"Complex Variables and Special Functions" 10/08/2021

Origin of Complex Numbers: $x^2+1=0$ - roots ?

$x = \pm i$,

$\boxed{i = \sqrt{-1}}$

$\boxed{x^2+x+1}$ roots $\dfrac{-1 \pm \sqrt{-3}}{2}$ $(\omega, \omega^2) = \dfrac{-1 \pm \sqrt{3}i}{2}$

$\mathbb{C} = \mathbb{R} + i\mathbb{R}$; $\mathbb{R}$ - Set of Real numbers

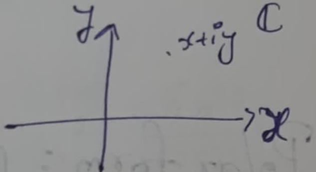
$z = x + iy$; $x, y \in \mathbb{R}$ $\quad x = Re(z)$

$y = Im(z)$

*(Algebraically closed - means set of polynomials having roots from $\mathbb{C}$)

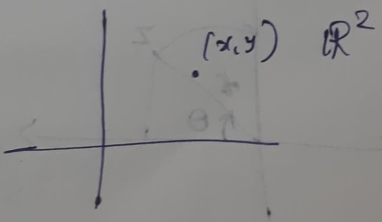
$\boxed{f : \mathbb{C} \to \mathbb{C}}$ $\quad \mathbb{C} = \mathbb{R} + i\mathbb{R}$

$= x + iy$



(Multivariables)

$f : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad f(x, y)$

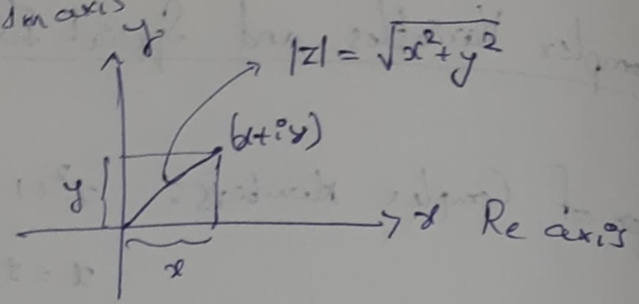


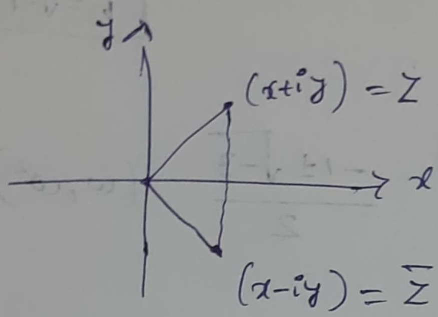
* All Real numbers are subset of $\mathbb{C}$; $\mathbb{R} = \mathbb{R} + i0$

$0 + i\mathbb{R}$ - Purely imaginary numbers.

$z = x + iy$

$\bar{z} = x - iy$





reflection of z along x-axis $= \bar{z}$

* $\bar{\bar{z}} = z$

$\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

$\overline{z_1 z_2} = \bar{z}_1 \, \bar{z}_2$

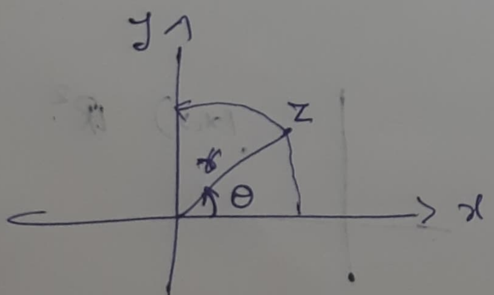
$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2} \quad (z_2 \neq 0) \Rightarrow (\bar{z}_2 \neq 0)$

* $z.\bar{z} = (x+iy).(x-iy) = (x^2+y^2) = |z|^2$

$z\bar{z} = |z|^2$; $|z|$ - Distance b/w point and origin

Polar form : (r, θ)

$\boxed{z = r\cos\theta + i r\sin\theta}$



$r = |z| = \sqrt{x^2+y^2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \arg(z)$

$z = r(\cos\theta + i\sin\theta) = r \operatorname{cis}\theta$

$\boxed{z = re^{i\theta}}$

* Principal value of Arg z :-

$\therefore -\pi < \theta \leq \pi \quad (0 < \theta \leq 2\pi)$

$\arg z = \theta_0 + 2n\pi \quad (-\pi < \theta_0 \leq \pi)$

$\operatorname{Arg} z = \theta_0$

$z_1 = r_1 e^{i\theta_1}$ & $z_2 = r_2 e^{i\theta_2}$

* $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$; $|z_1 z_2| = r_1 r_2 = |z_1|\,|z_2|$

$\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$

Proof: $e^{i\theta} = \cos\theta + i\sin\theta$ (We must prove e^z holds good as e^x) ~~other way~~

$\therefore e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots \infty$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$\therefore e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots \infty$

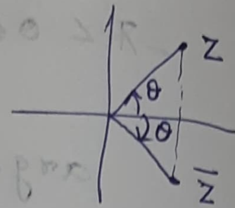
$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} \dots$

$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \dots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \dots\right) = \cos\theta + i\sin\theta$

11/08/2021

$Z = x + iy \quad | \quad Z = re^{i\theta},$

Conjugate : $\bar{Z} = x - iy \quad | \quad \bar{Z} = re^{-i\theta}$



$r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$Z + \bar{Z} = 2x = 2\,Re(z)$

$Z - \bar{Z} = 2iy = 2i\,Im(Z)$

Euler's formula : $e^{i\theta} = \cos\theta + i\sin\theta$

Proof: $f(\theta) = e^{-i\theta}[\cos\theta + i\sin\theta]$

$\Rightarrow f'(\theta) = e^{-i\theta}[-\sin\theta + i\cos\theta] - ie^{-i\theta}[\cos\theta + i\sin\theta]$

$= \cancel{e^{-i\theta}\{\cos\theta - \sin\theta + i(\cos\theta - \sin\theta)\}}$

$= 0$

$\boxed{\therefore f'(\theta) = 0} \Rightarrow f(\theta) = \text{constant}$

$\therefore f(0) = 1 \Rightarrow \boxed{f(\theta) = 1}$

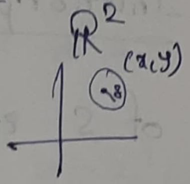
$\Rightarrow e^{-i\theta} \times (\cos\theta + i\sin\theta) = 1$

$\Rightarrow \boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

Hence, Proved

Topological Concepts in Complex Numbers $\mathbb{C}$

similar to $\mathbb{R}$ → neighbourhoods $\quad a-\delta \; (\; a \;) \; a+\delta \quad \mathbb{R}$



Circle: $(x-x_0)^2+(y-y_0)^2=a^2$

$\sqrt{(x-x_0)^2+(y-y_0)^2}=a$ $\qquad |z-z_0|=a$;

$z=x+iy$

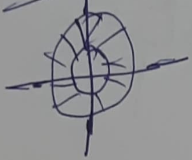
$z_0=x_0+iy_0$

Disk:

$|z-z_0|<a$ - open disk

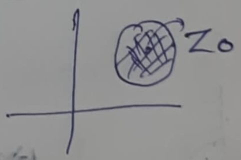
$|z-z_0|\le a$ - closed disk

Annular Disk



$a_1 \le |z-z_0| \le a_2$

(open) Neighbourhood: An δ-neighbourhood (in $\mathbb{C}$) of a point z_0 is the set of all points on the open disk $|z-z_0|<\delta$.

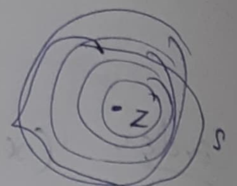


* $N_\delta(z_0)$

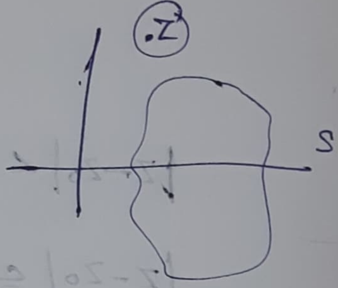
Deleted Neighbourhood: $0<|z-z_0|<\delta$

Interior point: $S \subseteq \mathbb{C}$, z is an interior point of S if $\exists\ \delta > 0$ such that $N_\delta(z) \subseteq S$

(atleast one neighbourhood inside S)



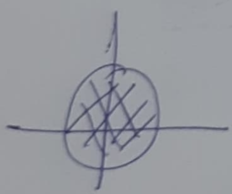
Exterior point: $\exists\ \delta > 0$ s.t $N_\delta(z) \cap S = \phi$



Boundary point: $\forall\ \delta > 0$, $N_\delta(z) \cap S \neq \phi$

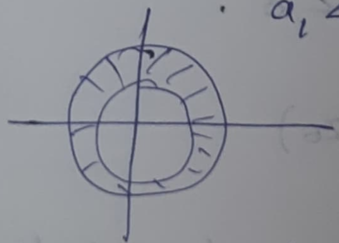
and $N_\delta(z) \cap S^c \neq \phi$

Open Set: A set S is said to be open if every $z \in S$ is an interior point of S.



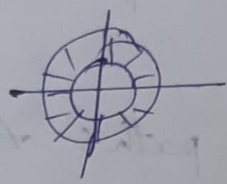
$|z - z_0| < \delta$

$a_1 < |z - z_0| < a_2$



Closed Set: A set S is said to be closed if it contains all the boundary points of S

$|z - z_0| \leq \delta$

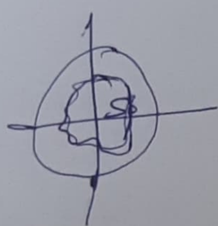


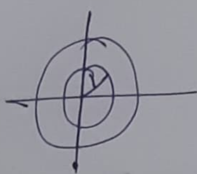
$a_1 \leq |z - z_0| \leq a_2$

* Every set need not to be either open or closed.

$a_1 < |z - z_0| \leq a_2$

Bounded Set: A set S is said to be bounded if $\exists\ M > 0$ such that $|z| \leq M \ \forall z \in S$





$|z| \leq 1, \forall z \in S$

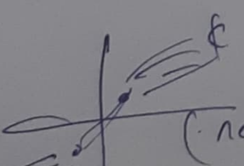
$\Rightarrow |z| < 2, \forall z \in S$

Unbounded Set:

Ex: $\{z : Re(z) = 0\} \Rightarrow \{z : x = 0\} = \{iy, y \in \mathbb{R}\}$

Connected Set: A set is said to be connected if for every pair of points in S, $\exists$ a polygonal path between them in S.

Ex:



(not connected)

Domain : (in $\mathbb{C}$) an open connected set.

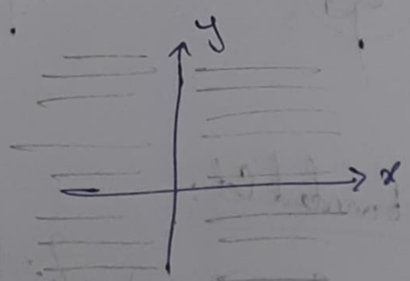
Eg: $\mathbb{C}$, open disk, 

(# Polygonal path: A path consisting of finite line-segments.)

Region: Domain with or having only some boundary points or without boundary points.

Every domain is a region but every region is not a domain.

Examples: ① $\{z : Re(z) \neq 0\}$

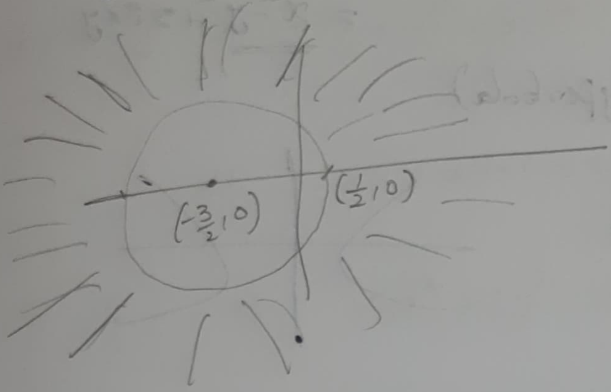


$\{z : x \neq 0\}$

$\rightarrow \mathbb{C} \setminus \{y \text{ axes}\}$

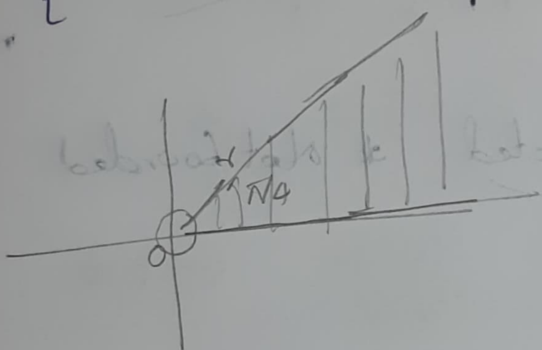
$\Rightarrow$ $\underbrace{\text{Open, not connected}}_{\text{Not a domain}}$, not bounded

② $\{ z : |2z+3| > 4 \}$ $\Rightarrow$ $|z + \frac{3}{2}| > 2$



* Open } Domain
* Connected }
* Not bounded

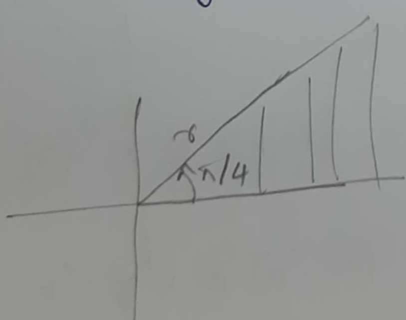
③ $\{ |z| > 0, \ 0 \leq \arg(z) \leq \pi/4 \}$



* Not opened
* not Closed
('0' is a boundary point but not in the set)
* Connected
* Not bounded.

* Not a domain.

④ $\{ 0 \leq \arg(z) \leq \pi/4 \}$



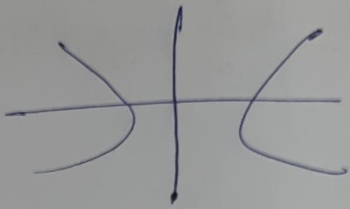
* Not opened
* closed
* Connected
* Not bounded
* Not a Domain

(5) $\{ Re(z^2) < 4 \}$: $(x+iy)^2 = x^2 + 2ixy - y^2$

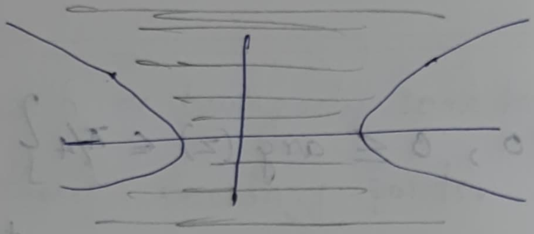
$= x^2 - y^2 + 2ixy$

$x^2 - y^2 < 4$ (hyperbola)

$\frac{x^2}{4} - \frac{y^2}{4} = 1$ $\Rightarrow$



$x^2 - y^2 < 4$ $\Rightarrow$



* Opened * Connected * Not bounded

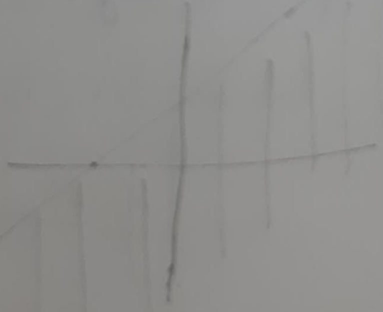
* Domain

H.W: (1) $Im(z-i) > Re(z+2+3i)$

$\Rightarrow Im(z-i) = y-1$ | $Re(z+2+3i) = x+2$

$\Rightarrow y-1 > x+2$

$\Rightarrow x-y < -3$



* Opened
* Connected } Domain
* Not bounded

(2) $|2z+1| < |z+4|$

Let $z=x+iy$ $\Rightarrow \sqrt{(2x+1)^2+(2y)^2} < \sqrt{(x+4)^2+y^2}$

$\Rightarrow 4x^2+4x+1+4y^2 < x^2+8x+16+y^2$

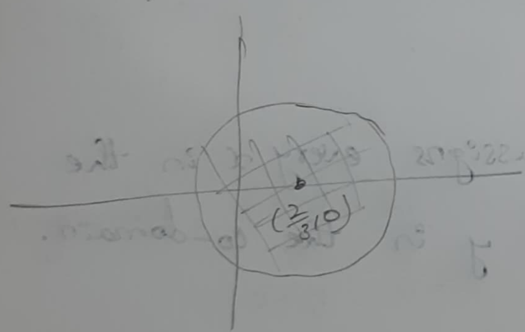
$\Rightarrow 3x^2+3y^2-4x-15 < 0$

$\Rightarrow 3\left(x^2-\frac{4}{3}x\right)+3y^2 < 15$

$\Rightarrow 3\left(x^2-\frac{4}{3}x+\frac{4}{9}\right)+3y^2 < 15+\frac{4}{3}$

$\Rightarrow 3\left(x-\frac{2}{3}\right)^2+3y^2 < \frac{49}{3}$

Circle: $\left(x-\frac{2}{3}\right)^2+y^2 < \frac{49}{9}$



* Opened
* Connected
* Bounded

} Domain

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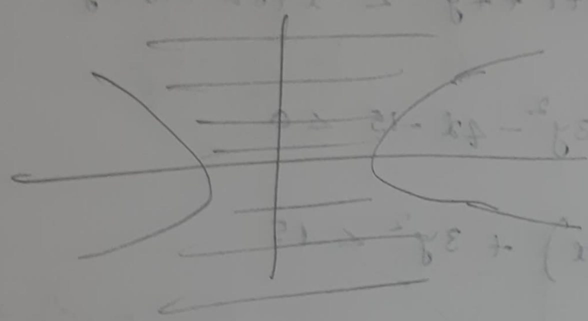
(3) $|z|^2+2\,\mathrm{Re}|z^2| < 3$

$\Rightarrow x^2+y^2+2(x^2-y^2) < 3$

$\Rightarrow 3x^2-y^2 < 3$ $\Rightarrow$ Hyperbola: $\frac{x^2}{1}-\frac{y^2}{3} < 1$

(3) $|z|^2 + 2\,\mathrm{Re}\,|z^2| < 3$

Hyperbola: $\frac{x^2}{1} - \frac{y^2}{3} < 1$



* Opened
* Connected
* Not bounded
* Domain

13/08/2021

Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$

$\theta = 0$: $1 = e^{i0}$

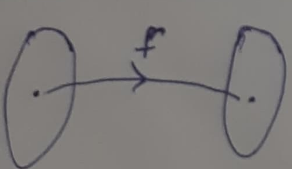
$\theta = \pi/2$: $i = e^{i\pi/2}$

$\theta = \pi$: $-1 = e^{i\pi}$ $\boxed{e^{i\pi} + 1 = 0}$

$\theta = 2\pi$: $\rightarrow e^{i0}$

Function: A rule that assigns every x in the domain to a unique y in the co-domain.

Domain Co-domain



$f(x) = y$

* Range $= \{ f(x) : x \in \text{domain} \}$ (Subset of Co-domain)

Complex function : A function where domain contains real no's and co-domain contains complex no.s.

Complex fn. of Complex Variables :

$f : \mathbb{C} \to \mathbb{C}$ i.e. $f : D \to D^*$

where $D, D^* \subseteq \mathbb{C}$

Range : $f(D) \subseteq D^*$

(Domain : where function can be defined
Range : Values of the function.)

Ex: $f(z) = z$ Domain : $\mathbb{C}$

$f(z) = \frac{1}{|z|}$ Domain : $\mathbb{C} \setminus \{0\}$
Range : $(0, \infty) \subseteq \mathbb{C}$

In general, $f(z) = W$ $u(x,y)$
$\downarrow$ $\downarrow$ $v(x,y)$
$x+iy$ $u+iv$

$f(z) = \omega \to u + iv$
$\downarrow$ $\downarrow$ $\downarrow$
$re^{i\theta}$ $u(r,\theta) + i v(r,\theta)$

Z - pre-image of W - cannot have multiple preimages

W - image of Z - can have multiple images.

Ex: Complex Polynomial : $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$

An expression of variable with some degree.

Rational functions : $f(z) = \frac{P(z)}{Q(z)}$ $(Q(z) \neq 0)$

$f(z) = \frac{1}{z^2+1}$ Domain $= \mathbb{C} / \{i, -i\}$

$(\because z^2+1=0$
$z = i, -i)$

Ex: $f(z) = 1$: Image of $\{i\} = 1$

Image of $\{3 - \frac{2}{3}i\} = 1$

Pre-image of $0 = \phi$

Pre-image of $1 = \mathbb{C}$

$f(z) = 2(z-i)$; Pre image of $\{3\} = \frac{3+2i}{2}$

$(\because 2z - 2i = 3 \Rightarrow z = \frac{3+2i}{2})$

H.W: (1) Domain of $f(z) = \frac{z^2+i}{|z|}$?

Sol: Domain : $\{|z| \neq 0\}$ — $\mathbb{C} | \{0\}$

$\{z \neq 0\}$

(2) Express as $u+iv$? Domain ?

$f(z) = \frac{1}{1-|z|^2}$; $f(z) = \frac{1}{Re(z)} - i[Im(z)]^2$

Sol: (i) $f(z) = \frac{1}{1-|z|^2}$ Domain : $\{|z| \neq 1\}$

$\{z \neq 1, -1, i, -i\}$

$\frac{\pm 1 \pm i}{\sqrt{2}}$

$$f(x+iy) = \frac{1}{1-|(x+iy)|^2} = \frac{1}{1-(\sqrt{x^2+y^2})^2} = \frac{1}{1-(x^2+y^2)}$$

$\therefore u = \frac{1}{1-x^2-y^2}$; $v = 0$

(ii) $f(z) = \frac{1}{Re(z)} - i[Im(z)]^2$ Domain: $\{Re(z) \neq 0\}$

$$f(x+iy) = \frac{1}{x} - iy^2$$

$\therefore u = \frac{1}{x}$; $v = -y^2$

Ex: $f(z) = \dfrac{z}{(z^4+16)^2}$ Domain ?

Sol

$z^4 + 16 = 0$

One way: $z^4 = -16$

$z^2 = \pm 4i$

$z = \sqrt{4i}\ ,\ \sqrt{-4i}$

4 solutions, $\because z^4$

the other way

$z^4 = (-1)(16)$

$z^4 = e^{i\pi}(16)$

$z = 2e^{i\pi/4}$

Ex: $f(z) = z^2$

Sol: Domain $= \mathbb{C}$; Co-domain .

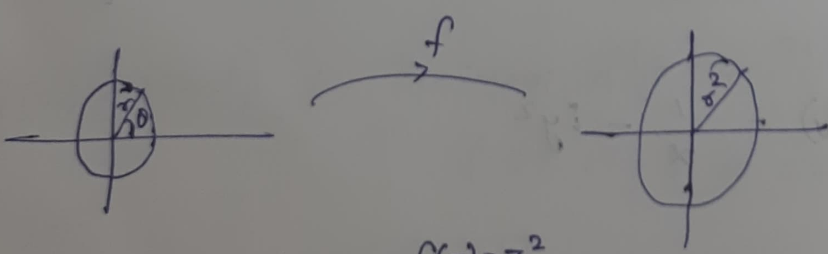
$f(x+iy) = (x^2-y^2) + i(2xy)$; $u = x^2 - y^2$

$v = 2xy$

$f(re^{i\theta}) = r^2 e^{i2\theta}$

$e^{i\theta} = \cos\theta + i\sin\theta$

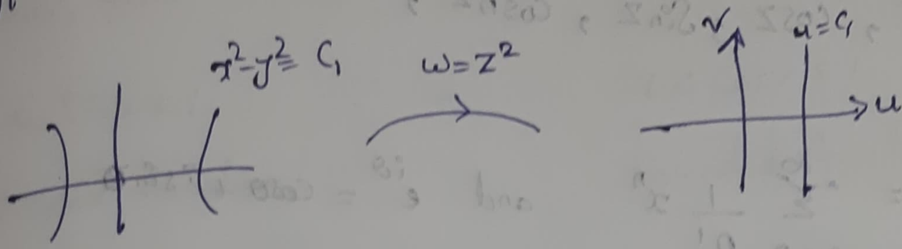
$|e^{i\theta}| = 1$



Circle of radius (r) $\xrightarrow{f(z)=z^2}$ Circle of radius r^2

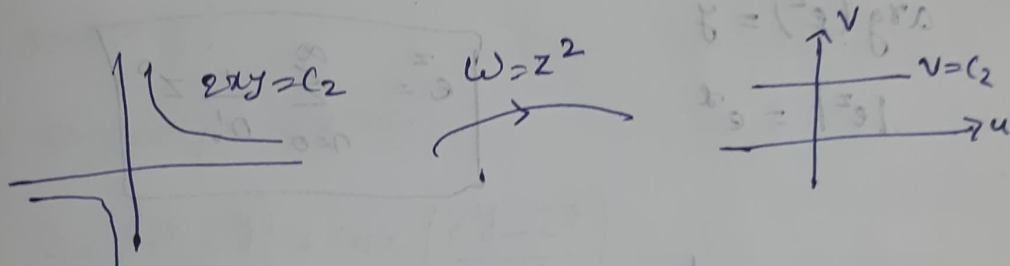
17/08/2021

Hyperbola : $x^2 - y^2 = c_1$, goes to $u = c_1$



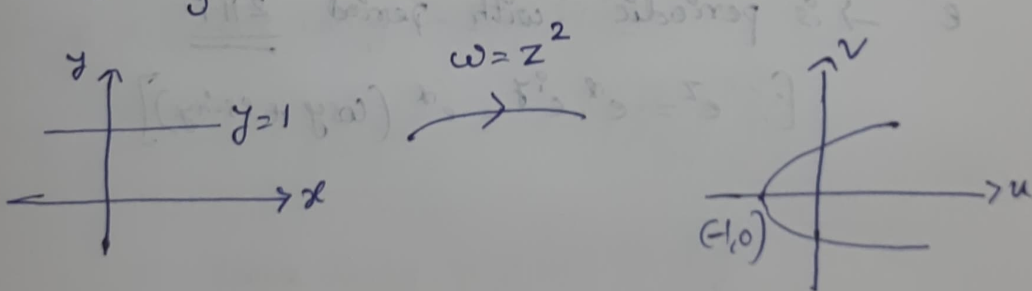
$u(x,y) = x^2 - y^2 = c_1$; $v(x,y) = 2xy$

Rectangular Hyperbola : $2xy = c_2$ goes to $v = c_2$



$\{z : \text{Im}(z) = 1\}$ goes to $(\omega = z^2)$

$\{z : y = 1\}$



$$\left.\begin{aligned} u &= x^2 - y^2 = x^2 - 1 \\ v &= 2xy = 2x \end{aligned}\right\} \quad \begin{aligned} u &= \left(\frac{v}{2}\right)^2 - 1 \\ u &= \frac{v^2}{4} - 1 \end{aligned}$$

$$\Rightarrow \boxed{v^2 = 4(u+1)}$$

Elementary Functions

e^z, $\cos z$, $\sin z$, $\cosh z$, $\sinh z$, $\ln z$.

