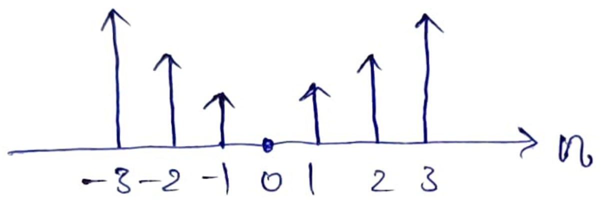
EC258:

15/12/2021

Discrete Time Sequence:

$$x(n) = \{+3, +2, 1, \underset{\uparrow}{0}, 1, 2, 3\}$$

arrow represent origin



$$x(t) = A\cos(2\pi F)t$$

$$x(n) = A\cos(2\pi(nT_s))F$$

$$x(n) = A\cos(2\pi(T_sF)n)$$

Digital frequency $\Rightarrow \boxed{\omega = T_s \times F}$

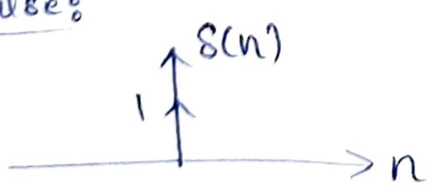
$$\boxed{\omega = T_s \times \Omega}$$

$-\infty < \Omega < \infty \rightarrow$ Range of Analog signal.

$0 < \omega < 2\pi$
$-\pi < \omega < \pi$ $\rightarrow$ Range of digital signal.

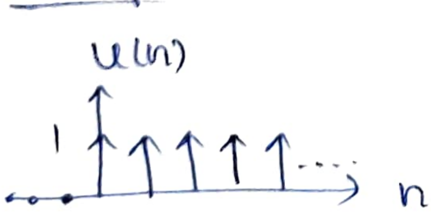
(*) Representation of Standard signals:

(1) Impulse:



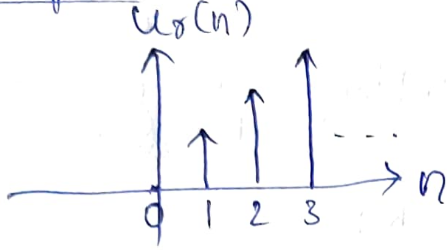
$$\delta(n) = \begin{cases} 1 & \text{for } n=0 \\ 0 & \text{otherwise} \end{cases}$$

(2) Unit step:



$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

③ Ramp:



$$u_r(n) = \begin{cases} n & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

✱ Classification of Discrete type sequences:

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$P = \frac{1}{2N+1} \lim_{N \to \infty} \sum_{n=-N}^{N} |x(n)|^2$$

eg: for unit step signal

$$E = \sum_{n=0}^{\infty} |1|^2 = 1+1+\cdots = \infty \text{ Energy } \times$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=0}^{N} |1|^2$$

$$= \lim_{N \to \infty} \frac{N+1}{2N+1}$$

$$= 1/2 \quad \rightarrow \cancel{\text{Energy}} \times \text{ Power signal } \checkmark$$

Convolution:

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) * h(n)$$

Convolution:

→ **Linear convolution:** Combining two signals using folding, shifting, multiplication and addition.

$$y(n) = x(n) * h(n)$$
$$= \sum_{K=-\infty}^{\infty} x(K)\, h(n-K)$$

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$$y(n) = T[x(n)]$$

$$x(n) = \sum_{K=-\infty}^{\infty} x(K)\, \delta(n-K)$$

$$y(n) = T[x(n)]$$
$$= T\left[\sum_{K=-\infty}^{\infty} x(K)\, \delta(n-k)\right]$$
$$= \sum_{K=-\infty}^{\infty} x(K)\, T[\delta(n-k)]$$

$$\boxed{y(n) = \sum_{K=-\infty}^{\infty} x(K)\, h(n-k)}$$

→ if $x(n) = \delta(n)$ [Impulse]
$y(n) = h(n)$ [Impulse response]

Ideal sampling: using Impulse signal.
Natural sampling: using pulse signal.

eg: Find linear convolution of two given sequences.

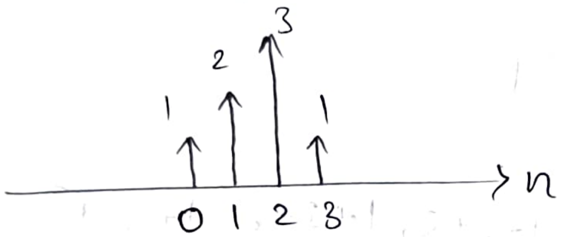
$x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\} \rightarrow N_1 = 4$

$h(n) = \{\underset{\uparrow}{1}, 1, 1\} \rightarrow N_2 = 3$

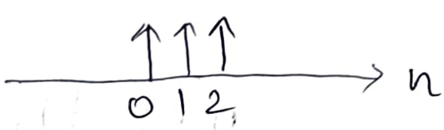
$y(n) = 4 + 3 - 1$
$= 6$ samples.

Sol:-

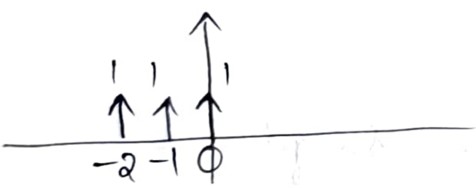
$x(n)$



$h(n)$



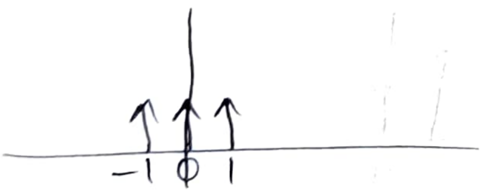
$h(-n)$



$y(0) = x(0)h(0)$

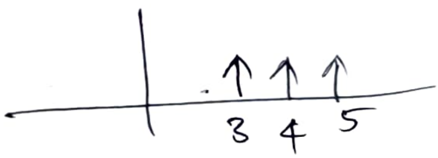
$\therefore y(0) = \underline{\underline{1}}$

$h(1-n)$

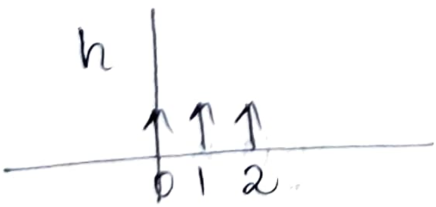


$y(1) = 1\times1 + 1\times2 = \underline{\underline{3}}$

$h(5-n)$



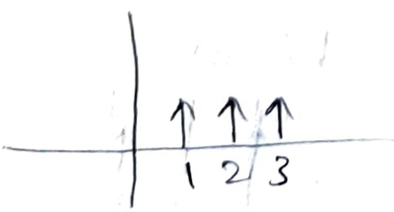
$h(2-n)$



$y(2) = 1 + 2 + 3$
$= \underline{\underline{6}}$

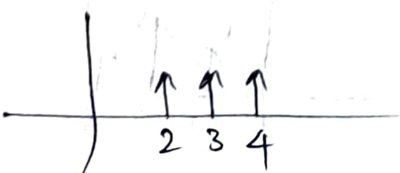
$y(5) = 1\times1$
$= \underline{\underline{1}}$

$h(3-n)$



$y(3) = 2 + 3 + 1$
$= \underline{\underline{6}}$

$h(4-n)$



$y(4) = 1\times3 + 1\times1$
$= \underline{\underline{4}}$

$y(n) = \{\underset{\uparrow}{1}, 3, 6, 6, 4, 1\}$

$y(n)$ \ $h(n)$ \ $x(n)$	1	2	3	1
1	1	2	3	1
1	1	2	3	1
1	1	2	3	1

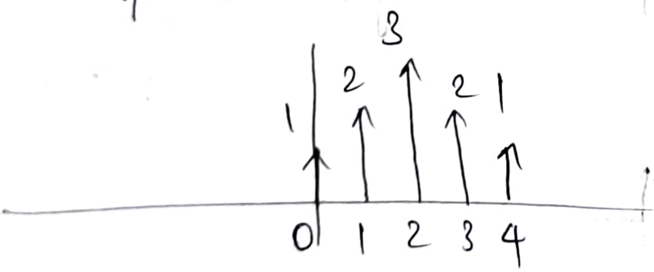
$y(n) = \{1, 1+2, 1+2+3, 1+3+2, 1+3, 1\}$

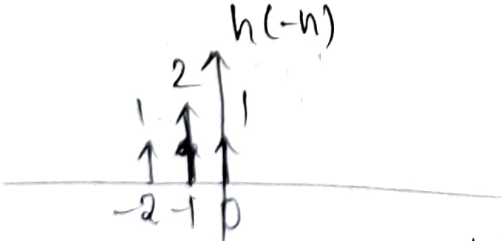
$y(n) = \{1, 3, 6, 6, 4, 1\}$

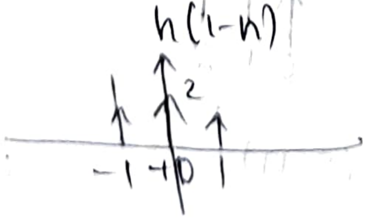
eg: $x(n) = \{\underset{\uparrow}{1}, 2, 3, 2, 1\}$

$h(n) = \{\underset{\uparrow}{1}, 2, 1\}$

$y(n) = 7$ samples

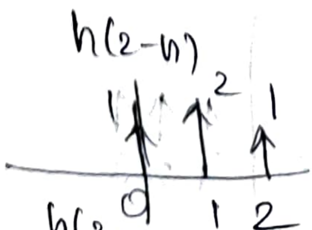






$y(0) = 1$

$y(1) = 2\times1 + 1\times2 = 4$

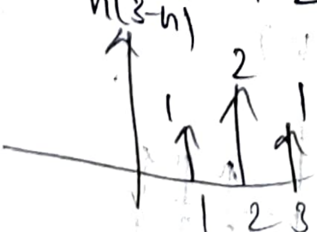


$y(2) = 1\times1 + 2\times2 + 1\times3$

$= 8$

$y(3) = 1\times2 + 2\times3 + 1\times2$

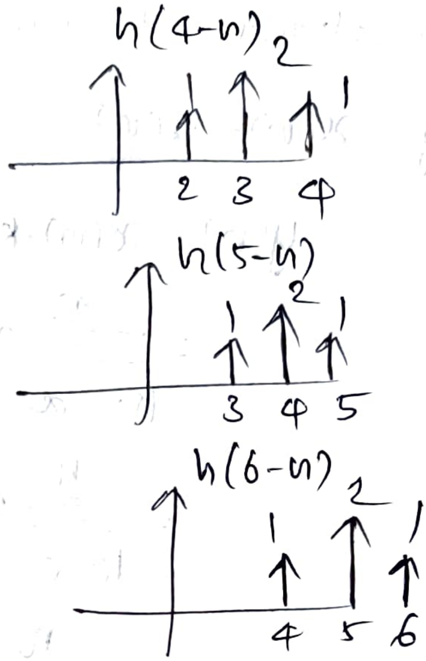
$= 10$



$y(4) = 1\times3 + 2\times2 + 1\times1$
$= 8$ //

$y(5) = 1\times2 + 2\times1$
$= 4$ //

$y(6) = 1\times1$
$= 1$ //



$y(n) = \{1, 4, 8, 10, 8, 4, 1\}$

$y(n)$

	1	2	3	2	1
1	1	2	3	2	1
2	2	4	6	4	2
1	1	2	3	2	1

$y(n) = \{1, 4, 8, 10, 8, 4, 1\}$

* Convolution of Infinite duration sequence:

eg: $x(n) = u(n)$ $\quad h(n) = \left(\frac{1}{2}\right)^n u(n)$

$$y(n) = x(n) * h(n)$$
$$= \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$
$$= \sum_{k=0}^{n} x(k) h(n-k)$$
$$= \sum_{k=0}^{n} \underline{u(k)} \left(\frac{1}{2}\right)^{n-k} \underline{u(n-k)}$$

*

$$\sum_{k=0}^{n} a^k = \frac{a^{n+1}-1}{a-1} \simeq \frac{1-a^{n+1}}{1-a}$$
$|a|>1$
$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$$
$|a|<1$

$$= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{-k}$$
$$= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} 2^k$$
$$= \left(\frac{1}{2}\right)^n \left[\frac{2^{n+1}-1}{2-1}\right]$$
$$y(n) = \left(\frac{1}{2}\right)^n (2^{n+1}-1) u(n)$$

eg: $x(n) = 2^n u(n)$ $\quad h(n) = \left(\frac{1}{2}\right)^n u(n)$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$
$$= \sum_{k=0}^{n} x(k) h(n-k)$$
$$= \sum_{k=0}^{n} 2^k u(k) \left(\frac{1}{2}\right)^{n-k} u(n-k)$$
$$= \left(\frac{1}{2}\right)^n \left(\frac{4^{n+1}-1}{3}\right) u(n)$$

eg: $x(n) = \{ \underset{\uparrow}{0}, 1, 2 \}$

$$h(n) = \left(\frac{1}{2}\right)^n u(n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

$$= \sum_{k=0}^{n} x(k) \left(\frac{1}{2}\right)^{n-k} u(n-k)$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} x(k)\, 2^k$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=0}^{2} x(k)\, 2^k$$

$$= \left(\frac{1}{2}\right)^n (1 \times 2 + 2 \times 4)$$

$$y(n) = 10 \left(\frac{1}{2}\right)^n$$

eg: $x(n) = u(n) - u(n-10)$

$$h(n) = \left(\frac{1}{2}\right)^n u(n)$$

$$y(n) = y_1(n) - y_2(n)$$

$$y_1(n) = \left(\frac{1}{2}\right)^n u(n) * u(n) = \left(\frac{1}{2}\right)^n \left(2^{n+1} - 1\right)$$

$$y_2(n) = \left(\frac{1}{2}\right)^n u(n) * u(n-10)$$

$$= \sum_{k=10}^{n} \left(\frac{1}{2}\right)^{n-k}$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=10}^{n} 2^k \qquad k - 10 = t$$

$$= \left(\frac{1}{2}\right)^n \sum_{t=0}^{n} 2^{t+10}$$

eg: $x(n) = u(n) - u(n-10)$

$h(n) = \left(\frac{1}{2}\right)^n u(n)$

$y(n) = y_1(n) - y_2(n)$

$y_1(n) = \left(\frac{1}{2}\right)^n u(n) * u(n) = \left(\frac{1}{2}\right)^n \left(\frac{2^{n+1}-1}{1}\right) u(n)$

$y_2(n) = \left(\frac{1}{2}\right)^n u(n) * u(n-10)$

$$= \sum_{k=-\infty}^{\infty} u(k-10) \left(\frac{1}{2}\right)^{n-k} u(n-k)$$

$k-10 = t$

$$= \sum_{t=-\infty}^{\infty} u(t) \left(\frac{1}{2}\right)^{n-t-10} u(n-t-10)$$

$$= \sum_{t=10}^{n} \left(\frac{1}{2}\right)^{n-10} 2^t$$

$$y_2(n) = \left(\frac{1}{2}\right)^{n-10} \cdot \frac{2^{n+9}-1}{2-1} \cdot u(n)$$

eg: $x(n) = u(n)$

$h(n) = \left(\frac{1}{2}\right)^n u(-n)$

$$y(n) = \sum_{k=-\infty}^{\infty} u(k) \left(\frac{1}{2}\right)^{n-k} u(-n-k)$$

$$= \sum_{k=-\infty}^{0}$$

EC253:

19/01/2022

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{j2\pi kn}{N}}$$

$K = 0, 1, \ldots N-1$

eg: $x(n) = \{0, 1, 0, 1\}$

$$X(0) = \sum_{n=0}^{3} x(n)\, e^{\cancel{\frac{-j2\pi kn}{4}}}\ 1$$

$= x(0) + x(1) + x(2) + x(3)$

$= 0 + 1 + 0 + 1$

$\underline{X(0) = 2}$

$$X(1) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j\cancel{2}\pi \times 1 \times n}{\cancel{4}\,2}}$$

$= x(0) + x(1) e^{\frac{-j\pi}{2}} + x(2) e^{-j\pi} + x(3) e^{-j\frac{3\pi}{2}}$

$= 0 + \cancel{-j} + 0 + \cancel{j}$

$\underline{X(1) = 0}$

$$X(2) = \sum_{n=0}^{2} x(n)\, e^{-j\cancel{2}\pi \times \frac{\cancel{2}}{\cancel{4}} \times n}$$

$= x(0) + x(1) e^{-j\pi} + x(2) e^{-j2\pi} + x(3) e^{-j3\pi}$

$= 0 + (-1) + 0 + (-1)$

$\underline{X(2) = -2}$

$$X(3) = \sum_{n=0}^{3} x(n)\, e^{-j\frac{2\pi \times 3 \times n}{\cancel{4}\,2}}$$

$$= x(0) + x(1)\, e^{-j\frac{3\pi}{2}} + x(2)\, e^{-j3\pi} + x(2)\, e^{-j\frac{9\pi}{2}}$$

$$= 0 + j + 0 + -j$$

$$\underline{\underline{X(3) = 0}}$$

$$\underbrace{\{0,1,0,1\}}_{4} \xleftrightarrow{\text{DFT}} \underbrace{\{2,0,-2,0\}}_{4}$$

---

eg: $x(n) = \{0,1,2,3\}$

$$X(0) = \sum_{n=0}^{3} x(n)\, \cancel{e^{-j\frac{2\pi \times 0 \times n}{4}}}^{\,1}$$

$$= x(0) + x(1) + x(2) + x(3)$$

$$= 0 + 1 + 2 + 3 = \underline{\underline{6}}$$

$$X(1) = \sum_{n=0}^{3} x(n)\, e^{-j\frac{2\pi \times 1 \times n}{\cancel{4}\,2}}$$

$$= x(0) + x(1)\, e^{-j\frac{\pi}{2}} + x(2)\, e^{-j\pi} + x(3)\, e^{-j\frac{3\pi}{2}}$$

$$= 0 + (-j) + 2(-1) + 3(j)$$

$$= \underline{\underline{-2+2j}}$$

$$X(2) = \sum_{n=0}^{3} x(n)\, e^{-j \times \frac{\cancel{2}\pi \times \cancel{2} \times n}{\cancel{4}}}$$

$$= x(0) + x(1)\, e^{-j\pi} + x(2)\, e^{-j(2\pi)} + x(3)\, e^{-j(3\pi)}$$

$$= 0 + 1(-1) + 2(1) + 3(-1)$$

$$X(2) = -2 /\!/$$

$$X(3) = \sum_{n=0}^{3} x(n) e^{\frac{-j \times 2\pi \times 3}{4} \times n}$$

$$= \cancel{x(0)e^{j3\pi}}$$

$$= x(0) + x(1) e^{-j\frac{3\pi}{2}} + x(2) e^{-j(3\pi)} + x(3) e^{-j\frac{9\pi}{2}}$$

$$= 0 + j + 2(-1) + 3(-j)$$

$$\underline{X(3) = -2-2j}$$

$$\{0, 1, 2, 3\} \xleftrightarrow{DFT} \{6, -2+2j, -2, -2-2j\}$$

<u>DFT</u>

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} \quad (T \leftrightarrow f)$$

<u>IDFT</u>

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{j\frac{2\pi}{N}kn} \quad (f \rightarrow T)$$

---

eg: $X(K) = \{6, -2+2j, -2, -2-2j\}$

$$x(0) = \frac{1}{4}\sum_{k=0}^{3} X(0)\, e^{\cancel{j\frac{2\pi}{4}\times k\times 0}} \; 1$$

$$= \frac{1}{4}[X(0)+X(1)+X(2)+X(3)]$$

$$= \frac{1}{4}[6-2+\cancel{2j}-2-2-\cancel{2j}]$$

$$\underline{\underline{x(0) = 0}}$$

$$x(1) = \frac{1}{4}\sum_{k=0}^{3} X(K)\, e^{j\frac{\cancel{2}\pi}{\cancel{4}2}\times K\times 1}$$

$$= \frac{1}{4}\left[X(0)+X(1)e^{\frac{j\pi}{2}}+X(2)e^{+j\pi}+X(3)e^{\frac{j3\pi}{2}}\right]$$

$$= \frac{1}{4}[6+(-2+2j)(j)+(-2)(-1)+(-2-2j)(-j)]$$

$$= \frac{1}{4}[6+-\cancel{2j}-2+\cancel{2}+\cancel{2j}-\cancel{2}] = \frac{1}{4}\times 4 = \underline{\underline{1}}$$

$$\underline{\underline{x(1) = 1}}$$

$$x(2) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, e^{\frac{j\cancel{2}\pi\times k\times\cancel{2}}{\cancel{4}}}$$

$$= \frac{1}{4}\left[X(0)+X(1)e^{j\pi}+X(2)e^{j(2\pi)}+X(3)e^{j(3\pi)}\right]$$

$$= \frac{1}{4}[6+(-2+2j)(-1)+(-2)(1)+(+2+2j)(+1)]$$

$$= \frac{1}{4}[6+\cancel{2}-\cancel{2j}-\cancel{2}+2+\cancel{2j}] = \frac{1}{4}\times 8 = 2 //$$

$$\underline{\underline{x(2) = 2}}$$

$$x(3) = \frac{1}{4} \sum_{k=0}^{3} X(k)\, e^{\frac{j \cancel{2}\pi \times k \times 3}{\cancel{4}\,2}}$$

$$= \frac{1}{4}\left[X(0) + X(1)\, e^{\frac{j3\pi}{2}} + X(2)\, e^{j3\pi} + X(3)\, e^{j\left(\frac{9\pi}{2}\right)} \right]$$

$$= \frac{1}{4}\left[6 + (-2+2j)(-j) + (-2)(-1) + (-2-2j)(j) \right]$$

$$= \frac{1}{4}\left[6 + \cancel{2j} + 2 + 2 - \cancel{2j} + 2 \right] = \frac{1}{4} \times 12 = 3$$

$$\underline{\underline{x(3) = 3}}$$

$$\underline{\{6, -2+2j, -2, -2-2j\}} \xrightarrow{\text{IDFT}} \underline{\{0, 1, 2, 3\}}$$

EC253: 31/01/2022

Properties of DFT:

1. Linearity:

$$x_1(n) \xrightarrow{DFT} X_1(K) \qquad x_2(n) \xrightarrow{DFT} X_2(K)$$

$$DFT\{a_1x_1(n)+a_2x_2(n)\} = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}}$$

$$\sum_{n=0}^{N-1}\{a_1x_1(n)+a_2x_2(n)\}e^{-j\frac{2\pi kn}{N}}$$

$$= a_1\sum_{n=0}^{N-1} x_1(n)e^{-j\frac{2\pi kn}{N}} + a_2\sum_{n=0}^{N-1} x_2(n)e^{-j\frac{2\pi kn}{N}}$$

$$= a_1X_1(K)+a_2X_2(K)$$

2. Periodicity:

$x(n+N) = x(n)$ for all 'n'.

$X(k+N) = X(k)$ for all 'k'.

