COURSE: *OPTICAL COMMUNICATION* (EC317)

UNIT-II

M. RAJARAO (Ad-hoc faculty)
Department of Electronics and Communication Engineering,
NIT Andhra Pradesh, Tadepalligudem,
Andhra Pradesh, INDIA.

MODES IN A PLANAR WAVEGUIDE

- Only a finite set of rays at certain discrete angles greater than or equal to the critical angle $\emptyset_c$ is capable of propagating along a fiber.
- These angles are related to a set of electromagnetic wave patterns or field distributions called ***modes*** that can propagate along a fiber.
- The ray model does not predict correctly that even after total internal reflection there will be some field in the cladding
- For an accurate and complete description of light propagation inside an optical fiber we have to go for the wave model**.** Here we treating light as an electromagnetic wave.

ELECTROMAGNETIC MODE THEORY FOR OPTICAL PROPAGATION

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = 0 \qquad \text{(no free charges)}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(no free poles)}$$

The four field vectors are related by the relations:

$$\mathbf{D} = \varepsilon \mathbf{E}$$
$$\mathbf{B} = \mu \mathbf{H}$$

$$\nabla^2 \mathbf{E} = \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\nabla^2 \mathbf{H} = \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

$$\nabla^2 \psi = \frac{1}{\upsilon_p^2} \frac{\partial^2 \psi}{\partial t^2}$$

Phase velocity

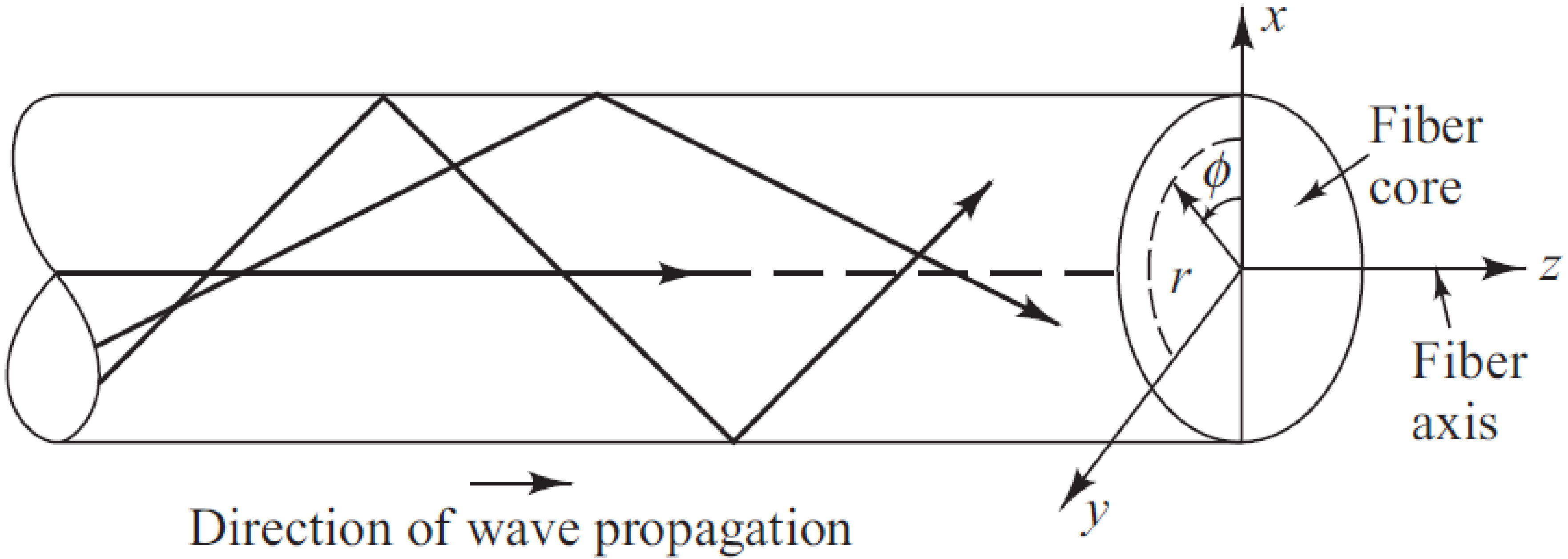
$$\upsilon_p = \frac{1}{(\mu\varepsilon)^{\frac{1}{2}}} = \frac{1}{(\mu_r \mu_0 \varepsilon_r \varepsilon_0)^{\frac{1}{2}}}$$

For an isotropic medium , the refractive index n is related as $v_p = c/n$

The planar wave guides described by rectangular coordinates (x,y,z) and the circular fibers described by cylindrical polar coordinates (r,Φ,z).

$$\Psi = \Psi_0(x, y)\, e^{j(\omega t - \beta z)}\ ; planar$$

$$\Psi = \Psi_0(r, \phi)\, e^{j(\omega t - \beta z)}; circular$$



$$\mathbf{E} = \mathbf{E}_0(r, \phi)\, e^{j(\omega t - \beta z)}$$

$$\mathbf{H} = \mathbf{H}_0(r, \phi)\, e^{j(\omega t - \beta z)}$$

- The electric field and magnetic filed are a vector quantities, each having three components. So we have total six field components.
- However all these components are not independent of each other, only two components are independent components and express the remaining four components in terms of the independent components.
- Since in this case the wave propagates along the axis of the fiber i.e. in the z direction, E_z and H_z (also called the longitudinal components) are taken as independent components and the other four transverse field components (E_r , $E\phi$, H_r , and $H\phi$) are expressed in terms of E_z and H_z components.