$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ and $e^{i\theta} = \cos\theta + i\sin\theta$

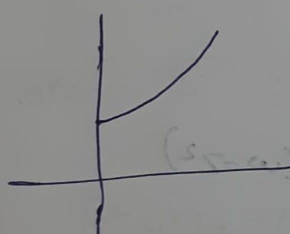
$e^z = e^{x+iy} = \underbrace{e^x \cdot e^{iy}}_{(r e^{i\theta})} = e^x[\cos y + i\sin y)$

$\arg(e^z) = y$

$|e^z| = e^x$

$$\boxed{e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n}$$

$\boxed{e^x}$



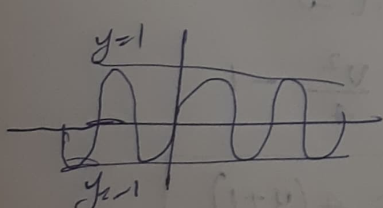
graph does not repeat

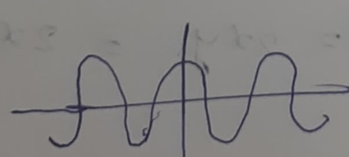
$\Rightarrow$ Not a Periodic function.

$e^z \rightarrow$ is periodic with period $\underline{\underline{2\pi i}}$

$[\because e^z = e^x \cdot e^{iy} = e^x(\cos y + i\sin y)]$

$\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$ # $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$





$\sin x$	$\cos x$
Periodic func.	Periodic fun.
$-1 \le \sin x \le 1$	$-1 \le \cos x \le 1$
Bounded function	Bounded function

$\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$ # $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

$$\sin z = \sin(x+iy)$$

$$= \sin x \cos iy + \cos x \sin iy$$

$$= \sin x \left(\frac{e^{y} + e^{-y}}{2}\right) + \cos x \left(\frac{e^{y} - e^{-y}}{2i}\right)$$

$$= \sin x \cosh y - i \cos x \sinh y$$

($\because$ $\sinh y$ & $\cosh y$ are not bounded)

$||^y$ $\cos z = \cos x \cos iy - \sin x \sin iy$

$$= \cos x \left(\frac{e^{y} + e^{-y}}{2}\right) - \sin x \left(\frac{e^{y} - e^{-y}}{2i}\right)$$

$$= \cos x \cosh y + i \sin x \sinh y$$

$\therefore$ $\sin z$ & $\cos z$ are not bounded

$\sin z = 0 \Rightarrow z = n\pi$

$\sin z = \sin x \cosh y - i \cos x \sinh y$

$z = n\pi$

$\left.\begin{array}{l} x = n\pi \\ y = 0 \end{array}\right\} \checkmark$

here, $\sin z = 0$

$z = n\pi i$

$\left.\begin{array}{l} x = 0 \\ y = n\pi \end{array}\right\}$ $\sinh(n\pi) = \dfrac{e^{n\pi} - e^{-n\pi}}{2} \neq 0$

Here, $\sin z \neq 0$

Ily $\cos z = 0 \Rightarrow z = (2n+1)\frac{\pi}{2}, \; n \in \mathbb{Z}$

---

* $\sinh z = \dfrac{e^{z} - e^{-z}}{2}$ # $\cosh z = \dfrac{e^{z} + e^{-z}}{2}$

$\sin ix = i \sinh x$

$\cos ix = \cosh x$

Proof:

$\sin ix = \dfrac{e^{i(ix)} - e^{-i(ix)}}{2i}$

$= \dfrac{e^{-x} - e^{x}}{2i} = \frac{i}{2}(e^{x} - e^{-x})$

$= i \sinh x$

$\cos ix = \dfrac{e^{i(ix)} + e^{-i(ix)}}{2}$

$= \dfrac{e^{-x} + e^{x}}{2}$

$= \cosh x$

$\therefore \boxed{\begin{array}{l} \sin ix = i \sinh x \\ \cos ix = \cosh x \end{array}}$

lly $\sin iz = i \sinh z$

$\cos iz = \cosh z$

$\sinh z = 0 \Rightarrow -i \sin iz = 0 \Rightarrow \sin iz = 0 \Rightarrow iz = n\pi$

$\Rightarrow \boxed{z = n\pi i}$

$n \in \mathbb{Z}$

$\cosh z = 0 \Rightarrow \cos iz = 0 \Rightarrow iz = (2n+1)\frac{\pi}{2}$

$\Rightarrow \boxed{z = (2n+1)\frac{i\pi}{2}} \quad n \in \mathbb{Z}$

$\ln z$

$\ln x = y \Rightarrow e^y = x$

$\ln z = w \Rightarrow e^w = z$

$\ln z = \ln(re^{i\theta}) = \ln r + \ln e^{i\theta}$

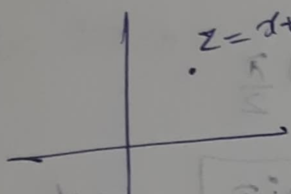
$= \ln r + i\theta$

$\therefore \ln z = \frac{1}{2}\ln(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$

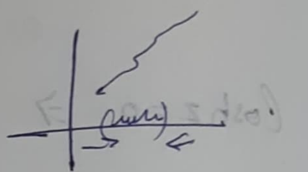
Limits:

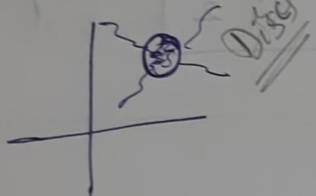
$$\lim_{x\to x_0} f(x)=L \qquad \lim_{x\to x_0^+} = \lim_{x\to x_0^-}$$

$|f(x)-L|<\varepsilon$, whenever $0<|x-x_0|<\delta$



$0<|z-z_0|<\delta$





* $|f(z)-L|<\varepsilon$, whenever $0<|z-z_0|<\delta$

→ Let $f(z)$ be a single valued function defined on a domain D. Let z_0 be such that $N_\delta(z_0)\subseteq D$. Then function $f(z)$ is said to have a limit $L\in\mathbb{C}$ at z_0 if for $\varepsilon>0$, $\exists\ \delta>0$ such that

$|f(z)-L|<\varepsilon$ whenever $0<|z-z_0|<\delta$,

* In this case, we write $\lim_{z\to z_0} f(z)=L$;

* If no such L exists then $f(z)$ has no limit as $z\to z_0$

Note: $f(z_0)$ may or may not be defined.

(1) If $\lim_{z \to z_0} f(z)$ exists, it is unique

(2) Let $f(z) = u(x,y) + iv(x,y)$. Then,

$\lim_{z \to z_0} f(z) = L = u_0 + iv_0$ iff

$\lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0$

(3) If $f(z)$ has a finite limit at z_0, then $f(z)$ is bounded in some neighbourhood of z_0.

$\lim_{z \to z_0} f(z) = L$

(4) If $|f(z)| \leq M$ for $0 < |z - z_0| < \delta$ and $\lim_{z \to z_0} g(z) = 0$, then $\lim_{z \to z_0} f(z)g(z) = 0$

(5) If $\lim_{z \to z_0} f(z) = L_1$ and $\lim_{z \to z_0} g(z) = L_2$, then

$f \pm g = L_1 \pm L_2$, $fg = L_1 L_2$, $f/g = \frac{L_1}{L_2}$, $(g \neq 0)$

$\alpha f = \alpha L_1$ ~~($\alpha \neq 0$)~~.
$(\alpha \in \mathbb{C})$

Limit at $z=\infty$

① $\lim\limits_{z\to\infty} f(z) = L$

※ $|f(x)-L| < \varepsilon$ whenever $|z| > \frac{1}{\delta}$ 	δ-is small

then $z \to \infty$ 	$\delta \to 0$

$\Rightarrow \frac{1}{\delta} \to \infty$

$z = \frac{1}{\xi}$; $z \to \infty$
$\xi \to 0$

② $\lim\limits_{\xi\to 0} f\left(\frac{1}{\xi}\right) = L$

Ex: $\lim\limits_{z\to i} z^2 = -1$ // | $\lim\limits_{z\to\infty} \frac{1}{z^2} = 0$ //

$\lim\limits_{z\to\infty} \frac{z}{2-iz} = \lim\limits_{z\to\infty} \frac{z}{z\left(\frac{2}{z}-i\right)} = \lim\limits_{z\to\infty} \frac{1}{\left(\frac{2}{z}\right)-i}$

$= \frac{1}{-i} = i$ //

$\lim\limits_{z\to 2i} [3x+iy^2] = \lim\limits_{(x,y)\to(0,2)} [3x+iy^2] = 0+i4 = 4i$

$u=3x$; $v=y^2$, $(x_0,y_0)=(0,2)$

$\lim\limits_{(x,y)\to(0,2)} u = 0$ & $\lim\limits_{(x,y)\to(0,2)} v = 4$

$$\boxed{\therefore \lim_{z\to 2i} [3x+iy^2] = 4i}$$ //

H.W ① $\lim_{z\to 1} \frac{z^2-1}{z-1}$

Sol: $\lim_{z\to 1} \frac{z^2-1}{z-1} = \lim_{z\to 1} \frac{(z-1)(z+1)}{z-1} = \lim_{z\to 1} (z+1)$

$= 2$ //

② $\lim_{z\to\infty} [\sqrt{z-2i} - \sqrt{z-i}]$

$= \lim_{z\to\infty} \frac{(\sqrt{z-2i} - \sqrt{z-i}) \times (\sqrt{z-2i} + \sqrt{z-i})}{(\sqrt{z-2i} + \sqrt{z-i})}$

$= \lim_{z\to\infty} \frac{z-2i-(z-i)}{\sqrt{z-2i} + \sqrt{z-i}}$

$= \lim_{z\to\infty} \frac{-i}{\sqrt{z-2i} + \sqrt{z-i}} = 0$ //

Let $P(z) = a_m z^m + a_{m-1} z^{m-1} + \dots + a_0$ $(a_m \neq 0)$,

$Q(z) = b_n z^n + b_{n-1} z^{n-1} + \dots + b_0$ $(b_n \neq 0)$,

then $\lim_{z\to 0} \frac{P(z)}{Q(z)} = ?$ and $\lim_{z\to\infty} \frac{P(z)}{Q(z)}$ (if $m=n$) ?

Sol: $\lim_{z\to 0} \frac{P(z)}{Q(z)} = \frac{a_0}{b_0}$ & $\lim_{z\to\infty} \frac{P(z)}{Q(z)} = \frac{a_m}{b_n}$

$\therefore \lim_{z\to\infty} \frac{a_m z^m + a_{m-1} z^{m-1} + \dots + a_0}{b_n z^n + \dots + b_0} = \lim_{z\to\infty} \frac{a_m + \frac{a_{m-1}}{z} + \dots + \frac{a_0}{z^m}}{b_n + \frac{b_{n-1}}{z} + \dots + \frac{b}{z^m}}$

$= \frac{a_m}{b_n}$ //

Show that following limits do not exist?

Need to prove w.r.t different curves limit values is not same

(i) Q. $\lim_{z \to 0} \frac{z}{|z|} = \lim_{(x,y)\to(0,0)} \frac{x+iy}{\sqrt{x^2+y^2}}$ let w.r.t $y = mx$ curve (m varies)

$= \lim_{x \to 0} \frac{x+imx}{\sqrt{x^2+m^2x^2}}$

$= \frac{1+im}{\sqrt{1+m^2}}$ (Limit value varies w.r.t m)

$\therefore$ Limit does not exist.

Q. $\lim_{z\to 0} \frac{[Re(z) - Im(z)]^2}{|z|^2}$

$= \lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2}$ Let $(y = mx)$

$= \lim_{x \to 0} \frac{(x-mx)^2}{x^2+m^2x^2}$

$= \lim_{x\to 0} \frac{(1-m)^2}{1+m^2} = \frac{(1-m)^2}{1+m^2}$ (Limit value varies w.r.t m)

$\therefore$ Limit does not exist.

Q. $\lim_{z \to 0} \left[\frac{1}{1-e^{Y x}} + i y^2 \right]$

Q. $\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{(x,y)\to(0,0)} \frac{x-iy}{x+iy}$ let $(y = mx)$

$= \lim_{x \to 0} \frac{x - imx}{x + imx}$

$= \lim_{x \to 0} \frac{1-im}{1+im}$

$= \frac{1-im}{1+im}$ (Limit value depends on m)

$\therefore$ Limit does not exist

Q. $\lim_{z \to 0} \frac{Im(z^2)}{|z|^2} = \lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$ (let $y = mx$)

$= \lim_{x \to 0} \frac{2x(mx)}{x^2+m^2x^2}$

$= \lim_{x \to 0} \frac{2m}{1+m^2}$

$= \frac{2m}{1+m^2}$ (Limit value depends on m)

$\therefore$ Limit does not exist.

Compute:

Q. $\lim_{z \to 1-i} [z^2 - \bar{z}^2] = \lim_{(x,y)\to(1,-1)} \left[(x^2-y^2+i2xy) - (x^2-y^2-i2xy)\right]$

$= \lim_{(x,y)\to(1,-1)} i4xy$

$= -4i$ //

$\therefore \lim_{z \to 1-i} [z^2 - \bar{z}^2] = -4i$

Q. $\lim_{z \to i} \frac{iz^3 - 1}{z - i} = \lim_{z \to i} \frac{iz^3 - i^4}{z - i}$

$[\because 1 = i^4]$ $\quad = \lim_{z \to i} \frac{i(z^3 - i^3)}{(z - i)}$

$= \lim_{z \to i} \frac{i(z-i)(z^2 + iz + i^2)}{(z-i)}$

$= \lim_{z \to i} i(z^2 + iz - 1)$

$= -3i$ //

$\therefore \lim_{z \to i} \frac{iz^3 - 1}{z - i} = -3i$

Q. $\lim_{z\to\infty} \sqrt{z}\left[\sqrt{z-2i} - \sqrt{z-i}\right]$

$$= \lim_{z\to\infty} \frac{\sqrt{z}\left(\sqrt{z-2i} - \sqrt{z-i}\right)\left(\sqrt{z-2i}+\sqrt{z-i}\right)}{\left(\sqrt{z-2i}+\sqrt{z-i}\right)}$$

$$= \lim_{z\to\infty} \frac{\sqrt{z}\left(z-2i-(z-i)\right)}{\left(\sqrt{z-2i}+\sqrt{z-i}\right)}$$

$$= \lim_{z\to\infty} \frac{-i}{\sqrt{1-\frac{2i}{z}}+\sqrt{1-\frac{i}{z}}}$$

$= \frac{-i}{2}$ // $\therefore \lim_{z\to 0} \sqrt{z}\left[\sqrt{z-2i} - \sqrt{z-i}\right] = -i/2$

Q. $\lim_{z\to 0}\left[\frac{1}{1-e^{1/x}} + iy^2\right]$ Show that Limit does not exist.

$$= \lim_{(x,y)\to(0,0)}\left[\frac{1}{1-e^{1/x}} + iy^2\right]$$

let $\left(y = \frac{m}{x}\right)$ $$= \lim_{x\to 0}\left[\frac{1}{1-e^{1/x}} + \frac{im^2}{x^2}\right]$$

there when $x\to 0$, Limit is not defined

$\boxed{\therefore \text{ Limit does not exist}}$

Q. $\lim_{z \to 1+2i} \left[\frac{z^2 - 2z + 5}{z^2 + 3 - 4i} \right]$

Sol: $z^2 + 3 - 4i \Rightarrow (1+2i)^2 + 3 - 4i$
$(z \to (1+2i))$
$= 1 - 4 + 4i + 3 - 4i$
$= 0$

$\therefore z^2 - 2z + 5 \Rightarrow z = \frac{2 \pm \sqrt{4 - 20}}{2}$

$z = 1 \pm 2i$

$\boxed{\therefore z^2 - 2z + 5 = (z - 1 - 2i)(z - 1 + 2i)}$

$\therefore z^2 + 3 - 4i \Rightarrow z = \frac{0 \pm \sqrt{0 - 4(3-4i)}}{2}$

$z = \pm i\sqrt{3-4i}$

$\sqrt{3-4i} = x + iy \Rightarrow \left.\begin{array}{l} x^2 - y^2 = 3 \\ xy = -2 \end{array}\right\}$ $\because x, y$ are real
$(x, y) = (2, -1)$
(or) $(-2, 1)$

$\therefore z = \pm(1 + 2i)$

$\boxed{\therefore z^2 + 3 - 4i = (z - 1 - 2i)(z + 1 + 2i)}$

$$\therefore \lim_{z \to 1+2i} \frac{(z-1-2i)(z-1+2i)}{(z-1-2i)(z+1+2i)}$$

$$= \lim_{z \to 1+2i} \frac{(z-1+2i)}{(z+1+2i)}$$

$$= \frac{4i}{2(1+2i)}$$

$$= \frac{2i}{1+2i} \quad //$$

$$\therefore \lim_{z \to 1+2i} \left[\frac{z^2-2z+5}{z^2+3-4i} \right] = \frac{2i}{1+2i}$$

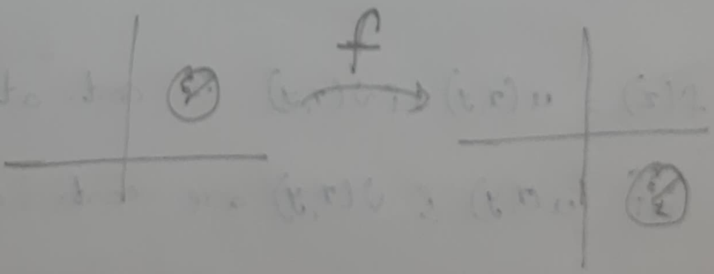
<u>Limit of a function:</u>

→ 'f' is said to have a limit 'L' at 'z_0' if

$\forall \, \varepsilon > 0, \exists \, \delta > 0$ such that $|f(z) - L| < \varepsilon$

whenever $0 < |z - z_0| < \delta$.

* $f(z_0)$ may or may not exist.



Continuity $\boxed{\lim_{z \to z_0} f(z) = f(z_0)}$

24/08/2021

→ f is said to be continuous at z_0 if $L = f(z_0)$ i.e. $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$

→ For f to be continuous at z_0,

(i) $f(z_0)$ should exist — that's why zero not included @ $|z-z_0| < \delta$

(ii) $\lim_{z \to z_0} f(z)$ should exist

(iii) $\lim_{z \to z_0} f(z) = f(z_0)$

* If $f(z)$ is continuous at all $z \in D$, then f is continuous everywhere on D (Domain)

Removable Discontinuity:

→ $f(z_0)$ exists, $\lim_{z \to z_0} f(z)$ exists but $\lim_{z \to z_0} f(z) \neq f(z_0)$

Change $f(z_0)$ to $\lim_{z \to z_0} f(z)$

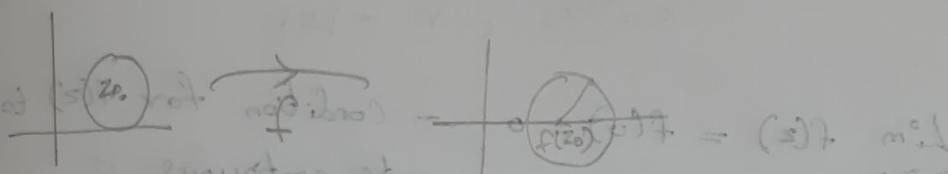
→ $f(z) = u(x,y) + i v(x,y)$ is cont. at $z_0 = x_0 + i y_0$; iff $u(x,y)$ & $v(x,y)$ are cont. at (x_0, y_0)

→ $f \pm g$, fg, f/g are cont. if f and g are continuous

→ $f(z)$ is continuous in a closed region S

$\Rightarrow f$ is bounded on S,

$$|f(z)| \le M \; ; \; \forall z \in S$$



(region → domain with or without boundary points)

→ $f(z)$ is continuous at $z = \infty$; if $f(1/\xi)$ is continuous at $\xi = 0$

→ $g(z)$ is continuous at $z = z_0$ and $f(z)$ is continuous at $g(z_0)$. Then ;

$$(f \circ g)(z) = f(g(z)) \text{ is continuous at } z_0.$$

→ $f(z)$ is continuous at z_0 $\Rightarrow$ $\overline{f(z)}$ is continuous at z_0. $\left(\because \left| \overline{f(z)} - \overline{f(z_0)} \right| = \left| \overline{f(z) - f(z_0)} \right| \right)$

Ex: e^z, $\sin z$, $\cos z$

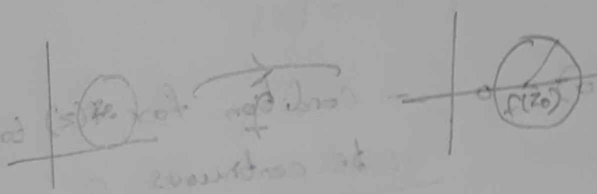
* Polynomial functions are continuous because they are ~~sum~~ of continuous functions, $f(z) = a_n z^n + \cdots + a_0$

$f(z) = z$, $f(z) = z^2$

→ $f(z)$ is continuous in a closed region S

$\Rightarrow f$ is bounded on S,

$|f(z)| \le M \; ; \forall z \in S$



(region → domain with or without boundary points)

→ $f(z)$ is continuous at $z = \infty$; if $f(1/\xi)$ is continuous at $\xi = 0$

$\delta > 0 \quad \infty$

$\frac{1}{\delta} \to \infty$

→ $g(z)$ is continuous at $z = z_0$ and $f(z)$ is continuous at $g(z_0)$. Then;

$(f \circ g)(z) = f(g(z))$ is continuous at z_0.

→ $f(z)$ is continuous at $z_0 \Rightarrow \overline{f(z)}$ is continuous at z_0. $\left(\because |f(z) - f(z_0)| = |\overline{f(z) - f(z_0)}| \right)$

Ex: e^z, $\sin z$, $\cos z$

* Polynomial functions are continuous because they are sum of continuous functions, $f(z) = a_n z^n + \dots + a_0$

$f(z) = z$, $f(z) = z^2$

3) Q. $f(z) = \begin{cases} \dfrac{Im(z)}{|z|} , & z \neq 0 \\ 0 & z = 0 \end{cases}$ Check Continuity ?

Sol: $\lim\limits_{z \to 0} f(z) = f(0) = 0$ - Condition for $f(z)$ to be continuous

$$\lim_{z \to 0} \frac{Im(z)}{|z|} = \lim_{z \to 0} \frac{y}{\sqrt{x^2+y^2}}$$

(Limit should be checked near to z_0 but not at z_0)

$$= \lim_{(x,y) \to (0,0)} \frac{y}{\sqrt{x^2+y^2}} \qquad (y = mx)$$

$$= \lim_{x \to 0} \frac{mx}{\sqrt{x^2+m^2x^2}}$$

$$= \lim_{x \to 0} \frac{m}{\sqrt{1+m^2}}$$

(So, Limit depends on 'm' value)

$\therefore$ Limit does not exist

$\Rightarrow$ function is not Continuous.

---

4) Q. $f(z) = \begin{cases} \dfrac{Re(z^3)}{|z|^2} ; & z \neq 0 \\ 0 ; & z = 0 \end{cases}$ Check Continuity ?

