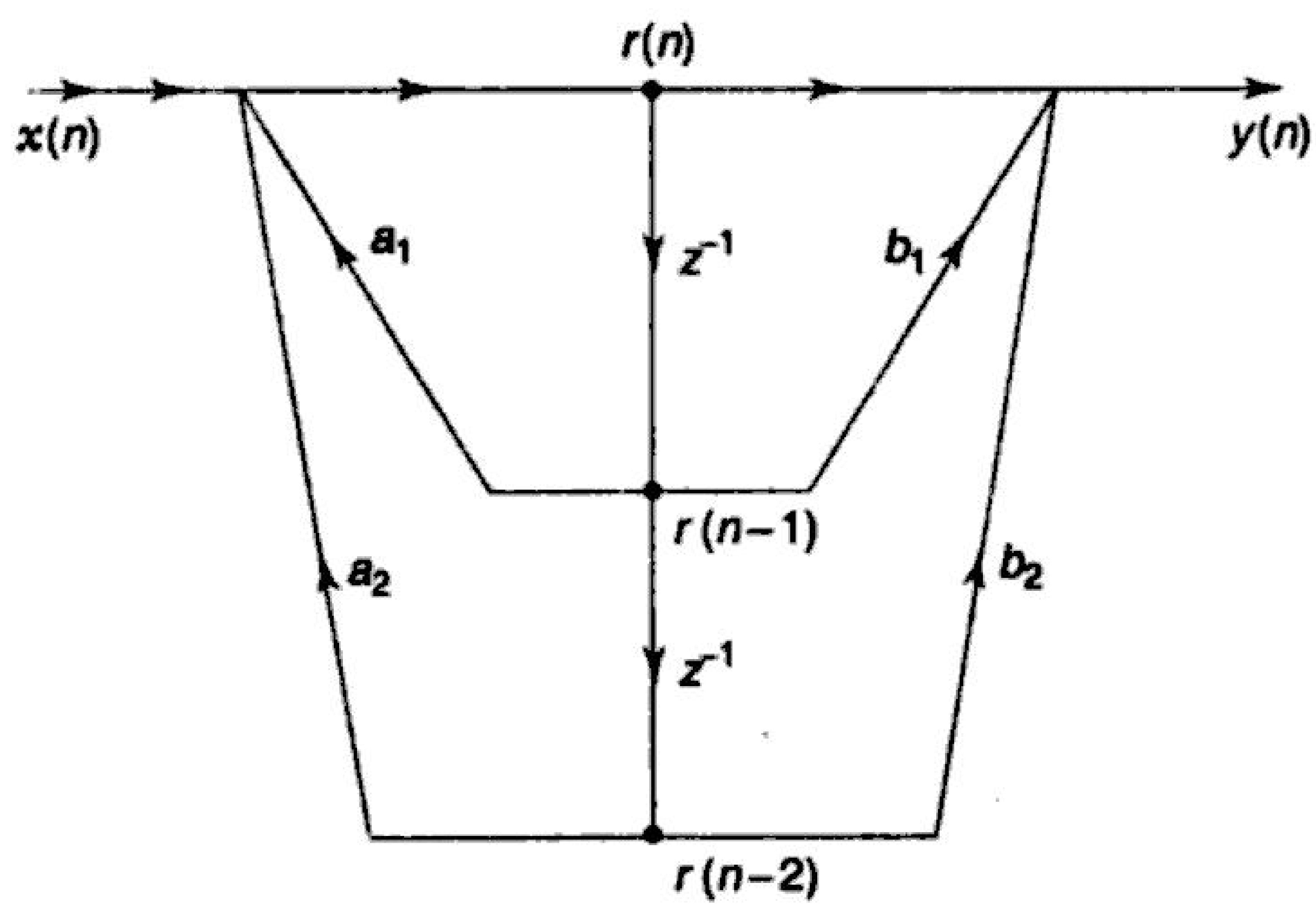


Fig. 11.10 *Signal Flow Graph for a Second-order System*

11.4.1 Manipulation of Signal Flow Graphs

Manipulation of signal flow graphs which is shown in Fig. 11.11, corresponds to the ways how the set of network equations are represented. In multirate systems, it is easier to modify the signal flow

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With this property, the number of multiplications can be reduced by a factor of two. If N is even,

$$y(n) = \sum_{k=0}^{\frac{N}{2}-1} h(k)\,\{x\,(n-k) + x\,(n-(N-1-k))\}$$

11.5.2 IIR Direct Form Structure

Consider an IIR filter with the difference equation represented by,

$$y(n) = \sum_{k=1}^{D} a_k y(n-k) + \sum_{k=0}^{N-1} b_k x(n-k) \tag{11.25}$$

Figure 11.13 shows the signal flow graph for the IIR filter. The system equation of the IIR filter is given by

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N-1} b_k z^{-k}}{1-\sum_{k=1}^{D} a_k z^{-k}} = \frac{N(z)}{D(z)} \tag{11.26}$$

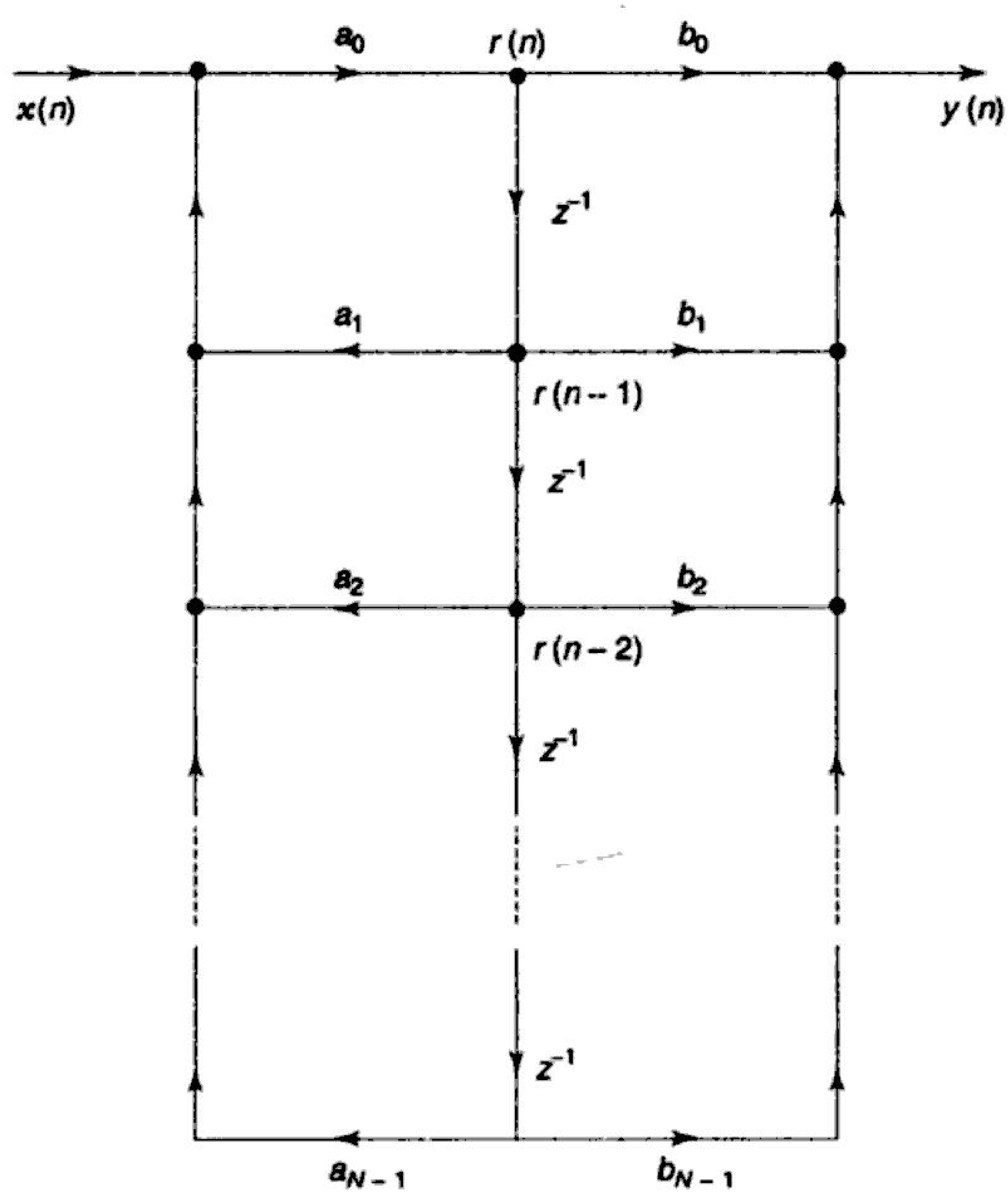

Fig. 11.13 *The Signal Flow Graph for an IIR Filter*

Solution

$$H(z) = E_0(z^2) + z^{-1} E_1(z^2)$$

where $E_0(z^2)$, $E_1(z^2)$ are polyphase components.

$$H(z) = \frac{1-4z^{-1}}{1+5z^{-1}}$$

$$= \frac{(1-4z^{-1})}{(1+5z^{-1})}\frac{(1-5z^{-1})}{(1-5z^{-1})}$$

$$= \frac{1-9z^{-1}+20z^{-2}}{1-25z^{-2}}$$

$$= \frac{1+20z^{-2}}{1-25z^{-2}} + z^{-1}\frac{-9}{1-25z^{-2}}$$

The polyphase components are

$$E_0(z) = \frac{1+20z^{-2}}{1-25z^{-2}} \text{ and } E_1(z) = \frac{-9}{1-25z^{-2}}$$

11.6.1 General Polyphase Framework

The z-transform of an anti-aliasing filter shown in Fig. 11.17 (a) with impulse response $h(n)$ is given by

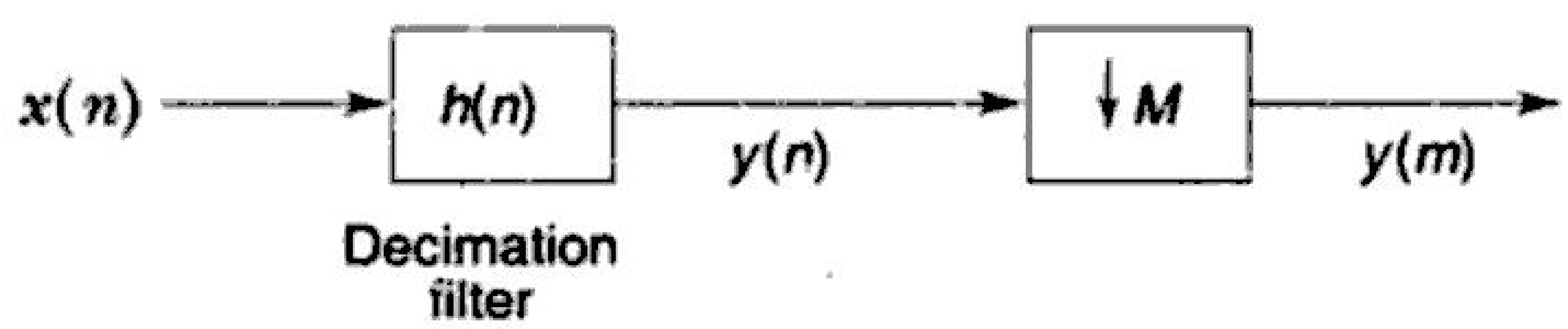

Fig. 11.17 *(a) Decimation by a factor M*

$$H(z) = \sum_{n=0}^{\infty} h(n)\, z^{-n}$$

$$= h(0) + h(1)\, z^{-1} + h(2)\, z^{-2} + \ldots$$

which can be partitioned into M sub-signals where M represents decimation factor. Hence,

$$\begin{aligned} H(z) &= h(0) + h(M)\, z^{-M} + h(2M)\, z^{-2M} + \ldots \\ &+ z^{-1}\{h(1) + h(M+1)\, z^{-M} + h(2M+1)\, z^{-2M} + \ldots\} \\ &+ z^{-(M-1)}\{h(M-1) + h(2M-1)\, z^{-M} + \ldots\} \qquad (11.32) \end{aligned}$$

