



A96-36386



**AIAA 96-1870**

**Thermodynamics, Phase Change,  
and Mass Transfer in Oscillatory  
Balloon Systems (Aerobots)**

R. E. Scheid, M. K. Heun,  
J. M. Cameron, and J. A. Jones

<http://robotics.jpl.nasa.gov/tasks/aerobot>

Jet Propulsion Laboratory  
California Institute of Technology  
4800 Oak Grove Drive  
Pasadena, CA 91109



**31st AIAA Thermophysics Conference**  
**June 17-20, 1996 / New Orleans, LA**

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Basic Balloon Physics</b>	<b>4</b>
2.1	Fundamental Governing Equations . . . . .	4
2.1.1	Development of the Energy Equation . . . . .	4
2.1.2	The Equation of Motion . . . . .	5
2.2	Balloon System Motion . . . . .	5
2.3	Balloon System Energy . . . . .	7
2.3.1	Energy Equations for Components of Balloon System . . . . .	8
2.3.2	Simplification of the Balloon Gas Energy Equation . . . . .	9
2.3.3	Summary of Assumptions . . . . .	10
2.3.4	A Generalized Form for the Energy Equation . . . . .	10
<b>3</b>	<b>Heat Transfer</b>	<b>11</b>
<b>4</b>	<b>Energy Conservation Equations for the Aerobot System</b>	<b>12</b>
<b>5</b>	<b>Dual-Phase Models for the Secondary Balloon System</b>	<b>12</b>
5.1	Thermodynamic Inputs . . . . .	12
5.2	Single-System Model . . . . .	13
5.3	Model with No Phase Holdup . . . . .	16
5.4	Model with Phase Holdup . . . . .	17
5.5	Summary of Dual-Phase Model . . . . .	18
<b>6</b>	<b>Summary of Equations for the Aerobot System</b>	<b>19</b>
<b>7</b>	<b>Simulation Results</b>	<b>21</b>
<b>8</b>	<b>Conclusion</b>	<b>22</b>
<b>Appendices</b>		
<b>A</b>	<b>Asymptotic Equivalence of Dual-Phase Models</b>	<b>23</b>
<b>References</b>		<b>26</b>

## Abstract

Oscillatory balloon systems are being studied to support future robotic space exploration by probes which will move up and down in a planetary atmosphere, land and explore numerous surface sites. Modeling of a reversible fluid which undergoes cyclical phase changes is essential to analyzing and simulating the oscillatory motion, but previous efforts have been limited by the inability to adequately handle these effects. This paper presents a non-equilibrium, differential-equation model for a two-balloon aerobot system including a dual-phase model for phase change **and** mass transfer. The derivation begins from fundamental governing equations for thermodynamics and motion and uses conservation principles to derive dual-phase models with and without latency effects in the mass transfer of phases between control volumes (phase holdup). Simulation results are presented to illustrate the qualitative features captured by the new models.

## 1 Introduction

One of the primary goals of the National Aeronautics and Space Administration (NASA) is to extend the scientific knowledge of our solar system. NASA's Jet Propulsion Laboratory (JPL), a division of the California Institute of Technology, supports research and technology development of robotic exploration probes to explore the planets of our solar system. A landing vehicle, as used in the Viking mission to Mars, allows the exploration of a small part of the surface of a planet but is not very flexible in terms of exploring the atmosphere or other parts of the planet. Alternatively, a planetary probe, as used in the Galileo mission to Jupiter, gathers data from a limited portion of the atmosphere for a very short time. To expand both the spatial and temporal ranges of data collection, a JPL research program focuses on the use of "smart" balloons which will move up and down in a planetary atmosphere, land and explore numerous surface sites. These smart balloons, or robotic balloons, are called *aerobots* [2].

