Optimal Trade Execution under Geometric Brownian Motion in the Almgren and Chriss Framework

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Abstract

With an alternative choice of risk criterion, we solve the HJB equation explicitly to find a closed-form solution for the optimal trade execution strategy in the Almgren-Chriss framework assuming the underlying unaffected stock price process is geometric Brownian motion.

1 Introduction

A problem of fundamental practical importance to algorithmic traders is how to optimally split up a large trade so as to minimize the effect of market impact. In a pair of seminal papers [2, 3], Almgren and Chriss showed how to frame this problem as a tradeoff between expected execution cost and risk; in their framework, execution costs are increasing (in fact linear) in the trading rate and the choice of risk criterion is variance (or more precisely, quadratic variation). Under these assumptions, there is a closed-form analytical solution to the optimal execution problem which turns out to be static. In particular, adaptive trade schedules that depend on the current level of the stock price are suboptimal.

A key question that then arises is to what extent the optimality of the static solution in the Almgren and Chriss framework depends on detailed modeling assumptions, specifically their assumption that the unaffected stock price process is arithmetic Brownian motion (ABM). In particular, if the true underlying process is geometric Brownian motion (GBM), how suboptimal would it be to follow the static strategy derived under the ABM assumption?

Under the GBM assumption for the unaffected stock price process with variance as risk criterion, the optimal execution problem does not appear to be analytically tractable. On the other hand, Peter Forsyth and his collaborators [8, 9] have investigated this version of the problem by solving the HJB equation numerically. They find that under the GBM assumption, the optimal trading rate increases with the stock price and that the cost-risk efficient frontier is almost identical under ABM and GBM assumptions. Thus, if the true underlying process is

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GBM, the static solution obtained by solving the problem under the ABM assumption is not very suboptimal.

In this paper, by adopting a different risk criterion, we find that we can solve the optimal execution problem under geometric Brownian motion analytically, obtaining an explicit closed-form expression for the trading rate. We are then able to make precise statements about properties of the optimal strategy under the GBM assumption, and explicitly compare the optimal strategy under GBM with the optimal strategy for the risk criterion under ABM.

2 The optimal trade execution problem

We use the following continuous-time formulation of the market impact model proposed by Almgren and Chriss [2, 3] and Almgren [1]. We assume that the number of shares in the trader's portfolio is described by an absolutely continuous trajectory $t \mapsto x_t$. Given this trading trajectory, the price at which transactions occur is

$$\widetilde{S}_t = S_t + \eta\, \dot{x}_t + \gamma\,(x_t - x_0) \tag{2.1}$$

where η and γ are constants and S_t is the unaffected stock price process. The term $\eta\,\dot{x}_t$ corresponds to the *temporary* or *instantaneous impact* of trading $\dot{x}_t\,dt$ shares at time t and only affects this current order. The term $\gamma(x_t - x_0)$ corresponds to the *permanent price impact* that has been accummulated by all transactions until time t. As mentioned earlier, in the original Almgren-Chriss model, the unaffected stock price follows a Bachelier process, or arithmetic Brownian motion (ABM),

$$S_t = S_0(1 + \sigma\, W_t), \tag{2.2}$$

where W is a standard Brownian motion. Here we will consider the case in which S is a geometric Brownian motion (GBM),

$$S_t = S_0 e^{\sigma\, W_t - \frac{1}{2}\,\sigma^2\, t}. \tag{2.3}$$

Let us now consider a *trade execution strategy* in which an initial long or short position of X shares is liquidated by time T. The asset position of the trader at time t is described by an absolutely continuous and adapted curve x_t satisfying the boundary condition $x_0 = X$ and $x_T = 0$. In such a strategy, $\dot{x}_t\,dt$ shares are bought at price $\widetilde{S}_t$ at each time t. Thus, the *costs* arising from the strategy x are

$$\begin{aligned} \mathcal{C}(x) \;&:=\; \int_0^T \widetilde{S}_t \dot{x}_t\,dt = \int_0^T S_t \dot{x}_t\,dt + \eta \int_0^T \dot{x}_t^2\,dt + \gamma \int_0^T x_t \dot{x}_t\,dt + \gamma X^2 \\ &=\; -XS_0 - \int_0^T x_t\,dS_t + \eta \int_0^T \dot{x}_t^2\,dt + \frac{\gamma}{2} X^2. \end{aligned}$$

The *optimal trade execution problem* consists in minimizing a certain objective function, which may involve both cost and risk terms, over the class of admissible trading strategies x with side conditions $x_0 = X$ and $x_T = 0$.

The easiest case corresponds to taking the expected costs,

$$\mathbb{E}[\,\mathcal{C}(x)\,] = -XS_0 + \frac{\gamma}{2} X^2 + \eta\, \mathbb{E}\Big[\int_0^T \dot{x}_t^2\,dt\Big], \tag{2.4}$$

as cost criterion. In this case, which was first considered by Bertsimas and Lo [6] in a discrete-time setting, a simple application of Jensen's inequality shows that the unique optimal strategy x^* is characterized by having the constant trading rate

$$\dot{x}_t^* = -\frac{X}{T},$$

regardless of the choice of the unaffected price process. When, as is usually assumed in practice, time is parameterized in volume time, such a constant trading rate corresponds to a VWAP strategy, where VWAP stands for volume-weighted average price.