$$E_r = -\frac{j}{q^2}\left(\beta\frac{\partial E_z}{\partial r} + \frac{\omega\mu}{r}\frac{\partial H_z}{\partial \phi}\right) \quad and \quad E_\phi = -\frac{j}{q^2}\left(\frac{\beta}{r}\frac{\partial E_z}{\partial \phi} - \omega\mu\frac{\partial H_z}{\partial r}\right)$$

$$H_r = -\frac{j}{q^2}\left(\beta\frac{\partial H_z}{\partial r} - \frac{\omega\epsilon}{r}\frac{\partial E_z}{\partial \phi}\right) \quad and \quad H_\phi = -\frac{j}{q^2}\left(\frac{\beta}{r}\frac{\partial H_z}{\partial \phi} + \omega\epsilon\frac{\partial E_z}{\partial r}\right)$$

Where $q^2 = \omega^2\mu\epsilon - \beta^2$

- The wave equation for the cylindrical homogeneous core waveguide

$$\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \phi^2} + q^2 E_z = 0$$

$$\frac{\partial^2 H_z}{\partial r^2} + \frac{1}{r}\frac{\partial H_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 H_z}{\partial \phi^2} + q^2 H_z = 0$$

- If the boundary conditions do not lead to coupling between the field components, mode solutions can be obtained in which either $E_z = 0$ or $H_z = 0$.
- When $E_z = 0$ the modes are called transverse electric or TE modes, and when $H_z = 0$ they are called transverse magnetic or TM modes.
- Hybrid modes exist if both E_z and H_z are non-zero. These are designated as HE or EH modes, depending on whether H_z or E_z, dominates to the transverse field.

WAVE EQUATIONS FOR STEP-INDEX FIBERS

- For a general solution of the wave equation, apply separation of variables

$$E_z = A\, F_1(r)F_2(\phi)F_3(z)F_4(t)$$

- From the general wave equation model, the time- and z-dependent factors are given by

$$F_3(z)F_4(t) = e^{j(\omega t-\beta z)}$$

- Because of the circular symmetry of the waveguide, each field component does not change when the coordinate f is increased by 2π. The fields must be periodic in ϕ with a period of 2π and which is of the form

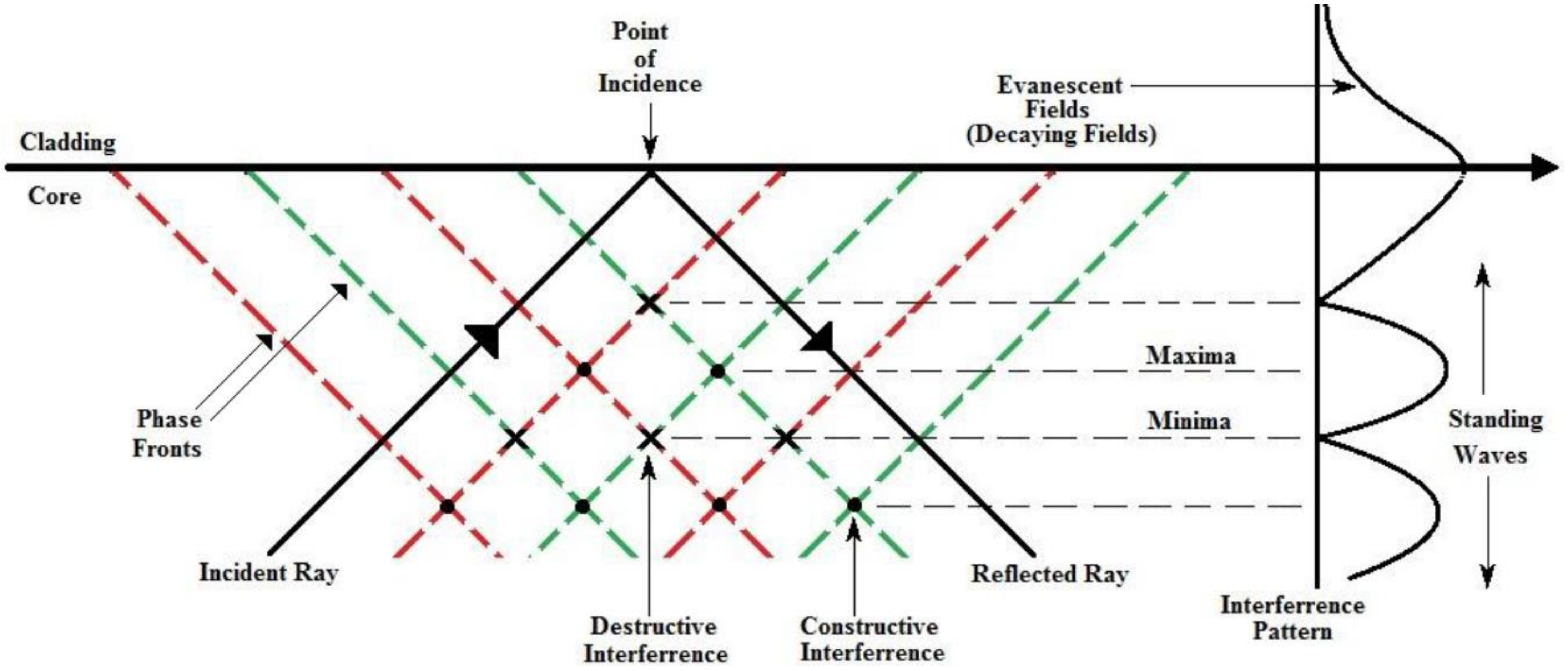
$$F_2(\phi) = e^{j\nu\phi}$$

- Where the constant *v* can be positive or negative, but it must be an integer

$$\frac{\partial^2 F_1(r)}{\partial r^2} + \frac{1}{r}\frac{\partial F_1(r)}{\partial r} + \left(-\frac{v^2}{r^2} + q^2\right)F_1(r) = 0$$

which is a well-known differential equation for Bessel functions.