$\longrightarrow$

Sol's $z = re^{i\theta}$ $\quad z^3 = r^3 e^{i3\theta}$

$Re(z^3) = r^3 \cos 3\theta$

$|z| = r \; ; \; |z|^2 = r^2$

$$\Rightarrow \frac{Re(z^3)}{|z|^2} = \frac{r^3 \cos 3\theta}{r^2} = r\cos 3\theta$$

We need to check; $\lim_{r \to 0} r\cos 3\theta = f(0) = 0$

(W.K.T $\lim_{x \to 0} f(x)\, g(x) = 0$ if $\lim_{x \to 0} f(x) = 0$ and $g(x)$ is bounded near zero)

Here, $\cos 3\theta$ is bounded

$\Rightarrow \lim_{r \to 0} r\cos 3\theta = 0$

$\therefore$ Given function is continuous.

H.W: ① Show that $Re\, z, \bar{z}$ are continous on $\mathbb{C}$

Sol: $Re(z) = x \quad (\because z = x+iy)$

$\lim_{z \to 0} Re(z) = \lim_{(x,y) \to 0} (x+i0) \Rightarrow \lim_{x \to 0} x = 0$

$\boxed{\therefore Re(z) \text{ is continuous on } \mathbb{C}}$

$$\lim_{z\to 0} \bar{z} = \lim_{(x,y)\to(0,0)} (x-iy) \Rightarrow u(x,y) = x$$
$$v(x,y) = -y$$

$$\lim_{x\to 0} x = 0 \quad \& \quad \lim_{y\to 0} (-y) = 0$$

$\boxed{\therefore \bar{z} \text{ is continuous on } \mathbb{C}}$

---

(2) Where is the following function not continuous?

$$f(z) = \frac{z-1}{z^2-4z+5}$$

Sol: $z^2-4z+5 \Rightarrow z = \dfrac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$

$\boxed{\therefore \text{given function is not continuous at } 2\pm i \text{ points}}$

Q $f(z) = \begin{cases} \dfrac{Re(z^3)}{|z|} & ; z \neq 0 \\ 0 & ; z = 0 \end{cases}$ (Try in terms of (x,y))

Check Continuity

Sol $Re(z^3) \Rightarrow Re((x+iy)^3) \Rightarrow x^3 - xy^2 - 2xy^2$
$\Rightarrow x^3 - 3xy^2$

$|z| = \sqrt{x^2+y^2}$

$(y = mx)$

$$\lim_{z\to 0} \frac{Re(z^3)}{|z|} = \lim_{(x,y)\to(0,0)} \frac{x^3 - xy^2 - 2xy^2}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{x^2(1-3m^2)}{\sqrt{1+m^2}}$$
$$= 0$$

$\boxed{\therefore \text{Given function is Continuous}}$

Q. $f(z) = \begin{cases} \dfrac{z^2+1}{z+i}, & z \neq -i \\ 0, & z = -i \end{cases}$ check for Continuity

Sol: $z = x+iy$, $\dfrac{z^2+1}{z+i} = \dfrac{(x^2-y^2+1)+i2xy}{x+i(y+1)}$

$$= \frac{(x^3+xy^2+2xy+x)+i[x^2y+y^3-y-x^2+y^2-1]}{x^2+(y+1)^2}$$

$$\lim_{z\to -i} \frac{z^2+1}{z+i} = \lim_{(x,y)\to(0,-1)} u(x,y)+i\,v(x,y)$$

$$\Rightarrow \lim_{(x,y)\to(0,-1)} \frac{x^3+xy^2+2xy+x}{x^2+(y+1)^2} \qquad (y=mx-1)$$

$$= \lim_{x\to 0} \frac{x^3+x(m^2x^2+1-2mx)+2x(mx-1)+x}{x^2+m^2x^2}$$

$$= \lim_{x\to 0} \frac{x^3(1+m^2)}{x^2(1+m^2)} = 0$$

$$\Rightarrow \lim_{(x,y)\to(0,-1)} \frac{x^2y+y^3-y-x^2+y^2-1}{x^2+(y+1)^2}$$

Q. $f(z) = \begin{cases} \dfrac{x^3 y^5 (x+iy)}{x^4+y^4} & , z \neq 0 \\ 0 & ; z = 0 \end{cases}$ $\quad z = re^{i\theta}$

Q. $f(z) = \begin{cases} e^{-1/z^2} & ; z \neq 0 \\ 0 & ; z = 0 \end{cases}$

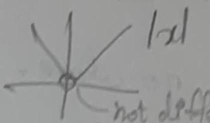
$\lim\limits_{z \to 0} e^{f(z)} = e^{\lim\limits_{z \to 0} f(z)}$

Q Find the value of $f(z_0)$ such that $\underline{4(i+1)}$

$f(z) = \dfrac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i}$ is continuous at $z = i$

Differentiability and Analyticity:

→ Continuous at '0'



→ not differentiable at '0'

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ exists.}$$

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} \; ; \; \frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

* Differentiability ⟹ Continuous but the converse is not always true.

* f, f_x, f_y : f_x, f_y exist and are continuous then f' exists.

* Let $f(z)$ be a single valued function defined on a domain D. The function $f(z)$ is said to be differentiable at $z_0 \in D$. if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists.

→ This limit is called the derivative of $f(z)$ at $z = z_0$ and is denoted by $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$

\# Differentiability implies continuity and not continuous ⇓ not differentiable

$[A \Rightarrow B] \Leftrightarrow [\text{not } B \Rightarrow \text{not } A]$

Ex: Let $f(z) = \bar{z}$ continuous on $\mathbb{C}$, but differentiable no where.

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta z \to 0} \left\{ \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} \right.$$

$(\Delta y = m\Delta x)$ (depends on m)

Ex: $f(z) = |z|^2 = z\bar{z}$

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z+\Delta z)(\bar{z}+\overline{\Delta z}) - z\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} z \cdot \frac{\overline{\Delta z}}{\Delta z} + \bar{z} + \overline{\Delta z}$$

$$= z \cdot \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} + \bar{z} + \overline{\Delta z}$$

$\therefore z = 0$, $f(z)$ is differentiable at $z = 0$.

$(f \pm g)' = f' \pm g'$; $(fg)' = f'g + g'f$; $\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$

$(f \circ g)'(z) = f'(g(z)) \, g'(z)$

Prove: z^n is differentiable on $\mathbb{C}$

$$\frac{d}{dz}(z^n) = n z^{n-1}$$

* Every polynomial function is differentiable.

Analytic Function: (Holomorphic function) 1②

* A function $f(z)$ is said to be analytic at z_0 if it is differentiable at z_0 and at each point in some neighbourhood of z_0.

* Analyticity ⇒ Differentiable in a neighbourhood.

* Diff. ⇏ Analytic

Ex: $f(z) = |z|^2$ differentiable at 0 but not analytic at 0.

* Analytic at every point in $\mathbb{C}$ - entire function.

* A function f is analytic at $z = \infty$ if $f(1/z)$ is analytic at $z=0$

Ex: $f(z) = \frac{z}{z+1}$ is analytic at $z = \infty$

$$f\left(\frac{1}{z}\right) = \frac{1/z}{1/z + 1} = \frac{1}{z+1} \checkmark \text{ at } z=0$$

Ex: $f(z) = z$ is not analytic at $z = \infty$

$$f\left(\frac{1}{z}\right) = \frac{1}{z} \times \text{ at } z=0$$

* If f & g are analytic, then $f \pm g$, fg, f/g, $f \circ g$ are also analytic.

Cauchy-Riemann Equations: (CR equations)

① **Necessary Conditions for a function to be analytic:**

→ Suppose $f(z) = u(x,y) + iv(x,y)$ is continuous at some neighbourhood of $z = x+iy$ and is differentiable at z.

→ Then u_x, u_y, v_x, v_y exist and satisfy

$$\boxed{u_x = v_y} \quad \text{and} \quad \boxed{u_y = -v_x} \quad \text{at } z.$$

② **Sufficient Conditions for a function to be analytic:**

→ Suppose $u(x,y)$ and $v(x,y)$ are continuous and have continuous first order partial derivatives on D.

→ If u & v satisfy CR equations at all points in D, then f is analytic in D and

$$\boxed{f'(z) = u_x + iv_x = v_y - iu_y}$$

//.

① $f = u + iv$ - continuous in nbhd & diff. at z_0

Then u_x, u_y, v_x, v_y exists and $u_x = v_y$ & $u_y = -v_x$ at z_0

② $u_x = v_y$ and $u_y = -v_x$ | u, v are continuous & have (continuous) 1st order partial derivatives.

26/08/2021

* Then 'f' is analytic &

$$f'(z) = u_x + i v_x = v_y - i u_y$$

Polar form:

$\boxed{v_\theta = r u_r}$ and $\boxed{u_\theta = -r v_r}$

$$f'(z) = \frac{1}{r} e^{-i\theta}(v_\theta - i u_\theta) = e^{-i\theta}(u_r + i v_r)$$

* CR eqns not satisfied ⇒ f not differentiable at z.
  ⇒ f not analytic at z.

* CR eqns may be satisfied but f may not be differentiable.

Ex: $f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

→ Satisfies CR eqns. at 0 but $f'(0)$ does not exist.

Sol: $u(x,y) = \dfrac{x^3 - y^3}{x^2+y^2}$ & $v(x,y) = \dfrac{x^3+y^3}{x^2+y^2}$

$$u_x(0,0) = \lim_{h \to 0} \frac{u(0+h,0) - u(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{(h^3 - 0)/(h^2+0^2) - 0}{h}$$

$$u_x(0,0) = \lim_{h\to 0} \frac{h}{h} = 1$$

$$v_y(0,0) = \lim_{k\to 0} \frac{v(0,0+k) - v(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{k^3/k^2 - 0}{k}$$

$$= \lim_{k\to 0} \frac{k}{k} = 1$$

$$\boxed{\therefore u_x = v_y \text{ at } (0,0)}$$

$$u_y(0,0) = \lim_{k\to 0} \frac{u(0,0+k) - u(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{-k^3/k^2 - 0}{k} = -1$$

$$v_x(0,0) = \lim_{h\to 0} \frac{v(0+h,0) - v(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h^3/h^2 - 0}{h} = 1$$

$$\boxed{\therefore u_y = -v_x \text{ at } (0,0)}$$

$$f'(0) = \lim_{z \to 0} \frac{f(0+z) - f(0)}{z}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\frac{x^3-y^3}{x^2+y^2} + i\,\frac{(x^3+y^3)}{x^2+y^2}}{x+iy}$$

(On rationalizing)

$$= \lim_{(x,y)\to(0,0)} \frac{x(x^3-y^3) + y(x^3+y^3)}{(x^2+y^2)^2}$$

$$+ \frac{i\left[x(x^3+y^3) - y(x^3-y^3)\right]}{(x^2+y^2)^2}$$

(Taking $y = mx$) — Limit does not exist

$\Rightarrow f(z)$ is not differentiable at $z = 0$

$u_x(x,y)$, $u_y(x,y)$, $v_x(x,y)$, $v_y(x,y)$ are not continuous

Q. Find a, b, c such that $f(z) = x - 2ay + i(bx - cy)$ is analytic.

Sol: (Polynomials are continuous & differentiable)

$$f(z) = x - 2ay + i(bx - cy)$$

$u(x,y) = x - 2ay$ | $u_x = 1$, $u_y = -2a$

$v(x,y) = bx - cy$ | $v_x = b$, $v_y = -c$

To satisfy CR equations:

$u_x = v_y$ & $u_y = -v_x$

$1 = -c$ & $\boxed{b = 2a}$

$\Rightarrow \boxed{c = -1}$

$$\therefore f(z) = x - 2ay + i(2ax + y)$$

$$= (x + iy) + i2a(x + iy)$$

$$= (1 + i2a) z$$

Q. If f' is continuous, then $\frac{\partial f}{\partial \bar{z}} = 0$ is equivalent to the CR equations!

Sol: $\frac{\partial f}{\partial \bar{z}} = 0 \Rightarrow$ 'f'' is independent of $\bar{z}$

CR eqns: $u_x = v_y$ and $u_y = -v_x$

$f = u + iv$, $z = x + iy$; $\bar{z} = x - iy \Rightarrow x = \frac{z + \bar{z}}{2}$

$y = \frac{z - \bar{z}}{2i}$

$= u(x,y) + iv(x,y)$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial z}$$

$$= \frac{\partial f}{\partial x} \cdot \frac{1}{2} + \frac{\partial f}{\partial y} \cdot \frac{1}{2i}$$

$$= \frac{1}{2} \left[\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right]$$

lly $\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left[\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right] = \frac{1}{2} (f_x + i f_y)$

$f_x = u_x + i v_x$; $f_y = u_y + i v_y$

$\frac{\partial f}{\partial \bar{z}} = 0 \Rightarrow \frac{1}{2} [f_x + i f_y] = 0 \Rightarrow \boxed{f_x = -i f_y}$

$f_x = -i f_y$

$\Rightarrow (u_x + i v_x) = -i(u_y + i v_y)$

$\Rightarrow u_x + i v_x = -i u_y + v_y$

$\Rightarrow (u_x - v_y) + i(u_y + v_x) = 0$

$\Rightarrow \boxed{u_x = v_y \quad ; \quad u_y = -v_x}$ — CR equations.

Q. If f is an analytic fn. and $|f|$ is a non-zero constant, then show that f is a constant.

Sol: $f = u + iv$; $|f| = \sqrt{u^2 + v^2} = C_1$

$\Rightarrow u^2 + v^2 = C_1^2 = C$

'f' is analytic:

$u_x = v_y$; $u_y = -v_x$

$u^2 + v^2 = C$

$\Rightarrow 2u.u_x + 2v.v_x = 0$

$\Rightarrow u u_x + v v_x = 0$

$\therefore u u_x + v v_x = 0$

lly $u u_y + v v_y = 0$

$u u_y + v v_y = 0$

$\Rightarrow -u v_x + v u_x = 0$

$u u_x + v v_x = 0 \quad \times v$

$v u_x - u v_x = 0 \quad \times u$

$\quad - \qquad +$

$(v^2 + u^2) v_x = 0$

$\Rightarrow \boxed{v_x = 0}$

$\Rightarrow V_x = 0$ $\Rightarrow V = \phi(y)$

lly $V_y = 0$ $\Rightarrow V = \text{constant}$

$\Rightarrow f$ is constant

Q. Suppose $f = u + iv$ is analytic and $u + v = (x+y)(2 - 4xy + x^2 + y^2)$. Find f.

Soln $(u+v)_x$

$\Rightarrow u_x + v_x = (2 - 4xy + x^2 + y^2) + (x+y)(-4y + 2x)$

$(u+v)_y$

$\Rightarrow \underbrace{u_y + v_y}_{u_x - v_x} = (2 - 4xy + x^2 + y^2) + (x+y)(-4x + 2y)$

$(\because u_x = v_y \; ; \; u_y = -v_x)$

$\Rightarrow 2u_x = 2[2 - 4xy + x^2 + y^2] + (x+y)2(x + y - 2x - 2y)$

$\Rightarrow u_x = 2 - 4xy + x^2 + y^2 + (x+y)(-x-y)$

$= 2 - 4xy + x^2 + y^2 - x^2 - y^2 - 2xy$

$\boxed{u_x = 2 - 6xy}$

lly $2v_x = (x+y)[2x - 4y + 4x - 2y] = 6(x^2 - y^2)$

$\boxed{v_x = 3(x^2 - y^2)}$

$u_x = 2 - 6xy = v_y$

$u = 2x - 3x^2y + \phi_1(y)$

$v = 2y - 3xy^2 + \phi_2(x)$

Ily $v_x = 3(x^2 - y^2) = -u_y$

$u = y^3 - 3x^2y + \phi_1(x)$

$v = x^3 - 3xy^2 + \phi_2(y)$

H.W Use CR equations to show: [CR: $u_x = v_y$, $u_y = -v_x$]

① $|z|^2$ is not analytic (but differentiable at origin)

Let $f(z) = |z|^2 \Rightarrow f(x+iy) = x^2+y^2$

$u(x,y) = x^2+y^2$; $v(x,y) = 0$

$\therefore u_x \neq v_y$ and $u_y \neq -v_x$ $\boxed{\therefore \text{It is not analytic}}$

② $\bar{z}$ is not analytic

Let $f(z) = \bar{z} \Rightarrow f(x+iy) = x-iy$

$u(x,y) = x$; $v(x,y) = -y$

$u_x = 1$; $u_y = 0$ $v_x = 0$, $v_y = -1$

$\therefore u_x \neq v_y$ and $u_y = v_x$ $\boxed{\therefore \text{It is not analytic}}$

③ $\frac{1}{z}$ $(z \neq 0)$ is analytic

Let $f(z) = \frac{1}{z} \Rightarrow f(x+iy) = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$

$\therefore u(x,y) = \frac{x}{x^2+y^2}$; $v(x,y) = \frac{-y}{x^2+y^2}$

$u_x = \frac{y^2-x^2}{(x^2+y^2)^2}$ $v_x = \frac{2xy}{(x^2+y^2)^2}$

$u_y = \frac{-2xy}{(x^2+y^2)^2}$ $v_y = \frac{y^2-x^2}{(x^2+y^2)^2}$

$\because u_x = v_y$ and $u_y = -v_x$ $\boxed{\therefore \text{It is analytic}}$

Also, 1st order partial derivatives are continuous

④ $\sin z$ is analytic

Let $f(z) = \sin z \Rightarrow f(x+iy) = \sin(x+iy)$

$= \sin x \cos iy + \cos x \sin iy$

$= \sin x \cosh y + i \cos x \sinh y$

$\therefore u(x,y) = \sin x \cosh y$; $v(x,y) = \cos x \sinh y$

$u_x = \cos x \cosh y$ $\quad v_x = -\sin x \sinh y$

$u_y = \sin x \sinh y$ $\quad v_y = \cos x \cosh y$

$\therefore u_x = v_y$ and $u_y = -v_x$

Also the 1st order partial derivatives are continuous.

$\boxed{\therefore \sin z \text{ is Analytic}}$

---

⑤ $\cos z$ is analytic

Let $f(z) = \cos z \Rightarrow f(x+iy) = \cos(x+iy)$

$= \cos x \cos iy - \sin x \sin iy$

$= \cos x \cosh y - i \sin x \sinh y$

$\therefore u(x,y) = \cos x \cosh y$; $v(x,y) = -\sin x \sinh y$

$u_x = -\sin x \cosh y$ $\quad v_x = -\cos x \sinh y$

$u_y = \cos x \sinh y$ $\quad v_y = -\sin x \cosh y$

$\therefore u_x = v_y$ and $u_y = -v_x$

Also, the 1^{st} order partial derivatives are continuous

$\boxed{\therefore \cos z \text{ is analytic}}$

(6) e^z is analytic

Let $f(z) = e^z$, $\Rightarrow f(x+iy) = e^{x+iy} = e^x \cdot e^{iy}$

$= e^x [\cos y + i\sin y]$

$\therefore u(x,y) = e^x \cos y$; $v(x,y) = e^x \sin y$

$u_x = e^x \cos y$ $\qquad v_x = e^x \sin y$

$u_y = -e^x \sin y$ $\qquad v_y = e^x \cos y$

$\therefore u_x = v_y$ and $u_y = -v_x$ $\qquad \boxed{\therefore e^x \text{ is Analytic}}$

Also the 1^{st} order partial derivatives are continuous

(7) Tan z and Sec z are analytic everywhere except $(2n+1)\pi/2$, $n \in \mathbb{Z}$; ~~$n\pi$, $n \in \mathbb{Z}$ respectively~~

Given, $f(z) = \tan z = \dfrac{\sin z}{\cos z}$, both $\sin z$ & $\cos z$ are analytic $\Rightarrow \dfrac{\sin z}{\cos z}$ is analytic for $\cos z \neq 0$

$\cos z = 0 \Rightarrow z = (2n+1)\pi/2$

$\boxed{\therefore \text{Tan } z \text{ is analytic everywhere except for } (2n+1)\pi/2,\ n \in \mathbb{Z}}$

Given, $f(z) = \sec z = \frac{1}{\cos z}$; here $\cos z$ is analytic

$\Rightarrow \frac{1}{\cos z}$ is also analytic

for $\cos z \neq 0$

$\cos z = 0 \Rightarrow z = (2n+1)\pi/2$, $n \in \mathbb{Z}$

$\boxed{\therefore \sec z \text{ is analytic for everywhere except for } (2n+1)\pi/2, n \in \mathbb{Z}}$

(8) $\cot z$ and $\operatorname{cosec} z$ are analytic everywhere except for $n\pi$, $n \in \mathbb{Z}$.

Given, $f(z) = \cot z = \frac{\cos z}{\sin z}$, here $\cos z$ and $\sin z$ are analytic

$\sin z = 0 \Rightarrow z = n\pi$, $n \in \mathbb{Z}$

$\Rightarrow \frac{\cos z}{\sin z}$ is also analytic

for $\sin z \neq 0$

lly $f(z) = \operatorname{cosec} z = \frac{1}{\sin z}$, here $\sin z$ is analytic

$\Rightarrow \frac{1}{\sin z}$ is also analytic

$\sin z = 0 \Rightarrow z = n\pi$, $n \in \mathbb{Z}$

for $\sin z \neq 0$

$\boxed{\therefore \cot z \text{ and } \operatorname{cosec} z \text{ are analytic everywhere except for } n\pi, n \in \mathbb{Z}}$

H.W: $u-v = e^{-x}\left[(x-y)\sin y - (x+y)\cos y\right]$ Find f?

Sol: $(u-v)_x$

$$\Rightarrow u_x - v_x = -e^{-x}\left[(x-y)\sin y - (x+y)\cos y\right]$$
$$+ e^{-x}\left[\sin y - \cos y\right] \rightarrow (1)$$

$(u-v)_y$

$$\Rightarrow u_y - v_y = e^{-x}\left[-\sin y - y\cos y - \cos y + y\sin y\right]$$
$$= e^{-x}\left[-\sin y - \cos y + y(\sin y - \cos y)\right]$$

$$-(u_x+v_x) = e^{-x}\left[(-\sin y - \cos y) + y(\sin y - \cos y)\right] \rightarrow (2)$$

$(\because u_x = v_y,\ u_y = -v_x)$

Solving (1) & (2)

$$u_x - v_x = -e^{-x}.x(\sin y - \cos y) + e^{-x}y(\sin y + \cos y)$$
$$+ e^{-x}(\sin y - \cos y)$$

$$-u_x - v_x = -e^{-x}(\sin y + \cos y) + e^{-x}y(\sin y - \cos y)$$

$$-2v_x = -xe^{-x}(\sin y - \cos y) + 2e^{-x}y\sin y - 2e^{-x}\cos y$$

H.W
If $f(z)$ and $\overline{f(z)}$ are analytic, then what is f?

Sol: $f = u+iv$, $\bar{f} = u-iv$

Both are analytic $\Rightarrow$ 'f' is a Constant function.

$\boxed{\therefore f \text{ is a Constant}}$

Harmonic Functions:

→ A real valued function $\phi(x,y)$ of two variable that has continuous second order partial derivatives that satisfies the Laplace equation $\phi_{xx} + \phi_{yy} = 0$ is said to be a Harmonic function.

* If $f(z) = u(x,y) + iv(x,y)$ is analytic, then u and v are harmonic

$\Rightarrow u_{xx} + u_{yy} = 0$ & $v_{xx} + v_{yy} = 0$

Also, v is called Harmonic Conjugate of u

* Converse not true (u, v harmonic $\not\Rightarrow u+iv$ analytic)

$\boxed{\text{\# Laplace Equation: } \phi_{xx} + \phi_{yy} = 0}$

Ex:

(1) $u = Re(z^2) = x^2 - y^2$

$u_x = 2x$; $u_{xx} = 2$ & $u_y = -2y$; $u_{yy} = -2$

$\therefore u_{xx} + u_{yy} = 2 - 2 = 0$

$\boxed{\therefore\ u = Re(z^2) \text{ is Harmonic}}$

(2) $V = Im(z^3) = 3x^2y - y^3$

$V_x = 6xy$, $V_y = -3y^2$

$V_{xx} = 6y$, $V_{yy} = -6y$

$\boxed{\therefore V_{xx} + V_{yy} = 0}$

$\because$ CR not satisfied at 0
$\Rightarrow$ not analytic 0.

$u + iv = (x^2 - y^2) + i(3x^2y - y^3)$ is not analytic

But $(u_y - v_x) + i(u_x + v_y)$ is always Analytic.

$\because U = u_y - v_x$; $V = u_x + v_y$

$U_x = u_{xy} - v_{xx}$ $\qquad V_x = u_{xx} + v_{xy}$

$U_y = u_{yy} - v_{xy}$ $\qquad V_y = u_{xy} + v_{yy}$

$U_x = V_y \Rightarrow v_{xx} + v_{yy} = 0$ & $U_y = -V_x \Rightarrow u_{xx} + u_{yy} = 0$

Q. Show that $v = e^x \sin y$ is harmonic and find the harmonic conjugate of v.

Sol. $v = e^x \sin y \Rightarrow v_x = e^x \sin y$ $\quad v_y = e^x \cos y$

$v_{xx} = e^x \sin y$ $\quad v_{yy} = -e^x \sin y$

$\therefore v_{xx} + v_{yy} = e^x \sin y - e^x \sin y = 0$

$\therefore v$ is harmonic

Let u is harmonic conjugate of v

iff $f(z) = v(x,y) + i\,u(x,y)$ - is analytic

$\Rightarrow$ Satisfies CR equations

$\therefore v_x = u_y$ and $v_y = -u_x$

$\Rightarrow u_y = e^x \sin y$ and $u_x = -e^x \cos y$

$u = -e^x \cos y + \phi$ | $u = -e^x \cos y$

$\boxed{\therefore \text{The harmonic conjugate of } v \text{ is } -e^x \cos y}$

HW:

Show that $u = 2x + y^3 - 3x^2y$ is Harmonic and find its harmonic conjugate.

