Introduction

The basics idea of graph theory were born in 1736 with Euler's paper in which he solved the Konigsberg bridge problem.In the last decades graph theory has established itself as a worthwhile mathematical disciplines and there are many applications of graph theory to a wide variety of subjects which include operation research, Physics, Chemistry, Economics, Genetics, Sociology, Engineering etc. We can associate several matrices which record information about vertices and how they interconnected. That is we can given an algebraic structure to every graph.Many interesting result can be proved about graphs using matrices and other algebraic properties. The main use of algebraic structure is that we can translate properties of graphs into algebraic properties and then using the results and methods of algebra, to deduce theorems about graphs.

We mainly concendrate on Eigenvalues of the adjacency matrices of some graphs. In early days, important progress was made in the development of algebraic theory by the investigation of some very concrete problems, for example, Kirchoffs study of electrical circuits and cayleys attempt to enumerates chemical isomers.

Application of matrices are not only graph theory. There are many ideas applied in field of secret in the banking, communication in military administration,confidential message transduction, computerized lockers etc.

Outline of the Project

Apart from the introductory chapter, We have described our work in three chapters.

Chapter 1: In this chapter we review the basic concepts in graph theory and linear algebra. In graph theory section we review some definitions [1] and notations and In the linear algebra section we review some useful results in matrix theory.

Chapter 2: This chapter begins by defining matrices associated with graphs.We shall restrict our attention to simple undirected graphs. So whenever the word graph is used it will be referring to simple undirected graph with n vertices.

chapter 3: In this chapter we discuss about energy of graph.

Chapter 4: This chapter shown that there exist an infinite number of values of n for which k-regular graphs exist whose energies are arbitrarily small compared to the known sharp bound $k + \sqrt{k(n-1)(n-k)}$ for the energy of k-regular graphs on n vertices.

Chapter 1

preliminaries

1.1 Graphs

Definition 1.1.1. A Graph G is a non empty set of vertices $V(G)$ and a set of edges $E(G)$ which consist of pair of elements of $V(G)$ [1].

(1) The graph G is said to be directed graph(diagraph) if elements of $E(G)$ are ordered pairs and undirected if they are not.

(2) If $\{v_i, v_j\} \in E(G)$, then v_i is said to be adjacent to v_j.

(3) If an edge joints a vertex to itself such an edge is known as loop.

(4) If some pair of vertices are connected by more than one edge, such edges are called parallel edges.

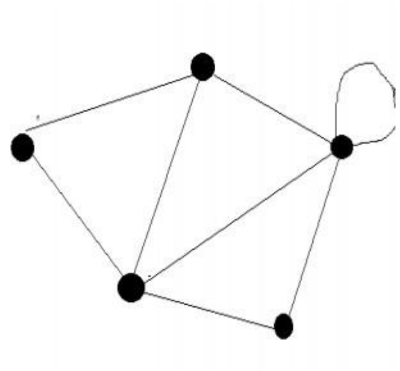

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Figure 1.1: A graph with multiple edges and loop

Definition 1.1.2. A graph G is simple if it has no loops and parallel edges.

Definition 1.1.3. If a graph G contains some loops or parallel edges then G is called multi graph.

Definition 1.1.4. The degree of vertex in a simple graph G is the number of edges incident with v.

A graph in which every vertex has equal degree K is called regular of degree K or K-regular.

Definition 1.1.5. A subgraph of a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of end points to edges in H is same as in G.

Definition 1.1.6. A path p_n is a graph with n vertices and edges set $\{v_i, v_{i+1} : 1 \leq i < n\}$.

Definition 1.1.7. A graph G is said to be connected if there is a path between every two of its vertices.

Definition 1.1.8. A cycle is a path in which all the vertices except the end vertices are disjoint.

Some special graphs

- A simple graph in which every pair of distinct vertices are adjacent is called a complete graph. A complete graph of n-vertices is denoted by k_n.

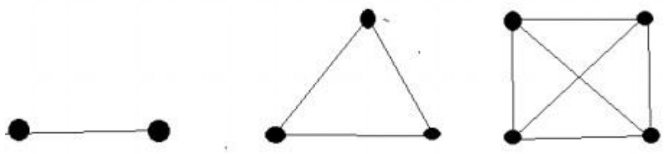

Figure 1.2: complete graph

- A cycle graph of order n is a connected graph of n-vertices whose edges form a cycle of length n and is denoted by c_n.

