MA 205 - Complex Integration

Note Title 26-08-2021

- Using integration, we shall show that an <u>analytic fn.</u> has derivatives of all orders. (not true in real case)
- Evaluate real integrals.

$f(x) = |x|$

f' does not exist at 0

$f(x) = |x|^2$

$f'(x) = 2|x|$

$f''(x)$ at $\cancel{0}$.

$f(x) = |x|^n$

$$\int_a^b f(x)\,dx \qquad\qquad \int_a^b f(z)\,dz$$

real variable

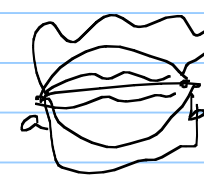
$x = a$ to $x = b$

single path

Complex variable

$z = a$ to $z = b$

infinite paths



$$\phi : \mathbb{R} \to \mathbb{C}$$

Definite Integrals:

$\phi(t) = \underbrace{\phi_1(t)}_{\text{real}} + i\, \underbrace{\phi_2(t)}_{\text{real}}$ — complex fn. of a real variable t.

ϕ is integrable if ϕ_1 and ϕ_2 are integrable.

$$\int_a^b \phi(t) = \int_a^b \phi_1(t)\,dt + i \int_a^b \phi_2(t)\,dt$$

a or b is infinite or if $\phi_1(t)$ or $\phi_2(t)$ has infinite discontinuity
– improper integral.

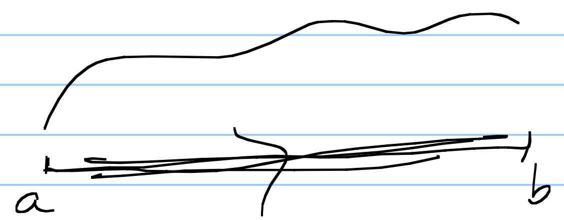
$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{s \to \infty} \int_{-s}^{s} f(x)\,dx \quad s > 0.$$

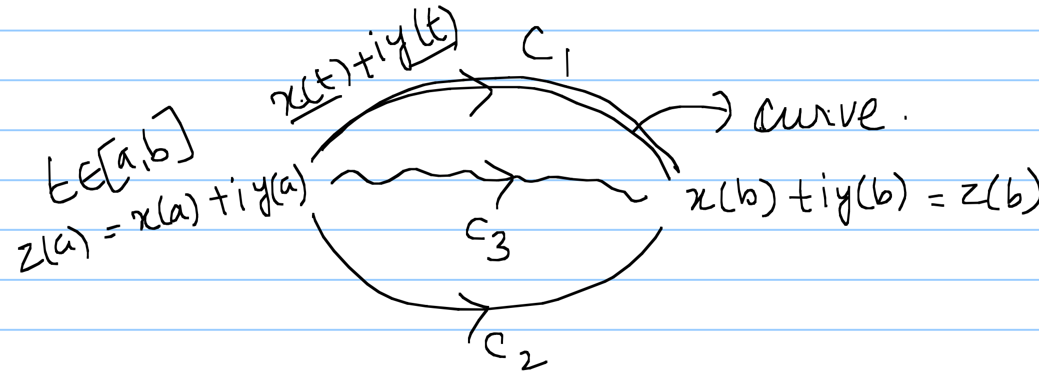
$t \in \mathbb{R}$

$$\int_0^1 (t+it^2)\,dt = \int_0^1 t\,dt + i\int_0^1 t^2\,dt = \left.\frac{t^2}{2}\right]_0^1 + i\left[\frac{t^3}{3}\right]_0^1 =$$

$$\int_0^1 \left(te^{-t^2} + \frac{2i}{\sqrt{t}}\right) dt = \int_0^1 te^{-t^2}\,dt + 2i\int_0^1 \frac{1}{\sqrt{t}}\,dt$$

Improper.