- A variety of solutions to the Bessel's equation depending upon the parameters '*v*' (integer and positive quantity) and '*q*' (real/ imaginary/ complex)



Total Internal Reflection of Light inside a fiber core

- For guided mode propagation, a mode is guided when its fields are confined to the guide (core), and outside the guide the fields decay monotonically.
- Thus, for r < a the solutions are Bessel functions of the first kind of order *v*. The expressions for Ez and Hz inside the core are

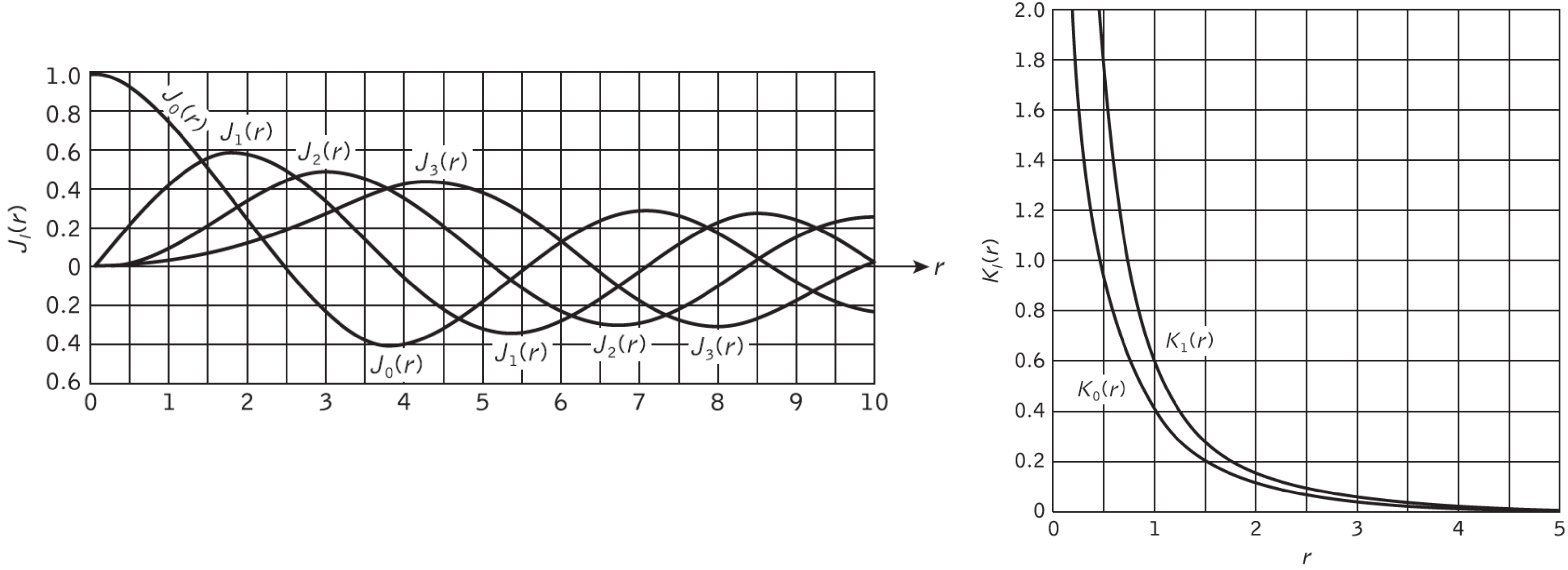
$$E_z(r < a) = A\, J_v(qr)\, e^{jv\phi} e^{j(\omega t - \beta z)}$$
$$H_z(r < a) = B\, J_v(qr)\, e^{jv\phi} e^{j(\omega t - \beta z)}$$

where A and B are arbitrary constants.

- For r > a (Outside of the core) the solutions are given by modified Bessel functions of the second kind order *v*. The expressions for Ez and Hz inside the core are

$$E_z(r > a) = C\, K_v(qr)\, e^{jv\phi} e^{j(\omega t - \beta z)}$$
$$H_z(r > a) = D\, K_v(qr)\, e^{jv\phi} e^{j(\omega t - \beta z)}$$

where C and D are arbitrary constants.



Variation of the Bessel function and modified Bessel function

Inside the core of the optical fiber **r < a** and $\boldsymbol{q^2 > 0}$ and therefore $\omega^2\mu\epsilon_1 > \beta^2$

$$\omega\sqrt{\mu\epsilon_1} = \frac{2\pi n_1}{\lambda} = kn_1 = k_1; where\ k = \frac{2\pi}{\lambda}$$

In the cladding of the optical fiber **r > a** and $\boldsymbol{q^2 < 0}$ and therefore $\omega^2\mu\epsilon_2 < \beta^2;\ \omega\sqrt{\mu\epsilon_2} = \frac{2\pi n_2}{\lambda} = kn_2 = k_2$

$$k_2 < \beta < k_1$$

Where we can defined

$$q = U = \sqrt{{k_1}^2 - \beta^2} \qquad for\ r < a$$

$$q = W = \sqrt{\beta^2 - {k_2}^2} \qquad for\ r > a$$

Thus the expressions for longitudinal components of electric and magnetic fields can be written as:
Inside the core $(r < a)$

$$E_{z1} = A\, J_v(Ur)\, e^{jv\phi}\, e^{j(\omega t - \beta z)}$$

$$H_{z1} = B\, J_v(Ur)\, e^{jv\phi}\, e^{j(\omega t - \beta z)}$$

Inside the cladding $(r > a)$

$$E_{z2} = C\, K_v(Wr)\, e^{jv\phi}\, e^{j(\omega t - \beta z)}$$

$$H_{z2} = D\, K_v(Wr)\, e^{jv\phi}\, e^{j(\omega t - \beta z)}$$

- The solutions for β must be determined from the boundary conditions.
- Since the core-cladding boundary is a purely dielectric-dielectric boundary, the following boundary conditions are applicable at the boundary (r=a) where 'a' is the radius of the optical fiber core.
- The tangential components of the electric field (E_ϕ and E_z) and magnetic field (E_ϕ and E_z) are continuous across the boundary.