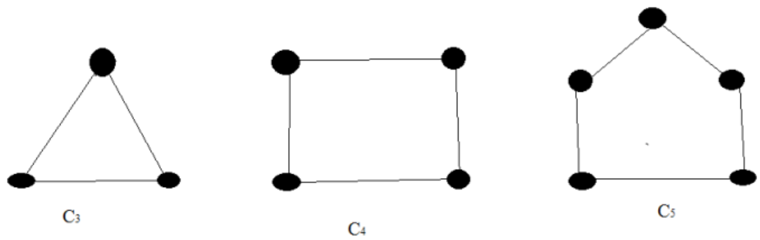

Figure 1.3: Example of a cycle

- A bipartite graph is a simple undirected graph whose vertex set V can be partitioned into two sets M and N in such a way that each edge joints a vertex M to a vertex in N and no edge joins either two vertices in M or two vertice in N. Example of a bipartite graph is given below.

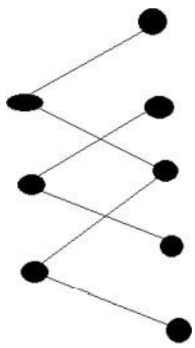

Figure 1.4: An example of a bipartite graph

- A bipartite graph in which every vertex in M is adjacent with every vertex in N is known as complete bipartite graph.A complete bipartite graph is denoted by $k_{m,n}$.

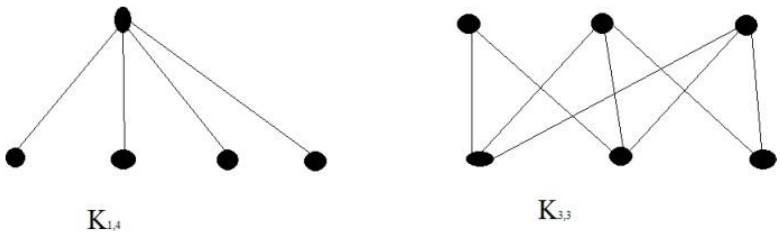

Figure 1.5: Example of a complete bipartite graph

Graph isomorphism

there are different ways to draw the same graph [1]. Consider following pair of graphs.

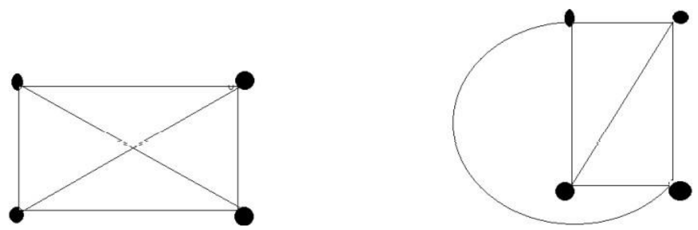

Figure 1.6: Pair of isomorphic graphs

Definition 1.1.9. Graph $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are isomorphic if there is a bijection $f : V_1 \to V_2$ and There is a bijection $g : E_1 \to E_2$ such that u and v are adjacent in G_1 if and only if $f(u)$ and $f(v)$ are adjacent to G_2.

