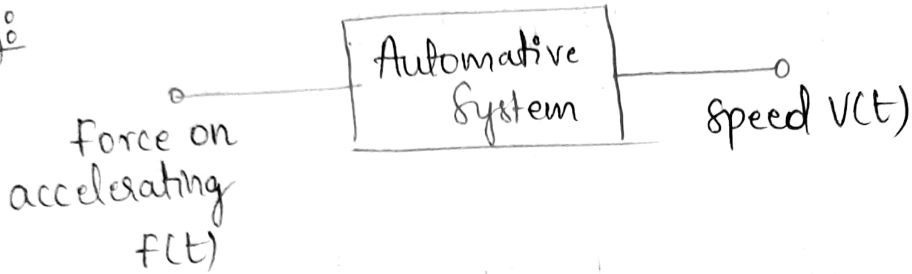
EC801

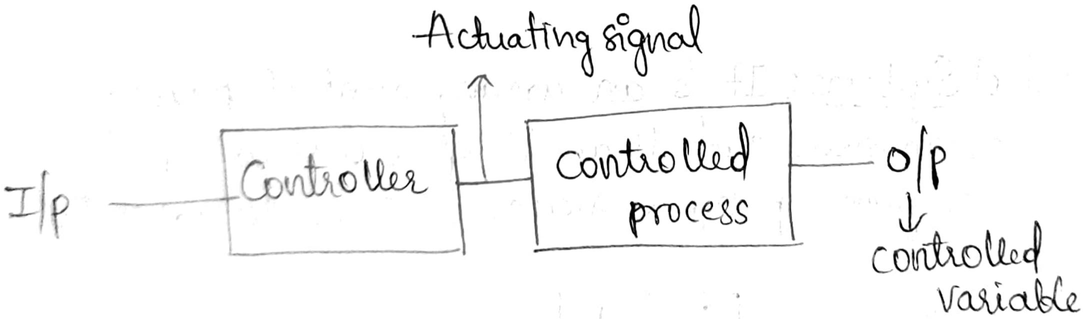
06/09/2022

eg:



① Open loop systems :

The systems in which desired output may exceed or decrease



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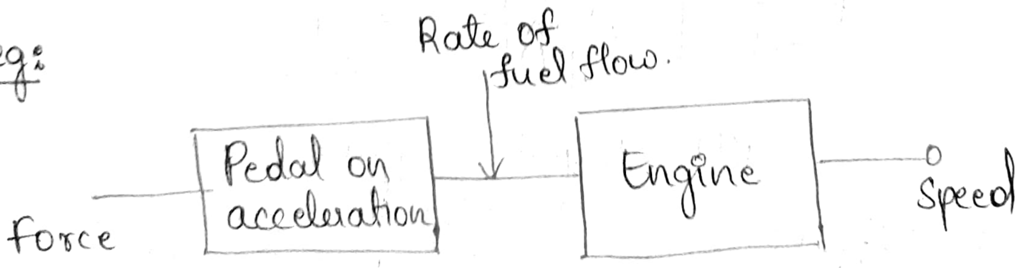
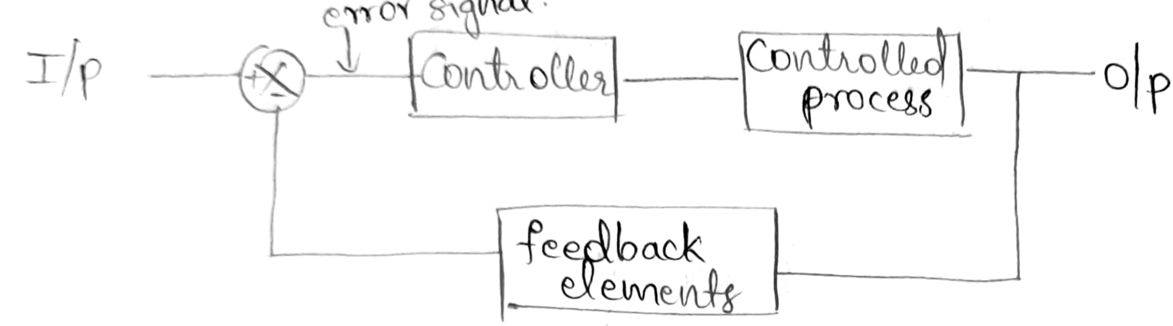


② Closed loop Systems:

The systems in which we achieve desired output.



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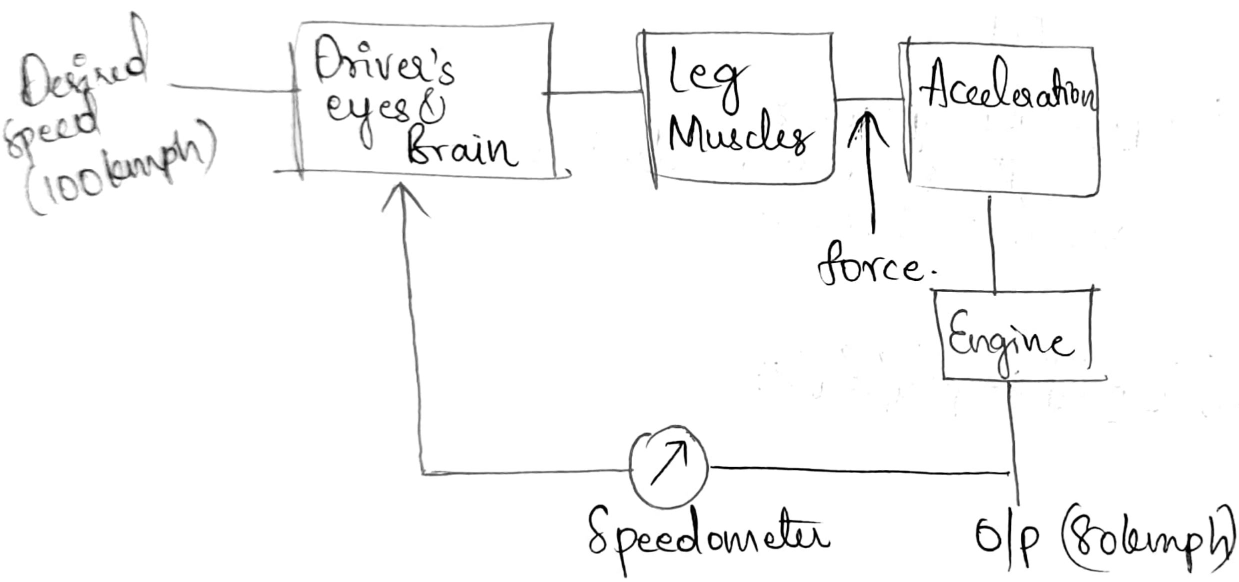
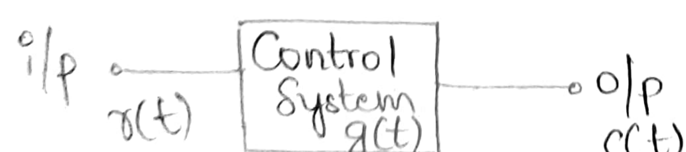


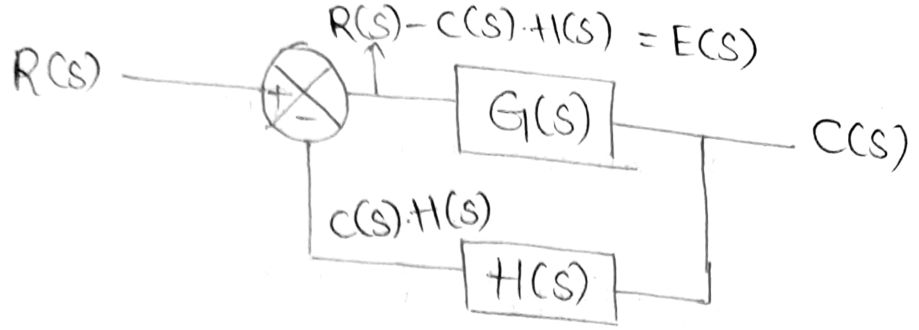
- Control Systems: It is an arrangement of physical components such that it gives desired o/p for a given i/p by a means of regulator (or) controller.



$g(t)$ - Impulse response
(When the i/p of the system is impulse signal)

$R(s) = L[r(t)]$
$G(s) = L[g(t)]$
$C(s) = L[c(t)]$

$$\boxed{G(s) = \frac{C(s)}{R(s)}}$$ → transfer function.



$$E(S) = R(s) - C(S) \cdot H(s) \quad \text{— (1)}$$

$$C(S) = E(S) \cdot G(s) \quad \text{— (2)}$$

$$C(S) = (R(s) - C(S) \cdot H(s))\, G(s)$$

transfer function

$$\boxed{\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s) H(s)}}$$

→ The feedback in the closed loop system reduces the gain by a factor of $\underline{1 + G(S) \cdot H(S)}$.

⊛ Laplace Transform:

If $f(t)$ satisfies $\int_0^\infty |f(t)\bar{e}^{\sigma t}| dt < \infty$

then,

$$F(S) = L[f(t)] = \int_0^\infty f(t) e^{-St} dt$$

$$S = \sigma + j\omega$$

eg: ① $f(t) = u(t) = 1 \quad , t > 0$

$$= 0 \quad , t < 0$$

$$\int_0^\infty |\bar{e}^{\sigma t}| dt < 0 \longrightarrow \text{for } \sigma > 0$$

region of convergence.

$$\boxed{L[u(t)] = \frac{1}{S} \; ; \; \sigma > 0}$$

② $f(t) = \bar{e}^{at} u(t) \quad , a > 0$

$$F(S) = \frac{1}{S+a} \; ; \; \sigma > -a$$

$$F(S) = \frac{1}{S-a} \; ; \; \sigma > a$$

- Theorems of Laplace T/F:

① Multiplication by a constant

$$L[Kf(t)] = KL[f(t)]$$

② Sum and Difference

$$L[f_1(t) \pm f_2(t)] = L[f_1(t)] \pm L[f_2(t)]$$

③ Differentiation

$$L\left[\frac{d^n f(t)}{dt}\right] = S^n F(S) - S^{n-1} f(0) - S^{n-2} f(0)$$

where,

$$f^K(0) = \left.\frac{d^K f(t)}{dt^K}\right|$$

④ Integration

$$L\left[\int_0^{t_n}\int_0^{t_{n-1}}\cdots\int_0^{t_1} f(t)\,d\tau\,dt_1\cdots dt_{n-1}\right] = \frac{F(s)}{s^n}$$

⑤ Shift in time

$$L[f(t-\tau)] = e^{-\tau s} F(s)$$

⑥ Complex-shift

$$L\left[e^{\mp\alpha t} f(t)\right] = F(s\pm\alpha)$$

$s \rightarrow$ complex number.

⑦ Real convolution

$$L[f_1(t) * f_2(t)] = F_1(s) \cdot F_2(s).$$

⑧ Complex convolution

$$L[f_1(t) f_2(t)] = F_1(s) * F_2(s)$$

Initial Value Theorem :

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} F(s)$$

- Final Value Theorem :

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

→ We can apply this theorem only if $sF(s)$ doesn't contain poles on right side or on imaginary axis of s-plane.

$$G(S) = \frac{P(S)}{Q(S)}$$

To find poles, $Q(S) = 0$

To find zeros, $P(S) = 0$

eg: $F(S) = \frac{5}{S(S^2+S+2)}$ Determine final value.

$$\lim_{t \to \infty} f(t) = \lim_{S \to 0} SF(S) = \lim_{S \to 0} \cancel{S} \times \frac{5}{\cancel{S}(S^2+S+2)}$$

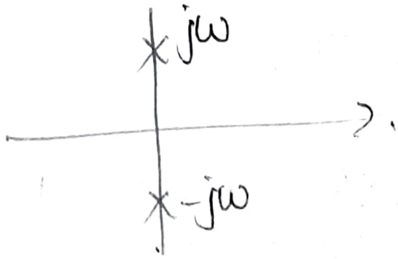
$= \underline{\underline{5/2}} \rightarrow$ final value of $F(S)$

eg: $F(S) = \frac{\omega}{S^2+\omega^2}$; Determine final value

$$\lim_{t \to \infty} f(t) = \lim_{S \to 0} S \times \frac{\omega}{S^2+\omega^2}$$

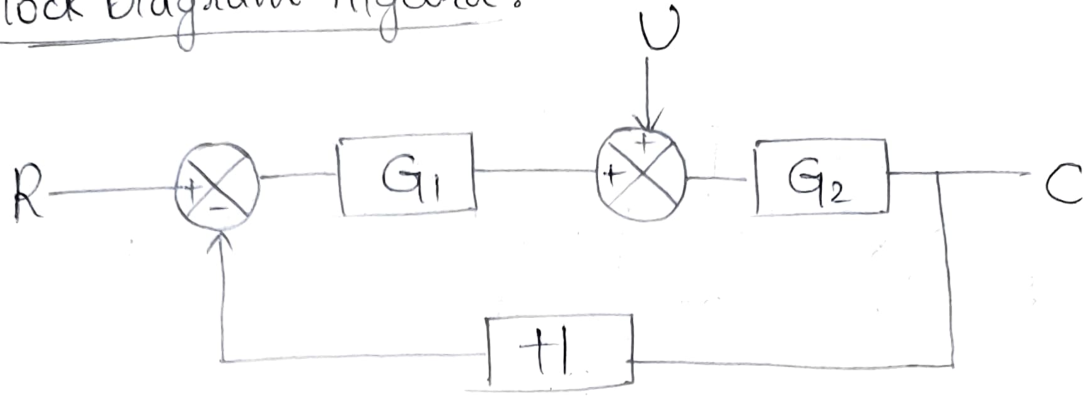
= undetermined

Poles: $S = \pm j\omega$ (Located on imaginary axis)

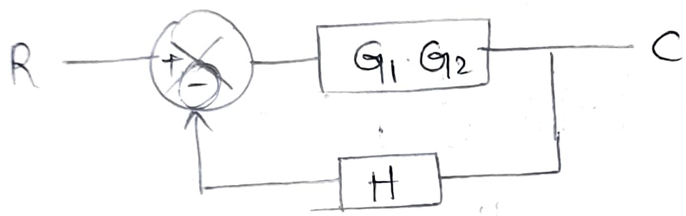


Block Diagram Algebra :

eg:

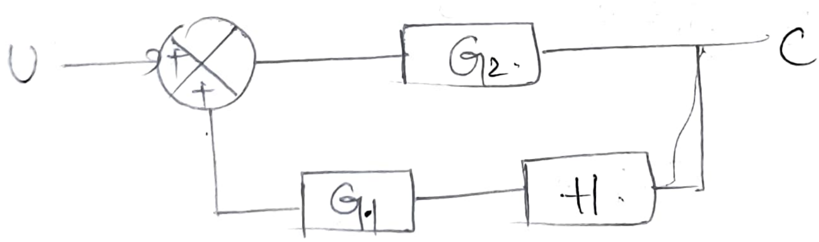


Assume, $U = 0$



$$\boxed{C_R = \frac{G_1 G_2}{1 + G_1 G_2 H} \times R}$$

Assume, $R = 0$

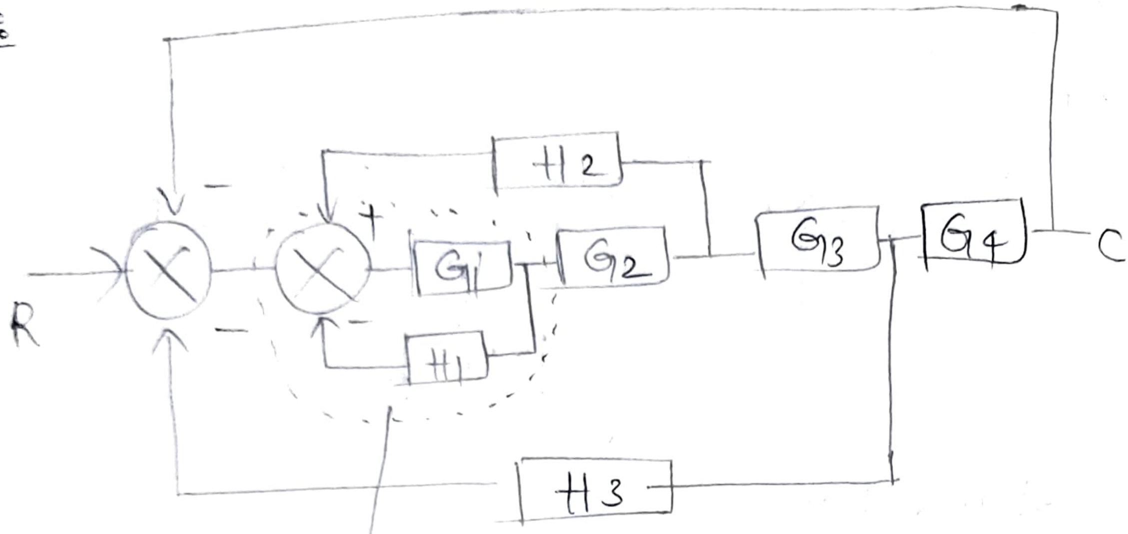


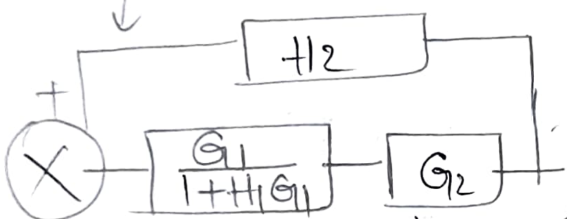
$$\boxed{C_U = \frac{G_2}{1 - G_1 G_2 H} U}$$

Overall output, $C = C_R + C_U$

$$\boxed{C = \frac{G_1 G_2 R}{1 + G_1 G_2 H} + \frac{G_2 U}{1 - G_1 G_2 H}}$$

eg:





$$\frac{G_1G_2}{1+H_1G_1}$$

$$\Downarrow$$

$$\frac{H_2}{1+\frac{G_1G_2H_2}{1+H_1G_1}}$$

$$\frac{\frac{G_1G_2}{1+H_1G_1}}{1-\left(\frac{G_1G_2}{1+H_1G_1}\right)H_2}$$

$$\left(\frac{G_1G_2}{1+H_1G_1-G_1G_2H_2}\right)\times G_3$$

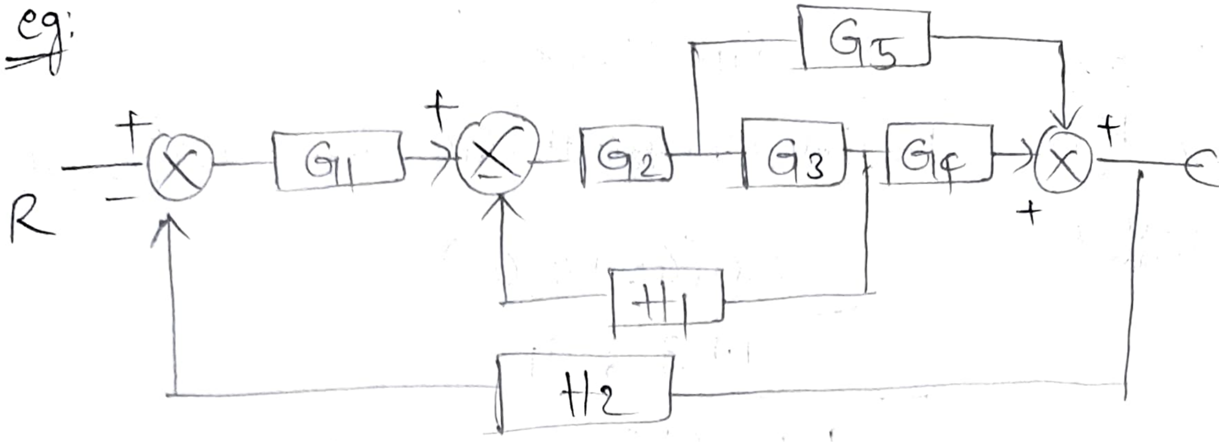
$$\left(\frac{\frac{G_1G_2G_3}{1+H_1G_1-G_1G_2H_2}}{1+\frac{G_1G_2H_3}{1+H_1G_1-G_1G_2H_2}}\right)\times G_4$$

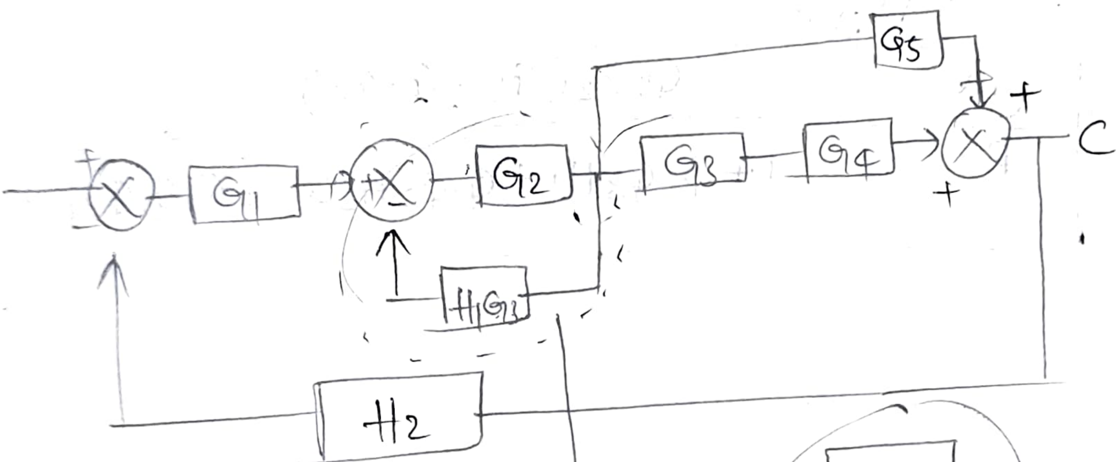
$$\frac{\frac{G_1G_2G_3G_4}{1+H_1G_1-G_1G_2H_2+G_1G_2H_3}}{1+\frac{G_1G_2G_3G_4}{1+H_1G_1-G_1G_2H_2+G_1G_2H_3}}$$

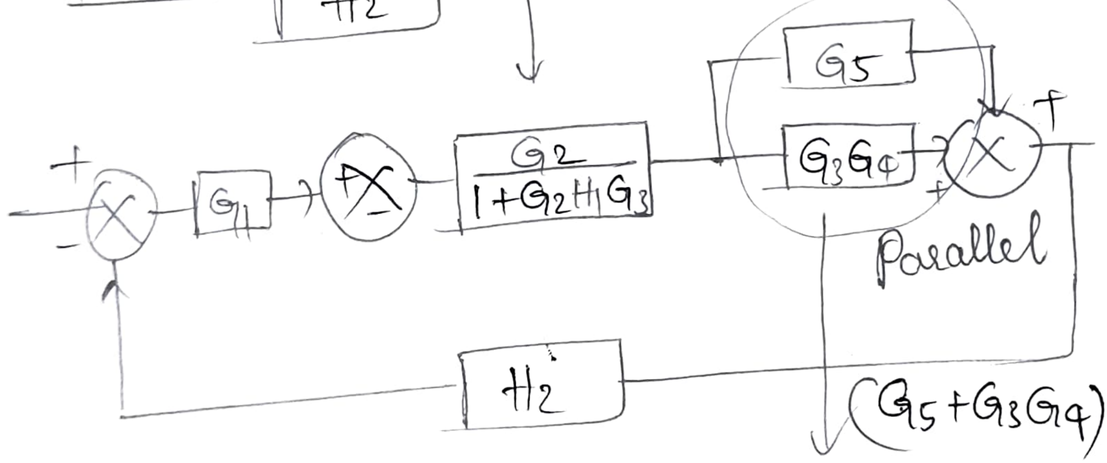
$$\frac{C}{R}=\frac{G_1G_2G_3G_4}{1+H_1G_1-G_1G_2H_2+G_1G_2H_3}$$

Final answer $\Rightarrow$ $$\boxed{\frac{G_1G_2G_3G_4}{1+H_1G_1-G_1G_2H_2+G_1G_2H_3+G_1G_2G_3G_4}}$$

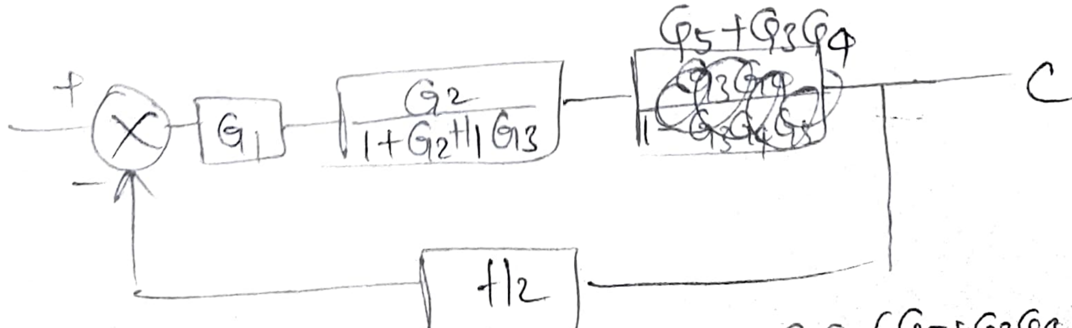
eg:







$$\cancel{\frac{G_3G_4}{1-G_3G_4G_5}}$$



$$\frac{G_1G_2(G_5+G_3G_4)}{(1+G_2H_1G_3)(1)}$$

$$\cancel{\frac{C}{R} = \frac{G_1G_2G_3G_4}{1 - G_3G_4G_5 + G_2G_3H_1 + G_1G_2G_3G_4H_2 - G_2G_3^2G_4G_5H_1}}$$

final answer.

$$\frac{\dfrac{G_1G_2(G_5 + G_3G_4)}{1 + G_2G_3H_1}}{1 + \dfrac{G_1G_2(G_5 + G_3G_4)H_2}{1 + G_2G_3H_1}}$$

final Answer

$$\boxed{\frac{C}{R} = \frac{G_1G_2(G_5 + G_3G_4)}{1 + G_2G_3H_1 + G_1G_2H_2G_5 + G_1G_2G_3G_4H_2}}$$

12/09/22

Signal-flow Graph :

→ Manson's gain formula.

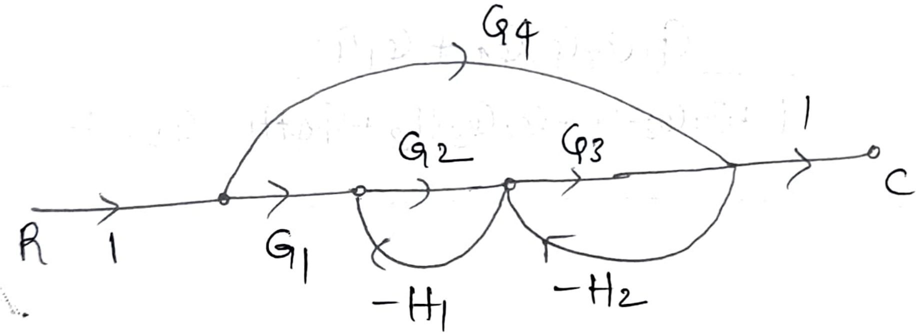
$$T = \frac{1}{\Delta} \sum_{K} P_K \Delta_K \qquad T - \text{transfer function.}$$

P_K - path gain of Kth forward path

$\Delta \rightarrow$ 1 - (Sum of loop gains of all individual loops)
+ (Sum of products of all possible combinations of two non-touching loops)
- (Sum of gain products of all possible combinations of three non-touching loops)
+ ......

