MA 205 – CR Equations and Harmonic Functions

Note Title 25-08-2021

Necessary Conditions for a function to be analytic:

$A \Rightarrow B$

not $B \Rightarrow$ not B.

- $f = u + iv$ – cont. in a nbhd. and diff at z_0.

Then u_x, u_y, v_x, v_y exist and $\boxed{u_x = v_y}$ and $\boxed{u_y = -v_x}$ at z_0.

Sufficient conditions for a fn. to be analytic:

- u, v are cont and have cont first order partial derivatives
- ✓ $u_x = v_y$ and $u_y = -v_x$

u_x, u_y, v_x, v_y – cont.

Then f is analytic and $f'(z) = u_x + iv_x = v_y - iu_y$.

Polar Form: $\boxed{v_\theta = r u_r}$ and $\boxed{u_\theta = -r v_r}$. $f'(z) = \frac{1}{r} e^{-i\theta}(v_\theta - i u_\theta) = e^{-i\theta}(u_r + i v_r)$

CR eqns not satisfied $\Rightarrow$ f not differentiable at z.
$\Rightarrow$ f not analytic at z.

CR eqns may be satisfied but f may not be diff.

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

satisfies CR eqns at 0 but $\underline{f'(0)}$ does not exist.

$f(z) = u + iv \quad \forall z \qquad f(0) = 0 = 0 + i0$

$$u(x,y) = \frac{x^3 - y^3}{x^2+y^2} \quad ; \quad v(x,y) = \frac{x^3+y^3}{x^2+y^2} \qquad u_x(x,y)$$

$$u_x(0,0) = \lim_{h \to 0} \frac{u(\overset{x_0+h,\, y_0}{0+h,\, 0}) - u(\overset{x_0,\, y_0}{0,0})}{h} = \lim_{h\to 0} \frac{h^3/h^2}{h} = 1 \quad ✓$$

✓ $\underline{u_x = v_y}$ & $\underline{u_y = -v_x}$. $(0,0)$.

$f'(0) = \lim_{z \to 0} \frac{f(0+z) - f(0)}{z} = \lim_{(x,y)\to(0,0)} \frac{\frac{x^3-y^3}{x^2+y^2} + i\,\frac{x^3+y^3}{x^2+y^2}}{x+iy} \times \frac{x-iy}{x-iy}$

$= \lim_{(x,y)\to(0,0)} \frac{x(x^3-y^3)+y(x^3+y^3)}{(x^2+y^2)^2} + i\,\frac{x(x^3+y^3)-y(x^3-y^3)}{(x^2+y^2)^2}$ $\qquad y = mx.$

$=$ $\qquad \frac{1}{x}$

$u_x(x,y)$ is not cont. at 0.

does not exist.

v_y

v_x

u_y.

Use CR eqns to show

$$\begin{bmatrix} u_x = v_y \\ u_y = -v_x \end{bmatrix} \text{CR.}$$

(1) $|z|^2$ is not analytic (diff at origin)

(2) $\bar{z}$ is not analytic

(3) $\frac{1}{z}$ $(z \neq 0)$ is analytic.

analytic

CR + cont. of partial deri.

(4) $\sin z$ is analytic ✓

f/g $g \neq 0$.

not analytic

not CR

(5) $\cos z$ is analytic ✓

$\tan z = \frac{\sin z}{\cos z}$

(6) e^z is analytic.

(7) $\tan z$ and $\sec z$ are analytic everywhere except $(2n+1)\pi/2$, $n \in \mathbb{Z}$

(8) $\cot z$ and $\operatorname{cosec} z$ $n\pi$, $n \in \mathbb{Z}$.

$u_x = v_y \; ; \; u_y = -v_x .$

Find a, b, c such that $f(z) = \underbrace{x - 2ay}_{u} + i(\underbrace{bx - cy}_{v})$ is analytic.

CR.

$u_x = 1 \quad v_x = b \quad u_y = -2a \quad v_y = -c$

$u_x = v_y \Rightarrow c = -1$

$v_x = -u_y \Rightarrow b = 2a$

$f(z) = x - 2ay + i(2ax + y)$

$= (x + iy) + 2a(i^2 y + ix) = \underbrace{(x+iy)}_{z} + i2a(\underbrace{x+iy}_{z}) = (1 + i2a)\, z$

HW: $f(z) = -x^2 + xy + y^2 + i(ax^2 + bxy + cy^2)$

$u_x = 0 = u_y$ $\quad u_x = \underline{\underline{v_y}} = 0 \quad v_x = 0.$

$\overset{=u+iv}{}$ $\quad\quad u$

<u>HW</u>: If f is <u>analytic</u> and $\mathrm{Re}(f)$ is constant, then show that f is a constant.

If f is cont., then $\dfrac{\partial f}{\partial \bar{z}} = 0$ is equivalent to the <u>CR eqns.</u>

$\Downarrow$

f is independent of $\bar{z}$

$\boxed{z \text{ \& } \bar{z} \text{ are independent}}$

$x = \frac{1}{2}(z+\bar{z})$

$u_x = v_y \quad ; \quad u_y = -v_x$

$f(z) = f = u + iv$
$= u(x,y) + iv(x,y)$

$z = x+iy \quad \bar{z} = x - iy \quad \underline{x = \dfrac{z+\bar{z}}{2}} \quad ; \quad \underline{y = \dfrac{z-\bar{z}}{2i}}$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial z} = \frac{\partial f}{\partial x}\cdot\frac{1}{2} + \frac{\partial f}{\partial y}\cdot\frac{1}{2i} = \frac{1}{2}\left[f_x - i f_y\right]$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left[f_x + i f_y\right] = 0 \qquad f_x = -i f_y \iff u_x + iv_x = -i(u_y + iv_y)$$

$$\Leftarrow \boxed{u_x = v_y \quad ; \quad v_x = -u_y.}$$

If f is an analytic fn. and $|f|$ is a non-zero constant, then show that f is a constant.