2.1 An alternative risk criterion

In this paper, we compute optimal strategies for a new risk criterion. Specifically, we choose to quantify the risk associated with a trading strategy as the time-averaged value-at-risk (VaR) associated with the P&L of the position. This choice of risk criterion closely mimics the practice at investment banks of imposing a daily risk capital charge on trading portfolios proportional to value-at-risk (VaR). Moreover, in contrast to VaR itself, which is a time-inconsistent risk measure, time-averaged VaR is time-consistent. At each point in time, we will compute the value-at-risk associated with the position $x_t \geq 0$. Under the GBM assumption, this value-at-risk is proportional to $x_t S_t$. Thus, under the GBM assumption, we choose our risk term to be the expectation of the time-average

$$\widetilde{\lambda} \int_0^T x_t \, S_t \, dt \tag{2.5}$$

of the risk measurements taken until the liquidation of our position. To solve the optimal liquidation problem, we study the minimization of an objective function of the form

$$\mathbb{E}[\,\mathcal{C}(x)\,] + \widetilde{\lambda}\,\mathbb{E}\Big[\int_0^T x_t S_t \, dt\Big] = -XS_0 + \frac{\gamma}{2}X^2 + \mathbb{E}\Big[\eta \int_0^T \dot{x}_t^2 \, dt + \widetilde{\lambda} \int_0^T x_t S_t \, dt\Big] \tag{2.6}$$

for some constant $\widetilde{\lambda} > 0$. More precisely, for $\lambda := \widetilde{\lambda}/\eta$ we will consider the problem

$$\text{minimize } \mathbb{E}\Big[\int_0^T \big(\dot{x}_t^2 + \lambda \, x_t \, S_t\big) \, dt\Big] \text{ over all strategies } x \in \mathcal{X}(T, X), \tag{2.7}$$

where $\mathcal{X}(T, X)$ consists of all adapted and absolutely continuous strategies that satisfy the side conditions $x_0 = X > 0$ and $x_T = 0$ and the integrability conditions $\int_0^T \dot{x}_t^2 \, dt < \infty$ P-a.s. and $\mathbb{E}\big[\int_0^T |x_t S_t| \, dt\big] < \infty$.

Remark 2.1 (Permanent price impact). The choice (2.5) of risk term corresponds to approximating the time average of value-at-risk (VaR), neglecting the effect on the stock price of permanent impact. This approximation is therefore valid when the permanent impact coefficient γ is either zero or small. In Section 3.1 we will add another risk term in (2.7) so as to take care of nonvanishing permanent impact.

Remark 2.2 (Risk measures alternative to VaR). The general form (2.5) of our risk term can also be motivated when risk is monitored by means of an alternative risk measure ρ such as the *mean-standard deviation risk measure*,

$$\rho(Y) = \mathbb{E}[-Y] + \alpha\sqrt{\mathbb{E}[Y^2] - \mathbb{E}[Y]^2},$$

or coherent risk measures such as *average value-at-risk* (AVaR, also called *conditional value-at-risk* (CVaR) or *expected shortfall* (ES)); see [5], [7]. The only property ρ needs to satisfy is positive homogeneity,

$$\rho(cY) = c\,\rho(Y) \qquad \text{for } c \geq 0.$$

With such a risk measure ρ we will measure the risk that the P&L of the currently held asset position would incur after a certain time span t_0 has passed. Thus, when $x_0 \geq 0$ shares of the risky asset are held at time $t = 0$, the P&L is $x_0(S_{t_0} - S_0)$, and the risk will be assessed as

$$\rho(x_0(S_{t_0} - S_0)) = x_0\rho(S_{t_0} - S_0) = x_0 S_0 \rho(e^{\sigma W_{t_0} - \frac{1}{2}\sigma^2 t_0} - 1).$$

When the risk of $x_t \geq 0$ is assessed at a later time $t > 0$ by means of the standard time-shift ρ_t of the risk measure ρ, we obtain

$$\rho_t(x_t(S_{t+t_0} - S_t)) = x_t\rho_t(S_{t+t_0} - S_t) = x_t S_t \rho(e^{\sigma W_{t_0} - \frac{1}{2}\sigma^2 t_0} - 1). \tag{2.8}$$

Thus, the risk of $x_t \geq 0$ is assessed as $x_t\, S_t\, \widetilde{\lambda}$, where $\widetilde{\lambda} = \rho(e^{\sigma\, W_{t_0} - \sigma^2\, t_0/2} - 1)$ is independent of t. Taking a time average thus also yields a risk term of the form (2.5). When taking ρ specifically as value-at-risk at confidence level $\alpha \in (0, 1)$,

$$\rho(Y) = \mathrm{VaR}_\alpha(Y) = \inf\big\{m \in \mathbb{R} \,|\, P[-Y \leq m] \geq \alpha\big\},$$

we find that

$$\widetilde{\lambda} = \rho(e^{\sigma W_{t_0} - \frac{1}{2}\sigma^2 t_0} - 1) = \mathrm{VaR}_\alpha(e^{\sigma W_{t_0} - \frac{1}{2}\sigma^2 t_0} - 1) = 1 - e^{-\sigma\sqrt{t_0}\Phi^{-1}(\alpha) - \sigma^2 t_0/2}, \tag{2.9}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$.