$\Delta_K \rightarrow$ The value of 'Δ' for the part of the graph not touching Kth forward path.

eg:



$P_1 = G_1 G_2 G_3$

$P_2 = G_4$

$\Delta_1 = 1 - 0$

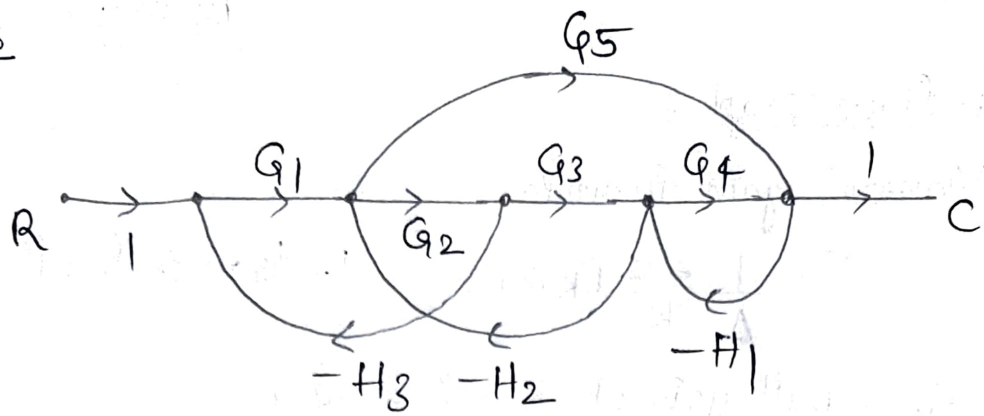
$\Delta_2 = 1 - (G_2(-H_1)) = 1 + G_2 H_1$

$\Delta = 1 - (G_2(-H_1) + G_3(-H_2))$

$\Delta = 1 + G_2 H_1 + G_3 H_2$

$$\boxed{T = \frac{1}{1 + G_2H_1 + G_3H_2}\left[G_1G_2G_3 + G_4 + G_2H_1G_4\right]}$$

eg:



$$P_1 = G_1G_2G_3G_4$$

$$P_2 = G_1G_5$$

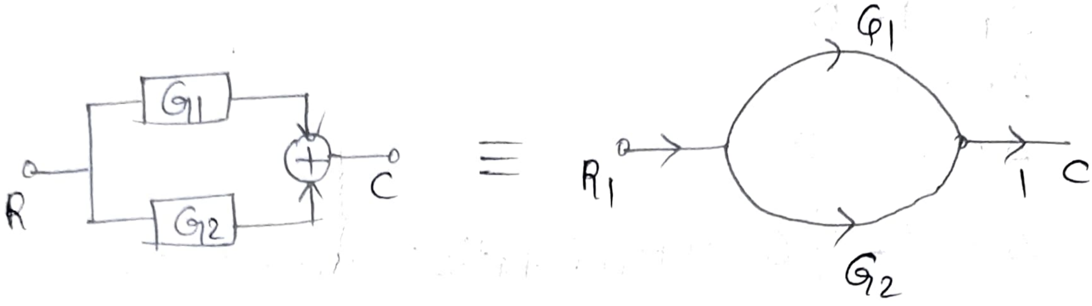
$$\Delta_1 = 1-0$$

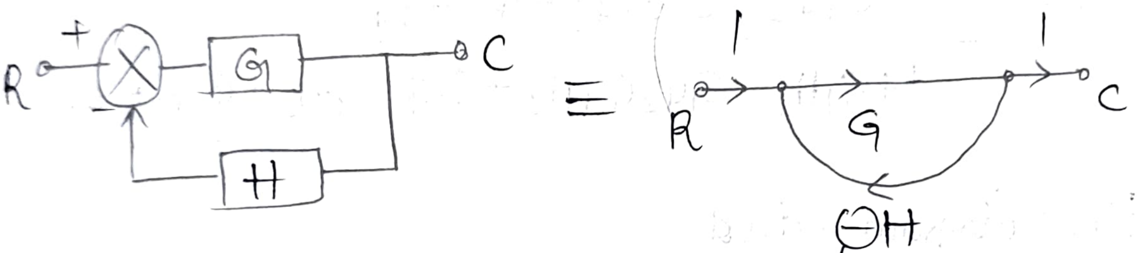
$$\Delta_2 = 1-0$$

$$\Delta = 1-\left(-G_1G_2H_3 - G_2G_3H_2 - G_4H_1 + G_5H_1H_2\right) + \left(G_1G_2H_3G_4H_1\right)$$

$$\boxed{T = \frac{G_1G_2G_3G_4 + G_1G_5}{1+G_1G_2H_3 + G_2G_3H_2 + G_4H_1 - G_5H_1H_2 + G_1G_2G_4H_3H}}$$

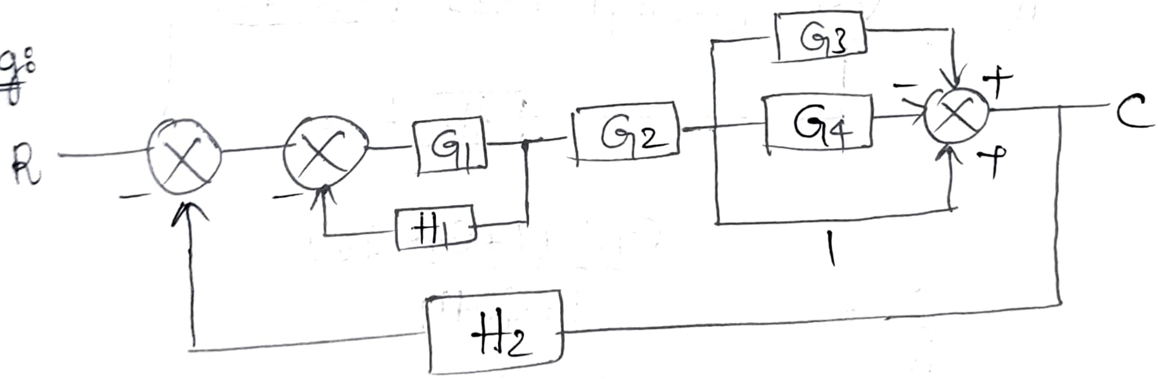
* Conversion of Block-diagrams into Signal-flow graphs:



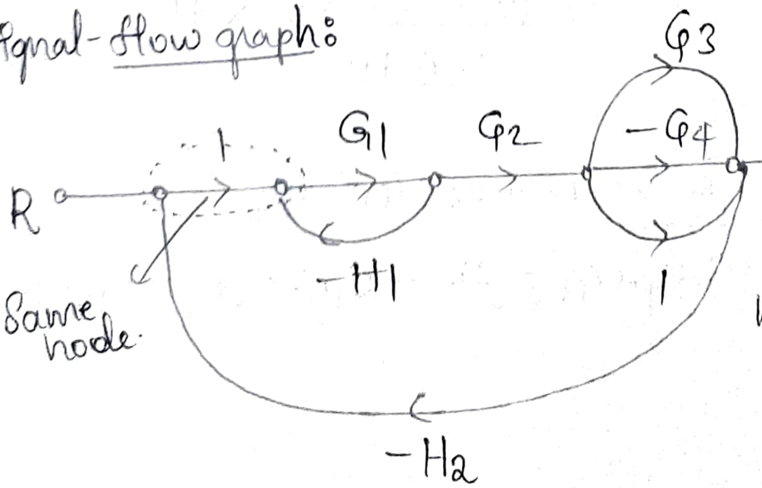


Due to negative feedback.

eg:



Signal-flow graph:



no. of paths = 3

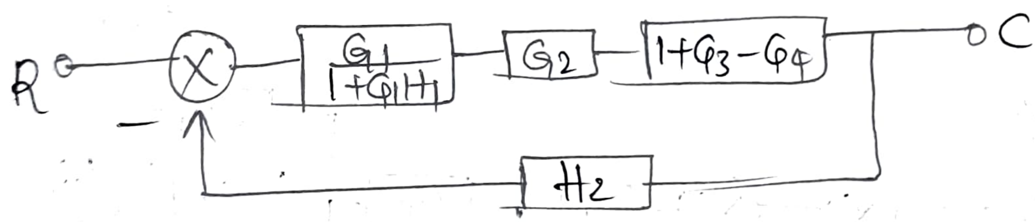
$$P_1 = -G_1G_2G_4$$
$$P_2 = G_1G_2G_3$$
$$P_3 = G_1G_2$$

$$\Delta_1 = 1 - 0$$
$$\Delta_2 = 1 - 0$$
$$\Delta_3 = 1 - 0$$

$$\Delta = 1 - (-G_1H_1 + G_1G_2G_4H_2 - G_1G_2G_3H_2 - G_1G_2H_2)$$

$$\boxed{T = \frac{G_1G_2 + G_1G_2G_3 - G_1G_2G_4}{1 + G_1H_1 - G_1G_2G_4H_2 + G_1G_2G_3H_2 + G_1G_2H_2}}$$

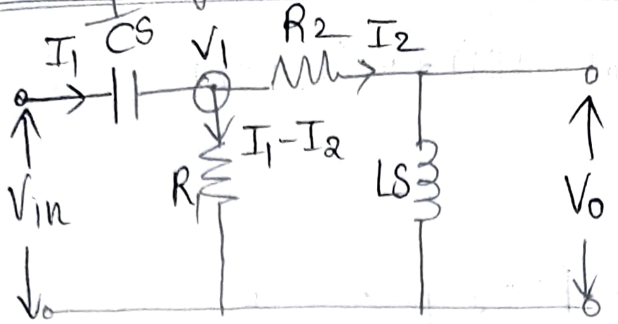
<u>Block-diagram method:</u>



$$\frac{C}{R} = \frac{\dfrac{G_1G_2(1+G_3-G_4)}{1+G_1H_1}}{1 + \dfrac{G_1G_2(1+G_3-G_4)}{1+G_1H_1} \times H_2}$$

$$\boxed{\frac{C}{R} = \frac{G_1G_2 + G_1G_2G_3 - G_1G_2G_4}{1 + G_1H_1 + G_1G_2H_2 + G_1G_2G_3H_2 - G_1G_2G_4H_2}}$$

Signal flow graphs for electrical N/w :



1. Identify node voltages and Branch currents.

1 node voltage – V_1.

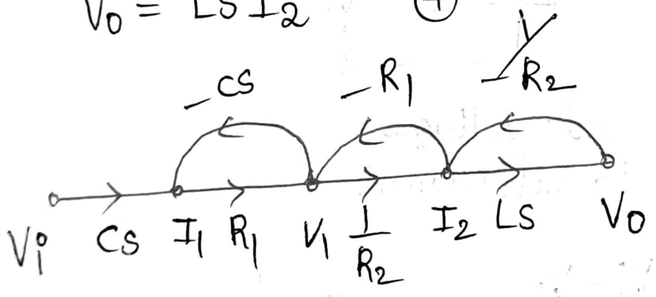
2 branch currents – I_1, I_2.

$$I_1 = \frac{V_i - V_1}{\frac{1}{CS}} = CS(V_i - V_1) \quad \text{—} (1)$$

$$I_2 = \frac{V_1 - V_o}{R_2} \quad \text{—} (2)$$

$$V_1 = (I_1 - I_2) R_1 \quad \text{—} (3)$$

$$V_o = LS I_2 \quad \text{—} (4)$$



only 1 path,

$$P_1 = \frac{CSR_1}{R_2} \times LS$$

$$\Delta_1 = 1 - 0$$

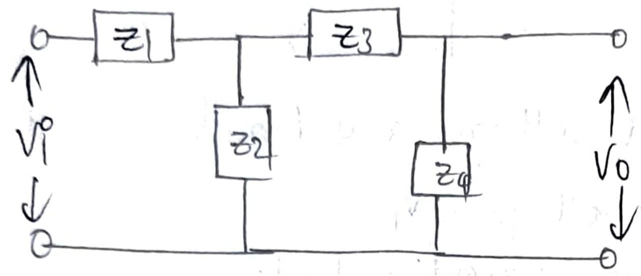
$$\Delta = 1 - \left(-CSR_1 - \frac{R_1}{R_2} - LS/R_2\right) + \left(CSLSR_1/R_2\right)$$

$$T = \frac{1}{\Delta} \sum_k P_k \Delta_k$$

$$T = \frac{\frac{CR_1LS^2}{R_2}}{1 + CSR_1 + \frac{R_1}{R_2} + LSR_2 + CS^2LR_1R_2}$$

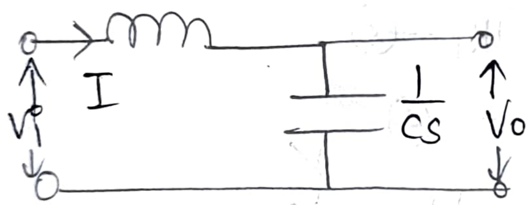
$$T = \frac{R_1 S L}{\frac{1}{SC}(R_1 + R_2 + SL) + R_1(R_2 + SL)}$$

eg:



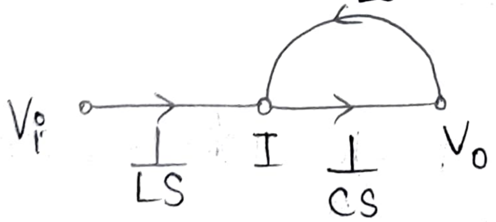
$$\boxed{\frac{V_o}{V_i} = \frac{Z_2 Z_4}{Z_1(Z_2 + Z_3 + Z_4) + Z_2(Z_3 + Z_4)}}$$

eg:



O - node voltages.
I - branch current.

$$I = \frac{V_i - V_o}{LS} \qquad V_o = \frac{I}{CS}$$

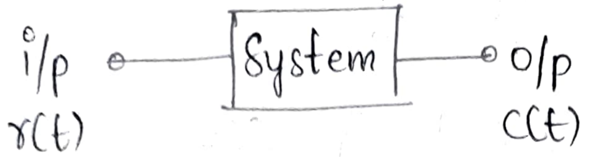


$$P_1 = \frac{1}{LCS^2} \qquad \Delta_1 = 1 - 0$$

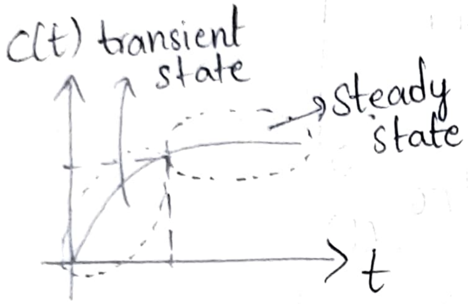
$$\Delta = 1 - \left(\frac{-1}{LCS^2}\right)$$

$$T = \frac{1}{1 + \frac{1}{LCS^2}}\left(\frac{1}{LCS^2}\right) \Rightarrow \boxed{T = \frac{1}{1 + S^2 LC}}$$

Time Domain Analysis:



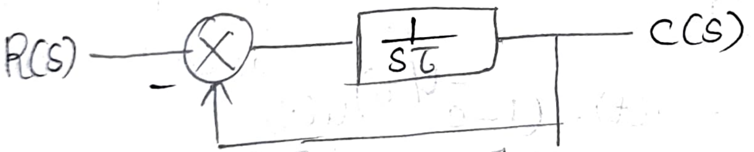
$$c(t) = C_{tr}(t) + C_{ss}(t)$$



Transient response : $\lim_{t \to \infty} C_{tr}(t) = 0$

eg: $c(t) = \underbrace{5 + 10\sin 2t}_{C_{ss}(t)} + \underbrace{e^{-10t}\cos 5t}_{C_{tr}(t)}$

Time response of first order System:



$$\boxed{\frac{C(S)}{R(S)} = \frac{1}{1+S\tau}}$$

$\tau \rightarrow$ time constant.

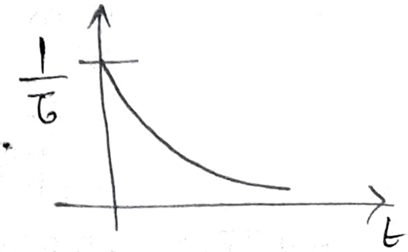
→ Impulse response: $r(t) = \delta(t)$

$R(S) = 1$

$$C(S) = \frac{1}{\tau\left[S + \frac{1}{\tau}\right]}$$

I.L.T

$$\boxed{c_{tr}(t) = \frac{1}{\tau} e^{-t/\tau} u(t)}$$

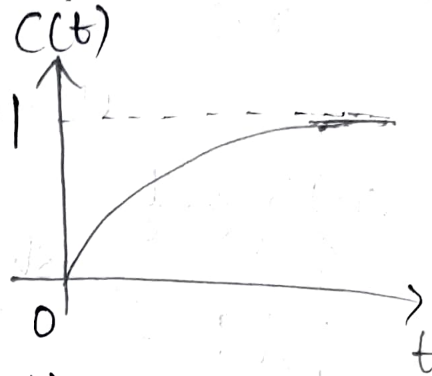


→ step response : $r(t) = u(t)$

$R(s) = \frac{1}{s}$

Widely used in Time Domain Analysis

$$C(s) = \frac{1}{s}\left(\frac{1}{s\tau+1}\right)$$



$$C(s) = \frac{1}{s} - \frac{\tau}{\tau s+1}$$

ILT ↓

$$c(t) = u(t) - e^{-t/\tau} u(t)$$

$$\boxed{c(t) = (1 - e^{-t/\tau})\, u(t)}$$

$C_{ss}(t)$ $C_{tr}(t)$

1st order control systems are not stable with ramp and parabolic bcoz these responses go on increasing even at infinite amount of time

$$\boxed{e(t) = r(t) - c(t)}$$

↓ System error

$$e(t) = u(t) - (1 - e^{-t/\tau})u(t)$$

$$e(t) = e^{-t/\tau} u(t)$$

1st order systems are stable with impulse and ramp bcoz they have bounded output

$$\boxed{e_{ss}(t) = \lim_{t\to\infty} e(t)}$$

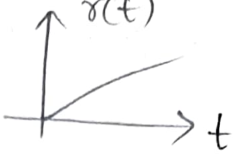
↓ Steady state error

$$\boxed{e_{ss}(t) = 0}$$

→ If the o/p of control system for an input varies wrt time, then it is called time response of the control system.

→ **Ramp input:** $r(t) = t\,u(t)$

$$R(s) = \frac{1}{s^2}$$



$\circledast$ $L\{t^n\} = \dfrac{n!}{s^{n+1}}$

$$C(s) = \frac{1}{s^2}\left(\frac{1}{\tau s+1}\right)$$

$$C(s) = \frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{\tau s+1}$$

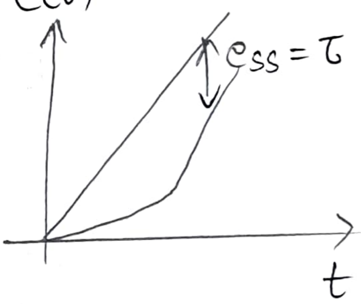
ILT ↓

$$\boxed{c(t) = (\underbrace{t-\tau}_{c_{ss}(t)} + \underbrace{\tau e^{-t/\tau}}_{c_{tr}(t)})\,u(t)}$$

$$e(t) = r(t) - c(t)$$

$$e(t) = \tau\,u(t)\left(1-e^{-t/\tau}\right)$$

$$e_{ss}(t) = \lim_{t\to\infty} e(t) = \tau$$



→ **parabolic i/p:** $r(t) = \dfrac{t^2}{2}u(t)$

$$R(s) = \frac{1}{s^3}$$

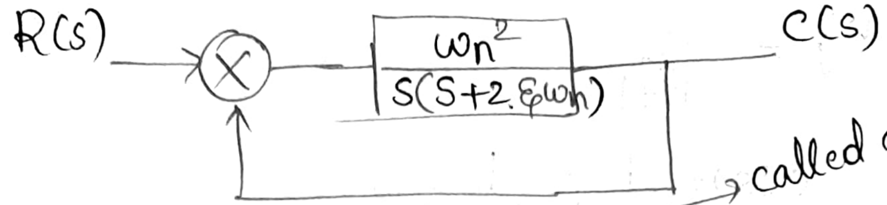
$$C(s) = \frac{1}{s^3}\left(\frac{1}{s\tau+1}\right)$$

$$C(s) = \frac{1}{s^3} - \frac{\tau}{s^2} + \frac{\tau^2}{s} + \frac{(-\tau^3)}{s\tau+1}$$

↓ ILT

$$\boxed{c(t) = \left(\underbrace{\frac{t^2}{2} - \tau t + \tau^2}_{c_{ss}(t)} - \underbrace{\tau^2 e^{-t/\tau}}_{c_{tr}(t)}\right)u(t)}$$

- Time response of Second order system :



called as gita

$\xi \rightarrow$ damping ratio

$\omega_n \rightarrow$ natural frequency

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$\rightarrow$ Step response : $r(t) = u(t)$

$$R(s) = \frac{1}{s}$$

$$C(s) = \frac{1}{s}\left(\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}\right)$$

$$C(s) = \frac{1}{s}\left(\frac{\omega_n^2}{(s+\xi\omega_n)^2+\omega_n^2(1-\xi^2)}\right)$$

$$C(s) = \frac{1}{s} + \frac{-s-2\xi\omega_n}{(s+\xi\omega_n)^2+\omega_n^2(1-\xi^2)}$$

$$\boxed{\omega_d = \omega_n\sqrt{1-\xi^2}} \rightarrow \text{Damped freq of oscillations.}$$

$$C(s) = \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} \quad \text{— (1)}$$

$$\left(\begin{aligned} L[e^{-bt}\sin at] &= \frac{a}{(s+b)^2+a^2} \\ L[e^{-bt}\cos at] &= \frac{(s+b)}{(s+b)^2+a^2} \end{aligned} \right)$$

By applying ILT to eq (1),

$$\boxed{c(t) = u(t)\left(1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi\omega_n}{\omega_d} e^{-\xi\omega_n t}\sin\omega_d t\right)}$$

$$c(t) = 1 - e^{-\xi\omega_n t}\left[\cos(\omega_d t) + \frac{\xi}{\sqrt{1-\xi^2}}\sin(\omega_d t)\right] \quad t>0$$

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sqrt{1-\xi^2}\cos(\omega_d t) + \xi\sin(\omega_d t)\right]$$

Assuming, $\xi = \cos\phi$, $\sqrt{1-\xi^2} = \sin\phi$

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sin\phi}\left[\sin\phi\cos(\omega_d t) + \cos\phi\sin(\omega_d t)\right]$$

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sin\phi}\sin(\phi + \omega_d t)$$

$$\boxed{c(t) = 1 - \frac{e^{-(\cos\phi)\omega_n t}}{\sin\phi}(\sin(\phi + \omega_d t))}$$

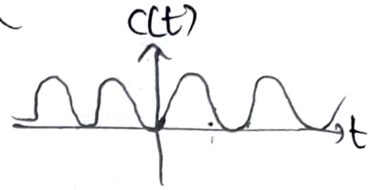
* Special cases:

1) Undamped System: ($\xi = 0$)

$$c(t) = 1 - \sin(\omega_d t + \pi/2) \quad \left(\omega_d = \omega_n\sqrt{1-\xi^2}\right)$$

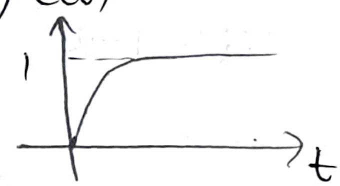
$$c(t) = 1 - \cos(\omega_d t)$$

$$\omega_d = \omega_n$$



$$\boxed{c(t) = 1 - \cos(\omega_n t)} \; ; t \geq 0$$

2) Critically damped System: ($\xi = 1$)



$$c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)$$

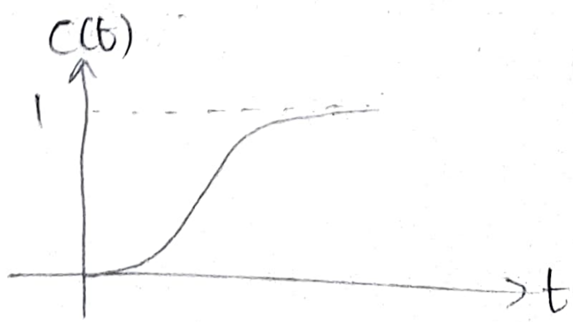
$$C(s) = \frac{\omega_n^2}{(s+\omega_n)^2} \times \frac{1}{s} \rightarrow C(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

$$\Big\downarrow \text{ILT}$$

$$\boxed{c(t) = \left(1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}\right) u(t)}$$

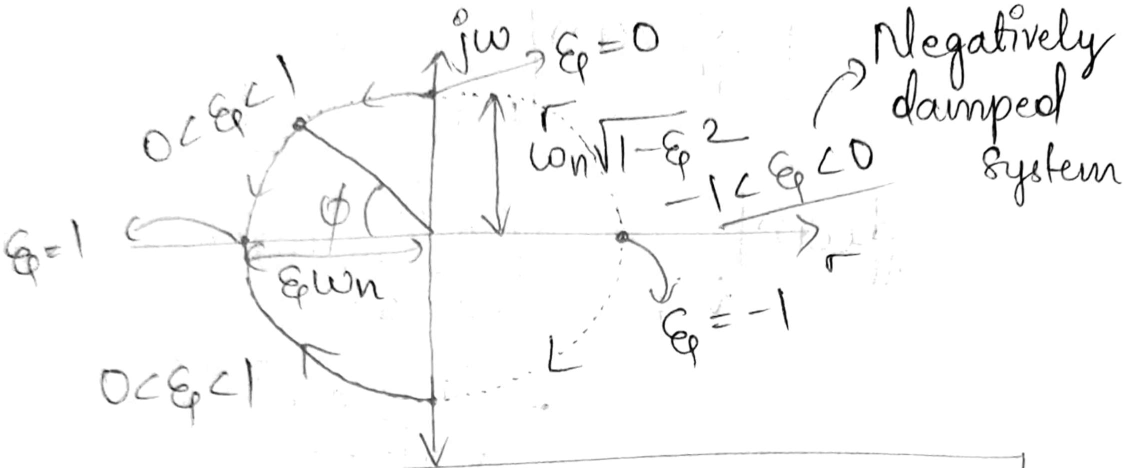
3) Overdamped System: ($\xi > 1$)

$$c(t) = 1 - \frac{e^{-(\xi - \sqrt{\xi^2-1})\omega_n t}}{2\sqrt{\xi^2-1}\left(\xi - \sqrt{\xi^2-1}\right)} + \frac{e^{-(\xi + \sqrt{\xi^2-1})\omega_n t}}{2\sqrt{\xi^2-1}\left(\xi + \sqrt{\xi^2-1}\right)}$$

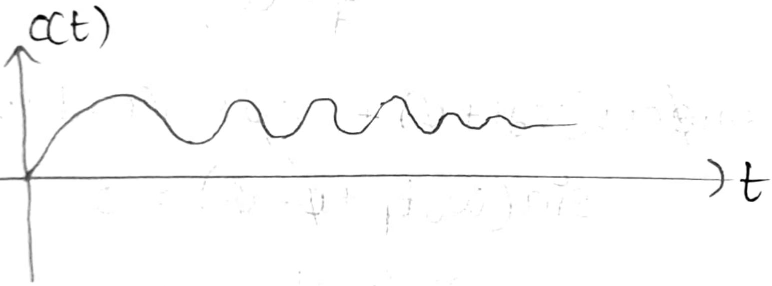


4) Underdamped System: ($0 < \xi < 1$)

poles: $s = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$



$$\boxed{c(t) = \left(1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \phi)\right)u(t)}$$

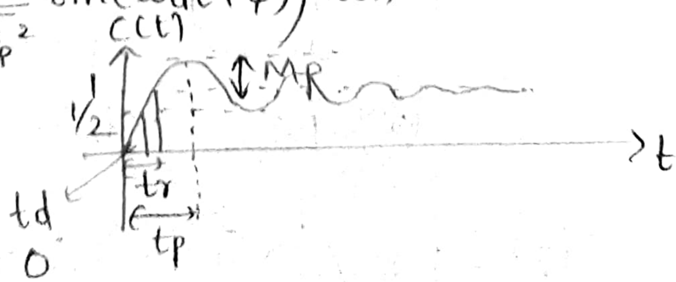


- Transient Response Specifications:

① Rise time: Time required to reach final value.
(t_r) $\left[c(t)\big|_{t=t_r} = 1 \right]$ ✱

eg: for underdamped,

$$c(t) = \left(1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \phi)\right)u(t)$$



$$\sin(\omega_d t_r + \phi) = 0$$

$$\omega_d t_r + \phi = \pi$$

$$\boxed{t_r = \frac{\pi - \phi}{\omega_d}}$$

② **Peak time** (t_p) : Time required to reach it's first peak.

eg: from the previous example

$$\boxed{\left.\frac{d}{dt}(c(t))\right|_{t=t_p} = 0}$$

$$\left.\frac{d}{dt}(c(t))\right|_{t=t_p} = -\left(\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\omega_d\cos(\omega_d t+\phi)\right) + \left(\frac{+\xi\omega_n}{\sqrt{1-\xi^2}}e^{-\xi\omega_n t}\sin(\omega_d t+\phi)\right)$$

$$0 = \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\omega_n\left(-\sqrt{1-\xi^2}\cos(\omega_d t+\phi) + \xi\sin(\omega_d t+\phi)\right)$$

$$\xi = \cos\phi$$

$$-\sin\phi\cos(\omega_d t+\phi) + \cos\phi\sin(\omega_d t+\phi) = 0$$

$$\sin(\omega_d t_p + \cancel{\phi} - \cancel{\phi}) = 0$$

$$\sin(\omega_d t_p) = 0$$

$$\omega_d t_p = \pi$$

$$\boxed{t_p = \frac{\pi}{\omega_d}}$$

③ **Peak Overshoot** (M_p) : It is the maximum peak value of the response curve measured from desired response of the system.

$$* \; [M_p = c(t_p) - c(\infty)]$$

$$c(t_p) = 1 + \left(\frac{e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}}\right)(+\sin\phi) \qquad \xi = \cos\phi, \quad \sqrt{1-\xi^2} = \sin\phi$$

$$\boxed{c(\infty) = 1} \qquad \boxed{c(t_p) = 1 + e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}} \qquad \boxed{M_p = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}}$$

④ Delay time (t_d) : Time required for response to reach half of it's final value.

$$c(t)\Big|_{t=t_d} = \frac{1}{2}$$

eg: from the previous example,

$$t_d = \frac{1+0.7\xi}{\omega_n}$$

⑤ Settling time (t_s) : The last movement of step response that enters 5% strip and never leaves from that point on.

$$t_s = \min\left\{ t>0,\ \frac{|c(t') - c(\infty)|}{c(\infty)} \leq 0.05\ \forall\, t' \geq t \right\}$$

eg: from previous example,

$$|c(t_s) - 1| \leq 0.05$$

$$\downarrow$$

$$\frac{e^{-\xi\omega_n t_s}}{\sqrt{}}\left[\cos\omega_d t_s + \frac{\xi\omega_n}{\omega_d}\sin\omega_d t_s\right] \leq 0.05$$

$$e^{-\xi\omega_n t_s} \leq 0.05$$

$$\boxed{t_s = \frac{-\ln(0.05)}{\xi\omega_n} = \frac{3}{\xi\omega_n}}$$

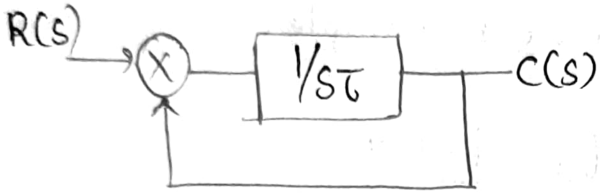
→ $t_s = \frac{3}{\xi\omega_n}$ (5% tolerance band)

→ $t_s = \frac{4}{\xi\omega_n}$ (2% tolerance band)

- Time constant :

The time required for the signal to attain 63.2 % of it's final value.