Sol:

$u = 2x + y^3 - 3x^2y$

$\Rightarrow u_x = 2 - 6xy \Rightarrow u_{xx} = -6y$

$\Rightarrow u_y = 3y^2 - 3x^2 \Rightarrow u_{yy} = 6y$

$\boxed{\therefore u_{xx} + u_{yy} = 0} \Rightarrow u$ is Harmonic.

Let v is Harmonic conjugate of u

iff $f(z) = u(x,y) + i\,v(x,y)$ is analytic

$\Rightarrow u_x = v_y$ and $u_y = -v_x$

$\Rightarrow v_y = 2 - 6xy$ | $v_x = 3x^2 - 3y^2$

$v = 2y - 3xy^2 + \phi$ | $v = x^3 - 3xy^2$

$\therefore v = x^3 - 3xy^2 + 2y$

$$\boxed{\begin{array}{c}\therefore \text{The harmonic conjugate of } u \text{ is} \\ x^3 - 3xy^2 + 2y\end{array}}$$

Laplace equation in Polar form:

$$\boxed{u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0} \; ; \; v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = 0$$

HW. Show that $u = r^2 \cos 2\theta$ is harmonic and find its harmonic conjugate.

Minor-I

Sol: Given, $u = r^2 \cos 2\theta$

$u_r = 2r \cos 2\theta$ & $u_{rr} = 2\cos 2\theta$

$u_\theta = -2r^2 \sin 2\theta \Rightarrow u_{\theta\theta} = -4r^2 \cos 2\theta$

$\therefore u_{rr} + \frac{1}{r} u_r + u_{\theta\theta} = 2\cos 2\theta + \frac{1}{r} \cdot 2r\cos 2\theta + \frac{1}{r^2}(-4r^2 \cos 2\theta)$

$= 0$

$\therefore u$ is harmonic

Let v is harmonic conjugate of $u \Rightarrow f = u(r,\theta) + iv(r,\theta)$ is analytic

$\Rightarrow v_\theta = r u_r$ and $u_\theta = -r v_r$

$\Rightarrow v_\theta = 2r^2 \cos 2\theta$ | $v_r = 2r \sin 2\theta$

$v = r^2 \sin 2\theta$ | $v = r^2 \sin 2\theta$

$$\boxed{\therefore v = r^2 \sin 2\theta}$$

$$\boxed{\therefore \text{The harmonic conjugate of } u \text{ is } r^2 \sin 2\theta}$$

Minor I 27/08/2021

Complex Integration

* Using integration, we shall show that an analytic function has derivatives of all orders. (not true in real case)

* We can only evaluate Real integrals

$$\int_a^b f(x)\,dx \;\not\longrightarrow\; \int_a^b f(z)\,dz$$

real variable — Complex variable

$x=a$ to $x=b$ – Singlepath — $z=a$ to $z=b$ infinite paths

Definite Integral:

$\phi(t) = \underbrace{\phi_1(t)}_{\text{real}} + i\,\underbrace{\phi_2(t)}_{\text{real}}$ – Complex function of a real variable 't'.

→ ϕ is integrable if ϕ_1 and ϕ_2 are integrable.

$$\int_a^b \phi(t) = \int_a^b \phi_1(t)\,dt + i\int_a^b \phi_2(t)\,dt \qquad (\phi : \mathbb{R} \to \mathbb{C})$$

* a or b is infinite or if $\phi_1(t)$ or $\phi_2(t)$ has infinite discontinuity – Improper Integral

Ex: $\int_0^1 (t+it^2)\,dt = \int_0^1 t\,dt + i\int_0^1 t^2\,dt$

$= \frac{t^2}{2}\Big|_0^1 + i\,\frac{t^3}{3}\Big|_0^1$

$= \frac{1}{2} + \frac{i}{3}$ // (Integrable)

∴ It is a Definite integrale – Function bounded and Limits are finite.

Ex: $\int_0^1 \left((te^{-t^2}) + 2\frac{i}{\sqrt{t}}\right) dt = \underbrace{\int_0^1 te^{-t^2}\,dt}_{\text{Proper Integral}} + 2i\underbrace{\int_0^1 \frac{1}{\sqrt{t}}\,dt}_{\text{Improper Integral}}$

Solving Improper Integrals

$\int_0^1 \frac{1}{\sqrt{t}}\,dt = \lim_{\varepsilon\to 0}\int_\varepsilon^1 \frac{1}{\sqrt{t}}\,dt = \lim_{\varepsilon\to 0} 2\sqrt{t}\Big|_\varepsilon^1$

$= \lim_{\varepsilon\to 0} (2-2\sqrt{\varepsilon})$

$= 2$

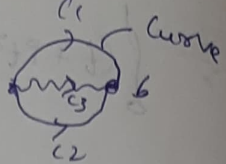
Curves:

* Let $x(t)$ and $y(t)$ be two continuous functions of 't', $a\le t\le b$.

$z = z(t) = x(t)+iy(t)$, $a\le t\le b$ traces a curve C in the complex plane starting at $z(a)$ and ending at $z(b)$.

$\overrightarrow{\quad}$ Real / ~~complex~~ ✓

$a \le t \le b$



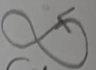
$\because$ We can't even compare two complex numbers.

Closed Curve: $z(a) = z(b)$

(Closed & Simple)

Simple Curve: does not intersect itself

$z(t_1) \neq z(t_2)$ if $t_1 \neq t_2$



(Closed but not Simple)

Let $z(t)$ be a simple curve:

(i) $\lim\limits_{t \to t^*} z(t) = \lim\limits_{t \to t^*} x(t) + i \lim\limits_{t \to t^*} y(t)$

(ii) $z(t)$ continuous on $[a,b]$ iff $x(t)$ & $y(t)$ are continuous on $[a,b]$

(iii) $z(t)$ piecewise continuous (continuous except for atmost finite number of jump discontinuities)

(iv) $z(t)$ differentiable iff $x(t)$ & $y(t)$ differentiable

$z'(t) = x'(t) + i y'(t)$ continuously differentiable if $z'(t)$ is continuous.

(v) Curve defined by $z(t)$ is <u>smooth</u> if $z(t)$ is continuously differentiable and $z'(t) \neq 0$ $\forall\, t \in [a,b]$

(vi) Curve defined by $z(t)$ is a <u>Contour</u> if it is smooth or piecewise smooth.

(vii) $C : z(t)$, $a \le t \le b$

$-C : z(-t)$, $-b \le t \le -a$

opposite direction

<u>Counter clockwise - Positive</u>

$-C :$ $a \xleftarrow{\quad} b$

$z(t) = t + i0$

$C :$ $a \xrightarrow{\quad} b$

$f(z) = u(x,y) + iv(x,y)$ is defined on D.

* Suppose $z(t)$, $a \le t \le b$ is contained in D;

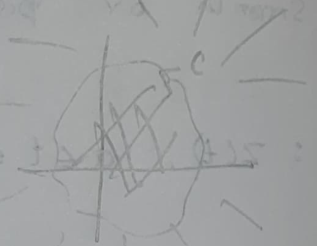
$z(t)$ is continuous.

* If f is continuous (analytic), then $f(z(t)) = u(x(t), y(t)) + iv(x(t), y(t))$

is continuous. (analytic)

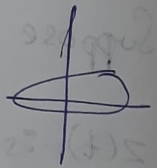
Jordan Curve Lemma :

→ Let c be a simple closed contour. Then c separates the complex plane into two distinct regions, the inside of c and the outside of C, one of which is bounded and the other is unbounded.



Parametric representation of $9x^2+y^2=9$
(i) Clockwise (ii) anticlockwise.

$9x^2+y^2=9 \Rightarrow \frac{x^2}{1}+\frac{y^2}{9}=1 \Leftarrow f(x,y)$ - Ellipse



$Z=Z(t)$, $t\in[a,b]$ $x(t)=\cos t$

$y(t)=3\sin t$, $t\in[0,2\pi]$

$C: z(t)=\cos t+i\,3\sin t$, $t\in[0,2\pi]$

Clockwise
$-C$: $z(-t)=\cos t-3i\sin t$, $-2\pi\le t\le 0$

The given curve:

$z(0) = 1 = z(2\pi) \Rightarrow z(a) = z(b)$

$\therefore$ closed curve

if $t_1 \neq t_2 \Rightarrow z(t_1) \neq z(t_2)$

$\therefore$ Simple Curve

$z'(t) = -\sin t + 3i\cos t$, $t \in (0, 2\pi)$

$\downarrow \quad \neq 0$

Continuously differentiable $\therefore$ Smooth

Q. $z(t) = \begin{cases} t & ; -1 \le t \le 1 \\ e^{i(t-1)} & ; 1 \le t \le \pi+1 \end{cases}$ is simple, closed, smooth or piece-wise smooth (P.S smooth)

Sol: $z(-1) = -1 = z(\pi+1)$ ✓ Closed Curve

$\lim_{t\to 1^+} z(t) = \lim_{t\to 1^-} z(t) = 1$

$t_1 \neq t_2$

$\Rightarrow z(t_1) \neq z(t_2)$

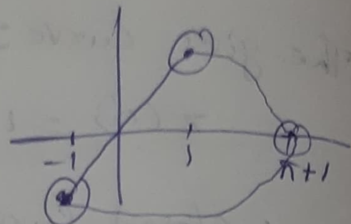
✓ Simple Curve

$z'(t) = \begin{cases} 1 & ; -1 < t < 1 \\ ie^{i(t-1)} & ; 1 < t < \pi+1 \end{cases}$

$\lim_{t\to 1^+} z'(t) = 1$ and $\lim_{t\to 1^-} z'(t) = 1$ $\neq$

$\Rightarrow$ not smooth

$\Rightarrow$ not continuously differentiable.



z- not differentiable at 1

$\Rightarrow$ z is differentiable at

$(-1,1) \cup (1, \pi+1)$

$\because \lim_{t\to 1^+} z'(t) \neq \lim_{t\to 1^-} z'(t)$

Also.

$\lim_{t\to -1^+} z'(t) = 1$ & $\lim_{t\to(\pi+1)^-} z'(t) = i$

$\lim_{t\to -1^+} z'(t) \neq \lim_{t\to(\pi+1)^-} z'(t)$

$z'(t) \neq 0$ ✓ Piece-wise Smooth

H.W: $z(t) = (1-\cos t)\, e^{it}$, $0 \le t \le 2\pi$ - closed, simple, smooth, p.w smooth?

Sol: $z(0) = 0 = z(2\pi)$ ✓ Closed Curve

$t_1 \neq t_2 \Rightarrow z(t_1) \neq z(t_2)$ ✓ Simple curve

$z'(t) = \begin{cases} \sin t\, e^{it} + i e^{it}(1-\cos t) \\ 0 < t < 2\pi \end{cases}$

$$\lim_{t \to 0^+} z'(t) = \lim_{t \to 0^+} \left[\sin t \, e^{it} + i e^{it} (1-\cos t) \right]$$

$$= 0$$

$$\lim_{t \to 2\pi^-} z'(t) = \lim_{t \to 2\pi^-} \left[\sin t \, e^{it} + i e^{it} (1-\cos t) \right]$$

$$= 0$$

$\Rightarrow$ continuously differentiable

✓ Smooth

$z'(t)$ can be equal to zero — not p.s smooth

---

* Curve: $x(t), y(t)$ - continuous functions $a \le t \le b$

$z = z(t) = x(t) + iy(t)$ traces a curve in complex plane from $z(a)$ to $z(b)$, $t \in [a, b]$

Simple Curve: $t_1 \neq t_2 \Rightarrow z(t_1) \neq z(t_2)$

Closed Curve: $z(a) = z(b)$

Contour

Smooth Curve: Continuously differentiable

$z'(t) \neq 0 \quad t \in [a, b]$

(or)

P.W Smooth

31/08/2021

Contour Integrals : (Line Integrals in the Complex plane)

* Let $z(t) = x(t) + iy(t)$, $a \le t \le b$ represent a simple smooth curve C.

* $f(t)$ – continuous on C. Partition $[a, b]$

$$a = t_0 < t_1 < \cdots < t_n = b$$

* Points $z_0, z_1, \ldots z_n$ on C.

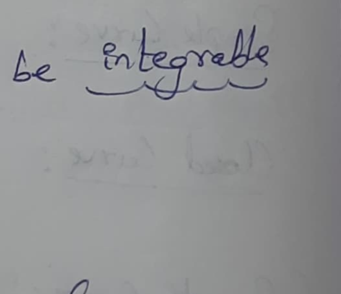
$$z_k = z(t_k) \; ; \; k = 0, 1, \ldots, n$$

* Let ξ_k be a point in (z_{k-1}, z_k)

Reimann Integration formula

* $S_n = \sum_{k=0}^{n} f(\xi_k)\, \Delta z_k \qquad (\Delta z_k = z_k - z_{k-1})$

as $n \to \infty$, $|\Delta z_k| \to 0$ $\qquad \lim_{n\to\infty} S_n = L$



* Regardless of the choice of z_k and ξ_k, if S_n approaches a limit, then $f(z)$ is said to be integrable along C.

* $\lim_{n\to\infty} S_n = L$ $\qquad$ * $\lim_{\max|\Delta z_k| \to 0} S_n = \int_C f(z)\, dz$

C-closed $\oint_C f(z)\, dz$ $\qquad$ Contour integral / Line integral of f along C.

C-path of integration

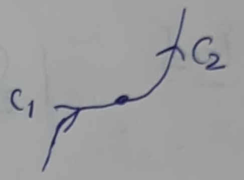
* Let $f(z)$ and $g(z)$ be two continuous functions defined on a piecewise smooth curve C.

$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$$

Illy $$\int_C (\alpha f + \beta g)(z)\,dz = \alpha \int_C f(z)\,dz + \beta \int_C g(z)\,dz$$

$(\alpha, \beta \in \mathbb{C})$

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$



Existence of the Contour integral:

* If $f(z)$ is continuous (or p.w. continuous) at every point on a smooth (or p.w. smooth) curve C, then $f(z)$ is integrable.

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy)$$

$$= \int_C (u\,dx - v\,dy) + i \int_C (v\,dx + u\,dy)$$

Suppose $C : z = z(t) = x(t) + iy(t)$, $a \le t \le b$

$$\int_C f(z)\,dz = \int_a^b f(z(t)) \cdot z'(t)\,dt$$

Theorem: ML inequality

Let 'c' be a p.w smooth curve

$$C : z = z(t) = x(t) + iy(t) , \quad a \leq t \leq b$$

Then $\left| \int_C f(z) dz \right| \leq ML$, where L is the length of C and $|f(z)| \leq M$ on C

Proof $\left| \int_C f(z) dz \right| \leq \int_C |f(z)| dz$

$$\leq M \int_C dz$$

$$\leq LM \; // \quad \boxed{\text{proved}}$$

* For a smooth curve C, $C : z = z(t) = x(t) + iy(t)$, $a \leq t \leq b$

$$L = \int_a^b |z'(t)| \, dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} \, dt$$

Using the definition of Contour integral, Evaluate

(1) $\int_C dz = z_m - z_0$!, where 'c' is any smooth curve b/w z_0 & z_m.

Sol:

$I = \int_C dz = \lim_{\max|\Delta z_k| \to 0} \left[\sum_{k=0}^{n} f(\xi_k)\, \Delta z_k \right]$;

Here; $f(z) = 1$

$\xi_k \in (z_{k-1}, z_k)$

$\rightarrow \xi_k = z_k$

(or)

$\rightarrow \xi_k = z_{k-1}$

$= \lim_{\max|\Delta z_k| \to 0} \left[\Delta z_1 + \Delta z_2 + \cdots + \Delta z_m \right]$

$+ z_m - z_{m-1}$

$\xi_k = z_k \Rightarrow I = z_1 - z_0 + z_2 - z_1 + \cdots$

~~$\xi_k = z_k$ Δz_k~~ $\Rightarrow \boxed{I = z_m - z_0}$

$$\boxed{\therefore \int_C dz = z_m - z_0}$$

(2) $\int_C z\,dz = \dfrac{z_m^2 - z_0^2}{2}$, 'c' is any smooth curve b/w z_0 & z_m.

Sol

$I = \int_C z\,dz = \lim_{\max|\Delta z_k| \to 0} \left[\sum_{k=0}^{n} f(\xi_k)\, \Delta z_k \right]$

Here, $f(z) = z$, $\xi_k \in (z_{k-1}, z_k)$

$$I = \int_C z\,dz = \lim_{\max|\Delta z_k| \to 0} \left[\xi_1 \Delta z_1 + \xi_2 \Delta z_2 + \dots + \xi_m \Delta z_m\right]$$

$$\xi_k = z_k \Rightarrow I = \left[z_1(z_1 - z_0) + \dots\dots + z_m(z_m - z_{m-1})\right]$$

$$\xi_k = z_{k-1} \Rightarrow I = \left[z_0(z_1 - z_0) + \dots\dots + z_{m-1}(z_m - z_{m-1})\right]$$

$$2I = z_m^2 - z_0^2$$

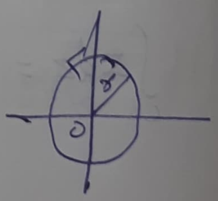
$$\therefore I = \frac{z_m^2 - z_0^2}{2}$$

$$\boxed{\therefore \int_C z\,dz = \frac{z_m^2 - z_0^2}{2}}$$

Evaluate: $\int_C z^n dz$, $n \in \mathbb{Z}$ where $C: |z| = r$

traversed counter clockwise.

Sol: $z = re^{i\theta}$ $\quad |z| = r$



$$\int_C z^n dz = \int_0^{2\pi} r^n e^{in\theta} (rie^{i\theta})\, d\theta$$

$$\left(\because \int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\, dt\right)$$

$$\Rightarrow \int_C z^n dz = i r^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} d\theta$$

$$= i r^{n+1} \left[\frac{e^{i(n+1)\theta}}{i(n+1)} \right]_0^{2\pi} \quad (n \neq -1)$$

$$= \frac{r^{n+1}}{n+1} \left[e^{i(2\pi)(n+1)} - 1 \right]$$

$$= 0 \;//$$

$\int_C z^n dz$ for $(n=-1) \Rightarrow \int_C \frac{1}{z} dz$

$$\int_C \frac{1}{z} dz = \frac{1}{r} \int_0^{2\pi} e^{-i\theta} (r i e^{i\theta}) d\theta$$

$$= i \int_0^{2\pi} d\theta$$

$$= 2\pi i \;//$$

$$\boxed{\therefore \int_C z^n dz = \begin{cases} 2\pi i \;; & n = -1 \\ 0 \;; & n \neq -1 \end{cases}}$$

Evaluate: $\int_C (z-z_0)^n\, dz$, $n \in \mathbb{Z}$ where $C: |z-z_0| = r$ is traversed counter clockwise.

Sol: Let $z - z_0 = w \Rightarrow dz = dw$

$$\int_C w^n\, dw = \begin{cases} 2\pi i & ; n = -1 \\ 0 & ; n \neq -1 \end{cases}$$

Evaluate: $I_k = \int_C f_k(z)\, dz$, for

(i) $f_1(z) = z$ (ii) $f_2(z) = \bar{z}$, (iii) $f_3(z) = (z-i)^n$

$n \in \mathbb{Z}$, $C: z = z(t) = i + e^{it}$, $0 \leq t \leq 2\pi$.

Sol: (i) $f_1(z) = z$

$$\int_C f_1(z)\, dz = \int_C z\, dz \quad (\text{or}) = \int_0^{2\pi} f_1(z(t)) \cdot z'(t)\, dt$$

$$= \int_0^{2\pi} (i + e^{it}) \cdot (i e^{it})\, dt$$

$$= -\int_0^{2\pi} e^{it} dt + i \int_0^{2\pi} e^{2it}\, dt$$

$$= -\left[\frac{e^{it}}{i}\right]_0^{2\pi} + i \left[\frac{e^{2it}}{2i}\right]_0^{2\pi}$$

$$\Rightarrow \int_C f_1(z)\,dz = \frac{1-e^{2\pi i}}{i} + \frac{1}{2}\left[e^{4\pi i} - 1\right]$$

$$= 0 //$$

$$\boxed{\therefore \; I_1 = \int_C f_1(z)\,dz = 0}$$

(ii) $f_2(z) = \bar{z}$

$$\int_C f_2(z)\,dz = \int_0^{2\pi} f_2(z(t)) \cdot z'(t)\,dt$$

$$= \int_0^{2\pi} \left(\overline{i + e^{it}}\right) \cdot \left(-i e^{-it}\right) dt$$

$$= \int_0^{2\pi} \left(-i + e^{-it}\right)\left(-i e^{-it}\right) dt$$

$$= \int_0^{2\pi} e^{-it}\,dt - i\int_0^{2\pi} e^{-2it}\,dt$$

$$= \left[\frac{e^{-it}}{-i}\right]_0^{2\pi} - i\left[\frac{e^{-2it}}{-2i}\right]_0^{2\pi}$$

$$= \frac{1-e^{-2\pi i}}{i} + \frac{1}{2}\left[e^{-2\pi i} - 1\right]$$

$$= 0 //$$

$$\boxed{\therefore \; I_2 = \int_C f_2(z)\,dz = 0}$$

(iii) $f_3(z) = (z-i)^n$

$$\int_C f_3(z)\,dz = \int_0^{2\pi} f_3(z(t)) \cdot z'(t)\,dt$$

$$= \int_0^{2\pi} (i + e^{it} - i)^n \cdot (ie^{it})\,dt$$

$$= i\int_0^{2\pi} e^{int}\, e^{it}\,dt$$

$$= i\int_0^{2\pi} e^{it(n+1)}\,dt$$

$$= i\left[\frac{e^{i(n+1)t}}{i(n+1)}\right]_0^{2\pi} \qquad (n \neq -1)$$

$$= \frac{1}{n+1}\left[e^{i2\pi(n+1)} - 1\right]$$

$$= 0 \; //$$

$(n=-1) \Rightarrow \displaystyle\int_C \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{e^{it}} \cdot (ie^{it})\,dt$

$$= i\int_0^{2\pi} dt = i\left[t\right]_0^{2\pi}$$

$$= 2\pi i \; //$$

$$\therefore \int_C f_3(z)\,dz = I_3 = \begin{cases} 2\pi i & ; n=-1 \\ 0 & ; n \neq -1 \end{cases}$$

Evaluate: $\int_C z^2 dz$; C: the arc of $|z|=2$ from $\theta=0$ to $\theta=\pi/3$.