For a balloon to rise or fall in an atmosphere by design, it must include a mechanism to change its buoyancy. For terrestrial planets such as Earth, Venus, and Titan (actually a moon of Saturn), a promising concept is to use two balloons: a primary balloon to provide necessary lift, and a secondary balloon which is filled with some type of phase-change fluid. For Earth applications, the primary balloon is usually filled with hydrogen or helium, but theoretically it could be filled with any fluid that has a molecular weight of less than the ambient atmosphere, and which remains a vapor for all anticipated temperatures and pressures during the flight.

Figure 1 shows a typical dual-phase aerobot configuration. The phase-change fluid in the secondary balloon is chosen to be gaseous on the ground but to condense above some altitude in the atmosphere (called the equilibrium altitude). This assumes the temperature in the atmosphere decreases with altitude. As the balloon rises above the equilibrium altitude, the secondary fluid will condense and the volume of the secondary balloon will decrease substantially, producing a change in buoyancy. With careful design, enough of the fluid will eventually become liquid so that the buoyancy of the system will become negative. Then the system will start descending. As the balloon goes below the equilibrium altitude, the phase change fluid (now mostly liquid) will start to vaporize, increasing the buoyancy. At some point it will reach positive buoyancy, reverse its motion, and start to ascend again—leading to a natural oscillation. For Earth applications, a convenient fluid for testing is 1,2-dichlorotetrafluoroethane or  $CClF_2CClF_2$ , also known as Refrigerant 114 (R114). This fluid has a molecular weight of 171, which is much higher than air at 28.97. Thus, when R114 evaporates at about 3–7 km altitude, it displaces air that is less than 1/6 of its mass.

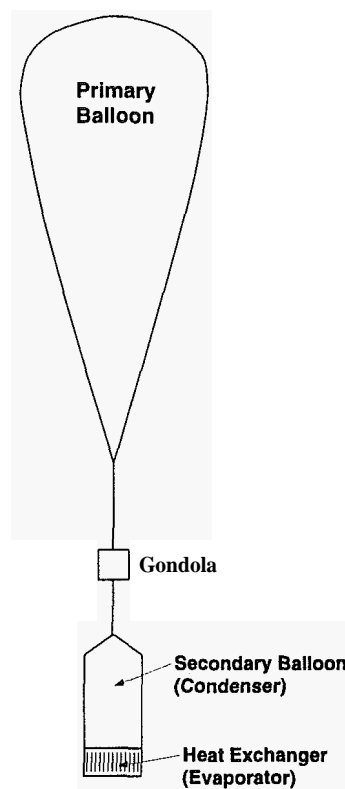


Figure 1: Aerobot Balloon System

The primary balloon of hydrogen or helium is necessary to provide the primary lift so that this small delta buoyancy can create a net difference in total system buoyancy to provide ascent versus descent.

Oscillatory dual-balloon systems are being considered by NASA at JPL for long-duration low-cost atmospheric and surface missions to Venus and Titan. See Jones [4] for a more detailed description of the basic concept. These exploratory balloon systems are being developed and tested here on earth.

The motion of oscillatory two-balloon systems is essentially determined by the thermodynamics of the system and environment. Before such systems can be deployed on other planets, the thermodynamic modeling involved must be thoroughly understood and validated. **Work** is underway at JPL to improve our understanding of the thermodynamics and heat transfer involved in these complex systems.

A previously developed thermal model of the system has resolved many relevant physical effects and has proved to be an excellent tool for design and analysis [7]. This balloon system model has three thermodynamic control volumes: the primary balloon, the secondary balloon, and a heat exchanger attached to the bottom of the secondary balloon. The heat exchanger is a necessary part of the system. Experiences in early earth-based test flights showed that without a heat exchanger, the heat transfer into the liquid part of the phase-change fluid was insufficient to vaporize all the fluid before the balloon struck the ground.

