MRM Converge - Problem Set 11

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Problem (Number Theory). (*Problem submitted by Jitendra Thoury*) Split $\{1, 2, \cdots, 2n\}$ randomly into two subsets X and Y, each containing n integers. Order the elements of X in increasing order as $x_1 < x_2 < \cdots < x_n$ and elements of Y in decreasing order as $y_1 > y_2 > \cdots > y_n$. What are the possible values that $|x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n|$ can take?

Problem (Probability). We uniformly choose two points inside a square of side length 1cm. Let the points be (x_1, y_1) and (x_2, y_2). The distance between these two points is given by $d = |(x_1 - x_2)| + |(y_1 - y_2)|$. What is the expected value of d?

Problem (Number Theory). Suppose we have infinitely many cards, each with some natural number written on it. Given any $n \in \mathbb{N}$, the number of cards which have a divisor of n written on it is exactly equal to n. Which numbers cannot appear on any card?

Problem (Number Theory). What is the value of $(-1)^{\sqrt{2}}$?

Solution - Problem Set 10

The trick to solve all the problems is **Recurrence**.

Solution (Probability). Let f_k represent the expected number of cards picked before the game ends when there are k cards at the start (assuming they include all four aces). Suppose there are n cards at the start then the following two cases can arise:

- The first draw is an Ace, probability $= \frac{4}{n}$
- The first draw is not an ace, probability of it occurring is $\frac{n-4}{n}$ and we end up having $n-1$ cards (which contains all 4 aces)

Then,

$$f_n = \frac{4}{n} \times 1 + \frac{n-4}{n} \times (1 + f_{n-1}) = 1 + \frac{n-4}{n} f_{n-1}$$

Solving the recurrence relation gives us $f_n = \frac{n+1}{5}$. Thus, $f_{52} = 10.6$.

Solution (Probability/Programming). Let $E(n)$ denote the expected number of cars that will be left after infinite time when we begin with n cars.

Then, the probability that the first car will be the slowest car of the lot is $\frac{1}{n}$. When the first car is the slowest one, the expected number of cars will be $1 + E(n-1)$, while in the other case, the number of cars will be $E(n-1)$, i.e the first car will be left alone in the former case while it will hit some other car in the front and will combine with it in the latter. Thus, we get the recurrence equation : $E(n) = \frac{1}{n}(1 + E(n-1)) + \frac{n-1}{n}(E(n-1))$. Simplifying, we get $E(n) = \frac{1}{n} + E(n-1)$. Recursively solving, we get $E(n) = \sum_{i=1}^{n} \frac{1}{i}$

Similarly, the programming question can also be solved by recurrence.

Solution (Probability). Let p be the probability that amoeba population will perish with infinite amount of time. Consider time $t = 0$, there is only one amoeba and these four events can happen with equal probability (0.25)

- Amoeba population perish and probability that new amoeba population will perish $=$ 1
- Amoeba population remains same and probability that new amoeba population will perish $= p$
- Amoeba population doubles and probability that new amoeba population will perish $= p^2$
- Amoeba population triples and probability that new amoeba population will perish $=$ p^3

This implies,

$$p = 0.25 + 0.25p + 0.25p^2 + 0.25p^3$$

Above equation have three different solutions for p as $1,\ -1+\sqrt{2},\ -1-\sqrt{2}$. One can show using a formal argument (by constructing a martingale), that $p = 1$ is not a feasible solution (*Left to the reader*). Thus, $p = -1 + \sqrt{2}$ is the feasible solution.

Solution (Probability). Let the expected number of trials required to get 2 consecutive heads be x. Then, x can be written as : $x = \frac{1}{2}(1 + x) + \frac{1}{4}(2) + \frac{1}{4}(2 + x)$. Here, the three terms account for the following mutually exclusive events:

- Event 1 : The first trial resulted in a tail (T) - Game resets
- Event 2 : The first two trials resulted in HH - Game won
- Event 3 : The first two trials resulted in HT - Game resets

Solving the above equation, we get $x = 6$. Similarly, for n consecutive heads, we get the expected number to be $2^{n+1} - 2$ (*use the same logic!*).