→ Determines how fast system reaches to final value.



$$\frac{C(S)}{R(S)} = \frac{1}{1+S\tau}$$ where $\tau \rightarrow$ time constant.

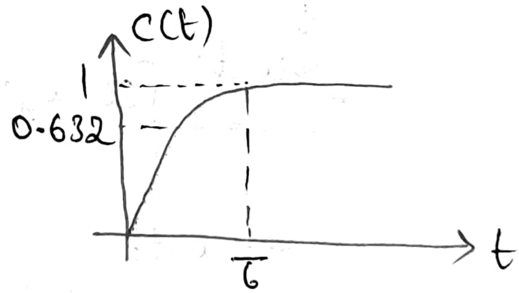
Assume, $R(S) = 1/S$

$$C(S) = \frac{1}{S(1+S\tau)}$$

ILT ↓

$$c(t) = (1-e^{-t/\tau})u(t)$$

$$c(t)\Big|_{t=\tau} = 1-e^{-1} = \underline{0.632}$$



eg: Given, $G(s) = \frac{25}{s(s+4)}$ and $H(s) = 1$

Find all time domain expressions.

Sol:- $$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{25}{s^2+4s}$$

$$\boxed{\omega_n = 5}$$

$$\boxed{\xi = 0.4}$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$$\omega_d = 4.5 \text{ rad/s}$$

$$t_d = \frac{1+0.7\xi}{\omega_n} = \underline{0.256 \text{ sec}}$$

$$t_r = \frac{\pi-\phi}{\omega_d} = \frac{\pi-\cos^{-1}(0.4)}{4.5} = \underline{0.44 \text{ sec}}$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{4.5} = \underline{0.893 \text{ sec}}$$

$$t_s = \frac{3}{\xi\omega_n} = \frac{3}{0.4\times 5} = \underline{1.5 \text{ sec}}$$

Steady state errors :

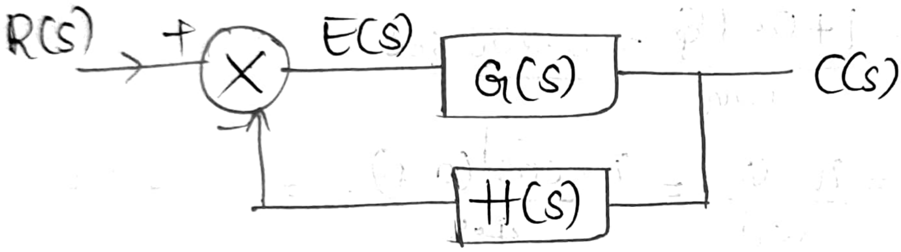
It is the deviation o/p from reference i/p at the steady state $(t \to \infty)$

$$\boxed{e_{ss} = \lim_{t \to \infty} e(t)}$$

where, $\boxed{e(t) = r(t) - c(t)}$

Using final value theorem

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$



$$E(s) = R(s) - C(s) \cdot H(s)$$

$$= R(s) - E(s) \cdot G(s) \cdot H(s)$$

$$E(s) = \frac{R(s)}{1 + G(s)H(s)}$$

$$e_{ss} = \lim_{s \to 0} s \left(\frac{R(s)}{1 + G(s)H(s)} \right)$$

→ Step input

$$r(t) = Au(t)$$

$$R(s) = \frac{A}{s}$$

$$K_p = \lim_{s \to 0} G(s)H(s)$$

Static position error constant.

$$e_{ss} = \lim_{s \to 0} \frac{A}{1 + G(s)H(s)}$$

$$\boxed{e_{ss} = \frac{A}{1 + \lim_{s \to 0} G(s)H(s)}}$$

→ ramp input

$$r(t) = A\,t\,u(t)$$

$$R(s) = \frac{A}{s^2}$$

$$e_{ss} = \underset{s \to 0}{\mathrm{lt}} \; \frac{1}{s}\left(\frac{A}{1 + G(s)H(s)}\right)$$

$$= \underset{s \to 0}{\mathrm{lt}} \; \frac{A}{s + s \cdot G(s) \cdot H(s)}$$

$$\boxed{e_{ss} = \frac{A}{\underset{s \to 0}{\mathrm{lt}}\, s\,G(s)H(s)} = \frac{A}{K_v}}$$

$$K_v = \underset{s \to 0}{\mathrm{lt}}\; s\,G(s)H(s)$$

↳ Velocity error constant.

→ parabolic input

$$r(t) = \frac{At^2}{2}u(t)$$

$$R(s) = \frac{A}{s^3}$$

$$e_{ss} = \underset{s \to 0}{\mathrm{lt}} \; \frac{1}{s^2 + s^2 G(s)H(s)}$$

$$\boxed{e_{ss} = \frac{1}{\underset{s \to 0}{\mathrm{lt}}\, s^2 G(s)H(s)} = \frac{A}{K_a}}$$

$$K_a = \underset{s \to 0}{\mathrm{lt}}\; s^2 G(s)H(s)$$

↳ acceleration error constant.

- Type and order of the System:

Type - no. of poles located at origin.
order - no. of poles of the system.

eg: $G(S) = \dfrac{10(S+2)}{S^2(S+4)(S+10)}$, $H(S) = 1$

↳ Type-2 with order '4'.

$$r(t) = \left(1 + 4t + \frac{t^2}{2}\right)u(t)$$

$$R(S) = \left(\frac{1}{S} + \frac{4}{S^2} + \frac{1}{S^3}\right)$$

$$e_{ss} = \lim_{S \to 0} S \left(\frac{\frac{1}{S} + \frac{4}{S^2} + \frac{1}{S^3}}{1 + \frac{10(S+2)}{S^2(S+4)(S+10)}} \right)$$

$$e_{ss} = \frac{1}{1+k_p} + \frac{4}{k_v} + \frac{1}{k_a}$$

$K_p = \infty$

$K_v = \infty$

$K_a = \frac{1}{2}$

$$e_{ss} = 0 + 0 + 2$$

$$\boxed{e_{ss} = 2}$$

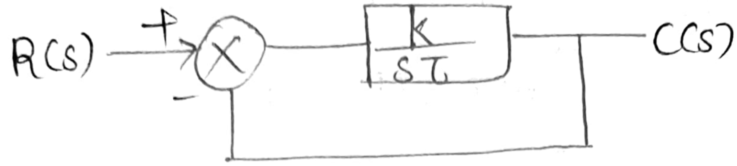
Type = Degree of i/p → ess = ?

Type > Degree of i/p → ess = 0

Type < Degree of i/p → $e_{ss} = \infty$

Steady state gain (or) DC gain:

Ratio of steady state o/p of a system to it's constant i/p, i.e steady state of unit step response.

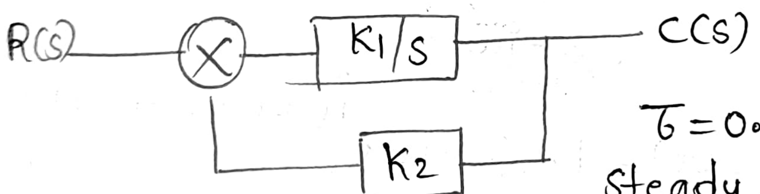


$$\frac{C(S)}{R(S)} = \frac{K}{K+S\tau}$$

$$C(S) = \frac{1}{S} \times \frac{K}{K+S\tau}$$

The steady state gain,

$$\lim_{t \to \infty} c(t) = \lim_{S \to 0} s\,C(S)$$

eg: 

$\tau = 0.4$

Steady state gain $= 2$

$K_1, K_2 = ?$

Sol:
$$\frac{C(s)}{R(S)} = \frac{K_1}{S+K_1K_2}$$

$$\tau = \frac{1}{K_1K_2} = 0.4$$

$$\text{Steady state gain} = \lim_{S \to 0} S \cdot \frac{1}{S} \cdot \frac{K_1}{S+K_1K_2}$$

* Response of the system for exponential input :

$$c(t) = r(t) * h(t)$$

$$= \int_{-\infty}^{\infty} h(\tau)\, r(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$= e^{st} \int_{-\infty}^{\infty} h(\tau) \cdot e^{-s\tau}\, d\tau$$

$$\boxed{c(t) = e^{st} \cdot H(s)}$$

$$\rightarrow H(s) = |H(s)| \cdot e^{j\angle H(s)}$$

$$\boxed{c(t) = |H(s)| \cdot e^{st + j\angle H(s)}}$$

$$s = j\omega$$

$$\boxed{c(t) = |H(j\omega)| \cdot e^{j(\omega t + \angle H(s))}}$$

eg: $r(t) = 10\cos(2t + 45°)$

$H(s) = \frac{C(s)}{R(s)} = \frac{s+1}{s+2}$. Determine output of system.

Sol:- $\boxed{c(t) = |H(j2)| \cdot r(t) \cdot e^{j\angle H(j2)}}$

$$H(j2) = \frac{2j+1}{2j+2} \times \frac{2j-2}{2j-2} = \frac{\cancel{8} + 2j + 6}{+8} = \frac{j+3}{4}$$

$$|H(j2)| = \sqrt{\frac{1}{16} + \frac{9}{16}} = \frac{\sqrt{10}}{4}$$

$$\boxed{c(t) = \frac{\sqrt{10}}{4} \cdot r(t) \cdot e^{j(18.43°)}}$$

$$\angle H(j2) = \tan^{-1}(1/3) = \underline{18.43°}$$

eg: $r(t) = \sin t$

$H(s) = \frac{C(s)}{R(s)} = \frac{1}{s+1}$. Determine output of system.

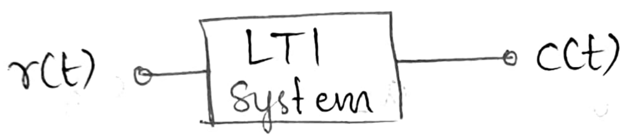
Sol: $c(t) = |H(j)| \cdot r(t) \cdot e^{j\angle H(j)}$

$H(j) = \frac{1}{j+1} \times \frac{j-1}{j-1} = \frac{j-1}{-2} = \frac{1}{2} - \frac{j}{2}$

$\angle H(j) = \tan^{-1}(-1) = -\pi/4$ $\qquad |H(j)| = \sqrt{\frac{1}{4}+\frac{1}{4}} = \frac{1}{\sqrt{2}}$

$$\boxed{c(t) = \frac{1}{\sqrt{2}} \sin(t - \pi/4)}$$

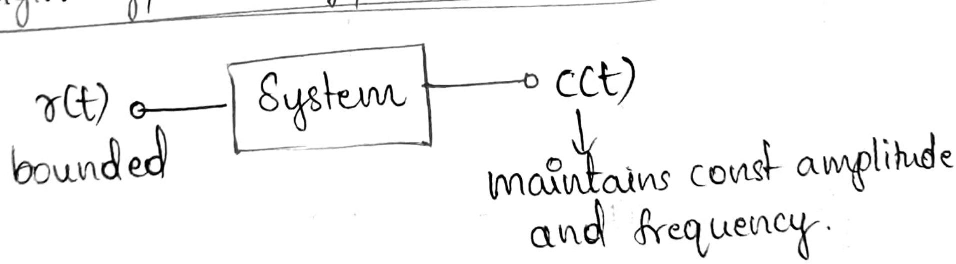
* Stability of the system:



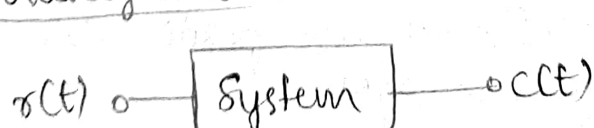
→ if $r(t)$ is bounded then $c(t)$ should be bounded

$$\left(\int_{-\infty}^{\infty} |r(t)| < \infty \implies \int_{-\infty}^{\infty} |c(t)|\, dt < \infty \right)$$

→ Marginally / Critically / Limited stable:



→ Absolutely stable:



- Both input and output are bounded.

→ Conditionally stable:

- It is stable only for certain range of system parameters.

$$H(s) = \frac{C(s)}{R(s)} = \frac{G}{1+GH}$$

$1+GH = 0$ → characteristic equation (Based on poles, we can determine stability).

⊛ Routh Hurwitz (RH) criteria:

→ Consider a nth order characteristic equation,

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_n = 0$$

s^n	a_0	a_2	a_4 ....
s^{n-1}	a_1	a_3	a_5 ....
s^{n-2}	$\frac{a_1a_2 - a_0a_3}{a_1}$	$\frac{a_1a_4 - a_0a_5}{a_1}$	....
⋮	⋮	⋮	⋮
s^0	a_n	....	....

→ If all coefficients are positive, then the system is stable.

eg: Characteristic equation is given by,

$$as^3 + bs^2 + cs + d = 0$$. find stability.

Sol:

$$\begin{array}{c|ccc} s^3 & a & c \\ s^2 & b & d \\ s^1 & \frac{bc-ad}{b} & 0 \\ s^0 & d & 0 \end{array}$$

The system is stable, if $a>0,\ b>0,\ d>0$, $bc>ad$

if $bc<ad \rightarrow$ unstable

if $bc=ad \rightarrow$ marginally stable

$\rightarrow$ C.E: $s^2+5s+10=0 \rightarrow$ stable

$\rightarrow$ C.E: $s^3+10s^2+3s+30=0 \rightarrow$ Marginally stable.

$\rightarrow$ C.E: $s^3+5s^2+2s+30=0 \rightarrow$ Unstable.

eg: C.E: $s^4+2s^3+6s^2+8s+10=0$. Determine stability.

$$\begin{array}{c|ccc} s^4 & 1 & 6 & 10 \\ s^3 & 2 & 8 & 0 \\ s^2 & 2 & 10 & 0 \\ s^1 & -2 & 0 & 0 \\ s^0 & 10 & 0 & 0 \end{array}$$

The system is unstable

2-poles on RH s-plane (due to sign change)

2-poles on LH s-plane

eg: C.E : $s^5 + s^4 + 3s^3 + 3s^2 + 2s + 2 = 0$. Determine stability.

s^5	1	3	2
s^4	1	3	2
s^3	$\cancel{0}$ 4	$\cancel{0}$ 6	
s^2	$\frac{3}{2}$	2	
s^1	$2/3$		
s^0	2		

A.E : $s^4 + 3s^2 + 2 \Rightarrow \psi$

$$\frac{\partial \psi}{\partial s} = 4s^3 + 6s$$

Replace zeros with these coefficients

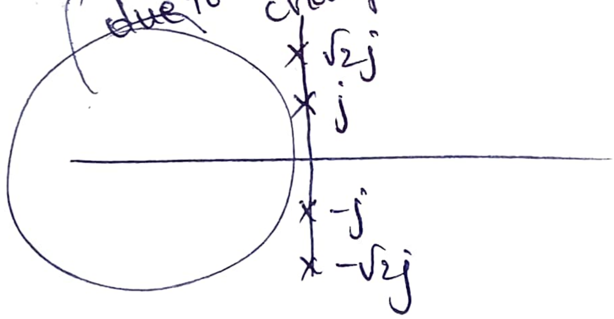
→ Whenever sequence of zero appears in a row, write auxiliary equation ~~[illegible]~~ (A.E)

$$A.E(\psi) \Rightarrow s^4 + 3s^2 + 2 = 0$$

$$(s^2+2)(s^2+1) = 0$$

$$s = \pm\sqrt{2}j \; ; \; s = \pm j$$

Another pole in LHS due to no sign change.



eg: C.E: $s^6+3s^5+4s^4+6s^3+5s^2+3s+2=0$

s^6	1	4	5	2
s^5	3	6	3	
s^4	2	4	2	
s^3	$\cancel{0}$ 8	$\cancel{0}$ 8	0	
s^2	2	2		
s^1	$\cancel{0}$ 4	0		
s^0	2			

A.E: $2s^4+4s^2+2=0$ (ψ_1)

$$\frac{\partial\psi_1}{\partial s}=8s^3+8s$$

A.E(ψ_2): $2s^2+2=0$

$$\frac{\partial\psi}{\partial s}=4s$$

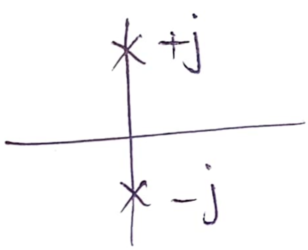
$\psi_1=2s^4+4s^2+2=0$

$(s^2+1)^2=0$

$\underline{s=\pm j\,;\,\pm j}$

$\psi_2=2s^2+2=0$

$\underline{s=\pm j}$



→ Whenever poles are repeating on Imaginary axis, the system is <u>unstable</u>.

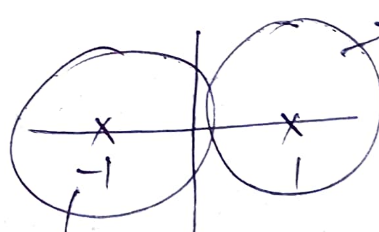
eg: C.E: $s^4+s^3-s-1=0$. Find no. of poles in LH s-plane.

s^4	1	0	-1
s^3	1	-1	0
s^2	1	-1	
s^1	$\cancel{0}$ 2	0	
s^0	-1		

A.E: $\psi \Rightarrow s^2-1=0$

$$\frac{\partial\psi}{\partial s}=2s$$

$$\boxed{s=\pm1}$$



2 poles in RH s-plane

2 poles in LH s-plane

eg: find the range of k for which system is stable,

C.E: $s^3+9s^2+4s+k=0$

$$\begin{array}{c|cc} s^3 & 1 & 4 \\ s^2 & 9 & k \\ s^1 & \frac{36-k}{9} & 0 \\ s^0 & k & \end{array}$$

$\frac{36-k}{9} > 0$

$\boxed{36 > k \text{ and } k > 0}$

if $\boxed{0 < k < 36}$ → stable

if $\boxed{k = 36}$ → marginally stable

⇓

$$\begin{array}{c|cc} s^3 & 1 & 4 \\ s^2 & 9 & 36 \\ s^1 & \cancel{0}\ 18 & 0 \\ s^0 & 36 & \end{array}$$

→ A.E: $9s^2+36$ (ψ)

$\frac{\partial \psi}{\partial s} = 18s$

$s = \pm j\omega_n$

$\psi \Rightarrow 9s^2+36=0$

$s = \pm 2j$

$\boxed{\omega_n = 2\,rad/s}$

$2\pi f_n = 2$

$\boxed{f_n = \frac{1}{\pi}\,Hz}$ → frequency of oscillation.

eg: Find value of 'k' for which system is stable,

$$GH = \frac{k}{s(s+2)(s+4)(s+6)}$$

CE: $1+GH=0$

$$1+\frac{k}{s(s+2)(s+4)(s+6)} = 0$$

$(s^2+2s)(s^2+10s+24)+k=0$

$s^4+12s^3+44s^2+48s+k=0$

$$\begin{array}{c|ccc} s^4 & 1 & 44 & K \\ s^3 & 12 & 48 & 0 \\ s^2 & 40 & K & 0 \\ s^1 & 48-\frac{3k}{10} & 0 & \\ s^0 & K & & \end{array}$$

$$\frac{12(40)}{12}$$

$$\frac{40(48)-12k}{40}$$

$$48-\frac{3k}{10}>0 \quad , \boxed{K>0}$$

$$\frac{\cancel{3}k}{10} < \overset{16}{\cancel{48}}$$

$$\boxed{K<160}$$

if $\boxed{0<K<160} \rightarrow$ stable

if $\boxed{K=160} \rightarrow$ marginally stable

$$\Downarrow$$

$$\begin{array}{c|ccc} s^4 & & & \\ s^3 & 1 & 44 & K \\ s^2 & 12 & 48 & 0 \\ s^1 & 40 & 160 & \\ s^0 & & & \end{array}$$

$$\begin{array}{c|ccc} s^4 & 1 & 44 & k \\ s^3 & 12 & 48 & 0 \\ s^2 & 40 & 160 & \\ s^1 & \cancel{0}\rightarrow 80 & 0 & \\ s^0 & 160 & & \end{array}$$

A·E

$$(\psi)=40s^2+160=0 \rightarrow \frac{\partial\psi}{\partial s}=80s$$

$$s=\pm 2j$$

$$\omega_n = 2\,\text{rad/s}$$

$$f_n=\frac{1}{\pi}\,Hz$$

eg: Determine value of K & P so that oscillates at 2 rad/s.

$$G(s) = \frac{K(s+1)}{s^3 + Ps^2 + 3s + 1}$$

Sol:

$\cancel{\frac{G(s)}{1+}}$

if $H(s)$ is not given, Assume unity feedback.

$\omega_n = 2\ \text{rad/s}$

C.E: $1 + GH = 0$

$$1 + \frac{K(s+1)}{s^3 + Ps^2 + 3s + 1} = 0$$

$p = 1$
$k = 3$

$$s^3 + Ps^2 + s(K+3) + (K+1) = 0$$

$$\begin{array}{c|cc} s^3 & 1 & K+3 \\ s^2 & P & K+1 \\ s^1 & K+3-\frac{K+1}{P} & \\ s^0 & K+1 & \end{array}$$

$$K+3 - \left(\frac{K+1}{P}\right) = 0$$

$$P = \frac{K+1}{K+3}$$

A.E: $(\varphi) \Rightarrow Ps^2 + (K+1) = 0$

$$\frac{\cancel{(K+1)}}{(K+3)} s^2 + \cancel{(K+1)} = 0$$

$$s^2 + (K+3) = 0$$

$s = \pm j\omega_n \longrightarrow s = \pm\sqrt{K+3}\, j$

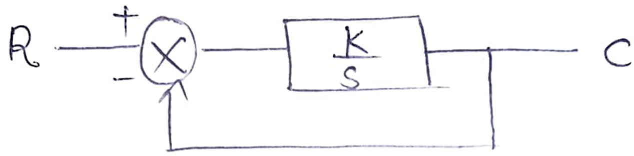
$\boxed{K = 1}$

$\boxed{P = 1/2}$

Root Locus :

→ Root locus is nothing but variation of closed loop system poles as the system gain(k) varies.

eg:

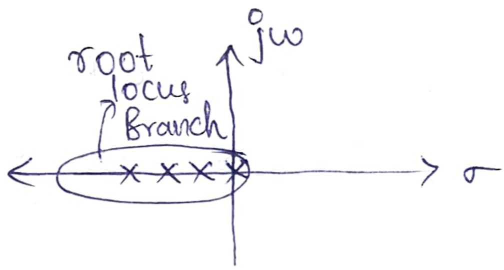


characteristic equation → $1+GH=0$

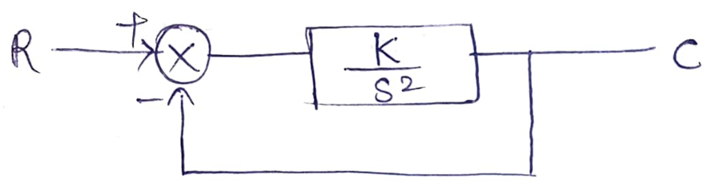
$$1+\frac{K}{S}(1)=0$$

$$\boxed{S=-k}$$

K	S
0	0
1	-1
2	-2
⋮	⋮
∞	-∞



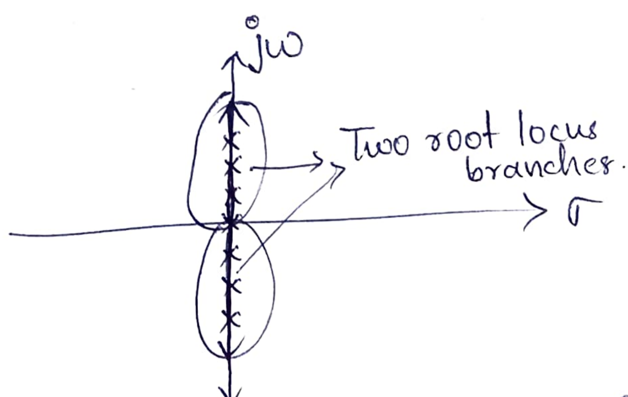
eg:



$$1+GH=0$$

$$1+\frac{K}{S^2}(1)=0 \rightarrow \boxed{S=\pm j\sqrt{K}}$$

K	S
0	0
1	$\pm j$
2	$\pm\sqrt{2}j$
⋮	⋮



$$\boxed{\text{No. of root locus branches} = \text{order of system}}$$

$GH \rightarrow$ open loop gain,

$1+GH=0 \rightarrow$ characteristic equation,

Let, $GH = \frac{K\,N(s)}{D(s)}$ —①

$1 + \frac{K\,N(s)}{D(s)} = 0$

from ①&②, $D(s) + K\,N(s) = 0$ —②

Case-①: If $D(s)=0$ and $K=0$

Then poles of open loop gain = poles of closed loop system

Case-②: if $K=\infty$ and $N(s)=0$

then zeroes of open loop gain = poles of closed loop system

→ The root locus diagram starts from $K=0$ (poles of GH) and ends at $K=\infty$ (zeroes of GH)

eg: find start and end points of the root locus diagram

$$GH = \frac{K(s+5)}{s(s+10)(s+20)}$$

Sol:- No. of branches = 3 (order of GH)

Start points: $s=0,-10,-20$

End points: $s=-5, \infty, \infty$

→ Angle condition,

To check whether $s = s_0$ is present in root locus or not,

$$1 + G(s_0)H(s_0) = 0$$

$$\boxed{GH = -1}$$

$$\angle GH = \angle -1$$

$$\boxed{\angle GH = \pm(2q+1)180^\circ \quad q = 0, 1, 2 \ldots}$$

eg: $GH = \dfrac{K}{s(s+2)(s+4)}$

check whether the following pts are in root locus or not

(a) $s = -0.75$ (b) $s = -1+4j$

Sol:- (a) $\angle GH\Big|_{s=-0.75} = \dfrac{\angle K}{\angle(-0.75)(1.25)(3.25)}$

$$= \frac{0^\circ}{\pm 180^\circ \cdot 0^\circ \cdot 0^\circ}$$

$$= \pm 180^\circ$$

So, $s = -0.75$ is in the RL.