Equation 11.32 can be written as

$$H(z) = \sum_{k=0}^{M-1} \sum_{m=0}^{\infty} h(mM+k)\, z^{-(mM+k)}$$

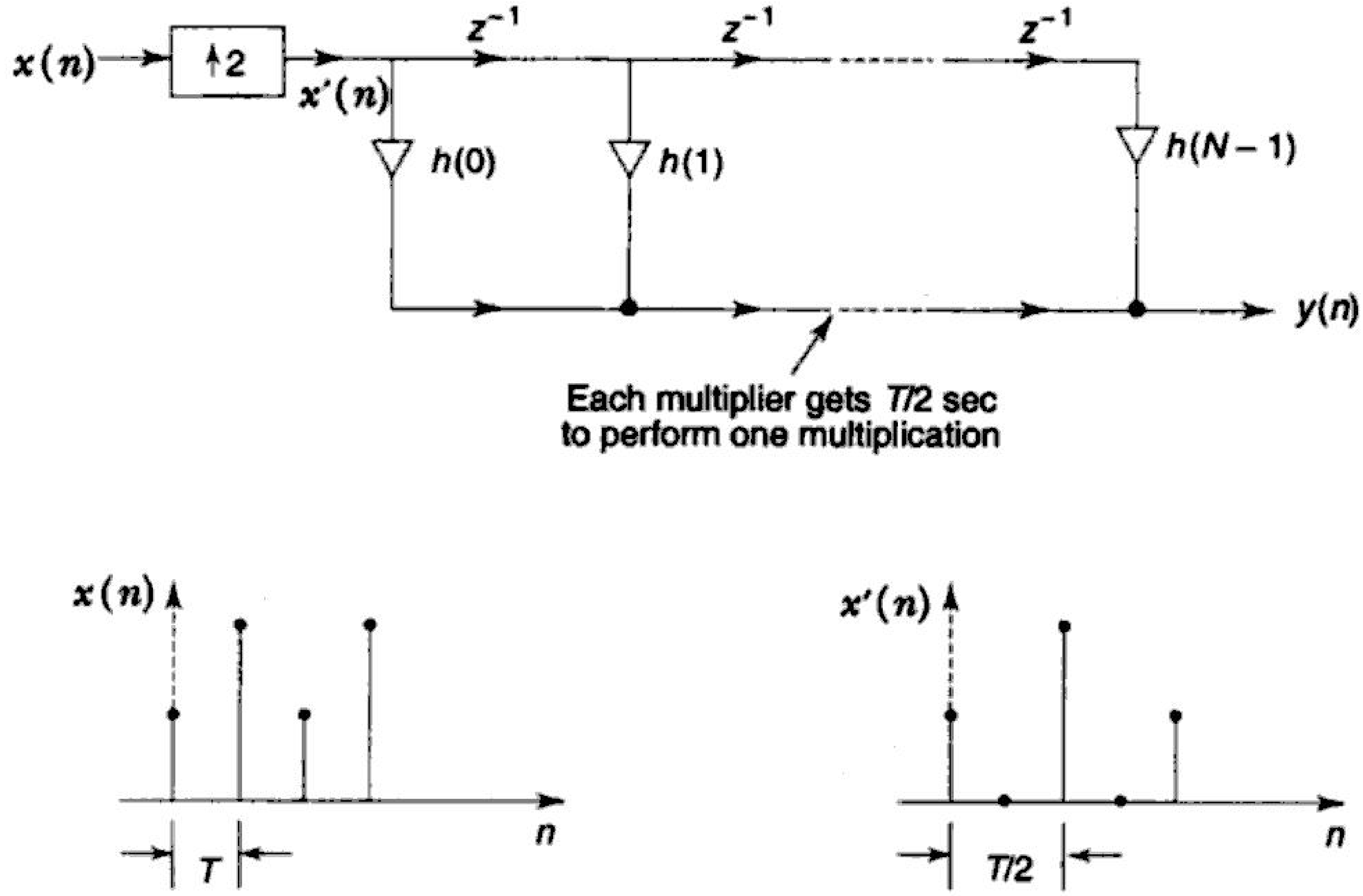

Fig. 11.18(e)

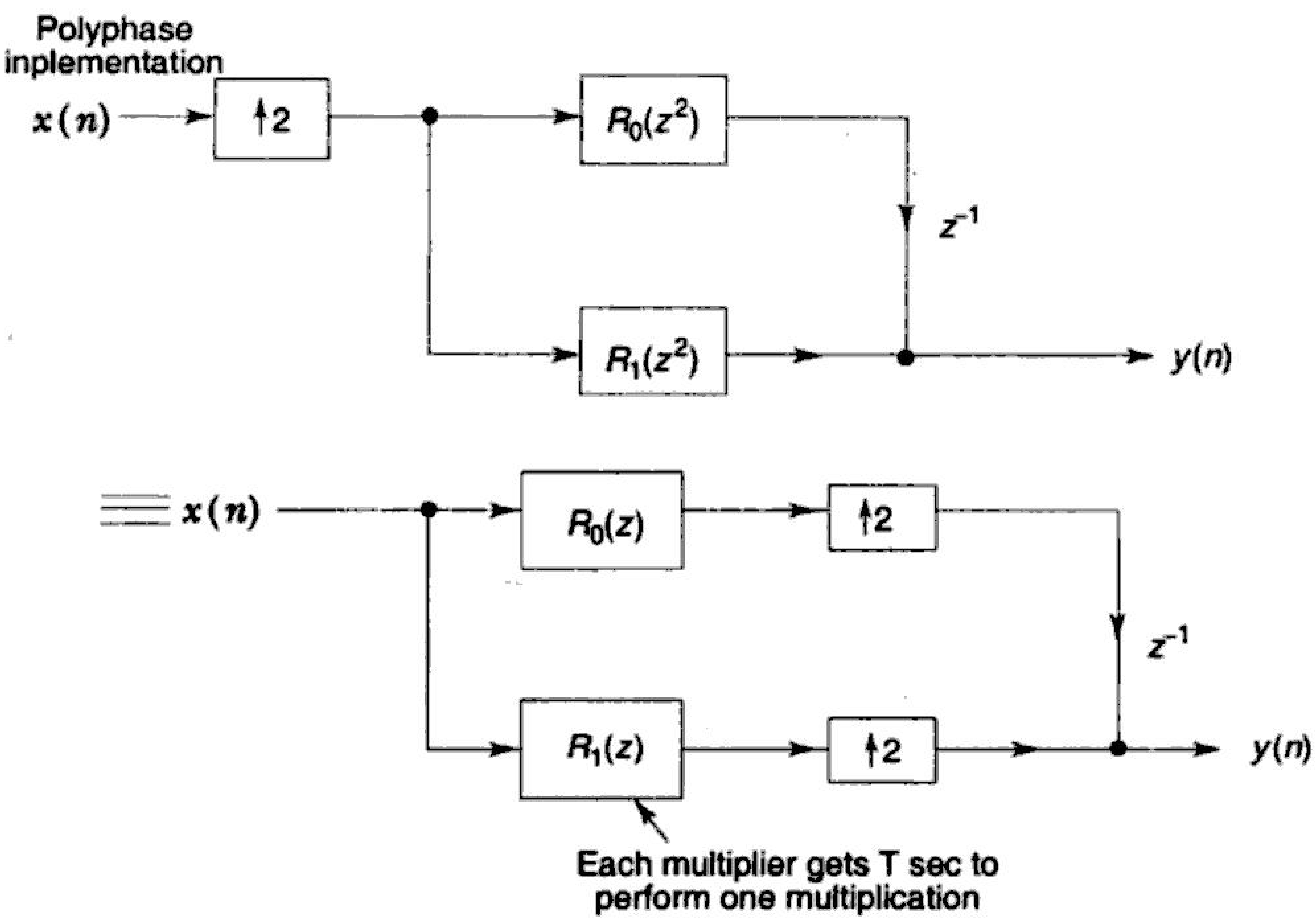

Fig. 11.18(f) *Polyphase Implementation of an Interpolator by a Factor of two*

IIR Structures for Decimators

The IIR filter is represented by the difference equation,

$$y(n) = \sum_{k=1}^{D} a_k y(n-k) + \sum_{k=0}^{N-1} b_k x(n-k) \tag{11.34}$$

The system function for the above difference equation is given by,

The demerits of the systems are that proper control structure is required in implementing the system and proper values of I should be chosen.

Example 11.5 Implement a two-stage decimator for the following specifications.

Sampling rate of the input signal = 20,000 Hz
$M = 100$
Passband = 0 to 40 Hz
Transition band = 40 to 50 Hz
Passband ripple = 0.01
Stop band ripple = 0.002

Solution

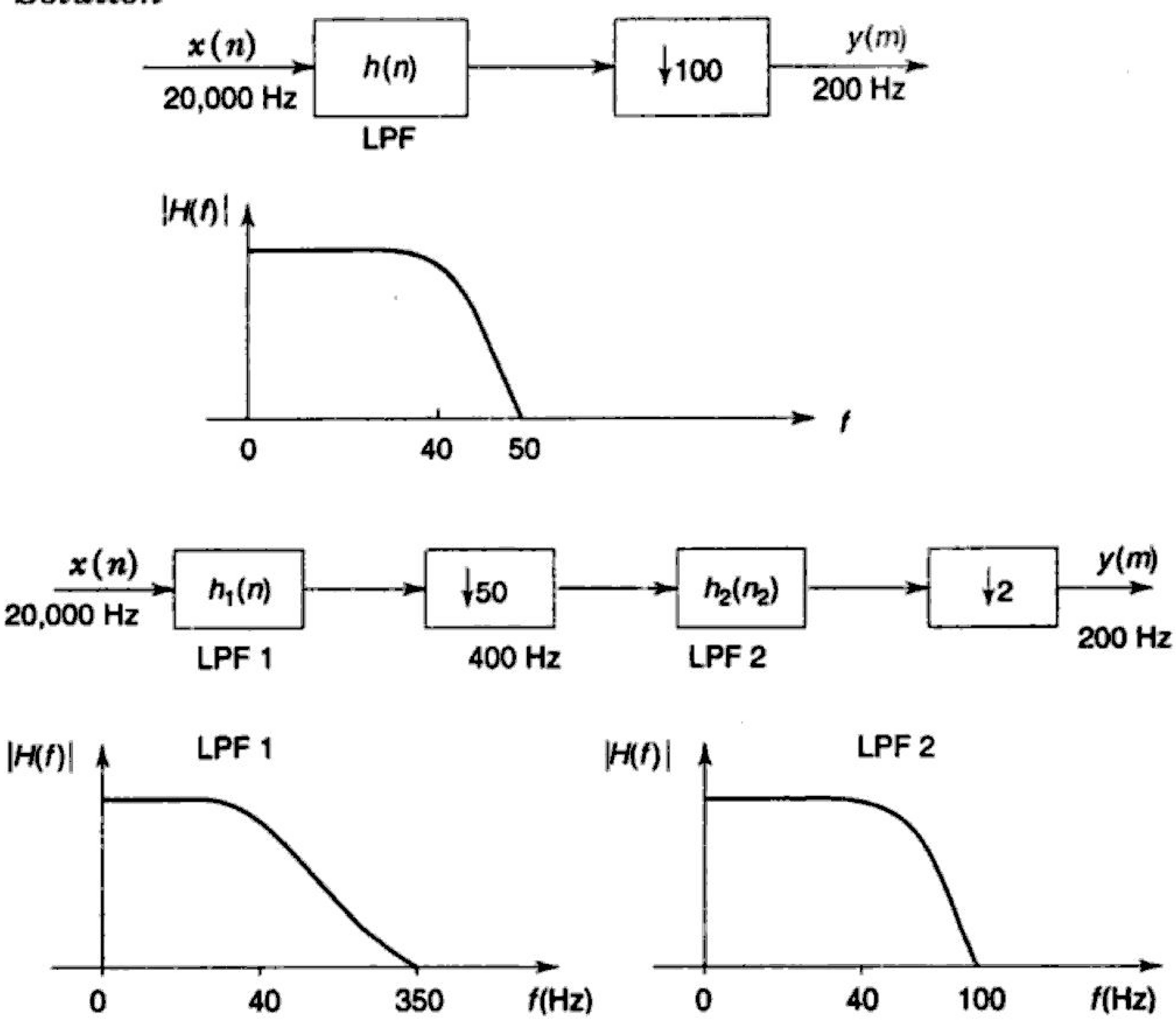

Fig. E11.5(a) *Single-stage and Two-stage Network for Decimator*

The implementation of the system is shown in Fig. E11.5(a).