Proof:

$$x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K)e^{j\frac{2\pi kn}{N}}$$

$$x(n+N) = \frac{1}{N}\sum_{K=0}^{N-1} X(k)e^{j\frac{2\pi k(N+n)}{N}}$$

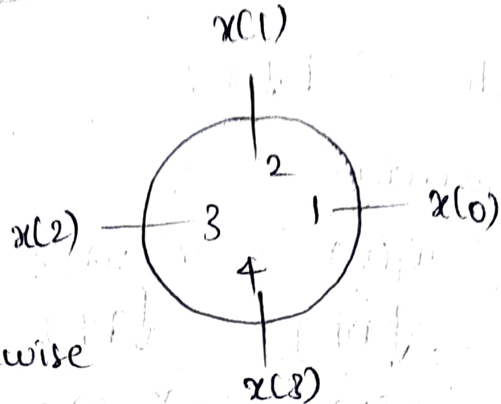
$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi kn}{N}}\underbrace{e^{j2\pi k}}_{1}$$

$$x(n+N) = x(n)$$

Circular shift of a sequence:

$x(n) = \{1, 2, 3, 4\}$

$\quad\; x(0)\; x(1)\; x(2)\; x(3)$



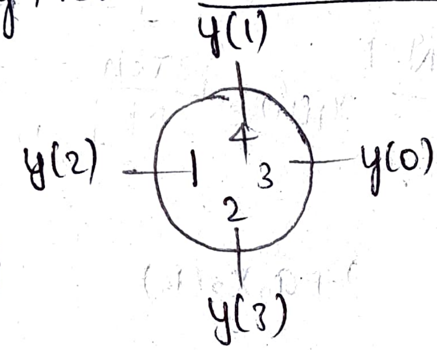
'−' delay → rotate anti-clockwise

'+' advanced → rotate clock-wise

$y(n) = \underline{x(n-2)}$

↓

Since delay, rotate anti-clockwise



$y(n) = \{3, 4, 1, 2\}$

$x(n) = \{1, 2, 3, 4\}$

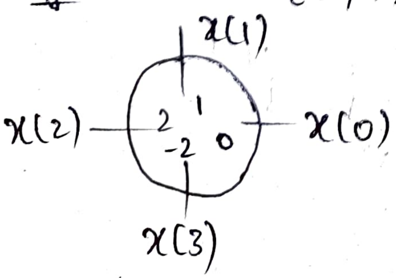
$x(n-1) = \{4, 1, 2, 3\}$

$x(n-2) = \{3, 4, 1, 2\}$ repeats

$x(n-3) = \{2, 3, 4, 1\}$

$x(n-4) = \{1, 2, 3, 4\}$

eg: $x(n) = \{0, 1, 2, -2\}$



$x(n-1) = \{-2, 0, 1, 2\}$

$x(n+2) = \{2, -2, 0, 1\}$

$$y(n) = x(n-n_0)\Big|_N$$

$$\underline{y(n) = x(N+n-n_0)}$$

eg: $x(n) = \{1, 0, 1, 0\}$

$$y(n) = x((n+1))_4$$

$$y(n) = \{0, 1, 0, 1\}$$

3. Circular shift property:

If $x(n) \xrightarrow{DFT} X(k)$

$$DFT\{x((n-n_0))_N\} \longrightarrow ?$$

IDFT: $$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi kn}{N}}$$

$$x(n-n_0) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi k}{N}(n-n_0)}$$

$$= e^{\frac{-j2\pi kn_0}{N}} \cdot \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi kn}{N}}$$

$$\underline{x(n-n_0) = e^{\frac{-j2\pi kn_0}{N}}\, x(n)}$$

$$DFT\{x(n-n_0)\} = e^{\frac{-j2\pi kn_0}{N}} \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$$

$$\underline{= X(k)\, e^{\frac{-j2\pi kn_0}{N}}}$$

- **DFT as linear Transformation:**

$$X(K) = \sum_{n=0}^{N-1} x(n)\, \omega_N^{kn} \qquad \boxed{\omega = e^{-j2\pi}}$$

$$X(K) = x(0)\cdot 1 + x(1)\, \omega_N^{k\cdot 1} + x(2)\, \omega_N^{k\cdot 2} + \cdots + \cdots x(N-1)\, \omega_N^{K(N-1)}$$

$$K = 0, 1, 2, \ldots N-1$$

<u>K=0</u>: $X(0) = x(0)\cdot 1 + x(1)\cdot 1 + x(2)\cdot 1 + \cdots + x(N-1)\cdot 1$

<u>K=1</u>: $X(1) = x(0)\cdot 1 + x(1)\cdot \omega_N^{1} + \cdots + x(N-1)\, \omega_N^{N-1}$

<u>K=2</u>: $X(2) = x(0)\cdot 1 + x(1)\cdot \omega_N^{2} + \cdots + x(N-1)\, \omega_N^{2(N-1)}$

$\vdots$

<u>K=N-1</u>: $X(N-1) = x(0)\cdot 1 + x(1)\cdot \omega_N^{N-1} + \cdots + x(N-1)\, \omega_N^{(N-1)^2}$

$$\Rightarrow \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega_N^{1} & \cdots & \omega_N^{N-1} \\ \vdots & & & \\ 1 & \omega_N^{N-1} & \cdots & \omega_N^{(N-1)(N-1)} \end{bmatrix} \times \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix} = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

$$\boxed{X(K) = \underset{N\times N}{\underline{\omega_N}} \times \underset{N\times 1}{\underline{x(n)}}}$$

$\omega_N \rightarrow$ Twiddle factor.

eg: $\cdot X(K) = \sum_{n=0}^{3} x(n)\, \omega_N^{kn}$ $(N=4)$

$$X(K) = x(0)\cdot 1 + x(1)\,\omega_{N=4}^{K\cdot 1} + x(2)\,\omega_4^{K\cdot 2} + x(3)\,\omega_4^{K\cdot 3}$$

$$\omega = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_4^1 & \omega_4^2 & \omega_4^3 \\ 1 & \omega_4^2 & \omega_4^4 & \omega_4^6 \\ 1 & \omega_4^3 & \omega_4^6 & \omega_4^9 \end{bmatrix}_{4\times 4}$$

$$\boxed{\begin{array}{l} (1)\ \ \omega_N^{K+N} = \omega_N^K \\ (2)\ \ \omega_N^{K+N/2} = -\,\omega_N^K \end{array}}$$

$$\omega = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_4^1 & \omega_4^2 & \omega_4^3 \\ 1 & \omega_4^2 & \omega_4^0 & \omega_4^2 \\ 1 & \omega_4^3 & \omega_4^2 & \omega_4^1 \end{bmatrix}$$

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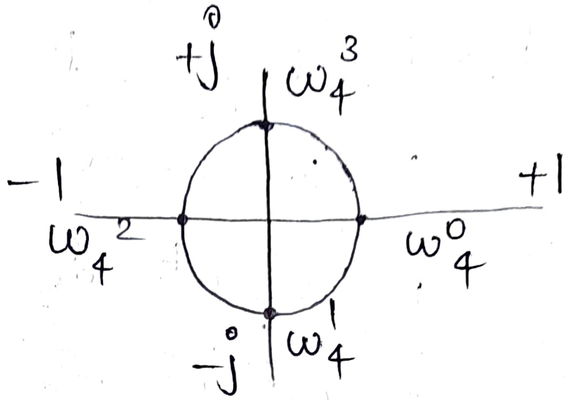
4-DFT:

$$X_N = W_N \cdot x_N$$

$$\boxed{\begin{array}{l} ① \; W_N^{k+N} = W_N^{k} \\ ② \; W_N^{k+\frac{N}{2}} = -W_N^{k} \end{array}}$$



$$W = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}$$

$$W = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix}$$

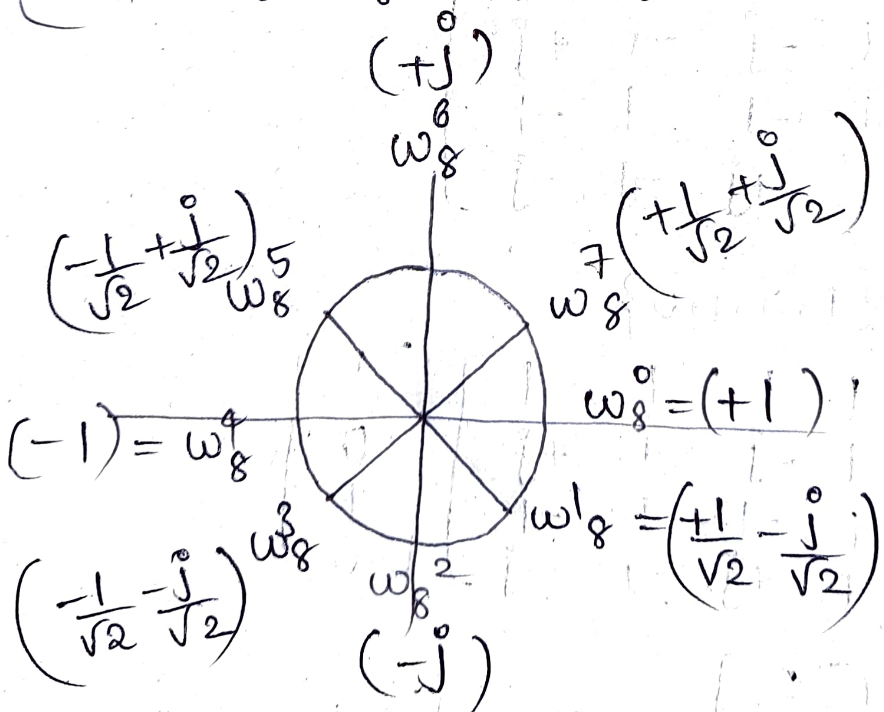
eg: $x(n) = \{1, 0, 1, 0\}$

$$X_N = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$X_N = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} \qquad X_N = \{2, 0, 2, 0\}$$

8-DFT:

$$
W = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & w_8^1 & w_8^2 & w_8^3 & w_8^4 & w_8^5 & w_8^6 & w_8^7 \\
1 & w_8^2 & w_8^4 & w_8^6 & w_8^8 & w_8^{10} & w_8^{12} & w_8^{14} \\
1 & w_8^3 & w_8^6 & w_8^9 & w_8^{12} & w_8^{15} & w_8^{18} & w_8^{21} \\
1 & w_8^4 & w_8^8 & w_8^{12} & w_8^{16} & w_8^{20} & w_8^{24} & w_8^{28} \\
1 & w_8^5 & w_8^{10} & w_8^{15} & w_8^{20} & w_8^{25} & w_8^{30} & w_8^{35} \\
1 & w_8^6 & w_8^{12} & w_8^{18} & w_8^{24} & w_8^{30} & w_8^{36} & w_8^{42} \\
1 & w_8^7 & w_8^{14} & w_8^{21} & w_8^{28} & w_8^{35} & w_8^{42} & w_8^{49}
\end{bmatrix}
$$



$$
W = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\
1 & W_8^2 & W_8^4 & W_8^6 & W_8^0 & W_8^2 & W_8^4 & W_8^6 \\
1 & W_8^3 & W_8^6 & W_8^1 & W_8^4 & W_8^7 & W_8^2 & W_8^5 \\
1 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 \\
1 & W_8^5 & W_8^2 & W_8^7 & W_8^4 & W_8^1 & W_8^6 & W_8^3 \\
1 & W_8^6 & W_8^4 & W_8^2 & W_8^0 & W_8^6 & W_8^4 & W_8^2 \\
1 & W_8^7 & W_8^6 & W_8^5 & W_8^4 & W_8^3 & W_8^2 & W_8^1
\end{bmatrix}
$$

eg: $x(n) = \{1, 1, 1, 1, 0, 0, 0, 0\}$

$X(K) = \{ \quad \ldots \quad \}$

$$
X(k) = \begin{Bmatrix}
4 \\
1 + W_8^1 + W_8^2 + W_8^3 \\
1 + W_8^2 + W_8^4 + W_8^6 \\
1 + W_8^3 + W_8^6 + W_8^1 \\
1 + W_8^4 + W_8^0 + W_8^4 \\
1 + W_8^5 + W_8^2 + W_8^7 \\
1 + W_8^6 + W_8^4 + W_8^2 \\
1 + W_8^7 + W_8^6 + W_8^5
\end{Bmatrix}
=
\begin{Bmatrix}
4 \\
1 - (1+\sqrt{2})j \\
0 \\
1 + (1-\sqrt{2})j \\
0 \\
1 + (-1+\sqrt{2})j \\
0 \\
1 + (1+\sqrt{2})j
\end{Bmatrix}
$$

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$$X_N = \omega_N \cdot x_N$$

$$X_N \cdot \omega_N^{-1} = \omega_N \cdot \omega_N^{-1} x_N$$

$$x_N = X_N \cdot \omega_N^{-1}$$

$$\boxed{x_N = \frac{1}{N} \omega_N^{*} \cdot X_N}$$

- Circular convolution:

$$x_1(n) \times x_2(n) \xrightarrow{DFT} X_1(K) \circledast X_2(K)$$

$$X_1(K) \times X_2(K) \xrightarrow{IDFT} x_1(n) \circledast x_2(n)$$

$\circledast = \textcircled{N}$ - circular convolution of max length N.

→ It is multiplication of any two DFTs.

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) * h(n)$$

$$y(n) = x(n) \textcircled{N} h(n)$$

$$X_1(K) \cdot X_2 \qquad X(K) \cdot H(K)$$

$$\downarrow IDFT$$

$$x(n) \textcircled{N} h(n)$$

Circular convolution

$$x_3(m) = IDFT\{X_1(K) \cdot X_2(K)\}$$

$$X_3(k) = X_1(k) \cdot X_2(k)$$

$$\downarrow \qquad \downarrow$$

$$x_1(n) \qquad x_2(n)$$

$$X_1(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{N}} \quad \text{— (1)}$$

$$X_2(k) = \sum_{l=0}^{N-1} x(l)\, e^{-j\frac{2\pi kl}{N}} \quad \text{— (2)}$$

$$\text{IDFT}\{X_3(k)\} = \frac{1}{N} \sum_{k=0}^{N-1} X_3(k)\, e^{j\frac{2\pi km}{N}}$$

$$x_3(m) = \frac{1}{N} \sum_{k=0}^{N-1} X_3(k)\, e^{j\frac{2\pi km}{N}} \quad \text{— (3)}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left(\sum_{n=0}^{N-1} x_1(n)\, e^{-j\frac{2\pi kn}{N}} \sum_{l=0}^{N-1} x_2(l)\, e^{-j\frac{2\pi kl}{N}} \right) e^{j\frac{2\pi km}{N}}$$

$$x_3(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \left\{ \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(m-n-l)} \right\}$$

$$\sum_{k=0}^{N-1} \Big(\underbrace{e^{j\frac{2\pi}{N}(m-n-l)}}_{a} \Big)^k$$

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & a = 1 \\ \dfrac{1-a^N}{1-a}, & a \neq 1 \end{cases}$$

$$\sum_{K=0}^{N-1} a^K = \begin{cases} N & \text{if } l = m-n+pN \\ & \end{cases}$$

$$\boxed{x_3(m) = \sum_{n=0}^{N-1} x_1(n) \cdot x_2((m-n))_N}$$

$$x(n) * h(n)$$

linear convolution $= \sum_{K=-\infty}^{\infty} x(k)\, h(n-k)$

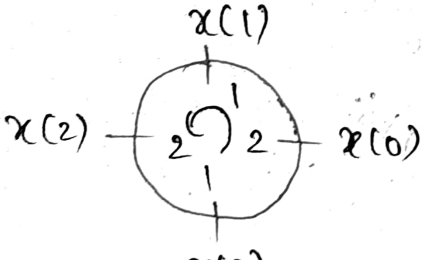
circular convolution $= \sum_{K=0}^{N-1} x_1(k) \cdot x_2((m-n))_N$

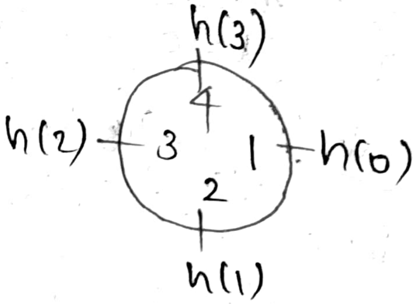
$$x(n) \; (N) \; h(n) = \sum_{K=0}^{N-1} x(k) \cdot h((m-n))_N$$

---

eg: $x(n) = \{2, 1, 2, 1\}$ $\quad h(n) = \{1, 2, 3, 4\}$

$$y(n) = x(n) \; (N) \; h(n) = \sum_{K=0}^{N-1} x(k)\, h(n-k))_N$$





$$y(0) = \sum_{K=0}^{3} x(k)\, h(-k)$$

$$= 1\times2 + 1\times4 + 2\times3 + 1\times2$$

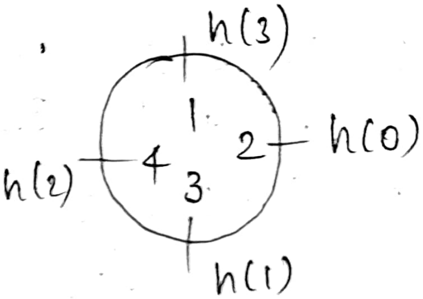
$$\therefore \underline{y(0) = 14}$$

$$y(1) = \sum_{k=0}^{3} x(k)\, h(1-k)$$

$h(-(k-1))$

$y(1) = 2\times2 + 1\times1 + 2\times4 + 1\times3$

$\therefore \underline{y(1) = 16}$

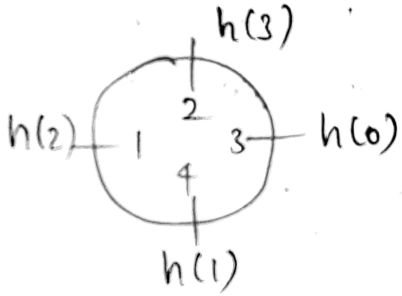


$$y(2) = \sum_{k=0}^{3} x(k)\, h(2-k)$$

$h(-(k-2))$

$y(2) = 2\times3 + 1\times2 + 2\times1 + 1\times4$

$\therefore \underline{y(2) = 14}$

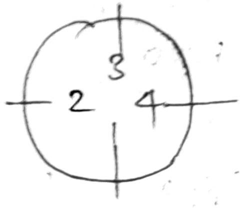


$$y(3) = \sum_{k=0}^{3} x(k)\, h(3-k)$$

$h(-(k-3))$

$y(3) = 2\times4 + 1\times3 + 2\times2 + 1\times1$

$\therefore \underline{y(3) = 16}$



$y(n) = \{14, 16, 14, 16\}$

eg: $x(n) = \{2, 1, 2, 1\}$ $h(n) = \{1, 2, 3, 4\}$

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 14 \\ 16 \\ 14 \\ 16 \end{bmatrix} = y(n)$$

eg: $x(n) = \{1, 1, 2, 2\}$ $h(n) = \{1, 2, 3, 4\}$

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 15 \\ 17 \\ 15 \\ 13 \end{bmatrix}$$

Circular convolution using DFT and IDFT:

$\text{IDFT}\{X_1(k) \cdot X_2(k)\} \Rightarrow$ circular convolution

eg: $x_1(n) = \{1, 1, 2, 2\}$ $x_2(n) = \{1, 2, 3, 4\}$

$$X_1(k) = \sum_{n=0}^{3} x_1(n) e^{-j2\pi kn/4}$$

$$X_1(k) = W_N \cdot x_1(n)$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

$$X_1(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & +1 & +j \\ 1 & -1 & +1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -1+j \\ 0 \\ -1-j \end{bmatrix}$$

$$X_2(k) = \sum_{n=0}^{3} x_2(n) e^{-j\frac{2\pi kn}{4}}$$

$$X_2(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & +1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

$$X_1(k) \cdot X_2(k) = \{60, (-1+j)(-2+2j), 0, (-1-j)(-2-2j)\}$$

$$X_1(k) \cdot X_2(k) = \{60, -4j, 0, 4j\}$$

$$\text{IDFT}\{X_1(k) \cdot X_2(k)\}$$

$$x_3(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_3(k) e^{\frac{j2\pi kn}{N}}$$

$$x_3(n) = \frac{1}{N} W_N^{*} \cdot X_N$$

$$x_3(n) = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & +j \end{bmatrix} \begin{bmatrix} 60 \\ -4j \\ 0 \\ 4j \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 60 \\ 68 \\ 60 \\ 52 \end{bmatrix}$$

$$\underline{x_3(n) = \{15, 17, 15, 13\}}$$

eg: $x(n) = \{1, 2, 2, 1\}$ $\quad h(n) = \{1, 2, 3\}$

$$y(n) = x(n) * h(n)$$

$\downarrow$

$4 + 3 - 1 = 6$ samples

$y(n)$	1	2	2	1
1	1	2	2	1
2	2	4	4	2
3	3	6	6	3

$$\underline{y(n) = \{1, 4, 9, 11, 8, 3\}}$$

$\Rightarrow$ To apply circular convolution,
length of given two sequences should be <u>same</u>.

$x(n) = \{1, 2, 2, 1, 0, 0\}$

$h(n) = \{1, 2, 3, 0, 0, 0\}$

Making the input length is equal to output length by adding zeroes.

$$y(n) = \begin{bmatrix} 1 & 0 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$y(n) = \begin{bmatrix} 1 \\ 4 \\ 9 \\ 11 \\ 8 \\ 3 \end{bmatrix}$$

$$\underline{y(n) = \{1, 4, 9, 11, 8, 3\}}$$

__eg:__ $x(n) = \{1, 1, 0, 1, 1\}$ $h(n) = \{1, -2, -3, 4\}$

y(n)	1	1	0	1	1
1	1	1	0	1	1
-2	-2	-2	0	-2	-2
-3	-3	-3	0	-3	-3
4	4	4	0	4	4

$$\underline{\underline{y(n) = \{1, -1, -5, 1, 3, -5, 1, 4\}}}$$

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Filtering of Long Data Sequences:

$$x(n) \longrightarrow \boxed{\overset{h(n)}{F}} \longrightarrow y(n) = h(n) * x(n)$$

$x(n)$ → long data sequence

$y(n)$ → filtering of $x(n)$

① Over-lap save method:

$x(n) \rightarrow$ long data sequence

$h(n) \Rightarrow$ size is 'M'

(a) Determine the size of $h(n)$

(b) N = size of each block $= 2^M$

$N = L+M-1$

$$x_1(n) = \{\underbrace{0, 0, \ldots}_{M-1}, \underbrace{x(0), x(1), \ldots x(L-1)}_{L}\}$$

$$x_2(n) = \{\underbrace{x(L-M+1), \ldots x(L-1)}_{M-1}, x(L), \ldots x(2L-1)\}$$

$$x_3(n) = \{\underbrace{x(2L-M+1), \ldots x(2L-1)}_{M-1}, x(2L), \ldots \ldots x(3L-1)\}$$

(c) $y_1(n) = x_1(n) \textcircled{N} h(n)$ Do zero padding if necessary.

$y_2(n) = x_2(n) \textcircled{N} h(n)$

$y_3(n) = x_3(n) \textcircled{N} h(n)$

Discard the M-1 samples from $y_1(n), y_2(n), \ldots$ and club into $y(n)$

---

eg: $x(n) = \{3, 0, -2, 0, 2, 1, 0, -2, -1, 0\}$

$h(n) = \{1, 1, 1\} \Rightarrow M = 3$

$N = 2^3 = 8$

$8 = L + 3 - 1 \Rightarrow \boxed{L = 6}$

$x_1(n) = \{\underbrace{0, 0}_{M-1}, \underbrace{3, 0, -2, 0, 2, 1}_{L}\}$

$x_2(n) = \{2, 1, 0, -2, -1, 0, 0, 0\}$

---

eg: $a(n) = \{3, -1, 0, 1, B, 2, 0, 1, 2, 1, 1, -1\}$

eg: $x(n) = \{3, -1, 0, 1, 3, 2, 0, 1, 2, 1\}$

$h(n) = \{1, 1, 1\} \rightarrow M = 3$

$N = 2^M = 2^3 = 8$

$8 = L + 3 - 1 \Rightarrow \boxed{L = 6}$

$x_1(n) = \{0, 0, 3, -1, 0, 1, 3, 2\}$

$x_2(n) = \{3, 2, 0, 1, 2, 1, 0, 0\}$

$y_1(n) = x_1(n) \; N \; h(n)$

$y_2(n) = x_2(n) \; N \; h(n)$

$$y_1(n) = \begin{bmatrix} 0 & 2 & 3 & 1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 2 & 3 & 1 & 0 & -1 & 3 \\ 3 & 0 & 0 & 2 & 3 & 1 & 0 & -1 \\ -1 & 3 & 0 & 0 & 2 & 3 & 1 & 0 \\ 0 & -1 & 3 & 0 & 0 & 2 & 3 & 1 \\ 1 & 0 & -1 & 3 & 0 & 0 & 2 & 3 \\ 3 & 1 & 0 & -1 & 3 & 0 & 0 & 2 \\ 2 & 3 & 1 & 0 & -1 & 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 6 \end{bmatrix}$$

$$y_1(n) = \{\overset{\times}{5}, \overset{\times}{2}, 3, 2, 2, 0, 4, 6\}$$

$$y_2(n) = \begin{bmatrix} 3 & 0 & 0 & 1 & 2 & 1 & 0 & 2 \\ 2 & 3 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 2 & 3 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 2 & 3 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 2 & 3 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$y_2(n) = \{3, 5, 5, 3, 3, 4, 3, 1\}$$

$$\therefore\ y(n) = \{3, 2, 2, 0, 4, 6, 5, 3, 3, 4, 3, 1\}$$

eg: $x(n) = \{3, 0, -2, 0, 2, 1, 0, -2, -1, 0\}$

$h(n) = \{2, 2, 1\} \rightarrow \underline{M=3}$

$N = 2^3 = 8 = L + 3 - 1$

$\boxed{L = 6}$

$x_1(n) = \{0, 0, 3, 0, -2, 0, 2, 1\}$

$x_2(n) = \{2, 1, 0, -2, -1, 0, 0, 0\}$

$y_1(n) = x_1(n) \; N \; h(n)$

$y_2(n) = x_2(n) \; N \; h(n)$

$$y_1(n) = \begin{bmatrix} 0 & 1 & 2 & 0 & -2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 & -2 & 0 & 3 \\ 3 & 0 & 0 & 1 & 2 & 0 & -2 & 0 \\ 0 & 3 & 0 & 0 & 1 & 2 & 0 & -2 \\ -2 & 0 & 3 & 0 & 0 & 1 & 2 & 0 \\ 0 & -2 & 0 & 3 & 0 & 0 & 1 & 2 \\ 2 & 0 & -2 & 0 & 3 & 0 & 0 & 1 \\ 1 & 2 & 0 & -2 & 0 & 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$y_1(n) = \{\cancel{4}, \cancel{1}, 6, 6, -1, -4, 2, 6\}$

$$y_2(n) = \begin{bmatrix} 2 & 0 & 0 & 0 & -1 & -2 & 0 & 1 \\ 1 & 2 & 0 & 0 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & -1 & -2 \\ -2 & 0 & 1 & 2 & 0 & 0 & 0 & -1 \\ -1 & -2 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & -1 & -2 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & -1 & -2 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$y_2(n) = \{\cancel{4}, \cancel{6}, 4, -3, -6, -4, -1, 0\}$$

$$y(n) = \{6, 6, -1, -4, 2, 6, 4, -3, -6, -4, -1, 0\}$$

EC253:

09/02/2022

② Overlap-add method:

$x(n) = \{x(0), x(1), \ldots\} \rightarrow$ large sequence.