Example of non-isomorphic graph is given in figure 1.7

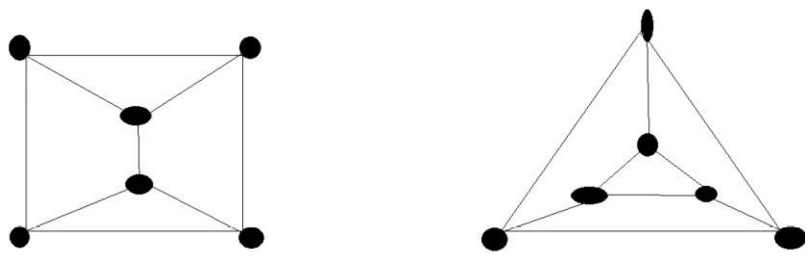

Figure 1.7: Non-isomorphic graphs

1.2 Linear algebra

Matrices

A matrix A over a field K or, simply a matrix A is a rectangular array of $m \times n$ matrix consists of mn real numbers arranged in m rows and n columns. The entry in row i and column j of a matrix A is denoted by $a_{i,j}$. An $m \times 1$ matrix is called a column vector of order m. Similarly, a $1 \times n$ matrix is a row vector of order n. An $m \times n$ matrix is called a square matrix if $m = n$. A diagonal matrix is a square matrix A such that $a_{i,j} = 0; i \neq j$. We denote the diagonal matrix by diag($\lambda_1, \lambda_2, ..\lambda_n$). when $\lambda_i = 1$ for all i, this matrix reduces to the identity matrix of order n, which is denoted by I_n.

Trace

Let A be a square matrix of order n. The entries $a_{11}, a_{22}, ...a_{nn}$ are the main diagonal of A. The trace of A is defined as $Trace(A) = a_{11}+, a_{22}, ...a_{nn}$

Example 1.2.1.

$$A = \begin{bmatrix} 2 & 2 & -4 \\ -1 & 1 & 3 \\ 2 & 0 & -1 \end{bmatrix}$$

$Trace(A) = 2 + 1 + -1 = 2$

Eigenvalues of symmetric matrices

Let A be a $n \times n$ matrix . The determinant $det(A - \lambda I)$ is a polynomial in the (complex) variable λ of degree n and is called the characteristic polynomials of A . The equation $det(A - \lambda I) = 0$ is called the characteristic equation of A. By fundamental theorem of algebra the equation as n complex roots and this roots are called Eigen value.

Chapter 2

Matrices Associated With Graphs

2.1 Adjacency matrix

In this section we introduce the concept of adjacency matrix [3] of graph. We also discuss some properties of adjacency matrix.

Definition 2.1.1. Let G be a graph with $V(G) = \{v_1, v_2 ...., v_n\}$ and $E(G) = \{e_1, e_2, ...e_m\}$ the **Adjacency matrix of G**, denoted by $A(G)$, is the $n \times n$ matrix defined as follows,

$$a_{i,j} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

We often denote A(G) by A

Example 2.1.1. Consider the graph G in figure 2.1

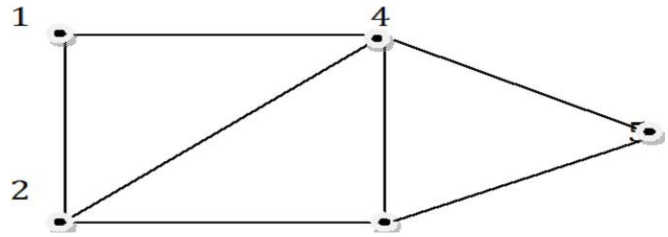

Figure 2.1: Graph G (a)

$$\text{Then, } A(G) = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Properties of adjacency matrix

Let A(G) be an adjacency matrix, then

(1) Elements of the diagonal entry of A(G) are zeroes.

(2) A is real symmetric matrix.

(3) Sum of entries in the *i* th row is the degree of the vertex.

(4) A is diagonalizable.

(5) Eigen value of A(G) are real

Lemma 2.1.1. Two graphs are isomorphic if and only if for some ordering of their vertices their adjacency matrices are equal.

Theorem 2.1.1. Let G be a graph and A(G) be the adjacency matrix of G. Then

1. Sum of the 1×1 principal minor of A is zero.
2. Sum of the 2×2 principal minor of A is equal to the number of edges of G.
3. Sum of the 3×3 principal minor of A is equal to twice the number of triangles in G

Proof. 1. The principle minor with one row and column is the corresponding diagonal entry. This implies that sum of the 1×1 principle minor of A is zero.

2. A principle minor with two row and columns, and which as a non zero entry, must be the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
There is one such mino r for each pair of adjacency vertices of G, and each has value -1. That is sum of principal minor of A of order two = $|E(G)| \times -1 = -|E(G)|$, giving the result.

3. There are essentially three possibilities for non trivial principal minor with three rows and columns. $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ and of these, the only nonzero one is the whose value is 2. This principal minor corresponds to three mutually adjoint vertices in G. That is corresponds to the triangle in G.

[1]