$$E_{z1} - E_{z2} = A\, J_v(Ua)\ - C\, K_v(Wa) = 0$$

$$H_{z1} - H_{z2} = B\, J_v(Ua)\ - D\, K_v(Wa) = 0$$

$$E_{\emptyset 1} - E_{\emptyset 2} = -\frac{j}{U^2}\left(A\frac{jv\beta}{a}J_v(Ua) - B\omega\mu U\, J_v{}'(Ua)\right) + \frac{j}{W^2}\left(C\frac{jv\beta}{a}K_v(Wa) - D\omega\mu W\, K_v{}'(Wa)\right) = 0$$

$$H_{\emptyset 1} - H_{\emptyset 2} = -\frac{j}{U^2}\left(B\frac{jv\beta}{a}J_v(Ua) + A\,\omega\epsilon_1\, U\, J_v{}'(Ua)\right) + \frac{j}{W^2}\left(D\frac{jv\beta}{a}K_v(Wa) + C\omega\epsilon_1 W\, K_v{}'(Wa)\right) = 0$$

From these four equation, the characteristic / eigen value equation can be expressed as

$$\left(\frac{J_v{}'(Ua)}{UJ_v(Ua)} + \frac{K_v{}'(Wa)}{WK_v(Wa)}\right)\left(k_1{}^2\frac{J_v{}'(Ua)}{UJ_v(Ua)} + k_2{}^2\frac{K_v{}'(Wa)}{WK_v(Wa)}\right) = \left(\frac{\beta v}{a}\right)^2\left(\frac{1}{U^2} + \frac{1}{W^2}\right)^2$$

Modes in step-index fiber

- For the dielectric fiber waveguide, all modes are hybrid modes except those for which $v = 0$.
- When $v = 0$,

$$\left(\frac{J_1(Ua)}{UJ_0(Ua)} + \frac{K_1(Wa)}{WK_0(Wa)}\right)\left(k_1{}^2\frac{J_1(Ua)}{UJ_0(Ua)} + k_2{}^2\frac{K_1(Wa)}{WK_0(Wa)}\right) = 0$$

Since the derivatives of the Bessel and the Modified Bessel functions can be related as

$$J_0{}'(x) = -J_1(x)$$

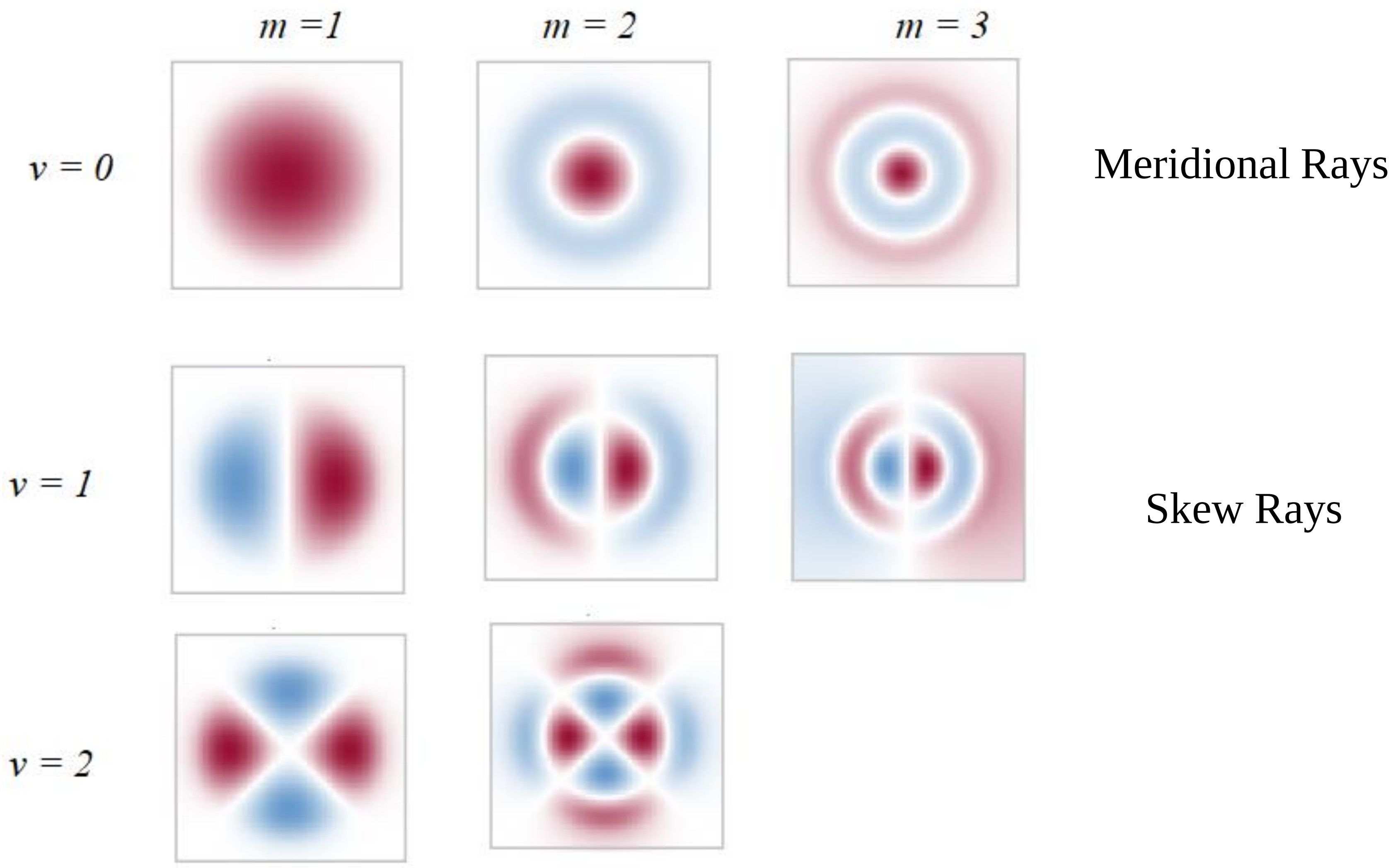
$$\frac{J_1(Ua)}{UJ_0(Ua)} + \frac{K_1(Wa)}{WK_0(Wa)} = 0; \qquad TE\ mode$$

$$k_1{}^2\frac{J_1(Ua)}{UJ_0(Ua)} + k_2{}^2\frac{K_1(Wa)}{WK_0(Wa)} = 0; \qquad TM\ mode$$