$F_p = 40$ Hz
$F_s = 50$ Hz
$\delta_p = 0.01$
$\delta_s = 0.002$
$F_T = 20$ KHz
$M = 100$

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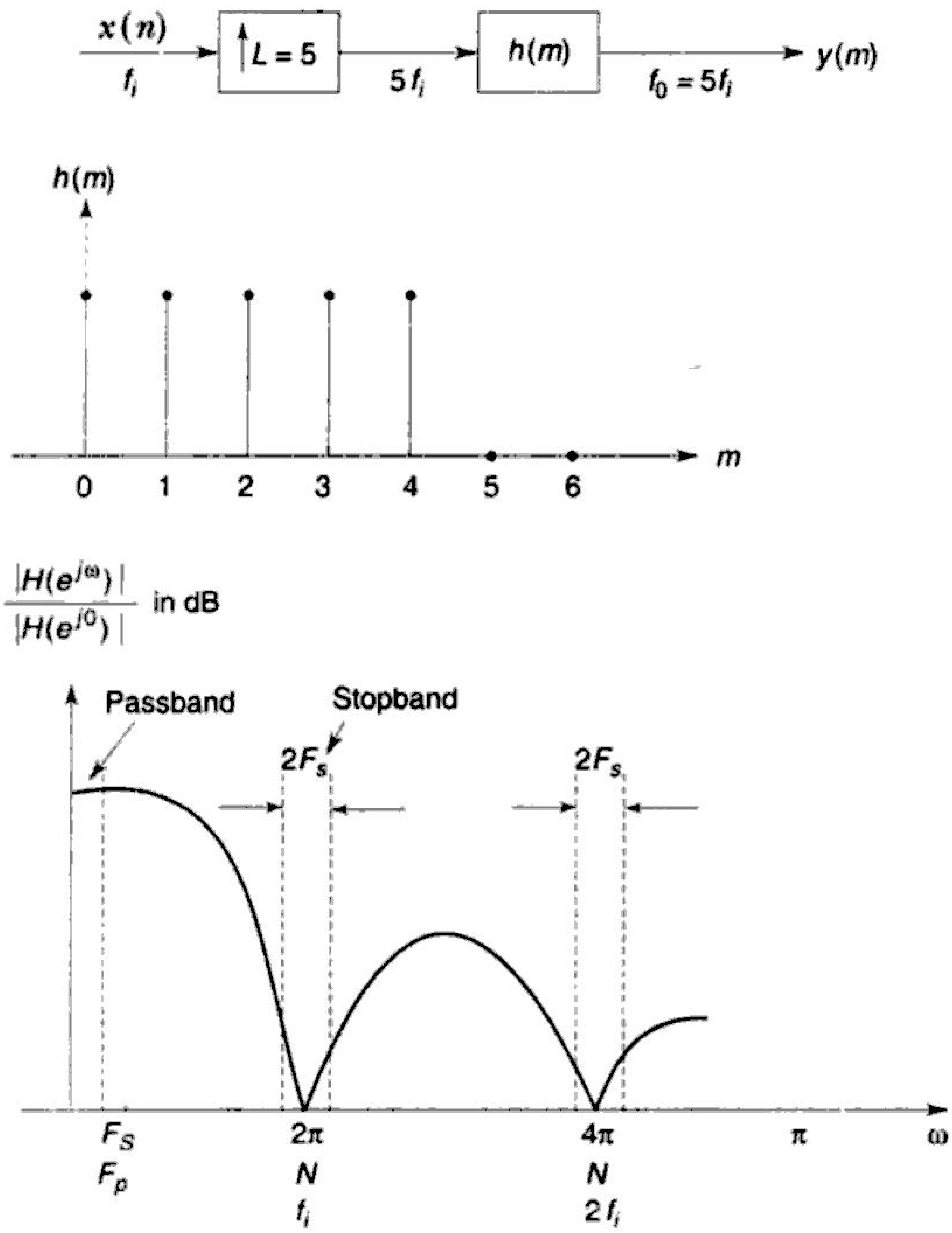

Fig. 11.28 *The Comb Filter for N = 5*

The number of multiplications per second is given by

$$R_{M,\,S} = 12 \times \frac{2000}{2} = 12{,}000$$

$$R_{M,\,R} = 30 \times \frac{1000}{5} = 60{,}000$$

The overall number of multiplications per second for a three-stage realisation is given by

$$R_{M,\,G} + R_{M,\,S} + R_{M,\,R} = 1{,}25{,}500$$

The number of multiplications per second for a three-stage realisation is more than that of a two-stage realisation. Hence higher than two-stage realisation may not lead to an efficient realisation.

Comb Filters

The impulse response of a comb filter (FIR filter) is given by,

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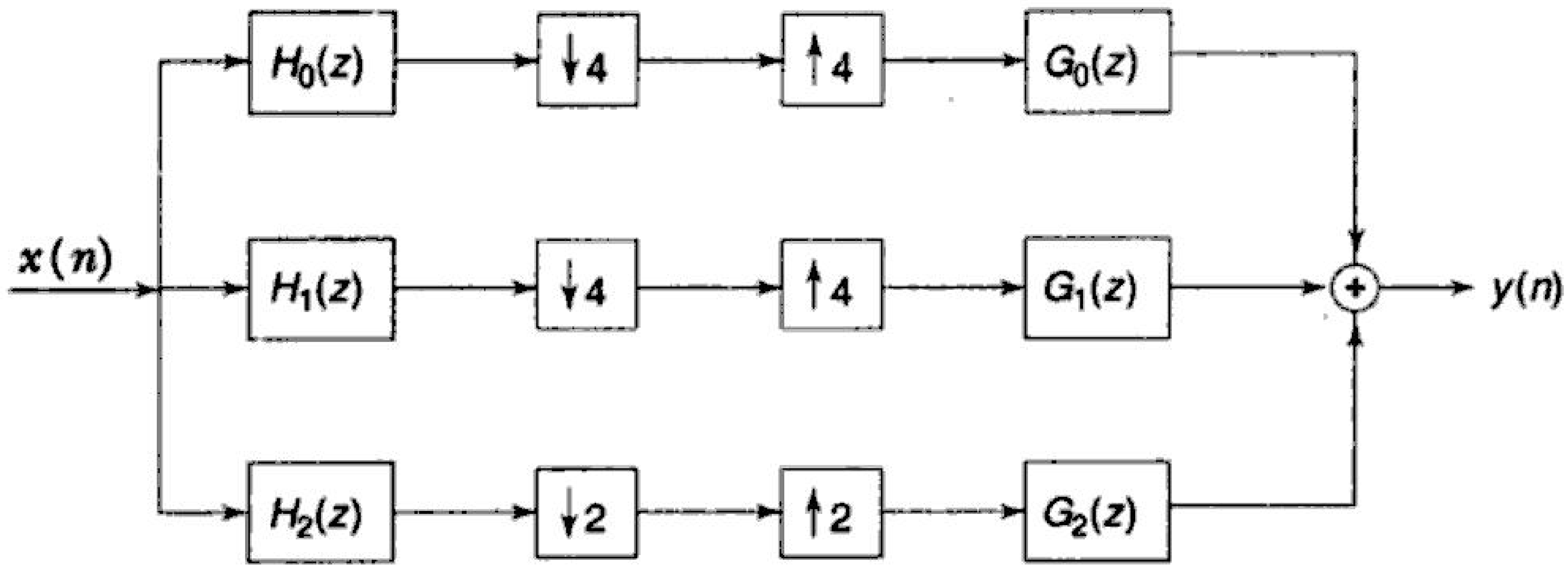

Fig. 11.41 *(c) Equivalent Three-channel QMF Filter Bank Realization*

On similar lines, we can derive a four-channel QMF filter bank from a three-channel QMF filter bank. These structures come under the class of non-uniform QMF filter banks. These types of filters find application in speech and image coding.

REVIEW QUESTIONS

11.1 What is the need for multirate signal processing ?
11.2 Give some examples of multirate digital systems.
11.3 Explain the interpolation process with an example.
11.4 Explain the decimation process with an example.
11.5 Write the input-output relationship for a decimation processing a factor of five.
11.6 With an example explain the sampling process.
11.7 What is meant by aliasing ?
11.8 How can aliasing be avoided ?
11.9 The signal x(n) is defined by $x(n) = \begin{cases} a^n, & n > 0 \\ 0, & \text{otherwise} \end{cases}$
   (a) Obtain the decimated signal with a factor of three.
   (b) Obtain the interpolated signal with a factor of three.
11.10 Explain polyphase decomposition process.
11.11 How can sampling rate be converted by a rational factor M / L ?
11.12 Draw the block diagram of a multistage decimator and integrator.
11.13 What are the characteristics of a comb filter ?
11.14 Explain with block diagram the general polyphase framework for decimators and interpolators.
11.15 What is a signal flow graph ?

(Contd.)

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$$= f_s \sum_{k=-\infty}^{\infty} X(f - kf_s) \tag{12.7}$$

If aliasing is avoided, i.e. $x(t)$ is band limited to a frequency less than $1/2T_s$ then,

$$X'(f) = f_s X(f) \tag{12.8}$$

Let $x(n)$ be the sampled version of $x(t)$. The Fourier transform of $x(n)$ is given by

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi fn} \tag{12.9}$$

The autocorrelation of the sampled signal $x(n)$ is given by

$$r_{xx}(k) = \sum_{n=-\infty}^{\infty} x^*(n)\, x(n+k) \tag{12.10}$$

The Fourier transform of $r_{xx}(k)$ from the Wiener-Khintchine theorem is

$$S_{xx}(f) = \sum_{k=-\infty}^{\infty} r_{xx}(k)\, e^{-j2\pi kf} \tag{12.11}$$

The other method for computing the energy density spectrum is obtained from the Fourier transform of $x(n)$,

$$S_{xx}(f) = |X(f)|^2$$

$$= \left| \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi fn} \right|^2 \tag{12.12}$$

Since finite energy signals possess Fourier transform, spectral analysis is done with the energy spectral density function.

12.3 ESTIMATION OF THE AUTOCORRELATION AND POWER SPECTRUM OF RANDOM SIGNALS

Consider signals which do not have finite energy. For these signals, Fourier transform is not possible. But these signals have finite average power. For these signals, spectral analysis is done with power spectral density function.

Let $x(t)$ be a stationary random process. The statistical autocorrelation function for this signal is,

$$\gamma_{xx}(\tau) = E\,[x*(t)\, x(t+\tau)] \tag{12.13}$$

The Fourier transform of the autocorrelation function of a stationary random process gives the power density spectrum,

$$\Gamma_{xx}(f) = \mathcal{F}(\gamma_{xx}(\tau))$$

$$= \int_{-\infty}^{\infty} \gamma_{xx}(\tau)\, e^{-j2\pi f\tau}\, d\tau \tag{12.14}$$

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Variance

$$\text{Var}\,[P_{xx}(f)] = \Gamma_{xx}^2(f) = \left[1 + \left(\frac{\sin 2\pi f N}{N \sin 2\pi f}\right)^2\right]$$

where $\quad W_{Bart}(f) = \dfrac{1}{N}\left(\dfrac{\sin \pi f N}{\sin \pi f}\right)^2$

When $N \to \infty$,

$$E\,[P_{xx}(f)] \to \Gamma_{xx}(f) \int_{-1/2}^{1/2} W_{Bart}(\theta)\mathrm{d}\theta$$

$$= W_{Bart}(0)\,\Gamma_{xx}(f)$$

$$= \Gamma_{xx}(f)$$

$\text{var}\,[P_{xx}(f)] \to \Gamma_{xx}^2(f)$

This is asymptotically unbiased estimate, but not consistent as variance does not approach zero when $N \to \infty$.

The quality factor is,

$$Q_p = \frac{\Gamma_{xx}^2(f)}{\Gamma_{xx}^2(f)} = 1$$

which is constant and independent of N specifies the poor quality.