Remark 2.3 (The risk term for arithmetic Brownian motion). Let us discuss how our risk criterion changes when we use ABM, $S_t = S_0(1 + \sigma W_t)$, instead of GBM. In this case, the P&L of the position $x_t \geq 0$ is $x_t(S_{t+t_0} - S_t) = x_t S_0 \sigma W_{t_0}$. Instead of (2.8) we hence get

$$\rho_t(x_t(S_{t+t_0} - S_t)) = x_t S_0 \rho(\sigma W_{t_0}). \tag{2.10}$$

The risk term thus becomes

$$\widehat{\lambda} S_0 \int_0^T x_t\, dt, \tag{2.11}$$

where $\widehat{\lambda} = \sigma\rho(W_{t_0})$. In the special case $\rho = \mathrm{VaR}_\alpha$, we get

$$\widehat{\lambda} = \sigma\rho(W_{t_0}) = \sigma\sqrt{t_0}\Phi^{-1}(\alpha). \tag{2.12}$$

Comparing this expression with (2.9), we see that

$$\widehat{\lambda} = \widetilde{\lambda} + O(\sigma^2 t_0) \qquad \text{when } \sigma\sqrt{t_0} \ll 1. \tag{2.13}$$

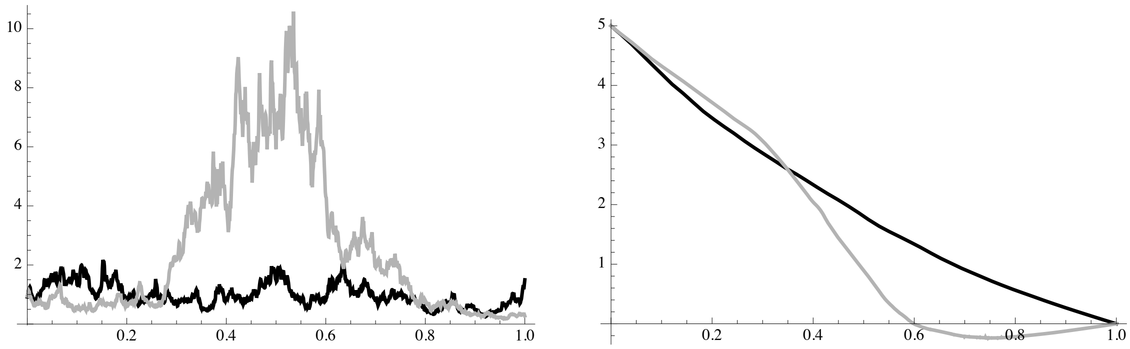

Figure 1: Two realizations of GBM and the corresponding optimal strategies.

3 Solution of the optimal execution problem

Here is our first main result:

Theorem 3.1. *The unique optimal trade execution strategy attaining the infimum in* (2.7) *is*

$$x_t^* = \Big(\frac{T-t}{T}\Big)\Big[X - \frac{\lambda T}{4}\int_0^t S_s\,ds\Big] \tag{3.1}$$

Moreover, the value of the minimization problem in (2.7) *is given by*

$$\mathbb{E}\Big[\int_0^T \big((\dot{x}_t^*)^2 + \lambda x_t^* S_t\big)\,dt\Big] = \frac{X^2}{T} + \frac{1}{2}\lambda\,T\,X\,S_0 - \frac{\lambda^2}{8\,\sigma^6}\,S_0^2\,\Big(e^{\sigma^2 T} - 1 - \sigma^2 T - \frac{1}{2}\,\sigma^4\,T^2\Big). \tag{3.2}$$

Remark 3.1. Note that, as illustrated in Figure 2, when selling stock the strategy x^* in (3.1) is *aggressive in-the-money* in the sense that it involves trading at a higher (slower) speed $v_t^* = -\dot{x}_t^*$ when stock prices increase (decrease). In particular, if the stock price becomes very high, it can be optimal to liquidate completely before time T. In this case, the optimal strategy (3.1) can actually become negative (see Figure 1). Such negative values of the strategy would be problematic since the interpretation of (2.5) as a risk term would be lost. But, as discussed in Section 4, a sign change of the optimal strategy is highly unlikely to happen in practice with reasonable parameters. The minimization problem (2.7) with strategies constrained to stay nonnegative will be solved in a forthcoming paper.

Theorem 3.1 relies on a formulation of the minimization problem (2.7) within the framework of stochastic optimal control. To this end, we parameterize strategies x by their rate of trading $v_t := -\dot{x}_t$ and introduce the class $\mathcal{V}(T,X)$ of all progressively measurable processes $(v_t)_{0\le t\le T}$ for which

$$x_t^v := X - \int_0^t v_s\,ds, \qquad 0 \le t \le T, \tag{3.3}$$

belongs to $\mathcal{X}(T,X)$. The value function $C(T,X,S)$ of our problem can then be expressed as

$$C(T,X,S_0) = \inf_{v\in\mathcal{V}(T,X)} \mathbb{E}\Big[\int_0^T v_t^2\,dt + \lambda\int_0^T x_t^v S_t\,dt\Big], \tag{3.4}$$

where we assume that the starting point S_0 of S is deterministic. Due to the state dependence of the class $\mathcal{V}(T,X)$ of admissible control processes, the problem (3.4) is a *finite-fuel control problem.*

Heuristic arguments suggest that $C(T,X,S)$ should satisfy the following Hamilton-Jacobi-Bellman (HJB) PDE:

$$C_T = \frac{1}{2}\sigma^2 S^2 C_{SS} + \lambda S X + \inf_{v\in\mathbb{R}}(v^2 - v\,C_X). \tag{3.5}$$