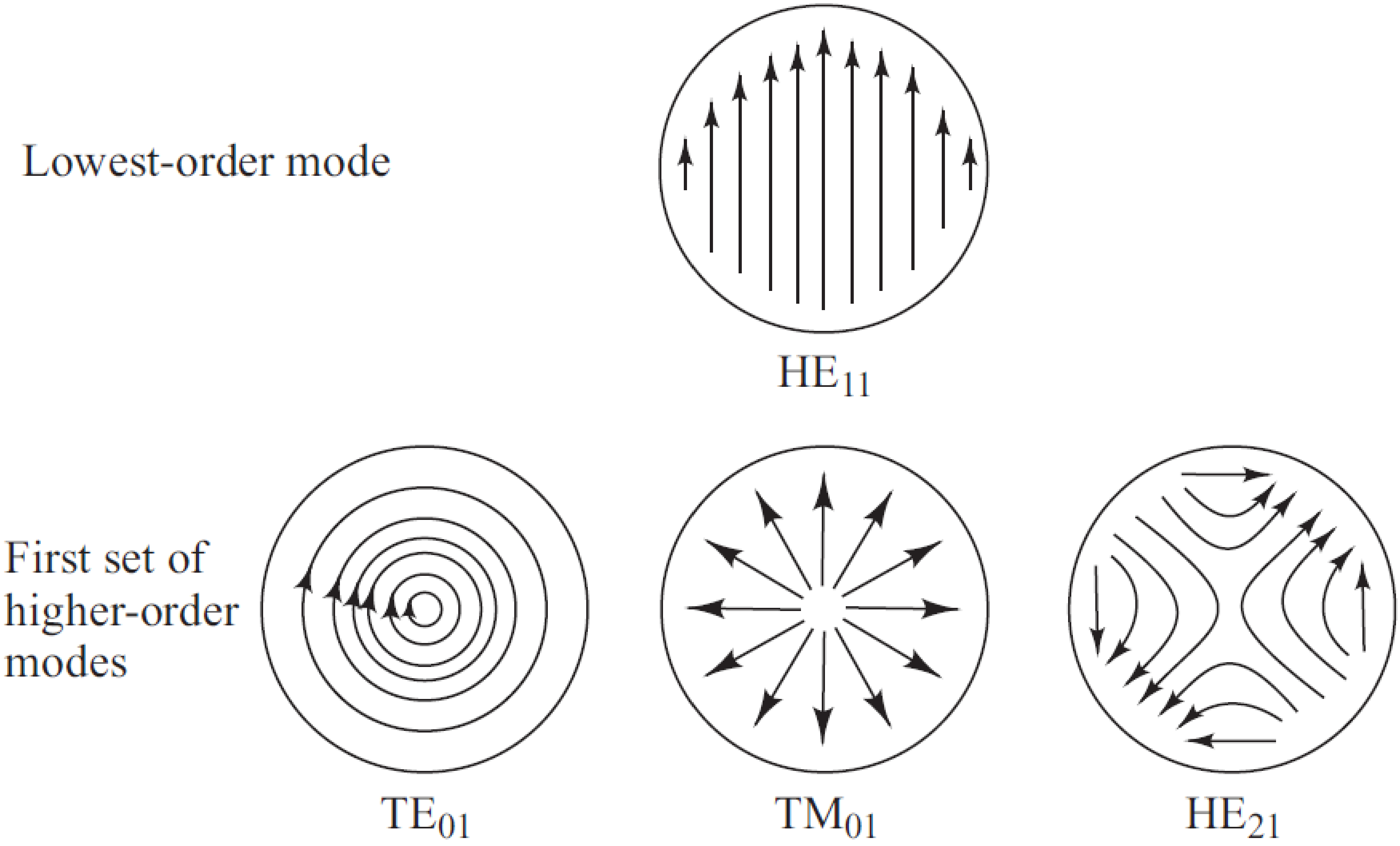
MODES IN STEP-INDEX FIBER: IDENTIFICATION OF THE FIELD PATTERN FOR ORDER (v, m)

- To help describe the modes, first examine the behavior of the J-type Bessel functions. Because of the oscillatory behavior of Bessel function J_v, there will be ***m* roots** for a given ***v* value**. These roots will be designated by β_{vm},
- The combination (v,m) helps us to identify a particular mode and its corresponding light intensity pattern and the corresponding modes are either TE_{vm}, TM_{vm}, EH_{vm}, or HE_{vm}
- The index v of the combination (v,m) represents the number of complete cycles of the field in the azimuthal plane (indicates the behavior of the field in the azimuthal plane, i.e. the variation of the field with respect to ϕ)
- the index 'm' represents the number of zero crossings in the azimuthal direction (maxima and minima in the azimuthal plane)
- For example TE_{02} would result in an intensity pattern that would be circularly symmetric about the axis with maximum intensity at the center of the fiber and there would be two concentric dark rings around the axis.