Sol: $\int_C z^2 dz = \int_0^{\pi/3} f(z(\theta))\, z'(\theta)\, d\theta$

$\boxed{z = 2e^{i\theta}}$

$$= \int_0^{\pi/3} (2e^{i\theta})^2 \cdot (2ie^{i\theta})\, d\theta$$

$$= 4i \int_0^{\pi/3} e^{i3\theta}\, d\theta$$

$$= 4i \left[\frac{e^{i3\theta}}{i3} \right]_0^{\pi/3}$$

$$= \frac{4}{3}\left[e^{i\pi} - 1\right] = \frac{4}{3}\left[-1-1\right]$$

$$= -\frac{8}{3} \,//$$

$$\boxed{\therefore \int_C z^2 dz = -\frac{8}{3}}$$

<u>Evaluate</u>: $\int_C Re(z^2)\,dz$ from 0 to $2+4i$

along (i) Line segment joining 0 & $2+4i$

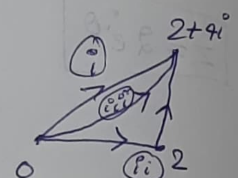
(ii) x-axis from 0 to 2 and vertically from 2 to $2+4i$

(iii) $y = x^2$

$\therefore Re(z^2)$ is not Analytic / smooth

$z'(t) \neq 0$

<u>Sol</u>: (i) Line segment joining 0 & $2+4i$



$C: \; z(t) = t + i2t \;;\; 0 \le t \le 2$

$$z^2(t) = (t^2 - 4t^2) + i4t$$

$$= -3t^2 + i\,4t$$

$$\int_C Re(z^2)\,dz = \int_0^2 (-3t^2)(1 + 2i)\,dt$$

$$\left[\because \int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt\right]$$

$$= -3(1+2i)\int_0^2 t^2\,dt$$

$$= -3(1+2i)\left[\frac{t^3}{3}\right]_0^2 = -8(1+2i)$$

(ii) x-axis from 0 to 2 & vertically from 2 to $2+4i$

$C_1 : z(t) = t$, $\quad z^2(t) = t^2$ $\quad\Big|\quad 0 \le t \le 2 \quad\Big|\quad C_2 : z(t) = 2 + 2ti$, $\quad z^2(t) = (4 - 4t^2) + 8ti$

$$\int_C Re(z^2)dz = \int_0^2 t^2(1)\,dt + \int_0^2 (4-4t^2)(2i)\,dt$$

$$= \left[\frac{t^3}{3}\right]_0^2 + 8i\left[t - \frac{t^3}{3}\right]_0^2$$

$$= \frac{8}{3} + 8i\left(2 - \frac{8}{3}\right)$$

$$= \frac{8}{3} - \frac{16i}{3} = \frac{8}{3}(1-2i) \,//$$

(iii) $y = x^2$; $\quad C : z(t) = t + it^2 \qquad 0 \le t \le 2$

$z^2(t) = (t^2 - t^4) + i2t^3$

$$\int_C Re(z^2)\,dz = \int_0^2 (t^2 - t^4)(1 + 2ti)\,dt$$

$$= \int_0^2 (t^2 - t^4)\,dt + 2i\int_0^2 (t^3 - t^5)\,dt$$

$$= \frac{8}{3} - \frac{32}{5} + 2i\left[\frac{16}{4} - \frac{64}{6}\right]$$

$$= \frac{-56}{10} - \frac{40i}{3} = -\left(\frac{28}{5} + i\frac{40}{3}\right) //$$

$$\boxed{\therefore \int_C Re(z^2)\,dz = \text{(i) } -8(1+2i) \quad \text{(ii) } \frac{8}{3}(1-2i) \quad \text{(iii) } -\left(\frac{28}{5} + i\frac{40}{3}\right)}$$

Evaluate: $\int_C (x+y^2-ixy)\,dz$

$$C: z = z(t) = \begin{cases} t-2i & ; 1 \le t \le 2 \\ 2-i(4-t) & ; 2 < t \le 3 \end{cases}$$

(Check smooth or p.w-smooth)

$C: z(t) = x(t) + iy(t)$

$C_1: z(t) = t + i(-2) \quad ; 1 \le t \le 2$

$C_2: z(t) = 2 + i(t-4) \quad ; 2 < t \le 3$

Sol:

$z'(t) = 1 \quad ; 1 \le t \le 3 \qquad z'(t) \neq 0$

$\therefore z(t)$ is continuously differentiable

$\therefore z(t)$ is smooth.

$$\int_C (x+y^2-ixy)\,dz = \int_{C_1} (x+y^2-ixy)\,dz + \int_{C_2} (x+y^2-ixy)\,dz$$

$$= \int_1^2 (t+4+i2t)(1)\,dt + \int_2^3 (2+(t-4)^2-2i(t-4))(i)\,dt$$

$$= \int_1^2 t\,dt + 4\int_1^2 dt + 2i\int_1^2 t\,dt + 2i\int_2^3 dt$$

$$+ i\int_2^3 (t-4)^2\,dt + 2\int_2^3 (t-4)\,dt$$

$$\int_c (x+y^2-ixy)\,dz$$

$$= (1+2i)\left[\frac{t^2}{2}\right]_1^2 + 4\,[t]_1^2 + 2i\,[t]_2^3$$

$$+\ i\left[\frac{(t-4)^3}{3}\right]_2^3 + 2\left[\frac{(t-4)^2}{2}\right]_2^3$$

$$= \frac{3}{2}(1+2i) + 4 + 2i + \frac{i}{3}\left[-1-(-8)\right]$$

$$+\left[1-4\right]$$

$$= \frac{3}{2}(1+2i) + 4+2i + \frac{7i}{3} - 3$$

$$= \frac{5}{2} + i\,\frac{22}{3} \;//$$

$$\boxed{\therefore \int_C (x+y^2-ixy)\,dz = \frac{5}{2} + i\,\frac{22}{3}}$$

Q. Find L for $C: Z = (1-i)t^2 ; -1 \le t \le 1$

Sol

$$L = \int_{-1}^{1} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$\because$ $z(t) = t^2 - it^2 \Rightarrow x(t) = t^2$ | $y(t) = -t^2$

$= x(t) + iy(t)$ $\quad x'(t) = 2t$ | $y'(t) = -2t$

$$\therefore L = \int_{-1}^{1} \sqrt{(2t)^2 + (-2t)^2}\, dt$$

$$= \int_{-1}^{1} \sqrt{4t^2 + 4t^2}\, dt$$

$$= 2\sqrt{2} \int_{-1}^{1} t\, dt$$

$$= 2\sqrt{2} \cdot \left[\frac{t^2}{2}\right]_{-1}^{1}$$

$= 0$ //

$\boxed{\therefore L = 0}$ //

01/09/2021

Cauchy Integral Theorem, Cauchy Goursat Theorem

$$\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

$$\int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\,dt \qquad C: \text{parameterized curve}$$

f - continuous (p.w. continuous) on C - smooth (p.w. smooth)

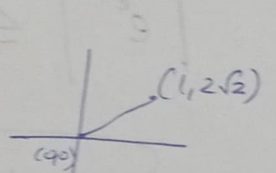
then $\int_C f(z)\,dz$ is integrable.

$$\left|\int_C f(z)\,dz\right| \le ML \;, \quad L\text{- Length of } C \text{ and } |f(z)| \le M \text{ on } C.$$

$$L = \int_a^b \sqrt{(x'(t))^2 + y'(t)^2}\;dt$$

Ex: Find an upper bound for the absolute value of $\int_C e^z\,dz$, C is the line segment going $(0,0)$ & $(1, 2\sqrt{2})$.

Sol: $\left|\int_C e^z\,dz\right| \le k$ C:



$$L = \sqrt{1^2 + (2\sqrt{2})^2} = \sqrt{1+8} = 3$$

$|f(z)| \le M$ on C. , $f(z) = e^z$

$f(z) = e^{z} = e^{x} e^{iy}$ $\quad [\because |e^{iy}| = 1]$

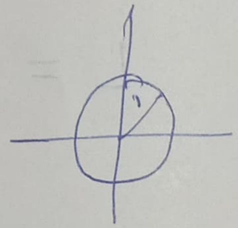
$|f(z)| = |e^{x} e^{iy}| = |e^{x}||e^{iy}| = e^{x} \le e^{1}$ on C

$|f(z)| \le e^{1}$ on C $\quad$ [C: Line Segment joing (0,0) & $(1, 2\sqrt{2})$]

$$\boxed{\therefore \left| \int_C e^{z}\, dz \right| \le 3e}$$

$\int_C e^{(\bar{z})^2}\, dz$; C : $|z| = 1$ traversed anti-clockwise

Sol: $|z| = 1 \Rightarrow z = e^{i\theta}$



$L = 2\pi(1)$

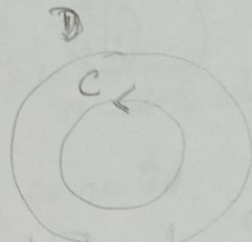
$|f(z)| = |e^{\bar{z}^2}| \le M$ on C

$\left| e^{e^{-2i\theta}} \right| \le e$

$$\boxed{\therefore \left| \int_C e^{(\bar{z})^2} \right| \le 2\pi e}$$

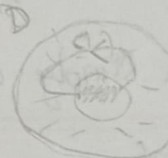
(a) Simply Connected Domain:

→ A domain is simply connected if every simple closed curve inside D enclose only points of D or any simple closed curve which lies in D can be shrunk to a point in D without leaving D.



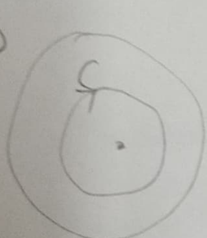
(b) Multiply Connected Domain:

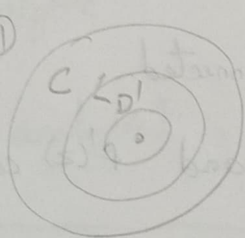
→ A domain which is not simply connected is called multiply connected.



* Multiply connected domains have holes.

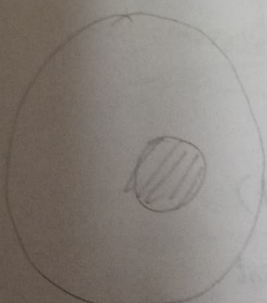
one hole - doubly connected ; two holes - triply connected

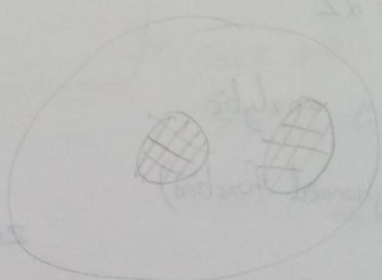
(a) 

(b) 

- we cannot shrink C to a point ∴ it goes to D'

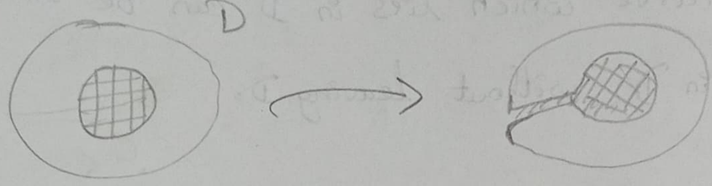
- Shrinking curve to a point without leaving D





* A multiply connected domain ... a

Simply connected domain by introducing cuts in the domain.



Cauchy Integral Theorem:

* Let $f(z)$ be analytic and $f'(z)$ be continuous in a simply connected domain D,

Then, $\int_C f(z)\,dz = 0$ along every simple smooth (p.w smooth) closed curve C in D.

#

- Domain simply connected
- $f(z)$ is analytic and $f'(z)$ continuous

Ex: $f(z) = z^2$

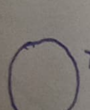
$$\int_{|z|=1} z^2\,dz = \int_0^{2\pi} e^{i2\theta}(i e^{i\theta})\,d\theta$$

$$= i\int_0^{2\pi} e^{i3\theta}\,d\theta$$

$$= 0 \quad \text{- By theorem also}$$

$f'(z) = 2z$

→ $f(z)$ is analytic (polynomial function)



$re^{i\theta} = z$

$$z(t) = x(t) + i y(t)$$

$$= r\cos t + i r\sin t$$

$$z'(t) = -r\sin t + i r\cos t \;;\; z'(t) \neq 0$$

smooth.

CIT: Conditions $\Rightarrow \int_C f(z)\,dz = 0$

- Domain - Simply connected
- f - analytic & $f'(z)$ - Continuous
- Curve - C - simple smooth closed curve

Cauchy - Goursat Theorem: ($f'(z)$ be continuous is not considered)

→ Let $f(z)$ be analytic in a simply connected domain D, Then $\oint_C f(z)\,dz = 0$ along every smooth closed curve C in D.

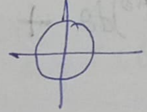
Ex: $\oint_C e^{\sin z^2}\,dz$; $C: |z| = 1$ - Domain - Simple Smooth closed curve

$f(z) = e^{\sin z^2}$ - analytic

(z^2, $\sin z$, e^z are analytic)

$$\boxed{\therefore \oint_C e^{\sin z^2}\,dz = 0}$$

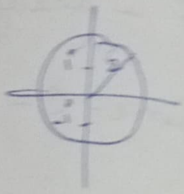
Ex: $\oint_C \tan z\,dz$; $f(z) = \tan z$ is analytic for $\cos z \neq 0$

$C: |z| = 1$  $z \neq (2n+1)\frac{\pi}{2}$

$n=0 \Rightarrow \pi/2 > 1$

$$\boxed{\therefore \oint_C \tan z\,dz = 0}$$

Ex: $\oint_C \frac{e^z}{z^2+1}\,dz$, $C: |z|=2$

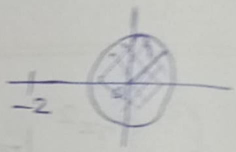


$f(z)=\frac{e^z}{z^2+1}$, $z \neq \pm i$

- Its analytic on C

$$\boxed{\therefore \oint_C \frac{e^z}{z^2+1}\,dz = 0}$$

Ex: $\oint_C \frac{3z+5}{z^2+2z}\,dz$; $C: |z|=1$



Sol: $z(z+2)$; $z=0,\ z=-2$ not analytic

$$\frac{3z+5}{z(z+2)} = \frac{A}{z} + \frac{B}{z+2} \Rightarrow A=\frac{5}{2}\ ;\ B=\frac{1}{2}$$

$$\therefore \oint_C \frac{3z+5}{z^2+2z}\,dz = \underline{\frac{5}{2}\int_C \frac{1}{z}\,dz} + \frac{1}{2}\int_C \frac{1}{z+2}\,dz$$

Analytic

$$= \frac{5}{2}\int_0^{2\pi} e^{-i\theta}\,(ie^{i\theta})\,d\theta + 0$$

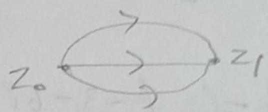
$(z=e^{i\theta})$

$$= \frac{i5}{2}\int_0^{2\pi} (1)\,d\theta = 5\pi i \,//$$

$$\boxed{\oint_C \frac{3z+5}{z^2+2z}\,dz = 5\pi i} \,//$$

Independence of Path

→ Let $f(z)$ be analytic in a simply connected domain D.



→ Let C be any path joining z_0 and z_1 in D.

The path lies entirely in D.

Then, $\int_C f(z)\,dz$ is independent of the path C and depends only on the end points z_0 and z_1.

$$\int_C f(z)\,dz = \int_{z_0}^{z_1} f(z)\,dz$$

Ex: $\int_C z^2\,dz = \frac{-1}{3}(11+2i)$

C: (i) Straight line joining $(0,0)$ and $(1,2)$

(ii) Straight line path from $(0,0)$ to $(1,0)$ followed by the straight line from $(1,0)$ to $(1,2)$

(iii) the parabolic path $y = 2x^2$

Sol: $C: z(t) = t + i2t$, $z^2(t) = t^2(-3+i4)$

$z'(t) = 1+2i$ $\qquad 0 \le t \le 1$

$$\int_C z^2 dz = \int_0^1 t^2(-3+4i)\cdot(1+2i)\, dt$$

$$= (-3-6i+4i-8)\int_0^1 t^2\, dt$$

$$= (-11-2i)\frac{1}{3}$$

$$= -\frac{1}{3}(11+2i)$$ //

(ii) $C_1 : z(t) = t$ & $C_2 : z(t) = 1+i2t$

$z^2(t) = t^2$ $(0 \le t \le 1)$ $z^2(t) = (1-4t^2+i4t)$

$$\int_C z^2 dz = \int_{C_1} z^2 dz + \int_{C_2} z^2 dz$$

$$= \int_0^1 t^2(1)\, dt + \int_0^1 (1-4t^2+i4t)(2i)\, dt$$

$$= \frac{1}{3} + 2i\int_0^1 dt - 8i\int_0^1 t^2 dt - 8\int_0^1 t\, dt$$

$$= \frac{1}{3} + 2i - \frac{8i}{3} - \frac{8}{2}$$

$$= \frac{1}{3} - 4 + i\left(2-\frac{8}{3}\right) = -\frac{1}{3}(11+2i)$$

$$= -\frac{1}{3}(11+2i)$$ //

(iii) C : $z(t) = t + i2t^2$; $0 \le t \le 1$

$z^2(t) = (t^2 - 4t^4 + i4t^3)$

$$\int_C z^2 dz = \int_0^1 (t^2 - 4t^4 + i4t^3)(1 + i4t)\,dt$$

$$= \int_0^1 (t^2 - 4t^4 + i4t^3)\,dt + 4i \int_0^1 (t^3 - 4t^5 + i4t^4)\,dt$$

$$= \int_0^1 t^2 - 20\int_0^1 t^4\,dt - 16i\int_0^1 t^5\,dt + 8i\int_0^1 t^3\,dt$$

$$= \frac{1}{3} - \frac{20}{5} - \frac{16i}{6} + \frac{8i}{4} = -\frac{55}{15} + i\left(2 - \frac{8}{3}\right)$$

$$= -\frac{1}{3}(11 + 2i) \quad //$$

$\therefore$ Here, $f(z) = z^2$ is an analytic function

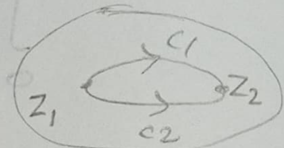
$\therefore$ It is independent of path 'C'.

$$\boxed{\int_C z^2 dz = -\frac{1}{3}(11 + 2i)}$$

02/09/[illegible]

Deformation of Path

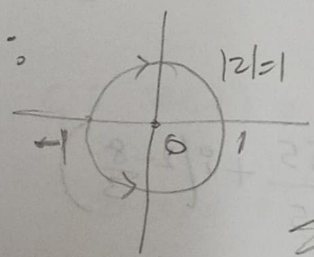
→ Let $z_1, z_2 \in D$, Consider C_1 & C_2 in D joining z_1 and z_2.



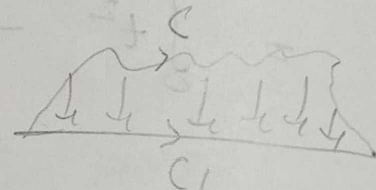
Continuous deformation

→ no point where $f(z)$ is not analytic appears in between the continuous deformation

Ex:



Ⓐ $\int_{C_1} \frac{1}{z} dz \neq \int_{C_2} \frac{1}{z} dz$



Ⓑ $\int_{C_1} \frac{1}{z+2} dz = \int_{C_2} \frac{1}{z+2} dz$

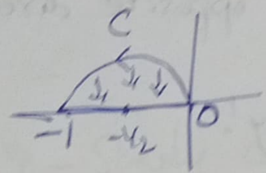
∵ in Ⓐ there is a point $(0,0)$ where $f(z) = \frac{1}{z}$ is not analytic.

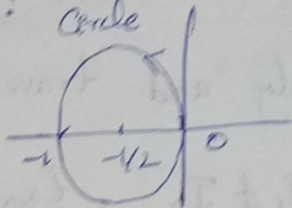
in Ⓑ, $f(z) = \frac{1}{z+2}$ is not analytic at $z=-2$

and limits : -1 to 1 (-2) not included //

Ex: Evaluate $\int_C \frac{z}{1+z^2}\,dz$ where C is the upper semi-circle of $|z+\frac{1}{2}| = \frac{1}{2}$ traversed counter clockwise, using deformation curve!

Sol.





C_1 : Line joining $(0,0)$ and $(-1,0)$

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz \qquad z = x+iy\,;$$

along C_1 : $z = x$

$$= \int_0^{-1} \frac{x}{1+x^2}\,dx$$

$$= \left[\frac{1}{2}\ln(1+x^2)\right]_0^{-1}$$

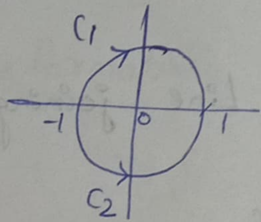
$$= \frac{1}{2}\ln(2) - \frac{1}{2}\ln(1)$$

$$= \frac{1}{2}\ln(2)$$

$$\boxed{\therefore \int_C \frac{z}{1+z^2}\,dz = \frac{1}{2}\ln(2)}$$

H.W: Evaluate $I_1 = \int_{C_1} \frac{dz}{z}$ and $I_2 = \int_{C_2} \frac{dz}{z}$ where C_1 & C_2 are upper & lower semi-circle of $|z|=1$ respectively and traversed in opposite directions. Show $I_1 \neq I_2$. Explain why!

Sol:



$$I_1 = \int_{C_1} \frac{dz}{z} \qquad z = re^{i\theta}$$

$$= \int_{\pi}^{0} \frac{1}{r} e^{-i\theta} \cdot ri \cdot e^{i\theta} \, d\theta$$

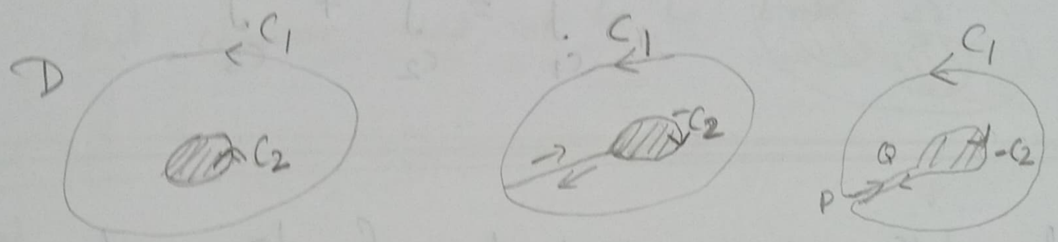
$$= i \int_{\pi}^{0} d\theta = -\pi i$$

$$I_2 = \int_{C_2} \frac{dz}{z} = \int_{\pi}^{2\pi} \frac{1}{r} e^{-i\theta} \cdot r i e^{i\theta} d\theta$$

$$= i \int_{\pi}^{2\pi} d\theta = \pi i$$

$\boxed{I_1 \neq I_2}$ $\because$ We cannot deform the path

Extension of CIT to Multiply connected domains



CIT $\rightarrow$ D - Simply connected

f - Analytic f' - continuous

$\oint_C f(z)\, dz = 0$ $\quad$ > C - Simple smooth closed curve

f - analytic

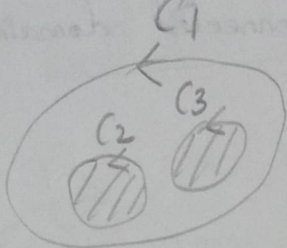
$C_1 \rightarrow P \rightarrow Q \rightarrow -C_2 \rightarrow Q \rightarrow P \rightarrow C_1$

Simple closed (pw) smooth curve in D.

$$\int_{C_1} + \cancel{\int_{PQ}} + \int_{-C_2} + \cancel{\int_{QP}} = 0$$

$$\int_{C_1} = \int_{C_2} \qquad \because \int_{-C_2} = -\int_{C_2}$$

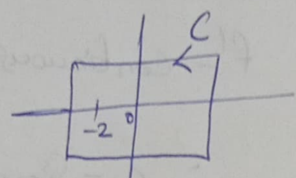
$\boxed{C_1 \text{ \& } C_2 \text{ Same direction}}$

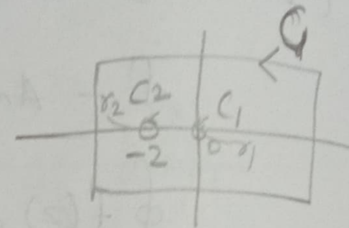


$$\int_{C_1} = \int_{C_2} + \int_{C_3}$$

Ex: Using the CIT, Evaluate $\oint \frac{dz}{z(z+2)}$ where

C :





Sol:

$$\int_C \frac{dz}{z(z+2)} = \int_C \frac{1}{2}\left[\frac{1}{z} - \frac{1}{z+2}\right] dz$$

$$\int_{|z|=r} z^n\,dz = \begin{cases} 2\pi i, & n=-1 \\ 0, & n\neq -1 \end{cases}$$

$||^{ly}$ $(z-z_0)^n \to |z-z_0|=r$

$$= \frac{1}{2}\int_{C_1} \frac{dz}{z} - \frac{1}{2}\int \frac{dz}{z+2}$$

$$+ \frac{1}{2}\int_{C_2} \frac{dz}{z} - \frac{1}{2}\int_{C_2} \frac{dz}{z+2}$$

[$\because$ $\int_{C_1} \frac{dz}{z+2} = 0$ $f(z) = z+2$ - analytic in C_1

$\int_{C_2} \frac{dz}{z} = 0$, $f(z) = z$ is analytic in C_2]

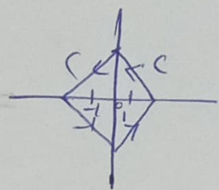
$$= \frac{1}{2}\left[2\pi i - 2\pi i\right] = 0 \;//$$

function - Analytic $\Rightarrow \int f = 0$

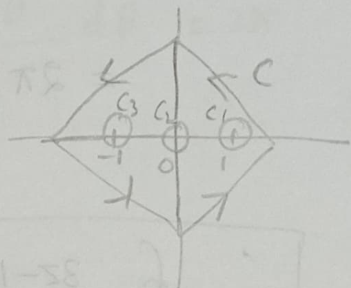
but converse need not to be true.