Indeed, assuming that C is sufficiently smooth, dynamic programming suggests that

$$C(T-t, x_t^v, S_t) + \int_0^t v_s^2\,ds + \lambda\int_0^t x_s^v S_s\,ds$$

should be a submartingale for every $v\in\mathcal{V}(T,X)$ and a martingale as soon as v is optimal. Thus, an application of Itô's formula suggests (3.5). In addition, the fuel constraint $\int_0^T v_t\,dt = +X$ required from strategies in $\mathcal{V}(T,X)$ suggests that the value function C should satisfy a singular initial condition of the form

$$\lim_{T\downarrow 0} C(T,X,S) = \begin{cases} 0 & \text{if } X = 0,\\ +\infty & \text{if } X \neq 0.\end{cases} \tag{3.6}$$

The intuitive explanation for this initial condition is that a state with a nonzero asset position with no time left for its liquidation means that the liquidation task has not been performed, so that this state should receive an infinite penalty. See also [12] for similar effects in exponential-utility maximization for order execution. As a first step in proving Theorem 3.1, one can check easily that the right-hand side of (3.2) solves the singular initial value problem (3.5), (3.6):

Lemma 3.1. *The function*

$$C^*(T,X,S) = \frac{X^2}{T} + \frac{1}{2}\lambda T X S - \frac{\lambda^2}{8\sigma^6}S^2\Big(e^{\sigma^2 T} - 1 - \sigma^2 T - \frac{1}{2}\sigma^4 T^2\Big), \tag{3.7}$$

is a classical solution of the singular initial value problem (3.5), (3.6). *Moreover, the strategy* x^* *defined in* (3.1) *satisfies the side conditions* $x_0^* = X$ *and* $x_T^* = 0$. *Its trading rate* $v_t^* = -\dot{x}_t^*$ *is such that*

$$\begin{aligned} v_t^* &= \frac{x_t^*}{T-t} + \frac{1}{4}\lambda S_t (T-t) = \frac{1}{2}C_X^*(T-t, x_t^*, S_t)\\ &= \arg\min\big\{v^2 - v\,C_X^*(T-t,x_t^*,S_t)\,\big|\,v\in\mathbb{R}\big\}. \end{aligned} \tag{3.8}$$

The proof of this lemma is straightforward. It allows us to prove Theorem 3.1 by means of the following verification argument.

Proof of Theorem 3.1: We first show that

$$C^*(T,X,S_0) \le \mathbb{E}\Big[\int_0^T \big(v_t^2 + \lambda\, x_t^v\, S_t\big)\, dt\Big] \tag{3.9}$$

for any given $v \in \mathcal{V}(X,T)$, where C^* is as in (3.7). To this end, we may clearly assume that the right-hand side of (3.9) is finite. Then in particular $\mathbb{E}\big[\int_0^T v_s^2\, ds\big] < \infty$ since the admissibility of v includes the the integrability condition $\mathbb{E}\big[\int_0^T |x_t^v S_t|\, dt\big] < \infty$. Note that the fuel constraint $\int_0^T v_s\, ds = X$ and Jensen's inequality imply that in addition

$$|x_t^v| = \Big|\int_t^T v_s\, ds\Big| \le \sqrt{(T-t)\int_0^T v_s^2\, ds} \in L^2(\mathbb{P}). \tag{3.10}$$

We next define

$$\tau_n := \inf\Big\{0 \le t \le T \,\Big|\, \int_0^t v_s^2\, ds \ge n\Big\} \wedge T$$

with the convention that $\inf\emptyset = +\infty$. Then (3.10) yields that

$$|x_{t\wedge\tau_n}^v| \le \sqrt{n(T-t)}. \tag{3.11}$$

For $0 < t < T$, Itô's formula and the fact that C^* satisfies the HJB equation (3.5) yield that

$$\begin{aligned} C^*(T - t\wedge\tau_n, x_{t\wedge\tau_n}^v, S_{t\wedge\tau_n}) - C^*(T,X,S_0) &= \int_0^{t\wedge\tau_n} C_S^*(T-s, x_s^v, S_s)\, dS_s \\ + \int_0^{t\wedge\tau_n} \Big[-C_T^*(T-s,x_s^v,S_s) - v_s C_X^*(T-s,x_s^v,S_s) &+ \frac{1}{2}S_s^2 C_{SS}^*(T-s,x_s^v,S_s)\Big]\, ds \\ \ge \int_0^{t\wedge\tau_n} C_S^*(T-s,x_s^v,S_s)\, dS_s - \int_0^{t\wedge\tau_n} &\big[v_s^2 + \lambda\, x_s^v\, S_s\big]\, ds. \end{aligned} \tag{3.12}$$

Differentiating (3.7), we have

$$C_S^*(T-s,x_s^v,S_s) = \frac{\lambda}{2}(T-s)x_s^v - \frac{\lambda^2}{4\sigma^2}S_s\Big(e^{\sigma^2(T-s)} - 1 - \sigma^2(T-s) - \frac{1}{2}\sigma^4(T-s)^2\Big).$$

By (3.11), the absolute value of the first term on the right is bounded by $\frac{\lambda}{2}\sqrt{n}(T-s)^{3/2}$ for $s \le t\wedge\tau_n$, and so $\int_0^{t\wedge\tau_n} C_S^*(T-s,x_s^v,S_s)\, dS_s$ is a true martingale. Taking expectations in (3.12) thus yields

$$C^*(T,X,S_0) \le \mathbb{E}\Big[C^*(T-t\wedge\tau_n, x_{t\wedge\tau_n}^v, S_{t\wedge\tau_n})\Big] + \mathbb{E}\Big[\int_0^{t\wedge\tau_n} \big(v_s^2 + \lambda x_s^v S_s\big)\, ds\Big]. \tag{3.13}$$