Each of the three control volumes has uniform thermodynamic properties throughout the control volume. The model includes appropriate heat transfer models for convection and radiation for each

control volume. Mass transfer resulting from phase changes can occur in two modes: (1) The flow of the condensed phase-change fluid from the secondary balloon down into the heat exchanger and (2) the flow of the vaporized phase-change fluid from the heat-exchanger up into the secondary balloon. The mass transfer in both directions has certain latencies, or "phase mixing holdup". Modeling these flows is important since the heat transfer to the secondary balloon will be different from that to the heat exchanger, and therefore each control volume will, in general, have different temperatures. Accurate modeling of the thermodynamics in the phase-change fluid is vital, since it controls the overall buoyancy of the system. The previously developed model uses discrete equations which do not adequately capture the effects resulting from phases changes of the phase-change fluid and mass transfer between the secondary balloon and the heat exchanger.

As a result of simplifications in treating the phase mixing, the previously developed model was developed as a hybrid system of differential equations and discrete equations. Differential equations were used to model heat transfer and balloon motion dynamics while discrete update equations were used to model the phase change processes in the secondary balloon. A disadvantage associated with this hybrid model is the inability to flexibly simulate and analyze the system dynamics since there is no comprehensive differential equation formulation. Such a complete formulation would allow use of variable step integration methods, the solution of dynamic optimization problems, as well as other analysis goals such as the physical characterization of aerobot oscillation frequency, amplitude, and limit cycle behavior.

In addition, a more flexible modeling framework allows extensions to more complex systems and the incorporation of previously ignored effects. For example, further development of aerobot technology will require accurate modeling of phase changes in fluid mixtures as well as pure substances. The use of fluid mixtures allows a balloon designer to select the equilibrium altitude by adjusting the mixture composition. In some cases, the mixtures are azeotropic and can be modeled as pure substances. In others, the mixtures are zeotropic, complicating the phase change thermodynamics considerably. Being able to deal with various mixtures provides the flexibility necessary to perform many different types of missions. However it introduces significant complexities into the model because the mixture composition is an additional variable as well as zeotropic phase change behavior.

In this paper we develop a non-equilibrium, differential-equation model for phase change and mass transfer in the secondary balloon and heat exchanger, extending the hybrid model previously developed. In Section 2 the basic governing equations of balloon physics are derived from fundamental principles of dynamics and thermodynamics. Relevant mechanisms for heat transfer are summarized in Section 3. In Section 4 the basic equations are summarized for all components except for the secondary balloon and the heat exchanger. Then in Section 5 a dual-phase modeling principle based on conservation of mass and energy is used to derive a coupled set of differential equations for these missing components. All relevant equations are collected and summarized in state-space form in Section 6. Preliminary simulation studies are discussed in Section 7 and final conclusions are given in Section 8.

This new model provides a significant improvement over previous models of phase-change heat and mass transfer involved in two-balloon systems. The differential equation formulation and the modeling of phase mixing holdup will advance the accuracy and flexibility of the models used to design and analyze these systems.

## 2 Basic Balloon Physics

Balloon performance can be described by the fundamental laws of nature. Specifically, the laws of motion and thermodynamics are sufficient to characterize balloon behavior. Archimedes' first principle of buoyancy (the buoyant force equals the weight of the displaced fluid) and Archimedes' second principle of buoyancy (a body floats if its average density is less than the density of the surrounding fluid) are particularly useful when considering balloons. Newton's second law of motion and the First Law of Thermodynamics are also necessary.

### 2.1 Fundamental Governing Equations

To develop an understanding of balloon behavior, let us examine equations of energy and motion applied to arbitrary balloon systems. In later sections these principles will be extended to oscillatory balloon systems, the focus of this study.