Bartlett Power Spectrum Estimate

Mean

$$E\left[P_{xx}^{Bart}(f)\right] = \int_{-1/2}^{1/2} \Gamma_{xx}(\theta) W_{Bart}(f-\theta)\mathrm{d}\theta$$

Variance

$$\text{Var}\left[P_{xx}^{Bart}(f)\right] = \frac{1}{K}\,\Gamma_{xx}^2(f)\left[1 + \left(\frac{\sin 2\pi f M}{M \sin 2\pi f}\right)^2\right]$$

where $\quad W_{Bart}(f) = \dfrac{1}{M}\left(\dfrac{\sin \pi f M}{\sin \pi f}\right)^2$

As $\quad N \to \infty,\ M \to \infty,\ k = \dfrac{N}{M}$ (fixed)

$$E\left[P_{xx}^{Bart}(f)\right] \to \Gamma_{xx}(f) \int_{-1/2}^{1/2} W_{Bart}(f)\mathrm{d}f$$

$$= \Gamma_{xx}(f) W_{Bart}(0)$$

$$= \Gamma_{xx}(f)$$

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$$= \frac{1}{N^2} \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} \sum_{n_3=0}^{N-1} \sum_{n_4=0}^{N-1} E\,[x(n_1)x(n_2)x(n_3)x(n_4)]\; e^{-j2\pi f_2 (n_2 - n_4)}$$

$E[P_{xx}(f_1)\, P_{xx}(f_2)]$

$$= \frac{1}{N^2} \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} \sum_{n_3=0}^{N-1} \sum_{n_4=0}^{N-1} \begin{Bmatrix} E\,[x(n_1)\,x(n_2)]\, E\,[x(n_3)\,x(n_4)] + \\ E\,[x(n_1)\,x(n_3)]\, E\,[x(n_2)\,x(n_4)] + \\ E\,[x(n_1)\,x(n_4)]\, E\,[x(n_2)\,x(n_3)] \end{Bmatrix} \begin{Bmatrix} e^{-j2\pi f_1 (n_1 - n_3)} \\ e^{-j2\pi f_2 (n_2 - n_4)} \times \end{Bmatrix}$$

$$= \frac{1}{N^2} \begin{Bmatrix} \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} \sigma_x^4 + \sum_{n_1=0}^{N-1} \sum_{n_3=0}^{N-1} \sigma_x^4\, e^{-j2\pi(f_1 - f_2)(n_1 - n_3)} \\ + \sum_{n_1=0}^{N-1} \sum_{n_3=0}^{N-1} \sigma_x^4\, e^{-j2\pi(f_1 + f_2)(n_1 - n_3)} \end{Bmatrix}$$

$$= \frac{\sigma_x^4}{N^2} \begin{Bmatrix} N^2 + \sum_{n_1=0}^{N-1} e^{-j2\pi(f_1 - f_2)n_1} \sum_{n_3=0}^{N-1} e^{j2\pi(f_1 - f_2)n_3} \\ + \sum_{n_1=0}^{N-1} e^{-j2\pi(f_1 + f_2)n_1} \sum_{n_3=0}^{N-1} e^{j2\pi(f_1 + f_2)n_3} \end{Bmatrix}$$

$$= \sigma_x^4 \begin{Bmatrix} 1 + \dfrac{1 - e^{-j2\pi(f_1 - f_2)N}}{1 - e^{-j2\pi(f_1 - f_2)}} * \dfrac{1 - e^{j2\pi(f_1 - f_2)N}}{1 - e^{j2\pi(f_1 - f_2)}} * \dfrac{1}{N^2} + \\ \dfrac{1 - e^{-j2\pi(f_1 + f_2)N}}{1 - e^{-j2\pi(f_1 + f_2)}} * \dfrac{1 - e^{j2\pi(f_1 + f_2)N}}{1 - e^{j2\pi(f_1 + f_2)}} * \dfrac{1}{N^2} \end{Bmatrix}$$

$$= \sigma_x^4 \begin{Bmatrix} 1 + \dfrac{2 - e^{-j2\pi(f_1 - f_2)N} - e^{j2\pi(f_1 - f_2)N}}{2 - e^{-j2\pi(f_1 - f_2)} - e^{j2\pi(f_1 - f_2)}} * \dfrac{1}{N^2} + \\ \dfrac{2 - e^{-j2\pi(f_1 + f_2)N} - e^{j2\pi(f_1 + f_2)N}}{2 - e^{-j2\pi(f_1 + f_2)} - e^{j2\pi(f_1 + f_2)}} * \dfrac{1}{N^2} \end{Bmatrix}$$

$$= \sigma_x^4 \left\{ 1 + \frac{1}{N^2} \left(\frac{2 - 2\cos 2\pi(f_1 - f_2)N}{2 - 2\cos 2\pi(f_1 - f_2)} \right) + \frac{1}{N^2} \left(\frac{2 - 2\cos 2\pi\,(f_1 + f_2)N}{2 - 2\cos 2\pi\,(f_1 + f_2)} \right) \right\}$$

$$= \sigma_x^4 \left\{ 1 + \left(\frac{\sin \pi\,(f_1 - f_2)N}{N \sin \pi(f_1 - f_2)} \right)^2 \left(\frac{\sin \pi\,(f_1 + f_2)N}{N \sin \pi(f_1 + f_2)} \right)^2 \right\}$$

(b)

$$\text{var}\quad [P_{xx}(f)] = E\left[P_{xx}^2(f)\right] - \left\{E\left[P_{xx}^2(f)\right]\right\}^2$$

But, $E\,[\,P_{xx}(f)] = \sigma_x^2$

where

$r_{xx}(k)$ is the autocorrelation sequence of $x(n)$

$r_{dl}(k)$ is the crosscorrelation sequence of $x(n)$ and $d(n)$

If $d(n) = x(n)$,

$$\sum_{k=1}^{N} a_N(k)\, r_{xx}(k-1) = -r_{xx}(l),\ l = 1, 2, \ldots, N$$

The above set of equations are called normal equations or the Yule-Walker equation.

Solve the equations recursively. First consider a predictor of order one, which is given by

$$a_1(1) = \frac{r_{xx}(1)}{r_{xx}(0)}$$

The least squares error becomes

$$\varepsilon_1 = \sum_{n=-\infty}^{\infty} [\, x(n) + a_1(1)\, x(n)]^2, \quad \text{since } x(n) = d(n)$$

$$= \sum_{n=-\infty}^{\infty} x^2(n) + 2a_1 \sum_{n=-\infty}^{\infty} x(n)\, x(n-1) + a_1^2 \sum_{n=-\infty}^{\infty} x^2(n-1)$$

$$\varepsilon_1 = r_{xx}(0) + 2a_1(1)\, r_{xx}(1) + a_1^2(1)\, r_{xx}(0)$$

substituting the value for $r_{xx}(1)$ in terms of $a_1(1)$, we get

$$\begin{aligned}\varepsilon_1 &= r_{xx}(0) + 2a_1(1)\,(-a_1(1)\, r_{xx}(0)) + a_1^2(1)\, r_{xx}(0)\\ &= r_{xx}(0) - 2a_1^2(1)\, r_{xx}(0) + a_1^2(1)\, r_{xx}(0)\\ &= r_{xx}(0)\,[1 - a_1^2(1)]\end{aligned}$$

Similarly considering the second-order predictor, we get

$$a_2(1)\, r_{xx}(0) + a_2(2)\, r_{xx}(1) = -r_{xx}(1)$$

$$a_2(1)\, r_{xx}(1) + a_2(2)\, r_{xx}(0) = -r_{xx}(2)$$

On solving,

$$a_2 = -\left(\frac{r_{xx}(2) + a_1(1)\, r_{xx}(1)}{r_{xx}(0) - a_1^2(1)\, r_{xx}(0)}\right)$$

$$= -\frac{r_{xx}(2) + a_1(1)\, r_{xx}(1)}{\varepsilon_1}$$

and $\qquad a_2(1) = a_1(1) + a_2(2)\, a_1(1)$

Now the second-order predictor coefficients are expressed in terms of first-order predictor coefficients.

In general, the m^{th} order predictor coefficients can be expressed in terms of $(M-1)^{\text{th}}$ order predictor coefficients.

$$\sum_{k=0}^{p} a_k e^{-j2\pi k f_l} = 1$$

In matrix form

$$E^{*t}(f_l)\, a = 1$$

where $E^t(f_l) = [\,1 \quad e^{j2\pi f_l} \ldots e^{j2\pi p f_l}]$

Minimise the variance σ_y^2 subjected to the constraint specified, yields an FIR filter which allows the f_l frequency components undistorted. Other frequency components are attenuated. This yields,

$$\hat{a} = \Gamma_{xx}^{-1} E^*(f_l) / E^t(f_l)\, \Gamma_{xx}^{-1} E^*(f_l)$$

The variance becomes

$$\sigma_{\min}^2 = \frac{1}{E^t(f_l)\, \Gamma_{xx}^{-1}\, E^*(f_l)}$$

The minimum variance power spectrum estimate at frequency f_l is represented in the above equation. By varying the frequency f_l from 0 to 0.5, the power spectrum estimate can be obtained. Even if f_l changes, Γ_{xx}^{-1} is computed only once. The denominator of $\sigma_{\min}^2$ can be computed using single DFT. If R_{xx} is the estimate of Γ_{xx}, R_{xx} can replace Γ_{xx} and the minimum variance power spectrum estimate of Capon's method is

$$P_{xx}^{mv}(f) = \frac{1}{E^t(f)\, R_{xx}^{-1}\, E^*(f)}$$

This estimate results in spectral peaks estimate proportional to the power at that frequency.

12.6.8 The Pisarenko Harmonic Decomposition Method

This method provides the estimate for signal components which are sinusoids corrupted by additive white noise.

A real sinusoid signal can be obtained from the difference equation

$$x(n) = -\,a_1\, x(n-1) - a_2\, x\,(n-2)$$

where $\quad a_1 = 2\cos 2\,\pi f_k$

$$a_2 = 1$$

initial conditions,

$$x(-1) = -1$$

$$x(-2) = 0$$

This system has complex-conjugate poles at $f = f_k$ and $f = -f_k$, obtaining the sinusiod $x(n) = \cos 2\pi f_k n, \quad n \geq 0$.

Consider p sinusoid components available in the signal,

$$x(n) = -\sum_{m=1}^{2p} a_m\, x(n-m)$$

The system function is given by

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$$\varepsilon_M = \sum_{l=0}^{n} w^{n-l} \, |e_M(l, n)|^2 \tag{13.37}$$

where

$$\begin{aligned} e_M(l, n) &= d(n) - \hat{d}(l - n) \\ &= d(n) - h_M^t(n) X_M(l) \end{aligned}$$

and w is the weighting factor lying in the range $0 < w \le 1$.

The weighting factor is used to give more weightage to the most recent points so that the filter coefficients can be properly adapted to the time varying statistical characteristics. Otherwise a finite duration sliding window with uniform weightage can be used.