It follows from our assumptions on v that

$$\lim_{n\uparrow\infty}\lim_{t\uparrow T}\mathbb{E}\Big[\int_0^{t\wedge\tau_n} \big(v_s^2 + \lambda x_s^v S_s\big)\, ds\Big] = \mathbb{E}\Big[\int_0^T \big(v_s^2 + \lambda x_s^v S_s\big)\, ds\Big].$$

Next, (3.10) yields that

$$\big|C^*(T-t\wedge\tau_n, x^v_{t\wedge\tau_n}, S_{t\wedge\tau_n})\big| \le \int_0^T v_s^2\,ds + \frac{\lambda}{2}T^{3/2}\sqrt{\int_0^T v_s^2\,ds}\max_{0\le s\le T}|S_s| + C\left(1+\max_{0\le s\le T}|S_s|^2\right),$$

where C is a suitable constant. Again by (3.10), the right-hand side is integrable and we obtain that

$$\lim_{n\uparrow\infty}\lim_{t\uparrow T}\mathbb{E}\Big[\,C^*(T-t\wedge\tau_n, x^v_{t\wedge\tau_n}, S_{t\wedge\tau_n})\,\Big] = \mathbb{E}\Big[\,\lim_{t\uparrow T}C^*(T-t, x^v_t, S_T)\,\Big] = 0.$$

Hence, (3.13) implies our claim (3.9).

Our next goal is to show that we get an equality in (3.9) if we take $v = v^*$, where v^* is as in (3.8) for x^* as in (3.1). First, however, we need to show that $v^* \in \mathcal{V}(T, X)$. But this is easy since

$$|x_t^*| \le X\left(\frac{T-t}{T}\right) + \frac{\lambda}{4}(T-t)t\max_{0\le s\le T}|S_s| \le X\left(\frac{T-t}{T}\right)^2 + \frac{\lambda T^2}{16}\max_{0\le s\le T}|S_s|.$$

It thus follows that $\mathbb{E}\big[\int_0^T|x_t^*\,S_t|\,dt\,\big] < \infty$ and in turn that $v^* \in \mathcal{V}(T, X)$.

We still need to show that the right-hand side of (3.9) is finite for v^*. To this end, we divide (3.1) by $T-t$ and thus see that $\sup_{t\le T}|x_t^*|$ belongs to L^2. Using the ODE (3.8) we obtain $\mathbb{E}[\int_0^T(v_t^*)^2\,dt\,] < \infty$, which in view of the established fact $v^* \in \mathcal{V}(T, X)$ proves that the right-hand side of (3.9) is finite for v^*.

Finally, we note that (3.8) implies that we get an $\mathbb{P}$-a.s. identity in (3.12) when choosing $v = v^*$. This implies an identity in (3.13) and in turn an identity in (3.9). Therefore v^* is an optimal strategy. In fact, v^* is the only strategy for which an identity in (3.12) holds $\mathbb{P}$-a.s. Therefore the inequality in (3.13) must be strict when $v \ne v^*$ and t and n are sufficiently large. This shows that v^* is indeed the only optimal strategy.

3.1 Including the effect of permanent impact

We now study the problem

$$\text{minimize } \mathbb{E}\Big[\int_0^T\left(\dot{x}_t^2+\kappa^2x_t^2+\lambda\,x_t\,S_t\right)dt\,\Big] \text{ over } x\in\mathcal{X}(T,X), \tag{3.14}$$

where κ is a positive constant. This problem arises, for instance, when instead of the unaffected stock price S_t we consider a stock price

$$\widehat{S}_t = S_t + \gamma(x_t - x_0)$$

including the effect of permanent price impact when computing time-averaged VaR in (2.5). We have the following result.

Theorem 3.2. *The unique optimal strategy for* (3.14) *is*

$$x_t^* = \sinh\big((T-t)\kappa\big)\left[\frac{X}{\sinh\big(T\kappa\big)} - \frac{\lambda}{2\kappa}\int_0^t \frac{S_s}{1+\cosh\big((T-s)\kappa\big)}\,ds\right] \tag{3.15}$$

Moreover, the value of the minimization problem is given by

$$\begin{aligned}&\mathbb{E}\Big[\int_0^T \big((\dot{x}_t^*)^2 + (\kappa x_t^*)^2 + \lambda x_t^* S_t\big)\,dt\Big] \qquad (3.16)\\ &\quad= \kappa X^2 \coth\big(T\kappa\big) + \frac{\lambda X S_0}{\kappa}\tanh\Big(\frac{T\kappa}{2}\Big) - \frac{\lambda^2 S_0^2 e^{\sigma^2 T}}{4\kappa^2}\int_0^T \Big[\tanh\Big(\frac{t\kappa}{2}\Big)\Big]^2 e^{-\sigma^2 t}\,dt.\end{aligned}$$

Remark 3.2. The rightmost integral in (3.16) cannot easily be computed in explicit form. By means of a computer algebra system such as Mathematica it is possible to obtain a lengthy representation in terms of hypergeometric functions. ♢