(b) $\angle GH\Big|_{s=-1+4j} = \dfrac{\angle K}{\angle -1+4j \; \angle 1+4j \; \angle 3+4j}$

$$= \frac{0^\circ}{104^\circ \cdot 76^\circ \cdot 53^\circ}$$

$$= -233^\circ$$

Since it is not the odd multiples of 180°, it is not in Root Locus.

→ Magnitude condition,

To find the system gain (K) at a point on RL

$$1 + GH = 0$$

$$GH = -1$$

$$\therefore \boxed{|GH| = 1}$$

eg: $GH = \dfrac{K}{S(S+4)}$ find system gain at a point, $S = -2 + 5j$

Sol:

$$|GH|_{S=-2+5j} = \left|\frac{K}{(-2+5j)(2+5j)}\right| = 1$$

$$= \left|\frac{K}{-25-4}\right| = 1$$

$$\therefore |K| = 29$$

$$\angle GH|_{S=-2+5j} = \frac{\angle K}{\angle -2+5j \ \angle 2+5j}$$

$$= \frac{0^\circ}{122.68^\circ}$$

$$= \underline{\pm 180^\circ}$$

180
-68
112

eg: $GH = \dfrac{(s+1)}{s(s+2)((s+1)^2+1)}$

$s = 0, -2, -1+j, -1-j$

→ If s is real then it is on the root locus if and only iff there are odd no. of real open poles and zeroes to the right side of 's'.

11/10

ASYMPTOTES: no of asymptotes = $n-m$

$$\phi = \frac{(2q+1)}{n-m}180^\circ \quad ; \quad q = 0,1,2,\dots (n-m+1)$$

$n \rightarrow$ no of poles of GH

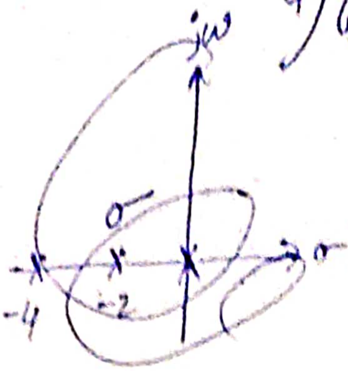
$m \rightarrow$ no of zeros of GH

CENTROID

The centroid can be computed as

$$\sigma = \frac{\text{sum of real part of poles of GH} - \text{sum of real part of zeros of GH}}{n-m}$$

1) $GH = \dfrac{k}{s(s+2)(s+4)}$



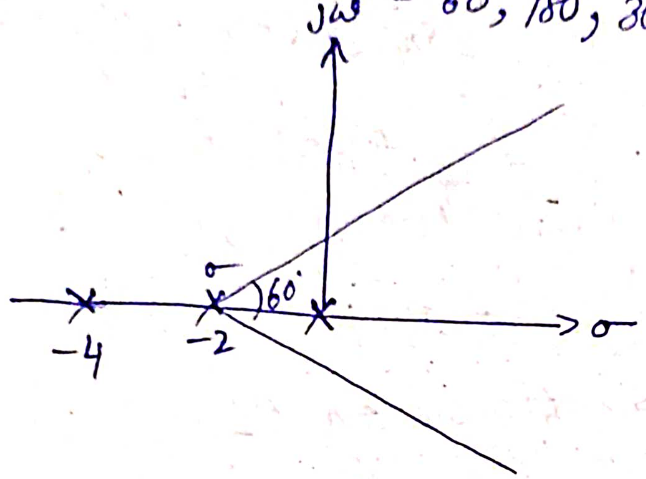
Sol^n: $\sigma = \dfrac{0-2-4-0}{3-0} = \dfrac{-6}{3} = -2 \rightarrow$ centroid.

no of poles = 3.

$\Rightarrow$ 3 asymptodes.

$\varphi = \dfrac{(2q+1)}{3} 180°$

$= 60; 180°; 300°$

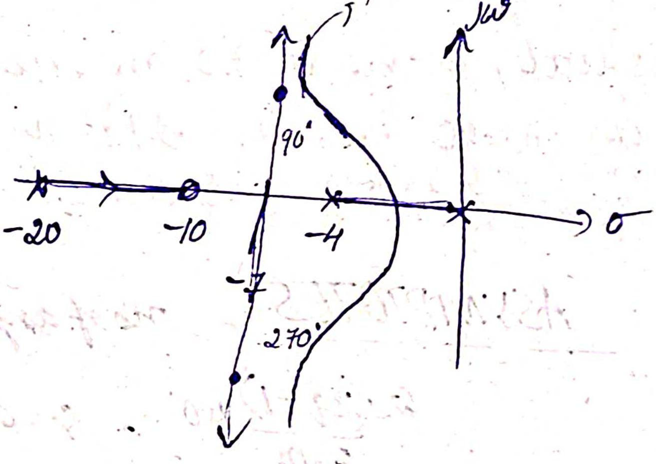


2) $GH = \dfrac{k(s+10)}{s(s+4)(s+20)}$

$\theta = \dfrac{2q+1}{2} \cdot 180°$

$= 90°; 270°$

$\sigma = -7$



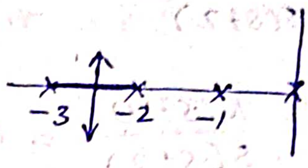
BREAK POINTS :-

Break away points - if 2 or more poles are located adjacent to each other

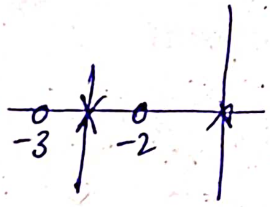
Break in points - if 2 or more zeros located adjct to each other.

$$\frac{dk}{ds} = 0$$

Q) $GH = \dfrac{k}{s(s+1)(s+2)(s+3)}$



$GH = \dfrac{k(s+2)(s+3)}{s^2}$



Q) Determine the coordinates of break points.

a) $GH = \dfrac{k}{s(s+2)}$

CH eq^n : $1+GH=0$

$$1+\frac{k}{s(s+2)} = 0$$

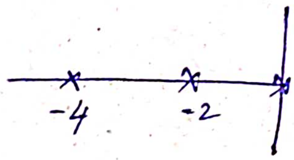
$$k = -s(s+2)$$

$$\frac{dk}{ds} = 0 \Rightarrow -2s-2=0$$

$$s=-1$$

b) $GH = \dfrac{k}{s(s+2)(s+4)}$

$$1+\frac{k}{s(s+2)(s+4)} = 0$$



$$k = -s(s+2)(s+4)$$

$$\cancel{dk} = (-s^2-2s)(s+4)$$

$$= -s^3-6s^2-8s$$

$$\frac{dk}{ds} = -3s^2 - 12s - 8 = 0$$

$$3s^2 + 12s + 8 = 0$$

$$3s^2 + 4s$$

24
12   8×3
2   4×6
6   2×12
4
8

$$s = -0.84,\ -3.15$$

c) $$GH = \frac{k(s+4)}{s(s+2)}$$

$$1 + \frac{k(s+4)}{s(s+2)} = 0$$

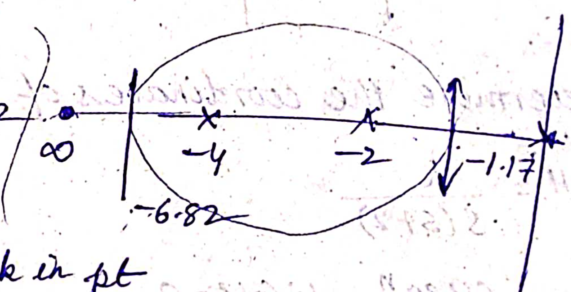
$$k(s+4) + s^2 + 2s = 0$$

$$k = \frac{-s^2 - 2s}{(s+4)}$$

$$\frac{dk}{ds} = 0\ ,$$

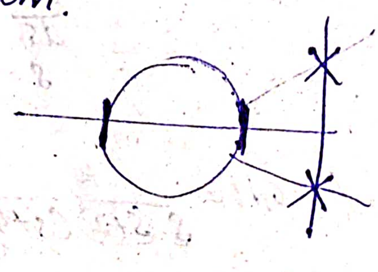
$$s = -1.17,\ -6.82$$

-1.17 → breakaway pt   -6.82 → break in pt



INTERSECTION POINT WITH IMAGINERY AXIS

some time root locus touch imaginery axis
& there need to compute gain of the system.
(by computing the value of 'k'
& get points by putting 'k' value in
the characteristic eqn)



Q. Calculate interse^n pt with img axis for the system

a) $GH = \dfrac{k}{s(s+1)(s+3)(s+5)}$

Sol^n: compute 'k' value by RH criteria

$$1+GH=0$$

$$s^4+9s^3+23s^2+15s+k=0 \longrightarrow ①$$

s^4	1	23	k
s^3	9	15	
s^2	21.3	k	
s	$\dfrac{21.3\times15-9k}{21.3}$		
s^0	k		

~~table~~

→ This table = 0 when a = 0

$\Rightarrow k = \dfrac{21.3\times15}{9}$

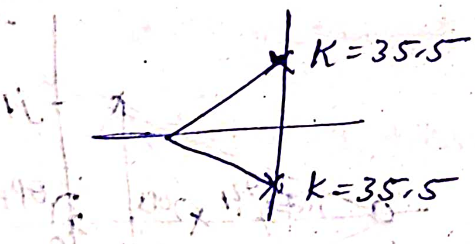
$k = 35.5$

$\Rightarrow$ At $k=35.5$ the system is margimally stable

(or)

At $k=35.5$ the root locus axis intersect the img axis.

here gain $= k = 35.5$



TO compute pts.

put k in ①

$$s^4+9s^3+23s^2+15s+35.5=0$$

not all sol^{n}s are accptd.

∵ roots are on img. axis ⇒ real part = 0

Substitute $s=j\omega$ we get (real + img = 0)

but ~~we~~ want pt on img axis so we choose real = 0

real term = 0

we get ~~even~~ real terms for even powers of s

$$s^4+23s^2+35.5=0$$

solve the above eq^n.

$s = \pm 1.28$

± 4.67

$\dfrac{-23\pm\sqrt{23^2-4(35.5)}}{2}$

$= \dfrac{-23\pm 19.67}{2}$

ANGLE OF DEPARTURE & ANGLE OF ARRIVAL

calculated if the pole is a complex number
not for the real pole.

∠le of arrival → at what pt the root locus starts → complex zero.
∠le of departure → complex pole

∠le of arrival — denoted by — θ_a

$\boxed{\theta_a = 180 + \phi}$

∠le of departure — $\boxed{\theta_d = 180^\circ - \phi}$

where $\phi = \sum_i \theta_{p_i} - \sum_j \theta_{z_j}$

Q Find the ∠le of ~~arrival~~ departure for the system.

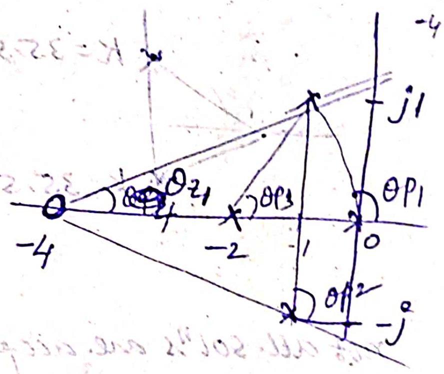
$$GH = \frac{K(s+4)}{s(s+2)(s^2+2s+2)}$$

Soln: complex pole for s^2+2s+2

$s^2+2s+2=0$

$$s = \frac{-2 \pm 2j}{2}$$

$s = -1 \pm j$



draw lines for complx poles from zeroes & other poles

find $\theta p_1, \theta p_2, \theta p_3, \theta p_4$

$\theta p_1 = 135^\circ$, $\theta p_2 = 90^\circ$ $\theta p_3 = 45^\circ$ $\theta p_4 = 18.43^\circ$

$\phi = \Sigma \theta p_i - \Sigma \theta p_j$

$\phi = (135^\circ + 90^\circ + 45^\circ) - 18.43^\circ$

$\phi = 251.5^\circ$

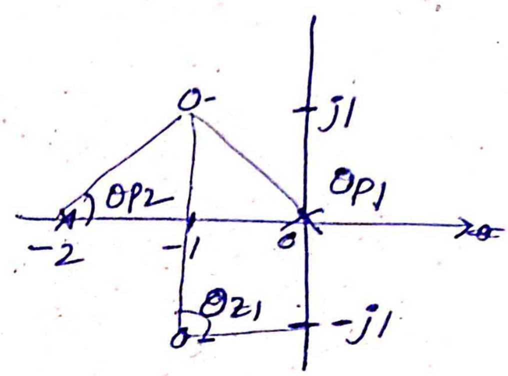
∴ The ∠le of departure is $-72^\circ = \theta d_1 = 180^\circ - 251.5^\circ$

For pole at $-1-j$ → we get

∠le of departure $= +72^\circ = \theta d_2$

Q] Find the ∠le of arrival of the system.

$$GH = \frac{(s^2+2s+2)K}{s(s+2)}$$



$\theta_{p1} = 135°$, $\theta_{p2} = 45°$, $\theta_{z1} = 90°$

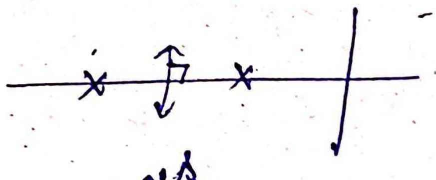
$\phi = 135° + 45° - 90°$

$\phi = 90°$

∠le of arrival $= 180° + 90° = \theta_{a1}$

$= 270°$

$\theta_{a2} = -270°$

→ 

for poles/zeroes located on real axis.

angle of arrival $\theta_a = \frac{180°}{n}$

angle of departure $\theta_d = \frac{180°}{n}$

Rules for construction of Root Locus:

Rule-1: Locate the open loop poles and zeroes in S-plane

Rule-2: Find no. of root locus branches.

↳ equal to open-loop poles / order of characteristic eq.

Rule-3: Identify and draw the real axis root locus branches

Rule-4: Find the centroid and the angle of asymptotes

Rule-5: Find the intersection pt of root locus branches with an imaginary axis.

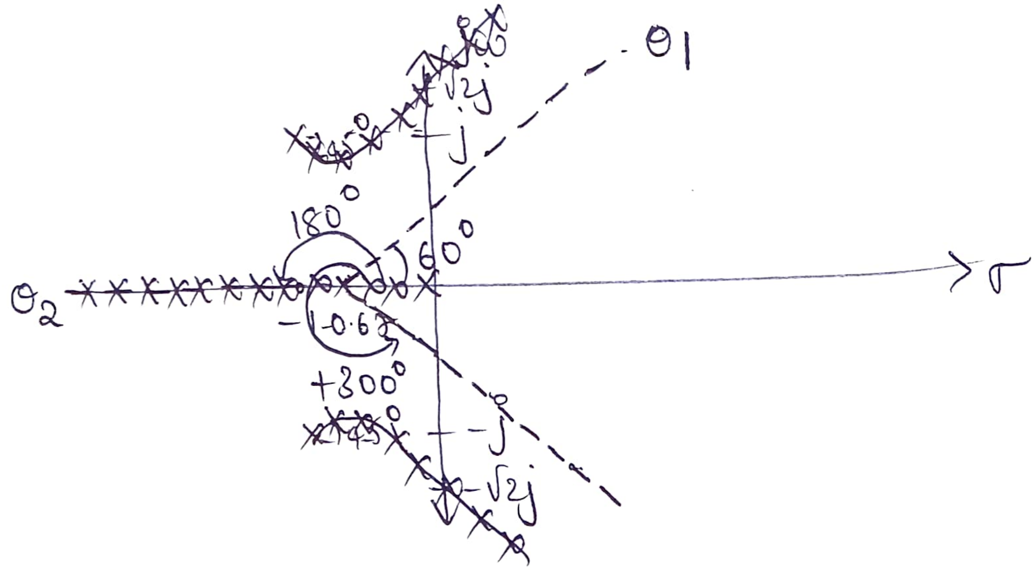
Rule-6: Find the Break-away & Break-in pts.

Rule-7: Find the angle of departure and the angle of arrival.

eg: Draw the root locus for the system,

$$GH = \frac{k}{s(s^2+2s+2)}$$

Sol:- poles of GH : $s=0$, $s=-1\pm j$



No. of roots locus branches = 3

no. of poles (n) = 3

no. of zeroes (m) = 0

$$\text{Centroid} = \frac{0+(-1+j)+(-1-j)}{3} = \frac{-2}{3} = -0.67$$

angle of asymptotes, $\frac{2q+1}{n-m} \times 180^\circ$ $(0,1,2)$

$\hookrightarrow 60^\circ, 180^\circ, 300^\circ$

ch. eq $\rightarrow 1+GH=0$

$$1+\frac{K}{s(s^2+2s+2)} = 0$$

$$s^3+2s^2+2s+k=0$$

s^3	1	2
s^2	2	K
s^1	$\frac{4-k}{2}$	0
s^0	K	

at $K=4 \rightarrow$ marginally stable.
(or)
$j\omega$ crossing pt

$$2s^2+K=0$$
$$2s^2+4=0$$
$$s^2=-2 \rightarrow \boxed{s=\pm\sqrt{2}j}$$

intersection pts

$$K=-s^3-2s^2-2s$$

$$\frac{dk}{ds}=0 \rightarrow +3s^2+4s+2=0$$

$$s=-0.67\pm0.47j$$

$\phi_d = 180-\phi$

where $\phi = \Sigma\theta_{poles} - \Sigma\theta_{zeroes}$

$$=(135^\circ+90^\circ)-0^\circ$$
$$=225^\circ$$

$\phi_{d_1} = -45^\circ$

$\phi_{d_2} = 45^\circ$

• eg: $GH = \frac{K(s+5)}{s^2}$

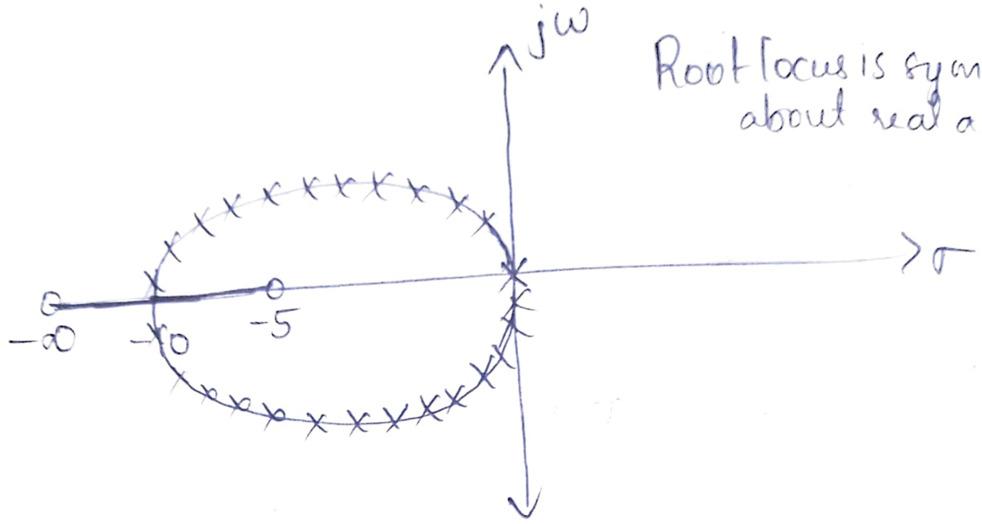
Poles of GH : $s = 0, 0$

No. of branches = 2

no. of poles (n) = 2

no. of zeroes (m) = 1

Root locus is symmetric about real axis



$$\text{Centroid} = \frac{0-(-5)}{1} = 5$$

$$\text{angle of asymptotes} = \left(\frac{2q+1}{1}\right)180^\circ \Rightarrow q = 0 \text{ only}$$

$$\theta = 180^\circ$$

$1 + GH = 0$

$1 + \frac{k(s+5)}{s^2} = 0$

$s^2 + ks + 5k = 0$

$$K = \frac{-s^2}{(s+5)}$$

$$\frac{dk}{ds} = 0$$

↳ $s = 0, -10$

↳ Break in pts

s^2	1	5k
s^1	k	0
s^0	5k	

*Bode plots:

- The purpose of bode plots are;
  (i) To analyze the frequency response of openloop gain.
  (ii) To analyze the closed loop system's stability
  (iii) To compute Gain margin & phase margin.

→ **Procedure for drawing Bodeplot:**

1. Replace $s = j\omega$ in GH function.
2. Find the magnitude (in dB) and phase of GH.

$$M_{dB} = 20\log_{10} |G(j\omega)H(j\omega)|$$

$$\phi = \tan^{-1}\left(\frac{\text{Img part}}{\text{Real part}}\right)$$

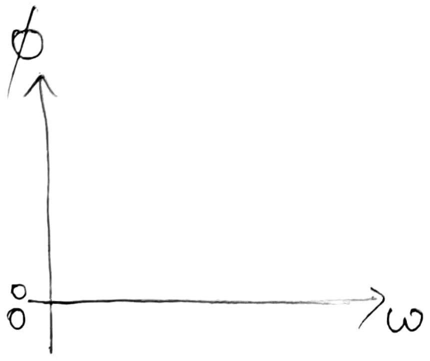
3. Draw magnitude and phase wrt 'ω'.

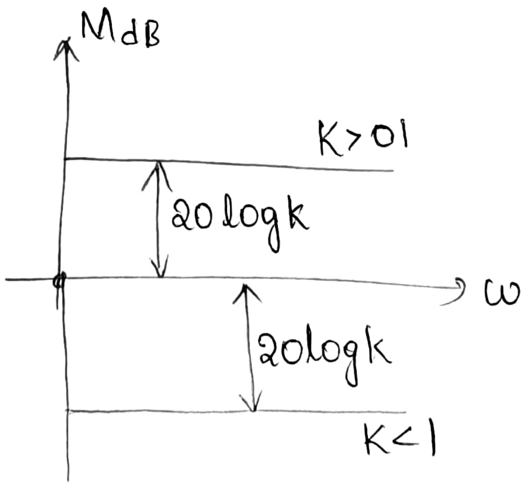
eg: $GH = K$

Sol:- $G(j\omega)H(j\omega) = K$

$$M_{dB} = 20\log_{10} K$$

$$\phi = 0^\circ$$





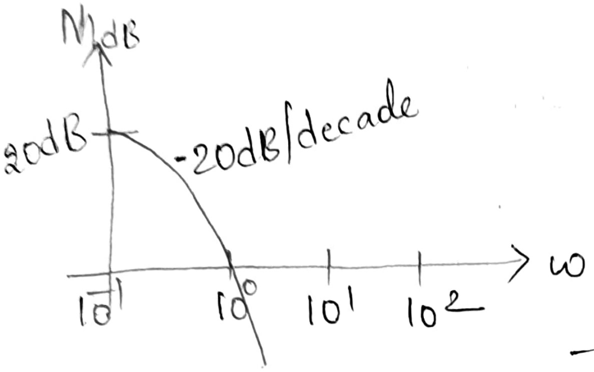
eg: $GH = \frac{1}{s}$

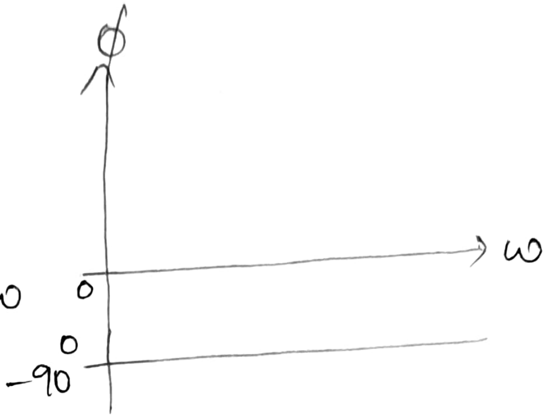
sol: $G(j\omega)H(j\omega) = \frac{1}{j\omega} = j\left(\frac{-1}{\omega}\right)$

$M_{dB} = 20\log\left(\frac{1}{\omega}\right) = -20\log(\omega)$

$\phi = -90^\circ$

$\text{slope} = \frac{d}{d\log_{10}\omega}(M_{dB}) = -20$





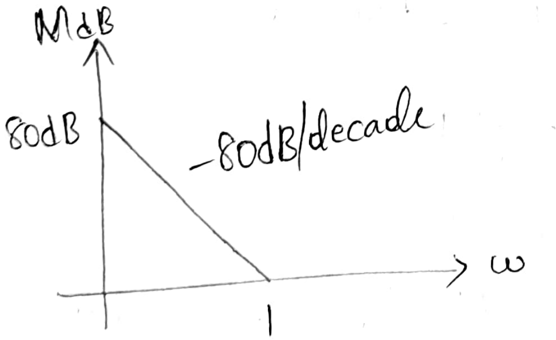
eg: $GH = \frac{100}{s^4}$

sol: $G(j\omega)H(j\omega) = \frac{100}{(j\omega)^4} = \frac{100}{\omega^4}$

$M_{dB} = 20\log_{10}\left(\frac{100}{\omega^4}\right) = 40 - 80\log_{10}\omega$

$\phi = 0^\circ$

slope = -80dB/decade

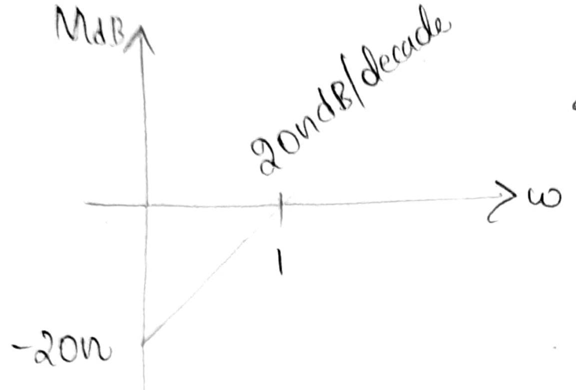


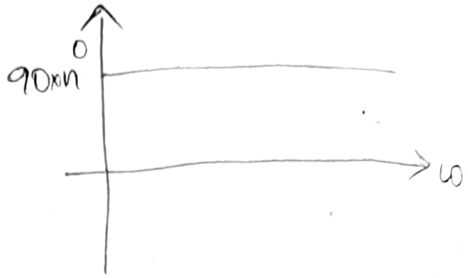
eg: $GH = s^n$

Sol:- $G(j\omega)H(j\omega) = (j\omega)^n$

$M_{dB} = 20\log_{10}|(j\omega)^n| = 20n\log_{10}\omega$

$\phi = (90n)^\circ$





eg: $GH = \frac{1}{1+s\tau}$

Sol:- $G(j\omega)H(j\omega) = \frac{1}{1+j\omega\tau}$

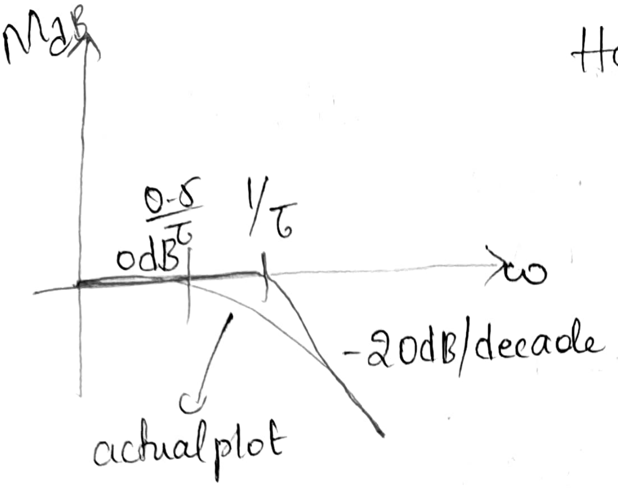
$M_{dB} = -20\log\sqrt{1+\omega^2\tau^2}$
$= -10\log(1+\omega^2\tau^2)$