MODE PATTERN



Lowest-order modes in a step-index fiber



MODE CUT OFF

- A mode is referred to as being cut off when it is no longer bound to the core of the fiber, so that its field no longer decays on the outside of the core.
- The propagation constant β for the guided mode has to be nearer to k_1 and larger than k_2 for the light energy of the mode to remain confined inside the core of the optical fiber. (i.e., $kn_2 < \beta < kn_1$)- indicative of the amount of energy in the mode that lies in the core and the cladding.
- when $\beta = kn_2$, then the mode phase velocity is equal to the velocity of light in the cladding and the mode is no longer properly guided. In this case the mode is said to be cut off and the value $W = 0$
- Unguided or radiation modes have frequencies below cutoff where $\beta < kn_2$, and hence W is imaginary (these called leaky modes)
- As β is increased above n_2k, less power is propagated in the cladding until at $\beta = n_1k$ all the power is confined to the fiber core.

Cutoff conditions for some lower-order modes

ν	Mode	Cutoff condition
0	TE_{0m}, TM_{0m}	$J_0(ua) = 0$
1	HE_{1m}, EH_{1m}	$J_1(ua) = 0$
≥ 2	$EH_{\nu m}$	$J_\nu(ua) = 0$
	$HE_{\nu m}$	$\left(\frac{n_1^2}{n_2^2} + 1\right) J_{\nu-1}(ua) = \frac{ua}{\nu - 1} J_\nu(ua)$

NORMALIZED FREQUENCY OR V-NUMBER OF OPTICAL FIBER

- Which is defined as

$$V^2 = a^2(U^2 + W^2) = \left(\frac{2\pi a}{\lambda}\right)^2 ({n_1}^2 - {n_2}^2)$$

$$V = \frac{2\pi a}{\lambda} \sqrt{{n_1}^2 - {n_2}^2} = \frac{2\pi a}{\lambda} NA$$

- which is a dimensionless number that determines how many modes a fiber can support.
- The total number of modes M entering the fiber

$$M = \frac{V^2}{2}; \qquad \text{for step index fiber}$$

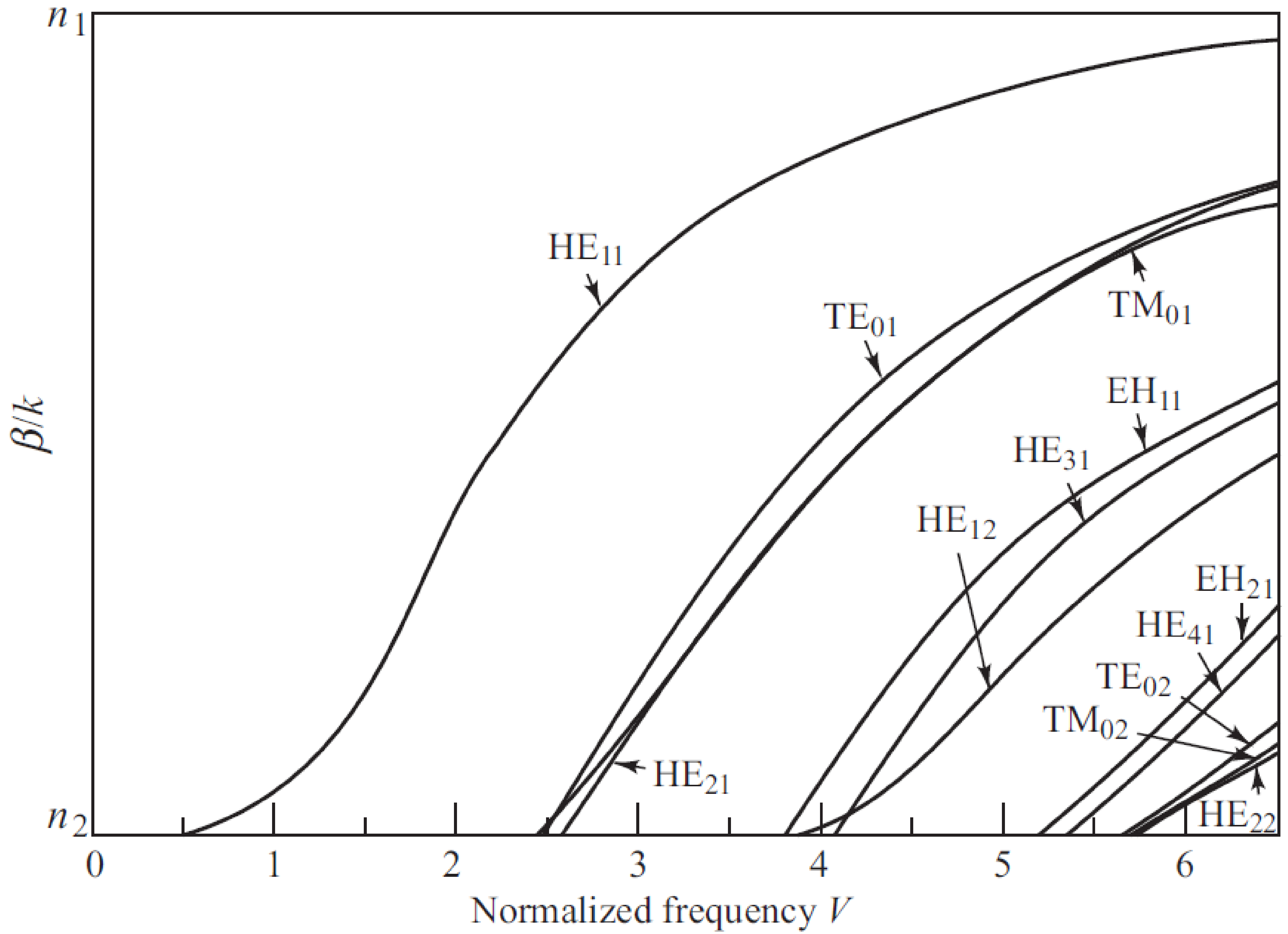
$$M = \frac{\alpha}{\alpha + 2} \frac{V^2}{2}; \qquad \text{for graded index fiber}$$