The strategy of proving Theorem 3.2 is similar to proving Theorem 3.1. We parameterize strategies x again by their rate of trading $v_t := -\dot{x}_t$ belonging to the class $\mathcal{V}(T,X)$. The value function $\Gamma(T,X,S)$ of our problem can then be expressed as

$$\Gamma(T,X,S_0) = \inf_{v\in\mathcal{V}(T,X)} \mathbb{E}\Big[\int_0^T v_t^2 + (\kappa x_t^v)^2 + \lambda x_t^v S_t\,dt\Big]. \tag{3.17}$$

Similar heuristic arguments as above suggest that $\Gamma(T,X,S)$ should satisfy the following HJB equation:

$$\Gamma_T = \frac{1}{2}\sigma^2 S^2 \Gamma_{SS} + \kappa^2 X^2 + \lambda S X + \inf_{v\in\mathbb{R}}(v^2 - v\Gamma_X) \tag{3.18}$$

with singular initial condition

$$\lim_{T\downarrow 0}\Gamma(T,X,S) = \begin{cases} 0 & \text{if } X=0,\\ +\infty & \text{if } X\neq 0.\end{cases} \tag{3.19}$$

A tedious, though straightforward, computation then shows that this singular initial value problem is solved by

$$\begin{aligned}&\Gamma^*(T,X,S) :=\\ &\quad \kappa X^2\coth\big(T\kappa\big) + \frac{\lambda X S}{\kappa}\tanh\Big(\frac{T\kappa}{2}\Big) - \frac{\lambda^2 S^2 e^{\sigma^2 T}}{4\kappa^2}\int_0^T\Big[\tanh\Big(\frac{t\kappa}{2}\Big)\Big]^2 e^{-\sigma^2 t}\,dt.\end{aligned}$$

Moreover, the strategy x^* defined in (3.15) satisfies the side conditions $x_0^* = X$ and $x_T^* = 0$. Its trading rate $v_t^* = -\dot{x}_t^*$ is such that

$$\begin{aligned} v_t^* &= x_t^*\kappa\coth\big(\kappa(T-t)\big) + \frac{\lambda S_t}{2\kappa}\cdot\frac{\sinh\big(\kappa(T-t)\big)}{1+\cosh\big(\kappa(T-t)\big)}\\ &= x_t^*\kappa\coth\big(\kappa(T-t)\big) + \frac{\lambda S_t}{2\kappa}\tanh\Big(\frac{\kappa(T-t)}{2}\Big)\\ &= \frac{1}{2}\,\Gamma_X^*(T-t,x_t^*,S_t) \qquad (3.20)\\ &= \arg\min\big\{v^2 - v\Gamma_X^*(T-t,x_t^*,S_t)\,|\,v\in\mathbb{R}\big\}.\end{aligned}$$

Having noticed these facts, one can repeat the verification argument from the proof of Theorem 3.1 with the appropriate modifications to obtain a proof for Theorem 3.2. The details are left to the reader.

Remark 3.3. As before, the optimal strategy involves trading at a higher (slower) rate $v_t^* = -\dot{x}_t^*$ when stock prices increase (decrease).

4 Sensitivity of risk-adjusted cost to strategy

Motivated by the form (3.8) of the optimal strategy, consider the one-parameter family of trading strategies with trading rate

$$v^\beta(t) = \frac{x_t}{T-t} + \beta\, S_t\,(T-t)$$

The choice $\beta = \lambda/4$ then corresponds to the optimal strategy.

Neglecting permanent impact (or alternatively with $\gamma = 0$), a straightforward computation gives the risk-adjusted expected cost (as of time $t = 0$) of this strategy as

$$\begin{aligned} C^\beta(T, X, S_0) &= \int_0^T dt\,\left\{\mathbb{E}\left[\left(\frac{x_t}{T-t} + \beta\,(T-t)\,S_t\right)^2\right] + \lambda\,\mathbb{E}[S_t\,x_t]\right\} \\ &= \frac{X^2}{T} + \frac{\lambda}{2}\,S_0\,X\,T + \left(\beta^2 - \beta\,\frac{\lambda}{2}\right)\frac{2\,S_0^2}{\sigma^6}\left\{e^{\sigma^2 T} - 1 - \sigma^2 T - \frac{1}{2}\sigma^4 T^2\right\}. \end{aligned} \tag{4.1}$$

which is minimized when

$$\beta = \frac{\lambda}{4} =: \beta^\star$$

consistent with our earlier claim. The choice $\beta = 0$ corresponds to a VWAP execution.

As discussed earlier in Remark 2.3, under the ABM assumption, the risk term corresponding to time-average VaR becomes

$$\widehat{\lambda}\int_0^T x_t\,S_0\,dt \tag{4.2}$$

and so to solve the optimal liquidation problem, we must study the minimization of a slightly different objective function of the form

$$\text{minimize}\ \mathbb{E}\Big[\int_0^T \big(\dot{x}_t^2 + \lambda\,x_t\,S_0\big)\,dt\Big]\ \text{over}\ x \in \mathcal{X}(T, X). \tag{4.3}$$

It is straightforward to verify that the optimal strategy that solves (4.3) is the static version of the dynamic strategy (3.8) obtained by replacing S_t with its expectation $\mathbb{E}[S_t] = S_0$, a strategy qualitatively similar to the popular Almgren-Chriss optimal strategy.