$\phi = \tan^{-1}\left(\frac{-\omega\tau}{1}\right) = -\tan^{-1}(\omega\tau)$

Asymptotic/Approximation plot,

Case-①: $\omega\tau << 1$
$(M_{dB})_{asy} = 0\,dB$
$\phi_{asy} = 0^\circ$

Case-②: $\omega\tau >> 1$
$(M_{dB})_{asy} = -20\log(\omega\tau)$
$(\phi)_{asy} = -90^\circ$



Here $\omega = \frac{1}{\tau}$ is called
corner frequency
(or)
Break frequency.

$$\text{Error}\Big|_{\omega=\frac{1}{\tau}} = M_{actual} - M_{asymptotic}$$

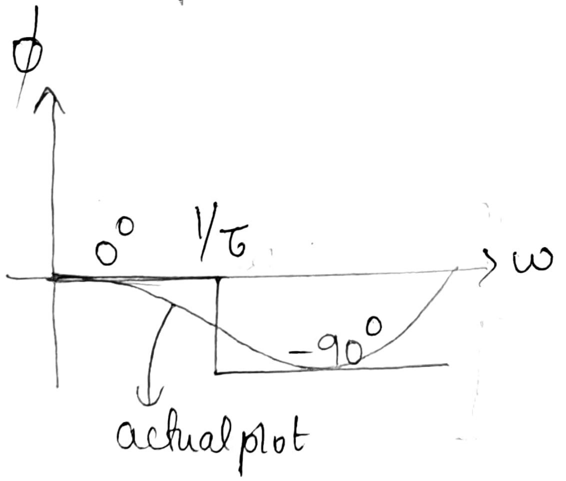
$$= -3dB - 0dB$$

$$\boxed{\text{Error} = -3dB}$$

$$\text{Error}\Big|_{\omega=\frac{0.5}{\tau}} = M_{actual} - M_{asymptotic}$$

$$= -0.96 - 0$$

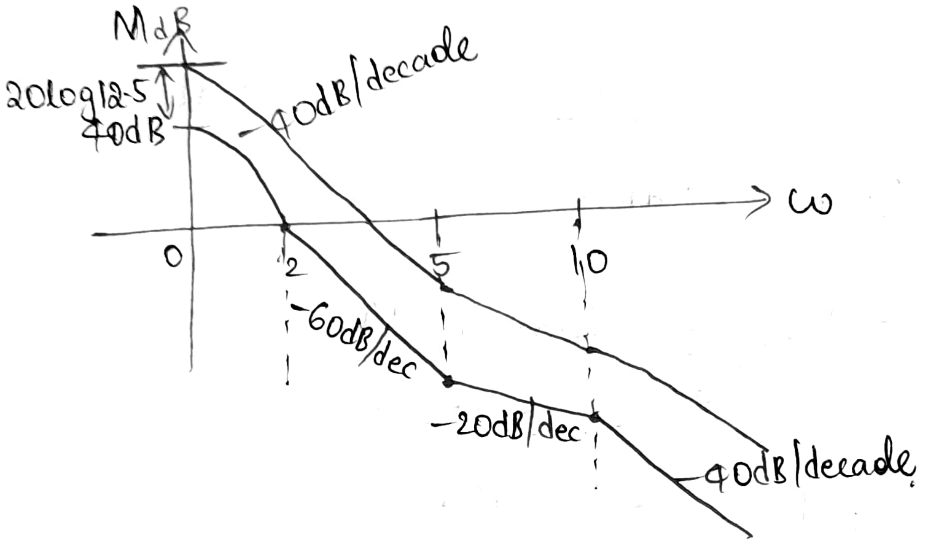
$$\boxed{E\Big|_{\omega=\frac{0.5}{\tau}} = -0.96dB}$$

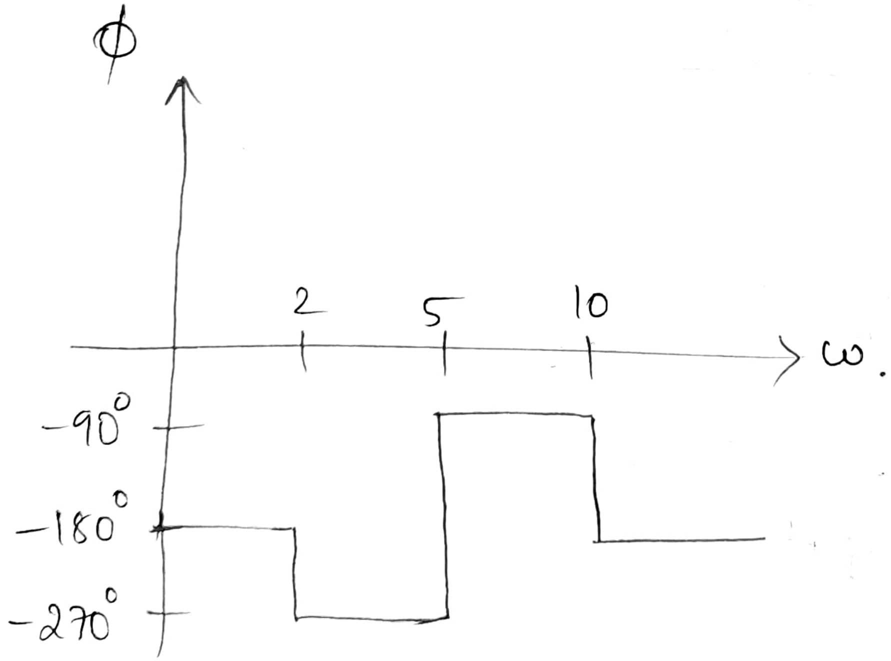


eg: $G(s)H(s) = \dfrac{10(s+5)^2}{s^2(s+2)(s+10)}$

Sol: $G(s)H(s) = \dfrac{K(1+sz_1)(1+sz_2)}{(1+sp_1)(1+sp_2)}$ → should be of form

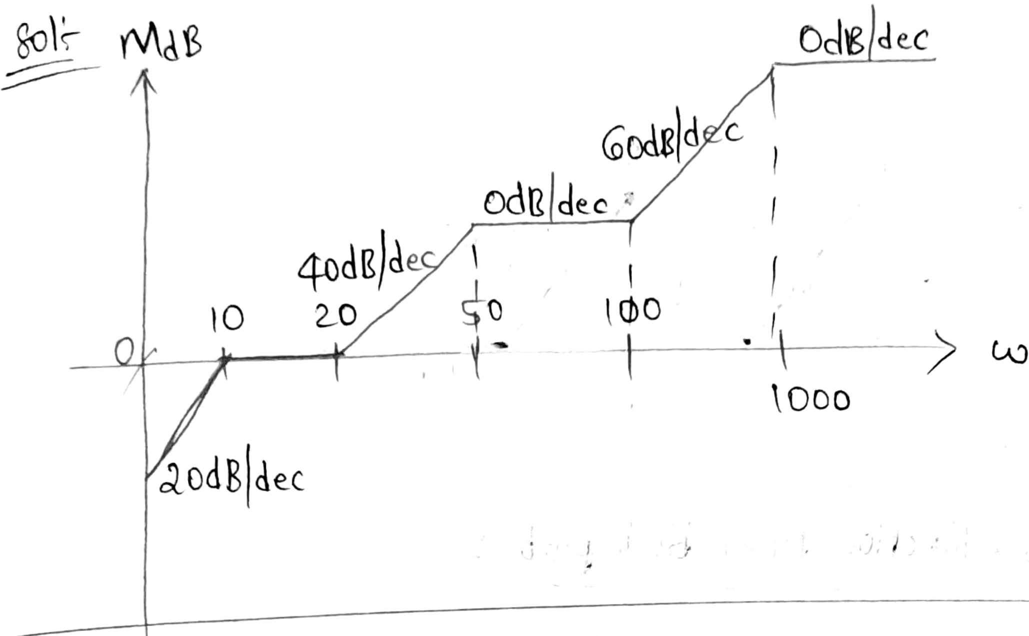
$$G(s)H(s) = \frac{12.5\left(1+\frac{s}{5}\right)^2}{s^2\left(1+\frac{s}{2}\right)\left(1+\frac{s}{10}\right)}$$





eg: $G(s)H(s) = \dfrac{s\left(1+\frac{s}{20}\right)^2\left(1+\frac{s}{100}\right)^3}{\left(1+\frac{s}{10}\right)\left(1+\frac{s}{50}\right)^2\left(1+\frac{s}{1000}\right)^3}$

Sol:



$$G(s) = \frac{K(1+sT_a)(1+sT_b)^x}{s^P(1+sT_1)(1+sT_2)^y}$$

① determine corner frequencies,

$0, \frac{1}{T_a}, \frac{1}{T_b}, \frac{1}{T_1}; \frac{1}{T_2}$

② Approximate magnitude in each freq range

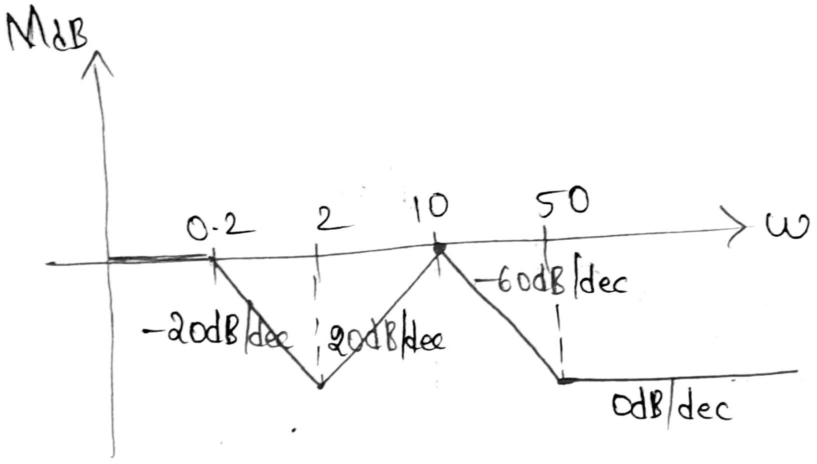
if $\omega > \omega_c$ → ignore 1 in mag

if $\omega < \omega_c$ → ignore $\frac{\omega}{\omega_c}$ in mag

③ find slope of magnitude in each range

④ Plot continuous curve of st. lines to get Bode plots.

eg: $G(s)H(s) = \dfrac{(1+S/2)^2(1+S/50)^3}{\left(1+\frac{S}{0.2}\right)\left(1+\frac{S}{10}\right)^4}$



- <u>Transfer function from Bode plot</u> :

1) Identify corner frequency.
2) At each corner frequency, determine change in frequency

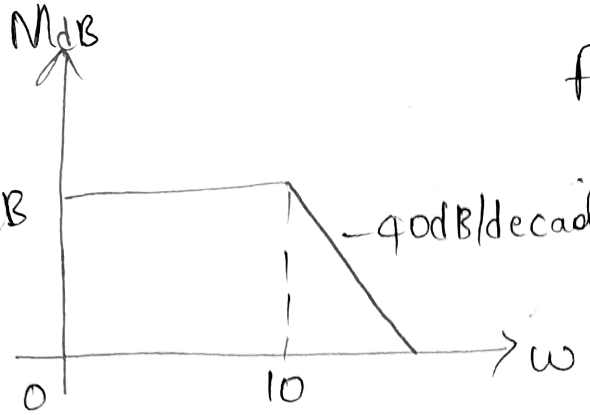
   change = + 20n dB/decade

   n → order of zero

   change = − 20n dB/decade

   n → order of pole
3) Write T.F with poles & zeroes & gain k.

eg:



find GH using M_{dB} Vs ω plot ?

Sol:- $GH = \dfrac{k}{\left(1+\frac{s}{10}\right)^2}$

$$M_{dB}\Big|_{\omega=0} = 80\text{ dB}$$

$$20\log k - 40\log\left|\left(1+\frac{s}{10}\right)\right| = 80$$

$$20\log k - 40\log\left|1+\frac{j\omega}{10}\right| = 80$$

$$20\log k - 40\log\sqrt{1+\frac{\omega^2}{100}} = 80$$

$$20\log k - 20\log\left(1+\frac{\omega^2}{100}\right) = 80$$

at $\omega=0$

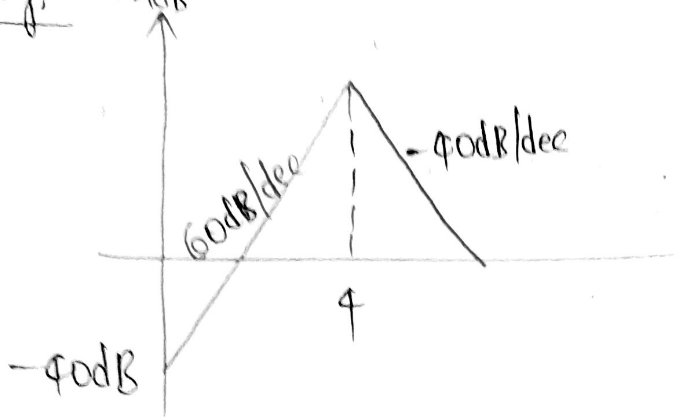
$$20\log k - 0 = 80$$

$$\log k = 1.5$$

$$\boxed{k = 31.6}$$

$$\therefore\ GH = \frac{31.6}{\left(1+\frac{s}{10}\right)^2}$$

e.g.



Sol:

$$GH = \frac{ks^3}{\left(1+\frac{s}{4}\right)^5} = \frac{10s^3}{\left(1+\frac{s}{4}\right)^5}$$

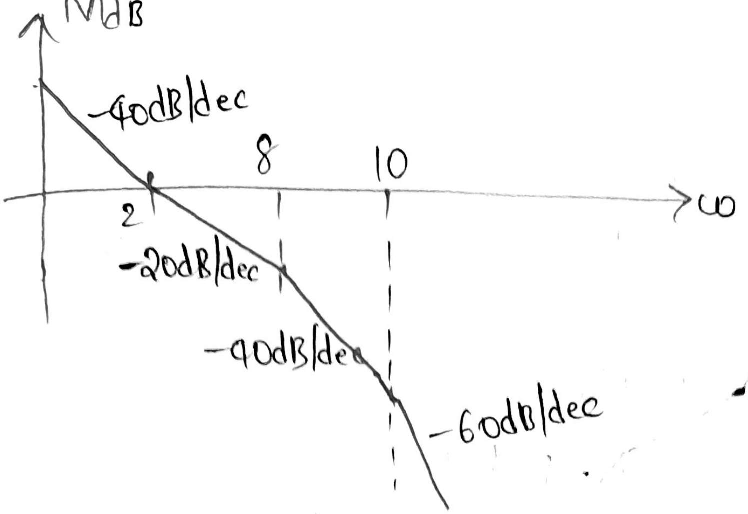
$$M_{dB}\Big|_{\omega=0.1} = -40$$

$$20\log k + 60\log 0.1 = -40$$

$$\log k = 1$$

$$\boxed{k=10}$$

eg:



Sol:
$$GH = \frac{K\left(1+\frac{s}{2}\right)}{s^2\left(1+\frac{s}{8}\right)\left(1+\frac{s}{10}\right)}$$

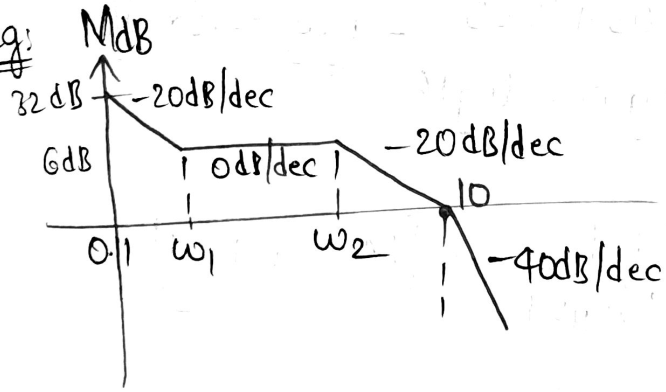
$$M_{dB}\big|_{\omega=2} = 0$$

$$20\log K - 40\log 2 + 20\log\sqrt{2} = 0$$

$\cancel{K=4}$ $\boxed{K = 2.81}$

---

eg:



Find ω_1, ω_2 and GH ?

$$GH = \frac{K\left(1+\frac{s}{2}\right)}{s\left(1+\frac{s}{5}\right)\left(1+\frac{s}{10}\right)}$$

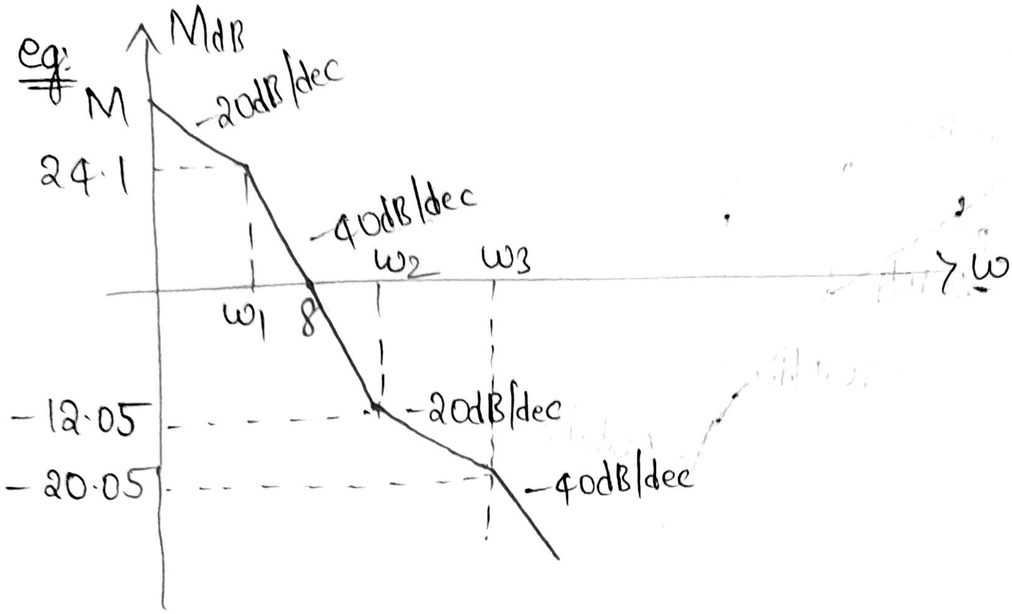
$$-20 = \frac{6-32}{\log\omega_1 - \log 0.1} \rightarrow \underline{\omega_1 = 2 \text{ rad/sec}}$$

$$-20 = \frac{0-6}{\log 10 - \log\omega_2} \rightarrow \underline{\omega_2 = 5 \text{ rad/sec}}$$

$$M_{dB}\big|_{\omega=0.1} = 32\text{ dB}$$

$$20\log K - 20\log 0.1 = 32 \rightarrow \boxed{K=4}$$

eg:



Sol:

$$-40 = \frac{0-24.1}{\log 8 - \log \omega_1}$$

$$\boxed{\omega_1 = 2 rad/s}$$

$$-40 = \frac{12.05}{\log 8 - \log \omega_2} \rightarrow \boxed{\omega_2 = 16 rad/s}$$

$$-20 = \frac{-20.05+12.05}{\log \omega_3 - \log 16} \rightarrow \boxed{\omega_3 = 40 rad/s}$$

$$-20 = \frac{M-24.1}{-\log 2 + \log 0.1} \rightarrow \boxed{M = 30.12 dB}$$

$$GH = \frac{K\left(1+\frac{S}{16}\right)}{S\left(1+\frac{S}{2}\right)\left(1+\frac{S}{40}\right)}$$

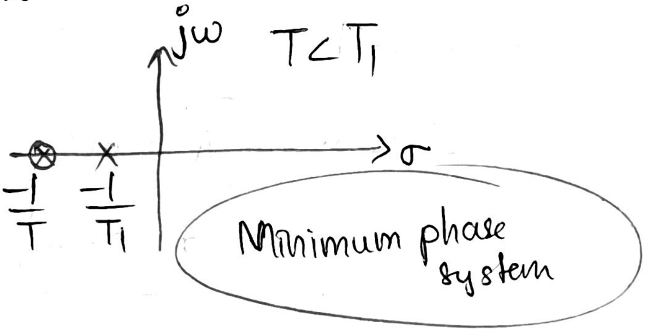
⊛ Minimum phase & Non-minimum phase system:

→ Transfer function having neither poles nor zeroes in the right half s-plane is called Minimum phase system.

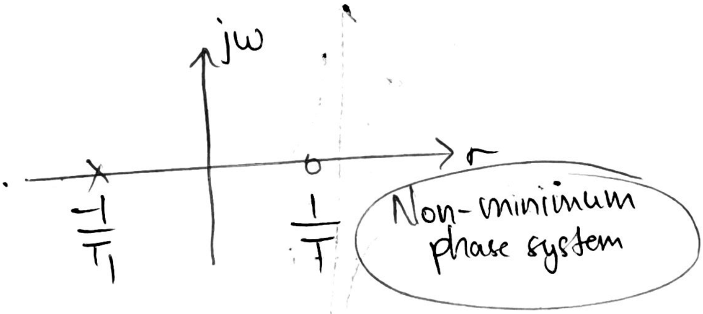
→ Transfer functions having poles/zeroes in the right half of s-plane is called Non-minimum phase system.

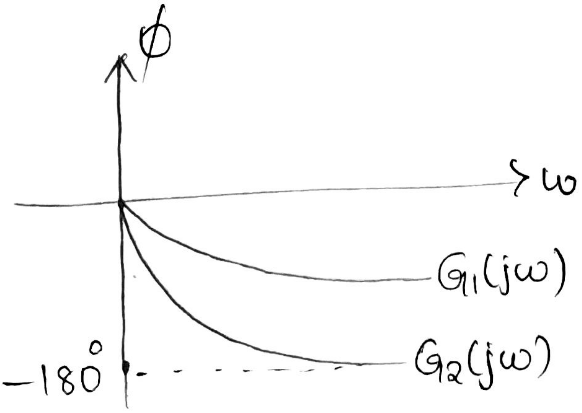
Allpass
∘ ~~α~~-system: The system in which zeroes lies on right half s-plane and poles lies on left half s-plane and the locus of pole-zero plane is symmetric about imaginary ~~ax~~ axis and that system is said to be α-system.

eg: $G_1(s) = \dfrac{1+Ts}{1+T_1 s}$

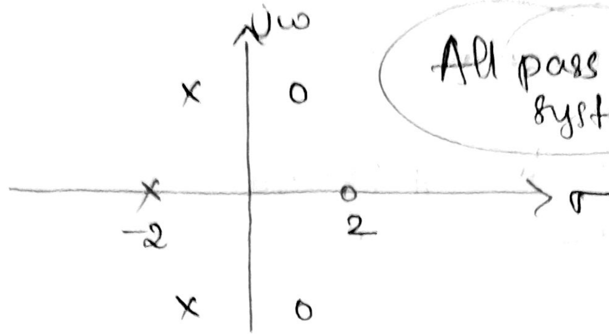


$G_2(s) = \dfrac{1-Ts}{1+T_1 s}$





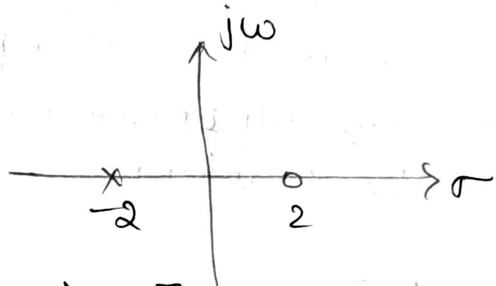
eg: $G_8(s) = \dfrac{(s-2)(s^2 - 2s + 2)}{(s+2)(s^2+2s+2)}$



All pass system

$|G(j\omega)| = 1$

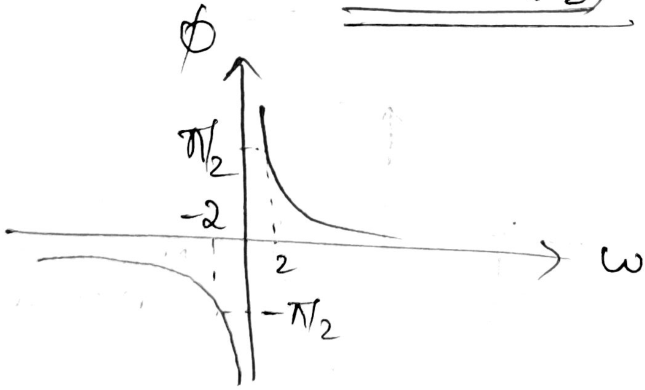
eg: $G_9(s) = \dfrac{s-2}{s+2}$



$|G(j\omega)| = 1$

$$\angle G_9(j\omega) = \left[\pi - \tan^{-1}\left(\frac{\omega}{2}\right)\right] - \tan^{-1}(\omega/2)$$

$$= \underline{\underline{\pi - 2\tan^{-1}\left(\frac{\omega}{2}\right)}}$$



eg: $G(s) = \dfrac{(s-1)(s+4)}{(s+2)(s+3)}$ → Non-minimum phase system.

