Introduction

In fixed point theory, researchers frequently develop and solve problems on the existence of fixed points using a variety of techniques in analysis, toplogy and geometry. If a topological space X is a metric space, or a linear topological space, then the fixed point theory in such spaces is very rich and easy to work in.

Let T be a self mapping on a set X. An element u in X is said to be a fixed point of mapping T if $Tu = u$. The fixed-Point-Theorem is a statement which asserts that under certain conditions, a mapping T of X into itself admits one or more fixed points.

This paper is an exposition of the Brouwer Fixed-Point Theorem of topology and the Three Points Theorem of transformational plane geometry. If we consider a set X and a function $f : X \to X$, a fixed point of f is a point $x \in X$ such that f(x) = x. [1] Brouwers Fixed-Point-Theorem states that every continuous function from the n-ball B^n to itself has atleast one fixed point. An isometry is a bijective function from $\mathbb{R}^2$ to itself which preserves distance. Although the Three Points Theorem is not itself a Fixed-Point-Theorem, it is a direct consequences of the following fixed-point-theorem: An isometry with three non collinear fixed points is the identity. The Three Points Theorem states that if two isometries agree at three non collinear points, they are equal.

Since the one-dimensional case of the Brouwer Fixed-Point Theorem is the most accessible and intuitive, we shall discuss it first. The development of this theorem is based heavily on topology, so our first task is to develop an under-standing of what topology is and how it works. The two-dimensional case of the Brouwer Fixed-Point Theorem and its proof are far less intuitive than their one-dimensional counterparts. First, we need to explore Cartesian products in detail, as well as the topologies they produce, since R^2 is infact the Cartesian product of R with itself.

Next we define homotopy, which can be understood as a weak form of “topo-logical equivalence” for continues functions. Intuitively, two continues functions are homotopic if one can “melt” into the other without sacrificing continuity during the process. After proving that homotopy is an equivalence relation, we address circle functions continuous functions which map the circle to itself-and we partition them into equivalence classes based on how many times the “wrap around the circle”.

The equivalence class to which a circle function belong is called its degree, and proving that the degree functions are representatives of the equivalence classes of circle functions is a very difficult problem. The degree of circle functions is used to prove the Two-Dimensional No-Retraction Theorem, which states that no retraction from the disk to its boundary exists. The Two-Dimensional No-

Retraction theorem is then used to prove the Two-Dimensional Brouwer Fixed-Point Theorem.

Finally, we show that the fixed-point property is a topological property. The Two Dimensional Brouwer Fixed-Point Theorem is shown to be a consequence of the No Retraction Theorem in two dimensions. Afterward, the No-Retraction Theorem is rewritten to accomodate the n-sphere homotopy classes.

As we into chapter 4, our first task is to define a transformation. We then look at a specific type of transformation, called an isometry, which preserves distance. Next, we show that if an isometry fixes two points, then it necessarily fixes the entire line defined by those two points, and if a third point not on that line is also fixed, then the entire plane is fixed! The Three Points Theorem is an immediate consequence. [5] We finish the main body of the paper by showing that reflections and rotations are isometries.

Chapter 1

Preliminaries

Let X be a set and let τ be a collection of subsets, called open subsets of X. Then τ is called a topology on X if the following axioms hold [4]:

(1) $X \in \tau$.

(2) $\theta \in \tau$.

(3) The intersection of any finite number of open sets is open.

(4) The union of any collection of open sets is open.

Let X be a set and B be a collection of subsets of X. We say B is basis for a topology on X and each B_n in B is called a basis element if B satisfies the following axioms:

(1) For every $x \in X$, there exists $B \in \mathrm{B}$ such that $x \in B$.

(2) if B_1 and B_2 are in B and $x \in B_1 \cap B_2$, then there exists B_3 in B such that $x \in B_3 \subseteq B_1 \cap B_2$

Let (X, τ) be a topological space and Y be a subset of X. The subspace topology on Y is defined to be $\tau_y = \{U \cap Y / U \in \tau\}$ and Y is called a subspace of X.

A topological space X is said to be disconnected if there exists two disjoint, non-empty open subsets whose union is X. If X is not disconnected, then it is connected.