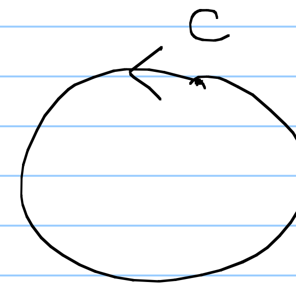
$a \le t \le b$ → ~~complex~~ or real ✓?

Curves:

Let $x(t)$ & $y(t)$ be two cont. fns. of t, $a \le t \le b$.

$z = z(t) = x(t) + i\,y(t)$, $a \le t \le b$ trace a curve C in the complex plane starting at $z(a)$ and ending at $z(b)$.

Closed curve: $z(a) = z(b)$

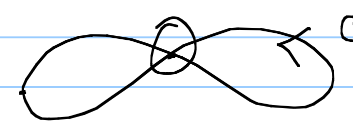


closed
✓simple

Simple curve: does not intersect itself

$z(t_1) \neq z(t_2)$ if $t_1 \neq t_2$

$t_1 \neq t_2 \implies z(t_1) \neq z(t_2)$



closed
not simple

$f = u + iv$

$z(t) = x(t) + iy(t)$

Let $z(t)$ be a simple curve.

(i) $\lim\limits_{t \to t^*} z(t) = \lim\limits_{t \to t^*} x(t) + i \lim\limits_{t \to t^*} y(t)$

(ii) $z(t)$ cont. on $[a,b]$ iff $x(t)$ & $y(t)$ are cont. on $[a,b]$

(iii) $z(t)$ piecewise cont. (cont. except for atmost finite number of jump discontinuities)

(iv) $z(t)$ diff iff $x(t)$ & $y(t)$ diff. $z'(t) = x'(t) + iy'(t)$

cont. diff if $z'(t)$ is cont.

Continuously differentiable.

$|x|$

$\begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$

(v) Curve defined by $z(t)$ is smooth if $z(t)$ is cont. diff and $z'(t) \neq 0 \;\; \forall\, t \in [a, b]$.

(vi) curve defined by $z(t)$ is a contour if it is smooth or piecewise smooth.

(vii) $C: \; z(t), \; a \le t \le b$ $\qquad -C: \; z(-t), \; -b \le t \le -a$

opp. direction

Counter clockwise - positive.

$z(t) = t + i0$ $\qquad C: \; a \xrightarrow{\qquad} b$ $\qquad -C: \; a \xleftarrow{\qquad} b$

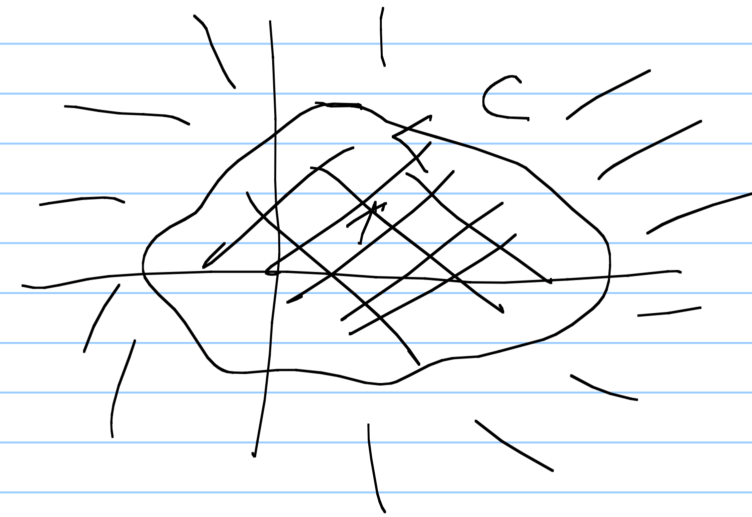
$f(z) = u(x,y) + iv(x,y)$ is defined on D. Suppose $z(t)$, $a \le t \le b$ is contained in D, $z(t)$ cont.