H.W

(1) Evaluate $\oint_C \frac{3z-1}{z^3-z}\,dz$; C :



Sol: f: not analytic at $z = -1, 0, 1$



$$\frac{3z-1}{z(z-1)(z+1)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z+1}$$

$$\Rightarrow 3z-1 = A(z^2-1) + B(z^2+z) + C(z^2-z)$$

$A+B+C=0$ | $B-C=3$ | $-A=-1$

$B+C=0$ | $\Rightarrow 2B=3$ | $A=1$

$B=3/2 \Rightarrow C=-3/2$

$$\oint_C \frac{3z-1}{z^3-z}\,dz = \int_C \frac{dz}{z} + \frac{3}{2}\int_C \frac{dz}{z-1} - \frac{3}{2}\int_C \frac{dz}{z+1}$$

$$= \int_{C_1} \frac{dz}{z} + \frac{3}{2}\int_{C_1} \frac{dz}{z-1} - \frac{3}{2}\int_{C_1} \frac{dz}{z+1} + \int_{C_2} \frac{dz}{z}$$

$$+ \frac{3}{2}\int_{C_2} \frac{dz}{z-1} - \frac{3}{2}\int_{C_2} \frac{dz}{z+1} + \int_{C_3} \frac{dz}{z} + \frac{3}{2}\int_{C_3} \frac{dz}{z-1}$$

$$- \frac{3}{2}\int_{C_3} \frac{dz}{z-1}$$

$\cancel{C_1 : |z|=1}$ $\quad$ $\cancel{C_2 : |z|=0}$

$$\int_C \frac{3z-1}{z^3-z^2}\,dz = \int_{C_2} \frac{dz}{z} + \frac{3}{2}\int_{C_1} \frac{dz}{z-1} - \frac{3}{2}\int_{C_3} \frac{dz}{z+1}$$

$$= 2\pi i + \frac{3}{2}(2\pi i) - \frac{3}{2}(2\pi i)$$

$$= 2\pi i \;//$$

$$\boxed{\therefore \oint_C \frac{3z-1}{z^3-z^2}\,dz = 2\pi i}$$

---

(2) By integrating $f(z) = \frac{1}{R-z}$, over $C : |z| = r$, $0 < r < R$, show that $\int_0^{2\pi} \frac{R\cos\theta}{R^2 - 2rR\cos\theta + r^2}\,d\theta = \frac{2\pi}{R^2-r^2}$

Sol: $f(z)$ - not analytic at $z = R$

$\int_C \frac{1}{R-z}\,dz$; $z = re^{i\theta}$ $\Rightarrow \int_C \frac{1}{R-z}\,dz = \int \frac{1}{R-re^{i\theta}} \cdot rie^{i\theta}\,d\theta$

$\therefore \int_C \frac{1}{R-z}\,dz = 2\pi i$

$= i\int \frac{re^{i\theta}}{R-re^{i\theta}}\,d\theta$

$$\Rightarrow i\int \frac{re^{i\theta}}{R-re^{i\theta}} \times \frac{R-re^{-i\theta}}{R-re^{-i\theta}} = 2\pi i$$

$$\Rightarrow \int \frac{Rre^{i\theta} - r^2}{R^2 - Rre^{i\theta} - Rre^{-i\theta} + r^2} d\theta = 2\pi$$

$$\Rightarrow \int \frac{Rr\cos\theta - r^2 + iRr\sin\theta}{R^2 - 2Rr\cos\theta + r^2} d\theta = 2\pi$$

$$\Rightarrow \int \frac{R\cos\theta\, d\theta}{R^2 - 2Rr\cos\theta + r^2} d\theta = \frac{2\pi r}{R^2 - r^2}$$

(or) $\cos\theta = \frac{z + 1/z}{2}$

$$\int_0^{2\pi} \frac{R\left(\frac{z+1/z}{2}\right)}{R^2 - 2rR\left(\frac{z+1/z}{2}\right) + r^2} \frac{dz}{iz}$$

$$= \frac{1}{i}\int \frac{Rz^2 + R}{R^2 - 2rRz - \frac{2rR}{z} + r^2} \frac{dz}{z^2}$$

$$= \frac{1}{i}\int \frac{R(1+z^2)}{R^2z - 2rRz^2 - 2rR + r^2} \frac{dz}{z}$$

(3) If $0 < r < R$, evaluate $\oint \frac{R+z}{z(R-z)} dz$, $C: |z| = r$ and deduce $\int_0^{2\pi} \frac{d\theta}{R^2 - 2Rr\cos\theta + r^2} = \frac{2\pi r}{R^2 - r^2}$

and $\int_0^{2\pi} \frac{\sin\theta \, d\theta}{R^2 - 2rR\cos\theta + r^2} = 0$

Sol: $f(z)$ is not analytic at $z = 0$, $z = R$

$$\int_C \frac{R+z}{z(R-z)} = \int_C \frac{1}{z} + \int_C \frac{2}{R-z} = \int_{|z|=r} \frac{1}{z} + \cancel{\int_{|z|=r} \frac{2}{R-z}}^{\,0}$$

by CIT

$$= 2\pi i \; //$$

$C: |z| = r \Rightarrow z = re^{i\theta}$

$$\int_0^{2\pi} \frac{R + re^{i\theta}}{re^{i\theta}(R - re^{i\theta})} \cdot (r \cdot i e^{i\theta}) \, d\theta = 2\pi i$$

$$\Rightarrow \int_0^{2\pi} \frac{R + re^{i\theta}}{R - re^{i\theta}} \, d\theta = 2\pi$$

$$\Rightarrow \int_0^{2\pi} \frac{R+re^{i\theta}}{R-re^{i\theta}} \times \frac{R-re^{-i\theta}}{R-re^{-i\theta}} \, d\theta = 2\pi$$

$$\Rightarrow \int_0^{2\pi} \frac{R^2 - Rre^{-i\theta} + Rre^{i\theta} - r^2}{R^2 - Rre^{-i\theta} - rRe^{i\theta} + r^2} \, d\theta = 2\pi$$

$$\Rightarrow \int_0^{2\pi} \frac{R^2 - r^2 + Rr(e^{i\theta} - e^{-i\theta})}{R^2 - 2Rr\cos\theta + r^2} \, d\theta = 2\pi$$

$$\Rightarrow \int_0^{2\pi} \frac{R^2 - r^2 + 2iRr\sin\theta}{R^2 - 2Rr\cos\theta + r^2} \, d\theta = 2\pi$$

$$\Rightarrow \int_0^{2\pi} \frac{d\theta}{R^2 - 2Rr\cos\theta + r^2} = \frac{2\pi r}{R^2 - r^2}$$

and $$\int_0^{2\pi} \frac{\sin\theta \, d\theta}{R^2 - 2rR\cos\theta + r^2} = 0 \quad //$$

Indefinite integral in the evaluation of line integrals:

→ $f(z)$ - analytic in a simply connected domain D

$z_0 \in D$ and z any other point in D.

Define $F(z) = \int_{z_0}^{z} f(z^*)\, dz^*$ → function of z

$F(z)$ analytic ; $F'(z) = f(z)$;

$$\int_{z_0}^{z_1} f(z)\, dz = F(z_1) - F(z_0)$$

Ex: $\int_{1}^{2-i} z\, dz = \left[\frac{z^2}{2}\right]_1^{2-i}$

(or) ↳ $f(z) = z$, $F(z) = \int z\, dz = \frac{z^2}{2}$

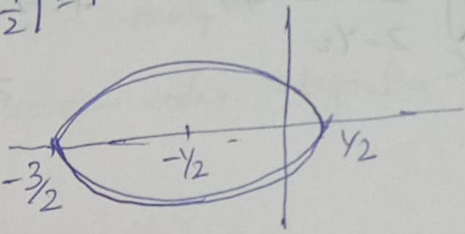
$$\int_{1}^{2-i} z\, dz = F(2-i) - F(1)$$

$\boxed{\text{CIT}}$

03/07/2021

Let $f(z)$ be an analytic fn. in a simply connected domain D. Let z_0 be any point in D and c be any simple closed curve in D enclosing z_0. Then $f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz$

Ex. $\oint_C \frac{e^z}{z+1}\, dz$ $\quad C : |z + \frac{1}{2}| = 1$

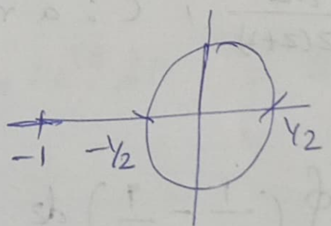


$f(z) = e^z$ - Analytic

$2\pi i \times f(-1) = \frac{2\pi i}{e}$

$\therefore \oint_C \frac{f(z)}{z - z_0}\, dz = \frac{2\pi i}{e}$

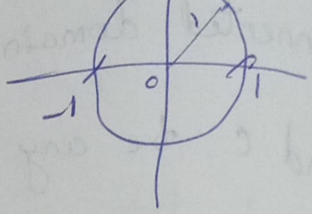
$C ; |z| = \frac{1}{2}$



$\oint_C \frac{e^z}{z+1}\, dz$. $\frac{e^z}{z+1}$ is analytic in C simple closed smooth curve

$\boxed{\therefore \oint_C \frac{e^z}{z+1}\, dz = 0}$ - CIT

Ex: $\oint_C \frac{dz}{2-\bar{z}}$; $C: |z|=1$



$$\frac{1}{2-\bar{z}} = \frac{z}{2z-z\bar{z}} = \frac{z}{2z-|z|^2} = \frac{z}{2z-1} = \frac{1}{2}\left[\frac{z}{z-1/2}\right]$$

$$\frac{1}{2}\oint_C \frac{z}{z-1/2}\,dz$$

$f(z) = z$ - analytic

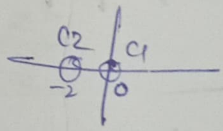
$z_0 = 1/2$ - lies in the area of integration

$$\therefore \frac{1}{2}\oint \frac{z}{z-1/2}\,dz = \frac{1}{2}\times 2\pi i \times f\left(\frac{1}{2}\right)$$

$$= \frac{\pi i}{2} \quad //$$

HW: Evaluate $\oint_C \frac{dz}{z(z+2)}$; C: a rectangle enclosing $z=0$ and $z=-2$

Sol: $\oint_C \frac{dz}{z(z+2)} = \oint_{C_1}\left(\frac{1}{z}-\frac{1}{z+2}\right)dz$



$+ \oint_{C_2}\left(\frac{1}{z}-\frac{1}{z+2}\right)dz$

$= \oint_{C_1}\frac{1}{z}\,dz + \oint_{C_2}\frac{-1}{z+2}\,dz$

(By CIT, $\oint_{C_1}\frac{-1}{z+2} = 0$, $\oint_{C_2}\frac{1}{z}\,dz = 0$)

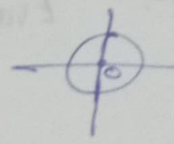
$= 2\pi i + (-2\pi i)$

$= 0$

Ex: Evaluate $\oint_{|z|=1} \frac{e^z}{z}\,dz$ and use it to

show $\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta)\,d\theta = 2\pi$ and

$\int_0^{2\pi} e^{\cos\theta} \sin(\sin\theta)\,d\theta = 0$



Sol $\oint_{|z|=1} \frac{e^z}{z}\,dz$; $f(z) = e^z$ - Analytic

$z = 0$ - under integration

$\therefore \oint_{|z|=1} \frac{e^z}{z}\,dz = 2\pi i \times f(0)$

$= 2\pi i$

$c: |z| = 1 \rightarrow z = e^{i\theta}$; $0 \le \theta \le 2\pi$

$\therefore \int_0^{2\pi} \frac{e^{e^{i\theta}}}{e^{i\theta}} \times i e^{i\theta}\,d\theta = 2\pi i$

$\Rightarrow \int_0^{2\pi} e^{e^{i\theta}}\,d\theta = 2\pi$

$\Rightarrow \int_0^{2\pi} e^{\cos\theta} \cdot e^{i\sin\theta}\,d\theta = 2\pi$

$\Rightarrow \int_0^{2\pi} e^{\cos\theta} \left[\cos(\sin\theta) + i\sin(\sin\theta)\right] d\theta = 2\pi$

$$\therefore \int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta)\, d\theta = 2\pi$$

$$\int_0^{2\pi} e^{\cos\theta} \sin(\sin\theta)\, d\theta = 0$$

Ex: Evaluate $\oint_C \frac{dz}{z^2+4}\, dz$ over

(i) $C: |z-2i|=1$ (ii) $C: |z+2i|=1$

(iii) $C: |z|=4$ (iv) C: a p.w smooth curve with initial point $z=0$ and terminal point $z=2$ and C does not pass through $\pm 2i$

(a) Parametrize
(b) CIF
(c) CIF for multiply conn

Sol: $z = \pm 2i$ – points

$$\oint_C \frac{dz}{z^2+4} = \oint_C \frac{dz}{(z+2i)(z-2i)}$$

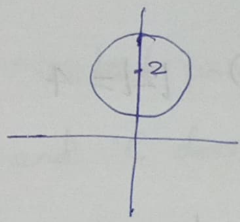
$C: |z-2i|=1$

$$= \frac{1}{4i} \oint_C \left(\frac{1}{z-2i} - \frac{1}{z+2i} \right) dz$$

$$= \frac{1}{4i} \int_C \frac{dz}{z-2i} - \frac{1}{4i} \int_C \frac{dz}{z+2i}$$

(or)

$$\oint_C \frac{dz}{z^2+4} = \int_C \frac{(1/z+2i)\,dz}{z-2i}$$



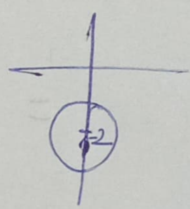
$$= 2\pi i \times f(2i) = 2\pi i \times \frac{1}{4i} = \frac{\pi}{2}$$

$$\left[\because \oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0) \right.$$

$$\left. \text{Here, } f(z) = \frac{1}{z+2i} \right]$$

$$\therefore \oint_C \frac{dz}{z^2+4} = \frac{\pi}{2} \;; \quad C : |z-2i| = 1$$

(ii) $C : |z+2i| = 1$



$$\oint_C \frac{dz}{z^2+4} = \oint_C \frac{(1/z-2i)\,dz}{z+2i}$$

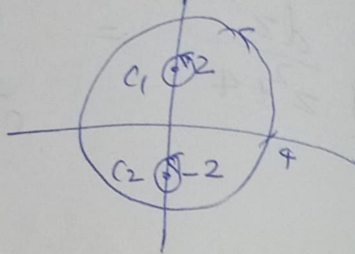
$$= 2\pi i \times f(-2i) \;; \quad f(z) = \frac{1}{z-2i}$$

$$= 2\pi i \times \frac{1}{-4i}$$

$$= -\frac{\pi}{2} \;//$$

$$\therefore \oint_C \frac{dz}{z^2+4} = \frac{-\pi}{2} \;; \quad C : |z+2i| = 1$$

(iii) $|z| = 4$



$$\oint_C \frac{dz}{z^2+4} = \int_C \frac{dz}{(z+2i)(z-2i)}$$

$$= \frac{1}{4i}\int_C \frac{dz}{z-2i} - \frac{1}{4i}\int_C \frac{dz}{z+2i}$$

(By CI Theorem)

$$= \frac{1}{4i}\int_{C_1} \frac{dz}{z-2i} - \frac{1}{4i}\int_{C_1} \frac{dz}{z+2i} + \frac{1}{4i}\int_{C_2} \frac{dz}{z-2i}$$

$$- \frac{1}{4i}\int_{C_2} \frac{dz}{z+2i}$$

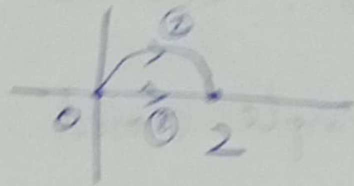
$$= \frac{1}{4i}\int_{C_1} \frac{dz}{z-2i} - \frac{1}{4i}\int_{C_2} \frac{dz}{z+2i}$$

$$= \frac{1}{4i} \times 2\pi i - \frac{1}{4i} \times 2\pi i$$

$= 0$ //

$$\therefore \oint_C \frac{dz}{z^2+4} = 0 \quad ; \quad C: |z| = 4$$

(IV) C : a p.w. smooth curve with initial point $z=0$ and terminal point $z=2$ and C does not pass through $\pm 2i$



$$|z-1| = 1$$

② $$\oint_C \frac{dz}{z^2+4} = \int_0^{\pi} \cdots$$

(or) ①

$$\oint_C \frac{dz}{z^2+4} = \int_0^2 \frac{dz}{z^2+4} = \frac{1}{2} \tan^{-1}\left(\frac{z}{2}\right)\Big]_0^2$$

$$= \frac{1}{2} \tan^{-1}(1)$$

$$= \frac{\pi}{8} \,//$$

$\therefore \oint_C \frac{dz}{z^2+4} = \frac{\pi}{8}$; for given C: $z=0$ to $z=2$ not passing through $\pm 2i$

Cauchy Integral formula for derivatives:

→ Let $f(z)$ be analytic in a simply connected domain D. Let z_0 be any point in D and C be any simple closed curve in D enclosing $z = z_0$,

→ Then $f(z)$ has derivatives of all orders in D which ~~which~~ are also analytic

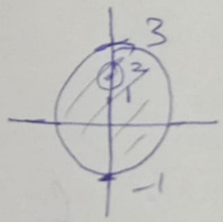
→ Further

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz \ ;$$

C : Traversed counter clockwise, $n \in \mathbb{N}$

Ex:

$$\oint_C \frac{dz}{(z^2+4)^2} \ ; \quad C : |z-i| = 2$$



$$= \oint_C \frac{\cancel{dz}/(z+2i)^2}{(z-2i)}\, dz$$

$$f(z) = \frac{dz}{z+2i}$$

$(n=1)$

$$= \frac{2\pi i}{1!} f'(2i)$$

$$= \frac{2\pi i}{1} \times \frac{1}{4i} = \frac{\pi}{2} //$$

Morera's Theorem (Converse of CIT)

If $f(z)$ is continuous on a simply connected domain D and if $\oint_C f(z)\,dz = 0$ for every simple smooth closed curve C in D, then $f(z)$ is analytic.

Cauchy Inequality:

Let $f(z)$ be analytic within and on $C: |z-z_0| = r$

Then $|f^{(n)}(z_0)| \leq \frac{M n!}{r^n}$, where $|f(z)| \leq M$ on C.

Ex: $f(z) = e^z$; $C: |z| = 1$ $\qquad |f^{(n)}(0)| \leq \frac{n!\,e}{1}$

$\boxed{C: |z| = 1}$ $|f(z)| \leq M$

$|e^z| = e^x \leq e^1$

$|f^{(n)}(0)| \leq e\,n!$

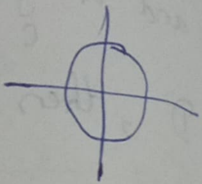
Maximum/Minimum Modulus theorem:

If $f(z)$ is non-constant and analytic within and on a simple closed curve C, then $|f(z)|$ assume its maximum/minimum on C.

Max/Min $f(z)$ occurs on the boundary C.

HW: Verify the above theorem for $f(z) = e^z$ on

$C: |z| = 1$



Sol: $|f(z)| \leq M$

$\Rightarrow |e^z| \leq M$

$|e^z| \leq e^x \Rightarrow \boxed{|e^z| \leq e^1}$ At $(1,0)$ is a boundary point where the function has max. Value

$|f(z)| \geq N$

$\Rightarrow |e^z| \geq N$

$\Rightarrow |e^z| \geq e^x$

$\Rightarrow \boxed{|e^z| \geq \frac{1}{e}}$ At $(-1,0)$ is a boundary point where function has min. value.

$\boxed{-1 \leq x \leq 1}$

Hence, Proved.

Ex: $f(z) = z^2 + 1 = x^2 - y^2 + 1 + i2xy$ on $|z| = 1$

$|f(z)| = \sqrt{(x^2 - y^2 + 1) + 4x^2y^2}$

$\Rightarrow \max|f(z)| \rightarrow \max|f(z)|^2$

max. of $g(x,y) = (x^2-y^2+1)+4x^2y^2$

$g_x = 0 \Rightarrow 2x + 8xy^2 = 0$

$\Rightarrow 2x(1+4y^2) = 0$

$x = 0$ (or) $y^2 = -\frac{1}{4} \Rightarrow y = \pm \frac{i}{2}$

$g_y = 0 \Rightarrow -2y + 8x^2y = 0$

$2y(-1+4x^2) = 0$

$y = 0$ (or) $x^2 = \frac{1}{4} \Rightarrow x = \pm \frac{1}{2}$

$g_{xx} = 2+8y^2$; $g_{yy} = -2+8x^2$

$(g_{xx}g_{yy} - g_{xy}^2 > 0)$

$g_{xy} = 16xy$; $(2+8y^2)(8x^2-2) - (16xy)^2$

$\Rightarrow 16x^2 - 16y^2 - 4$

$\Rightarrow 16(x+y)(x-y)+4$

At $\left(\pm\frac{1}{2}, \pm\frac{i}{2}\right) \Rightarrow 16x^2 - 16y^2 - 4 > 0$

$g_{xx} = 2 \pm \frac{8}{2} = -2 < 0 \Rightarrow \left(\pm\frac{1}{2}, \pm\frac{i}{2}\right)$ - Max. value

Liouville's Theorem:

If an entire function $f(z)$ is bounded $\forall z \in \mathbb{C}$

then $f(z)$ must be a constant.

Fundamental Theorem of Algebra: (Complex plane is algebraic

If $p(z)$ is a non-constant polynomial of degree n, then $\exists\ \xi \in \mathbb{C}$ such that $p(\xi) = 0$

(equivalently, $p(z)$ has exactly n roots)

07/09/202

Power Series, Taylor Series and Laurent Series:

→ An infinite series of the form $\sum\limits_{n=0}^{\infty} a_n (\underbrace{z-z_0}_{\text{Centre of Power Series}})^n$ is called a Power Series.

→ Every complex function $f(z)$ which is analytic in a domain can be represented as a power series in some circular region R about a point z_0.

R and z_0 lie inside D.

→ Conversely every power series represents an analytic function

$f_n : \mathbb{N} \to \mathbb{R}$ – Sequence

$1 \to a_1$

$2 \to a_2$

$n \to a_n$

– A function whose domain is $\mathbb{N}$ and co-domain $\mathbb{R}$

– Real Sequence.

Series Convergent: $\sum a_n = S$

Absolutely Convergent: $\sum |a_n| < \infty$

Ex: $e^x = \sum \frac{x^n}{n!} = \sum \frac{1}{n!}(x-0)^n$

Power Series:

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + a_n(z-z_0)^n + \cdots$$

a_n ↓ Finite number

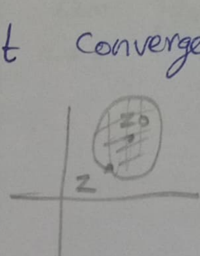
→ Power Series about $z = z_0$, a_n's are real / Complex constants

→ z_0 – Centre of the Power series.

→ Converges at $z = z_0$ $\left(\because \sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0\right)$

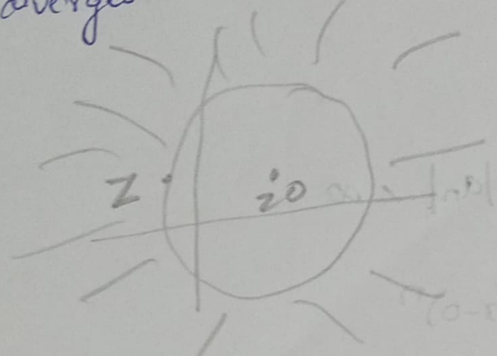
→ If $z_0 = 0$, we have $\sum_{n=0}^{\infty} a_n z^n$

Theorem: If the power series $\sum a_n (z-z_0)^n$ converges for som $z = z^* \ (\neq z_0)$, then it converges $\forall z$ in $|z-z_0| < |z^* - z_0|$

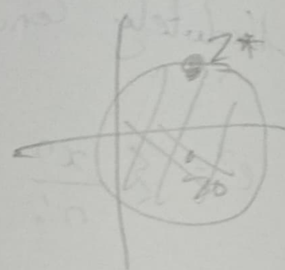


Theorem:

→ If the power series diverges for some z^*, then it diverges $\forall\ z$ such that $|z-z_0| > |z^*-z_0|$.

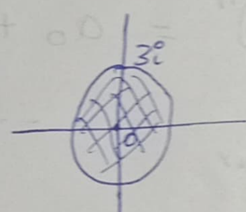


Divergent



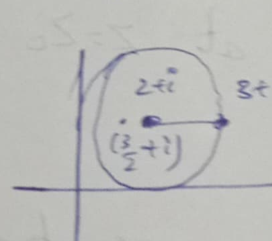
Ex: $\sum a_n z^n \to$ Series is convergent at $z = 3i$

Everywhere inside the disc, the power series is convergent.



Ex: $\sum a_n(z-(2+i))^n$ - Series is convergent at $z = (3+i)$

Within the circle, the power series is convergent everywhere

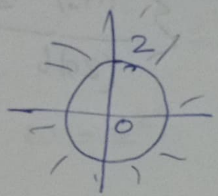


Is $z=$ … convergent

Yes, $\left(\frac{3}{2}+i\right)$ is convergent, $\because$ it lies inside the circle

Ex: $\sum a_n z^n$ - Series is divergent at 2

Outside the circle, everywhere the power series is divergent



Let R be the radius of the circle with centre at z_0 and contains all points where the series is convergent.