Minimisation of ε_M with respect to $h_M(n)$

$$\begin{aligned} \text{Min}\{\varepsilon_M\} &= \text{Min}\left\{\sum_{k=0}^{n} w^{n-l} |e_M(l, n)|^2\right\} \\ &= \text{Min}\left\{\sum_{l=0}^{n} w^{n-l} |\text{d}(l) - h_M^t(n) X_M(l)|^2\right\} \end{aligned} \tag{13.38}$$

The minimisation of ε_M results in

$R_M(n) \, h_M(n) = D_M(n)$

where

$$R_M(n) = \sum_{l=0}^{n} w^{n-l} X_M^*(l) X_M^t(l)$$

—estimated signal correlation matrix

$$D_M(n) = \sum_{l=0}^{n} w^{n-l} X_M(l) \, \text{d}(l)$$

—estimated crosscorrelation vector

The solution can be obtained as

$$h_M = R_M^{-1}(n) \, D_M(n) \tag{13.39}$$

$R_M(n)$ and $D_M(n)$ can be computed recursively by

$$R_M(n) = w \, R_M(n-1) + X_M^*(n) X_M^t(n) \tag{13.40}$$

This is known as the time update equation for $R_M(n)$

$$D_M(n) = w \, D_M(n-1) + \text{d}(n) X_M^*(n) \tag{13.41}$$

Matrix Inversion Lemma

Let A and B be two positive definite $M \times M$ matrices, D is a positive definite $N \times N$ matrix and C is an $M \times N$ matrix.

$$A = B^{-1} + C D^{-1} C^T \tag{13.42}$$

A^{-1} can be obtained from matrix inversion lemma as

$$A^{-1} = B - BC\,[D + C^T B\, C]^{-1} C^T B \tag{13.43}$$

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$$\sum_{i=1}^{n-1} w^{n-1-i}\, b_{m-1}^2(i) = \sum_{i=1}^{n} w^{n-i}\, b_{m-1}^2(i-1) \tag{13.65}$$

Hence,

$$K_{m-1}(n) = \sum_{i=1}^{n-1} w^{n-i} f_{m-1}(i)\, b_{m-1}(i-1) + f_{m-1}(n)\, b_{m-1}(n-1)$$

$$= \sum_{i=1}^{n-1} w^{n-1-i} f_{m-1}(i)\, b_{m-1}(i-1) + f_{m-1}(n)\, b_{m-1}(n-1)$$

$$= w\, K_{m-1}(n-1) + f_{m-1}(n)\, b_{m-1}(n-1) \tag{13.66}$$

Similarly,

$$E_{m-1}^{(f)}(n) = w\, E_{m}^{(f)}{}_{-1}\, f_{m-1}^2(n) \tag{13.67}$$

and

$$E_{m-1}^{(b)}(n) = w\, E_{m-1}^{(b)}(n-1) + b_{m-1}^2(n) \tag{13.68}$$

Order update recursions The prediction errors are given by

$$\begin{aligned} f_m(i) &= f_{m-1}(i) + \gamma_m^{(f)}(n)\, b_{m-1}(i-1), \quad 1 \le i \le n \\ b_m(i) &= b_{m-1}(i-1) + \gamma_m^{(b)} f_{m-1}(i), \quad 1 \le i < n \end{aligned} \tag{13.69}$$

where

$\gamma_m^{(f)}(n)$ – forward reflection coefficient of m^{th} stage
$\gamma_m^{(b)}(n)$ – backward reflection coefficient of m^{th} stage
and $\gamma_m^{(f)}(n)$ and $\gamma_m^{(b)}(n)$ are considered as constants for the time interval $1 \le i < n$.

Estimate for forward and backward prediction error

$$E_m^{(f)}(n) = \sum_{i=1}^{n} w^{n-i} f_m^2(i)$$

$$= \sum_{i=1}^{n} w^{n-i} \left[f_{m-1}(i) + \gamma_m^{(f)}(n)\, b_{m-1}(i-1)\right]^2$$

$$= \sum_{i=1}^{n} w^{n-i} f_m^2(i) + 2\gamma_m^{(f)}(n)$$

$$\sum_{i=1}^{n} w^{n-i} f_{m-1}(i)\, b_{m-1}(i-1) + \left[\gamma_m^{(f)}(n)\right]^2 \sum_{i=1}^{n} w^{n-i} b_{m-1}^2(i-1)$$

$$= E_{m-1}^{(f)} - \frac{2K_{m-1}(n)}{E_{m-1}^{(b)}(n-1)} K_{m-1}(n) + \left[\frac{K_{m-1}(n)}{E_{m-1}^{(b)}(n-1)}\right]^2 E_{m-1}^{(b)}(n-1)$$

$$= E_{m-1}^{(f)}(n) - \frac{K_{m-1}^2(n)}{E_{m-1}^{(b)}(n-1)} \tag{13.70}$$

Chapter 14

Applications of Digital Signal Processing

14.1 INTRODUCTION

Digital signal processing techniques are used in a variety of areas which include speech, radar, sonar, image, etc. These techniques are applied in spectral analysis, channel vocoders, homomorphic processing systems, speech synthesisers, linear prediction systems, analysing the signals in radar tracking, etc.

14.2 VOICE PROCESSING

There are different areas in voice processing like encoding, synthesis and recognition. In a speech signal some amount of redundancy is present, which can be removed by encoding. Synthesis is required at the receiver side because in the transmitter, compression/coding is done. Recognition involves recognising both the speech and the speaker.

14.2.1 Speech Signal

A speech signal consists of periodic sounds, interspersed with bursts of wide band noise and sometimes short silences. The vocal organs are in motion continuously, hence the signal generated is not stationary. But short segments of 50 ms are treated to be approximately stationary.

These signals are generated by the vibration of the vocal cords. The muscles in the larynx stretch these cords, which vibrate when air is forced, thus producing sound in the form of a pulse train. This passes through the pharynx cavity and tongue and is expelled either at the mouth or nasal cavity depending on the position of velum. Fig. 14.1 shows the mechanism of human speech production.

When air passes through narrow constrictions in the vocal tract, a turbulent flow is produced. Otherwise pressure is built up along the tract behind a point of total constriction.

The vocal system can be modelled with a periodic signal excitor, a variable filter representing the vocal tract, switch to pass the signal

The short time transform at a finite set of N frequencies, over the band $0 \le \omega T \le 2\pi$. $h(nT)$ is the impulse response of an FIR filter. It is non-zero for $0 \le n \le M-1$ and the centre frequencies are

$$\omega_k = \frac{2\pi k}{NT}, \quad k = 0, 1, \ldots, N-1 \tag{14.6}$$

$$\text{Therefore, } X(\omega_k, nT) = \sum_{r=n-M+1}^{n} x(rT)\, h(nT - rT)\, e^{-j\omega_k rT}$$

$$= \sum_{m=0}^{\left[\frac{M}{N}\right]+1} \sum_{r=n-(m-1)N+1}^{n-mN} x(rT)\, h(nT - rT)\, e^{-j\omega_k rT} \tag{14.7}$$

where $[M/N] + 1$ is the greatest integer less than or equal to M/N.
Let $l = n - mN - r$
$X(\omega_k, nT) =$

$$\sum_{m=0}^{\left[\frac{M}{N}\right]+1} \sum_{l=0}^{N-1} x(nT - rT - mNT)\, h(lT - mNT)\, e^{-j\omega_k (l-n+mN)T}$$

$$= e^{-j\left(\frac{2\pi}{N}\right)kn} \sum_{l=0}^{N-1} \sum_{m=0}^{\left[\frac{M}{N}\right]+1} \begin{Bmatrix} x(nT - lT - mNT)x \\ h(lt + mNT)\, e^{j\left(\frac{2\pi}{N}\right)kl} \end{Bmatrix}$$

$$h(lT + mNT)\, e^{j\left(\frac{2\pi}{N}\right)kl} \tag{14.8}$$

$$X(\omega_k, nT) = e^{-j\left(\frac{2\pi}{N}\right)kn} \sum_{l=0}^{N-1} g(l, n)\, e^{j\left(\frac{2\pi}{N}\right)lk} \tag{14.9}$$

where

$$g(l, n) = \sum_{m=0}^{\left[\frac{M}{N}\right]+1} x(nT - lT - mNT)\, h(lT + mNT) \tag{14.10}$$

This analysis is done usually with a bank of digital filters.

14.2.4 Speech Analysis Synthesis System

The main objective is to measure the outputs of bandpass filter banks and reconstruct the speech from these signals. Figure 14.5 shows the analysis-synthesis system. Let $x(nT)$ be the speech input and $y(nT)$ be the reconstructed synthetic waveform. The impulse response of the bandpass filter bank is $h_k(nT)$, $k = 1, 2, \ldots, M$. $y(nT)$ is obtained by summing the individual bandpass filter outputs

$$y_k(nT), k = 1, 2, \ldots, M$$

$$h_k(nT) = h(nT)\cos(\omega_k nT) \tag{14.11}$$

where $h(nT)$ is the impulse response of a lowpass filter.

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λ - wavelength

Pulsed Doppler signals can be used to obtain both range and velocity resolution.

14.3.1 Signal Design

Transmitting narrow pulse provides good range but poor velocity measurement. A wide pulse of single frequency gives good velocity but bad range information.

Consider the radar model shown in Fig. 14.16. Let the signal be generated digitally and transmitted through an analog filter. The transmitted signal is $s(t)$. The received signal is $s(t-\tau)\,e^{j2\pi f(t-\tau)}$ which is delayed and frequency shifted. The received signal is passed through

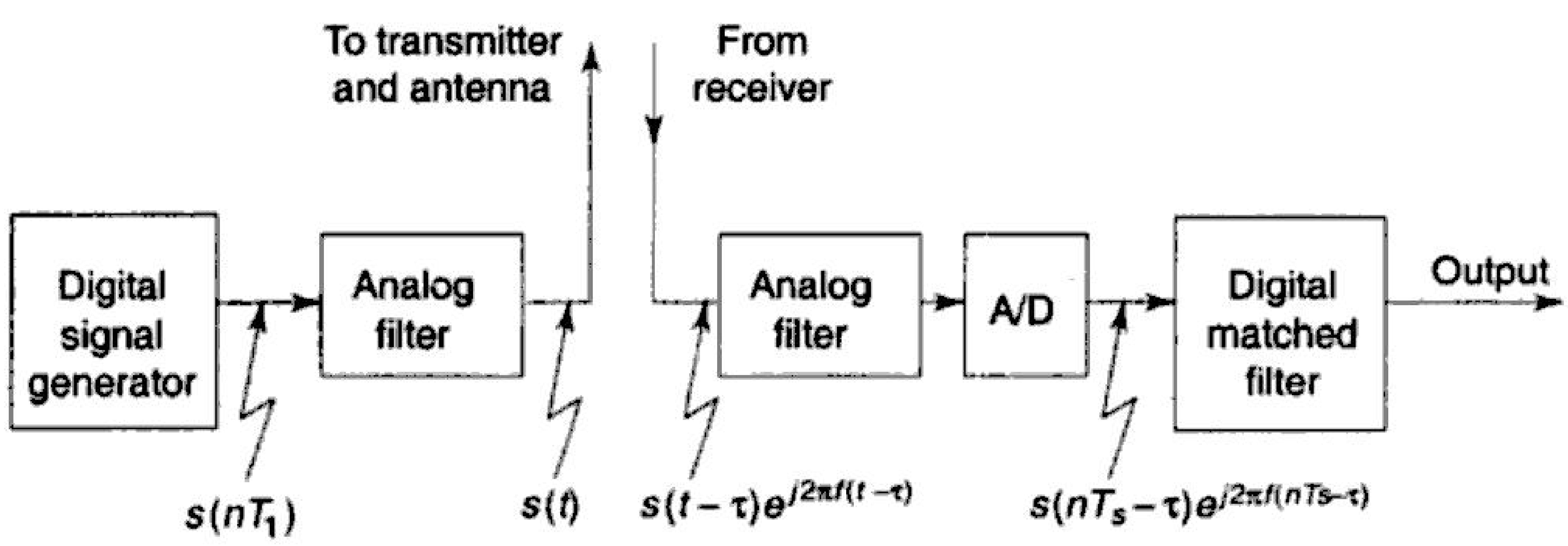

Fig. 14.16 *Block Diagram of a Radar Model*

an analog filter, A/D converter and then through a digital matched filter. The input signal to the matched filter is $s(nTs-\tau)\,e^{j2\pi f(nTs-\tau)}$.

A long duration signal is required for preserving radar power. But for preserving range resolution, narrow signals are required. This problem can be resolved by designing long duration signals with short duration correlation functions. When the received signal is passed through the appropriate matched filter, a sharp pulse will be available at the filter output. The digital filter can be matched to the signal return for zero range and zero Doppler. Hence its impulse response can be $s^*(-n\,\tau_s)$.

14.4 APPLICATIONS TO IMAGE PROCESSING

2D signal processing is helpful in processing the images. The different processing techniques are image enhancement, image restoration and image coding.

Image enhancement focuses mainly on the features of an image. The various feature enhancements are sharpening the image, edge enhancement, filtering, contrast enhancement, etc. Linear filtering emphasises some spectral regions of the signal. Histogram modification is done on pixel-by-pixel basis and finds its application in contrast equalisation or enhancement. Figure 14.17 shows some image enhancement operations.

in time and centered at the location of the Dirac. For small a's, the transform "zooms-in" to the Dirac with good localisation for very small scales.