$h(n) \Rightarrow$ no of samples $= M$

$N = L + M - 1$

$$x_1(n) = \{\underbrace{x(0), x(1) \ldots, x(L-1)}_{L}, \underbrace{0, 0, \ldots 0}_{M-1}\}$$

$$x_2(n) = \{\underbrace{x(L), x(L+1), \ldots x(2L-1)}_{L}, \underbrace{0, 0, \ldots 0}_{M-1}\}$$

We are doing zero padding to every sub-sequence at last.

eg: $x(n) = \{3, -1, 0, 1, 3, 2, 0, 1, 2, 1\}$

$h(n) = \{1, 1, 1\} \longrightarrow \boxed{M = 3}$

$y(n) = 12$ samples

$N = 2^M = 2^3 = 8$

$8 = L + 3 - 1 \Rightarrow \boxed{L = 6}$

$x_1(n) = \{3, -1, 0, 1, 3, 2, 0, 0\}$

$x_2(n) = \{0, 1, 2, 1, 0, 0, 0, 0\}$

$h(n) = \{1, 1, 1, 0, 0, 0, 0, 0\}$

$y_1(n) = x_1(n) \; N \; h(n)$

$y_2(n) = x_2(n) \; N \; h(n)$

$$y_1(n) = \begin{bmatrix} 3 & 0 & 0 & 2 & 3 & 1 & 0 & -1 \\ -1 & 3 & 0 & 0 & 2 & 3 & 1 & 0 \\ 0 & -1 & 3 & 0 & 0 & 2 & 3 & 1 \\ 1 & 0 & -1 & 3 & 0 & 0 & 2 & 3 \\ 3 & 1 & 0 & -1 & 3 & 0 & 0 & 2 \\ 2 & 3 & 1 & 0 & -1 & 3 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 2 & 3 & 1 & 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$y_1(n) = \{3, 2, 2, 0, 4, 6, 5, 2\}$$

$$y_2(n) \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$y_2(n) = \{0, 1, 3, 4, 3, 1, 0, 0\}$$

discard last $M-1$ samples

$$y(n) = \{3, 2, 2, 0, 4, 6, \underline{5, 2}\}$$

$$\{\underline{0, 1}, 3, 4, 3, 1, \underline{\cancel{0}, 0}\}$$

$+$

$$\{3, 2, 2, 0, 4, 6, 5, 3, 3, 4, 3, 1\}$$

eg: $x(n) = \{1, 2, -1, 3, -2, -3, -1, +1, +1, 2, -1\}$

$h(n) = \{1, 2\} \rightarrow M = 2$

$y(n) = 12$ samples

$N = 2^2 = 4$

$4 = L + 2 - 1 \Rightarrow \boxed{L = 3}$

$x_1(n) = \{1, 2, -1, 0\}$ $\quad h(n) = \{1, 2, 0, 0\}$

$x_2(n) = \{3, -2, -3, 0\}$

$x_3(n) = \{-1, 1, 1, 0\}$

$x_4(n) = \{2, -1, 0, 0\}$

$$y_1(n) = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 0 & -1 \\ -1 & 2 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$y_1(n) = \{1, 4, 3, -2\}$

$$y_2(n) = \begin{bmatrix} 3 & 0 & -3 & -2 \\ -2 & 3 & 0 & -3 \\ -3 & -2 & 3 & 0 \\ 0 & -3 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$y_2(n) = \{3, 4, -7, -6\}$

$$y_3(n) = \begin{bmatrix} -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$y_3(n) = \{-1, -1, 3, 2\}$$

$$y_4(n) = \begin{bmatrix} 2 & 0 & 0 & -1 \\ -1 & 2 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$y_4(n) = \{2, 3, -2, 0\}$$

$$y(n) = \{1, 4, 3, -2\}$$

$$+ \{3, 4, -7, -6\}$$

$$\{-1, -1, 3, 2\}$$

$$+$$

$$\{2, 3, -2, \cancel{0}\}$$

$$+$$

$$y(n) = \{1, 4, 3, 1, 4, -7, -7, -1, 3, 4, 3, -2\}$$

• Properties:

4. Circular frequency shift property:

$$x(n) \xleftrightarrow{DFT} X(k)$$

$$x(n) \cdot e^{\frac{j2\pi l n}{N}} \xleftrightarrow{DFT} X((k-l))_N$$

Proof:

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}}$$

$$DFT\left[x(n) \cdot e^{\frac{j2\pi l n}{N}}\right] = \sum_{n=0}^{N-1} x(n) \cdot e^{\frac{j2\pi l n}{N}} \cdot e^{\frac{-j2\pi kn}{N}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-\frac{j2\pi}{N}(k-l)n}$$

$$= \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi}{N}(N+k-l) \cdot n}$$

$$= X(N+k-l)$$

$$= X((k-l))_N$$

5. Time-Reversal:

$$x(n) \xleftrightarrow{DFT} X(k)$$

$$x(-n) \xleftrightarrow{DFT} X(-k)$$

$$DFT\{x((-n))_N\} = DFT\{x(N-n)\}$$

$$X((-k))_N = X(N-k)$$

Proof: Assume $N-n=m$

$$\text{DFT}[x(N-n)] = \sum_{n=0}^{N-1} x(N-n)\, e^{-j\frac{2\pi kn}{N}}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{-j\frac{2\pi}{N}k(N-m)}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{+j\frac{2\pi}{N}km}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{-j\frac{2\pi}{N}k(-m)}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{-j\frac{2\pi}{N}m(N-k)}$$

$$= X(N-k) = X((-k))_N$$

6. Complex-conjugate property:

$$x(n) \xleftrightarrow{\text{DFT}} X(k)$$

$$\text{DFT}\{x^*(n)\} \longleftrightarrow X^*((-k))_N = X^*(N-k)$$

$$= \sum_{n=0}^{N-1} x^*(n)\, e^{-j\frac{2\pi kn}{N}}$$

$$= \left[\sum_{n=0}^{N-1} x(n)\, e^{j\frac{2\pi kn}{N}}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}n(N-k)}\right]^*$$

$$= [X(N-k)]^* = X^*(N-k) = X^*((-k))_N$$

$$x^*(n) \xrightarrow{DFT} X^*((-k))_N$$
$$x^*(-n) \xrightarrow{DFT} X^*((k))_N$$

→ $x(n)$ is a real valued sequence then $x(n) = x^*(n)$

Apply DFT on both side's,

$$X(k) = X^*((-k))_N$$

$$X_R(k) + jX_I(k) = X_R(N-k) - jX_I(N-k)$$

$$\left.\begin{aligned} &\bullet\ X_R(k) = X_R(N-k) \\ &\bullet\ X_I(k) = -X_I(N-k) \end{aligned}\right\} \text{ If } x(n) \text{ is a real sequence}$$

eg: Let $X(k)$ be a 14-point DFT, the first 8 samples are $\{\underset{0}{12}, \underset{1}{-1+j3}, \underset{2}{3+j4}, \underset{3}{1-j5}, \underset{4}{-2+j2}, \underset{5}{6+j3}, \underset{6}{-2-j3}, \underset{7}{10}\}$

Determine remaining samples of $X(k)$ if $x(n)$ is real

Sol:- $X(k) = X^*(N-k) \rightarrow$ if $x(n)$ is real

$X(8) = X^*(14-8) = X^*(6) = \underline{-2+3j}$

$X(9) = X^*(14-9) = X^*(5) = \underline{6-j3}$

$X(10) = X^*(14-10) = X^*(4) = \underline{-2-j2}$

$X(11) = X^*(14-11) = X^*(3) = \underline{1+j5}$

$X(12) = X^*(14-12) = X^*(2) = \underline{3-j4}$

$X(13) = X^*(14-13) = X^*(1) = \underline{-1-j3}$

Remaining samples = $\{\underset{8}{-2+3j}, \underset{9}{6-j3}, \underset{10}{-2-j2}, \underset{11}{1+j5}, \underset{12}{+3-j4}, \underset{13}{-1-j3}\}$

7. **Parseval's Energy theorem:**

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

Proof:

$$\sum_{n=0}^{N-1} x(n) \cdot x^*(n) = \sum_{n=0}^{N-1} x(n) \cdot \left[\frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi kn}{N}} \right]^*$$

$$= \sum_{n=0}^{N-1} \frac{1}{N} x(n) \left[\sum_{k=0}^{N-1} X^*(k)\, e^{-\frac{j2\pi kn}{N}} \right]$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X^*(k) \cdot \underbrace{\sum_{n=0}^{N-1} x(n)\, e^{-\frac{j2\pi kn}{N}}}_{X(k)}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X^*(k) \cdot X(k)$$

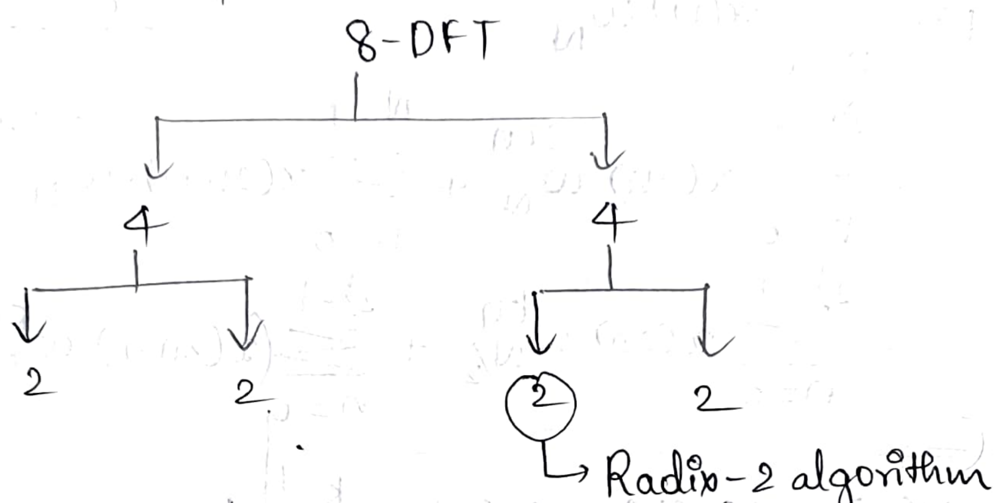
$$= \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

16/02/2022

- **Fast-fourier Transform:**

→ To reduce computational complexity, we use FFT.

→ Radix – base/minimum value



→ It is also called as Radix-algorithms
(or)
Butterfly diagram.

eg: $N=8$

$x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$

$$\left.\begin{array}{l} x_e(n) = \{x(0), x(2), x(4), x(6)\} \\ x_o(n) = \{x(1), x(3), x(5), x(7)\} \end{array}\right] - 4$$

$$\left.\begin{array}{l} x_{ee}(n) = \{x(0), x(4)\} \\ x_{eo}(n) = \{x(2), x(6)\} \\ x_{oe}(n) = \{x(1), x(5)\} \\ x_{oo}(n) = \{x(3), x(7)\} \end{array}\right] - 2$$

→ Let N = no. of samples in $x(n)$

$$x_e(n) = x(2n) \qquad n = 0, 1, 2, \ldots \frac{N}{2} - 1$$

$$x_o(n) = x(2n+1) \qquad n = 0, 1, 2, \ldots \frac{N}{2} - 1$$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, \omega_N^{kn} \qquad \underline{\omega = e^{-j2\pi}}$$

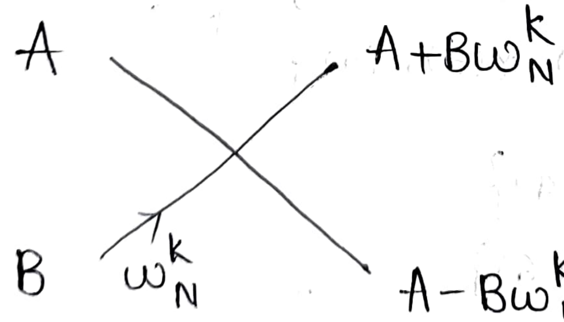
$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n)\, \omega_N^{2kn} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) \cdot \omega_N^{k(2n+1)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n)\, \omega_{N/2}^{kn} + \sum_{n=0}^{\frac{N}{2}-1} \left(x(2n+1)\, \omega_{N/2}^{kn} \right) \cdot \omega_N^{k}$$

$$\boxed{X(k) = X_e(k) + X_o(k) \cdot \omega_N^{k}} \qquad k = 0, 1, 2, \ldots, \frac{N}{2} - 1$$

if $k \geq N/2$ i.e. $k = \frac{N}{2}, \ldots, N-1$

$$\boxed{X(k) = X_e(k - N/2) - \omega_N^{(k-N/2)} \cdot X_o(k - N/2)}$$



DIT – Decimation in Time Domain

DIT – Radix-2 algorithm.

8-point DFT:

① $N = 8 = 2^3 \rightarrow M = 3 \rightarrow$ no. of stages

② Bit Reversal

$000 \rightarrow 000 \quad x(0)$
$001 \rightarrow 100 \quad x(4)$ } 2 DFT
$010 \rightarrow 010 \quad x(2)$
$011 \rightarrow 110 \quad x(6)$ } 2 DFT
$100 \rightarrow 001 \quad x(1)$
$101 \rightarrow 101 \quad x(5)$ } 2 DFT
$110 \rightarrow 011 \quad x(3)$
$111 \rightarrow 111 \quad x(7)$ } 2 DFT

③ $N/2$ Butterflies in each stage.

④ 2^{l-1} samples

⑤ Twiddle factor $\quad K = \dfrac{N \times t}{2^l}$

$l \rightarrow$ stage $\qquad t = 0, 1, \ldots (2^{l-1} - 1)$.

Stage-1: $t = 0 \qquad K = \dfrac{8 \times 0}{2^1} = 0$

$K = 0$

Stage-2: $t = 0, 1 \qquad K = \dfrac{8 \times 0}{2^2} = 0 \qquad K = \dfrac{8 \times 1}{2^2} = 2$

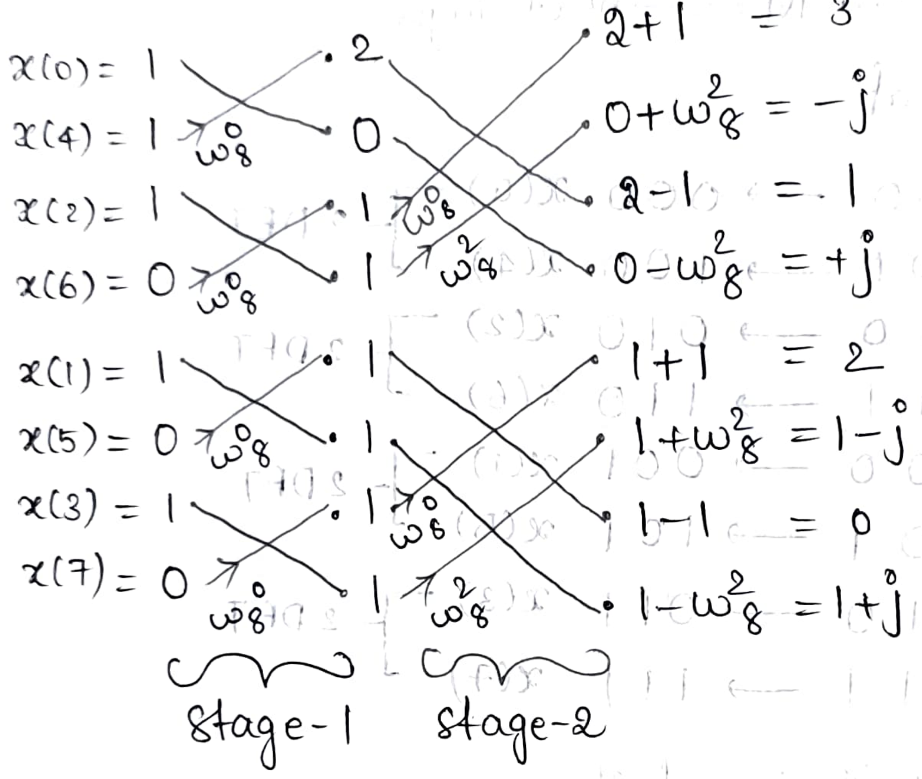
$K = 0, 2$

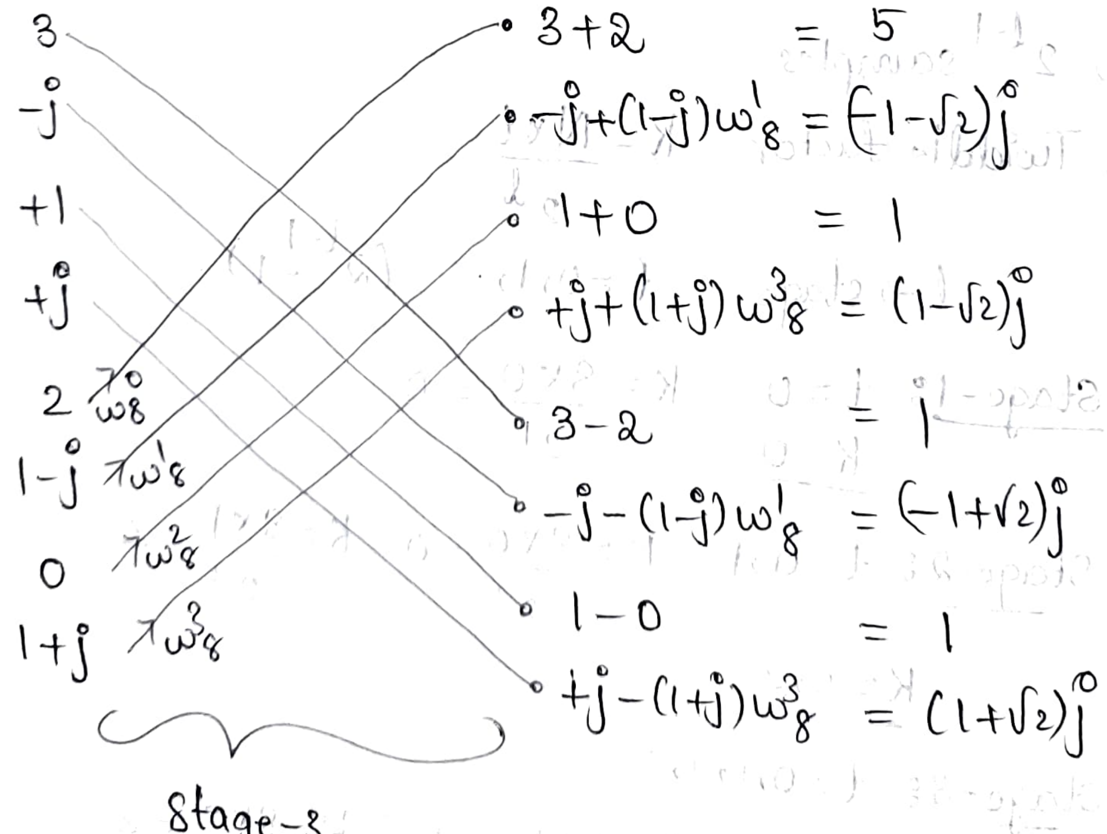
Stage-3: $t = 0, 1, 2, 3$

$K = \dfrac{8 \times 0}{8} = 0 \qquad K = \dfrac{8 \times 1}{2^3} = 1 \qquad K = \dfrac{8 \times 2}{2^3} = 2$

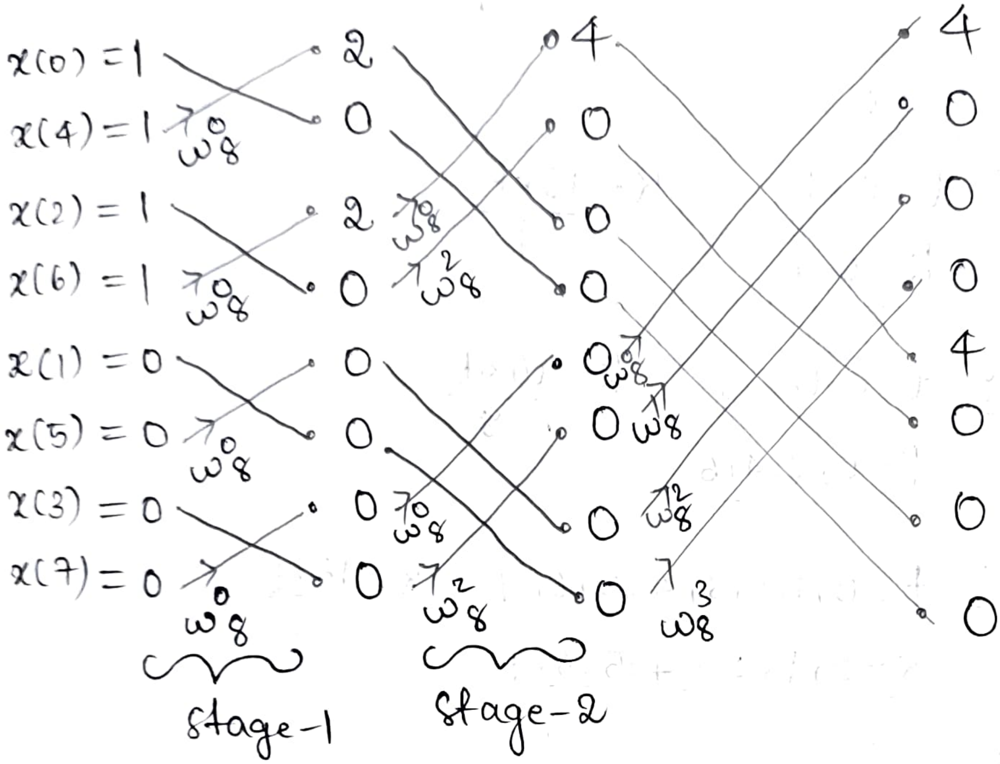
$K = \dfrac{8 \times 3}{2^3} = 3 \qquad K = 0, 1, 2, 3$

eg: $x(n)=\{1,1,1,1,1,0,0,0\}$





eg: $x(n) = \{\underset{0}{1}, \underset{1}{0}, \underset{2}{1}, \underset{3}{0}, \underset{4}{1}, \underset{5}{0}, \underset{6}{1}, \underset{7}{0}\}$



$\frac{N}{2} \times \log_2 N \rightarrow$ multiplications in FFT

$N \log_2 N \rightarrow$ additions in FFT

$N^2 \rightarrow$ multiplications in DFT

$N(N-1) \rightarrow$ additions in DFT

* 16-point DFT:

$K = \frac{Nt}{2m}$ $m \to$ no. of stage

$t = 0, 1, 2, \ldots (2^{m-1} - 1)$

Stage-1: $t = 0$

$k = 0$

Stage-2: $t = 0, 1$ $K = \frac{16 \times t}{4}$

$k = 0, 4$

Stage-3: $t = 0, 1, 2, 3$ $k = \frac{16 \times t}{8}$

$k = 0, 2, 4, 6$

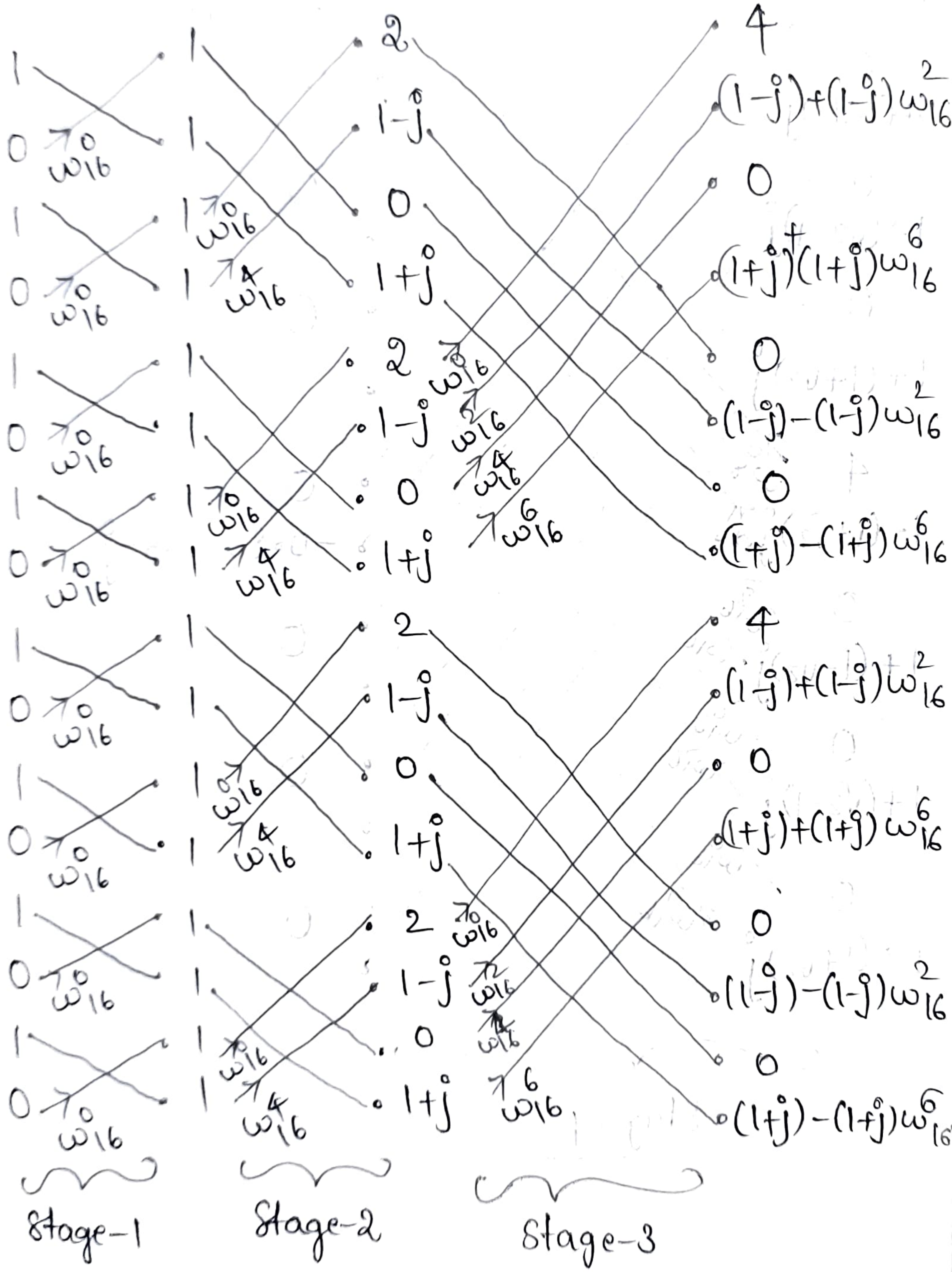
Stage-4: $t = 0, 1, 2, 3, 4, 5, 6, 7$ $K = \frac{16 \times t}{16}$