Let $A \subseteq \mathbb{R}$ with at least two distinct points. Then A is connected if and if only if A is an interval.

Let X, Y be a topological spaces and $V \subseteq Y$. If $f : X \to Y$ and V open in Y implies $f^{-1}(V)$ is open in X, then f is continuous.

A function $f : X \to Y$ is continuous if and if only if for every $x \in X$ and every open set U containing $f(x)$, there exists a neighborhood V of x such that $f(V) \subseteq U$.

Let X, Y be topological spaces. Consider a bijection function $f : X \to Y$ and its inverse $f^{-1} : Y \to X$. Then f is a homomorphism if f and f^{-1} are both continuous. If such a function exists, then X and Y are homeomorphic or topologically equivalent, in that case we write $X \cong Y$.

Let X be a topological space and let $A \subseteq X$. Then X is compact if every open cover of X has a finite subcover.

Lebesgue number lemma on $\mathbb{R}$

Let $\mathcal{O}$ be cover of the closed bounded interval $[a, b]$ by sets that are open in $\mathbb{R}$. Then there is a lebesgue number $\lambda > 0$ such that for every $x \in [a, b]$, there exists $O \in \mathcal{O}$ such that $(x - \lambda, x + \lambda) \subseteq O$

Intermediate value Theorem (IVT)

Suppose $a < b$. Let $f : [a, b] \to \mathbb{R}$ be continuous and $f(a) \quad f(b)$. Then for each point $y \in \mathbb{R}$, there exists a point $c \in [a, b]$ such that $y = f(c)$.

The identity function id : $(X, \tau) \to (X, \tau)$, given by $id(x) = x$, is continuous.

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and let $c \in \mathbb{R}$. Then $cf : \mathbb{R} \to \mathbb{R}$ defined by $(cf)(x) = c.f(x)$ is continuous.

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous. Then $(f + g) : \mathbb{R} \to \mathbb{R}$, defined by $(f + g)(x) = f(x) + g(x)$ is continuous.

Let $f : [a, b] \to \mathbb{R}$, $a < b$. If $f(a)$ and $f(b)$ have opposite signs , then there exist a point $c \in [a, b]$ such that $f(c) = 0$.

Chapter 2

One-Dimensional Case of the Brouwer Fixed Point Theorem

To visualize the One-Dimensional Brouwer Fixed-Point Theorem, imagine a unit square situated in the first quadrant of the Cartesian plane with one of its vertices at the orgin. Now, suppose we draw the diagonal connecting the bottom-left and upper-right vertices of our square. This diagonal line segment is contained in the line $y = x$. The Brouwer Fixed-Point Theorem tells us that we cannot draw a curve from the left side of the square to the right side of the square without either crossing this diagonal or picking up our pencil (Figure 1). Thinking of our curve as the graph of a continuous function, every point at which the curve crosses the diagonal $y = x$ is a fixed point.