Frequency localisation

Consider the sinc wavelet, i.e. a perfect bandpass filter. Its magnitude spectrum is 1 for $|\omega|$ between π and 2π. Consider a complex sinusoid of unit magnitude and at frequency ω_0. The highest frequency wavelet that passes the sinusoid having a scale factor of π/ω_0 (gain of $\sqrt{\pi/\omega_0}$) while the low frequency wavelet that passes the sinusoid having a scale factor of $2\pi/\omega_0$ (gain of $\sqrt{2\pi/\omega_0}$).

(viii) Reproducing Kernel

The CWT is a very redundant representation since it is a 2-D expansion of a 1-D function. Consider the space V of a square integrable function over the plane (a, b) with respect to $da\, db/a^2$. Only a subspace H of V corresponds to wavelet transforms of functions from $L^2(R)$.

If a function $W(a, b)$ belongs to H, i.e. it is the wavelet transform of $f(t)$, then $W(a, b)$ satisfies

$$W(a_0, b_0) = \frac{1}{C} \iint K(a_0, b_0, a, b)\, W(a, b)\, \frac{da\, db}{a^2} \tag{14.34}$$

where $K(a_0, b_0, a, b) = \langle \psi_{a_0, b_0}, \psi_{a, b} \rangle$ is the reproducing kernel.

Discrete Wavelet Transform

The discrete wavelet transform (DWT) corresponding to a CWT function $W(a, b)$ can be obtained by sampling the co-ordinates (a, b) on a grid as shown in Fig. 14.22. This process is called the ***dyadic sampling*** because the consecutive values of discrete scales as well as the corresponding sampling intervals differs by a factor of two. Then the dilation takes the values of the form $a = 2^k$ and translation takes the values of the form $b = 2^k l$ where k and l are integers. The values

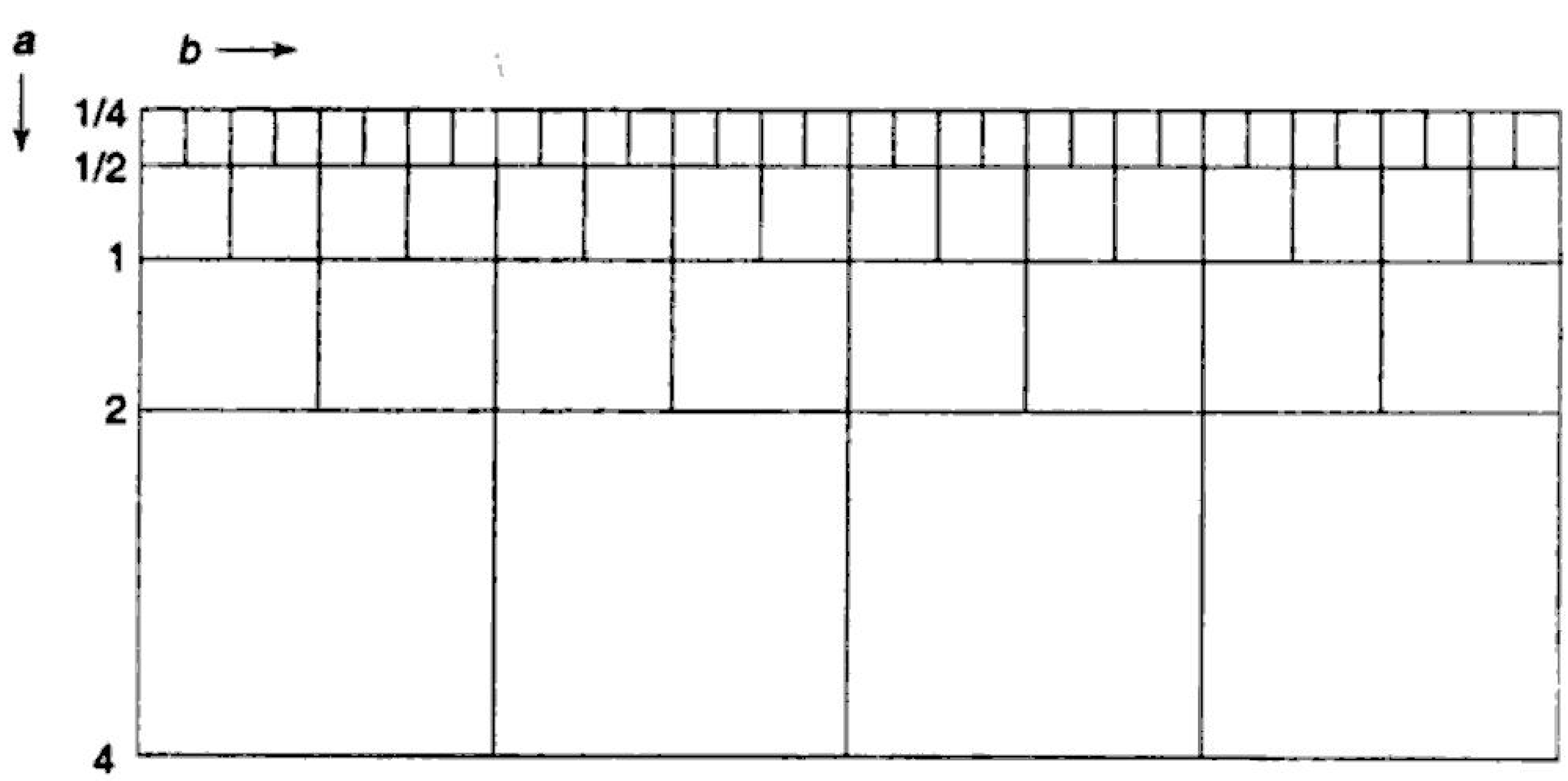

Fig. 14.22 *Time-Frequency Cells that Correspond to Dyadic Sampling*

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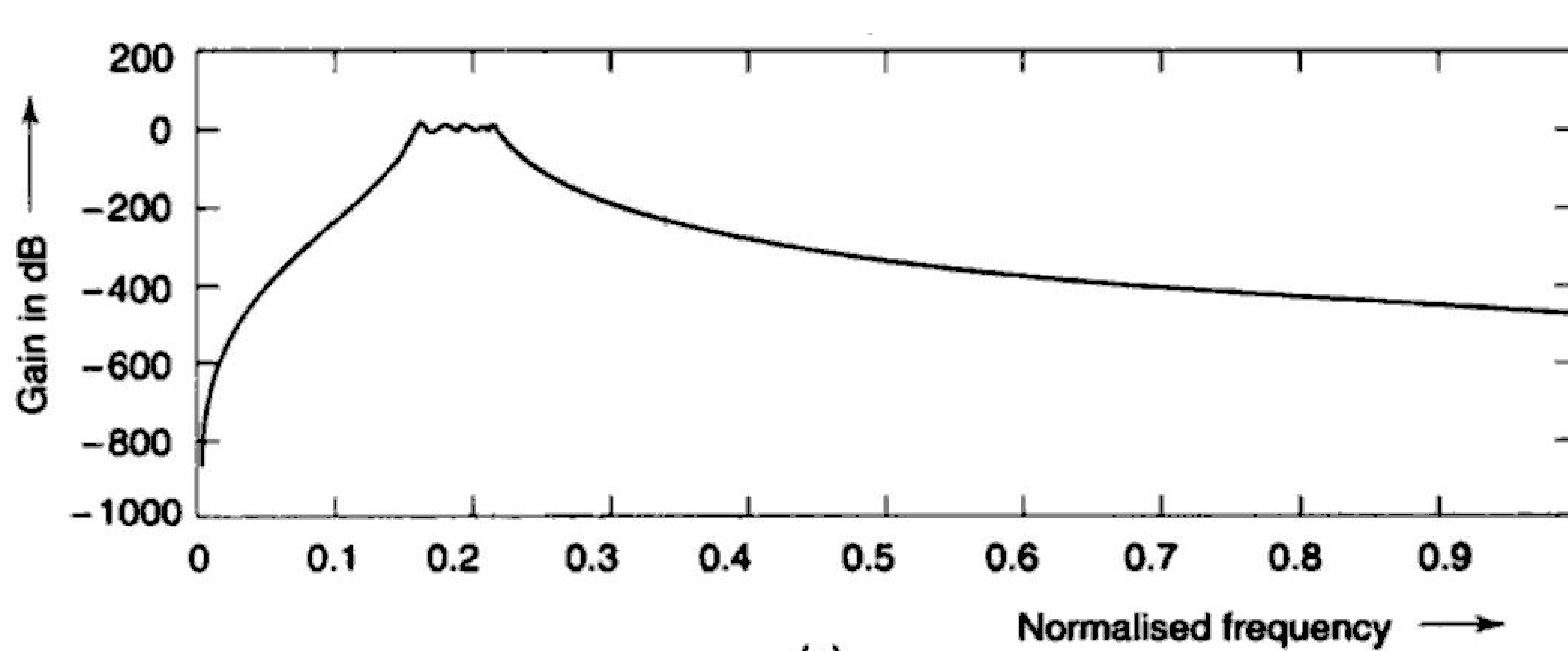
%Program for the design of Butterworth analog Bandpass filter

```
clc;
close all;clear all;
format long
rp=input('enter the passband ripple...');
rs=input('enter the stopband ripple...');
wp=input('enter the passband freq...');
ws=input('enter the stopband freq...');
fs=input('enter the sampling freq...');
w1=2*wp/fs;w2=2*ws/fs;
[n]=buttord(w1,w2,rp,rs,'s');
wn=[w1 w2];
[b,a]=butter(n,wn,'bandpass','s');
w=0:.01:pi;
[h,om]=freqs(b,a,w)     ;
m=20*log10(abs(h));
an=angle(h);
subplot(2 ,1, 1);plot(om/pi,m);
ylabel('Gain in dB -->');xlabel('(a)   Normalised frequency -->');
subplot(2 ,1 ,2);plot(om/pi,an);
xlabel('(b)                    Normalised frequency -->');
ylabel('Phase in radians -->');
```

As an example,

```
enter the passband ripple...    0.36
enter the stopband ripple...      36
enter the passband freq...      1500
enter the stopband freq...      2000
enter the sampling freq...      6000
```

The amplitude and phase responses of Butterworth bandpass analog filter are shown in Fig. 15.9.