$= \underbrace{\dfrac{(s+1)(s+4)}{(s+2)(s+3)}}_{\text{Min-phase system}} \times \underbrace{\dfrac{(s-1)}{s+1}}_{\text{All pass system}}$

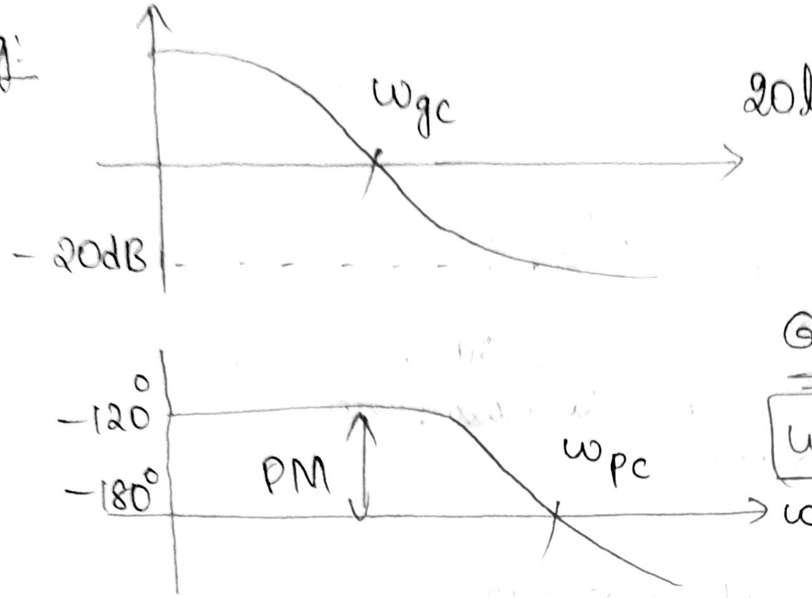
⊛ Gain margin & phase margin:

→ Gain margin (GM) $= -20\log M \big|_{\omega=\omega_{pc}}$ $(M = |GH|)$ (ω_{pc} → phase cross over frequency → $\phi = -180^\circ$)

→ Phase margin (PM) $= 180 + \phi \big|_{\omega=\omega_{gc}}$
(ϕ – phase angle of GH)
(ω_{gc} → gain cross over frequency) → $|M| = 0\,dB = 1$

- Stability → GM, PM should be positive
  $\omega_{pc} > \omega_{gc}$
- Marginally stable → GM = 0 dB, PM = 0°
  $\omega_{pc} = \omega_{gc}$
- Unstable → GM, PM are negative
  $\omega_{pc} < \omega_{gc}$

eg:



$20\log M = -20dB$

$GM = -20\log M$

$= -(-20dB)$

$GM = 20dB$

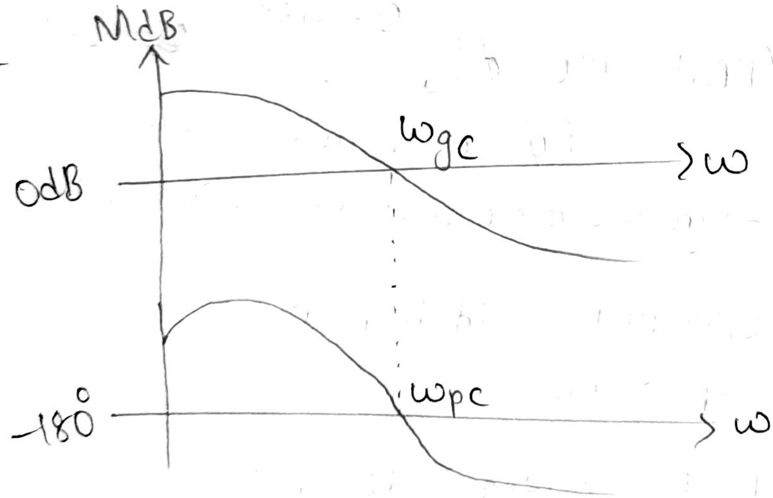
$\boxed{\omega_{pc} > \omega_{gc}}$

stable

$PM = 180 + (-120°)$

$PM = 60°$

eg:



$GM = 0dB$

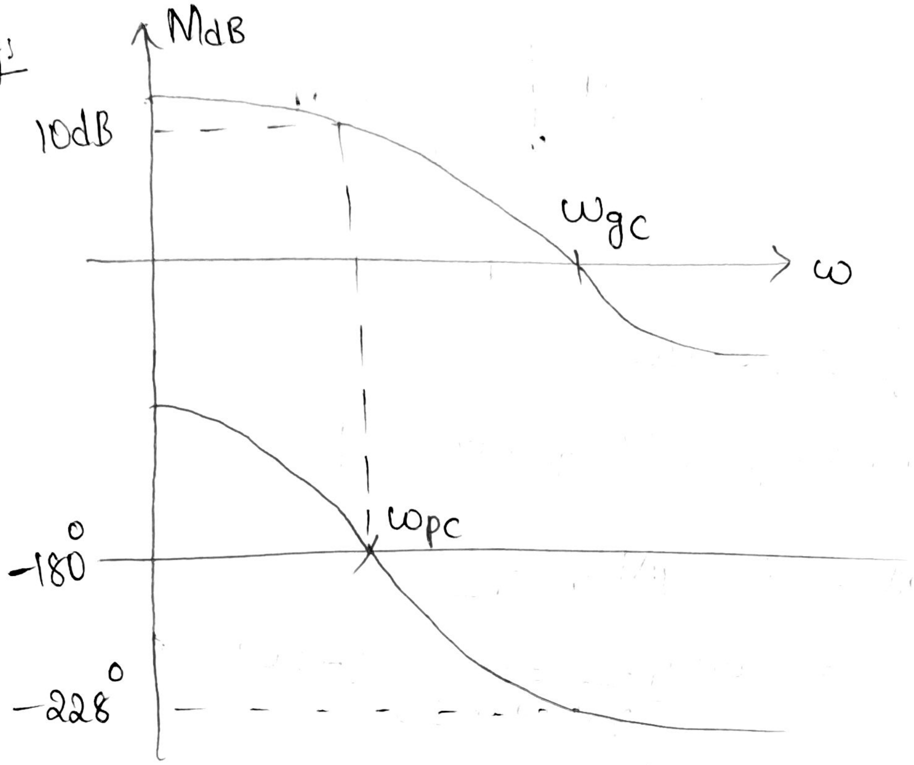
$PM = +180 + (-180)$

$PM = 0°$

$\boxed{\omega_{gc} = \omega_{pc}}$

Marginally stable

eg:



GM = −10dB

PM = 180 + (−228)

PM = −48°

$\boxed{\omega_{gc} > \omega_{pc}}$

Unstable

eg: $GH = \dfrac{1}{s(s+5)(s+10)}$. Compute GM and PM.

Soln:
$$\angle GH \big|_{\omega=\omega_{pc}} = -180^\circ$$

$$-90^\circ - \tan^{-1}\left(\frac{\omega_{pc}}{5}\right) - \tan^{-1}\left(\frac{\omega_{pc}}{10}\right) = -180^\circ$$

$$\tan^{-1}\left(\frac{\frac{\omega_{pc}}{5} + \frac{\omega_{pc}}{10}}{1 - \frac{\omega_{pc}^2}{50}}\right) = 90^\circ$$

$$1 - \frac{\omega_{pc}^2}{50} = 0$$

$$\omega_{pc}^2 = 50 \rightarrow \boxed{\omega_{pc} = 5\sqrt{2}\ \text{rad/s}}$$

$$|M|_{\omega=\omega_{gc}} = 1 \quad \text{or} \quad 20\log M \Big|_{\omega=\omega_{gc}} = 0\,dB$$

$\downarrow$

$$\frac{1}{\omega_{gc}\sqrt{\omega_{gc}^2+25}\,\sqrt{\omega_{gc}^2+100}} = 1$$

$$(\omega_{gc}^2)(\omega_{gc}^2+25)(\omega_{gc}^2+100) = 1$$

$$GM = -20\log M \big|_{\omega=5\sqrt{2}}$$

$$= 20\log \frac{1}{\sqrt{50\times75\times150}} \quad (+ve)$$

$$PM = 180^\circ + \phi \big|_{\omega=\omega_{gc}}$$

eg: The openloop transfer function of system is

$$GH = \frac{k}{s(s+2)(s+4)}$$

. Determine the value of k such that phase margin $= 60^\circ$.

Sol:

$$PM = 180^\circ + \phi\big|_{\omega=\omega_{gc}} = 60^\circ$$

$$\angle GH\big|_{\omega=\omega_{gc}} = -90^\circ - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{4}\right)$$

$$-180^\circ = -90^\circ - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{4}\right)$$

$$\boxed{\omega_{gc} = 0.72 \text{ rad/s}}$$

$$|GH|_{\omega=\omega_{gc}} = 1$$

$$20\log M\big|_{\omega=\omega_{gc}} = 0\,dB$$

$$20\log \frac{k}{\sqrt{(0.72)^2((0.72)^2+4)((0.72)^2+16)}} = 0\,dB$$

* Polar plot:

→ Rules for drawing polar plot:

$$GH = K\frac{N(s)}{D(s)}$$

1) Substitute $s = j\omega$ in GH
2) Find magnitude and phase of GH

$|M|_{\omega=0}$ ↓ Starting magnitude

$\angle GH|_{\omega=0}$ ↓ Starting phase angle.

3) Find $|M|_{\omega=\infty}$, $\angle GH|_{\omega=\infty}$

$|M|_{\omega=\infty}$ ↓ Ending magnitude

$\angle GH|_{\omega=\infty}$ ↓ Ending phase angle.

4) Identify starting direction,
   - finite pole of GH near imaginary axis – starting direction (clockwise)
   - finite zero of GH near imaginary axis, – starting direction (anti-clockwise)

5) Ending direction,

$$\phi = \angle GH|_{\omega=0} - \angle GH|_{\omega=\infty}$$

- if ϕ is +ve then ending direction is clockwise.
- if ϕ is -ve then ending direction is Anticlockwise.

eg: $GH = \frac{1}{1+s\tau}$

$G(j\omega)H(j\omega) = \frac{1}{1+j\omega\tau}$

at $\omega = 1/\tau$

$|GH| = 0.707$

$\angle GH = -45^\circ$

at $\omega = 0$,

$|GH| = 1$

$\angle GH = 0^\circ$

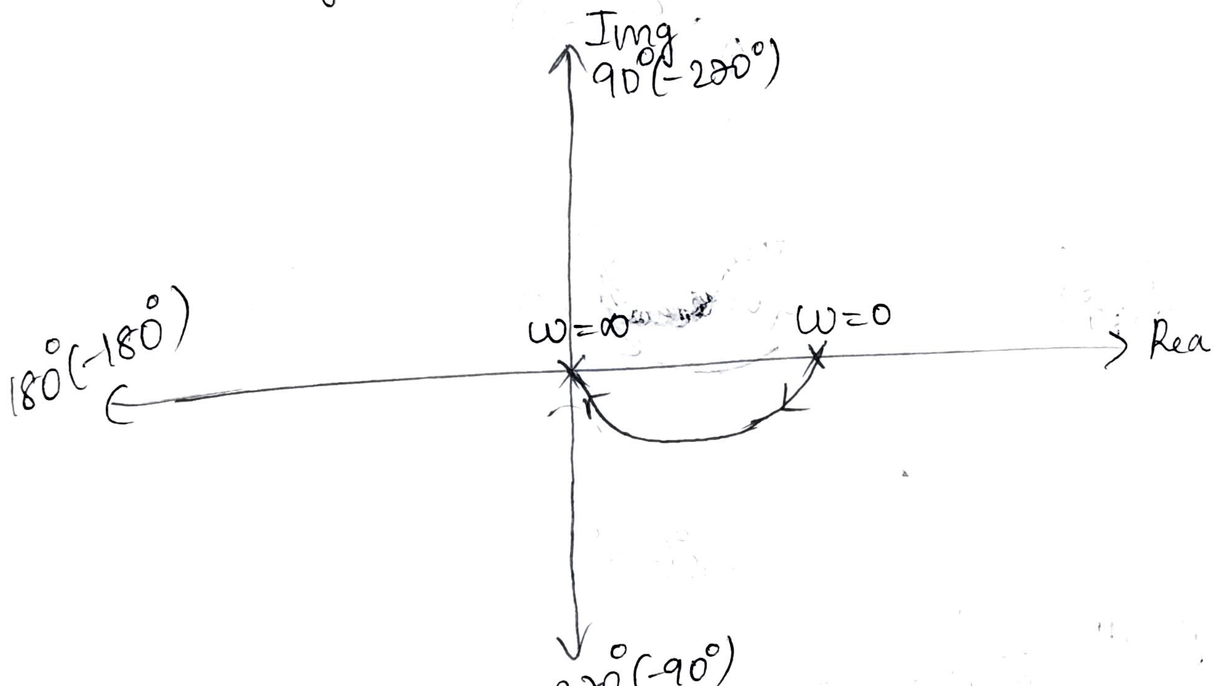
at $\omega = \infty$,

$|GH| = 0$

$\angle GH = -90^\circ$

starting direction → clockwise

Ending direction → clockwise (Since, $\phi = +ve$)



eg: $GH = \frac{1}{(s+1)(s+2)}$ $\quad \phi = -\tan^{-1}(\omega) - \tan^{-1}(\omega/2)$

$$G(j\omega)H(j\omega) = \frac{1}{(1+j\omega)(2+j\omega)}$$

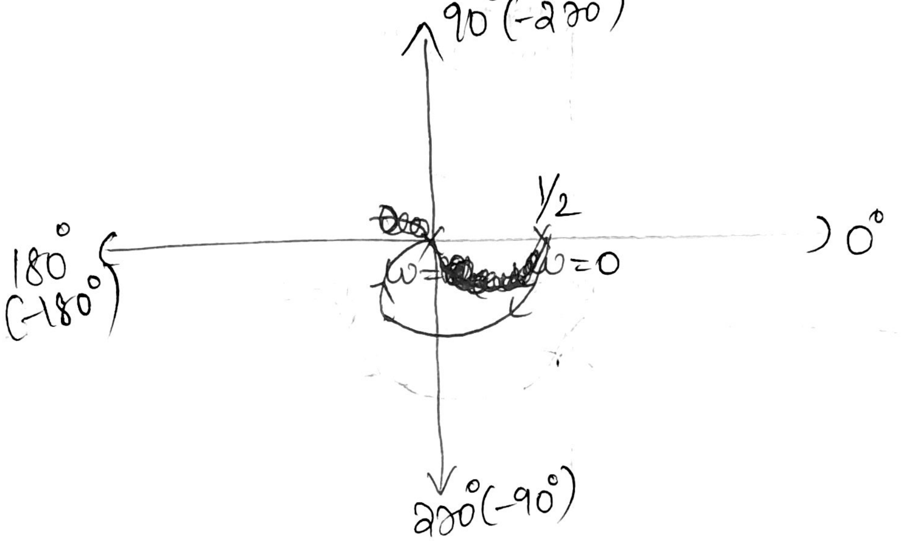
at $\omega = 0$ $\quad$ at $\omega = \infty$,

$|GH| = 1/2$ $\quad$ $|GH| = 0$

$\angle GH = 0^\circ$ $\quad$ $\angle GH = -180^\circ$

starting direction → clockwise

Ending direction → clockwise (since ϕ = +ve)



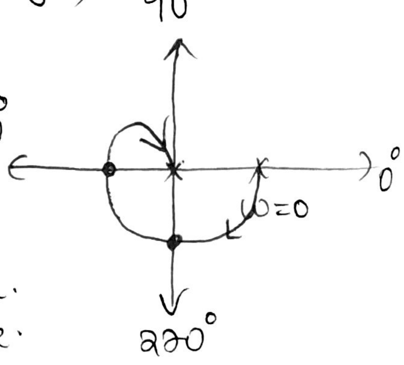
eg: $GH = \frac{1}{(s+1)(s+2)(s+3)}$ $\quad \phi = -\tan^{-1}(\omega) - \tan^{-1}(\omega/2) - \tan^{-1}(\omega/3)$

$$G(j\omega)H(j\omega) = \frac{1}{(1+j\omega)(2+j\omega)(3+j\omega)}$$

at $\omega = 0$ $\quad$ at $\omega = \infty$

$|GH| = 1/6$ $\quad$ $|GH| = 0$

$\angle GH = 0^\circ$ $\quad$ $\angle GH = -270^\circ$



starting direction → clockwise.

Ending direction → clockwise.

$\angle\phi = 270^\circ$

$$-\tan^{-1}(\omega) - \tan^{-1}(\omega/2) - \tan^{-1}(\omega/3) = -90^\circ$$

$$-\tan^{-1}(\omega) - \tan^{-1}\left(\frac{\omega/2 + \omega/3}{1 - \frac{\omega^2}{6}}\right) = -90^\circ$$

$$-\tan^{-1}\left(\frac{\omega+\frac{5\omega}{6-\omega^2}}{\frac{5\omega}{6-\omega^2}}\right) = -90^\circ$$

$$+\tan^{-1}\left(\frac{11\omega-\omega^3}{5\omega}\right) = +90^\circ$$

$$\boxed{\omega = \pm 1} \text{ rad/s}$$

$$\angle\phi = 180^\circ$$

$$-\tan^{-1}(\omega)-\tan^{-1}\left(\frac{\omega}{2}\right)-\tan^{-1}\left(\frac{\omega}{3}\right) = 180^\circ$$

$$\underline{\omega = \sqrt{11} \text{ rad/s}}$$

eg) $GH = \dfrac{1}{s^2(s+1)}$

Soln $G(j\omega)H(j\omega) = \dfrac{1}{(j\omega)^2(1+j\omega)}$ $\quad \phi = -180^\circ - \tan^{-1}(\omega)$

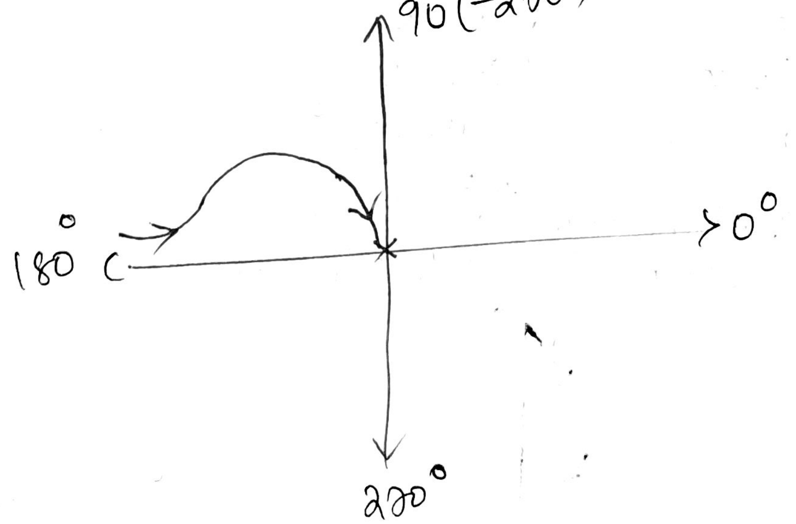
at $\omega = 0$, $\quad$ at $\omega = \infty$,

$|GH| = \infty$ $\quad |GH| = 0$

$\angle GH = -180^\circ$ $\quad \angle GH = -270^\circ$

Starting direction → clockwise

Ending direction → clockwise



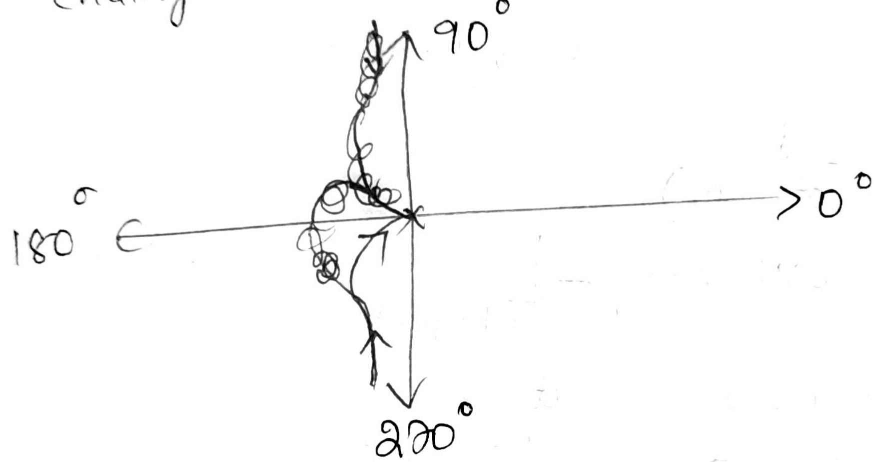
eg: $GH = \frac{1}{s(s+1)}$

Sol: $G(j\omega)H(j\omega) = \frac{1}{j\omega(1+j\omega)}$ $\phi = -90 - \tan^{-1}(\omega)$

at $\omega = 0$,
$|GH| = \infty$
$\angle GH = -90^\circ$

at $\omega = \infty$,
$|GH| = 0$
$\angle GH = -180^\circ$

Starting direction → clockwise
Ending direction → clockwise



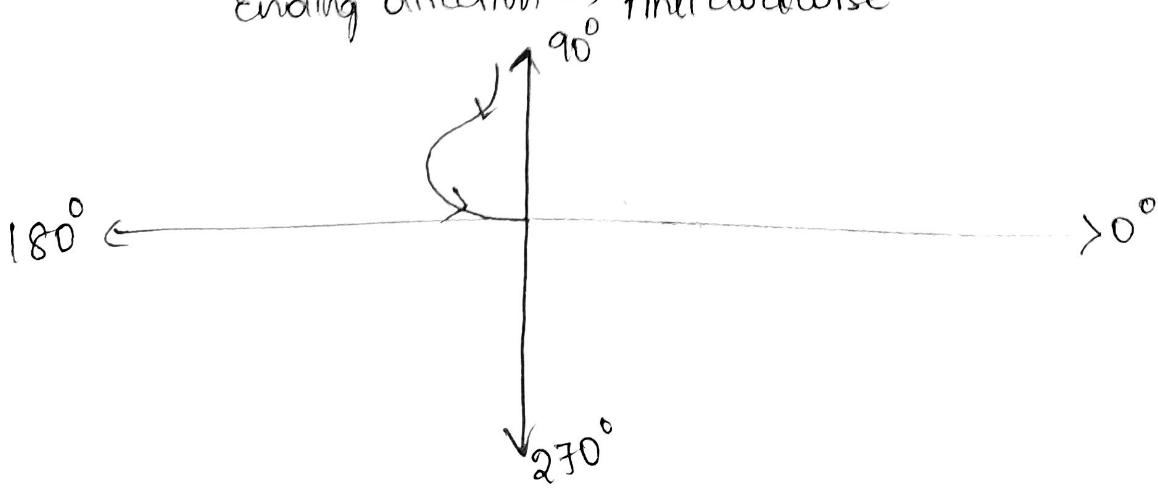
eg: $GH = \frac{s+1}{s^3}$

Sol: $G(j\omega)H(j\omega) = \frac{j\omega+1}{(j\omega)^3}$ $\phi = \tan^{-1}(\omega) - 270^\circ$

at $\omega = 0$
$|GH| = \infty$
$\angle GH = -270^\circ$

at $\omega = \infty$
$|GH| = 0$
$\angle GH = -180^\circ$

Starting direction → Anti clockwise
Ending direction → Anti clockwise



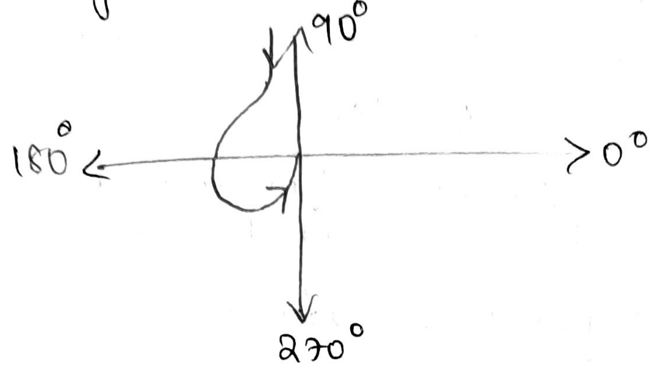
eg: $GH = \frac{(s+1)(s+2)}{s^3}$

at $\omega = 0 \rightarrow \infty \angle -270^\circ$ $\quad \phi = \tan^{-1}(\omega) + \tan^{-1}\left(\frac{\omega}{2}\right) - 270^\circ$

at $\omega = \infty \rightarrow 0 \angle -90^\circ$

Starting direction $\rightarrow$ ACW

Ending direction $\rightarrow$ ACW



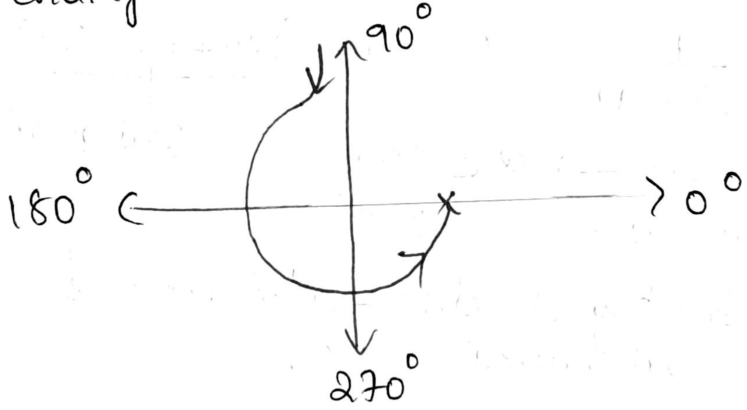
eg: $GH = \frac{(s+1)(s+2)(s+3)}{s^3}$ $\quad \phi = \tan^{-1}(\omega) + \tan^{-1}\left(\frac{\omega}{2}\right) + \tan^{-1}\left(\frac{\omega}{3}\right) - 270^\circ$

at $\omega = 0 \rightarrow \infty \angle -270^\circ$

at $\omega = \infty \rightarrow 1 \angle +0^\circ$

Starting direction $\rightarrow$ ACW

Ending direction $\rightarrow$ ACW



eg: $GH = \frac{1}{s(s+1)}$

at $\omega = 0 \rightarrow \infty \angle -90^\circ$ } Done ✓ already

---

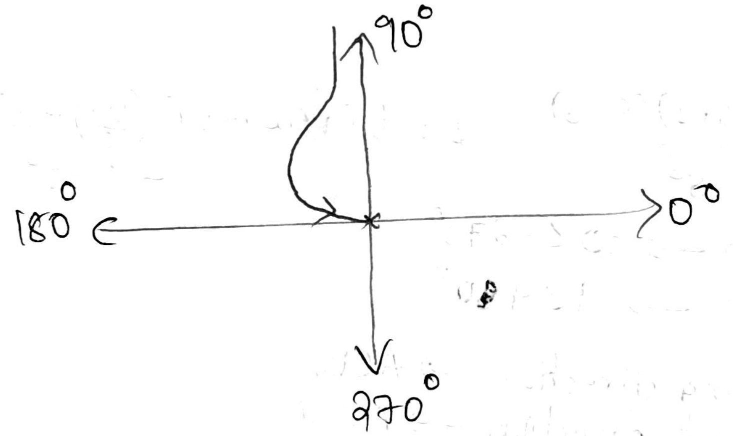
eg: $GH = \frac{1}{s(s-1)}$ $\quad \phi = -90^\circ - (\underbrace{180 - \tan^{-1}\omega}_{\text{2nd quadrant}})$

Sol: at $\omega = 0 \rightarrow \infty \angle -270^\circ$

at $\omega = \infty \rightarrow 0 \angle -180^\circ$

starting direction $\rightarrow$ Not required (if it contains -ve terms)

Ending direction $\rightarrow$ ACW



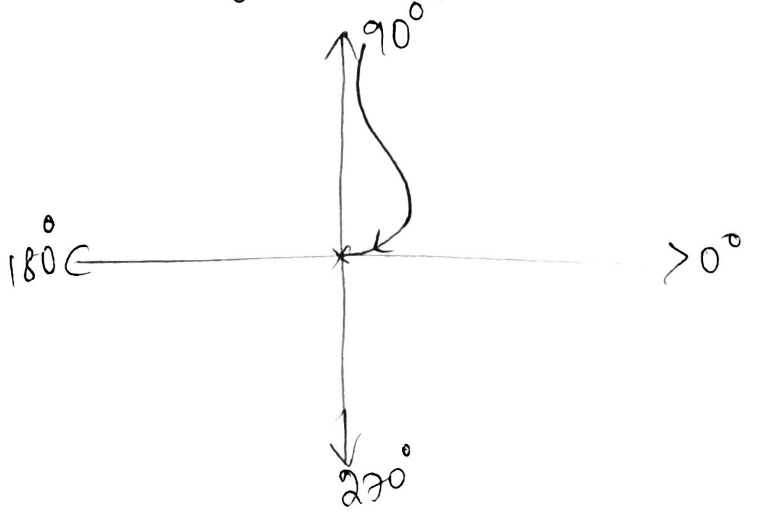
eg: $GH = \frac{1}{s(-s-1)}$ $\quad \phi = -90^\circ - (\underbrace{180 + \tan^{-1}(\omega)}_{\text{3rd quadrant}})$

Sol: at $\omega = 0 \rightarrow \infty \angle -270^\circ$

at $\omega = \infty \rightarrow 0 \angle -360^\circ$

Starting direction $\rightarrow$ Not required

Ending direction $\rightarrow$ CW



eg: $GH = \frac{1}{s(-s+1)}$ $\phi = -90^\circ - (-\tan^{-1}(\omega))$

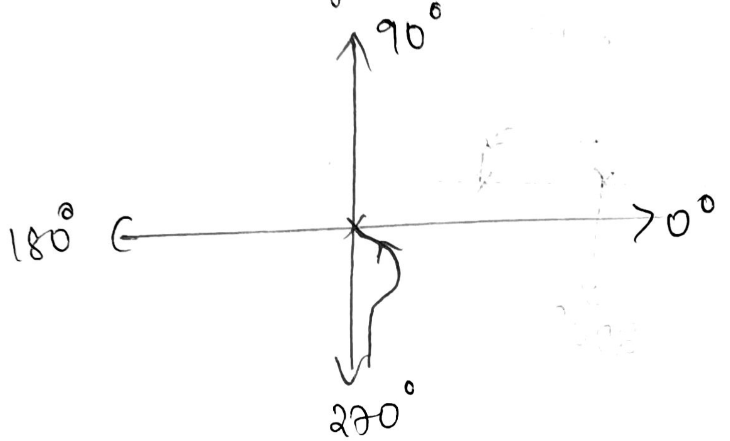
4th quadrant

Sol: at $\omega = 0 \rightarrow \infty \angle -90^\circ$

at $\omega = \infty \rightarrow 0 \angle 0^\circ$

Starting direction → Not required

Ending direction → ACW



$-\frac{\omega}{\omega^2+1}$

-2

eg: $GH = \frac{(S+2)}{(S+1)(s-1)} = \frac{s+2}{(s^2-1)} = \frac{j\omega+2}{-\omega^2-1}$

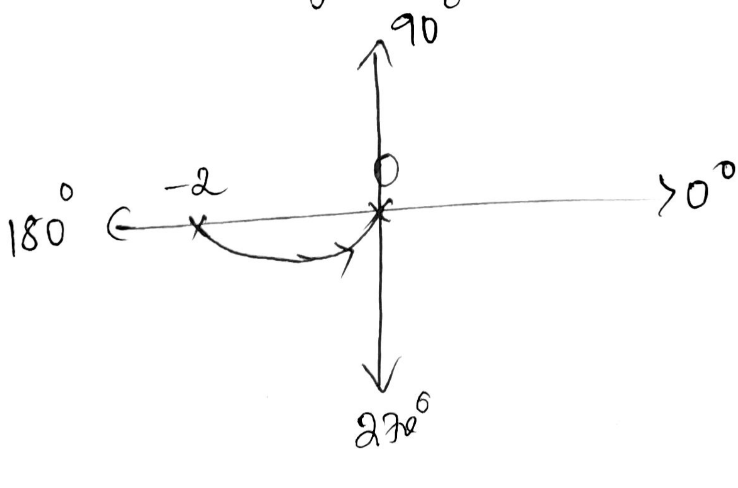
$\phi = \tan^{-1}(\omega/2) - \tan^{-1}(\omega) - (180 - \tan^{-1}(\omega))$

Sol: at $\omega = 0 \rightarrow$ ~~[illegible]~~ $-2 \angle -180^\circ$

at $\omega = \infty \rightarrow 0 \angle -90^\circ$

Starting direction → Not required

Ending direction → ACW



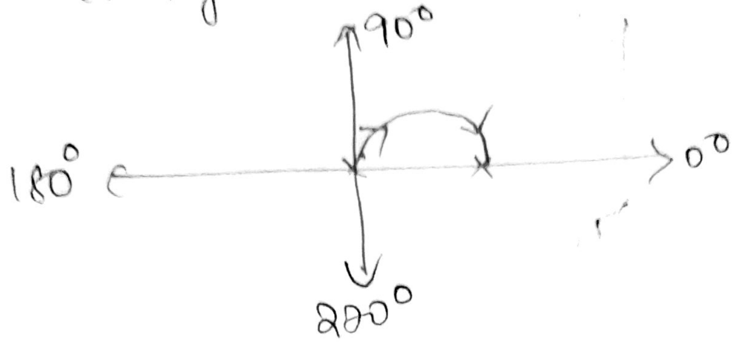
eg: $GH = \frac{s}{s+1}$ $\qquad \phi = 90° - \tan^{-1}(\omega)$

at $\omega = 0 \rightarrow 0\angle 90°$

at $\omega = \infty \rightarrow 1\angle 0°$

Starting direction → CW

Ending direction → CW



Nyquist plot:

→ Rules for drawing Nyquist plot:

1) Draw the polar plot.
2) Draw the mirror image of polar plot about real axis.
3) The no. of infinite radius half circles will be equal to no. of poles or zeroes at origin.

