MRM Weekly Converge - Week 6

May 31, 2019

"We share a philosophy about linear algebra : we think basis free,
but when the chips are down we close the office door
and compute with matrices like fury."
- Irving Kaplansky

Problem (Chess - Endgame). White to play and mate in exactly 3 moves.

Figure 1: White to move

Problem (Brain Teaser). (*Problem submitted by Ashwin Kumar*) There are 17 prisoners in solitary cells. There's a central living room with one light bulb; this bulb is initially off. No prisoner can see the light bulb from his or her own cell. Everyday, the warden picks a prisoner equally at random, and that prisoner visits the living room. While there, the prisoner can toggle the bulb if he or she wishes. Also, the prisoner has the option of asserting that all 100 prisoners have been to the living room by now. If this assertion is false, all 100 prisoners are shot. However, if it is indeed true, all prisoners are set free. The prisoners are allowed to get together one night in the courtyard, to discuss a plan. What plan should they agree on, so that eventually, someone will make a correct assertion?

Problem (Programming). Find the total number of numbers less than 100000 whose prime factors are less than the square root of the number.

Problem (Number Theory). Derive an expression for the number of co-prime numbers less than n as a function of n and its prime divisors.This is known as Euler Totient function. Any appeal to chinese remainder theorem must necessarily include its proof.

Solution - Week 5

Solution (Chess). *(Solution - Kousik Krishnan)*

- g8=N! b5 (Forced)
- Ne7!! Kxb4 (Forced)
- Nc6++

Solution (Combinatorics). *(Solution - Kashish Garg/ Priyadarshi Abhishek/ Chandan Kumar)* The total number of way in which the cube can be reassembled is given by $S_1 = (27!)(24^{27})$. Of which, $S_2 = (8!)(6!)(12!)(8^8)(6^6)(12^{12})(24)$ will lead to red faced cube. Assuming uniform probability, we get the result as S_2/S_1

Solution (Brain Teaser). *(Solution - Richard Harris/ Archana Tatavarti/ Kashish Garg/ Sushant Vijayan/ Priyadarshi Abhishek/ Chandan Kumar)* Take one coin from each bag and mark them so you know which bag they came from. Divide into three sets of 171, 171 and 170 coins and weigh the first against the second. If they weigh equal then one of the coins in the third set is fake, otherwise itll be in the heavier of the first two sets. Repeating this way, we have $3^5 \leq 512 \leq 3^6$. Thus, the balance must be used 6 times.

Solution (Simulation). *(Solution - Sushant Vijayan)* We use the standard spherical coordinates of r, θ & ϕ. Consider the distribution of r^3 when you have a uniform distribution over the sphere. It simply is the ratio of the volumes. Hence :

$$P(r \leq x) = P(r^3 \leq x^3) = \left(\frac{x}{R}\right)^3 \tag{1}$$

This imples that the random variable $\frac{r^3}{R^3}$ is uniformly distributed. Similarly one can prove that $\frac{\theta}{2\pi}$ is uniformly distributed. Φ is a bit more trickier but it is closely related to the solid angle of the sphere. Using such ideas one can show that $\frac{(1-cos(\Phi))}{2}$ is also uniformly distributed. Of course one can also show by conisdering their joint distributuion that under uniform probability over the sphere these these three quantities are independent. These facts now hints at the following simulation strategy:

- Simulate $U_1 \sim unif(0, 1)$. Set $r = R * U^{1/3}$.
- Simulate $U_2 \sim unif(0, 1)$ independently from U_1.Set $\theta = 2\pi U_2$.
- Simulate $U_3 \sim unif(0, 1)$ independenly from both U_1 and U_2. Set $\Phi = cos^{-1}(2U_3 - 1)$.