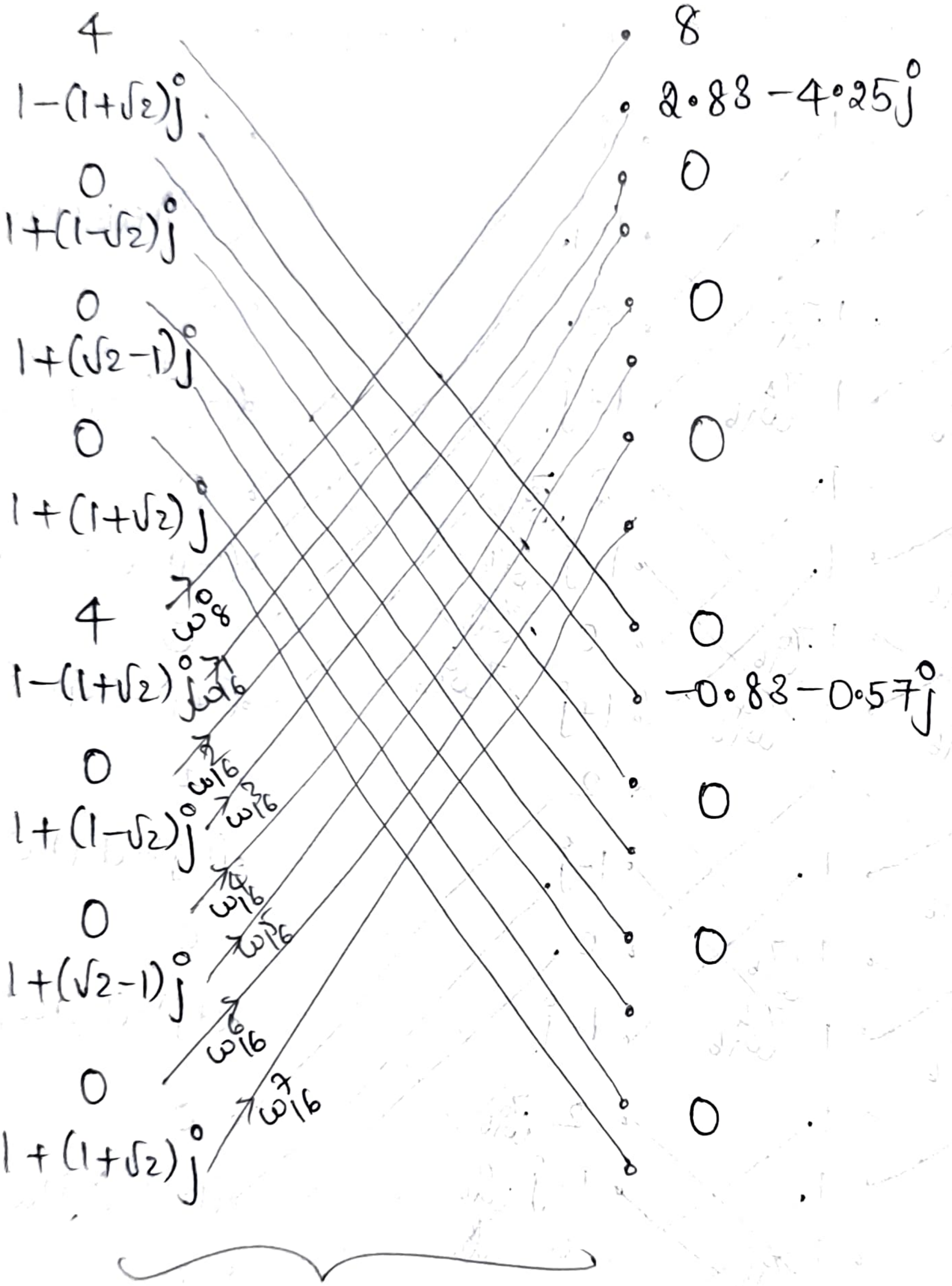
$K = 0, 1, 2, 3, 4, 5, 6, 7$

Bit Reversal

0000 → 0000 → $x(0)$
0001 → 1000 → $x(8)$
0010 → 0100 → $x(4)$
0011 → 1100 → $x(12)$
0100 → 0010 → $x(2)$
0101 → 1010 → $x(10)$
0110 → 0110 → $x(6)$
0111 → 1110 → $x(14)$
1000 → 0001 → $x(1)$
1001 → 1001 → $x(9)$
1010 → 0101 → $x(5)$
1011 → 1101 → $x(13)$
1100 → 0011 → $x(3)$
1101 → 1011 → $x(11)$
1110 → 0111 → $x(7)$
1111 → 1111 → $x(15)$

eg: $x(n) = \{\underset{0}{1}, \underset{1}{1}, \underset{2}{1}, \underset{3}{1}, \underset{4}{1}, \underset{5}{1}, \underset{6}{1}, \underset{7}{1}, \underset{8}{0}, \underset{9}{0}, \underset{10}{0}, \underset{11}{0}, \underset{12}{0}, \underset{13}{0}, \underset{14}{0}, \underset{15}{0}\}$




4
1-(1+√2)j
0
1+(1-√2)j
0
1+(√2-1)j
0
1+(1+√2)j
4
1-(1+√2)j
0
1+(1-√2)j
0
1+(√2-1)j
0
1+(1+√2)j
W8^0
W16^1
W16^2
W16^3
W16^4
W16^5
W16^6
W16^7
8
2.83-4.25j
0
0
0
0
-0.83-0.57j
0
0
0

Stage-4

04/03/2022

• Radix-2 Decimation in frequency algorithm :

$$X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$x_1(n) = x(n) \text{ for } n = 0, 1, 2, \dots \tfrac{N}{2} - 1$$

$$x_2(n) = x\left(n + \tfrac{N}{2}\right) \text{ for } n = 0, 1, 2, \dots \tfrac{N}{2} - 1$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\tfrac{N}{2}\right) \cdot W_N^{k\left(n+\frac{N}{2}\right)}$$

$$= \sum_{n=0}^{N/2-1} x_1(n) \cdot W_N^{kn} + \underbrace{W_N^{kN/2}}_{(-1)^k} \sum_{n=0}^{\frac{N}{2}-1} x_2(n) \cdot W_N^{kn}$$

→ if k is even,

$$X(2r) = \sum_{n=0}^{N/2-1} \left(x_1(n) + x_2(n)\right) \cdot W_N^{2rn}$$

$$\downarrow$$

$$X(2r) = \sum_{n=0}^{N/2-1} \left(x_1(n) + x_2(n)\right) \cdot W_{N/2}^{rn} \quad \text{— (1)}$$

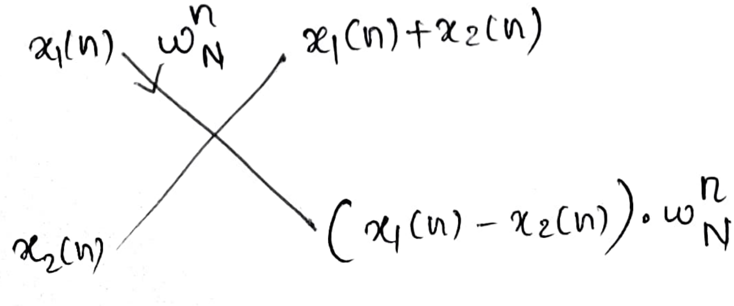
→ if k is odd,

$$X(2r+1) = \sum_{n=0}^{N/2-1} \left[x_1(n) - x_2(n)\right] W_N^{(2r+1)n}$$

$$\downarrow$$

$$X(2r+1) = \sum_{n=0}^{N/2-1} \left(x_1(n) - x_2(n)\right) W_N^{n} \cdot W_{N/2}^{rn} \quad \text{— (2)}$$

(*)



* 8-point DFT using Radix-2 DIF algorithm:

(i) $N = 8 = 2^M \rightarrow M = $ no. of stages.

(ii) $N/2$ Butterflies in each stage.

(iii) Twiddle factor,

$m \rightarrow$ stage no.
$M =$ no. of stages.

$$K = \frac{Nt}{2^{M-m+1}}$$

$$t = 0, 1, \cdots 2^{M-m} - 1$$

(iv) Stage-1: $m = 1 \quad M = 3$

$t = 0, 1, 2, 3 \quad , \quad K = 0, 1, 2, 3$

$$K = \frac{8 \times 0}{2^{3-1+1}} = 0, \quad K = \frac{8 \times 1}{2^{3-1+1}} = 1$$

$$K = \frac{8 \times 2}{2^{3-1+1}} = 2, \quad K = \frac{8 \times 3}{2^{3-1+1}} = 3$$

Stage-2: $m = 2, M = 3$

$t = 0, 1 \longrightarrow K = 0, 2$

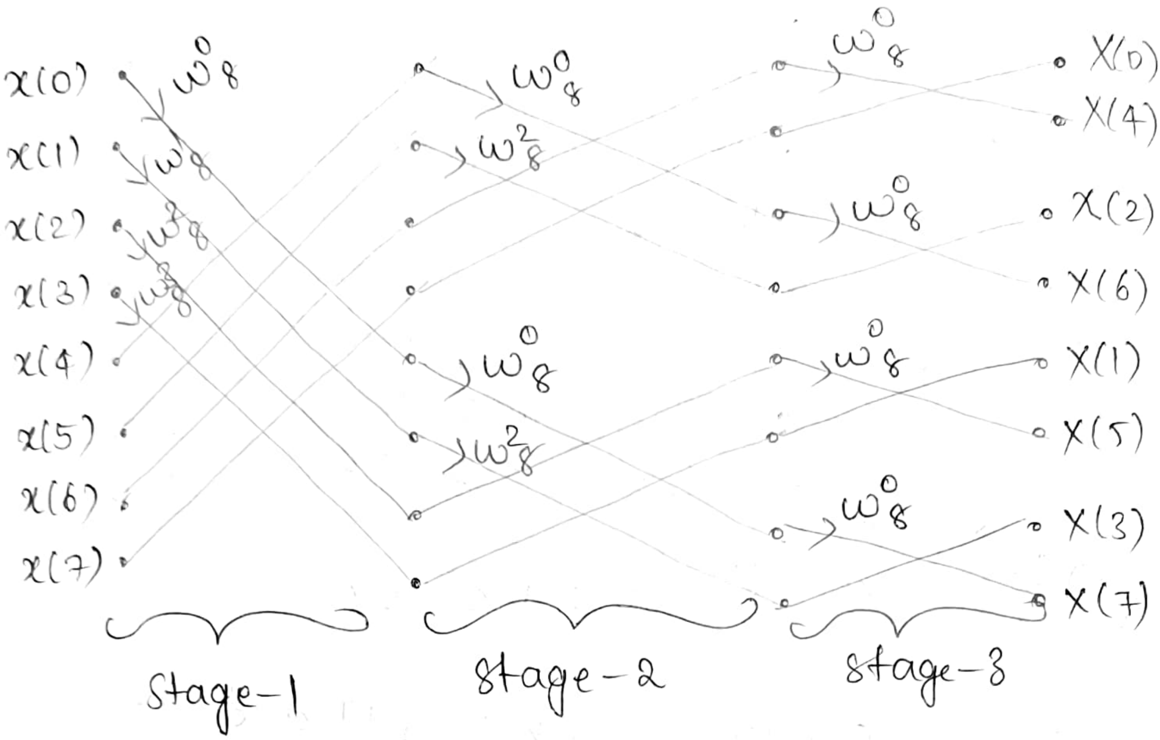
$$K = \frac{8 \times 0}{2^{3-2+1}} = 0 \qquad K = \frac{8 \times 1}{2^{3-2+1}} = 2$$

Stage-3: $m = 3, M = 3$

$t = 0$

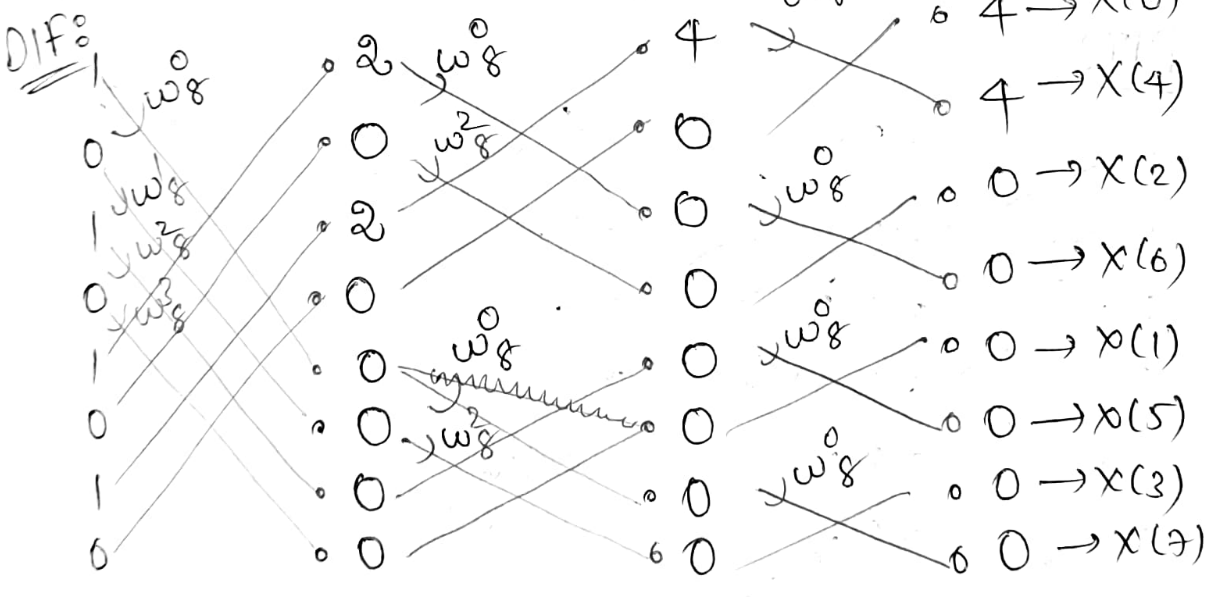
$K = 0$

(v) No Bit Reversal in DIF algorithm at starting
But Bit Reversal in DIF algorithm at ending.

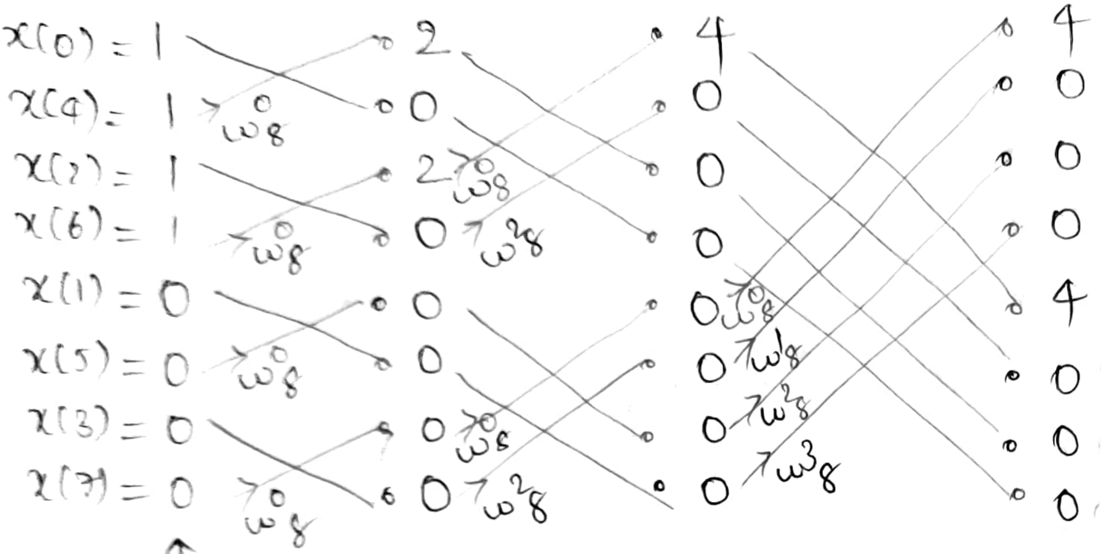


eg: $x(n) = \{1,0,1,0,1,0,1,0\}$

DIF:



$X(k) = \{4,0,0,0,0,4,0,0,0\}$

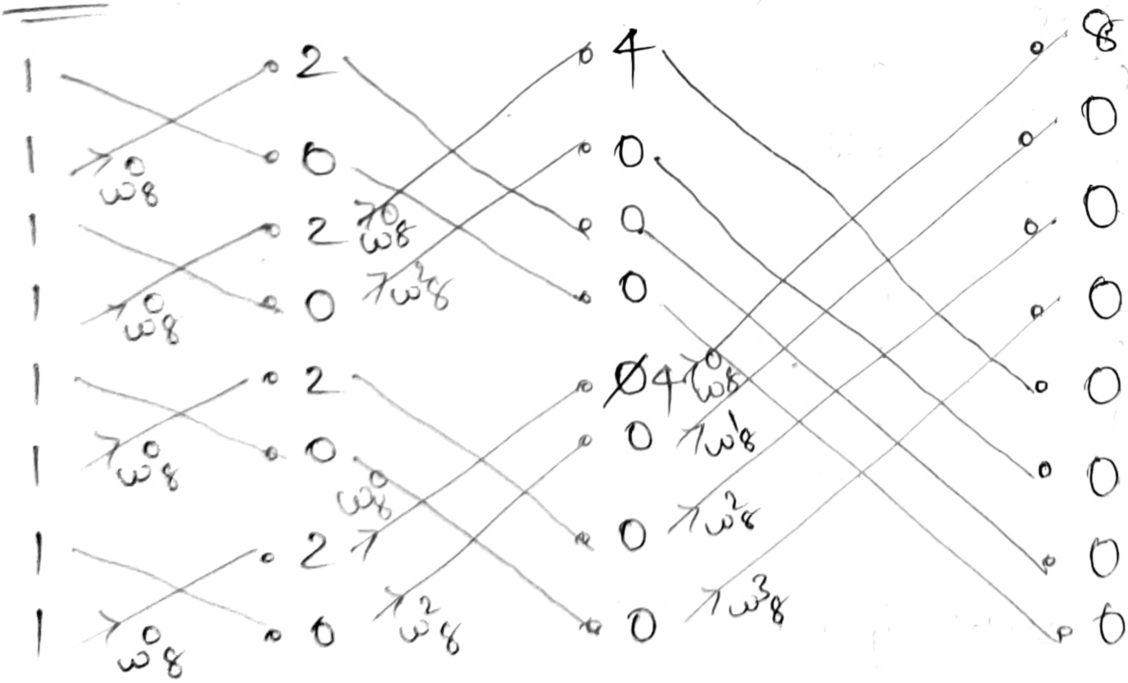


DIT:

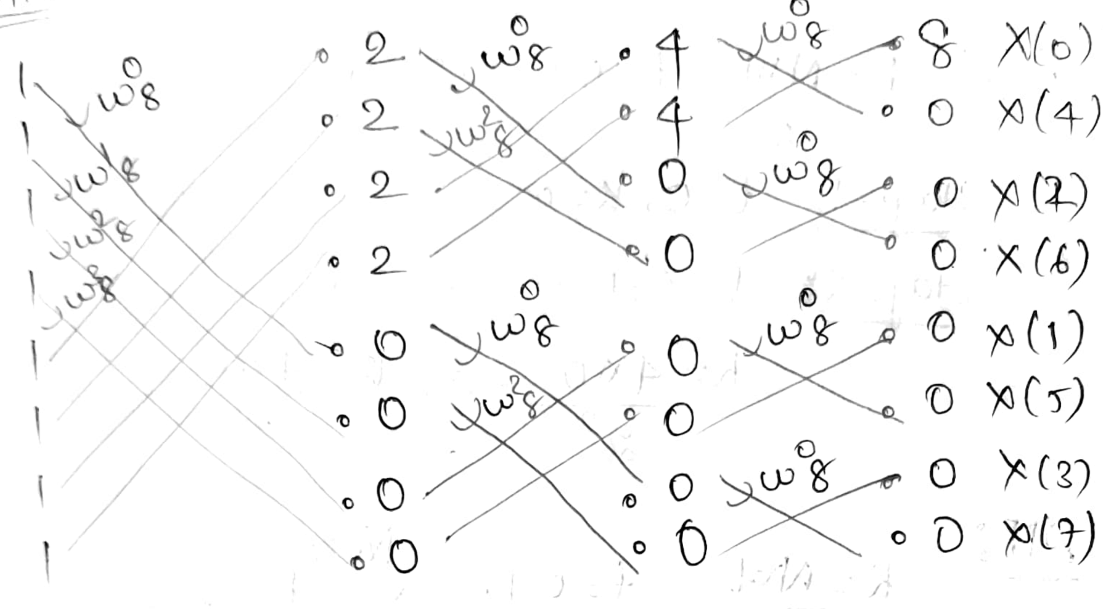
$X(k) = \{4, 0, 0, 0, 4, 0, 0, 0\}$

eg: Calculate 8pt DFT using both DIT and DIF from
$x(n) = 1$ for $n \leq 7$

DIT:



DIF:



eg: $x(n) = \{0, 1, 2, 3\}$

DIT: $k = \frac{N \times t}{2^m}$ $t = 0, 1, \cdots, 2^{m-1} - 1$

Stage-1: $t = 0$, $k = 0$

Stage-2: $t = 0, 1 \rightarrow k = 0, 1$

$k = \frac{4 \times 0}{2^2} = 0$, $k = \frac{4 \times 1}{2^2} = 1$

DIF:

$k = \frac{N \times t}{2^{m+M+1}}$ $t = 0, 1, \ldots 2^{M-m} - 1$ $M = 2$

Stage-1: $t = 0, 1 \rightarrow k = 0, 1$

$k = \frac{4 \times 0}{2^{2-1+1}} = 0$, $k = \frac{4 \times 1}{2^{2-1+1}} = 1$

Stage-2: $t = 0$

$k = 0$

09/03/2022

* IDFT Calculation using FFT:

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot W_N^{-kn}$$

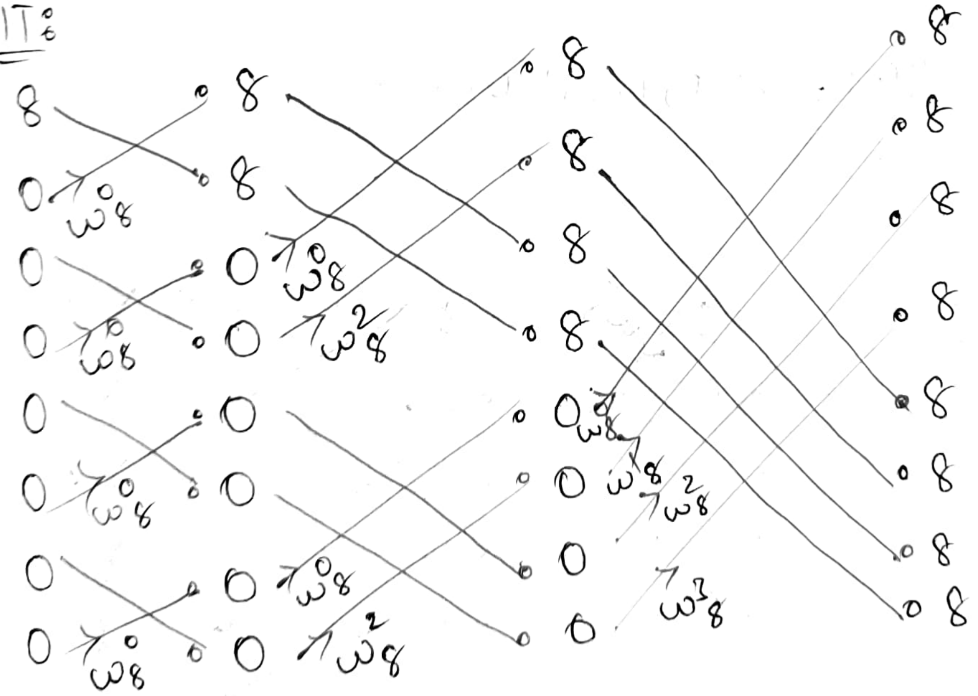
$$= \frac{1}{N} \underbrace{\left[\sum_{k=0}^{N-1} X^*(k) W_N^{kn} \right]^*}_{\text{DFT of } X^*(k)}$$

$$\therefore \boxed{x(n) = \frac{1}{N} \times \text{FFT of } X^*(k)}$$

eg: $X(k) = \{8, 0, 0, 0, 0, 0, 0, 0\}$

$X^*(k) = \{8, 0, 0, 0, 0, 0, 0, 0\}$

DIT:



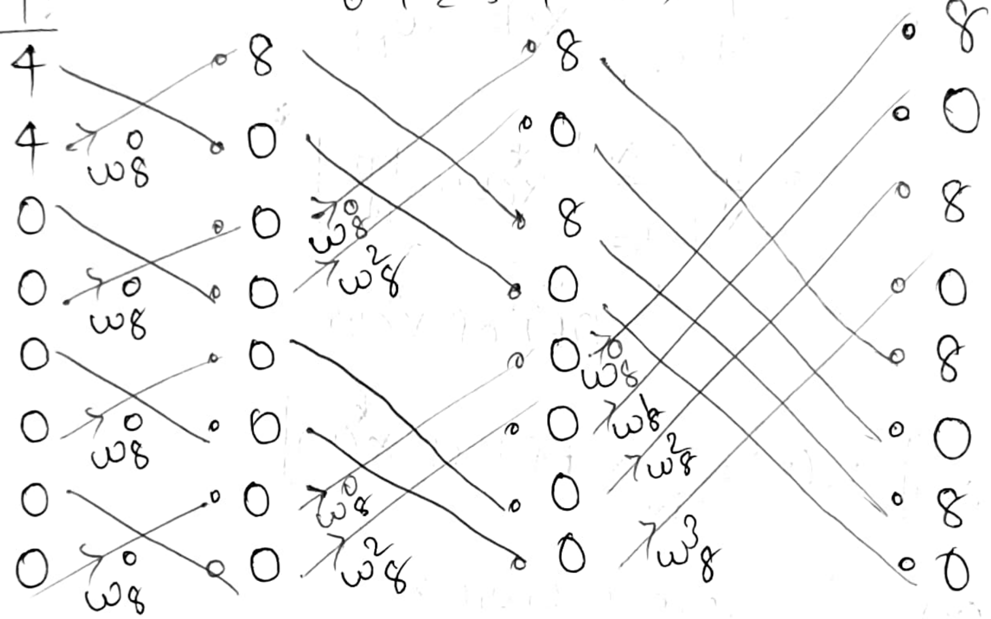
$$x(n) = \frac{1}{8} \times \text{FFT of } X^*(k)$$

$$= \{1, 1, 1, 1, 1, 1, 1, 1\}$$

eg: $X(k) = \{4, 0, 0, 0, 4, 0, 0, 0\}$

$$X^*(k) = \{\underset{0}{4}, \underset{1}{0}, \underset{2}{0}, \underset{3}{0}, \underset{4}{4}, \underset{5}{0}, \underset{6}{0}, \underset{7}{0}\}$$

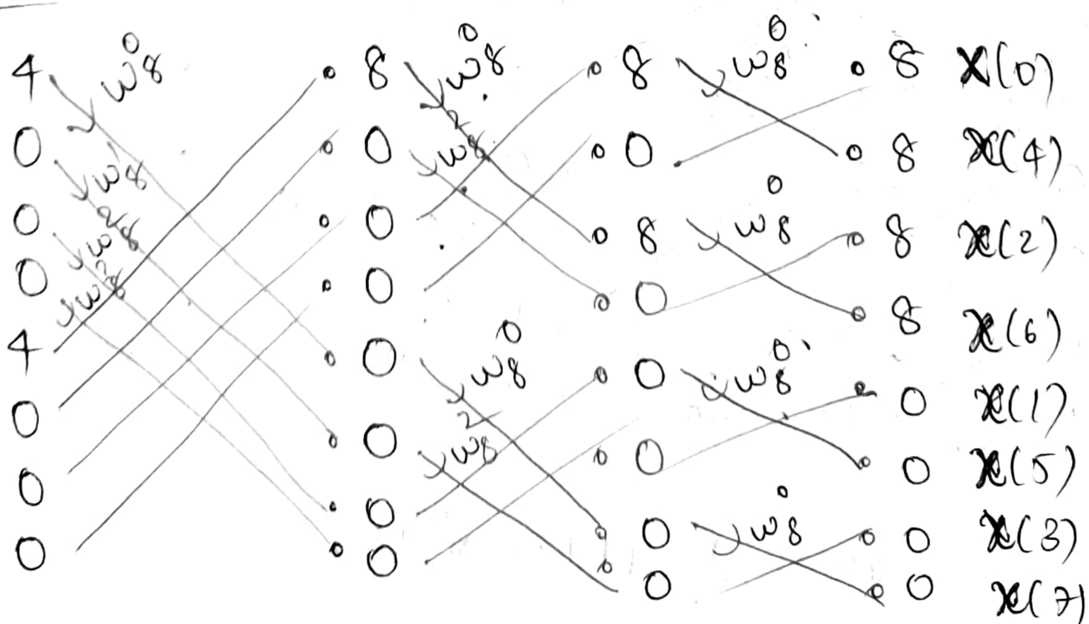
DIT:



$$x(n) = \frac{1}{8} \{8, 0, 8, 0, 8, 0, 8, 0\}$$

$$x(n) = \{1, 0, 1, 0, 1, 0, 1, 0\}$$

DIF:



~~X(k)~~ $x(n) = \frac{1}{8} \{8, 0, 8, 0, 8, 0, 8, 0\}$

$$= \{1, 0, 1, 0, 1, 0, 1, 0\}$$

11/08/2022

⊛ Divide and Conquer method of evaluating DFT:

$$X(k) = \sum_{n=0}^{N-1} x(n)\, \omega_N^{kn}$$

$x(n) \longrightarrow x(l,m)$

$X(k) \longrightarrow X(p,q)$

$l \to$ row index
$m \to$ column index

$p \to$ row index
$q \to$ column index

$N \to 16$

Assume, $N = L \times M$

where, $0 \le l \le L-1$
$0 \le m \le M-1$

$0 \le p \le L-1$
$0 \le q \le M-1$

$n = Ml + m$ (or) $Lm + l$
$k = Mp + q$ (or) $Lq + p$

$$X(p,q) = \sum_{l=0}^{L-1} \sum_{m=0}^{M-1} x(l,m)\, \omega_N^{(Mp+q)(Lm+l)}$$

$$\omega_N^{(Mp+q)(Lm+l)} = \cancel{\omega_N^{LMmp}} \cdot \omega_N^{Lmq} \cdot \omega_N^{Mpl} \cdot \omega_N^{ql}$$

$$= \omega_{N/L}^{mq} \cdot \omega_{N/M}^{pl} \cdot \omega_N^{ql}$$

$$\therefore X(p,q) = \sum_{l=0}^{L-1} \sum_{m=0}^{M-1} x(l,m) \cdot \omega_{N/L}^{mq} \cdot \omega_{N/M}^{pl} \cdot \omega_N^{ql}$$

$$\boxed{X(p,q) = \sum_{l=0}^{L-1} \omega_N^{lq} \left[\sum_{m=0}^{M-1} x(l,m) \cdot \omega_M^{mq} \right] \cdot \omega_L^{lp}}$$

Assume,

① $f = M\text{-pt DFT} \rightarrow \left[\sum_{m=0}^{M-1} x(l,m) \cdot \omega_M^{mq}\right]$

② $f \times \omega_N^{lq} = G$

③ L-pt DFT for G

eg: $x(n) = \{1,0,1,0,1,0,1,0\}$

$N = 8 = L \times M$, $L \downarrow 2$, $M \downarrow 4$, i.e. 2×4

$n = Ml + m$

$K = Mp + q$

$\underline{x(l,m)}$ $\quad 0 \leq l \leq 1$

$0 \leq m \leq 3$

$$X(p,q) = \sum_{l=0}^{1} \omega_N^{lq} \left[\sum_{m=0}^{3} x(l,m) \cdot \omega_M^{mq}\right] \cdot \omega_L^{lp}$$

$x(0,0) \rightarrow x(0)$

$x(0,1) \rightarrow x(1)$

$x(0,2) \rightarrow x(2)$

$x(0,3) \rightarrow x(3)$

$x(1,0) \rightarrow x(4)$

$x(1,1) \rightarrow x(5)$

$x(1,2) \rightarrow x(6)$

$x(1,3) \rightarrow x(7)$

Filters:

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$$y(n) = x(n) * h(n)$$
$$= T[x(n)]$$

(No past outputs
Non-recursive filters)

if $h(n)$ is finite then it is finite impulse Response (FIR)

if $h(n)$ is infinite, then it is Infinite Impulse Response (IIR)

(Present, past input
and
present, past output)

→ Recursive filters.

- **Realization of digital filters:**

(delay, multipliers and adders)

1. Direct form – I
2. Direct form – II
3. Cascade
4. Parallel
5. Lattice Structure

1 Direct form – I:

$$y(n) + a_1 y(n-1) + \cdots\cdots = b_0 x_n + b_1 x(n-1) + \cdots$$

N no. of i/p's

$$y(n) + \sum_{k=1}^{M} a_k y(n-k) = \sum_{k=0}^{N} b_k x(n-k)$$

$$\boxed{y(n) = \sum_{k=0}^{N} b_k x(n-k) - \sum_{k=1}^{M} a_k y(n-k)}$$

⇓

IIR filters.

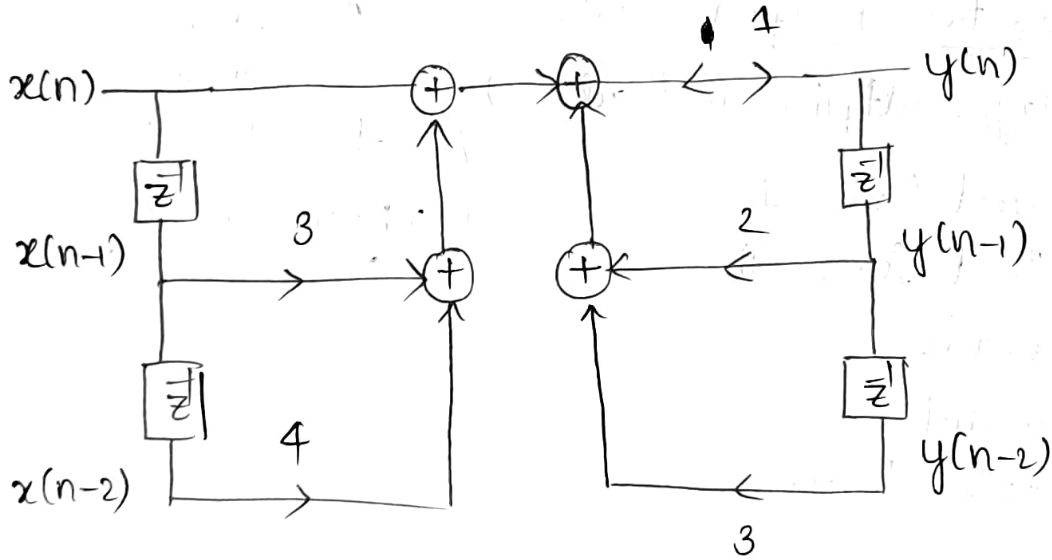
(i) One delay unit $\boxed{z^{-1}}$

(ii) Multipliers $x(n) \xrightarrow{a} a \cdot x(n)$

(iii) Adders $\oplus$

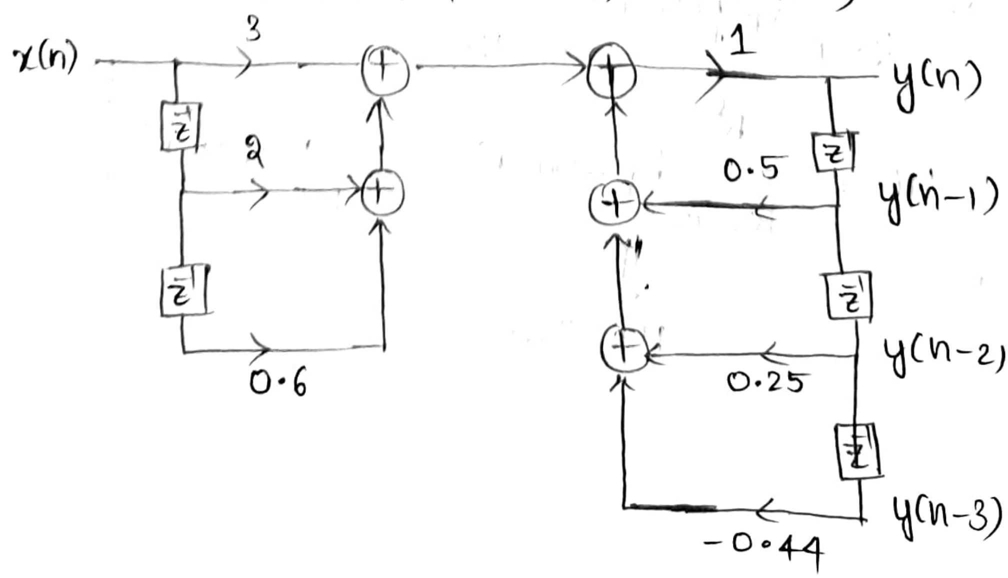
eg: $y(n) = 2y(n-1) + 3y(n-2) + \underbrace{x(n) + 3x(n-1) + 4x(n-2)}_{\text{Assume } w(n) \text{ as input}}$

$w(n) = x(n) + 3x(n-1) + 4x(n-2)$



eg: $y(n) = 0.5y(n-1) + 0.25y(n-2) - 0.44y(n-3)$
$\quad + \underbrace{3x(n) + 2x(n-1) + 0.6x(n-2)}_{w(n)}$

$w(n) = 3x(n) + 2x(n-1) + 0.6x(n-2)$



2. Direct form – II :

$$y(n) = \sum_{k=0}^{N} b_k\, x(n-k) - \sum_{k=1}^{M} a_k\, y(n-k)$$

$$Y(z) = \sum_{k=0}^{N} b_k X(z) z^{-k} - \sum_{k=1}^{M} a_k Y(z) \cdot z^{-k}$$

$$Y(z)\left[1 + \sum_{k=1}^{M} a_k z^{-k}\right] = \sum_{k=0}^{N} b_k X(z) \cdot z^{-k}$$

$$\therefore \boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b_k \cdot z^{-k}}{1 + \sum_{k=1}^{M} a_k z^{-k}}}$$

$$\rightarrow \frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \times \frac{W(z)}{X(z)}$$

$$\frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)} = \sum_{k=0}^{N} b_k \cdot z^{-k} \cdot \frac{1}{1 + \sum_{k=0}^{M} a_k z^{-k}}$$

(i) $$\frac{Y(z)}{W(z)} = \sum_{k=0}^{N} b_k \cdot z^{-k}$$

$$y(n) = b_0 w(n) + b_1 w(n-1) + \cdots\cdots$$

(ii) $$\frac{W(z)}{X(z)} = \frac{1}{1 + \sum_{k=1}^{M} a_k \cdot z^{-k}}$$

$$x(n) = w(n) + a_1 w(n-1) + \cdots\cdots$$

eg: $y(n) = 2y(n-1) + 3y(n-2) + x(n) + 3x(n-1) + 4x(n-2)$

Applying Z-transform,

$$Y(z)\left[1 - 2z^{-1} - 3z^{-2}\right] = X(z)\left[1 + 3z^{-1} + 4z^{-2}\right]$$

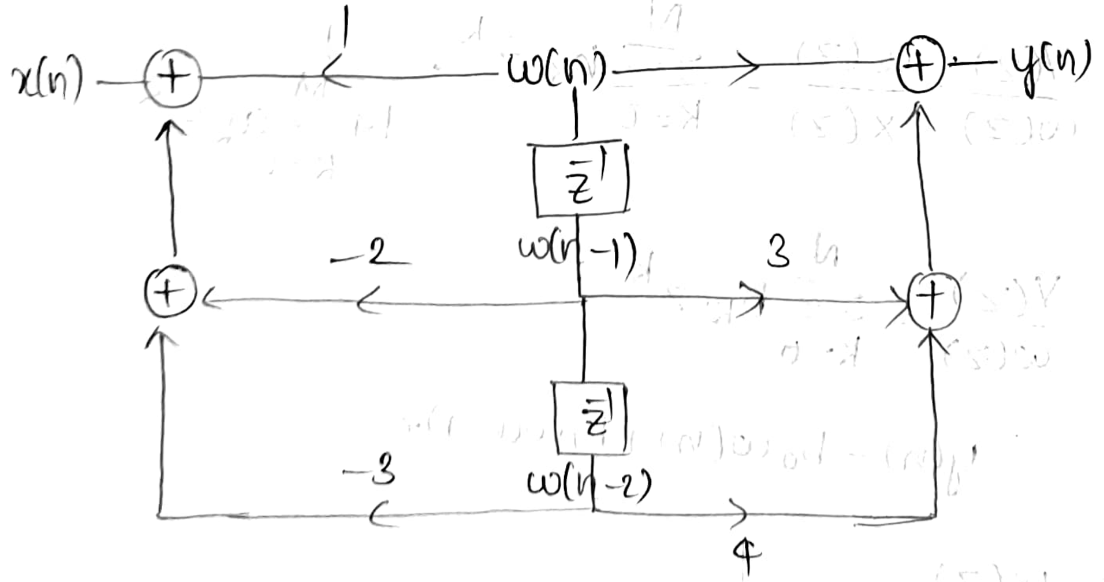
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 3z^{-1} + 4z^{-2}}{1 - 2z^{-1} - 3z^{-2}}$$

(i) $\frac{Y(z)}{W(z)} = 1 + 3z^{-1} + 4z^{-2}$ (IZT) → Inverse Z-transform.

$$y(n) = w(n) + 3w(n-1) + 4w(n-2) \quad \text{①}$$

(ii) $\frac{W(z)}{X(z)} = \frac{1}{1 - 2z^{-1} - 3z^{-2}}$ (IZT)

$$x(n) = w(n) - 2w(n-1) - 3w(n-2) \quad \text{②}$$



eg: $y(n) = -0.1\,y(n-1) + 0.2\,y(n-2) + x(n) + 0.33\,x(n-1) + 0.44\,x(n-2)$

$$Y(z)\left[1+0.1z^{-1}-0.2z^{-2}\right] = X(z)\left[1+0.33z^{-1}+0.44z^{-2}\right]$$

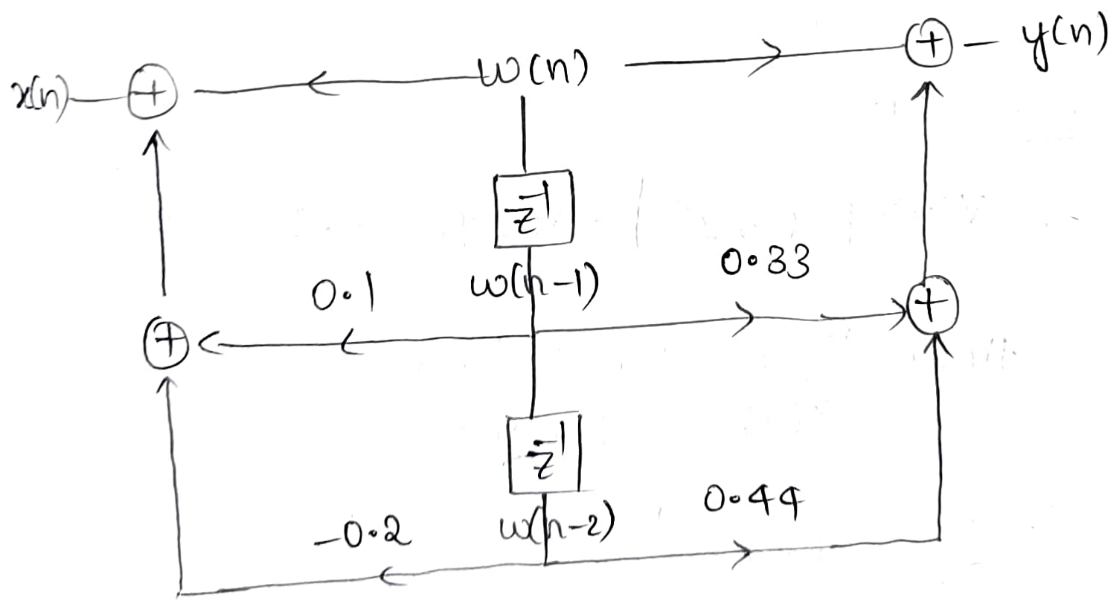
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+0.33z^{-1}+0.44z^{-2}}{1+0.1z^{-1}-0.2z^{-2}}$$

(i) $\dfrac{Y(z)}{W(z)} = 1+0.33z^{-1}+0.44z^{-2}$

$$y(n) = w(n) + 0.33\,w(n-1) + 0.44\,w(n-2)$$

(ii) $\dfrac{W(z)}{X(z)} = \dfrac{1}{1+0.1z^{-1}-0.2z^{-2}}$

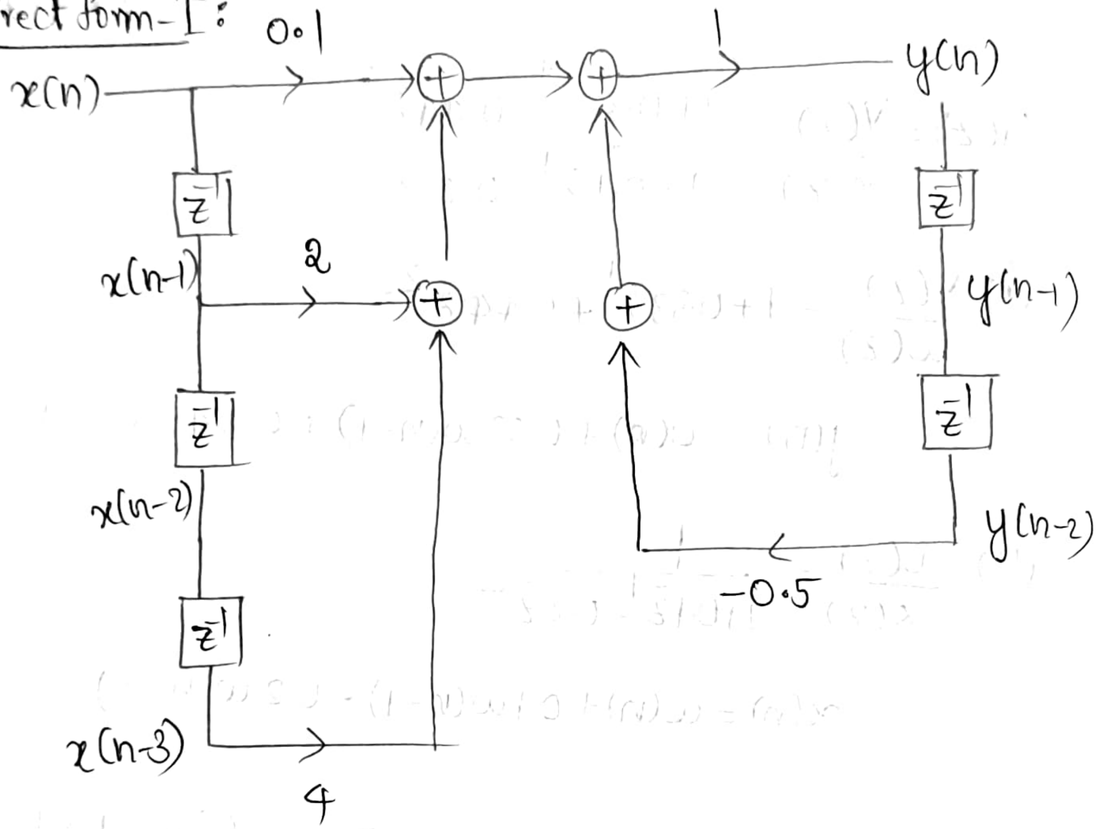
$$x(n) = w(n) + 0.1\,w(n-1) - 0.2\,w(n-2)$$



eg: $y(n) - 0.5y(n-2) = 0.1\,x(n) + 2\,x(n-1) + 4x(n-3)$

$$w(n) = 0.1x(n) + 2x(n-1) + 4x(n-3)$$

Direct form-I:



Direct form-II:

$$Y(z)\left[1 - 0.5z^{-2}\right] = X(z)\left[0.1 + 2z^{-1} + 4z^{-3}\right]$$

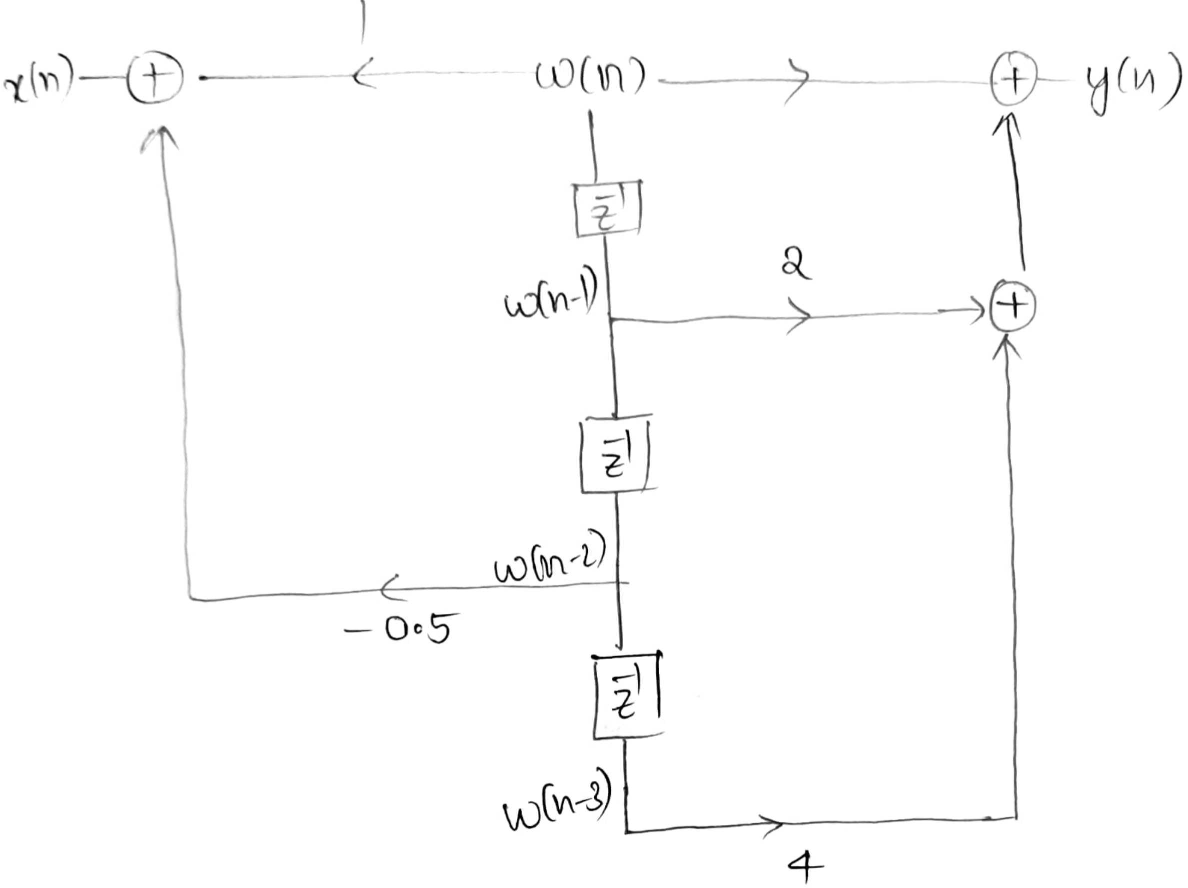
$$H(z) = \frac{Y(z)}{X(z)} = \frac{0.1 + 2z^{-1} + 4z^{-3}}{1 - 0.5z^{-2}}$$

(i) $\dfrac{Y(z)}{W(z)} = 0.1 + 2z^{-1} + 4z^{-3}$

$$y(n) = w(n) + 2w(n-1) + 4w(n-3)$$

(ii) $\dfrac{W(z)}{X(z)} = \dfrac{1}{1 - 0.5z^{-2}}$

$$x(n) = w(n) - 0.5w(n-2)$$

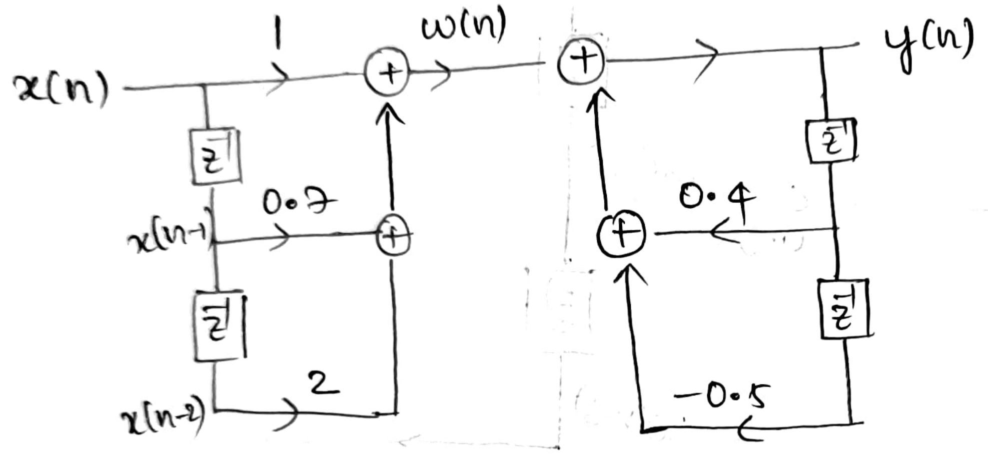

x(n)
w(n)
y(n)
z^{-1}
w(n-1)
2
z^{-1}
w(n-2)
-0.5
z^{-1}
w(n-3)
4

eg: $y(n) - 0.4y(n-1) + 0.5y(n-2) = x(n) + 0.7x(n-1) + 2x(n-2)$

Direct-form-I:

Sol: $y(n) = x(n) + 0.7x(n-1) + 2x(n-2) + 0.4y(n-1) - 0.5y(n-2)$

$$w(n) = x(n) + 0.7x(n-1) + 2x(n-2)$$



Direct-form-II:

Apply Z-transform,

$$Y(z)\left[1 - 0.4z^{-1} + 0.5z^{-2}\right] = X(z)\left[1 + 0.7z^{-1} + 2z^{-2}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 0.7z^{-1} + 2z^{-2}}{1 - 0.4z^{-1} + 0.5z^{-2}}$$

(i) $\dfrac{Y(z)}{W(z)} = 1 + 0.7z^{-1} + 2z^{-2}$

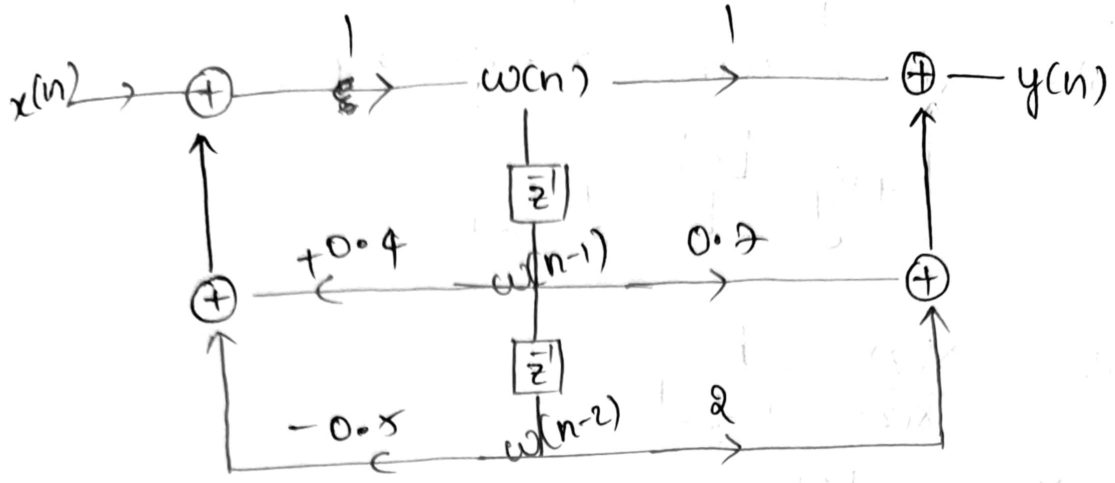
$\downarrow$ (IZT)

$$y(n) = w(n) + 0.7w(n-1) + 2w(n-2)$$

(ii) $\dfrac{W(z)}{X(z)} = \dfrac{1}{1 - 0.4z^{-1} + 0.5z^{-2}}$ $\xrightarrow{}$ (IZT)

$$x(n) = w(n) - 0.4w(n-1) + 0.5w(n-2)$$

$$w(n) = x(n) + 0.4w(n-1) - 0.5w(n-2)$$



③ Cascade form:

$$H(z) = H_1(z) \times H_2(z) \times H_3(z) \cdots\cdots$$

④ Parallel form:

$$H(z) = C + H_1(z) + H_2(z) + \cdots\cdots$$

eg: Realise the following transfer function;

$$H(z) = \frac{1+z^{-1}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$$

Sol:

$$H(z) = \frac{1+z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}$$

$$H(z) = \underbrace{\frac{1+z^{-1}}{1-\frac{1}{4}z^{-1}}}_{H_1(z)} \times \underbrace{\frac{1}{1-\frac{1}{2}z^{-1}}}_{H_2(z)}$$

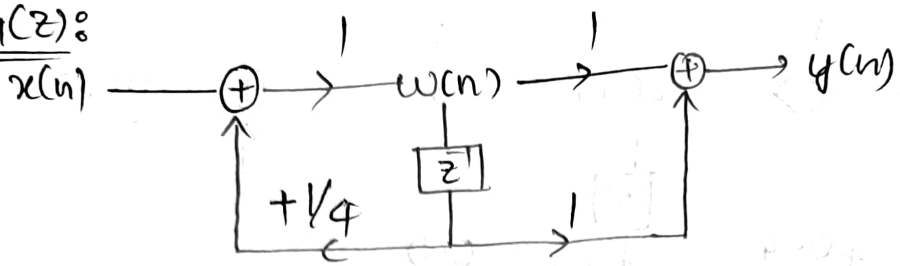
$$H_1(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1-\frac{1}{4}z^{-1}}$$

(i) $\frac{Y(z)}{W(z)} = 1+z^{-1} \xrightarrow{IZT} y(n) = w(n) + w(n-1)$

(ii) $\frac{W(z)}{X(z)} = \frac{1}{1-\frac{1}{4}z^{-1}} \xrightarrow{IZT} x(n) = w(n) - 0.25\,w(n-1)$

$$w(n) = x(n) + 0.25\,w(n-1)$$

$H_1(z)$:



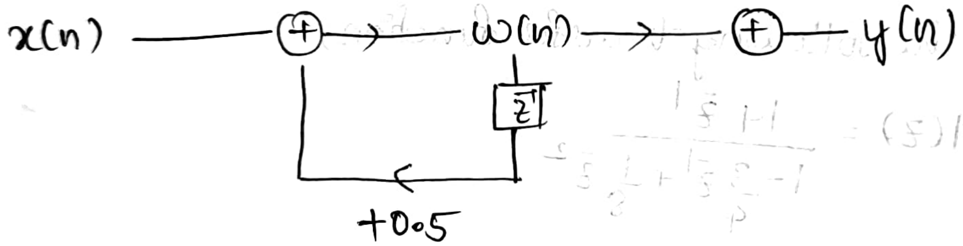
$$H_2(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-\frac{1}{2}z^{-1}}$$

(i) $\frac{Y(z)}{W(z)} = 1 \xrightarrow{IZT} y(n) = w(n)$

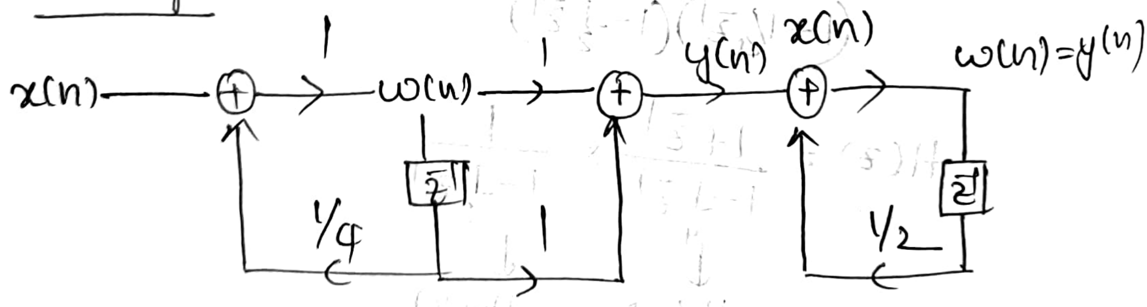
(ii) $\frac{W(z)}{X(z)} = \frac{1}{1-\frac{1}{2}z^{-1}} \xrightarrow{IZT} x(n) = w(n) - 0.5\,w(n-1)$

$$w(n) = x(n) + 0.5\,w(n-1)$$

$H_2(z)$:



Cascading:



eg: Realise the following function.

$$H(z) = \frac{1-z^{-1}}{(1+z^{-1})^8}$$

Sol:

$$H(z) = \frac{1-z^{-1}}{(1+z^{-1})^3}$$

$$H(z) = \underbrace{\frac{1-z^{-1}}{1+z^{-1}}}_{H_1(z)} \times \underbrace{\frac{1}{(1+z^{-1})^2}}_{H_2(z)}$$

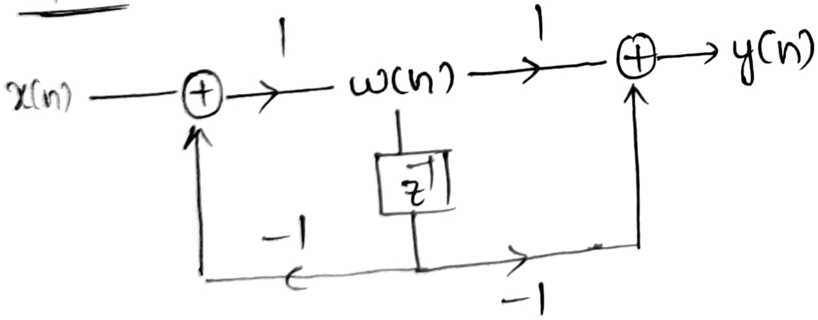
$$H_1(z) = \frac{Y(z)}{X(z)} = \frac{1-z^{-1}}{1+z^{-1}}$$

(i) $\frac{Y(z)}{W(z)} = 1 - z^{-1} \xrightarrow{IZT} y(n) = w(n) - w(n-1)$

(ii) $\frac{W(z)}{X(z)} = \frac{1}{1+z^{-1}} \xrightarrow{IZT} x(n) = w(n) + w(n-1)$

$w(n) = x(n) - w(n-1)$

$H_1(z)$:



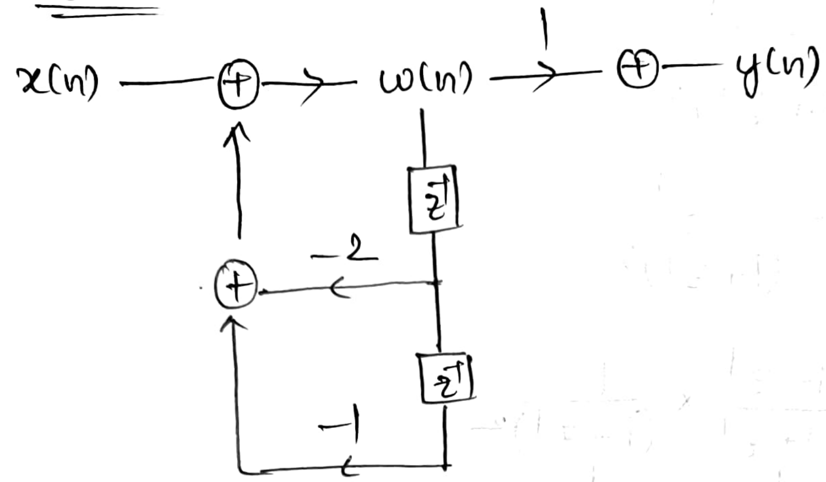
$$H_2(z) = \frac{1}{(1+z^{-1})^2} = \frac{1}{1+z^{-2}+2z^{-1}}$$

(i) $\frac{Y(z)}{W(z)} = 1 \xrightarrow{IZT} y(n) = w(n)$

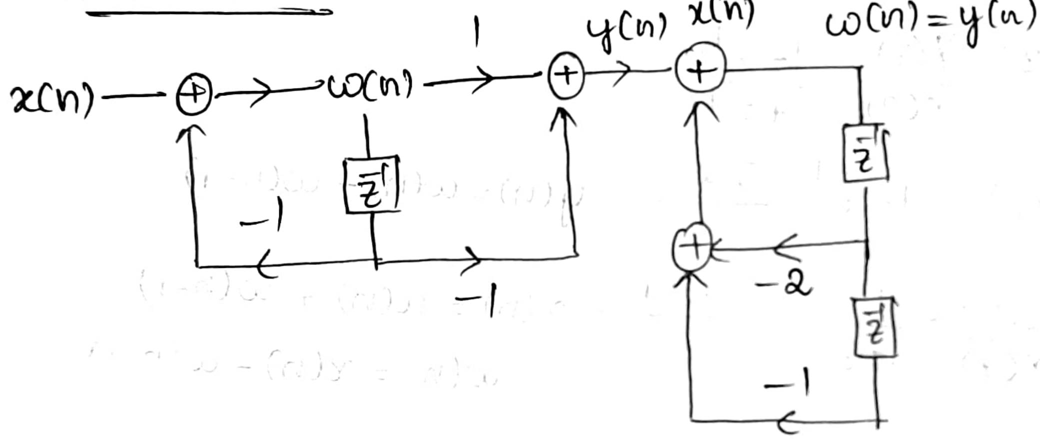
(ii) $\frac{W(z)}{X(z)} = \frac{1}{1+z^{-2}+2z^{-1}} \xrightarrow{IZT} x(n) = w(n) + 2w(n-1) + w(n-2)$

$w(n) = x(n) - 2w(n-1) - w(n-2)$

$H_2(z)$:



Cascaded form:



④ Parallel form:

$$H(z) = C + \sum_{k=1}^{N} H_k(z)$$

where, $H_k(z) = \dfrac{C_k}{1 - P_k z^{-1}}$

eg: $y(n) = -0.1y(n-1) + 0.2y(n-2) + 3x(n) + 3.6x(n-1) + 0.6x(n-2)$

Sol:
$$\frac{Y(z)}{X(z)} = H(z) = \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}}$$

$$-0.2z^{-2} + 0.1z^{-1} + 1 \,)\, 0.6z^{-2} + 3.6z^{-1} + 3 \,(\, -3$$
$$\underset{(-)}{0.6z^{-2}} \; \underset{(+)}{-0.3z^{-1}} \; \underset{(+)}{-3}$$
$$\overline{\quad 3.9z^{-1} + 6 \quad}$$

$$H(z) = -3 + \frac{3.9z^{-1} + 6}{1 + 0.1z^{-1} - 0.2z^{-2}}$$

$$H(z) = -3 + \frac{3.9z^{-1} + 6}{\underbrace{(1 - 0.4z^{-1})(1 + 0.5z^{-1})}}$$

↓ finding partial fractions

$$\frac{A}{1 - 0.4z^{-1}} + \frac{B}{1 + 0.5z^{-1}} = \frac{3.9z^{-1} + 6}{1 + 0.1z^{-1} - 0.2z^{-2}}$$

$$(A+B) + \frac{1}{2}z^{-1}A - 0.4z^{-1}B = 3.9z^{-1} + 6$$

→ $(A+B) = 6$

→ $2 \times (0.5A - 0.4B) = 3.9 \times 2$
$\quad (-) \qquad (+) \qquad (-)$

$1.8B = -1.8 \rightarrow \boxed{B = -1}$

$\boxed{A = 7}$

$$H(z) = -3 + \underbrace{\frac{7}{1-0.4z^{-1}}}_{H_1(z)} + \underbrace{\frac{-1}{1+0.5z^{-1}}}_{H_2(z)}$$

$H_1(z)$:

$$\frac{Y(z)}{X(z)} = \frac{7}{1-0.4z^{-1}}$$

(i) $\frac{Y(z)}{W(z)} = 7 \xrightarrow{IZT} y(n) = 7w(n)$

(ii) $\frac{W(z)}{X(z)} = \frac{1}{1-0.4z^{-1}} \xrightarrow{IZT} x(n) = w(n) - 0.4w(n-1)$

$w(n) = x(n) + 0.4w(n-1)$

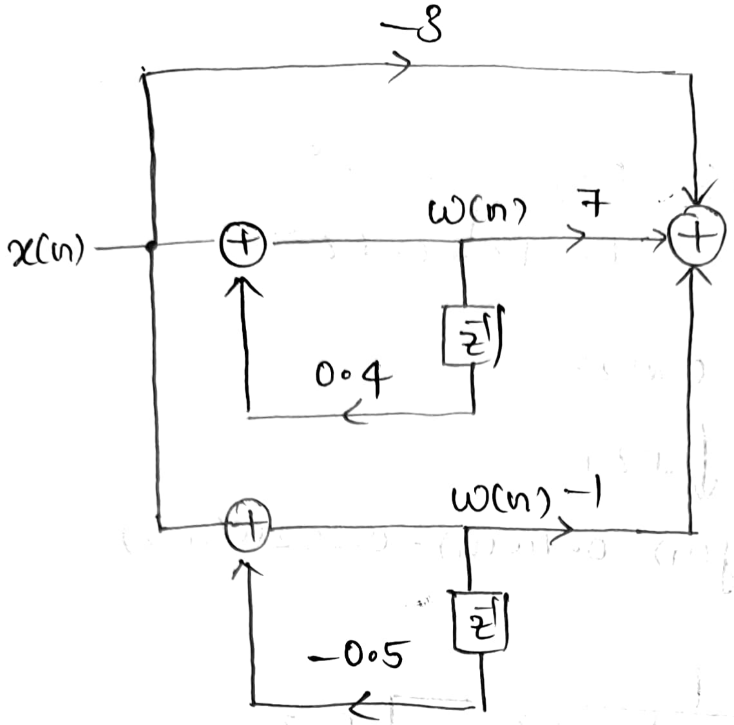
$H_2(z)$:

$$\frac{Y(z)}{X(z)} = \frac{-1}{1+0.5z^{-1}}$$

(i) $\frac{Y(z)}{W(z)} = -1 \xrightarrow{IZT} y(n) = -w(n)$

(ii) $\frac{W(z)}{X(z)} = \frac{1}{1+0.5z^{-1}} \xrightarrow{IZT} x(n) = w(n) + 0.5w(n-1)$

$w(n) = x(n) - 0.5w(n-1)$

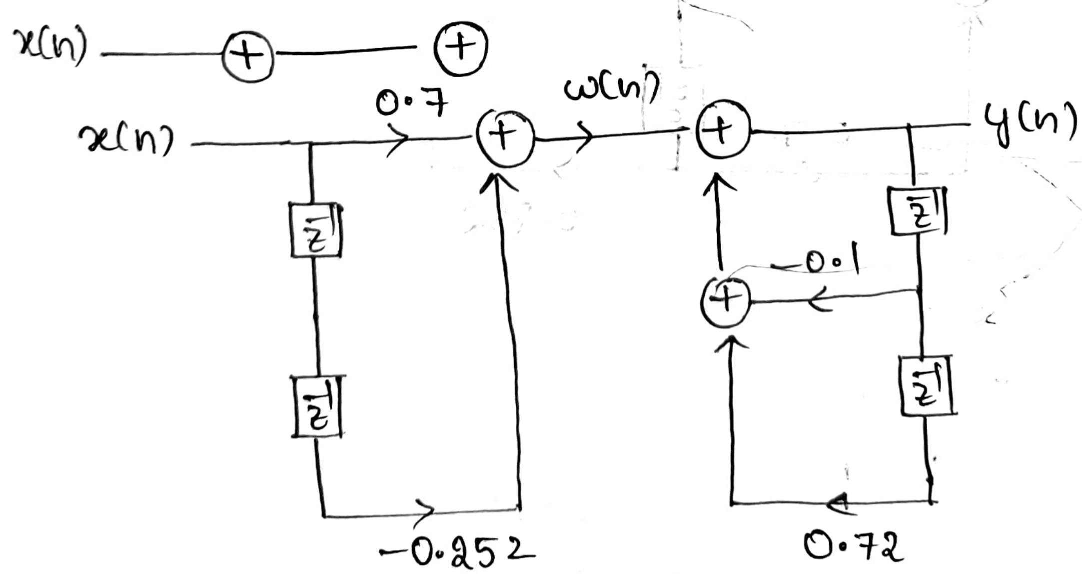


eg: Realise the following filter using Direct form - I & II, cascade and parallel form.