- The infinite radius half circles started at the point where mirror image of polar plot ends. and this infinite radius half circle will end at where polar plot starts.

→ Stability analysis using Nyquist plot:

$$GH = K\frac{N(s)}{D(s)} \quad \text{—— (1)}$$

The closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G}{1+G} = \frac{G(s)\,D(s)}{D(s) + K\cdot N(s)} \quad \text{—— (2)}$$

characteristic eq, $1+GH=0 \rightarrow 1+G=0$

$$= 1 + K\frac{N(s)}{D(s)}$$

$$= \frac{D(s) + K(N(s))}{D(s)}$$

- poles of ch. eq = poles of open loop T.F
- zeroes of ch. eq = poles of closed loop T.F

} right s-plane

Nyquist stability criteria: $N = P - Z$

N – no. of encirclements of point $-1+j0$.
P – poles of ch. eq
Z – zeroes of ch. eq

1) Open loop system is said to be stable, when $\boxed{N = -Z}$

2) Closed loop system is said to be stable, when $\boxed{N = P}$

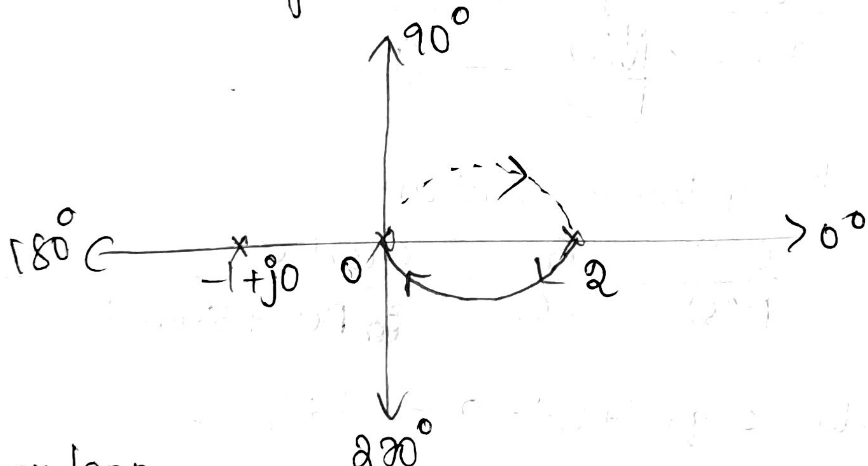
eg: $GH = \dfrac{10}{s+5}$ $\phi = -\tan^{-1}(\omega/5)$

Sol: at $\omega = 0 \rightarrow 2\angle 0^\circ$

at $\omega = \infty \rightarrow 0\angle -90^\circ$

Starting direction $\rightarrow$ CW

Ending direction $\rightarrow$ CW



Open loop,

Here, $Z = 0$

$N = 0$

Since $N = -Z$ (stable)

Closed loop,

Here, $P = 0$

$N = 0$

Since, $N = P$ (stable)

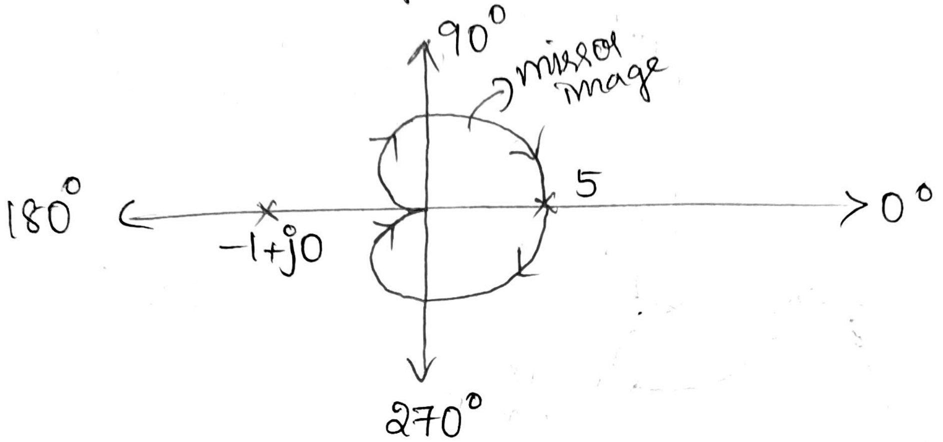
eg: $GH = \frac{10}{(s+1)(s+2)}$ $\phi = -\tan^{-1}(\omega) - \tan^{-1}(\omega/2)$

Sol:- at $\omega = 0 \rightarrow 5\angle 0^\circ$

at $\omega = \infty \rightarrow 0\angle 180^\circ$

Starting direction $\rightarrow$ CW

Ending direction $\rightarrow$ CW



Open loop system is stable when $N = -Z$,

Here, $Z = 0$ (No zeroes on right s-plane)

$N = 0$

Since $N = -Z$ (Here open loop system is stable)

Closed loop system is stable when $N = P$,

$P = 0$ (No poles on right s-plane)

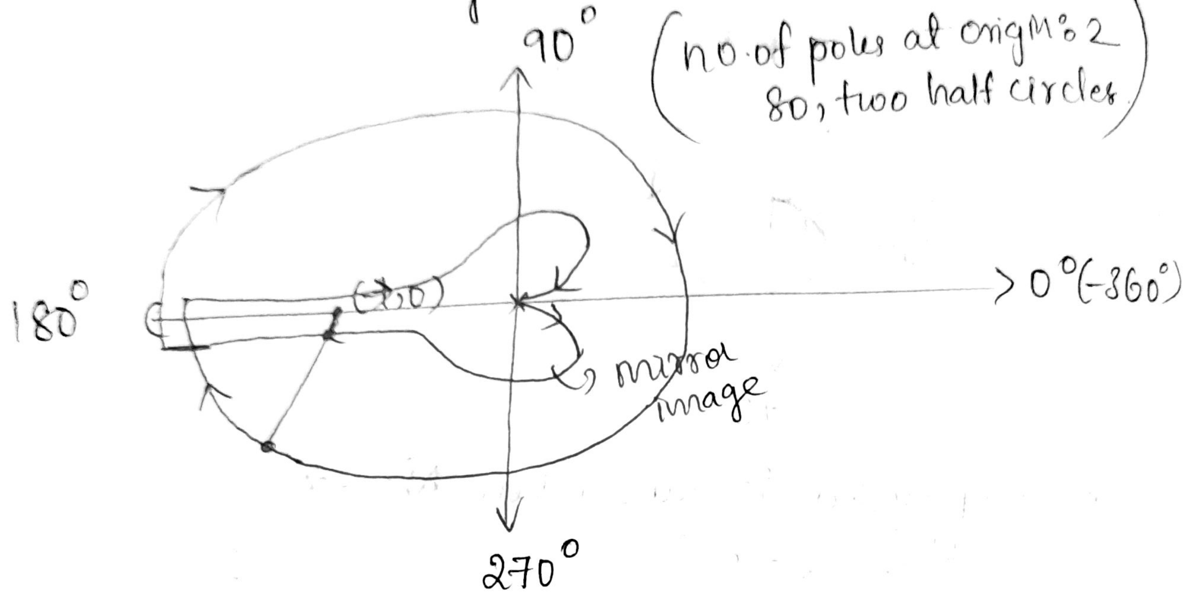
$N = 0$

Since $N = P$ (Here closed loop system is also stable) eg: $GH = \dfrac{10}{s^2(s+1)(s+2)}$ $\quad \phi = -180^\circ - \tan^{-1}(\omega) - \tan^{-1}(\omega/2)$

Sol: at $\omega = 0 \rightarrow \infty \angle -180^\circ$
at $\omega = \infty \rightarrow 0 \angle -360^\circ$

Starting direction → CW
Ending direction → CW

(no. of poles at origin: 2
so, two half circles)



Open loop system,

$N = P - Z$

$-2 = 0 - Z$

$\boxed{Z = 2}$

$N = -Z$
(stable)

closed loop system,

$P = 0$ (-ve)
$N = -2$ (Since clockwise)

$N \neq P$
(Unstable)

eg: $GH = \frac{1}{s^3(s+1)}$ $\phi = -270^\circ - \tan^{-1}(\omega)$

Sol: at $\omega = 0 \rightarrow \infty \angle -270^\circ$

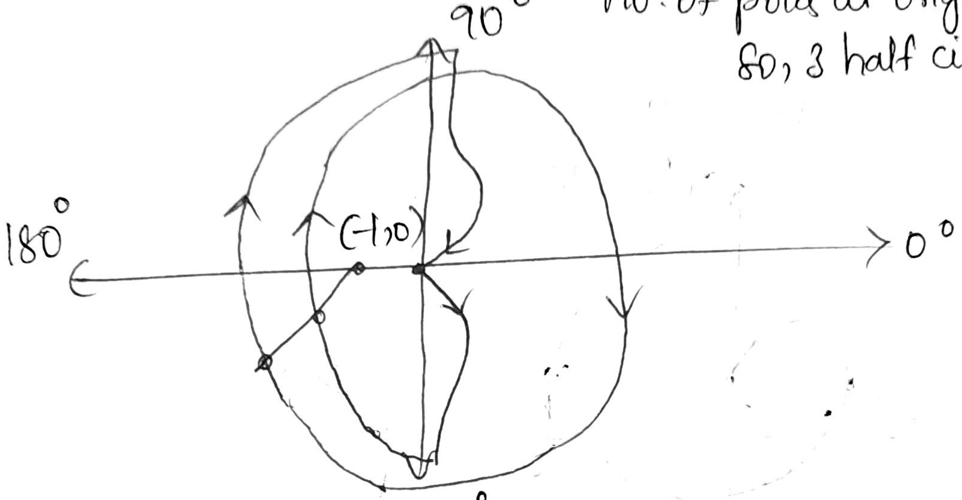
at $\omega = \infty \rightarrow 0 \angle -360^\circ$

S.D $\rightarrow$ CW

E.D $\rightarrow$ CW

no. of poles at origin: 3

So, 3 half circles



$N = -2$ (Since clockwise) (it is negative)

$P = 0$ (No poles on right s-plane)

$N \neq P$ (Unstable) $\rightarrow$ closed loop

$N = P - Z$

$-2 = 0 - Z \rightarrow \boxed{Z = 2}$

$N = -Z$ (stable open loop system)

eg: $GH = \dfrac{k}{(s+1)(s+2)(s+3)}$

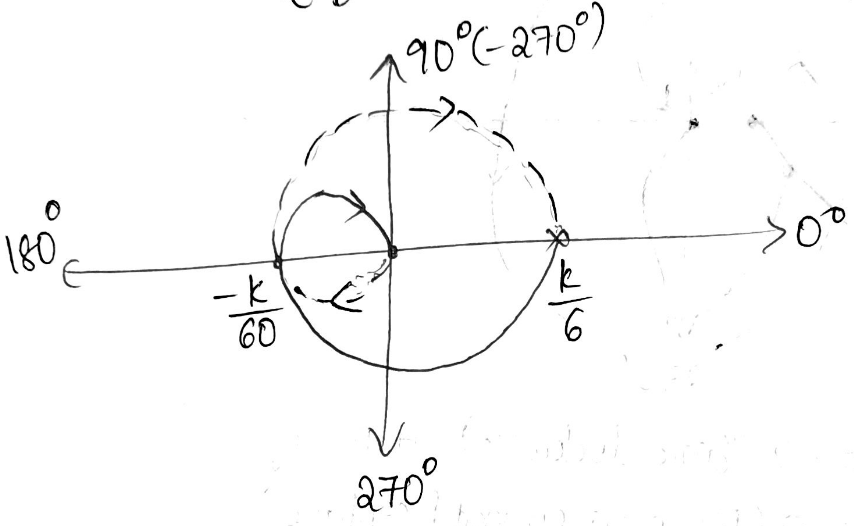
Determine 'k' for which closed loop system is s

Sol:- at $\omega = 0 \rightarrow \frac{k}{6} \angle 0^\circ$ $\quad \phi = -\tan^{-1}(\omega) - \tan^{-1}(\omega/2) - \tan^{-1}(\omega/3)$

at $\omega = \infty \rightarrow 0 \angle -270^\circ$

SD → CW

E.D → CW



$P = 0$
$N = 0$ } closed loop system is stable.

↓

$\dfrac{-k}{60} > -1$ $\qquad\qquad$ $\dfrac{k}{6} > -1$

$\boxed{K < 60}$ $\qquad\qquad$ $\boxed{K > -6}$

$\boxed{-6 < K < 60}$ → for which system is stable.

eg: $GH = \frac{K(s+2)}{(s+1)(s-1)}$. Determine 'k' for which system is stable.

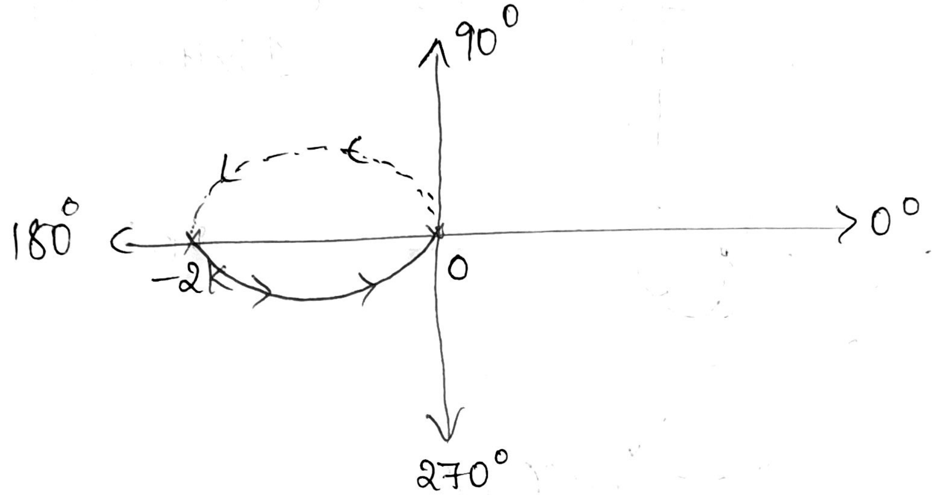
Sol: $\phi = -\tan^{-1}(\omega) - (180 - \tan^{-1}(\omega)) + \tan^{-1}\left(\frac{\omega}{2}\right)$

at $\omega = 0 \rightarrow -2K\angle -180^\circ$

at $\omega = \infty \rightarrow 0\angle -90^\circ$

S.D $\rightarrow$ ~~ACW~~ (Not required when -ve terms are present)

E.D $\rightarrow$ ACW



$N = P$

Here $P = 1$

so, $N = 1$

$\downarrow$

$-2K < -1$

$\boxed{K > 1/2}$

eg: $GH = \dfrac{K(s+3)}{s(s-1)}$ $\qquad \phi = \tan^{-1}\left(\frac{\omega}{3}\right) - 90^\circ - (180 - \tan^{-1}\omega)$

Sol: at $\omega = 0 \rightarrow \infty \angle -270^\circ$

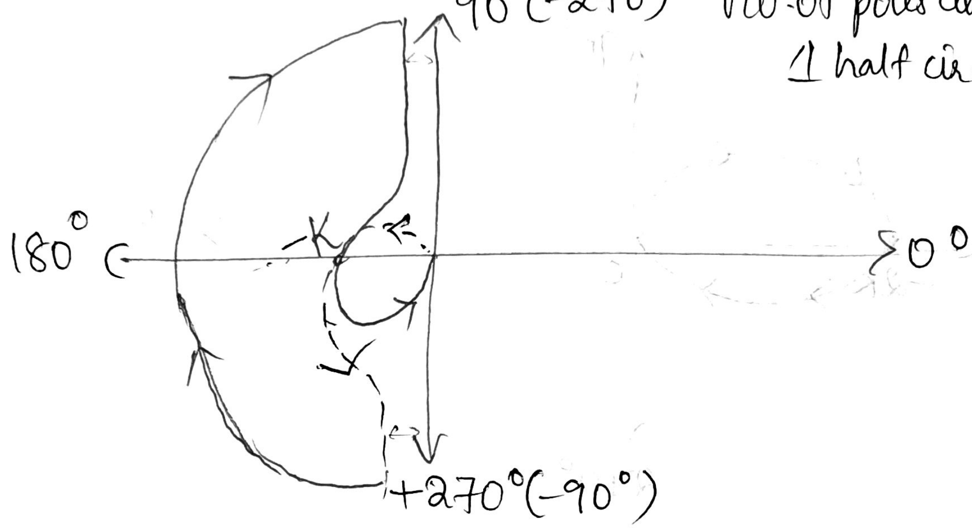
at $\omega = \infty \rightarrow 0 \angle -90^\circ$

S.D → Not required

E.D → ACW

no. of poles at origin: 1

1 half circle.



$$\phi = 180^\circ$$

$$-270^\circ + \tan^{-1}\left(\frac{\omega + \omega/3}{1 - \frac{\omega^2}{3}}\right) = 180^\circ$$

$$\tan^{-1}\left(\frac{4\omega}{3 - \omega^2}\right) = 360^\circ + 90^\circ$$

$$3 - \omega^2 = 0$$

$$\boxed{\omega = \pm\sqrt{3}}$$

$$M\Big|_{\omega = \pm\sqrt{3}} = \frac{K\sqrt{9+3}}{\sqrt{3}\cdot\sqrt{3+1}} = K$$

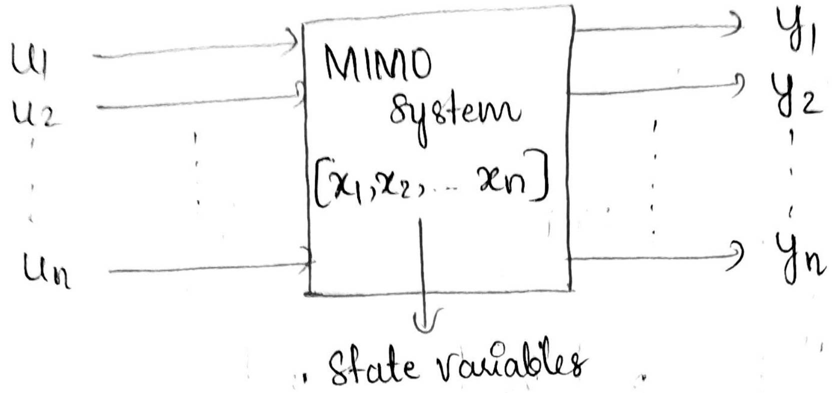
Here, $P = 1$ (One pole on right s-plane)

So, N should be '1' to be stable.

$(N = 1)$

* State-Space Analysis :

(15/11/22)



- **Standard form of state model:**

$$\underset{n\times 1}{\overset{\circ}{X}} = \underset{n\times n}{A}\,\underset{n\times 1}{X} + \underset{n\times m}{B}\,\underset{m\times 1}{U}$$

where,

$\overset{\circ}{X} \rightarrow$ state equation (or) Differential state vector $(n\times 1)$

$A \rightarrow$ state matrix $(n\times n)$

$X \rightarrow$ state vector $(n\times 1)$

$B \rightarrow$ input matrix $(n\times m)$

$U \rightarrow$ input vector $(m\times 1)$

$$\underset{P\times 1}{Y} = \underset{P\times n}{C}\,\underset{n\times 1}{X} + \underset{P\times m}{D}\,\underset{m\times 1}{U}$$

where,

$Y \rightarrow$ o/p vector

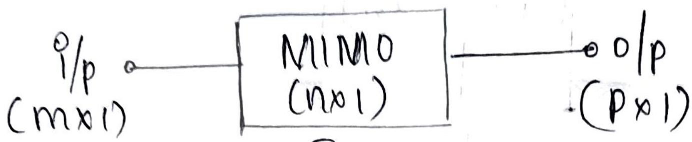
$C \rightarrow$ o/p matrix

$D \rightarrow$ transition matrix

eg: $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 5y = 10\,u(t)$

Sol: One i/p $\rightarrow m = 1$

one o/p $\rightarrow p = 1$



$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}$$

No. of state variables = Degree of D.E

So, $\boxed{n = 2}$

$$\dot{X}_{2\times1} = A_{2\times2} \cdot X_{2\times1} + B_{2\times1} \cdot U_{1\times1}$$

$$Y_{1\times1} = C_{1\times2} \cdot X_{2\times1} + D_{1\times1} \cdot U_{1\times1}$$

---

eg: $y''' + 2y'' + 3y' + y = u$

one i/p $\rightarrow m = 1$

one o/p $\rightarrow p = 1$

$\boxed{n = 3} \rightarrow x_1, x_2, x_3$

$$\dot{X}_{3\times1} = A_{3\times3} \cdot X_{3\times1} + B_{3\times1} \cdot U_{1\times1}$$

$$Y_{1\times1} = C_{1\times3} \cdot X_{3\times1} + D_{1\times1} \cdot U_{1\times1}$$

Assume, $x_1 = y$

$\dot{x}_1 = \dot{y} = x_2$

$\dot{x}_2 = \ddot{y} = x_3$

$\dot{x}_3 = \dddot{y}$

$\dot{x}_3 + 2\dot{x}_2 + 3\dot{x}_1 + x_1 = u \longrightarrow \dot{x}_3 = u - 2\dot{x}_2 - 3\dot{x}_1 - x_1$

$$\dot{X} = AX + BU$$

$$\underset{3\times1}{\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}} = \underset{3\times3}{\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -3 & -2\end{bmatrix}} \underset{3\times1}{\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}} + \underset{3\times1}{\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}} [u]_{1\times1}$$

$$Y = CX + DU$$

$$[y] = \underset{1\times3}{[1 \ 0 \ 0]} \underset{3\times1}{\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}} + \underset{1\times1}{[0]} [u]_{1\times1}$$

---

eg: $y'''' + 10y''' - 6y'' + 7y' + 5y = 10u(t)$

Sol: one i/p $\Rightarrow m = 1$

one o/p $\Rightarrow p = 1$

$\boxed{n = 4} \rightarrow x_1, x_2, x_3, x_4$

Assume $x_1 = y$

$\dot{x}_1 = \dot{y} = x_2$

$\dot{x}_2 = \ddot{y} = x_3$

$\dot{x}_3 = \dddot{y} = x_4$

$\dot{x}_4 = \ddddot{y}$

$$\dot{x}_4 + 10\dot{x}_3 - 6\dot{x}_2 + 7\dot{x}_1 + 5x_1 = 10u$$

$$\dot{X} = AX + BU$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -5 & -7 & 6 & -10\end{bmatrix} \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ 10\end{bmatrix} [u]$$

$$Y = CX + DU$$

$$[y] = [1 \ 0 \ 0 \ 0] \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + [0][u]$$

eg: $\dfrac{Y(s)}{U(s)} = \dfrac{10s+5}{s^3+6s^2+7s+8}$

Sol: Equate numerators and denominators separately.

$$U(s) = s^3+6s^2+7s+8 \;;\; Y(s) = 10s+5$$

Assume, $\dot{x}_n = s^n$

$$\dot{x}_1 = s = x_2 \qquad x_1 = s^0$$

$$\dot{x}_2 = s^2 = x_3$$

$$\dot{x}_3 = s^3$$

$$U(s) = \dot{x}_3 + 6x_3 + 7x_2 + 8x_1$$

$$\dot{x}_3 = U(s) - 8x_1 - 7x_2 - 6x_3$$

$$\underset{3\times1}{\dot{X}} = \underset{3\times3}{A}\,\underset{3\times1}{X} + \underset{3\times1}{B}\,\underset{1\times1}{U}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -7 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [u]$$

$$Y(s) = 10x_2 + 5x_1$$

$$\underset{1\times1}{Y} = \underset{1\times3}{C}\,\underset{3\times1}{X} + \underset{1\times1}{D}\,\underset{1\times1}{U}$$

$$[y] = [5 \;\; 10 \;\; 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0][u]$$

eg: $\frac{Y(s)}{U(s)} = \frac{s^2+5s+10}{s^4+3s^3+6s^2+5}$

Sol:

$$x_1 = s^0$$
$$\dot{x}_1 = s^1 = x_2$$
$$\dot{x}_2 = s^2 = x_3$$
$$\dot{x}_3 = s^3 = x_4$$
$$\dot{x}_4 = s^4$$

$$u(s) = s^4 + 3s^3 + 6s^2 + 5$$

$$u(s) = \dot{x}_4 + 3x_4 + 6x_3 + 5x_1$$

$$\dot{x}_4 = u(s) - 3x_4 - 6x_3 - 5x_1$$

$$\dot{X} = AX + BU$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & 0 & -6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} [u]$$

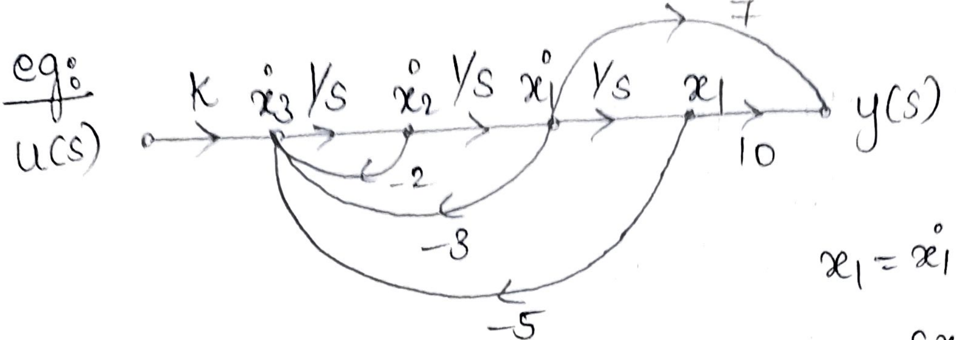
$$y(s) = s^2 + 5s + 10$$

$$y(s) = x_3 + 5x_2 + 10x_1$$

$$Y = CX + DU$$

$$[y] = [10 \;\; 5 \;\; 1 \;\; 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + [0][u]$$

eg:



$$x_1 = \dot{x}_1 \cdot \frac{1}{s}$$
$$= s x_1 \cdot \frac{1}{s} = x_1$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -5x_1 - 8x_2 - 2x_3 + Ku(s)$$

$$y(s) = 10x_1 + 7x_2$$

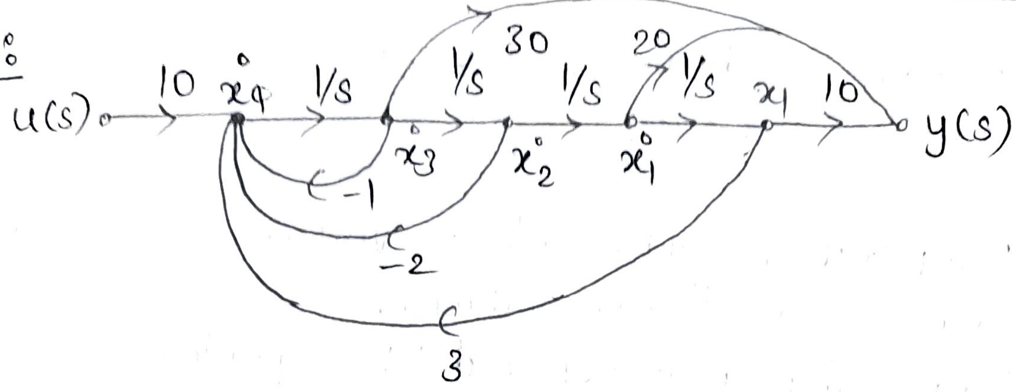
$$\dot{X} = AX + BU$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix} [u]$$

$$Y = CX + DU$$

$$[y] = [10 \;\; 7 \;\; 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0][u]$$

eg:



$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = 3x_1 - 2x_3 - x_4 + 10u(s)$$

$$y(s) = 10x_1 + 20x_2 + 30x_4$$

$$\dot{X} = AX + BU$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \end{bmatrix} [u]$$

$$y = CX + DU$$

$$[y] = \begin{bmatrix} 10 & 20 & 0 & 30 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + [0][u]$$

(*) Procedure for obtaining state equation for electrical Network:

1) Select state variables such as voltage across capacitor and current through inductor.
2) Write independent KCL and KVL equations for the electrical N/w.
3) The resultant equation must consist of state variables, differential state variables, input and output variables.

