National Institute Of Technology Andhra Pradesh
School of Sciences (Mathematics)
Minor-II Examination, Nov. 2021
2^{nd} Year B.Tech.(ECE) (Odd Semester)
MA205-Complex Variables and Special Functions

Date: 05.11.2021 **Max. Marks: 10**

Please answer all the questions and submit a neatly written assignment by 12/11/2021.

1. Describe the following regions as open/closed, bounded/unbounded, connected. Also, are they domains?
   (a) $Re(z) \geq |z-i|^2$.
   (b) $2 \leq |z-1-i| \leq 3$.
2. Find the domain of definition of the function $f(z) = \dfrac{z-1}{2z+1}$. Also, write $f(z)$ in the form $u(x,y) + iv(x,y)$.
3. Find the values of z satisfying $e^z = -i$.
4. Evaluate $\displaystyle\lim_{z \longrightarrow 0} \left[\frac{1}{1 - e^{\frac{1}{x}}} + iy^2 \right]$.
5. Discuss the continuity of $f(z) = \begin{cases} Re(z^2), & z \neq 0 \\ 0, & z = 0 \end{cases}$ at $z = 0$.
6. Let $P(z)$ and $Q(z)$ be polynomials of degrees m and n respectively. Evaluate $\displaystyle\lim_{z \longrightarrow 0} \frac{P(z)}{Q(z)}$ and $\displaystyle\lim_{z \longrightarrow \infty} \frac{P(z)}{Q(z)}$.
7. Show that $f(z) = |z|^2$ is not analytic at any point and $g(z) = \frac{1}{z}$ is analytic everywhere except at one point in the complex plane.
8. State the CR equations in polar form. Show that if the $arg(f)$ is a constant, then f is a constant.
9. Show that $r^2 cos 2\theta$ is a harmonic function and find its harmonic conjugate. Write down the corresponding analytic function.
10. Evaluate $\displaystyle\int_0^1 \phi(t)dt$, where $\phi(t) = t + \frac{i}{\sqrt{t}}$.
11. Evaluate $\displaystyle\int_C z^n dz$, $n = 0, \pm 1, \pm 2, ...$, where C is the circle with centre 0 and radius r traveresed counter clockwise.

12. Evaluate $\int_C \frac{z}{\bar{z}} dz$, where C is the boundary of the half annulus (annulus in the upper half plane) $2 \leq |z| \leq 3$.

13. Obtain an upper bound for the absolute value of $\int_C \frac{z}{z+1} dz$, where C is the upper half of the circle $|z| = 2$.

14. State the Cauchy Integral theorem and the Cauchy Goursat theorem.

15. Evaluate $\int_C [Re(z) + z] dz$, where $C : |z| = 2$ using the Cauchy Integral theorem.

16. Use the extension of Cauchy Integral theorem to multiply connected domains and evaluate $\int_C \frac{3z-1}{z^3 - z} dz$, where C is a square with centre at 0 and side length 3.

17. Evaluate $\int_0^1 z^2 e^{z^3} dz$.

18. State the Cauchy Integral formula and use it to evaluate $\int_C \frac{z^2+1}{z(2z-1)} dz$, where C is the unit circle.

19. State the Cauchy Integral formula for derivatives, Morera's theorem, Liouville's theorem and Maximum Modulus theorem.

20. Obtain the Taylor series expansion of $f(z) = \frac{1}{z^2 + (1+2i)z + 2i}$ about $z = 0$. Find the radius of convergence of this series.

21. Find all possible series expansions of $f(z) = \frac{1}{(z+1)(z+2)^2}$ about $z = 1$.

22. Classify the singularities of $f(z) = tan(\frac{1}{z})$.

23. Classify the singularities of $f(z) = \frac{z^2 + iz + 2}{(z^2+1)^2(z+3)}$ and calculate the residues at those points.

24. Compute the residues at the singularities of $f(z) = sec\ z$.

25. Using the Residue theorem, evaluate

    (a) $\int_C \frac{dz}{z^4+1}$, where $C : |z-1| = 1$.

    (b) $\int_0^{2\pi} \frac{d\theta}{1 - 2a\ cos\ \theta + a^2}$, where $a \in \mathbb{C}$ is such that $|a| \neq 1$.

    (c) $\int_0^{\infty} \frac{x^2+2}{(x^2+1)(x^2+4)} dx$.

(d) $\displaystyle\int_{-\infty}^{\infty} \frac{cos\ ax}{x^2+b^2}dx$ and $\displaystyle\int_{-\infty}^{\infty} \frac{sin\ ax}{x^2+b^2}dx,\ a, b > 0.$

(e) $\displaystyle\int_{-\infty}^{\infty} \frac{3x+5}{x(x+2)(x^2+1)}dx$

26. Find the image of $Im(z) < 0$ and $|z| > 1$ under the mapping $w = \dfrac{i}{z-i}$.

27. Find all bilinear transformations whose fixed points are 1 and -1. Also, find the bilinear transformation that maps $z = 1, i, 2+i$ onto the points $w = i, 1, \infty$.

28. Determine the points where $coz\ z$ and $cosh\ z$ are not conformal.

29. Classify the singular points of the following differential equations:

    (a) $x^2y'' + (x + x^2)y' - y = 0.$

    (b) $x^2y'' + (sin\ x)y' + (cos\ x)y = 0.$

30. Find the power series solution about $x = 0$ of the differential equation $(1-x^2)y'' - 4xy' + 2y = 0$.

31. Find the power series solution about $x = 2$ of the IVP $4y'' - 4y' + y = 0$, $y(2) = 0$, $y'(2) = \frac{1}{e}$.

32. Find a fourth degree polynomial approximation (a power series about $x = 0$) to the IVP $y'' - y = 0$, $y(0) = 2$, $y'(0) = 0$.

33. Find two linearly independent solutions of $2x^2y'' + xy' - (x^2 - 1)y = 0$ using the Frobenius method.

34. Find the series solution of $xy'' + y' - xy = 0$ using the Frobenius method.

35. Find a series solution of $x^2y'' + x^3y' + (x^2 - 2)y = 0$ using the Frobenius method.

36. State the Legendre differential equation, Rodrigue's formula and generating function of Legendre polynomials. Also, show that $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$.

37. State the orthogonality property of Legendre polynomials and use it to expand $f(x) = x^3 + x$, $-1 \le x \le 1$, as a Fourier-Legendre series.

38. State the Bessel's differential equation and evaluate $\displaystyle\int x^3 J_0(x)dx$. Also, show that $\displaystyle\int xJ_0^2(x)dx = \frac{x^2}{2}[J_0^2(x) + J_1^2(x)]$.

39. Show that $2\displaystyle\sum_{n=0}^{\infty} J_n^2 = \frac{1+J_0^2}{2}$.

40. Show that $J_0(x) = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} cos(x\ cos\ \theta)d\theta$.