#### 2.1.1 Development of the Energy Equation

We begin with the simplest form of the energy inventory (the First Law of Thermodynamics) for a control volume (Figure 2). The energy equation can be expressed as

$$\Delta(T E) + \Delta(K E) = q + \Delta(E_{\text{mass\_exchange}}) + W_{\text{expansion}} + W_{\text{body\_forces}}, \quad (1)$$

where

$$\begin{aligned} \Delta(T E) &= \text{rate of change of thermal energy [W]}, \\ \Delta(K E) &= \text{rate of change of kinetic energy [W]}, \\ q &= \text{rate of heat transfer to the control volume [W]}, \\ \Delta(E_{\text{mass\_exchange}}) &= \text{rate of change in energy due to mass exchange [W]}, \\ W_{\text{expansion}} &= \text{rate of work done on the control volume during} \\ &\quad \text{expansion or contraction [W]}, \text{ and} \\ W_{\text{body\_forces}} &= \text{rate of work done by body forces acting on the control volume [W]}. \end{aligned}$$

Equation 1 utilizes the form of the energy equation that treats gravitational force as a body force, not as potential energy. Evaluating each term in Equation 1 gives the following equation for the conservation of energy in a control volume with heat transfer  $q$ , mass exchange with the surroundings  $\frac{dm}{dt}$ , and body forces  $F_{\text{tot}}$ :

$$\frac{d}{dt} \left( mu + \frac{mV^2}{2} \right) = q + \sum_{\text{C.S.}} \frac{dm_i}{dt} \left( h_i + \frac{V_i^2}{2} \right) - P \frac{dV}{dt} + \mathbf{F}_{\text{tot}} \cdot \mathbf{V} \quad (2)$$

where

$$\begin{aligned} m &= \text{mass in the control volume [kg]}, \\ u &= \text{internal energy [J/kg]}, \\ V &= \text{magnitude of the velocity of the control volume [m/s]}, \\ q &= \text{rate of heat transfer to the control volume [W]}, \\ \frac{dm_i}{dt} &= \text{rate of mass transfer between control volume and} \\ &\quad \text{surroundings through stream "i" [kg/s]},^1 \\ h_i &= \text{enthalpy of mass leaving the control volume through stream "i" [J/kg]}, \end{aligned}$$

$V_i$	= magnitude the velocity of the fluid being transferred out of the control volume through stream “i” [ $m/s$ ],
$P$	= balloon pressure (same as atmospheric pressure for zero-pressure) [ $Pa$ ],
$V$	= balloon volume [ $m^3$ ],
$\mathbf{F}_{tot}$	= vector sum of forces on the control volume [ $N$ ],
$\mathbf{V}$	= velocity vector for the control volume [ $m/s$ ], and
$C.S.$	= control surface.

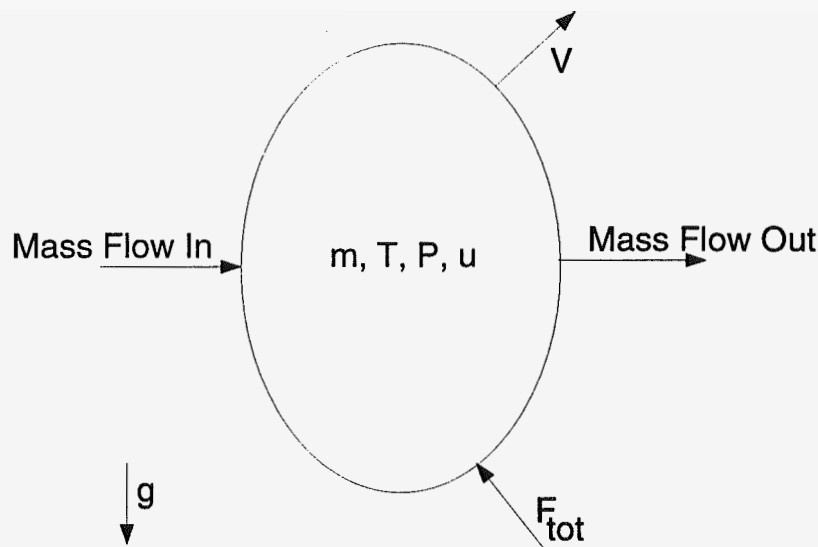


Figure 2: Arbitrary Control Volume

### 2.1.2 The Equation of Motion

The equation of motion for any rigid system under the influence of body forces is

