MRM Weekly Converge - Week 10

September 6, 2019

Note : All the below problems can be solved with the use of a single trick.

Problem (Probability). Given a deck of playing cards, we pick a card at random. If the card is an Ace, the game stops. Else, we continue the game with the set of remaining cards (i.e no replacement). Find the expected number of cards that you will pick before the game ends.

Problem (Probability/ Programming). In a single lane highway, we have n cars which are all travelling at different speeds and consecutive cars are separated by some distance. If a car hits another, the two cars will move together as a single car with the speed of the slower one.

- (Programming) Given the velocities of the cars (i.e $[v_1, v_2, ..., v_n]$). Find the number of cars that will be left after an infinite amount of time
- (Probability) Assuming any sequence of velocities, what is the expected number of cars that will be left after an infinite amount of time.

Problem (Probability). At any time t, if we have an amoeba, at $t+1$, the amoeba can either perish or remain the same or double or triple with equal probabilities (0.25). In the beginning there is one amoeba. Assuming independence, what is the probability that the amoeba population will perish?

Problem (Probability). Given a fair coin, let X be the random variable denoting the number of trials till we get 2 consecutive heads and Y be the random variable denoting the number of trials till we get 3 consecutive heads. (Both the events are independent of each other)

- Find the expectation of X
- Find the expectation of Y
- Find $P(X > Y)$

Solution - Week 9

Solution (Pattern Recognition). *(Solution - Ojaswee Dhayal, Archana Tattavarti, Abhishek Priyadarshi, Protyaya Halder, Seghal Ankit)*
If a power of 2 starts with 1, the next power of 2 can either start with 2 or 3. If it starts with 2, then the next power can either be 4 or 5 while if it starts with 3, the next power is 6 or 7. All powers of 2 starting with 5, 6 and 7 will have its next power of 2 to start with 1. (All this is done using the fact that in powers of 2 calculation, there cannot be two carry forwards) The possible combinations are:

- $1 \to 2 \to 4 \to 8$
- $1 \to 2 \to 4 \to 9$
- $1 \to 3 \to 6$
- $1 \to 2 \to 5$
- $1 \to 3 \to 7$

It must be noted that only the length 4 tree contains a 4. Every time there is a digit increase, it implies one of these trees have happened. This gives us a system of equations. Let there be a length 4 trees and b length 3 trees in powers of 2 from 0 to 2003. Solving $a + b = 603$ and $4a + 3b = 2004$ (including 2^0), we get $a = 195$.

Solution (Number Theory). *(Solution - Kousik Krishnan)*
Counting the powers of 2 in LHS and RHS using legendre's formula gives us the inequality that $k > \frac{(n)(n-1)}{2}$. Also, $2^{n^2} = (2^n)(2^n)...(2^n) > (2^n - 2^{n-1})(2^n - 2^{n-2})...(2^n - 2^0) > (\frac{(n)(n-1)}{2})!$. But we know that $2^{n^2} < (\frac{(n)(n-1)}{2})!\ \forall n \geq 6$ (Proof by Induction). Thus the equation cannot hold true for $n \geq 6$. Computing the other cases we get the solutions as $(0, 1), (1, 1)$ and $(3, 2)$.

Solution (Combinatorics). *(Solution - Kashish Garg, Ojaswee Dhayal, Archana Tattavarti, Abhishek Priyadarshi, Protyaya Halder, Seghal Ankit)*
$\binom{22}{10}$. Choose any set of 10 people, let there be x boys and y girls in the selection, Rather than selecting those y girls, choose the girls who have been left $(10 - y)$.

Solution (Number Theory). *(Solution - Ojaswee Dhayal, Archana Tattavarti, Abhishek Priyadarshi, Protyaya Halder, Ayan Halder)*
Let $10^{20n} = (10^n + 3)m + c$, where c is the remainder and $n = 100$ and the greatest integer less than the given number is m. Now m can be represented as $10^{19n} - 3 * 10^{18n} + 9 * 10^{17n} - ....$ This implies $(10^n + 3) * m = 10^{20n} - 3$. Each term of m is a multiple of 10 except -3^{19} Therefore, $m \equiv -3^{19} \equiv -3^3 \equiv -7 \equiv 3 \mod 10$. Hence, the unit digit is 3.