Remark 4.1. In the GBM framework, had we chosen to model price impact in the following multiplicative way instead of the additive definition (2.1):

$$\widetilde{S}_t = S_t\big[1 + \eta\,\dot{x}_t + \gamma\,(x_t - x_0)\big], \tag{4.4}$$

the costs of a strategy $x \in \mathcal{X}(T,X)$ would become

$$\mathcal{C}(x) = \int_0^T \widetilde{S}_t x_t \, dt = -X S_0 - \int_0^T x_t \, dS_t + \eta \int_0^T S_t \dot{x}_t^2 \, dt + \gamma \int_0^T S_t x_t \dot{x}_t \, dt.$$

Adding our risk term (2.5) would then lead to the problem of minimizing

$$\mathbb{E}\Big[\int_0^T S_t \big(\eta \dot{x}_t^2 + \gamma x_t \dot{x}_t + \widetilde{\lambda} x_t\big) \, dt \Big] \tag{4.5}$$

over $x \in \mathcal{X}(T,X)$. Since S is a martingale, the expectation in (4.5) is equal to

$$\mathbb{E}\Big[S_T \int_0^T \big(\eta \dot{x}_t^2 + \gamma x_t \dot{x}_t + \widetilde{\lambda} x_t\big) \, dt \Big],$$

the minimization of which is equivalent to the ABM minimization problem (4.3) (generalized to include permanent market impact). The optimal strategy is thus also static.

This motivates us to define another one-parameter family of static strategies, independent of S_t:

$$v_0^\beta(t) = \frac{x_t}{T-t} + \beta \, S_0 \, (T-t)$$

Once again, the choice $\beta = \lambda/4$ corresponds to the optimal strategy.

Another straightforward computation gives the expected cost (as of time $t = 0$) of this strategy as

$$\begin{aligned} C_0^\beta(T,X,S_0) &= \int_0^T dt \left\{ \mathbb{E}\left[\left(\frac{x_t}{T-t} + \beta \, (T-t) \, S_0 \right)^2 \right] + \lambda \, S_0 \, \mathbb{E}[x_t] \right\} \\ &= \frac{X^2}{T} + \frac{\lambda}{2} \, S_0 \, X \, T + \left(\beta^2 - \beta \, \frac{\lambda}{2} \right) \, S_0^2 \, \frac{T^3}{3}. \end{aligned} \tag{4.6}$$

Now define the characteristic timescale[1]

$$T^\star = \sqrt{\frac{X}{\beta^\star \, S_0}} = \sqrt{\frac{4 \, X}{\lambda \, S_0}}$$

and choose the liquidation time T to be $T^\star$. Further define $\theta := \beta/\beta^\star$. Substitution into (4.6) gives

$$C_0^\beta(T^\star, X, S_0) = \frac{X^2}{T^\star} \left(3 + \frac{\theta^2 - 2\,\theta}{3} \right) \tag{4.7}$$

which is minimized when $\theta = 1$ (*i.e.* when $\beta = \beta^\star$).

With $T = T^\star$, the optimal trading rate under ABM becomes

$$v^A(t) = \frac{x_t}{T-t} + \frac{X}{T^2} \, (T-t) = \frac{2\,X}{T} \, \big(1 - \frac{t}{T}\big) \tag{4.8}$$

[1] $T^\star$ is a critical point of the cost function and so for T close to $T^\star$ the cost is approximately independent of T.

and the optimal trading rate under GBM becomes

$$v^G(t) = \frac{x_t}{T-t} + \frac{X}{T^2}\,\frac{S_t}{S_0}\,(T-t) \tag{4.9}$$

(here we assume implicitly that we use the same parameter λ for both GBM and ABM, which can be justified by (2.13)).

We observe that with this particular choice of liquidation time, the cost function $C_0^\beta(T^\star, X, S_0)$ has a sharp minimum with respect to the parameter θ which parameterizes the sensitivity of the trading rate to the stock price. In particular, the risk-adjusted cost of a VWAP execution ($\theta = 0$) is 1/9 (or over 10%) greater than the risk-adjusted cost of an optimal execution.