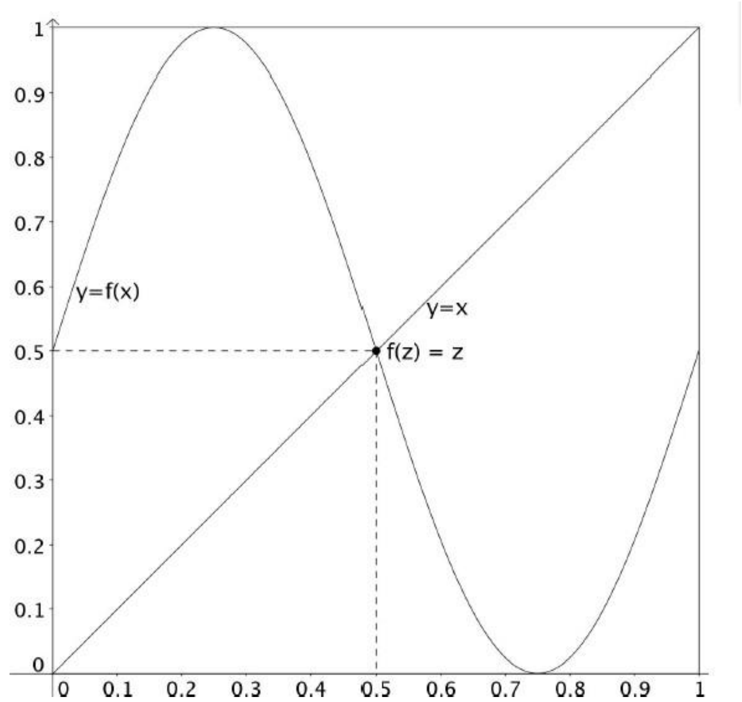

Figure 1

2.1 The Fixed Point Theorem

Theorem 2.1.1. One-Dimensional Brouwer Fixed Point Theorem

Every continuous function $f : [0, 1] \to [0, 1]$ has a fixed point.

Proof. : Let $f : [0, 1] \to [0, 1]$ be continuous. The trivial cases where $f(0) = 0$ or $f(1) = 1$ obviously satisfy the conclusion. Therefore, we will consider only functions f such that $f(0) > 0$ and $f(1) < 1$. Define a function $g : [0, 1] \to \mathbb{R}$ by

$$g(x) = x - f(x)$$

Since f(x) is continuous on [0, 1] by hypothesis, and the identity function $id(x) = x$ is continuous on [0, 1], g(x) is also continuous on [0, 1]. Now, if we evaluate g at 0 and 1, we have

$$g(0) = 0 - f(0) = -f(0) < 0$$

and,

$$g(1) = 1 - f(1) > 1 - 1 = 0.$$

Thus, $g(0)$ is negative and $g(1)$ is positive. It follows that there exists a point $z \in [0, 1]$ such that $g(z) = f(z) - z = 0$. Consequently, $f(z) = z$, which proves that z is precisely the fixed point we required □

Chapter 3

Two-Dimensional Case of the Brouwer Fixed Point Theorem

Although the two-dimensional case is not quite as apparent as the one dimensional case anyone who has ever been lost at an amusement park should be familiar with it. Consider a function f which assigns every point in the amusement park to a point on a map of the park. We can think of f as a continuous function from the park to itself . [3]. Assuming the park is topologically equivalent to a two-dimensional disk, the two-dimensional case of the Brouwer Fixed-Point Theorem states that f has a fixed point. On a map posted on the ground at a stationary location in the park, this fixed point is usually marked with a yellow star, as well as a caption which says “You are here!”

3.1 Homotopy

Definition 3.1.1. Let X and Y be a topological spaces, and let $f, g : X \to Y$ be a continuous functions. Assume $I = [0, 1]$ has the standard topology, $X \times I$ has the product topology. Then f and g are homotopic if there exists a continuous function $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. F is called a homotopy from f to g, and $f \simeq g$ denotes that f and g are homotopic.

Homotopy is a powerful tool for studying functions. We can use it to continuously “melt” one function into another (Figure 2), but more inportantly, we can use homotopy to sort continuous functions into equivalent classes, since $\simeq$ happens to be an eqivalence relation

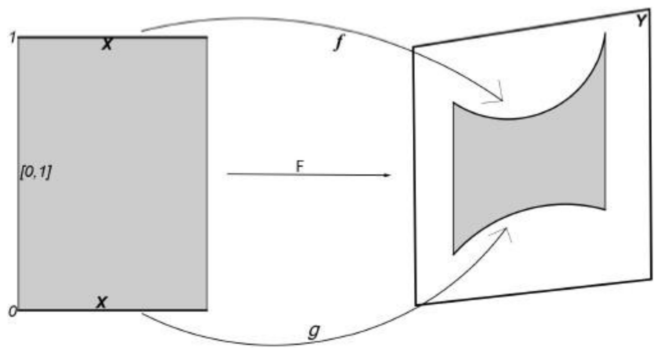

Figure 2

Definition 3.1.2. Let $C(X, Y)$ denote the set of all continuous functions $f : X \to Y$. The equivalence classes under $\simeq$ are called homotopy classes in $C(X, Y)$. The homotopy class containing a function f is denoted $[f]$