MRM Weekly Converge - Week 2

April 12, 2019

Mathematics is the handwriting on the human
consciousness of the very spirit of life itself
- Claude Bragdon

Problem (Chess). The position in Fig 1 was reached in a game. Here white to play and win in one move. What is the winning move and why is it feasible?

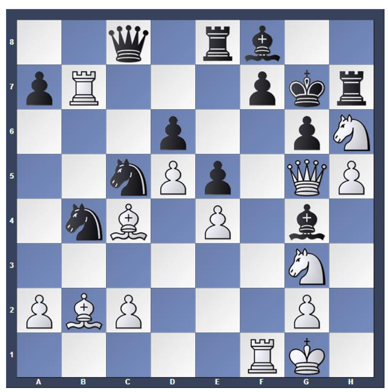

Figure 1: White to play and win in one move.

Problem (Numerical Puzzle). To divide 8,101,265,822,784 by 8, we transfer the 8 from the beginning to the end, i.e, 8,101,265,822,784 = 8 × 1012658227848. Find a number beginning with 4 that can be divided by 4 in the same manner. (Hint : The answer is a simple 6 digit number)

Problem (Probability & Stats). Given 5 i.i.d uniform random variables with support $(0, 10)$, what is the probability that their sum is less than 5?

Problem (Math - Coffin). Given a sequence of 0s and 1s, the complement of the sequence is obtained by flipping all 0s to 1s and vice-versa. Let $s_0 = 0$, consider the complement of s_0 appended to it : s_1= 01. Now define $s_n := s_{n-1} \cup s_{n-1}^c$, where the union is the append operation. Consider the decimal $0.s_n$ as $n \to \infty$. Is that decimal rational? Why? (E.g. $s_0 = 0$, $s_1 = 01$, $s_2 = 0110$, $s_3 = 01101001$, etc.)

Solution - Week 1

Solution (Programming 1). The best time complexity is O(N). In the first iteration, the max difference of two elements such that the min occurs before the max is calculated. In the next iteration the max of that is chosen.

Solution (Programming 2 - Online Median). The best time complexity is O(log(N)), where N is the number of elements that the stream had seen till time t. A simple balanced binary search tree like Red-Black tree will suit the purpose.

Solution (Puzzle - 1). The max number of queens is 8. A position is given by : a5, b3, c1, d7, e2, f8, g6, h4. Follow-up question - how many such positions are possible?

Solution (Puzzle - 2). Minimum number of locks required - $\binom{40}{30}$. For every combination of 30 people, a lock will be created and the keys will be shared with all.

Solution (Puzzle - Coffin). Convert the problem into a dual problem of minimizing the area of a triangle whose sides are greater than 1 and the sides are reciprocals of the altitude of the original triangle. A constrained optimization with these constraints reveal that the optima is obtained in an equilateral triangle. This implies the original problem is maximized by an equilateral triangle.

Solution (Stat - 1). Toss a coin twice, if the event that occurs is HT, the new random variable will take the value H. If the event that occurs is TH, the new random variable will be T. Otherwise, repeat. The probability of H will be the same as T for the new random variable.

For the other way, consider H and T as 0 and 1. We can simulate uniform random variable(Why? - Binary Representation) using this. From a uniform random variable, a biased coin can be simulated.

Solution (Stat - Non-Markov Process). Take a stochastic process with following properties:

- It is an AR2 process.
- Each random variable X_n is a bernoulli distribution.
- Assume the transition distributions $P(X_n|X_{n-1})$ and $P(X_n|X_{n-1}, X_{n-2})$ are independent of n.Construct these distributions such that they backward compatible.
- The probability distributionn of X_0 follows the stationary distribution of the one stage transition matrix.

We can show with appropriate chosen constants that this is a non Markov process with time homogeneity.

Solution (Stat - Quantile Question). Let $k \in (0,1)$ be the quantile given. Consider the random variable Y, which is defined as :

$$\begin{cases} (x-u)(k-1) & x \leq u \\ (x-u)(k) & x \geq u \end{cases}$$

Then, the function $f(X,u) = E(Y)$, is minimized when u is the k-th quartile of X. This function f, does not induce a metric. We still have not proved/disproved the existence of a metric.

Solution (Puzzle - Quant finance). Refer to theorem 4.5.1 in Shreve Vol1 for proof of the claim.