→ The Series is convergent for all z in $|z-z_0| < R$ and divergent $\forall$ z in $|z-z_0| > R$,

Then, R ($\in \mathbb{R}$) is called Radius of convergence and $|z-z_0| = R$ is called the Circle of Convergence

If $R = 0$, $\sum a_n (z-z_0)^n$ converges only at z_0

If $R = \infty$, converges $\forall\, z \in \mathbb{C}$

$\boxed{\sum a_n (z-z_0)^n}$

$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left(\frac{1}{|a_n|^{1/n}} \right)$

Interval of Convergence
↓
$[(x_0 - R, (x_0 + R)]$

Ratio Test: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ < 1 - cgt

> 1 - dgt

$= 1$ - test fails

Root Test:

$\lim_{n\to\infty} |a_n|^{1/n}$ < 1 - cgt

> 1 - dgt

$= 1$ test fails

Ex:

$$\sum \frac{(n!)^2 z^n}{(2n)!} \qquad a_n = \frac{(n!)^2}{(2n)!} \qquad \text{Centre} : (0,0)$$

$$R = \lim_{n\to\infty} \left| \frac{(n!)^2}{(2n)!} \times \frac{(2n+2)!}{((n+1)!)^2} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(2n+2)(2n+1)}{(n+1)^2} \right| = 4$$

Radius of Convergence $= 4$, Circle of Convergence; $|z| = 4$

Ex: $\sum \left(1+\frac{2}{n}\right)^{n^2} z^n$; $a_n = \left(1+\frac{2}{n}\right)^{n^2}$

$$R = \lim_{n\to\infty} \frac{1}{\left(1+\frac{2}{n}\right)^n} \qquad \because \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

$$= \lim_{n\to\infty} \frac{1}{\left[\left(1+\frac{1}{n/2}\right)^{n/2}\right]^2}$$

$$= e^{-2} \;//$$

$$\boxed{\therefore |z| = e^{-2}}$$

Ex: Show that $\sum \frac{z^{2n}}{4^n n^\alpha}$, $\alpha > 0$ is cgt. in $|z|<2$

08/09/2021

Sol: $b_n = \frac{z^{2n}}{4^n n^\alpha} \Rightarrow \sum b_n$ Let $w = z^2$

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{1}{4^n n^\alpha} \cdot \frac{4^{n+1}(n+1)^\alpha}{1}\right|$$

$$= \lim_{n\to\infty} \left|4 \cdot \left(1+\frac{1}{n}\right)^\alpha\right|$$

$$= 4$$

$|w| = 4 \Rightarrow |z^2| = 4 \Rightarrow \boxed{|z| = 2}$

Theorem: A power series $\sum a_n (z-z_0)^n$ represents an analytic function within its circle of convergence.

Ex: $\sum z^n$; $a_n = 1$; $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = 1 \Rightarrow |z| = 1$

$= 1 + z + z^2 + \ldots = \frac{1}{1-z}$ $\quad |z| < 1$

Ex: Find the radius of convergence of $\sum \frac{1}{n!}\left(\frac{iz-1}{2+i}\right)^n$ and express it as an analytic function in the region of convergence.

Sol:

Sol: $\sum \frac{1}{n!}\left(\frac{iz-1}{2+i}\right)^n = \sum \frac{1}{n!} \frac{i^n}{(2+i)^n}(z+i)^n$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{i^n}{n!\,(2+i)^n} \times \frac{(n+1)!\,(2+i)^{n+1}}{i^{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|(n+1)\left(\frac{2}{i}+1\right)\right|$$

$$R = \infty$$

(or)

$\because e^z = \sum \frac{z^n}{n!}$

$\Rightarrow \sum \frac{1}{n!}\left(\frac{iz-1}{2+i}\right)^n = e^{w}$ – is analytic at

$\Rightarrow R = \infty$

$\therefore \sum \frac{1}{n!}\left(\frac{iz-1}{2+i}\right)^n = e^{\frac{iz-1}{2+i}}$ – also analytic

$\boxed{\therefore R = \infty}$

$\therefore$ Its convergent at all

Theorem: A power series $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ can be integrated term by term within its circle of convergence i.e., $\int_C f(z)\,dz = \sum_{n=0}^{\infty} \int_C a_n (z-z_0)^n \,dz$

for every contour C lying within the circle of convergence.

→ The radius of convergence of the integrated series is same as radius of convergence of the original series.

Theorem: A power series can be differentiated term by term within its circle of convergence. The radius of convergence of the differentiated series :

$\sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$ is same as that of original series.

$f(x) = \sum a_n$

$\int f(x) = \int \sum a_n \neq \sum \int a_n$ (at all times)

but if $f(x)$ is power series

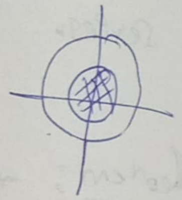
then $\int f(x) = \sum \int a_n$

When a_n is too complex, then we can differentiated for many times to simplify it.

→ Suppose two series $\sum a_n (z-z_0)^n$ and $\sum b_n (z-z_0)^n$ have the same radii of convergence R and converge to the same sum within the circle of convergence $|z-z_0| = R$; then the 2 Series are identical

* $f(z) = \sum a_n (z-z_0)^n$, $g(z) = \sum b_n (z-z_0)^n$

with radius of convergence R_1 & R_2



$$f(z) \pm g(z) = \sum a_n (z-z_0)^n \pm \sum b_n (z-z_0)^n$$

$$R : \min \{ R_1, R_2 \}$$

Cauchy Product:

$$f(z) \cdot g(z) = \sum_{n=0}^{\infty} \left(\sum_{r=0}^{n} a_r b_{n-r} \right) (z-z_0)^n$$

$$= \sum_{n=0}^{\infty} C_n (z-z_0)^n \quad ; \quad C_n = \sum_{r=0}^{n} a_r b_{n-r}$$

Function — Analytic – it has Taylor series

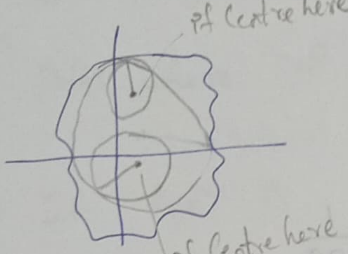
— not Analytic – it has Laurent Series

$p \to q$ Converse: $q \to p$
Contrapositive: $\sim q \to \sim p$

Taylor's Series:

* Every power series convergent in a circle of convergence define an analytic function.

Converse: Every analytic function can be written/represented as in the form of power series in the circle of convergence.



Radius - Centre to nearest point where function is divergent.

Theorem: A function $f(z)$ is analytic inside a circle $|z-z_0|=R$ may be represented inside the circle of convergence as a (unique) convergent power series

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n}, \quad \boxed{a_n = \frac{1}{n!} f^{(n)}(z_0)}$$

Ex: Expand $f(z) = \frac{1}{z}$ about $z=2$ in Taylor's Series. obtain the radius of convergence.

Sol: Given function, $f(z) = \frac{1}{z}$: $z=0 \to$ Laurent's Series
$\notin \{0\} \to$ Taylor's Series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad a_n = \frac{f^{(n)}(z_0)}{n!}$$

Here, $z_0 = 2$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-2)^n \quad ; \quad a_n = \frac{f^{(n)}(2)}{n!}$$

$$a_0 = \frac{f^{(0)}(2)}{0!} = f(2) = \frac{1}{2}$$

$$a_1 = \frac{f^{(1)}(2)}{1!} = f'(2) = \frac{-1}{2^2}$$

$$a_2 = \frac{f^{(2)}(2)}{2!} = \frac{f''(2)}{2!} = \frac{1}{2!} \times \frac{2}{2^3} = \frac{1}{2^3}$$

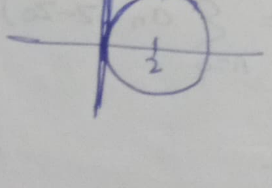
$$a_3 = \frac{f^{(3)}(2)}{3!} = \frac{f'''(2)}{3!} = \frac{1}{3!} \times \frac{-6}{2^4} = \frac{-1}{2^4}$$

$$\therefore \; a_n = (-1)^n \frac{1}{2^{n+1}}$$

$$\therefore \; \frac{1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (z-2)^n$$

$$\left(R = \lim_{n\to\infty} \left| \frac{(-1)^n}{2^{n+1}} \cdot \frac{2^{n+2}}{(-1)^{n+1}} \right| = 2 \right)$$

$$\Rightarrow \; |z-2| < 2$$



* If f is not analytic at z_0, then f cannot have a Taylor series expansion around z_0.

→ Suppose f is analytic everywhere except at $z_1, z_2, \ldots z_n$. Suppose $f(z)$ is expanded as a Taylor series around $z_0 (\neq z_i's)$. Then the radius of convergence = distance between z_0 and the nearest point where f is not analytic.

→ If $f(z)$ is analytic in a domain D and z_0 lies in D, the radius of convergence of the power series representation of $f(z)$ around z_0 is not less than the distance of z_0 from the boundary of D.

→ $M^* = \max |f(z)|$ on $C_r : |z - z_0| = r$, then

$$|a_n| \leq \frac{M^*}{r^n}$$

→ $z_0 = 0$, $f(z) = \sum a_n z^n$ is called as <u>Maclaurin's Series</u>

→ Suppose $f(z)$ and $g(z)$ are analytic at $z = z_0$ and $f(z_0) = g(z_0) = 0$. Suppose $g'(z_0) \neq 0$, then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

(Prove using Taylor series expansion)

Proof

Consider, $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ & $g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$

Given, $f(z_0) = g(z_0) = 0 \Rightarrow a_0 = 0$ & $b_0 = 0$

$$\therefore \lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots}{b_0 + b_1(z-z_0) + b_2(z-z_0)^2 + \cdots}$$

(By applying L-H Rule)
$$= \lim_{z \to z_0} \frac{a_1 + 2a_2(z-z_0) + \cdots}{b_1 + 2b_2(z-z_0) + \cdots}$$

$$= \lim_{z \to z_0} \frac{f'(z)}{g'(z)}$$

$$= \frac{f'(z_0)}{g'(z_0)} \quad //$$

Hence, Proved

(Only $|x| < 1$)

$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$(1-x)^{-2} = 1 + 2x + \frac{2.3}{2!}x^2 + \frac{2.3.4}{3!}x^3 + \cdots$$

$$= 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} (n+1)x^n$$

Q. $f(z) = \frac{1}{z}$ about $z = 2$

Sol. $f(z) = \frac{1}{(z-2)+2} = \frac{1}{2\left[1+\frac{z-2}{2}\right]}$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} (-1)^n \left(\frac{z-2}{2}\right)^n\right]$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(z-2)^n}{2^{n+1}} \quad //$$

H.W Expand e^z about $z=0$ & $z=2$ and obtain the radius of convergence in each case

Sol $f(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!}(z-0)^n$

(i)

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{1}{n!} \times \frac{(n+1)!}{1}\right| = \infty$$

(ii) $f(z) = e^z = \sum_{n=0}^{\infty} \frac{1}{n!}(z-2+2)^n$

?

Q $f(z) = \dfrac{1}{z^2+(1+2i)z+2i}$ about $z=0$.

obtain radius of convergence.

Sol. $z^2+(1+2i)z+2i = 0$

$$z = \frac{-(1+2i) \pm \sqrt{(1+2i)^2 - 4(2i)}}{2}$$

$$z = \frac{-(1+2i) \pm (1-2i)}{2} = -2i, -1$$

$\therefore f(z) = \dfrac{1}{(z+2i)(z+1)}$

$$= \frac{1}{1-2i}\left[\frac{1}{z+2i} - \frac{1}{z+1}\right]$$

$$= \frac{1}{1-2i}\left[\frac{1}{2i}\cdot\frac{1}{1+\frac{z}{2i}} - \frac{1}{1+z}\right]$$

$$= \frac{1}{1-2i} \times \frac{1}{2i} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{2i}\right)^n - \frac{1}{1-2i}\sum_{n=0}^{\infty} (-1)^n z^n$$

$$\boxed{\begin{array}{l} |z| < R_1 \\ |z| < R_2 \\ \therefore |z| < \min\{R_1, R_2\} \end{array}}$$

$$R_1 = \lim_{n\to\infty} \left| \frac{(-1)^n}{(2i)^n} \times \frac{(2i)^{n+1}}{(-1)^{n+1}} \right| = 2$$

$$R_2 = \lim_{n\to\infty} \left| \frac{(-1)^n}{1} \times \frac{1}{(-1)^{n+1}} \right| = 1$$

$R = \min\{R_1, R_2\} = \min\{2, 1\} = 1$ //

Laurent Series:

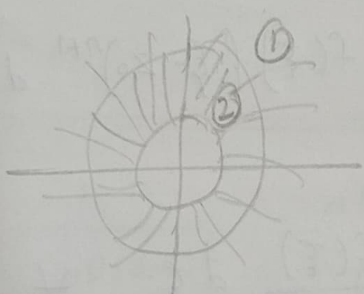
→ If $f(z)$ is not analytic at z_0, then we can express $f(z)$ about z_0 in a series which contains both positive and negative powers of $(z-z_0)$. This series is called Laurent's Series.

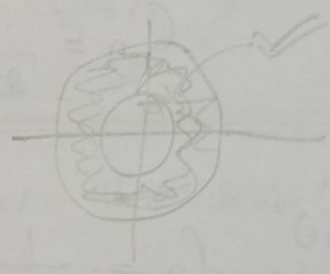
$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z-z_0)^n = \underbrace{\sum_{n=0}^{\infty} C_n (z-z_0)^n}_{1} + \underbrace{\sum_{n=1}^{\infty} \frac{C_{-n}}{(z-z_0)^n}}_{2}$$

z_0 - fixed

→ Region of convergence is the intersection of region of convergence of each sum.

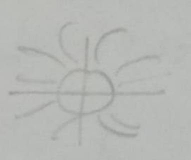
$r < |z-z_0| < R$





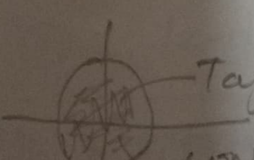
→ 1 circle convergent inside circle, 2 circle convergent outside circle everywhere

∴ Laurent series convergent within annular region.

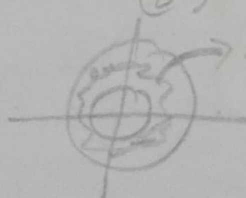


convergent outside circle

(or)



Taylor Series convergent



Laurent series convergent

Theorem: Let $f(z)$ be analytic inside the annular $r < |z - z_0| < R$ and on the circle $C_1 : |z - z_0| = r$ and $C_2 : |z - z_0| = R$. Then $f(z)$ can be represented as a Laurent Series.

$$\rightarrow f(z) = C_0 + C_1(z - z_0) + \cdots + \frac{C_{-1}}{z - z_0} + \frac{C_{-2}}{(z - z_0)^2} + \cdots$$

$$= \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n$$

where, $C_n = \dfrac{1}{2\pi i} \oint_C \dfrac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi$;

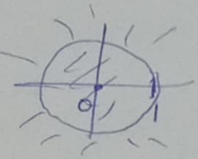
$$C_{-n} = \frac{1}{2\pi i} \oint_C f(\xi)(\xi - z_0)^{n+1} \, d\xi$$

(or) $C_n = \dfrac{1}{2\pi i} \oint_C \dfrac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi \quad \forall n \in \mathbb{Z}$

$C_1, C_2, C : |z - z_0| = \rho$, $r < \rho < R$ are traversed counter clockwise.

Q. Find all possible Taylor's & Laurent series expansion of $f(z) = \frac{1}{1-z}$ about $z=0$

Sol:



$|z| < 1$ – Taylor's Series

$|z| > 1$ – Laurent Series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad ; \quad |z| < 1 \text{ — Taylor's Series}$$

— Laurent Series

$$\frac{1}{1-z} = \frac{-1}{z}\cdot\left(\frac{1}{1-\frac{1}{z}}\right) = \frac{-1}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \quad ; \quad \frac{1}{|z|} < 1$$

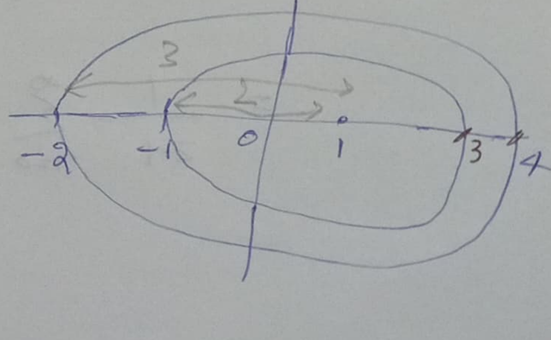
$$= -\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} \quad ; \quad |z| > 1$$

Q. $f(z) = \frac{1}{(z+1)(z+2)^2}$ about $z=1$ Give Series!

(not analytic at $z = -1, -2$)

Sol: $f(z) = \frac{1}{3}\frac{1}{z+1} - \frac{1}{3}\frac{1}{z+2} + \frac{1}{3}\frac{1}{(z+2)^2}$

$$= \frac{1}{3}\left[\frac{1}{(z-1)+2} - \frac{1}{(z-1)+3} + \frac{1}{[(z-1)+3]^2}\right]$$



(1) $|z-1| < 2$

(2) $2 < |z-1| < 3$

(3) $|z-1| > 3$

(1) $|z-1|<1$ — only (+ve) po

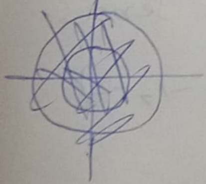
$$\frac{1}{(z-1)+2} = \frac{1}{2\left(1+\frac{z-1}{2}\right)} \qquad \left(|z-1| < 2 \Rightarrow \left|\frac{z-1}{2}\right| < 1 \right)$$

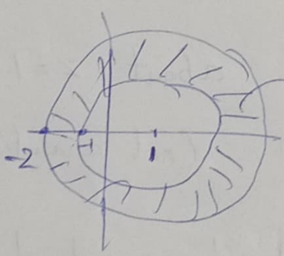
$$= \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \left(\frac{z-1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n (z-1)^n}{2^{n+1}}$$

$$\frac{1}{(z-1)+3} = \frac{1}{3\left(1+\frac{z-1}{3}\right)} = \frac{1}{3}\sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^n}{3^n} \qquad \left(\because |z-1| < 2 < 3 \quad \frac{|z-1|}{3} < 1 \right)$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^n}{3^{n+1}}$$

$$\frac{1}{[(z-1)+3]^2} = \frac{1}{9\left[1+\left(\frac{z-1}{3}\right)\right]^2} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1) [z-1]^n}{3^{n+2}}$$





annular region
— analytic
— have both +ve & -ve powers

$3 > |z-1| > 2$ (2) — both +ve & -ve powers

$$\frac{1}{(z-1)+2} = \frac{1}{(z-1)\left(1+\frac{2}{z-1}\right)} \qquad \left(|z-1| > 2 \Rightarrow \frac{2}{|z-1|} < 1 \right)$$

$$= \frac{1}{z-1}\sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{z-1}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(z-1)^{n+1}}$$

$$\frac{1}{(z-1)+3} = \frac{1}{3\left(1+\frac{z-1}{3}\right)} \qquad \left(|z-1| < 3 \Rightarrow \left|\frac{z-1}{3}\right| < 1 \right)$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^n}{3^{n+1}}$$

$$\frac{1}{((z-1)+3)^2} = \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^n}{3^{n+2}}$$

---

(3) $|z-1| > 3$ — fn. is analytic — only (-ve) powers

$|z-1| > 3 > 2$

$|z-1| > 2$

$\Rightarrow \frac{2}{|z-1|} < 1$

$\Rightarrow \frac{3}{|z-1|} < 1$

$$\therefore \frac{1}{(z-1)+2} = \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{(z-1)^{n+1}}$$

$$\frac{1}{(z-1)+3} = \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{(z-1)^{n+1}}$$

$$\frac{1}{[(z-1)+3]^2} = \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{(z-1)^{n+2}}$$

Zeros, Singularities

15/09/2

Zeros: Let $f(z)$ be a complex fn. $z = z_0$ is a zero of $f(z)$ if $f(z)$ is analytic at $z = z_0$ and $f(z_0) = 0$

→ If $f'(z_0) = 0 = f''(z_0) = \cdots\cdots = f^{(m-1)}(z_0)$ and $f^{(m)}(z_0) \neq 0$, then z_0 is a zero of order m.

$m = 1$ - simple zero

Order of the zero = order of the 1st non-vanishing derivative

---

Q $f(z) = \sin z$

Sol: $f(z)$ is analytic, $f(z) = 0 \Rightarrow$ 'z' values are zeroes of the function

$\sin z = 0$

$\Rightarrow \boxed{z = n\pi}, n \in \mathbb{Z}$

$f'(n\pi) \neq 0 \Rightarrow$ order of zero $= 1$

$\Rightarrow$ Simple zeroes

Q $f(z) = (1-e^z)^2$

Sol fn. is analytic

$f(z) = 0 \Rightarrow (1-e^z)^2 = 0$

$\Rightarrow 1 - e^z = 0$

$\Rightarrow e^z = 1 = e^{2n\pi i}$

$\Rightarrow \boxed{z = 2n\pi i}$, $n \in \mathbb{Z}$

$f'(z) = 2(1-e^z)$ & $f'(2n\pi i) = 0$

$f''(z) = -2e^z$ & $f''(2n\pi i) \neq 0$

$\therefore$ Order of zeroes $= 2$

Suppose $z = z_0$ is a zero of order m

Then f is analytic at z_0 and so $f(z)$ has a Taylor series expansion around z_0

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots$$

$$\cdots = \sum_{n=0}^{\infty} a_n (\cancel{z} - z_0)^2$$

$$\left[a_n = \frac{f^n(z_0)}{n!}\right]$$

$$= 0 + 0 + 0 + \cdots + \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m$$

$\Rightarrow f(z) = (z-z_0)^m \left[g(z) \right]$; $g(z_0) \neq 0$

$g(z)$ - analytic function

$\downarrow$

order of zero $= m$

$\rightarrow$ When $f(z)$ has a zero at $z = z_0$ but no other zero in some nbhd of z_0, z_0 is said to be an Isolated zero.

$\rightarrow$ If every neighbourhood of z_0 has another zero of $f(z)$, then $z = z_0$ is a non-isolated zero.

Singular Points:

$\rightarrow$ $z = z_0$ is a singular point of $f(z)$, if $f(z)$ is not defined at z_0 or $f(z)$ is not analytic at z_0.

where, $\dfrac{1}{f(z)} = 0$

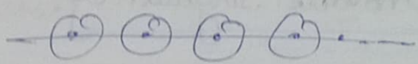
Ex: $f(z) = \dfrac{P(z)}{Q(z)}$; Zeros : $P(z) = 0$

Singularities : $Q(z) = 0$ and $P(z) \neq 0$

Isolated - Singular Point :  - no other singular point in its nbhd.

Non-Isolated Singular  - have other singular points in its nbhd.

Ex: $f(z) = \frac{1}{e^z - 1}$ $\qquad e^z - 1 = 0 \Rightarrow z = 2n\pi i$



Isolated - Singular points: $z = 2n\pi i$

Ex: $f(z) = \tan\left(\frac{1}{z}\right) = \frac{\sin(\frac{1}{z})}{\cos(\frac{1}{z})}$; $\cos\left(\frac{1}{z}\right) = 0$

$\Rightarrow z = \frac{1}{(2n+1)} \cdot \frac{2}{\pi}$

$\lim_{n \to \infty} \frac{1}{(2n+1)} \cdot \frac{2}{\pi} = 0$ $\qquad$ # $\tan\left(\frac{1}{z}\right)$ is not defined at $z = 0$

$\Rightarrow$ in nbhd of "0" there are infinite numbers.

"0" - non-isolated - Singular point.

17/09/2021

→ Laurent series expansion can be used to identify isolated singular points.

Suppose, z_0 is an isolated singular point, then $\exists$ a nbhd $0 < |z - z_0| < R$ s.t. $f(z)$ is analytic in this deleted nbhd.

→ So, we can expand $f(z)$ as a Laurent's series

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z - z_0)^n}_{\text{Analytic part}} + \underbrace{\sum_{n=1}^{\infty} a_{-n} \frac{1}{(z - z_0)^n}}_{\text{Principal part}}$$

Removable Singularity:

→ Suppose $f(z)$ is not defined at $z = z_0$, but $\lim_{z \to z_0} f(z)$ exists, then z_0 is a Removable Singularity.

→ Principal part of Laurent Series - 0

and $\lim_{z \to z_0} f(z) = a_0$

→ If we define $f(z_0) = a_0$, then $f(z)$ is an analytic fn.

Ex: $f(z) = \begin{cases} \frac{\sin z}{z} & ; z \neq 0 \\ 3 & ; z = 0 \end{cases}$

$\lim_{z \to z_0} f(z) = f(z_0)$ → $\lim_{z \to 0} f(z) = 1 \neq 3 = f(0)$

∴ $f(z)$ is not continuous

$z = 0$ is a Removable singularity.

Pole: If the principal part of the Laurent series expansion has finitely many terms, then $f(z)$ is said to have a pole at z_0.

$\frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}$ — Pole of order "m"

Thus $z = z_0$ is a pole if $\lim_{z \to z_0} f(z) = \infty$ and

$\lim_{z \to z_0} (z-z_0)^k f(z)$ exists $\forall\ k \geq 1$

→ The smallest value of "k" s.t. $\lim_{z \to z_0} (z-z_0)^k f(z)$

exists is the order of the pole z_0.

Ex: $f(z) = \frac{4}{(z-2)^5}$; pole : $z = 2$

Order : $m = 5$

$f(z) = \frac{g(z)}{(z-z_0)^m}$; Here $g(z) = 4$

↳ Analytic ⇒ Taylor Series expansion.