(a)

Fig. 15.9 *(Contd.)*

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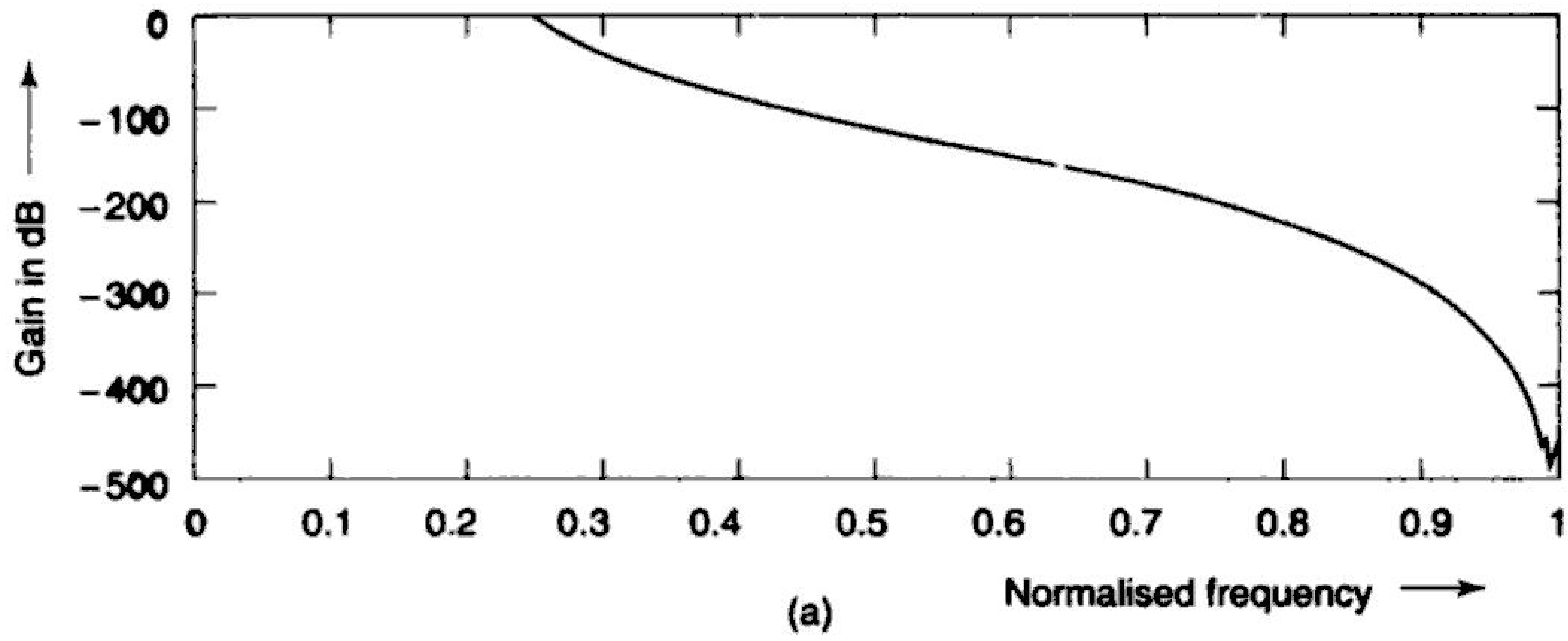

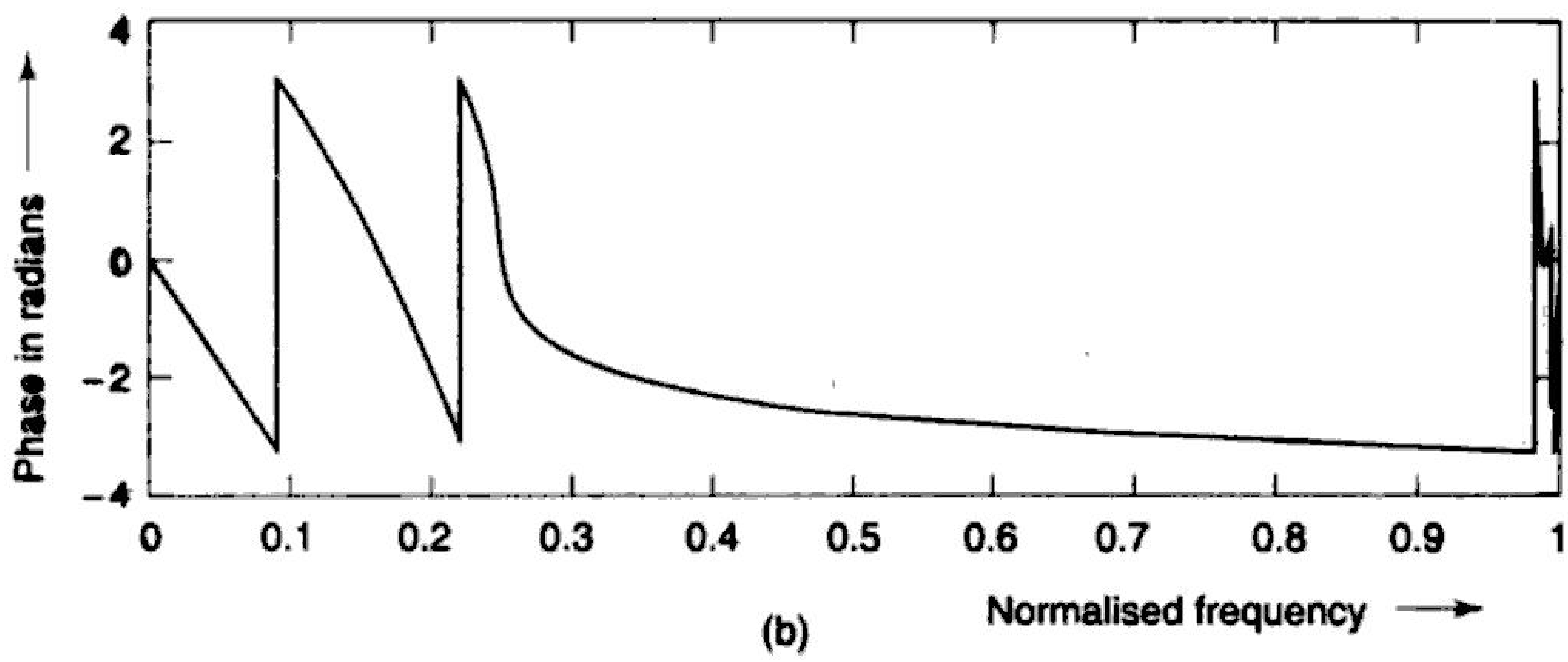

Fig. 15.23 *Chebyshev Type - I Low-pass Digital Filter (a) Amplitude Response and (b) Phase Response*

```
[n,wn]=cheb1ord(w1,w2,rp,rs);
[b,a]=cheby1(n,rp,wn,'high');
w=0:.01/pi:pi;
[h,om]=freqz(b,a,w);
m=20*log10(abs(h));
an=angle(h);
subplot(2,1,1);plot(om/pi,m);
ylabel('Gain in dB -->');xlabel('(a)    Normalised frequency -->');
subplot(2,1,2);plot(om/pi,an);
xlabel('(b)                   Normalised frequency -->');
ylabel('Phase in radians -->');
```

As an example,

```
enter the passband ripple...   0.3
enter the stopband ripple...    60
enter the passband freq...    1500
enter the stopband freq...    2000
enter the sampling freq...    9000
```

The amplitude and phase responses of Chebyshev type - 1 high-pass digital filter are shown in Fig. 15.24.

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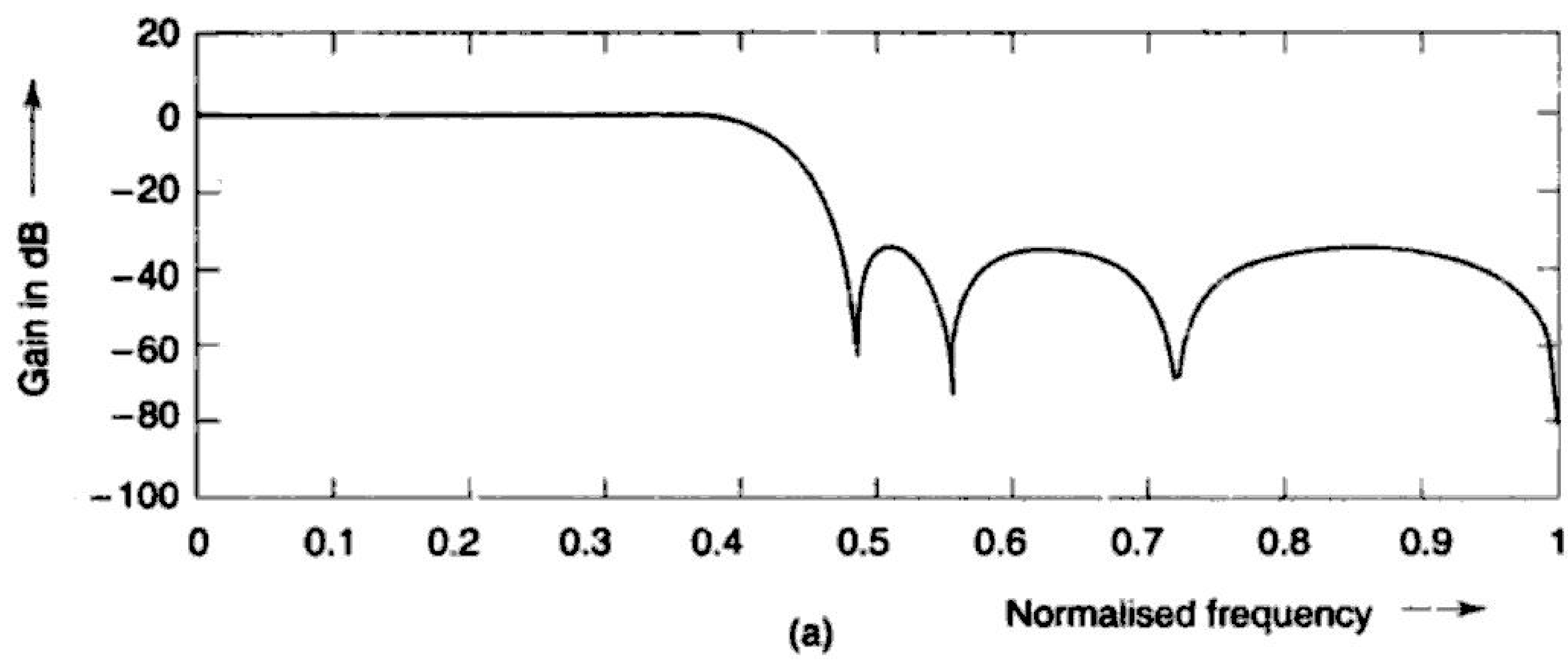
```
an=angle(h);
subplot(2,1,1);plot(om/pi,m);
ylabel('Gain in dB -->');xlabel('(a)    Normalised frequency -->');
subplot(2,1,2);plot(om/pi,an);
xlabel('(b)                    Normalised frequency -->');
ylabel('Phase in radians -->');
```

As an example,

```
enter the passband ripple...   0.35
enter the stopband ripple...     35
enter the passband freq...     1500
enter the stopband freq...     2000
enter the sampling freq...     8000
```

The amplitude and phase responses of Chebyshev type - 2 low-pass digital filter are shown in Fig. 15.27.



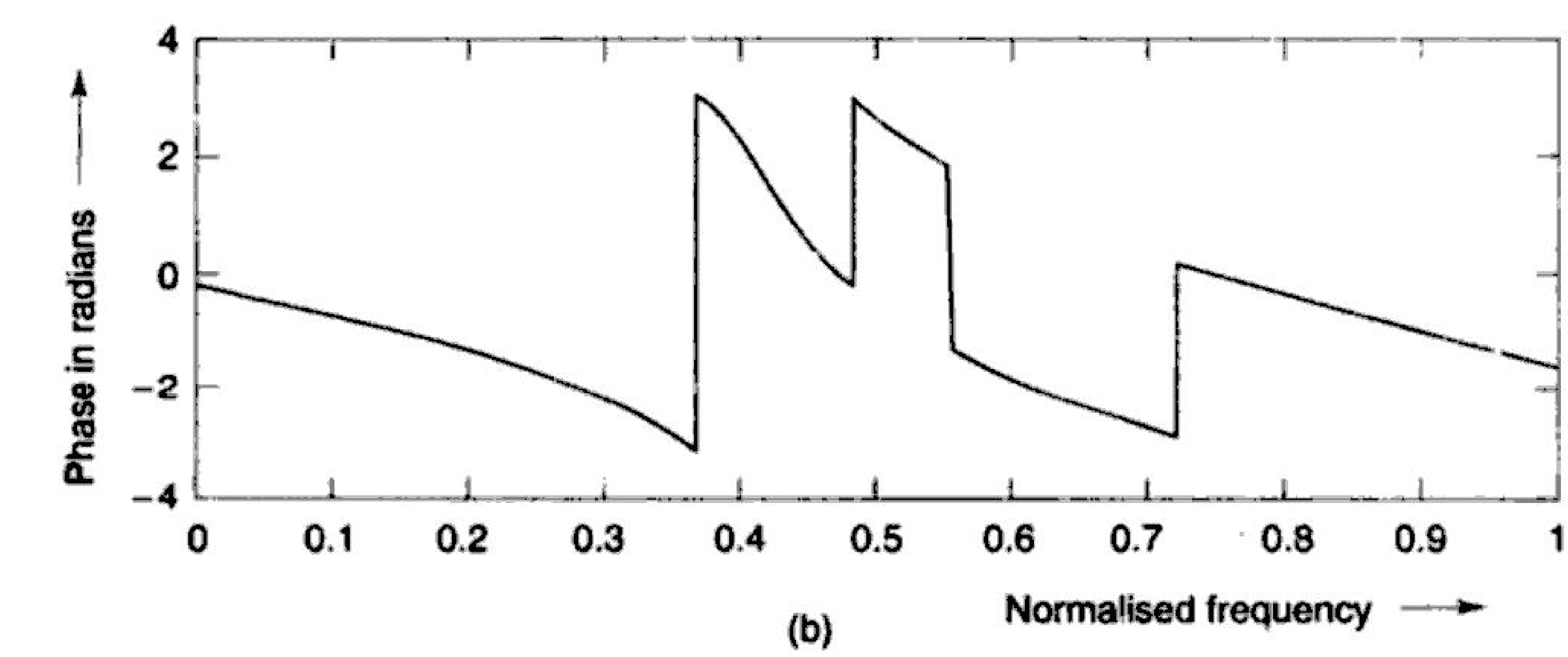


Fig. 15.27 *Chebyshev Type - 2 Low-pass Digital Filter (a) Amplitude Response and (b) Phase Response*

