MRM Weekly Converge - Week 7

June 19, 2019

"Mathematics is a game played according to certain simple rules with meaningless marks on paper."
- David Hilbert

Problem (Chess). White to play and mate in exactly 5 moves.

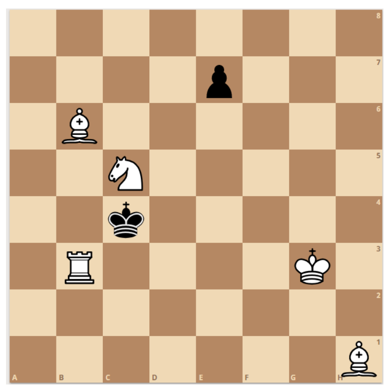

Figure 1: White to move

Problem (Statistics). We have N chocolates numbered 1 to N, where N is unknown. We randomly steal some chocolates and observe their numbers. Give an estimate of the number N from this random sequence.

Problem (Probability). Ocean and his 3 other friends are presented with the following game of chances: There are 4 boxes each containing one of their names with no repetition. Each person is allowed to open 2 boxes. An individual wins if their name is present in any one of the boxes that they open. The game is won if all the individual wins. The players are allowed to discuss a strategy before the beginning of the game. They are not allowed to discuss anytime in the middle of the game.

- What is the probability of winning the game if each individual opens the boxes at random?
- If they are only allowed to open the boxes simultaneously , what is the best strategy that maximizes the chances of winning the game?

- If each individual can sequentially open the box, what is the best strategy that maximizes the chances of winning the game?
- Consider the game with $2n$ people, where each can open n boxes. In the sequential case, is there any strategy that gives greater than 30% chance of winning the game?

Solution - Week 6

Solution (Chess).

- Nxa7+ Bxa7(forced)
- Qxc6+ bxc6(forced)
- Ba6++

Solution (Brain Teaser). *(Solution - Priyadarshi Abhishek/ Mikolaj Fido)* The team choose a leader. If a normal prisoner enters the dark living room for the first time, he lights up the room. If its already turned on, the prisoner does nothing. Only leader is entitled to turn off the light. Once a normal prisoner has turned on the light, he does not do anything for the rest of the game. When the leader turns the light off for 99th time, he has assurance that all 100 prisoners have been to living room.

Solution (Programming). *(Solution - Priyadarshi Abhishek* There are 26,612 numbers less than 100000 such that their largest prime factor is less than their square root.

Solution (Number Theory). We shall use inclusion exclusion principle to prove the result. Suppose n has three prime factors:

$$n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$$

with all exponents greater or equal to 1. Now let us give an estimate of the co-prime numbers to n. As a first estimate we can simply take all numbers $\leq$ n.Thus our first estimate of $\phi(n)$ is n. But this is clearly an overestimate as it includes all numbers divisible by p_1, p_2 and p_3. These numbers are individually given by $\frac{n}{p_1}$, $\frac{n}{p_2}$ and $\frac{n}{p_3}$. Thus the next estimate is

$$\phi(n) \approx n - \left(\frac{n}{p_1} + \frac{n}{p_2} + \frac{n}{p_3}\right)$$

But now this estimate is an underestimate because it now subtracts all numbers divisible by both p_1, p_2 twice while for $\phi(n)$ we require it to be subtracted once. So consequently we add terms like $\frac{n}{p_1 p_2}$. Thus our estimate becomes :

$$\phi(n) \approx n - \left(\frac{n}{p_1} + \frac{n}{p_2} + \frac{n}{p_3}\right) + \left(\frac{n}{p_1 p_2} + \frac{n}{p_2 p_3} + \frac{n}{p_3 p_1}\right)$$

However this estimate includes numbers which are multiples of p_1, p_2 and p_3. So our last and final estimate (since this process can't go further as n by our assumption has only three unique prime factors) is

$$\phi(n) = n - \left(\frac{n}{p_1} + \frac{n}{p_2} + \frac{n}{p_3}\right) + \left(\frac{n}{p_1 p_2} + \frac{n}{p_2 p_3} + \frac{n}{p_3 p_1}\right) - \frac{n}{p_1 p_2 p_3}$$

This maybe re-written as

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_3}\right) = n\prod_{i=1}^{3}\left(1 - \frac{1}{p_i}\right)$$

Clearly this argument could be generalised when n has k many unique prime factors and hence:

$$\phi(n) = n\prod_{i=1}^{k}\left(1 - \frac{1}{p_i}\right)$$