$g(z) = 4 = \sum a_n (z-z_0)^n \longrightarrow a_n = \frac{f^{(n)}(z_0)}{n!}$

Essential Singularity:

→ If the principal part of the Laurent Series expansion has infinitely many terms, then $z = z_0$ is said to be an essential Singular point.

Ex: $e^{1/z} = 1 + \frac{1}{z}\cdot\frac{1}{1!} + \frac{1}{z^2}\cdot\frac{1}{2!} + \cdots + \frac{1}{z^n}\cdot\frac{1}{n!} + \cdots$

- Essential Singularity : $z = 0$

$z = \infty$ is a singular point of $f(z)$ if

$z = 0$ is a singular point of $f(\frac{1}{z})$

Ex:

$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

$z = \infty$ is singular point of e^z if $z = 0$ is a singular point of $f(\frac{1}{z}) = e^{1/z}$

Proof: $e^{1/z}$ has $z = 0$ as singular point

$\Rightarrow$ e^z has $z = \infty$ as Singular Point

$f(z)$ has a zero at $z = z_0$ of order "n"

$\Rightarrow \frac{1}{f(z)}$ has a pole of order "n"

$f(z)$ has a zero of order "m", then $f^2(z)$ has zero of order "$2m$"

* $f(z)$ has a zero of order m, then

$f^2(z)$ has zero of order "$2m$"

$f'(z)$ " " " " "$(m-1)$"

$f^{(n)}(z)$ " " " " "$(m-n)$"

$f^n(z)$ " " " " "nm"

→ A function which is analytic everywhere except at a finite no. of poles is called a Meromorphic function

(Holomorphic function / Analytic function)

→ Entire function - function analytic on $\mathbb{C}$

Liouville's Thm: If an entire fun. $f(z)$ is bounded $\forall z \in \mathbb{C}$ then $f(z)$ must be a constant.

→ "Bounded entire function is a constant"

Ex: $f(z) = \int_0^z e^{\xi} \sin^3 \xi \, d\xi$ has a zero of order 4 at $z=0$

Prove !

Proof: $f'(z) = e^z \sin^3 z \Rightarrow f'(0) = 0$

$f''(z) = e^z \sin^3 z + 3\sin^2 z \cos z e^z \Rightarrow f''(0) = 0$

∴ It has zero of order 4 at $z=0$

$f'''(z) = e^z \sin^3 z + 6\sin^2 z \cos z e^z + 6 \sin z \cos z e^z - 3 \sin^3 z e^z$

$\Rightarrow f'''(0) = 0$

$f^{IV}(z) = e^z \sin^3 z + 12 \sin^2 z \cos z e^z \ldots + 6\cos^2 z e^z \ldots \Rightarrow f^{IV}(0) \neq 0$

Q) Classify the singular points of $f(z) = \frac{z^2 + iz + 2}{(z^2+1)^2(z+3)}$

Sol:

Singular Points : $\pm i$, 3

$$f(z) = \frac{z^2 + iz + 2}{(z+i)^2(z-i)^2(z+3)}$$

$\boxed{-3}$

$\lim_{z \to -3} f(z) = \infty$ & $\lim_{z \to -3} (z+3) f(z)$ exists.

$f(z) = \frac{g(z)}{z+3}$; $g(z) = \frac{z^2 + iz + 2}{(z^2+1)^2}$

$g(z) \Rightarrow$ Taylor Series Expansion : $g(-3) + \frac{g'(-3)}{1!}(z+3) + \frac{g''(-3)}{2!}(z+3)^2$

$+ \ldots\ldots$

Here, $g(-3) \neq 0$ $\Rightarrow$ Laurent's Series have only one negative powers of $(z+3)$

$\Rightarrow$ $\boxed{\text{order} = 1}$

By looking at the function, we think $z = \pm i$ are poles of order 2 ? $\longrightarrow$

$\boxed{z=i}$ $\lim_{z\to i} f(z) = \infty \Rightarrow \lim_{z\to i} (z+i) f(z)$

$$= \lim_{z\to i} \frac{z^2+iz+2}{(z+i)^2(z-i)(z+3)}$$

$$= \lim_{z\to i} \frac{(z-i)(z+2i)}{(z+i)^2(z-i)(z+3)}$$

$$= \lim_{z\to i} \frac{z+2i}{(z+i)^2(z+3)} \quad \text{- exists}$$

$\therefore$ $z=i$ is pole order of "1"

$$f(z) = \frac{g(z)}{(z-i)} \qquad g(z) = \frac{(z+i)(z+2i)}{(z+i)^2(z-i)(z+3)} = \frac{z+2i}{(z+i)^2(z+3)}$$

taylor Series expansion: $g(i) + \frac{g'(i)}{1!}(z-i) + \frac{g''(i)}{2!}(z-i)^2 + \cdots$

Here, $g(i) \neq 0 \Rightarrow$ order $= 1$

$\boxed{z=-i}$ $\Rightarrow$ Pole of order 2

$\lim_{z\to -i} f(z) = \infty \Rightarrow \lim_{z\to -i} (z+i)^2 f(z)$ exists

$\therefore$ $z=-i$ is pole of order 2

Ex: $\dfrac{e^z}{z+\sin z}$ Singularities of given function?

Sol: $f(z)$ - singularities $\Leftrightarrow$ $f\left(\frac{1}{z}\right)$ - zeroes

$$\frac{z+\sin z}{e^z}$$

$\Rightarrow$ $z+\sin z = z + \dfrac{z}{1!} - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dfrac{z^7}{7!} + \cdots$

$$= z\left(2 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right)$$

$\boxed{f(z) = (z-z_0)^m \, g(z) \; ; \; g(z_0) \neq 0}$ $\Rightarrow$ $z = z_0$ is zero of order m

$\therefore$ $z=0$ is the zero of order 1 for $\dfrac{z+\sin z}{e^z}$

$\therefore$ $z=0$ is the pole of order 1 for $\dfrac{e^z}{z+\sin z}$

\# $z=1$ - order of pole - Simple Pole

Residues:

→ Cauchy Integral Theorem: " $\oint_c f(z) = 0$ ";

c - simple closed curve, simply connected domain, analytic fn.

* When the fn. is not analytic at certain points, then CIT cannot be applied.

→ Each of the isolated singular points inside c contributes to the value of the complex integral, these are called "Residues."

Residue of an analytic fn. at an isolated singular point:

* Coefficient of $\frac{1}{z-z_0}$ i.e. b_1 - $\underset{z=z_0}{\text{Res}} f(z)$

→ Residue at a removable singular point: Laurent's series does not have negative powers $\Rightarrow b_1 = 0$

→ Residue at a simple pole:

$$b_1 = \lim_{z \to z_0} (z-z_0) f(z)$$

$$f(z) = \frac{g(z)}{h(z)} \; ; \; b_1 = \frac{g(z_0)}{h'(z_0)} \; ; \; h'(z_0) \neq 0$$

Residue at a pole of order m

$$\rightarrow b_1 = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left((z-z_0)^m . f(z) \right).$$

Residue at an isolated essential singular point

→ From the Laurent's Series

Residue at $z = \infty$

$$\operatorname*{Res}_{z=\infty} f(z) = - \operatorname*{Res}_{w=0} g(w) \; ; \text{ where } g(w) = \frac{1}{w^2} f\left(\frac{1}{w}\right)$$

21/09/2021

Residues:

Simple pole : $\lim_{z \to z_0} (z-z_0) f(z)$

Pole of order m : $\frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z)$

Removable Singularity : 0

Essential Singularity : Laurent Series b_1

$z = \infty$: $- \lim_{w \to 0} g(w) \; ; \; g(w) = \frac{1}{w^2} f\left(\frac{1}{w}\right)$

$$f(z) = \frac{\phi(z)}{\psi(z)} \quad \underset{\substack{(\text{at} \\ \text{simple pole})}}{\text{Res}} f(z) = \frac{\phi(z_0)}{\psi'(z_0)}$$

Evaluation of Contour integrals using residues:

Residue Theorem:

Let C be a simple, piecewise smooth closed curve and let $f(z)$ be analytic inside and on C except for a finite number of isolated singularities $z_1, z_2, \ldots z_n$.

Then $\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{\text{Res}} f(z)$

→ $z_1, z_2, \ldots z_n$ - isolated singularities of $f(z)$ in the finite complex plane.

$r_1, r_2, \ldots r_n$ - residues

r_∞ - res. at ∞ $\quad \boxed{\therefore r_\infty = -\frac{1}{2\pi i}\oint_C f(z)\,dz}$

C - enclosed all singularities $\quad \boxed{\sum_{k=1}^{n} r_k + r_\infty = 0}$

$r_1 + r_2 + r_\infty = 0 \Rightarrow r_\infty = -(r_1+r_2)$

Process

1. Identify Singularities ~~Parameterize~~
2. Evaluate the residues at the singularities.
3.

Ex: $\oint_C \frac{dz}{e^z - 1}$ $\quad C: |z| = 1$

Sol: Residue theorem:

$f(z) = \frac{1}{e^z - 1}$ ← not analytic ⇒ Singular points ⇒ $e^z - 1 = 0$

$\boxed{z = 2n\pi i} \; \forall n \in \mathbb{Z}$

Sol Singular points $\therefore$ $z = 2n\pi i \quad \forall n \in \mathbb{Z}$

$f(z) = \dfrac{1}{e^z - 1}$ $\qquad z=0$ is the Pole - Simple pole

$\left[\because \text{ Zeroes of } f(z) \iff \text{Poles of } \dfrac{1}{f(z)}\right]$

$$2\pi i \operatorname{Res}_{z=0} f(z) = 2\pi i \lim_{z\to 0} z f(z)$$

$$= 2\pi i \lim_{z\to 0} \frac{z}{e^z - 1}$$

$$= 2\pi i \lim_{z\to 0} \frac{1}{e^z}$$

$$= 2\pi i \;//$$

$\boxed{\therefore \oint_C f(z)\,dz = 2\pi i \sum \text{Res}}$

Ex: $\displaystyle\oint_C \frac{e^z - 1}{z(z-1)(z-i)^2}\,dz \quad ; \quad C : |z| = 2$

Sol $z = 0, 1, i$ $\qquad z = i$ - Pole of order 2

$z = 1$ - Pole of order 1 - Simple pole

$$z=0 \Rightarrow e^z - 1 = \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$= z\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots\right)$$

$z = 0 \rightarrow$ Removable Singularity

$$\underset{z=i}{\text{Res}}\, f(z) = \frac{1}{(2-1)!} \lim_{z \to i} \frac{d}{dz}\left[(z-i)^2 f(z)\right]$$

$$= 1 \cdot \lim_{z \to i} \frac{d}{dz}\left[\frac{e^z - 1}{z(z-1)}\right]$$

$$= \lim_{z \to i}\left[\frac{e^z}{z(z-1)} + (e^z - 1)\left(\frac{-1}{z^2 - z}\right)(2z-1)\right]$$

$$= \frac{e^i}{i(i-1)} - \frac{(e^i - 1)}{-1-i}(2i-1)$$

$$= \frac{e^i}{i(i-1)} + \frac{(e^i-1)(2i-1)}{i+1} \quad //$$

$$\underset{z=1}{\text{Res}}\, f(z) = \lim_{z \to 1} (z-1) f(z) = \lim_{z \to 1} \frac{e^z - 1}{z(z-i)^2}$$

$$= \frac{e-1}{(1-i)^2} \quad //$$

$$\underset{z=0}{\text{Res}}\, f(z) = 0 \quad \{\because z=0 \text{ is a removable Singularity}$$

$$\therefore \oint_C \frac{e^z - 1}{z(z-1)(z-i)^2} = 2\pi i \sum \text{Res}$$

$$= 2\pi i\left[\frac{e^i}{i(i-1)} + \frac{(e^i-1)(2i-1)}{i+1} + \frac{e-1}{(1-i)^2}\right]$$

Ex: $\oint z e^{1/z}\, dz$; $C: |z|=1$

Sol ① Identify Singularities

$\Rightarrow \dfrac{1}{z e^{1/z}} = 0 \quad \Rightarrow \quad z=0, \; z=+\infty$

② $C: |z|=1 \quad \Rightarrow \quad z=0$

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!}\cdot\frac{1}{z^2} + \cdots$$

$$z e^{1/z} = z + 1 + \frac{1}{2!}\cdot\frac{1}{z} + \frac{1}{3!}\cdot\frac{1}{z^3} + \cdots$$

$z=0$ - Laurent Series have infinite no. of terms in principal part

$\therefore \operatorname{Res}_{z=0} f(z) = b_1$ - coefficient of $\frac{1}{z} = \frac{1}{2}$

$$\therefore \oint z e^{1/z}\, dz = 2\pi i \operatorname{Res}_{z=0} f(z)$$

$$= 2\pi i \times \frac{1}{2}$$

$$= \pi i \;//$$

22/09/2021

Evaluation of Real integral using Residue Theorem:

(I) Real definite integrals involving trignometric functions

$$\boxed{\int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta}$$

$z = e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow \cos\theta = \dfrac{z + \frac{1}{z}}{2}$

$\frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta$

$= \dfrac{z^2+1}{2z}$ //

$\Rightarrow \sin\theta = \dfrac{z - \frac{1}{z}}{2i} = \dfrac{z^2-1}{2iz}$ //

$z = e^{i\theta}$

$dz = ie^{i\theta}\, d\theta$

$\Rightarrow d\theta = \dfrac{1}{iz} dz$

$$\Rightarrow \boxed{\oint_C F\left(\frac{z^2-1}{2iz}, \frac{z^2+1}{2z}\right) \frac{dz}{iz}}$$

$(C : |z| = 1)$

Ex: $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\sin\theta}$; $\sin\theta = \dfrac{z^2-1}{2iz}$

$$= \oint_{|z|=1} \frac{1}{2 + \frac{z^2-1}{2iz}} \cdot \frac{dz}{iz} = \oint_{|z|=1} \frac{2}{z^2+4iz-1}\, dz$$

$$= 2 \oint_{|z|=1} \frac{dz}{z^2+4iz-1}$$

Sol

$z^2+4iz-1=0 \Rightarrow z = \dfrac{-4i \pm \sqrt{-16+4}}{2}$

$$z = \frac{-4i \pm 2\sqrt{3}i}{2}$$

$$z = (-2 \pm \sqrt{3})i = (\sqrt{3}-2)i \,;\, (-2-\sqrt{3})i$$

$C: |z|=1 \Rightarrow z = (\sqrt{3}-2)i$ lies in given region

$\downarrow$

Simple pole

$$\operatorname*{Res}_{z=(\sqrt{3}-2)i} f(z) = \lim_{z\to(\sqrt{3}-2)i} (z-\sqrt{3}i+2i)\, f(z)$$

$$= \lim_{z\to(\sqrt{3}-2)i} \frac{1}{z+(2+\sqrt{3})i}$$

$$= \frac{1}{2\sqrt{3}i}$$

$$\therefore 2\oint_{|z|=1} \frac{dz}{z^2+4iz-1} = 2\cdot 2\pi i \left\{ \operatorname*{Res}_{z\to c_1} (z-c_1)\, f(z) \right.$$

$$= 2\cdot 2\pi i \times \frac{1}{2\sqrt{3}i}$$

$$= \frac{2\pi}{\sqrt{3}} \;//$$

$$\boxed{\therefore \int_0^{2\pi} \frac{d\theta}{2+\sin\theta} = \frac{2\pi}{\sqrt{3}}}$$

Ex: $\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}$ (i) $|a|<1$ (ii) $|a|>1$

Sol: $\Rightarrow \oint_{|z|=1} \frac{1}{1-2a\left(\frac{z^2+1}{2z}\right)+a^2} \cdot \frac{dz}{iz}$

$= \oint_{|z|=1} \frac{z}{z-az^2-a+a^2z} \cdot \frac{dz}{iz}$

$= \oint_{|z|=1} \frac{dz}{i(-az^2+(a^2+1)z-a)}$

$= \frac{1}{-ai} \oint_{|z|=1} \frac{dz}{z^2-\left(\frac{a^2+1}{a}\right)+1}$

$\therefore z^2-\left(\frac{1+a^2}{a}\right)z+1=0$

$z = \frac{\left(\frac{1+a^2}{a}\right) \pm \sqrt{\left(\frac{1+a^2}{a}\right)^2-4}}{2}$

$= \left[\left(\frac{1+a^2}{a}\right) \pm \left(\frac{1-a^2}{a}\right)\right] \times \frac{1}{2}$

$\boxed{z = \frac{1}{a}, a}$

(i) a - Simple pole (ii) $\frac{1}{a}$ - Simple pole

$$\operatorname*{Res}_{z=a} f(z) = \lim_{z\to a} (z-a) f(z)$$

$$= \lim_{z\to a} \frac{1}{\left(z-\frac{1}{a}\right)}$$

$$= \frac{a}{a^2-1} \quad //$$

$$\boxed{\therefore \oint_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2} = \frac{2\pi i \times a}{a^2-1}}$$

(ii) $$\operatorname*{Res}_{z=\frac{1}{a}} f(z) = \lim_{z\to \frac{1}{a}} \left(z-\frac{1}{a}\right) f(z)$$

$$= \lim_{z\to\frac{1}{a}} \frac{1}{(z-a)}$$

$$= \frac{a}{1-a^2} \quad //$$

$$\boxed{\therefore \int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2} = \frac{2\pi i \times a}{1-a^2}}$$

\# Res

$$\boxed{\oint f(z)\, dz = 2\pi i \sum \operatorname*{Res}_{z\to z_k} f(z)}$$

(II) Improper Integrals of the form $\boxed{\int_{-\infty}^{\infty} f(x)\,dx}$

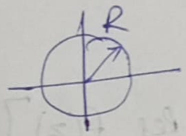
(Usually, $\lim_{s \to \infty} \int_{-s}^{s} f(x)\,dx$)

→ (All the singularities of $f(z)$ in the upper half plane are poles).

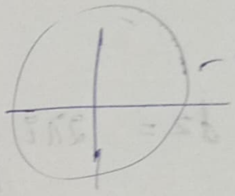
Assumptions

1.# $f(z)$ is analytic in the upper half plane except at a $\underbrace{\text{finite no. of poles}}$

2.# $z f(z) \to 0$ as $(C_R : |z| = R)$; $R \to \infty$

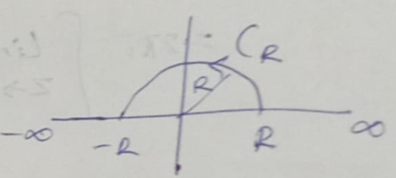


$R \to \infty$



entire $\mathbb{C}$

$$\underbrace{\int_{C_R} f(z)\,dz}_{①} + \underbrace{\int_{-R}^{R} f(z)\,dz}_{②}$$



$$= \oint_C f(z)\,dz \qquad C : \text{Semi Circle}$$

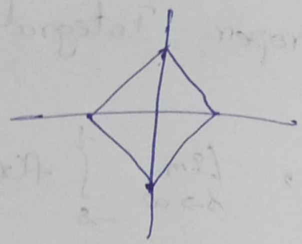
① $\to 0$ as $R \to \infty$

② $\Longrightarrow \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \operatorname{Res} f(z)$ //

$$\left| \int_{C_R} f(z)\,dz \right| = \left| \int_0^{\pi} f(Re^{i\theta})\, iRe^{i\theta}\,d\theta \right| = \left| \int_0^{\pi} z f(z)\, i\,d\theta \right|$$

$$\leq \int_0^{\pi} |z f(z)|\,d\theta \to 0$$

Ex: $\displaystyle\int_{-\infty}^{\infty} \frac{x^2+2}{(x^2+1)(x^2+4)}\,dx$



$\int_C f(z)\,dz$

Sol:

$$\oint_C f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx$$

$$f(z) = \frac{z^2+2}{(z^2+1)(z^2+4)} \qquad z = \pm i, \pm 2i$$

$C: \Rightarrow z = i, 2i$ — Simple poles

$$\oint_C f(z)\,dz = 2\pi i \left[\operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=2i} f(z) \right]$$

$$= 2\pi i \left[\lim_{z\to i} \frac{z^2+2}{(z+i)(z^2+4)} + \operatorname*{Res}_{z\to 2i} \frac{z^2+2}{(z^2+1)(z+2i)} \right]$$

$$= 2\pi i \left[\frac{-1+2}{(2i)(-1+4)} + \frac{-4+2}{(-4+1)(4i)} \right]$$

$$= 2\pi i \left[\frac{1}{6i} + \frac{2}{12i} \right]$$

$$= \frac{2\pi}{3} \,//$$

$$\therefore \oint_C f(z)\,dz = \frac{2\pi}{3}$$

$\boxed{C_R}$ $$\left|\frac{z^2+2}{(z^2+1)(z^2+4)}\right| \leq \frac{R^2+2}{(R^2-1)(R^2-4)}$$

$$\left(\because \quad |z_1| - |z_2| \leq |z_1+z_2| \leq |z_1| + |z_2| \right.$$

$$\left. \Rightarrow z^2+1 \geq R^2 - 1 \quad \Rightarrow \quad \frac{1}{z^2+1} \leq \frac{1}{R^2-1} \right)$$

$$\left|\frac{z^2+2}{(z^2+1)(z^2+4)}\right| \leq \left|\frac{R^2+2}{(R^2-1)(R^2-4)}\right| \longrightarrow 0$$

as $R \to \infty$

$$\therefore \oint_{C_R} f(z)\,dz = 0$$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \oint_{C} f(z)\,dz$$

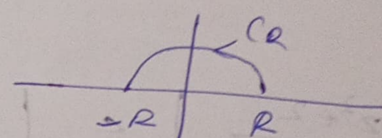
$$\boxed{\therefore \int_{-\infty}^{\infty} \frac{x^2+2}{(x^2+1)(x^2+4)}\,dx = \frac{2\pi}{3}}$$

$\int_0^{\infty}$ even fn. $= \frac{1}{2} \int_{-\infty}^{\infty}$ even fn.

$\int_0^{\pi}$ even fn. $= \frac{1}{2} \int_0^{2\pi}$ $\left[\because \int_0^{2\pi} = \int_0^{\pi} + \int_{\pi}^{2\pi} = 2\int_0^{\pi} \right.$

$\therefore \int_0^{\pi}$ even fn. $= \frac{1}{2} \int_0^{2\pi}$ even fn.

Ex: $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$



Soln

$f(z) = \frac{1}{1+z^4}$, $\quad z^4+1=0 \Rightarrow z^4 = -1$

$\Rightarrow z^2 = \pm i = e^{i\pi/2}, e^{i3\pi/2}$

$\Rightarrow z = \pm e^{i\pi/4}, \pm e^{i3\pi/4}$

$z^2 = \pm i$

$\Rightarrow z = \sqrt{i}, -\sqrt{i}, +\sqrt{-i}, -\sqrt{-i}$

$(x+iy)^2 = i$

$\therefore$

$z = \frac{\pm(1+i)}{\sqrt{2}}, \frac{\pm(1-i)}{\sqrt{2}}$

$C: \Rightarrow z = \frac{1+i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}$ - Simple poles

$$\oint_C f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx$$

$$\int_{C_R} f(z)\,dz \to 0 \quad \text{as } R \to \infty$$

$$\oint_C f(z)\,dz = 2\pi i \left[\operatorname*{Res}_{z \to e^{i\pi/4}} \left(z - e^{i\pi/4}\right) f(z) + \operatorname*{Res}_{z \to e^{i3\pi/4}} \left(z - e^{i3\pi/4}\right) f(z) \right]$$

$$= 2\pi i \left[\lim_{z \to \frac{(1+i)}{\sqrt{2}}} \frac{1}{\left(z + \frac{1+i}{\sqrt{2}}\right)\left(z - \left(\frac{1-i}{\sqrt{2}}\right)\right)\left(z + \left(\frac{1-i}{\sqrt{2}}\right)\right)} \right.$$

$$\left. + \lim_{z \to \frac{-1+i}{\sqrt{2}}} \frac{1}{\left(z - \left(\frac{1+i}{\sqrt{2}}\right)\right)\left(z + \frac{1+i}{\sqrt{2}}\right)\left(z - \frac{1-i}{\sqrt{2}}\right)} \right]$$

$$= 2\pi i \left[\frac{1}{\sqrt{2}(1+i)(\sqrt{2}i)(\sqrt{2})} + \frac{1}{(-\sqrt{2})(\sqrt{2}i)(\sqrt{2})(-1+i)} \right]$$

23/09/2021

III Improper real integrals of the form

$$\boxed{\int_{-\infty}^{\infty} \cos ax \, f(x)\, dx} \quad \text{(or)} \quad \boxed{\int_{-\infty}^{\infty} \sin ax \, f(x)\, dx}$$ - Fourier Integrals

#
$$\int_{-\infty}^{\infty} \cos ax \, f(x)\, dx = \int_{-\infty}^{\infty} Re\left(e^{iax}\right) f(x)\, dx$$

$$= Re \int_{-\infty}^{\infty} \underbrace{e^{iax} f(x)}_{J_1}\, dx$$

$g(x)$ - case II

$$\int_{-\infty}^{\infty} \sin ax \, f(x)\, dx = Im \int_{-\infty}^{\infty} e^{iax} f(x)\, dx$$

Consider, $I = \int_{-\infty}^{\infty} e^{iax} f(x)\, dx.$