NORMALIZED PROPAGATION CONSTANT

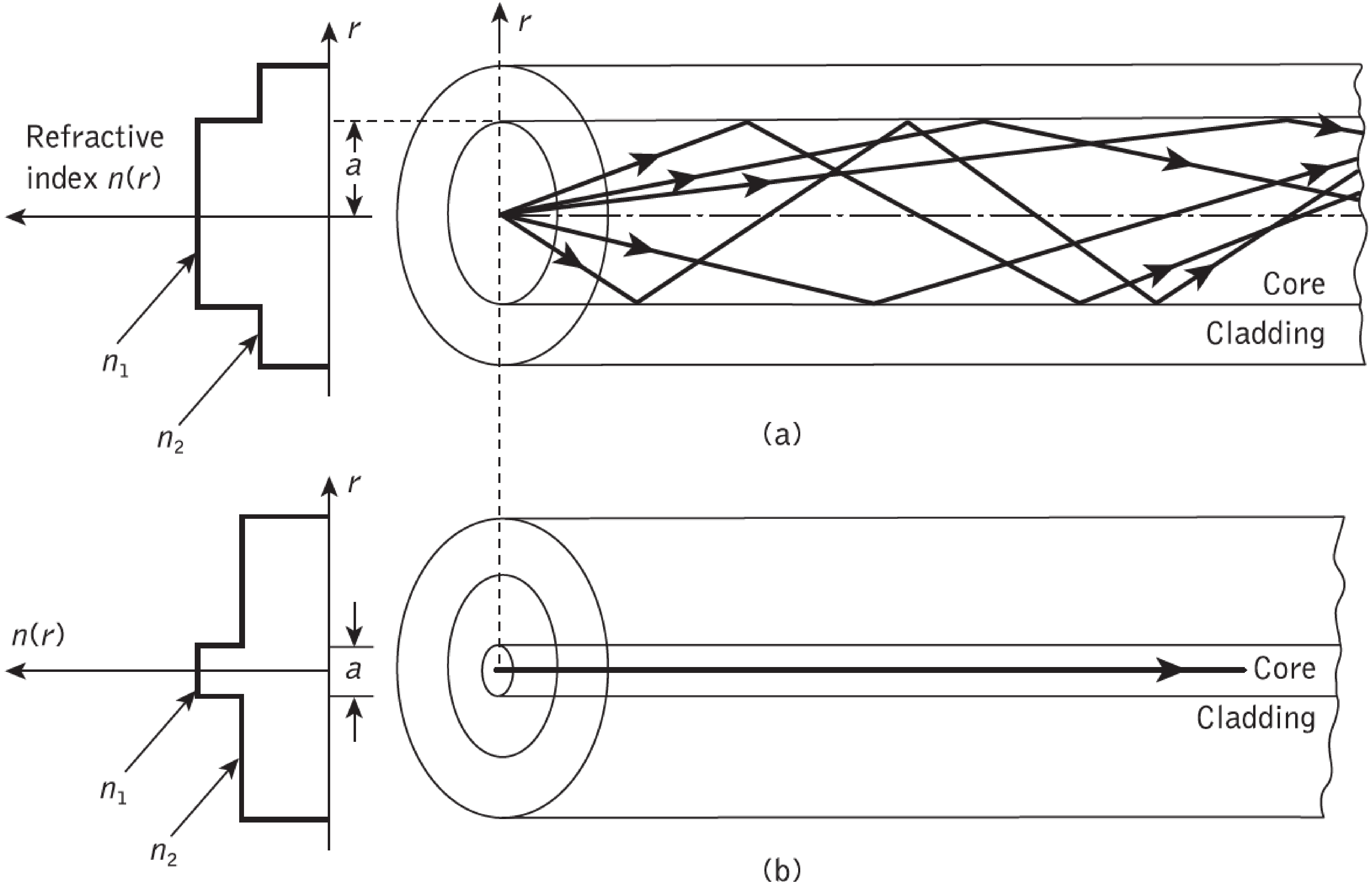
- The number of modes that can exist in a waveguide as a function of V may be conveniently represented in terms of a normalized propagation constant b defined

$$b = \frac{a^2 W^2}{V^2} = \frac{\left(\frac{\beta}{k}\right)^2 - {n_2}^2}{{n_1}^2 - {n_2}^2}$$

- The curve between $\frac{\beta}{k}$ and the normalized frequency V shows that for a particular mode to propagate, the V-number of the fiber corresponding to that mode must be greater than certain value which is the cut-off value for the mode.
- The HE_{11} mode has no cutoff and ceases to exist only when the core diameter is zero. This is the principle on which the single-mode fiber is based.
- For a fiber to be single mode: $V \leq 2.405$



n_1
n_2
β/k
HE11
TE01
TM01
EH11
HE31
HE12
HE21
EH21
HE41
TE02
TM02
HE22
0
1
2
3
4
5
6
Normalized frequency V