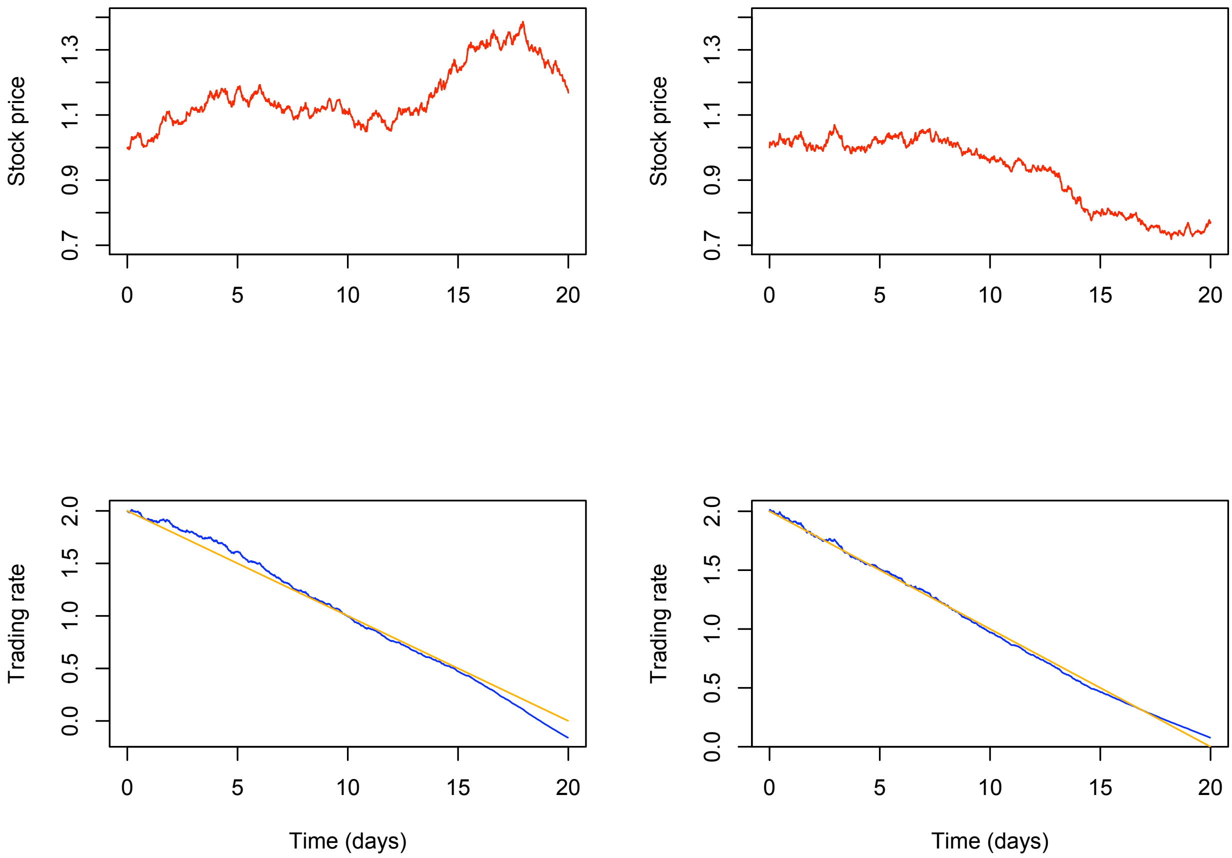

Figure 2: The upper plots show rising and falling stock price scenarios respectively; the trading period is 20 days and daily volatility is 4%. The lower plots show the corresponding optimal trading rates from (4.8) and (4.9); the optimal trading rate under ABM is in gray and the optimal trading rate under GBM is in black. Even with such extreme parameters and correspondingly extreme changes in stock price, the differences in optimal trading rates are minimal.

Moreover, for reasonable values of $\sigma^2 T \ll 1$, there is almost no difference in expected costs and risks between the optimal strategies under ABM and GBM assumptions. We can see this directly by comparing (4.1) and (4.6) and noting that $\mathbb{E}[S_t^2] = S_0^2\,(1+\epsilon)$ with $\epsilon \ll 1$. Intuitively, although the optimal strategy is stock price-dependent under GBM assumptions but not under ABM assumptions, when $\sigma^2 T \ll 1$, the difference in optimal frontiers is tiny because the stock-price S_t cannot diffuse very far away from S_0 in the short time available. Equivalently, as in Figure 2, there can only be a small difference in optimal trading rates under the two assumptions.

In particular, since the expected costs and risks are almost identical under ABM and GBM, so must be the optimal frontiers, confirming the observations of [8, 9]. This observation also formally justifies the original choice of Almgren and Chriss [2, 3] of ABM as a modeling assumption. Another corollary is that with $\sigma^2 T \ll 1$, the cost function $C^\beta(T^\star, X, S_0)$ associated with the one-parameter family of strategies under the GBM assumption is also very sensitive to the parameter θ.

From (4.8) and (4.9), we see that the typical deviation of the optimal trading rate $v^G(t)$ under GBM from its static counterpart is roughly

$$v^G(t) - v^A(t) = \frac{X}{T^2}\,\frac{S_t - S_0}{S_0}\,(T-t) \sim \frac{X}{T}\,\sigma\,\sqrt{T}.$$

which is tiny, again consistent with Figure 2. On the other hand, in practice, a typical aggressive-(passive-)in-the-money strategy may double(halve) the trading rate if the stock price goes in-the-money by more than one standard deviation, a strategy that is extremely suboptimal in this framework.

The sharp minimum in (4.7) thus appears to contrast with the numerical results of [8] where the risk criterion is variance but is consistent with the numerical results of [9] where the risk criterion is quadratic variation.

4.1 Numerical example

In Figure 2, we plot the optimal trading rates in two scenarios, one where the stock price is rising and one where the stock price is falling. We assume that liquidation of a stock position takes place over 20 trading days and that the underlying stock has a *daily* volatility of 4%. The price of risk λ is chosen such that the characteristic time $T^\star$ is 20 days. Obviously these are extreme parameters that are only rarely applicable in practice. It is immediately apparent from the plots of the optimal trading strategies that the difference between the optimal strategy under GBM and ABM assumptions is minimal, even with these extreme parameters. With more reasonable volatilities and trading horizons, the differences in practice between the optimal trading rates is negligible.

5 Summary and conclusions

In summary, with an alternative financially-motivated choice of risk criterion, we have found an explicit closed-form solution to the optimal execution problem in the Almgren and Chriss framework when the underlying stock price follows geometric Brownian motion. We find that if the liquidation horizon is chosen optimally, the minimum of the cost function is rather sharp in contrast to the results of [8] with a pre-commitment mean-variance objective function but consistent with the results of [9] with a mean-quadratic-variation objective function. One practical consequence of this is that, in the Almgren and Chriss framework, trading at a rate significantly different from the optimal rate is significantly suboptimal.

We also find that for even quite extreme choices of volatility and trading horizon, there is very little difference between the optimal trading rates under ABM and GBM assumptions and thus the difference in efficient frontiers is also very small.

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