15.13.2 High-pass Filter

Algorithm

1. Get the passband and stopband ripples
2. Get the passband and stopband edge frequencies

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```
%HIGH-PASS FILTER
b=fir1(n-1,wp,'high',y);
[h,o]=freqz(b,1, 256);
m=20*log10(abs(h));
subplot(2,2,2);plot(o/pi,m);ylabel('Gain in dB -->');
xlabel('(b)                    Normalised frequency -->');
%BAND-PASS FILTER
wn=[wp ws];
b=fir1(n-1,wn,y);
[h,o]=freqz(b,1,256);
m=20*log10(abs(h));
subplot(2,2,3);plot(o/pi,m);ylabel('Gain in dB -->');
xlabel('(c)                    Normalised frequency -->');
%BAND-STOP FILTER
b=fir1(n-1,wn,'stop',y);
[h,o]=freqz(b,1,256);
m=20*log10(abs(h));
subplot(2,2,4);plot(o/pi,m);ylabel('Gain in dB -->');
xlabel('(d)                    Normalised frequency -->');
```

As an example,

```
enter the passband ripple      0.03
enter the stopband ripple      0.02
enter the passband freq        1800
enter the stopband freq        2400
enter the sampling freq        10000
enter the ripple value(in dBs) 40
```

The gain responses of low-pass, high-pass, bandpass and bandstop filters using Chebyshev window are shown in Fig. 15.34.

15.14.5 Hamming Window

Algorithm

1. Get the passband and stopband ripples
2. Get the passband and stopband edge frequencies
3. Get the sampling frequency
4. Calculate the order of the filter
5. Find the window coefficients using Eq. 7.40
6. Draw the magnitude and phase responses.

%Program for the design of FIR Low pass, High pass, Band pass and Bandstop filters using Hamming window

```
clc;clear all;close all;
rp=input('enter the passband ripple');
rs=input('enter the stopband ripple');
fp=input('enter the passband freq');
fs=input('enter the stopband freq');
f =input('enter the sampling freq');
```

```
a=input('enter the donomiator polynomials=');
c=input('enter the gain of the filter=');
[n,m]=size(b);
a=a(:,2:3);b=b(:,2,:3);
u=zeros(n,2);
for i=1 :length(x),
  for k=1 :n,
     unew=x(i)-sum(u(k,:).*a(k,:));
     x(i)=-unew+sum(u(k,:).*b(k,:))
     u(k,:)=[unew,u(k,1)];
  end
     y(i)=c*x(i);
  end
```

15.31 DECIMATION BY POLYPHASE DECOMPOSITION

%Program for computing convolution and m-fold decimation by polyphase decomposition

```
function y = ppdec(x,h,M);
x=input('enter the input sequence=');
h=input('enter the FIR filter coeficients=');
M=input('enter the decimation factor=');
lh = length(h); lp = floor ((lh-1)/M) + 1;
p = reshape([reshape(h,1,lh),zeros(1,lp*M-lh)],M,lp);
lx=length(x); ly = floor ((lx + lh-2)/M)+1;
lu=floor((lx+M-2)/M)+1; %length of decimated sequences
u = [zeros(1,M-1),reshape(x,1,lx), zeros(1,M*lu-lx-M+1)];
y = zeros(1,lu+lp-1);
for m = 1:M, y = y + conv(u(m,:),p(m,:));end
y = y(1:ly);
```

15.32 MULTIBAND FIR FILTER DESIGN

%Program for the design of multiband FIR filters

```
function h = firdes(N,spec,win);
N = input('enter the length of the filter=');
spec = input('enter the low, high cutoff frequencies and
  gain=');
win = input('enter the window length=');
flag = rem(N,2);
[K,m] = size(spec);
n = (0:N) - N/2;
if (~flag), n(N/2+1) = 1;
end,h=zeros(1,N+1);
for k=1:K,
  temp= (spec(k,3)/pi)*(sin(spec(k,2)*n)-sin(spec(k,1)*n))./n;
  if(~flag),temp(N/2+1) = spec(k,3)*(spec(k,2)-spec(k,1))/pi;
```

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```
rs1=filter(h,1,y);
%for i=1;length(ns),
% rs1(i)=rs6(i);
%end
for i=1:length(ns),
  rs(i)=rs1(i);
end
subplot(2,2,2);plot(rs);title('noise signal');
xlabel('Time ——');ylabel('Amplitude ——');
%Far end signal
fs1=[5 5 2 5 2 5 2 5 2 5 2 5 2 5 5 5 5 5 5 5 5 5 2 2 2 2 2 2 2 2 2 2 2 2 2 5];
%rs=sign(rs2);
%Far end signal is digitally modulated and plotted
z1 = dmod(fs1, fc, fd, fs,'psk');
for i = 1 :length(ns),
  z(i) = z1(i);
end
subplot (2, 2, 3);plot(z);title ('far-end signal');
xlabel('Time ——;);ylabel('Amplitude ——');
%Echo and the far end modulated signal is added in the hybrid
q1=z1 + rs1;
for i=1:length(ns),
  q(i)=q1(i);
end
subplot(2,2,4);plot(q);title('received signal');
xlabel('Time ——'); ylabel('Amplitude ——');
q2=xcorr(q);
%Auto correlation is taken for the near end signal
ar=xcorr(ns);
%cross correction is taken for the near end and far end signal
crd=xcorr(rs,ns);
l1=length(ar);j=1;
for i=round(l1/2):l1,
  ar1(j)=ar(i)
  j=j+1;
end
%Toeplitz matrix is taken for the auto correlated signal
r=toeplitz(ar1);
l2=length(crd);j=1;
for i=round (l2/2):l2,
  crd1(j)=crd(i);
  j=j+1;
end
p=crd1';
%Maximum and minimum eigen values are calculated from the
toeplitz matrix
lam=max(eig(r));la=min(eig(r));l=lam/la;
```

(c) *Repeat parts (a) and (b) using the Hamming window*

(d) *Repeat parts (a) and (b) using the Bartlett window.*

15.20 *Design an FIR Linear Phase, bandstop filter having the ideal frequency response*

$$H_d(\omega) = \begin{cases} 1, & \text{for } |\omega| \le \pi/6 \\ 0, & \text{for } \pi/6 < |\omega| \le \pi/3 \\ 1, & \text{for } \pi/3 \le |\omega| \le \pi \end{cases}$$

(a) *Determine the coefficient of a 25 tap filter based on the window method with a rectangular window.*

(b) *Determine and plot the magnitude and phase response of the filter.*

(c) *Repeat parts (a) and (b) using the Hamming window*

(d) *Repeat parts (a) and (b) using the Bartlett window.*

15.21 *A digital low-pass filter is required to meeth the following specfications:*

Passband ripple ≤ 1 *dB*

Passband edge 4 KHz

Stopband attenuation ≥ 40 *dB*

Stopband edge 6 KHz

Sample rate 24 KHz

The filter is to be designed by performing a bilinear transformation on an analog system function. Determine what order Butterworth, Chebyshev and elliptic analog design must be used to meet the specifications in the digital implementation.

15.22 *An IIR digital low-pass filter is required to meet the following specfications*

Passband ripple ≤ 0.5 *dB*

Passband edge 1.2 KHz

Stopband attenuation ≥ 40 *dB*

Stopband edge 2 KHz

Sample rate 8 KHz

Use the design formulas to determine the filter order for

(a) *Digital Butterworth filter*

(b) *Digital Chebyshev filter*

(c) *Digital elliptic filter*

15.23 *An analog signal of the form* $x_a(t) = a(t)\cos(2000\pi t)$ *is bandlimited to the range* $900 \le F \le 1100$ *Hz. It is used as an input to the system shown in Fig. Q15.23.*

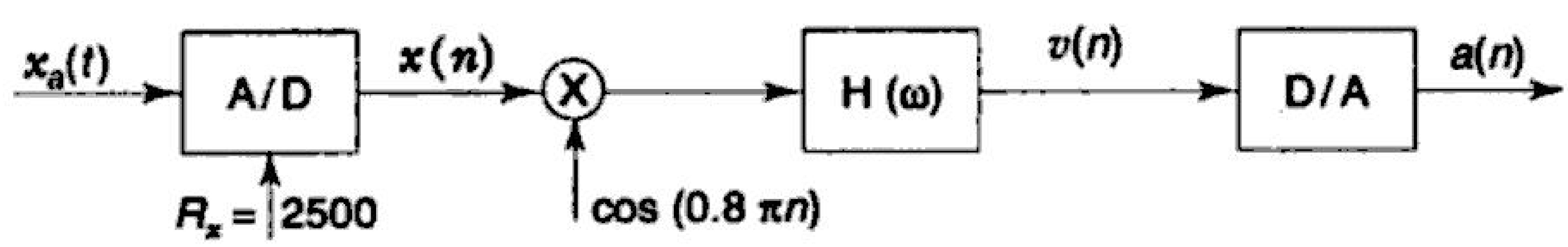

Fig. Q 15.23

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```
  if (M< = 0)
  {
    M = fabs(M);
  }
  M1 = fabs(20 * log10(M));
  if (( F > = edge[1]) && (F < = edge[2]) && (fx[1] == 1))
  { M1 = 0;}
  else if ((F > = edge[3]) && (F < = edge[4]) && (fx[2] == 1))
  { M1 = 0;}
  else if((F > = edge[5]) && (F < = edge[6]) && (fx[3] == 1))
  { M1 = 0;}
  if ( M1 > 85)
  {M1 = 85;}
  X1 = 25 + 1000 * F;
  Y1 = 25 + M1;
  X3 = 25 + 1000 * F;
  Y3 = 170 - 15 * N;
  line(X1,Y1,X2,Y2);
  line(X3,Y3,X4,Y4);
}
getch( );
closegraph( );
if (nfilt != 0)
{ goto ag1;}
return ;
}

/* FUNCTION FOR REMEZ EXCHANGE ALGORITHM */
remez(edge, nband)
int nband;
float edge[20];
{
  int j, k, l, nz, nm1, neg, kup, lband;
  int luck, itrmax, kkk;
  int jm1, jp1, nzz, niter, nu, nut, nut1, jet;
  double a[66], p[66], q[66];
  double dnum, dden, dtemp;
  double d( ), gee( );
  float y1, err, jchnge, devl, delf, fsh, gtemp, cn;
  float k1, kn, ynz, knz, klow, comp, aa, bb, ft, xt, xe;
  /* The max.no.of iterations is 25*/
  itrmax = 25;
  devl = -1.0;
  nz = nfcns + 1;
  nzz = nfcns +2;
  niter = 0;
```

Index

DIGITAL SIGNAL PROCESSING

This book comprehensively covers the undergraduate course on Digital Signal Processing. Computer usage is integrated into the text in the form of problem solving using MATLAB. Solved examples and critical-thinking exercises and review questions enhance the reader's comprehension of the concepts.

Salient Features

- Overview of Signals and Systems concepts.
- Comprehensive coverage of key topics, including Infinite Impulse Response Filters (IIR), Finite Impulse Response Filters (FIR), effects of finite word length in digital filters, and multirate digital signal processing.
- Review of the essential mathematical concepts such as Fourier Analysis, Laplace Transforms and Z Transforms.
- A chapter devoted to Applications of Digital Signal Processing in Speech Processing, Image Processing and RADAR.
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ISBN 0-07-463996-X

9780074639962