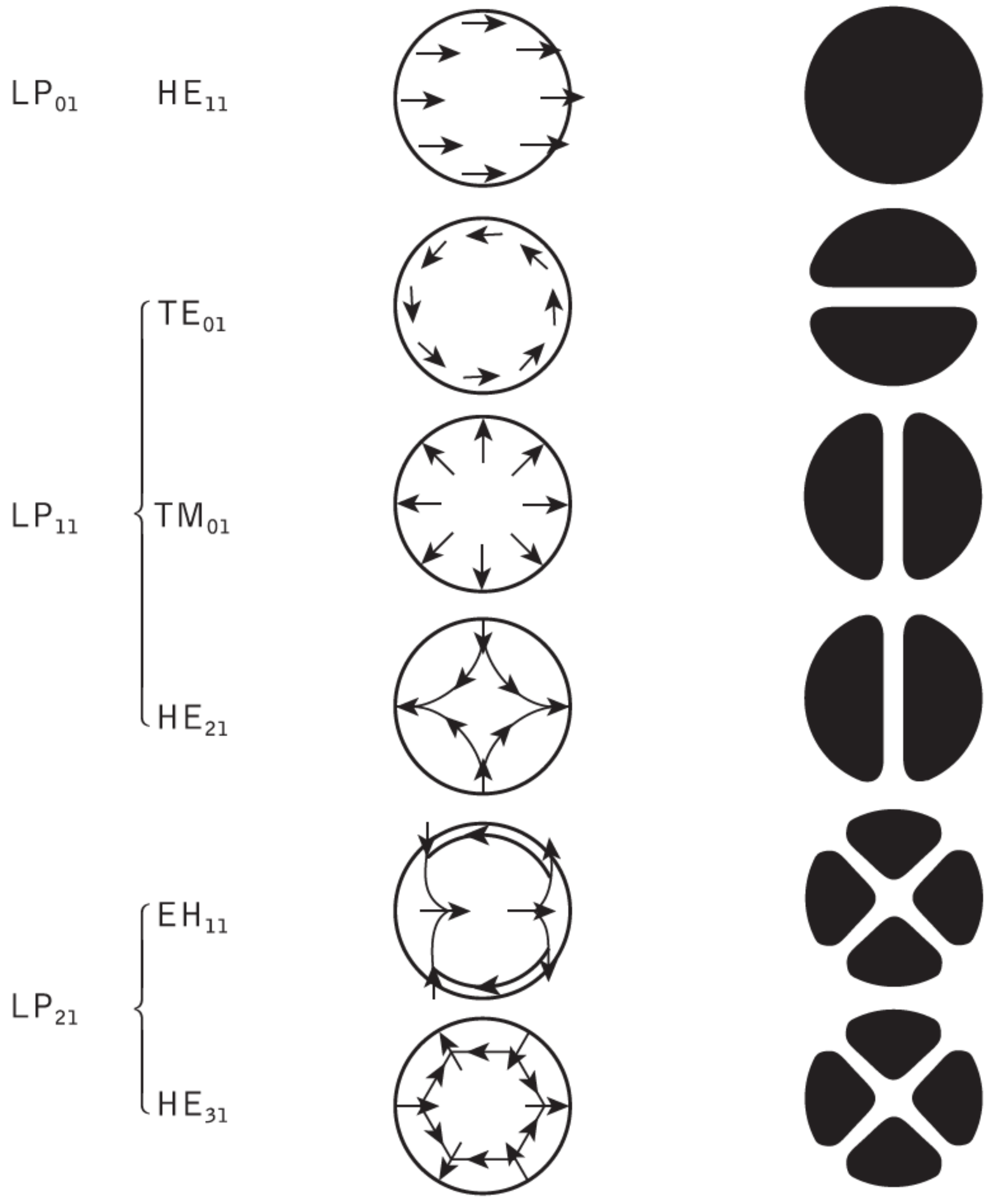


(a) multimode step index fiber; (b) single-mode step index fiber

LINEAR POLARIZED (LP) MODES:

- Consider the weakly guiding approximation where the relative index difference $\Delta << 1$.
- For very small relative refractive index difference, the HE–EH mode pairs have almost identical propagation constants. Such modes are said to be degenerate. Such degenerate modes are called linearly polarized (LP) modes, and are designated LP_{jm} modes
- The superpositions of these degenerating modes characterized by a common propagation constant correspond to particular LP modes regardless of their HE, EH, TE or TM field configurations.
  - Each LP_{0m} mode is derived from an HE_{1m} mode.
  - Each LP_{1m} mode comes from TE_{0m}, TM_{0m}, and HE_{2m} modes.
  - Each LP_{vm} mode ($v \geq 2$) is from an $HE_{v+1,\, m}$ and an $EH_{v-1,\, m}$ mode.

Linear polarized (LP) modes



PROBLEMS

- A multimode step index fiber with a core diameter of 80 μm and a relative index difference of 1.5% is operating at a wavelength of 0.85 μm. If the core refractive index is 1.48, estimate: (a) the normalized frequency for the fiber; (b) the number of guided modes. **(Ans: (a) 75.8, (b) 2873)**
- A manufacturing engineer wants to make an optical fiber that has a core index of 1.480 and a cladding index of 1.478. What should the core size be for single-mode operation at 1550 nm? **(Ans: 7.7 μm)**
- An applications engineer has an optical fiber that has a 3.0 μm core radius and a numerical aperture of 0.1. Will this fiber exhibit single-mode operation at 800 nm? **(Ans: Yes, V=2.356)**
- Suppose we have a 50 μm diameter graded-index fiber that has a parabolic refractive index profile (α = 2). If the fiber has a numerical aperture NA = 0.22, what is the total number of guided modes at a wavelength of 1310 nm? **(Ans: 174)**

Problems

- A graded index fiber with a parabolic index profile supports the propagation of 742 guided modes. The fiber has a numerical aperture in air of 0.3 and a core diameter of 70 μm. Determine the wavelength of the light propagating in the fiber. Further estimate the maximum diameter of the fiber which gives single-mode operation at the same wavelength. **(Ans: 1.2 μm, 4.4 μm)**
- A single-mode step index fiber which is designed for operation at a wavelength of 1.3 μm has core and cladding refractive indices of 1.447 and 1.442 respectively. When the core diameter is 7.2 μm, confirm that the fiber will permit single-mode transmission and estimate the range of wavelengths over which this will occur. **(Ans: <1139 nm)**

TEXT BOOKS

- Gerd Keiser, Optical Fiber Communications, TMH India, Fourth Edition, 2010.
- Senior John M., Optical Fiber Communications, Pearson Education India, Third Edition, 2009.
- R.P. Khare, *Fiber optics and optoelectronics, Oxford University Press 2004*