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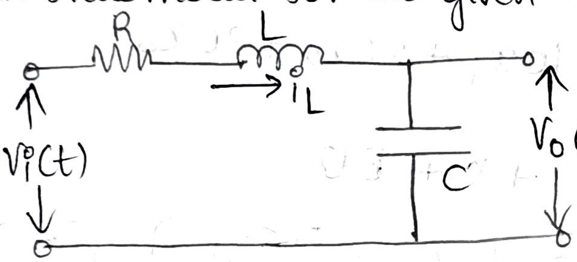
eg: Obtain state model for the given electrical N/w.



Sol:- State variables → $V_c(t)$, $i_L(t)$

$$i_L = C \overbrace{\frac{dV_c}{dt}}^{\dot{V}_c} \quad \text{—} ①$$

$$V_i(t) = i_L(t) \times R + L \overbrace{\frac{di_L}{dt}}^{\dot{i}_L} + V_c(t) \quad \text{—} ②$$

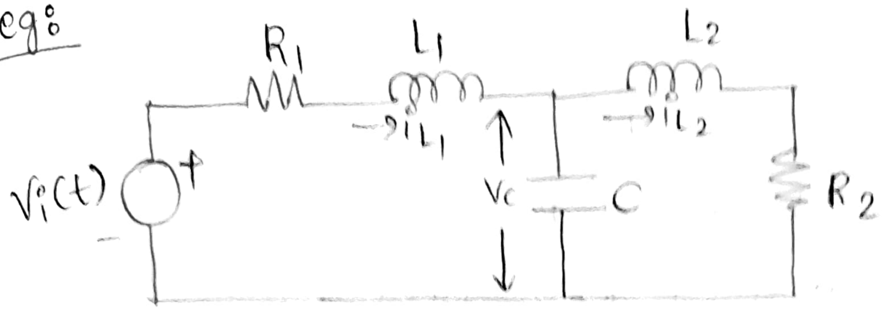
$$\dot{V}_c = \frac{i_L(t)}{C} \quad \text{—} ③$$

$$\dot{I}_L(t) = \frac{V_i(t)}{L} - \frac{R}{L} i_L(t) - \frac{V_c(t)}{L} \quad \text{—} ④$$

$$\begin{bmatrix} \dot{V}_c(t) \\ \dot{I}_c(t) \end{bmatrix} = \begin{bmatrix} 0 & 1/c \\ \frac{-1}{L} & \frac{-R}{L} \end{bmatrix} \begin{bmatrix} V_c \\ I_c \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} [V_i(t)]$$

$$[V_o(t)] = [1 \quad 0] \begin{bmatrix} V_c \\ I_c \end{bmatrix} + [0]\, V_i(s)$$

eg:



Sol:- State variables → $\dot{i}_{L_1}, \dot{i}_{L_2}, V_c$

$$i_{L_1} = i_{L_2} + C\frac{dV_c}{dt}$$

$$\dot{V}_c = \frac{1}{C}[i_{L_1} - i_{L_2}] \quad \text{— (1)}$$

$$V_i(t) = i_{L_1} \times R + L_1\frac{di_{L_1}}{dt} + V_c$$

$$\dot{i}_{L_1} = \frac{V_i(t) - V_c - i_{L_1} \times R}{L_1} \quad \text{— (2)}$$

$$V_c(t) = L_2\frac{di_{L_2}}{dt} + i_{L_2} \times R_2$$

$$\dot{i}_{L_2} = \frac{V_c(t) - i_{L_2} \times R_2}{L_2} \quad \text{— (3)}$$

$$\dot{X} = AX + BU$$

$$\begin{bmatrix} \dot{V}_c \\ \dot{i}_{L_1} \\ \dot{i}_{L_2} \end{bmatrix} = \begin{bmatrix} 0 & 1/C & -1/C \\ -1/L_1 & \frac{-R}{L_1} & 0 \\ 1/L_2 & 0 & \frac{-R_2}{L_2} \end{bmatrix} \begin{bmatrix} V_c \\ i_{L_1} \\ i_{L_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L_1 \\ 0 \end{bmatrix} [V_i(t)]$$

$$\dot{X} = AX + BU$$

$\downarrow$ on Laplace Transform

$$SX(S) = AX(S) + BU(S)$$

$$X(S)(SI - A) = BU(S)$$

$$X(S) = (SI - A)^{-1} BU(S) \quad \text{— (1)}$$

$$Y = CX + DU$$

$\downarrow$ on Laplace Transform

$$Y(S) = CX(S) + DU(S)$$

$$Y(S) = C(SI - A)^{-1} BU(S) + DU(S)$$

$$Y(S) = (\cancel{B}C(SI - A)^{-1}B + D)\,U(S)$$

$$\boxed{\frac{Y(S)}{U(S)} = \cancel{B}C(SI - A)^{-1}B + D}$$

$\hookrightarrow$ Transfer function of MIMO system.

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eg:
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \end{bmatrix} [u]$$

$$[y] = [1 \;\; 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(i) Obtain nature of system
(ii) Obtain stability
(iii) Obtain Transfer function

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Sol:-

$$\frac{Y(s)}{U(s)} = C(sI-A)^{-1}B + D$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \left[\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 2 & 3 \\ -4 & -2 \end{pmatrix} \right]^{-1} \begin{bmatrix} 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s-2 & -3 \\ 4 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \times \frac{1}{s^2+8} \underbrace{\begin{bmatrix} s+2 & 3 \\ -4 & s-2 \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} 3 \\ 5 \end{bmatrix}}_{2\times 1}$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \times \frac{1}{s^2+8} \times \begin{bmatrix} 3s+6+15 \\ -12+5s-10 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{1\times 2} \times \frac{1}{s^2+8} \underbrace{\begin{bmatrix} 3s+21 \\ 5s-22 \end{bmatrix}}_{2\times 1}$$

$$= \frac{3s+21}{s^2+8} + \frac{5s-22}{s^2+8}$$

$$\boxed{\frac{Y(s)}{U(s)} = \frac{8s-1}{s^2+8}} \rightarrow \text{T.F}$$

→ Nature of system is undamped (since poles are lying on imaginary axis)

→ System is marginally stable.

Solution to the state equation:

$$x(t) = L^{-1}[\underbrace{(SI-A)^{-1}X(0)}_{ZIR}] + L^{-1}[\underbrace{(SI-A)^{-1}BU(S)}_{ZSR}]$$

$$= e^{At}X(0) + \int_0^t e^{A(t-\tau)}BU(\tau)\,d\tau$$

(System impulse response) (free response) (Natural respone) ← ZIR → Zero input Response.

ZSR → Zero state response.

↓

(forced response)

e^{At} → state transition matrix

↓

$$\phi = e^{At} = L^{-1}[(SI-A)^{-1}]$$

↓ L.T

$$\phi(s) = (SI-A)^{-1}$$

$$ZSR = L^{-1}[\phi(s)BU(s)]$$

$$ZSR = \int_0^t \phi(t-\tau)BU(\tau)\,d\tau$$

- Properties of state transition matrix:

1) $\phi(0) = I$
2) $\phi^{k}(t) = (e^{At})^{k} = e^{A(kt)} = \phi(kt)$
3) $\phi^{-1}(t) = \phi(-t)$
4) $\phi(t_1+t_2) = \phi(t_1)\cdot\phi(t_2)$
5) $\phi(t_2-t_1)\phi(t_1-t_0) = \phi(t_2-t_0)$

eg: $\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} x$ → where $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$y = [1 \ -1] x$

(i) find solution of state equation
(ii) find response $y(t), x(t)$

Sol:

$$x(t) = L^{-1}\left[(SI - A)^{-1} x(0)\right] + 0$$

$$x(t) = L^{-1}\left[\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}\right)^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

$$x(t) = L^{-1}\left[\begin{bmatrix} s & -1 \\ 2 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

$$= L^{-1}\left[\frac{1}{s^2+2} \underset{2\times2}{\begin{bmatrix} s & 1 \\ -2 & s \end{bmatrix}} \underset{2\times1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}\right]$$

$$= L^{-1}\left[\frac{1}{s^2+2} \begin{bmatrix} s+1 \\ s-2 \end{bmatrix}\right]$$

$$= L^{-1}\begin{bmatrix} \frac{s+1}{s^2+2} \\ \frac{s-2}{s^2+2} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} \cos\sqrt{2}t + \frac{1}{\sqrt{2}}(\sin\sqrt{2}t) \\ -\sqrt{2}\sin\sqrt{2}t + \cos\sqrt{2}t \end{bmatrix}_{2\times1}$$

$$\left\{ L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at \qquad L^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at \right\}$$

$y(t) = c\,x(t)$

$= \underset{1\times2}{[1 \ -1]} \ \underset{2\times1}{x(t)}$

$$y(t) = \frac{3}{\sqrt{2}} \sin\sqrt{2}t$$

- Controllability:

→ A system is said to be controllable if it is possible to transfer initial state to desired state in a finite time interval by the controlled i/p.

$$Q_c = [B \quad AB \quad A^2B \cdots\cdots A^{n-1}B]$$

↳ kalman's test

if $|Q_c| \neq 0$ → controllable.

eg: Check the controllability of T.F $= \dfrac{1}{s^3+2s^2+3s+4}$

Sol:- n-order of T.F

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -2 \end{bmatrix}_{3\times3} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{3\times1}$$

$$AB = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}_{3\times1} \qquad A^2B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$|Q_c| = -1 \neq 0$$

So system is controllable.

- **Observability:**

→ A system is said to be observable if it is possible to determine initial state of the system by observing the o/p in a finite time interval.

$$Q_o = \left[C^T \quad A^T C^T \quad (A^T)^2 C^T \quad \cdots \quad (A^T)^{n-1} C^T \right]$$

if $|Q_o| \neq 0 \rightarrow$ Observable.

eg: check the observability of the system,

$$\dot{x} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

$$C^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad A^T C^T = \underset{2\times 2}{\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$A^T C^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$Q_o = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

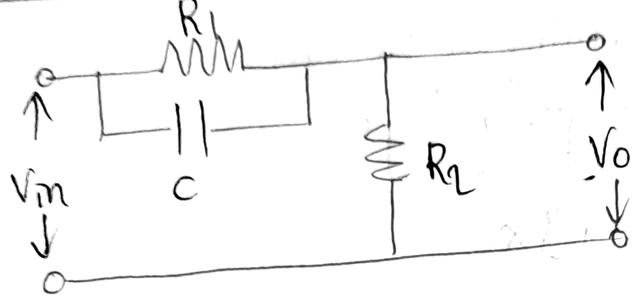
Since $|Q_o| = 0 \rightarrow$ System is not observable.

⊛ Compensators:

→ Redesigning the system by the use of external device is called compensation

→ The external physical device connected to system for redesigning purpose is called compensator.

∘ Lead Compensator: (High pass filter)



$$\frac{V_o(s)}{V_{in}(s)} = \frac{R_2}{R_1 \parallel \frac{1}{sC} + R_2}$$

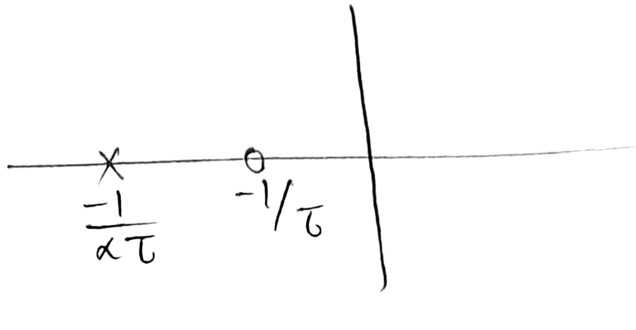
$$\frac{V_o(s)}{V_{in}(s)} = \frac{\left(s + \frac{1}{CR_1}\right)}{\left[s + \frac{1}{\left(\frac{R_2}{R_1+R_2}\right) CR_1}\right]}$$

$0 \le \alpha \le 1$ $\quad \alpha = \frac{R_2}{R_1+R_2}$ $\quad \tau = CR_1$

$$\frac{V_o(s)}{V_{in}(s)} = \frac{\alpha(s\tau+1)}{\alpha s\tau + 1}$$

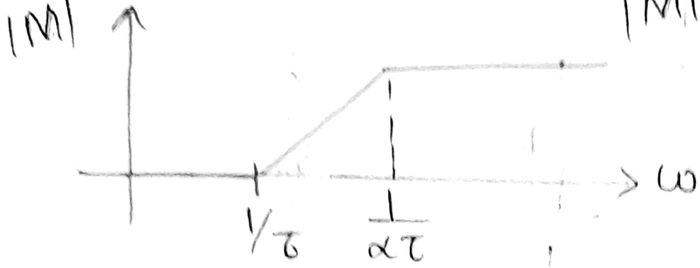
Pole: $s = -1/\alpha\tau$

Zero: $s = -1/\tau$

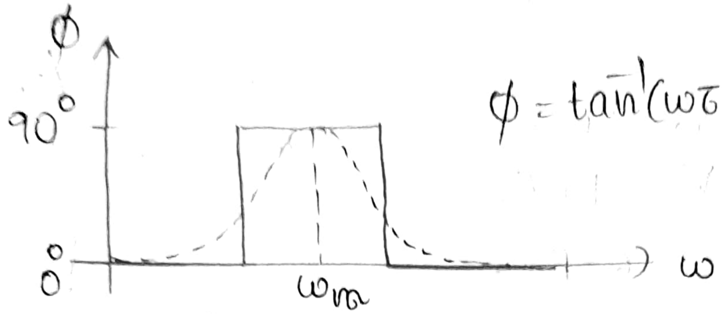


→ Now draw the Bode plots,

$$|M| = \sqrt{\frac{1+\omega^2\tau^2}{1+\alpha^2\omega^2\tau^2}}$$



$$\phi = \tan^{-1}(\omega\tau) - \tan^{-1}(\alpha\omega\tau)$$



→ Lead compensator is called <u>HPF</u>.

$$\phi = \tan^{-1}(\omega\tau) - \tan^{-1}(\alpha\omega\tau)$$

$$\frac{d\phi}{d\omega} = \frac{\tau}{1+(\omega\tau)^2} - \frac{\alpha\tau}{1+(\alpha\omega\tau)^2} = 0$$

$$\tau(1+\alpha^2\omega^2\tau^2 + 2\alpha\omega\tau)$$
$$-\alpha\tau(1+\omega^2\tau^2 + 2\omega\tau) = 0$$

$$\tau(1-\alpha) + \omega^2\tau^3\alpha(\alpha-1) = 0$$

$$(1-\alpha)\tau.(1-\omega^2\tau^2\alpha) = 0$$

$$1 = \omega^2\tau^2\alpha$$

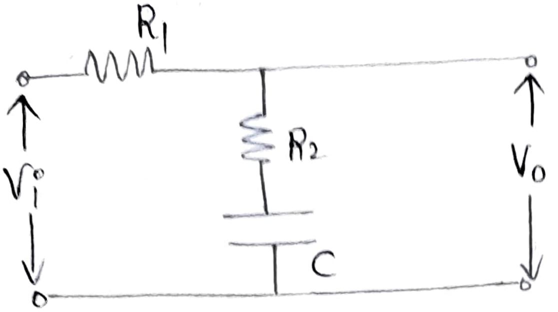
$$\omega = \frac{1}{\tau\sqrt{\alpha}}$$

$$\tan\omega_m = \frac{1-\alpha}{2\sqrt{\alpha}}$$

$$\omega_m = \tan^{-1}\left(\frac{1-\alpha}{2\sqrt{\alpha}}\right)$$

$$\boxed{\omega_m = \frac{1}{\tau\sqrt{\alpha}}}$$

- **Lag compensator: (Low-pass filter)**



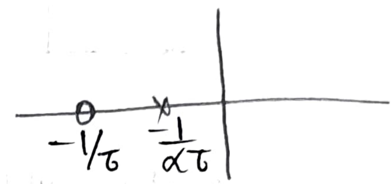
$$\frac{V_o}{V_i} = \frac{1+\tau s}{1+\alpha\tau s}$$

Zero: $s = -1/\tau$

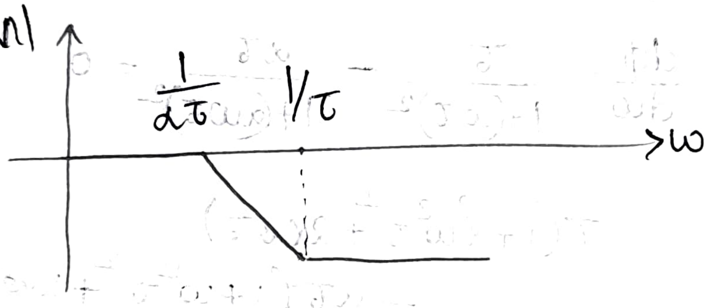
Pole: $s = -1/\alpha\tau$

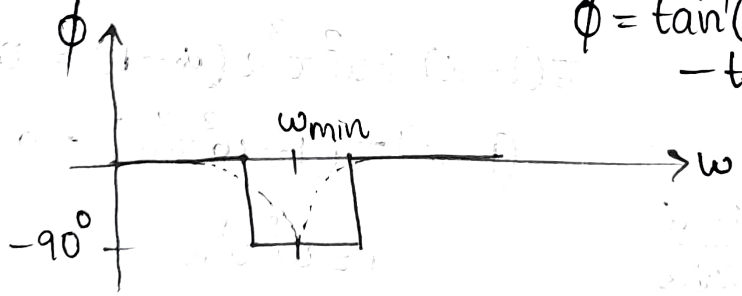
$$\tau = R_2 C$$

$$\alpha = \frac{R_1+R_2}{R_2} > 1$$



→ Now draw the bode plots,



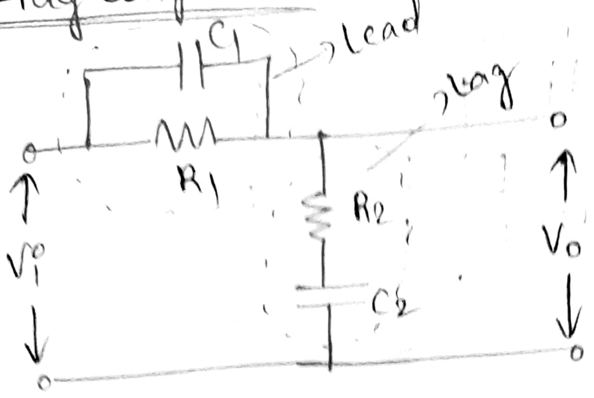


$$\phi = \tan^{-1}(\omega\tau) - \tan^{-1}(\alpha\omega\tau)$$

$$\omega_{min} = \frac{1}{\tau\sqrt{\alpha}}$$

$$\phi_{min} = \sin^{-1}\left(\frac{\alpha-1}{\alpha+1}\right)$$

- **Lead-lag compensator**: (Band pass filter)



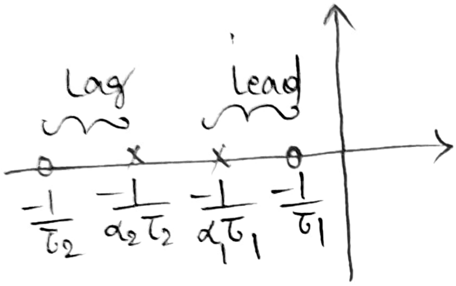
$$\frac{V_o}{V_i} = \frac{1+\tau_1 s}{1+\alpha_1\tau_1 s} \cdot \frac{1+\tau_2 s}{1+\alpha_2\tau_2 s}$$

where $\tau_1 = R_1C_1$

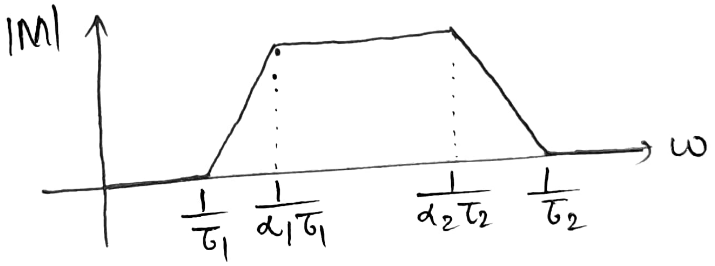
$\tau_2 = R_2C_2$

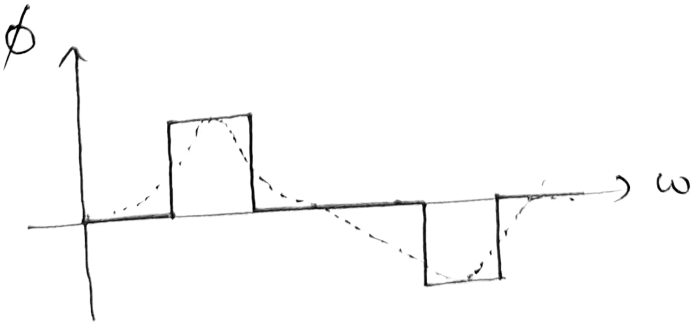
$\alpha_1 = \dfrac{R_2}{R_1+R_2} < 1$

$\alpha_2 = \dfrac{R_1+R_2}{R_2} > 1$

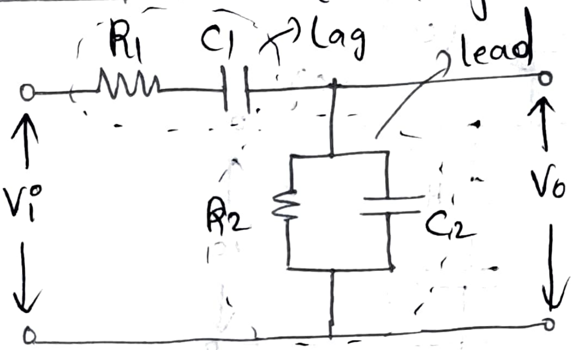


→ Now draw bode plots,





- Lag-lead compensator: (Band reject filter)



- Resonant peak, Resonant freq & B.W of 2nd order system:

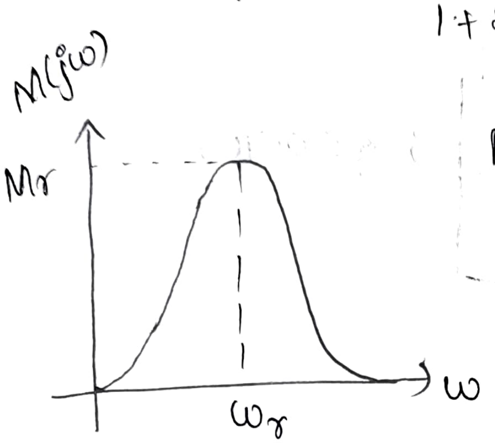
$$M(S) = \frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2 + 2\xi\omega_n S + \omega_n^2}$$

$\omega_n \to$ natural freq

$\xi \to$ damping ratio.

$S = j\omega$

$$M(j\omega) = \frac{\cancel{\omega_n^2}\ 1}{1 + 2j\left(\frac{\omega}{\omega_n}\right)\xi - \left(\frac{\omega}{\omega_n}\right)^2}$$



$M_r \to$ resonant peak

$\omega_r \to$ resonant freq.

$$u = \frac{\omega}{\omega_n}$$

$$|M(ju)| = \frac{1}{\sqrt{(1-u^2)^2 + (2\xi u)^2}}$$

$$\angle M(ju) = \phi_m(ju) = -\tan^{-1}\left(\frac{2\xi u}{1-u^2}\right)$$

To find M_r, ω_r,

$$\frac{d|M(ju)|}{du} = 0$$

$$u_r = \sqrt{1-2\xi^2}$$

$$\frac{\omega_r}{\omega_n} = \sqrt{1-2\xi^2}$$

$$\boxed{\omega_r = \omega_n\sqrt{1-2\xi^2}}$$

$$2\xi^2 < 1$$

$$\xi < 0.707$$

$$|M| = \frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{(1-u^2)^2+(2\xi u)^2}} = \frac{1}{\sqrt{2}}$$

$$\boxed{B.W = \omega_n\left[(1-2\xi^2) + \sqrt{4\xi^4 - 4\xi^2 + 2}\right]^{1/2}}$$

$$\boxed{M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}} \quad ; \ \xi \le 0.707$$

$$t_r = \frac{\pi - \theta}{\omega_d}$$

if ω_n ↑es, B.W ↑es, t_r ↓es

(the system responds faster)

Sensitivity:

- It is a measure of change in system response wrt change in input or any other parameter of the system
- Sensitivity wrt variations in G(s)

$$\boxed{S_G^T = \frac{\partial T/T}{\partial G/G} = \frac{\partial T}{\partial G} \cdot \frac{G}{T}}$$

where, $T \rightarrow$ Transfer function.

- Sensitivity wrt variations in H(s)

$$\boxed{S_H^T = \frac{\partial T/T}{\partial H/H} = \frac{\partial T}{\partial H} \cdot \frac{H}{T}}$$

- For closed loop systems,

$$T = \frac{C(S)}{R(S)} = \frac{G}{1+GH}$$

$$S_G^T = \frac{1}{1+GH}$$

$$S_H^T = \frac{-GH}{1+GH}$$

- for open loop systems,

$$S_G^T = 1$$