$f = u + iv$ $\qquad |f| = \sqrt{u^2+v^2} = c_1 \Rightarrow u^2+v^2 = c_1^2 = c$ $\qquad u^2+v^2 = c.$

$u_x = v_y \quad ; \quad u_y = -v_x$ $\qquad 2u \cdot u_x + 2v \cdot v_x = 0$ $\qquad \sin^2 x$

$u u_x + v v_x = 0.$ $\qquad 2\sin x \cos x.$

$\boxed{u u_y + v v_y = 0}$

$-u v_x + v u_x = 0$

$v u u_x + v^2 v_x = 0$

$-uv u_x + u^2 v_x = 0$

$(u^2+v^2) v_x = 0 \Rightarrow \left.\begin{array}{l} v_x = 0 \\ v_y = 0 \end{array}\right\} \Rightarrow v = \text{Constant} \Rightarrow f$ is a constant

$u_x = v_y$; $v_x = -u_y$.

u, v

Suppose $f = u+iv$ is analytic and $u+v = (x+y)(2-4xy+x^2+y^2)$. Find f.

$= 2x+2y-4x^2y-4xy^2+x^3+x^2y+y^3+xy^2$

$= 2x+2y-3x^2y-3xy^2+x^3+y^3$

$u_x+v_x = 2-6xy-3y^2+3x^2$

$\underbrace{u_y+v_y}_{-v_x+u_x} = 2-3x^2-6xy+3y^2$

$2u_x = \cancel{2}\,4-12xy$

$u_x = 2-6xy$

$u = 2x-3x^2y+y^3+c$

$v = x^3-3y^2x+\phi_2(y)$

$f = u+iv$.

$u_y = -\cancel{3x^2}+\phi_1'(y) = -v_x = -\cancel{3x^2}+3y^2$

$\phi_1'(y) = 3y^2 \Rightarrow \phi_1(y) = y^3+C$

$v_y = u_x$.

HW: $u - v = e^{-x}\left[(x-y)\sin y - (x+y)\cos y\right]$

$f(z)$ and $\overline{f(z)}$ are analytic. What is f?

$f = u + iv$ $\quad\quad$ $\bar{f} = u - iv$ $\quad\quad$ (Constant)

Harmonic Functions:

A real valued fn. $\phi(x,y)$ of two variables that has cont. second order partial derivatives that satisfies the Laplace equation $\phi_{xx} + \phi_{yy} = 0$ is said to be a harmonic function.

- If $f(z) = u(x,y) + iv(x,y)$ is analytic, then u and v are harmonic.

$u_{xx} + u_{yy} = 0$ & $v_{xx} + v_{yy} = 0$.

$$\begin{cases} \text{analy} \Rightarrow u \,\&\, v \text{ har} \\ u \text{ or } v \text{ not har} \Rightarrow \text{not analy.} \end{cases}$$

$A \Rightarrow B$

not $B \Rightarrow$ not A.

$u_x = v_y$

$u_y = -v_x$

$u_{xx} = v_{xy}$

$u_{yy} = -v_{yx}$

$u_{xx} + u_{yy} = \underbrace{v_{xy} - v_{yx}}_{=0}$

If $f = u + iv$ is analytic, then u and v are harmonic and v is called the $\boxed{\text{harmonic conjugate}}$ of u.

Converse not true (u, v harmonic $\not\Rightarrow$ $u + iv$ analytic)

$u = \mathrm{Re}(z^2) = x^2 - y^2$; $v = \mathrm{Im}(z^3) = 3x^2y - y^3$

$$\boxed{u_{xx} + u_{yy} = 0}$$

$u_{xx} + u_{yy} = 0$; $v_{xx} + v_{yy} = 0$ (Check!)

$\boxed{u + iv = (x^2 - y^2) + i(3x^2y - y^3)}$ is not analytic (not satisfying CR eqns).

CR not satisfied at 0.

not analytic at 0.

u, v

$$\text{But } \boxed{\overbrace{(u_y - v_x)}^{U} + i\overbrace{(u_x + v_y)}^{V}} \text{ is always analytic.} \longrightarrow U_x = u_{yx} - v_{xx} \quad V_y = u_{xy} + v_{yy}$$

$U_x = V_y$?

$v_{xx} + v_{yy} = 0.$

S.T. $v = e^x \sin y$ is harmonic and find the harmonic conjugate of v.

u

$v_{xx} + v_{yy} = 0.$ ✓

$$\left.\begin{aligned} v_x &= -u_y \\ v_y &= u_x \end{aligned}\right\} \Rightarrow u$$

HW: S.T. $u = 2x + y^3 - 3x^2y$ is harmonic and find its harmonic conjugate

Laplace Eqn. in Polar form:

$$\boxed{u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0} \quad ; \quad v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = 0.$$

HW: S.T. $u = r^2\cos 2\theta$ is harmonic and find its harmonic conjugate.

$u(r,\theta)$

CR polar form