$$y(n) = -0.1\,y(n-1) + 0.72\,y(n-2) + 0.7\,x(n) + -0.252\,x(n-2)$$

Sol: Direct form - I:

$$w(n) = 0.7\,x(n) - 0.252\,x(n-2)$$



Direct form-II:

$$\frac{Y(z)}{X(z)} = H(z) = \frac{0.7 - 0.252z^{-2}}{1+0.1z^{-1} - 0.7z^{-2}}$$

(i) $$\frac{Y(z)}{W(z)} = 0.7 - 0.252z^{-2}$$

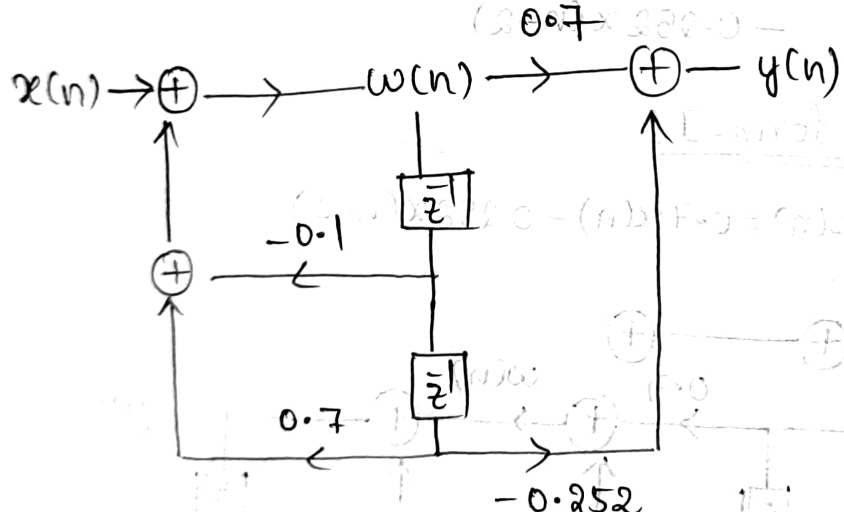
$\downarrow$ IZT

$$y(n) = 0.7w(n) - 0.252w(n-2)$$

(ii) $$\frac{W(z)}{X(z)} = \frac{1}{1+0.1z^{-1} - 0.7z^{-2}}$$ $\downarrow$ IZT

$$x(n) = w(n) + 0.1w(n-1) - 0.7w(n-2)$$

$$w(n) = x(n) - 0.1w(n-1) + 0.7w(n-2)$$



Cascade form:

$$H(z) = \frac{0.7 - 0.252 z^{-2}}{1 + 0.1 z^{-1} - 0.72 z^{-2}}$$

$$H(z) = \frac{0.7 - 0.252 z^{-2}}{(1 - 0.8 z^{-1})(1 + 0.9 z^{-1})}$$

$$H(z) = \underbrace{\frac{0.7 - 0.252 z^{-2}}{1 - 0.8 z^{-1}}}_{H_1(z)} \times \underbrace{\frac{1}{1 + 0.9 z^{-1}}}_{H_2(z)}$$

$H_1(z)$:

$$\frac{Y(z)}{X(z)} = \frac{0.7 - 0.252 z^{-2}}{1 - 0.8 z^{-1}}$$

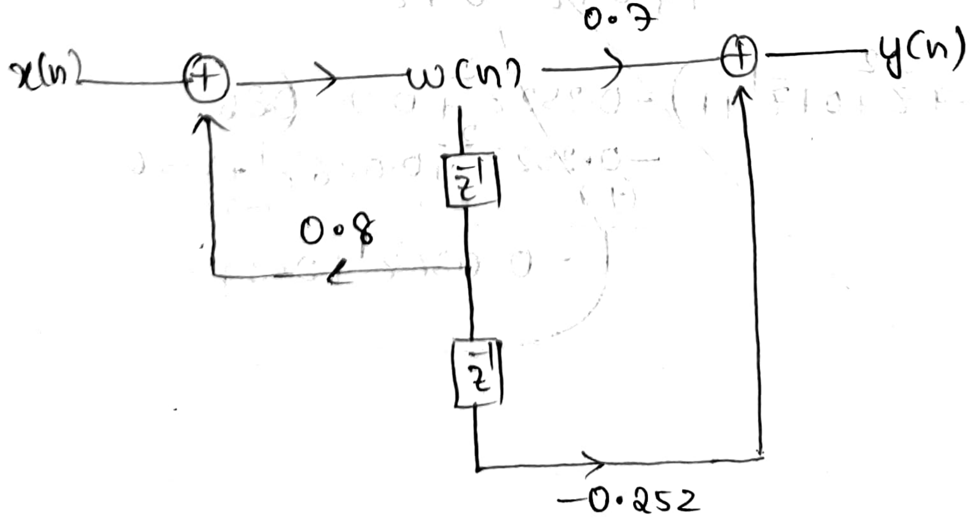
(i) $\frac{Y(z)}{W(z)} = 0.7 - 0.252 z^{-2}$

$\downarrow$ IZT

$y(n) = 0.7 w(n) - 0.252 w(n-2)$

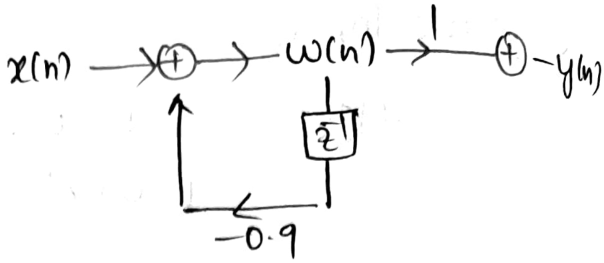
(ii) $\frac{W(z)}{X(z)} = \frac{1}{1 - 0.8 z^{-1}} \xrightarrow{IZT} x(n) = w(n) - 0.8 w(n-1)$

$w(n) = x(n) + 0.8 w(n-1)$



$H_2(z)$:

$$\frac{Y(z)}{X(z)} = \frac{1}{1+0.9z^{-1}}$$

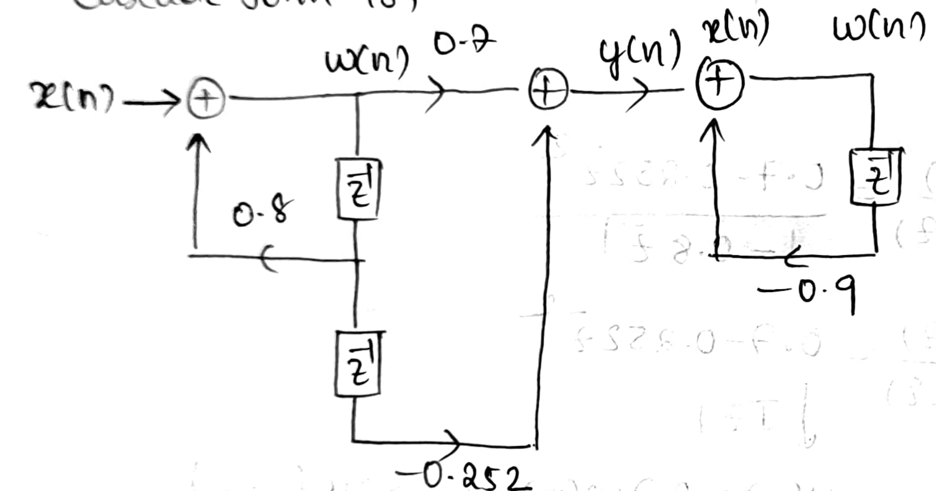


(i) $\frac{Y(z)}{W(z)} = 1 \xrightarrow{IZT} y(n) = w(n)$

(ii) $\frac{W(z)}{X(z)} = \frac{1}{1+0.9z^{-1}} \xrightarrow{IZT} x(n) = w(n) + 0.9w(n-1)$

$$w(n) = x(n) - 0.9w(n-1)$$

→ Cascade form is,



$\begin{array}{r}36\\ 7\\ \hline 252\end{array}$

Parallel form:

$$H(z) = \frac{0.7 - 0.252z^{-2}}{1 + 0.1z^{-1} - 0.7z^{-2}}$$

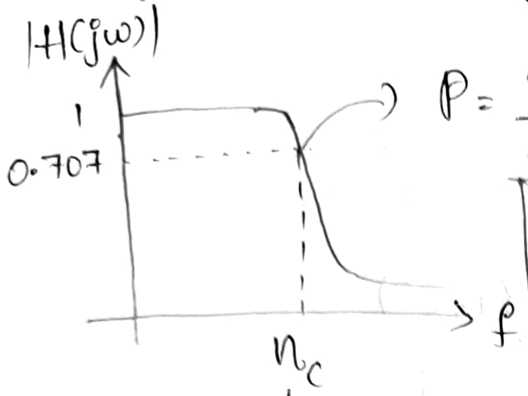
$-0.7z^{-2} + 0.1z^{-1} + 1 \,\big)\, -0.252z^{-2} + 0.7 \,\big(\, 0.36$

$\quad -0.252z^{-2} + 0.036z^{-1} + 0.36$

$\quad (+) \qquad (-) \qquad (-)$

$\quad -0.036z^{-1} + 0.34$

✱ Design of IIR filters: (Butterworth filters)

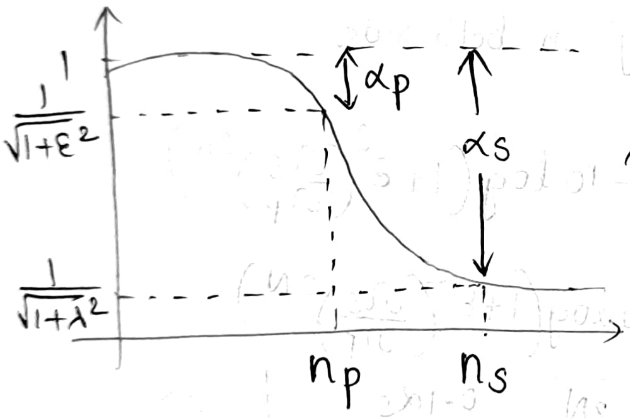


$$P = \frac{V^2}{R} = \frac{(V/\sqrt{2})^2}{R} = \frac{V^2}{2R} = \frac{V^2}{R} \times \frac{1}{2} = \frac{P}{2}$$

$$\boxed{|H(j\omega)|^2 = \frac{1}{1+\left(\frac{n}{n_c}\right)^{2N}}}$$

$n_c \rightarrow$ cut-off frequency $\rightarrow$ The frequencies at which Power is $\frac{1}{2}$ of P_{tot}.

$$10\log\frac{o/p}{i/p} = 10\log\left(\frac{P/2}{P}\right) = 10\log(0.5)$$
$$= -3\,dB$$



$\varepsilon \rightarrow$ attenuation const wrt passband

$\lambda \rightarrow$ attenuation const wrt stop band.

α_p – passband attenuation

α_s – stopband attenuation.

$$\boxed{|H(j\omega)|^2 = \frac{1}{1+\varepsilon^2\left(\frac{n}{n_p}\right)^{2N}}}$$

Here $(n = \Omega)$

$$\boxed{|H(j\omega)|^2 = \frac{1}{1+\lambda^2\left(\frac{\Omega}{\Omega_s}\right)^{2N}}}$$

Case-(i): $\Omega = \Omega_p$

$$|H(j\omega_p)|^2 = \frac{1}{1+\varepsilon^2}$$

Apply 'log' on both sides.

$$\underbrace{20 \log |H(j\omega_p)|}_{\downarrow} = -10 \log (1+\varepsilon^2)$$

$$-\alpha_p = -10 \log_{10} (1+\varepsilon^2)$$

$$\boxed{\varepsilon^2 = 10^{0.1\alpha_p} - 1} \text{ --- } (1)$$

$$\boxed{\lambda^2 = 10^{0.1\alpha_s} - 1} \text{ --- } (*)$$

case-(ii): $\Omega = \Omega_s$

$$|H(j\omega_s)|^2 = \frac{1}{1+\varepsilon^2 \left(\frac{\Omega_s}{\Omega_p}\right)^{2N}}$$

Apply 'log' on both sides

$$\underbrace{20 \log (|H(j\omega_s)|)}_{\downarrow} = -10 \log \left(1+\varepsilon^2 \left(\frac{\Omega_s}{\Omega_p}\right)^{2N}\right)$$

$$+\alpha_s = +10 \log \left(1+\varepsilon^2 \left(\frac{\Omega_s}{\Omega_p}\right)^{2N}\right)$$

$$\boxed{\varepsilon^2 \left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = 10^{0.1\alpha_s} - 1} \text{ --- } (2)$$

$$\frac{(2)}{(1)} \Rightarrow \left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}$$

$$N \log \left(\frac{\Omega_s}{\Omega_p}\right) = \log \left(\sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}\right)$$

$$\boxed{N = \frac{\log \left(\sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}\right)}{\log \left(\frac{\Omega_s}{\Omega_p}\right)}}$$

eg: Determine order of filter with following specifications

$\Omega_p = 200\ rad/s \qquad \alpha_p = 1\,dB$

$\Omega_s = 600\ rad/s \qquad \alpha_s = 30\,dB$

Sol:

$$N = \frac{\log\left(\sqrt{\dfrac{10^{\alpha_s \times 0.1} - 1}{10^{0.1\alpha_p} - 1}}\right)}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)}$$

$$N = \frac{\log\left(\sqrt{\dfrac{10^3 - 1}{10^{0.1} - 1}}\right)}{\log\left(\dfrac{\cancel{600}^{3}}{\cancel{200}}\right)}$$

$$N = \log\left(\sqrt{\frac{999}{1.25}}\right) \Big/ \log(3)$$

$$\underline{\underline{N = 3.75}}$$

order $\Rightarrow \underline{\underline{N \cong 4}}$

eg: $\alpha_p = 3\,dB$; $\alpha_s = 40\,dB$

$f(\Omega_p) = 500\,Hz \rightarrow \Omega_p = 2\pi f_p = 2\pi(500)\ rad/s$

$f(\Omega_s) = 1000\,Hz \rightarrow \Omega_s = 2\pi f_s = 2\pi(1000)\ rad/s$

$$N = \log\left(\sqrt{\frac{10^4 - 1}{10^{0.3} - 1}}\right) \Big/ \log(2)$$

$$N = 6.64$$

$\therefore \underline{\underline{N \simeq 7}}$

- Derivation of transfer function in S-domain

$$|H(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}} \qquad \begin{aligned} S &= j\Omega \\ \Omega &= S/j \end{aligned}$$

$$H(\Omega^2) = H\left(\left(\frac{S}{j}\right)^2\right)$$

$$= H(-S^2)$$

$$= H(S)\cdot H(-S)$$

$$H(S)\cdot H(-S) = \frac{1}{1+\left(\frac{S/j}{\Omega_c}\right)^{2N}} = \frac{1}{1+\left(\frac{-S^2}{\Omega_c^2}\right)^N}$$

For determining poles,

$$1+\left(\frac{-S^2}{\Omega_c^2}\right)^N = 0$$

$$\left(\frac{-S^2}{\Omega_c^2}\right) = (-1)^{1/N}$$

$$S^2 = j^2\,\Omega_c^2\,(-1)^{1/N}$$

$$S = j\Omega_c\,(-1)^{1/2N}$$

$$S_K = \Omega_c\, e^{j\pi/2}\, e^{j\frac{(2k-1)\pi}{2N}}$$

$$S_K = \Omega_c\, e^{j\left(\pi/2 + \frac{(2k-1)\pi}{2N}\right)}$$

$$\boxed{S_K = \Omega_c e^{j\phi_K}}$$

where $\phi_K = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}$ $\quad K = 1, 2, \ldots N$

$$\therefore \boxed{H(S) = \frac{1}{(S-S_1)(S-S_2)\cdots}}$$

eg: for order ($N = 3$) determine transfer function of butterworth filter. Assume, $\Omega_c = 1$

Sol:- $S_K = \Omega_c e^{j\phi_K}$

$$\phi_1 = \frac{\pi}{2} + \frac{(2-1)\pi}{6} = 2\pi/3$$

$$\phi_2 = \frac{\pi}{2} + \frac{3\pi}{6} = \pi$$

$$\phi_3 = \frac{\pi}{2} + \frac{5\pi}{6} = \frac{4\pi}{3}$$

$$S_1 = \Omega_c e^{j\phi_1} = e^{j(2\pi/3)} = \frac{-1}{2} + \frac{j\sqrt{3}}{2}$$

$$S_2 = \Omega_c e^{j\phi_2} = e^{j(\pi)} = -1$$

$$S_3 = \Omega_c e^{j\phi_3} = e^{j\left(\frac{4\pi}{3}\right)} = -\frac{1}{2} - \frac{j\sqrt{3}}{2}$$

$$(s+1)(s-(-0.5+j(0.86))(s-(-0.5-j(0.86)))$$

$$(s+1)(s+0.5-j(0.86))(s+0.5+j(0.86)))$$

$$(s+1)((s+0.5)^2+(0.86)^2)$$

- Relation b/w cut-off freq, pass band freq, stopband freq of low-pass filter

$$|H(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}} = \frac{1}{1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}}$$

$$\cancel{1}+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N} = \cancel{1}+\left(\frac{\Omega}{\Omega_c}\right)^{2N}$$

$$\varepsilon^2 = \left(\frac{\Omega_p}{\Omega_c}\right)^{2N}$$

$$\left(10^{0.1\alpha_p}-1\right)^{1/2N} = \frac{\Omega_p}{\Omega_c}$$

$$\therefore \boxed{\Omega_c = \frac{\Omega_p}{\left(10^{0.1\alpha_p}-1\right)^{1/2N}}}$$

$$\therefore \boxed{\Omega_c = \frac{\Omega_s}{\left(10^{0.1\alpha_s}-1\right)^{1/2N}}}$$

eg: Determine transfer function of Order $N=2$

$$S_k = \Omega_c e^{j\phi_k} \qquad \underline{\Omega_c = 1}$$

where $\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}$

$K = 1, 2$

$$\phi_1 = \frac{\pi}{2} + \frac{\pi}{4} = 3\pi/4$$

$$\phi_2 = \frac{\pi}{2} + \frac{3\pi}{4} = \frac{5\pi}{4}$$

$$S_1 = e^{j\phi_1} = e^{j(3\pi/4)} = -\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}$$

$$S_2 = e^{j\phi_2} = e^{j(5\pi/4)} = -\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$$

$$H(s) = \frac{1}{(s-s_1)(s-s_2)}$$

$$= \frac{1}{\left(s+\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= \frac{1}{s^2 + \sqrt{2}s + \frac{1}{2} + \frac{1}{2}}$$

$$\therefore \boxed{H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}}$$

Conversion of Analog filter to Digital filter:

1. Impulse Invariance method.
2. Bilinear Transformation method.

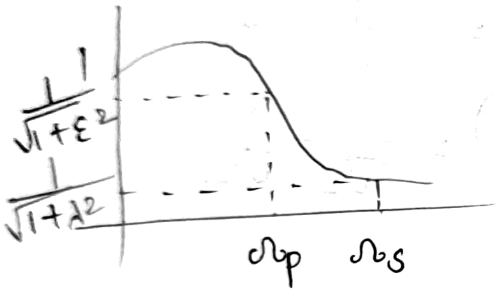
$\Big\} \Rightarrow H(s) \longrightarrow H(z)$

1. Impulse Invariance method:

eg: $0.9 < |H(j\omega)| \le 1$ for $0 \le \Omega \le 0.2\pi$ → Ω_p

$|H(j\Omega)| \le 0.2$ for $0.4\pi \le \Omega \le \pi$

↓ Ω_s



$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.9 \qquad \Bigg| \qquad \frac{1}{\sqrt{1+\lambda^2}} = 0.2$$

$\varepsilon^2 = 0.2321$ $\qquad$ $\lambda = \sqrt{24}$

$\underline{\varepsilon = 0.48}$ $\qquad$ $\underline{\lambda = 4.89}$

$\varepsilon^2 = 10^{0.1\alpha_p} - 1$ $\qquad$ $\lambda^2 = 10^{0.1\alpha_s} - 1$

$\varepsilon^2 + 1 = 10^{0.1\alpha_p}$ $\qquad$ $\lambda^2 + 1 = 10^{0.1\alpha_s}$

$\underline{\alpha_p = 0.915}$ $\qquad$ $\underline{\alpha_s = 13.979}$

$$N = \frac{\log\left(\sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}\right)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} \Rightarrow N \ge 3.3$$

$\underline{N \simeq 4}$

$$S_K = e^{j\phi_k}\Omega_c$$

Assume $\Omega_c = 1$

$$S_k = e^{j\phi_k}$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}$$

$$\phi_1 = \frac{\pi}{2} + \frac{\pi}{8} = 5\pi/8 \longrightarrow S_1 = e^{j5\pi/8} = -0.8 + j(0.92)$$

$$\phi_2 = \frac{\pi}{2} + \frac{3\pi}{8} = 7\pi/8 \longrightarrow S_2 = e^{j7\pi/8} = -0.9 + j(0.38)$$

$$\phi_3 = \frac{\pi}{2} + \frac{5\pi}{8} = 9\pi/8 \longrightarrow S_3 = e^{j9\pi/8} = -0.9 - j(0.38)$$

$$\phi_4 = \frac{\pi}{2} + \frac{7\pi}{8} = 11\pi/8 \longrightarrow S_4 = e^{j11\pi/8} = -0.8 - j(0.92)$$

$$\Rightarrow (S-S_1)(S-S_2)(S-S_3)(S-S_4)$$

$$= (s+038 - j(0.92))(s+0.9 - j(0.38))(s+0.9 + j(0.38))$$
$$(s+0.8 + j(0.92))$$

$$= ((s+038)^2 + (0.92)^2)((s+0.9)^2 + (0.38)^2)$$

$$= (s^2 + 1.6s + 0.64 + 0.8464)$$

$$= (s^2 + 0.76s + 0.1444 + 0.8464)$$
$$(s^2 + 0.81 + 1.8s + 0.1444)$$

8100
1444
9544

$$= (s^2 + 0.76s + 0.9908)(s^2 + 0.9544 + 1.8s)$$

$$\therefore H(s) = \frac{1}{(s^2+0.76s+1)(s^2+1.8s+1)}$$

1. Impulse Invariance method:

$$H(s) = \sum_{k=1}^{N} \frac{C_k}{S - P_k}$$

$\downarrow$ converting s-domain to time domain.

L.T

$$\underset{\text{Analog}}{h(t)} \longrightarrow \underset{\text{Discrete}}{h(n)} \longrightarrow \underset{\text{Digital}}{H(z)}$$

$$h(t) = C_k e^{P_k t}$$

$$\left(e^{at} \xrightarrow{L.T} \frac{1}{s-a} \right)$$

$t = nT$ (Sampling)

$$h(n) = \sum_{k=1}^{N} C_k e^{P_k nT}$$

$\downarrow$ Applying z-transform

$$H(z) = \sum_{n=0}^{\infty} \sum_{k=1}^{N} C_k e^{P_k nT} \cdot z^{-n}$$

$$= \sum_{k=1}^{N} C_k \sum_{n=0}^{\infty} \left(e^{P_k T} z^{-1} \right)^n$$

$$\therefore \boxed{H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{P_k T} z^{-1}}}$$

(*) $H(s) = \sum_{k=1}^{N} \frac{C_k}{S - P_k} \longrightarrow H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{P_k T} z^{-1}}$

$z = e^{sT}$

$s = \sigma + j\Omega$

$z = e^{(\sigma + j\Omega)T}$

$re^{j\omega} = e^{(\sigma + j\Omega)T}$

$\boxed{|r| = e^{\sigma T}}$

$\boxed{\omega = \Omega T}$

ω → digital frequency

Ω → analog frequency

T → Sampling Time period.