If f is cont. (analytic), then $f(z(t)) = u(x(t), y(t)) + i\,v(x(t), y(t))$ is cont. (analytic)

$f(z(t))$

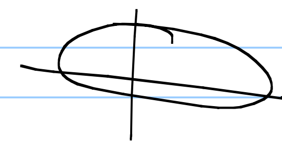
Jordan Curve Lemma:

Let C be a simple closed contour. Then C separates the complex plane into two distinct regions, the inside of C and the outside of C, one of which is bounded and the other is unbounded.



Parametric Representation of $9x^2+y^2=9$. (i) clockwise (ii) anticlockwise

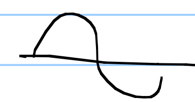
$f(x,y)$ = ellipse $\quad \frac{x^2}{1^2}+\frac{y^2}{3^2}=1$

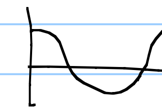


$z = z(t) \quad t\in[a,b]$

$x(t) = \cos t$

$y(t) = 3\sin t \quad t\in[0,2\pi]$





$C: z(t) = \boxed{\cos t + 3i\sin t}\,, \quad t\in[0,2\pi]$

$z(0) = 1 = z(2\pi)$

clockwise : $\quad z(-t) \quad -2\pi \le t \le 0$

$-C \quad \cos t - 3i\sin t$

$t_1 \ne t_2 \Rightarrow z(t_1) \ne z(t_2)$

$z'(t) = -\sin t + 3i\cos t \quad t\in(0,2\pi)$

$\ne 0$

$$z(t) = \begin{cases} t & -1 \le t \le 1 \\ e^{i(t-1)} & 1 \le t \le \pi+1 \end{cases}$$ is simple closed ~~smooth~~ or p.w. smooth.

$\underset{-1}{z(-1)} = \underset{-1}{z(\pi+1)}$ ✓ closed

cont. diff $\quad \lim_{t \to 1^+} z(t) = \lim_{t \to 1^-} z(t) = 1$

$$z'(t) = \begin{cases} 1 ✓ & -1 < t < 1 \\ ie^{i(t-1)} ✓ & 1 < t < \pi+1 \end{cases}$$

z - not diff at $1, -1, \pi+1$

z is diff at $(-1, 1) \cup (1, \pi+1)$

$\underset{i}{\lim_{t \to 1^+} z'(t)} \neq \underset{1}{\lim_{t \to 1^-} z'(t)}$

$\lim_{t \to -1^+} z'(t) = 1 \qquad \lim_{t \to (\pi+1)^-} z'(t) = i$

$z'(t) \neq 0$ ✓

HW : $z(t) = (1-\cos t)\, e^{it}$, $0 \le t \le 2\pi$ – closed simple smooth. p.w. smooth ?

Contour Integrals : (Line integrals in the complex plane)

Let $z(t) = x(t) + i\,y(t)$, $a \le t \le b$ represent a simple smooth curve.
Suppose $f(z)$ is a cont. fn. defined on a domain containing C.

Let $a = t_0 < t_1 < \dots < t_n = b$ be a partition of $[a, b]$.

Let $z_k = z(t_k)$. $\Delta z_k = z_k - z_{k-1}$ $|\Delta z_k| = |z_k - z_{k-1}|$

Let ξ_k be any point in (t_{k-1}, t_k). $S_n = \sum_{k=1}^{n} f(\xi_k)\, \Delta z_k$

As $n \to \infty$, $|\Delta z_k| \to 0$. $\lim_{\max|\Delta z_k| \to 0} S_n = \int_C f(z)\, dz$

$\int_C f(z)\,dz$ is called the contour integral or line integral of $f(z)$ along C

If C is closed $\int_C f(z)\,dz \longrightarrow \oint_C f(z)\,dz$. $\quad C$ - path of integration

$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$$

$$\int_C [\alpha f(z) + \beta g(z)]\,dz = \alpha \int_C f(z)\,dz + \beta \int_C f(z)\,dz$$

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$