$\sigma < 0$; $|r| < 1$ (poles are mapped to inside the circle)

$\sigma = 0$; $r = 1$ (poles are mapped on the boundary)

$\sigma > 0$; $|r| > 1$ (poles are mapped outside the circle) eg: $H(s) = \dfrac{2}{(s+1)(s+2)}$

Sol:-

$$\frac{A}{s+1} + \frac{B}{s+2} = \frac{2}{(s+1)(s+2)}$$

$$(A+B)s + (B+2A) = 2$$

$$A+B = 0 \rightarrow \underline{A = -B}$$

$$2A + B = 2$$

$$\boxed{B = -2}$$

$$\boxed{A = 2}$$

$\left(s - p_k \leftrightarrow 1 - e^{p_k T} z^{-1}\right)$

Assume $(T=1)$

$$H(s) = \frac{2}{s+1} - \frac{2}{s+2}$$

$$\therefore H(z) = \frac{2}{1 + -e^{-1} z^{-1}} - \frac{2}{1 - e^{-2} \cdot z^{-1}}$$

2. Bilinear Transformation:

$$x_a(t) \longrightarrow \boxed{\int dt} \xrightarrow{1/s} y_a(t)$$

$$y_a(t) = \int x_a(t)\, dt$$

$\downarrow$ sampling

$$y_a(nT) = y_a((n-1)T) + \int_{(n-1)T}^{nT} x_a(nT)\, dt$$

$$\simeq y_a((n-1)T) + \frac{T}{2}\left[x_a(n-1)T + x_a(nT)\right]$$

$$y_a(nT) - y_a((n-1)T) = \frac{T}{2}\left[x_a(n-1)T + x_a(nT)\right]$$

Assume, $T = 1$

$$y_a(n) - y_a(n-1) = \frac{T}{2}\left[x_a(n-1) + x_a(n)\right]$$

$\downarrow$ Z-transform

$$Y(z) - Y(z).z^{-1} = \frac{T}{2}\left[X(z).z^{-1} + X(z)\right]$$

$$Y(z)\left[1 - z^{-1}\right] = \frac{T}{2}\left[X(z)\left[1 + z^{-1}\right]\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{T}{2}\left[\frac{1+z^{-1}}{1-z^{-1}}\right]$$

$$\frac{1}{s} = \frac{T}{2}\left[\frac{1+z^{-1}}{1-z^{-1}}\right]$$

$$\boxed{\therefore\ s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

eg Convert the following using Bilinear Transformation.

$$H(s) = \frac{2}{(s+1)(s+2)(s-1)}$$

Assume, T = 1

$$s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$$

$$s = 2\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$$

$$H(s) = \frac{2}{\left(\frac{2(1-z^{-1})}{1+z^{-1}}+1\right)\left(\frac{2(1-z^{-1})}{1+z^{-1}}+2\right)\left(\frac{2(1-z^{-1})}{1+z^{-1}}-1\right)}$$

$$S = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$$

$$\therefore \boxed{z = \frac{1+ST/2}{1-ST/2}}$$

$$S = \sigma + j\Omega$$

$$|z| = \left|\frac{1+(\sigma+j\Omega)T/2}{1-(\sigma+j\Omega)T/2}\right| = \frac{\sqrt{(1+\sigma T/2)^2+(\Omega T/2)^2}}{\sqrt{(1-\sigma T/2)^2+(\Omega T/2)^2}}$$

$\sigma < 0 \Rightarrow |z| < 1$

$\sigma = 0 \Rightarrow |z| = 1$

$\sigma > 0 \Rightarrow |z| > 1$

Limitation

① Warping method

$$S = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$$

Assume, $z = e^{j\omega}$

$$S = \frac{2}{T}\left[\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right]$$

$$j\Omega = \frac{2}{T} \times \frac{e^{-j\omega/2}}{e^{-j\omega/2}} \times \left(\frac{e^{j\omega/2} - e^{-j\omega/2}}{e^{j\omega/2} + e^{-j\omega/2}}\right)$$

$$j\Omega = \frac{2}{T} j \tan\left(\frac{\omega}{2}\right)$$

$$\therefore \boxed{\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)}$$

ω → Digital frequency

Ω → Analog frequency

$$\boxed{\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)}$$

$$\frac{1}{\sqrt{2}} \le |H(j\Omega)| \le 1 \qquad 0 \le \Omega \le 2 \rightarrow \Omega_p$$

$$|H(j\Omega)| \le 0.2 \qquad \Omega \geq 4 \rightarrow \Omega_s$$

$\Omega_p = 2$

$\Omega_s = 4$

$$\frac{1}{\sqrt{1+\varepsilon^2}} = \frac{1}{\sqrt{2}} \qquad \frac{1}{\sqrt{1+\lambda^2}} = 0.2$$

$$1+\varepsilon^2 = 2 \qquad 1+\lambda^2 = 25$$

$$\boxed{\varepsilon = \pm 1} \qquad \boxed{\lambda = \sqrt{24}}$$

$$\varepsilon^2 = 10^{0.1\alpha_p} - 1 \qquad \lambda^2 = 10^{0.1\alpha_s} - 1$$

$$1+\varepsilon^2 = 10^{0.1\alpha_p} \qquad 1+\lambda^2 = 10^{0.1\alpha_s}$$

$$2 = 10^{0.1\alpha_p} \qquad 25 = 10^{0.1\alpha_s}$$

$$\alpha_p = 10 \log_{10} 2 \qquad \alpha_s = 10 \log_{10} 25$$

$$= 10 \times (0.301) \qquad = 20 \log_{10} 5$$

$$= \underline{3.01} \qquad = 20 \times (0.6989)$$

$$\underline{\alpha_s = 13.97}$$

$$N = \log\left(\sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}\right) \Big/ \log\left(\frac{\Omega_s}{\Omega_p}\right)$$

$$= \log\left(\frac{\lambda}{\varepsilon}\right) \Big/ \log\left(\frac{\Omega_s}{\Omega_p}\right)$$

$$\underline{N \geq 2.29}$$

$$\therefore \underline{N = 3}$$

$$S_k = \Omega_c \cdot e^{j\phi_k}$$

$$\Omega_c = 1$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}$$

$$\phi_1 = \frac{\pi}{2} + \frac{\pi}{6} = 2\pi/3$$

$$\phi_2 = \frac{\pi}{2} + \frac{3\pi}{6} = \pi$$

$$\phi_3 = \frac{\pi}{2} + \frac{5\pi}{6} = 4\pi/3$$

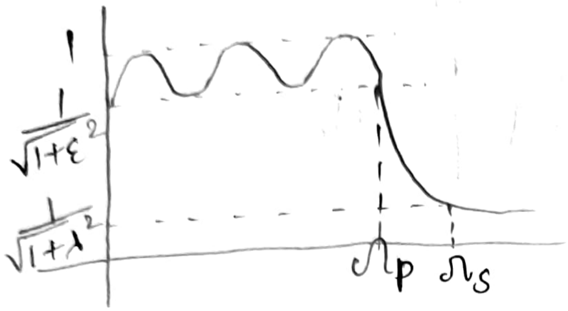
$$S_k = e^{j(2\pi/3)} = -\frac{1}{2} + \frac{j\sqrt{3}}{2}$$

$$S_2 = e^{j(\pi)} = -1$$

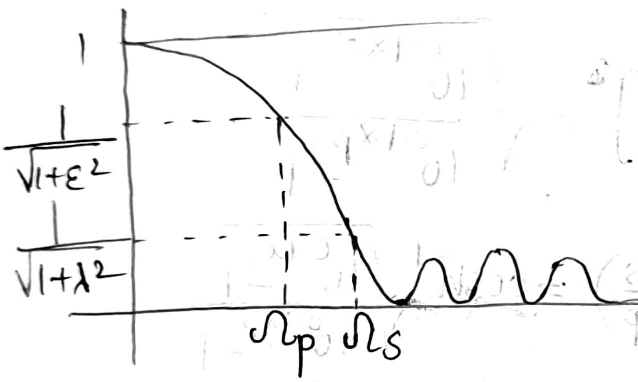
$$S_3 = e^{j(4\pi/3)} = -\frac{1}{2} - \frac{j\sqrt{3}}{2}$$

$$H(s) = \frac{1}{(s-s_1)(s-s_2)(s-s_3)}$$

* Chebyshev filters :



Type-I



Type-II

- **Derivation for order of chebyshev filter-Type-I :**

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 C_N^2\left(\frac{\Omega}{\Omega_p}\right)}$$

where, $C_N(x)$ is chebyshev polynomial,

$$\boxed{\begin{aligned} C_N(x) &\simeq \cos(N\cos^{-1}x) & x \leq 1 \\ &\simeq \cosh(N\cosh^{-1}x) & x > 1 \end{aligned}}$$

$\therefore C_N(1) \simeq 1$

Case-(i): $\Omega = \Omega_p$

$$20\log|H(j\Omega_p)| = -\alpha_p = -10\log(1+\varepsilon^2)$$

$$\boxed{\varepsilon^2 = 10^{0.1\alpha_p} - 1} \quad \text{— (1)}$$

Case-(ii): $\Omega = \Omega_s$

$$\alpha_s = 10\log\left(1+\varepsilon^2\left[\cosh\left(N\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)\right)\right]^2\right)$$

$$\cosh\left[N\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)\right] = \sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}$$

$$N\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right) = \cosh^{-1}\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}$$

$$\boxed{\therefore \quad N = \cosh^{-1}\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}} \Big/ \cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

eg: $\alpha_p = 3\,dB$; $\alpha_s = 15\,dB$; $\omega_p = 20$, $\Omega_s = 60$

$$N \geq \cosh^{-1}\sqrt{\frac{10^{0.1\times 15}-1}{10^{0.1\times 3}-1}} \Big/ \cosh^{-1}\left(\frac{60}{20}\right)$$

$$N \geq 1.36$$

$\therefore$ Order $= 2$

$$H(s) = \frac{C}{D(s)}$$

- N is odd $\rightarrow S=0 \Rightarrow D(s) = D(0) = Nr = C$
- N is even $\rightarrow S=0 \Rightarrow \frac{D(s)}{\sqrt{1+\varepsilon^2}} = \frac{D(0)}{\sqrt{1+\varepsilon^2}} = Nr = C$

- Poles for chebyshev Type-I filter:

$$S_k = a\cos\phi_k \pm jb\sin\phi_k$$

where $\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}$ $\quad K=1,2,\ldots,N$

$$a = \Omega_p\left[\frac{(\mu)^{1/N} - (\mu)^{-1/N}}{2}\right]$$

$$b = \Omega_p\left[\frac{(\mu)^{1/N} + (\mu)^{-1/N}}{2}\right]$$

$$\mu = \varepsilon^{-1} + \sqrt{\varepsilon^{-2}+1}$$

$$\varepsilon = \sqrt{10^{0.1\alpha p} - 1}$$

eg: find poles for $N=2$ $\left[\begin{array}{l}\alpha p = 3dB, \alpha s = 15dB \\ \omega p = 20, \Omega s = 60\end{array}\right]$

$\varepsilon = \sqrt{10^{0.1\alpha p} - 1} \rightarrow \varepsilon = 0.99$

$\mu = \varepsilon^{-1} + \sqrt{\varepsilon^{-2}+1} \rightarrow \mu = 2.4$

$a = \frac{20}{2}\left[(2.4)^{1/2} - (2.4)^{-1/2}\right] \rightarrow a = 9.03$

$b = \frac{20}{2}\left[(2.4)^{1/2} + (2.4)^{-1/2}\right] \rightarrow b = 21.94$

$\underline{N=2}$

$$\phi_1 = \frac{\pi}{2} + \frac{\pi}{4} = 3\pi/4$$

$$\phi_2 = \frac{\pi}{2} + \frac{8\pi}{4} = 5\pi/4$$

$$S_1 = 9.03 \cos\phi_1 + j21.94 \cos\phi_1$$

$$S_1 = 9.03 \cos\left(\frac{3\pi}{4}\right) + j21.94 \sin\left(3\pi/4\right)$$

$$S_2 = 9.03 \cos\phi_2 + 21.94 \sin\phi_2$$

$$S_2 = 9.03 \cos(5\pi/4) + j21.94 \sin(5\pi/4)$$

$$\therefore S_1 = -6.38 + j(15.51)$$

$$\therefore S_2 = -6.38 - j(15.51)$$

- Chebyshev-Type-II filter:

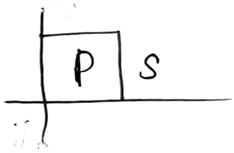
$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 \dfrac{C_N^2\left(\frac{\Omega_s}{\Omega_p}\right)}{C_N^2\left(\frac{\Omega_s}{\Omega}\right)}}$$

$$S_K = j\frac{\Omega_s}{\sin\phi_K}$$

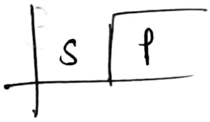
$$\phi_K = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \qquad k = 1, 2, \dots N$$

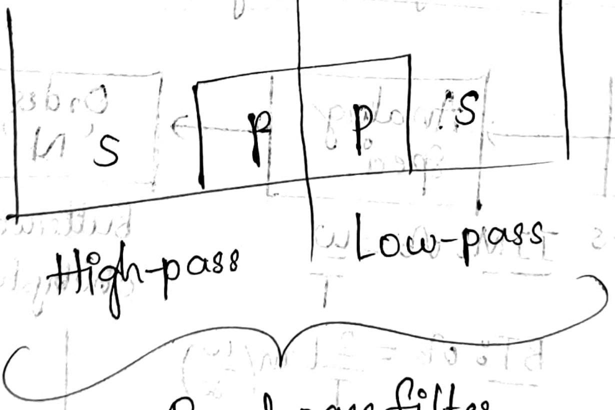
* Filter Transformations in analog domain:

$$S \longrightarrow \frac{S}{\Omega_c} \text{ (low-pass filter)}$$



$$S \longrightarrow \frac{\Omega_c}{S} \text{ (High-pass filter)}$$





Band-pass filter.

$$S \rightarrow \frac{\Omega_0}{B.W} \times \left(\frac{S}{\Omega_0} + \frac{\Omega_0}{S} \right)$$

$$\frac{\cancel{\Omega_0}}{\Omega_u - \Omega_l} \times \left(\frac{S^2 + \Omega_0^2}{S \cancel{\Omega_0}} \right)$$

$$\frac{S^2 + \Omega_0^2}{S(\Omega_u - \Omega_l)}$$

$$\boxed{\Omega_0 = \sqrt{\Omega_u \cdot \Omega_l}}$$

$$S \xrightarrow{\text{Lowpass}} \frac{S^2 + \Omega_u \Omega_l}{S(\Omega_u - \Omega_l)}$$

$$S \xrightarrow{\text{Highpass}} \frac{S(\Omega_u - \Omega_l)}{(S^2 + \Omega_u \Omega_l)}$$

- <u>Architecture for design of digital filter:</u>

Digital speci → Analog speci → Order 'N' → Poles → H(s)

Digital speci: $\alpha_p, \alpha_s, \Omega_p, \Omega_s$

Analog speci:

IIM: $\Omega = \frac{\omega}{T}$

BT: $\Omega = \frac{2}{T} \tan\left(\frac{\omega}{2}\right)$

Order 'N':

Butterworth – log

chebyshev – $\cosh^{-1}$

Poles:

B: $S_k = e^{j\phi_k} \Omega_c$

C: $S_k = a\cos\phi_k + j\, b\sin\phi_k$

H(s):

B: $H(s) = \frac{1}{D(s)}$

C: $H(s) = \frac{N}{D(s)}$

Analog Transformation $\longrightarrow$ $H(z)$ $\longrightarrow$ Realisation.

$S \longrightarrow$

IIM: $S - P_k \rightarrow 1 - e^{P_k T} z^{-1}$

BT: $S = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$

Questions: (5M)

1. Design an analog Butterworth lowpass filter.
2. Design an analog chebyshev low pass filter
3. Design an digital filter using impulse invariance.
4. Design " " " Bilinear Transformation
5. Design HPF/ BPF using digital specifications.

eg: Design a digital butter-worth and chebyshev filters with the following specifications.

$\alpha_p = 3\,dB \quad ; \quad \omega_p = 0.2\pi$

$\alpha_s = 25\,dB \quad ; \quad \omega_s = 0.45\pi$

Sol:- (1) Digital specifications to Analog

BLT: $\Omega_p = \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right)$

Assume, $T = 1$

$\underline{\Omega_p = 0.65}$

$\Omega_s = \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right)$

$\underline{\Omega_s = 1.7}$

(2) Order N

(i) Butterworth $\Rightarrow N = \log\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}} \Big/ \log\left(\frac{\Omega_s}{\Omega_p}\right)$

$N \geq 2.99 \longrightarrow \boxed{N=3}$

(ii) Chebyshev $\Rightarrow N = \cosh^{-1}\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}} \Big/ \cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)$

$N \geq 2.21 \longrightarrow \boxed{N=3}$

(3) Poles

(i) Butterworth

$S_k = \Omega_c e^{j\phi_k}$

$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}$

$\phi_1 = \frac{\pi}{2} + \frac{\pi}{6} = 2\pi/3$

$\phi_2 = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

$\phi_3 = \frac{\pi}{2} + \frac{5\pi}{6} = \frac{4\pi}{3}$

$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}}$

$\underline{\Omega_c = 0.65}$

$S_1 = 0.65 e^{j(2\pi/3)} = 0.65\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} j\right)$

$S_2 = 0.65 e^{j\pi} = -0.65$

$S_3 = 0.65 e^{j(4\pi/3)} = 0.65\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} j\right)$

$S_1 = -0.325 + j(0.56)$

$S_2 = -0.65$

$S_3 = -0.325 - j(0.56)$

(ii) chebyshev

$\varepsilon = \sqrt{10^{0.1\alpha p} - 1} = \underline{\underline{0.997}}$

$S_k = a\cos\phi_k + j b\sin\phi_k$; $\mu = \varepsilon^{-1} + \sqrt{\varepsilon^{-2} + 1} = \underline{\underline{2.6}}$

$\phi_1 = 2\pi/3,\ \phi_2 = \pi,\ \phi_3 = 4\pi/3$

$a = 0.65\left[\frac{(2.6)^{1/3} - (2.6)^{-1/3}}{2}\right] = \underline{\underline{0.21}}$

$b = 0.65\left[\frac{(2.6)^{1/3} + (2.6)^{1/3}}{2}\right] = \underline{\underline{0.68}}$

$S_1 = 0.21\cos\frac{2\pi}{3} + j(0.68)\sin(2\pi/3)$

$S_1 = -0.105 + j(0.588)$

$S_2 = 0.21\cos\pi + j(0.68)\sin\pi$

$S_2 = -0.21$

$S_3 = 0.21\cos\left(\frac{4\pi}{3}\right) + j(0.68)\sin\left(\frac{4\pi}{3}\right)$

$S_3 = -0.105 - j(0.588)$

(4) H(s)

(i) Butterworth

$$H(s) = \frac{1}{(s-s_1)(s-s_2)(s-s_3)}$$

$$= \frac{1}{(s+0.32-j(0.56))(s+0.65)(s+0.32+j(0.56))}$$

$$= \frac{1}{((s+0.32)^2+(0.56)^2)(s+0.65)}$$

$$= \frac{1}{(s^2+0.1024+0.64s+0.3136)(s+0.65)}$$

$$H(s) = \frac{1}{(s^2+0.64s+0.416)(s+0.65)}$$

(ii) Chebyshev

$$H(s) = \frac{N}{D(s)}$$

$$D(s) = (s+0.21)((s+0.1)^2+(0.58)^2)$$

$$= (s+0.21)(s^2+0.34+0.2s)$$

$$N = D(0) = \underline{\underline{0.07}}$$

$$H(s) = \frac{0.07}{(s+0.21)(s^2+0.34+0.2s)}$$

(5) S-domain to Z-domain

(i) Butterworth

$$H(s)\Big|_{s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

Assume $(T=1)$

$$\cancel{H(s)}\ H(z) = \frac{1}{\left(\frac{2(1-z^{-1})}{1+z^{-1}}+0.65\right)\left(\left(\frac{2(1-z^{-1})}{(1+z^{-1})}\right)^2+0.64\left(\frac{2(1-z^{-1})}{1+z^{-1}}\right)+0.416\right)}$$

$$\frac{H(z)}{\cancel{H(s)}} = \frac{1}{\left(\frac{2(1-z^{-1})+0.65(1+z^{-1})}{1+z^{-1}}\right)\left(\frac{4+4z^{-2}-8z^{-1}+1.28(1-z^{-2})+0.416(1+z^{-2}+2z^{-1})}{(1+z^{-1})^2}\right)}$$

$$\frac{H(z)}{\cancel{H(s)}} = \frac{(1+z^{-1})^3}{(2.65-1.35z^{-1})(3.13z^{-2}-7.16z^{-1}+5.69)}$$

$$\therefore\ \frac{H(z)}{\cancel{H(s)}} = \frac{(z+1)^3}{(2.65z-1.35)(5.69z^2-7.16z+3.13)}$$

$$H(z) = \frac{z^{-3}+3z^{-2}+3z^{-1}+1}{-4.23z^{-3}+17.99z^{-2}-26.71z^{-1}+15.03}$$

2.720
0.416
3.136

−8.000
0.832
−7.168

5.28
0.416
5.696

(ii) Chebyshev

$$H(s) = \frac{0.07}{(s+0.21)(s^2+0.2s+0.35)}$$

$$s = 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$\cancel{H(s)}\ H(z) = \frac{0.07}{\left(\frac{2(1-z^{-1})+0.21(1+z^{-1})}{(1+z^{-1})}\right)\left(\frac{4+4z^{-2}-8z^{-1}+0.2-0.2z^{-2}+0.35(1+z^{-2}+2z^{-1})}{(1+z^{-1})^2}\right)}$$

$$\cancel{H(s)}\ H(z) = \frac{0.07(1+z^{-1})^3}{(2.21-1.79z^{-1})(4.15z^{-2}-7.3z^{-1}+4.55)}$$

$$\cancel{H(s)}\ H(z) = \frac{0.07\times(z+1)^3}{(2.21z-1.79)(4.55z^{+2}-7.3z^{1}+4.15)}$$

$$\therefore H(z) = \frac{0.07\times(z^{-3}+3z^{-2}+3z^{-1}+1)}{-7.42z^{-3}+22.23z^{-2}-24.27z^{-1}+10.05}$$

(6) Realisation

(i) Butterworth

$$H(z) = \frac{z^{-3} + 3z^{-2} + 3z^{-1} + 1}{-4.23z^{-3} + 17.99z^{-2} - 26.71z^{-1} + 15.03}$$

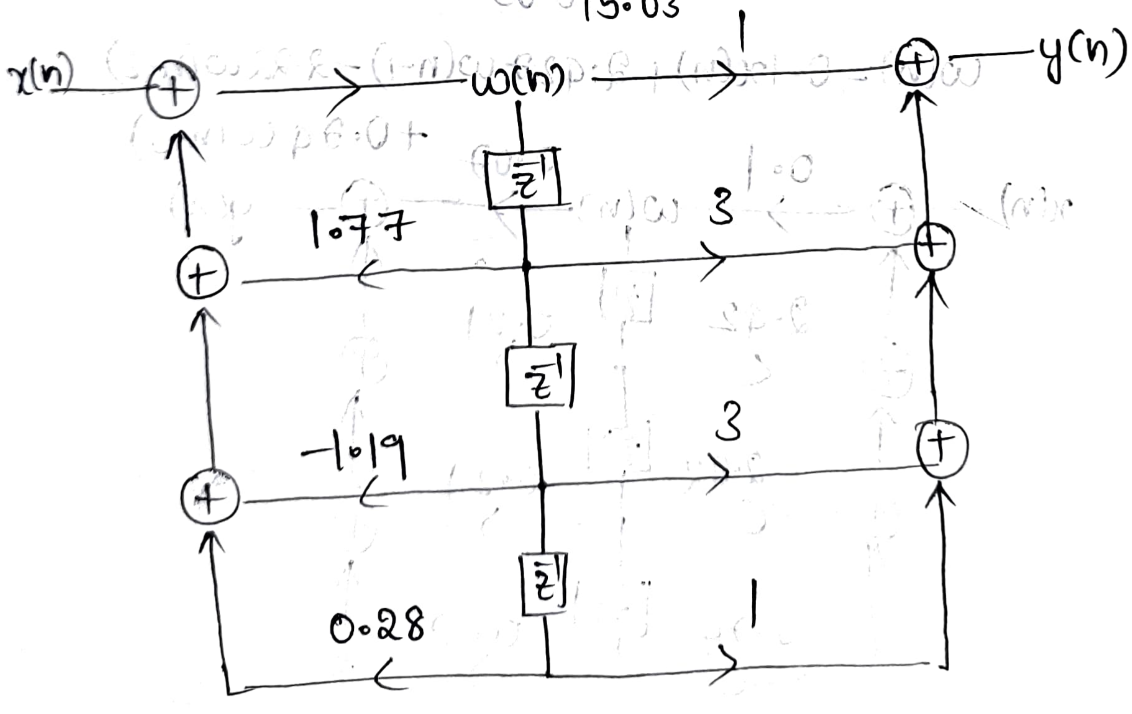
$$\rightarrow \frac{Y(z)}{W(z)} = z^{-3} + 3z^{-2} + 3z^{-1} + 1$$

$$y(n) = w(n-3) + 3w(n-2) + 3w(n+-1) + w(n)$$

$$\rightarrow \frac{W(z)}{X(z)} = \frac{1}{-4.23z^{-3} + 17.99z^{-2} - 26.71z^{-1} + 15.03}$$

$$x(n) = -4.23w(n-3) + 17.99w(n-2) - 26.71w(n-1) + 15.03w(n)$$

$$w(n) = \frac{x(n) + 4.23w(n-3) - 17.99w(n-2) + 26.71w(n-1)}{15.03}$$



(ii) Chebyshev

$$H(z) = \frac{0.07\times(z^{-3}+3z^{-2}+3z^{-1}+1)}{-7.42z^{-3}+22.23z^{-2}-24.27z^{-1}+10.05}$$

$$\rightarrow \frac{Y(z)}{W(z)} = 0.07(z^{-3}+3z^{-2}+3z^{-1}+1)$$

$$y(n) = 0.07w(n) + 0.21w(n-1) + 0.21w(n-2) + 0.07w(n-3)$$

$$\rightarrow \frac{W(z)}{X(z)} = \frac{1}{-7.42z^{-3}+22.23z^{-2}-24.27z^{-1}+10.05}$$

$$x(n) = 10.05w(n) - 24.27w(n-1) + 22.23w(n-2) - 7.42w(n-3)$$

$$w(n) = \frac{x(n) + 24.27w(n-1) - 22.23w(n-2) + 7.42w(n-3)}{10.05}$$

$$w(n) = 0.1x(n) + 2.427w(n-1) - 2.22w(n-2) + 0.74w(n-3)$$