$$\mathbf{F}_{tot} = m\mathbf{a}_{tot}. \quad (3)$$

Although balloons are not considered rigid bodies, to first order, Equation 3 provides correct predictions of balloon dynamical behavior. Our task is to develop Equations 2 and 3 for use in modeling a simple, zero-pressure balloon system. By “zero-pressure” we mean that the pressure inside each balloon is the same as the outside environmental pressure. Important effects to consider are lifting gas leakage, motion of an accelerated body in a fluid medium, and all modes of heat transfer (in a typical balloon, both convective and radiative modes). We shall see each of these effects in the following section.

## 2.2 Balloon System Motion

Figure 3 is a diagram of a simple balloon system. The system consists of a balloon envelope

---

<sup>1</sup>The mass exchange can take place over any portion of the control surface. Thus, the mass transfers represented by  $\frac{dm_i}{dt}$  are integrated over the entire control surface. The summation is taken over “i” unique mass transfer streams.

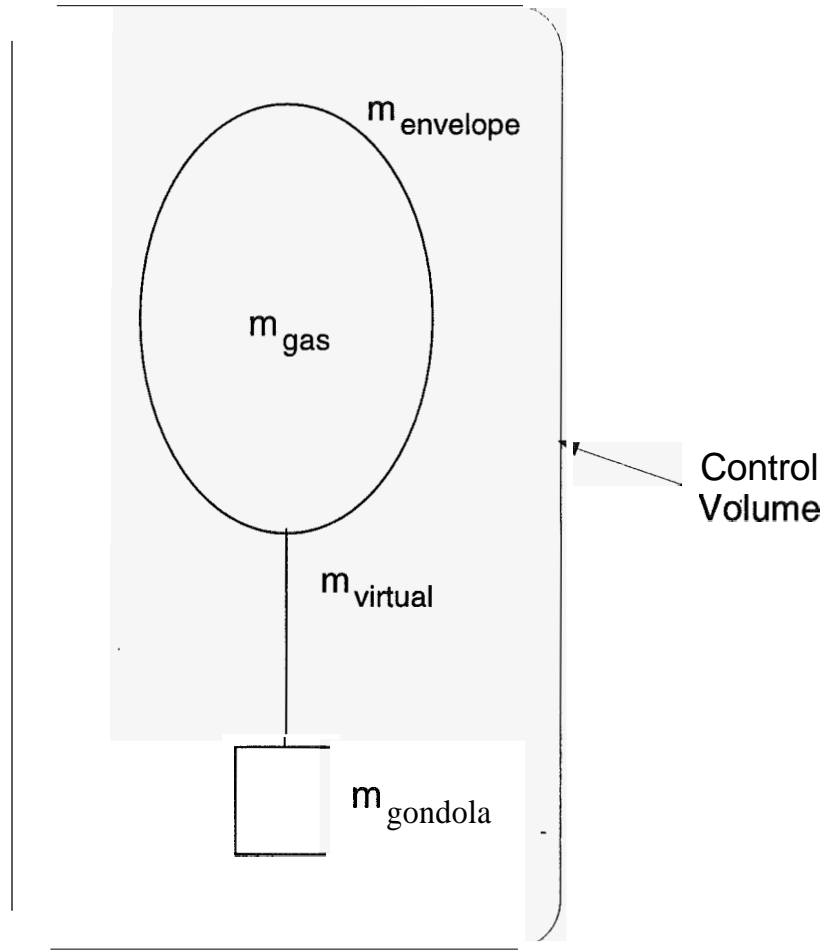


Figure 3: Schematic Diagram of a Simple Balloon System

with mass  $m_{\text{envelope}}$ , balloon gas with mass  $m_{\text{gas}}$ , the gondola and assorted other hardware with mass  $m_{\text{gondola}}$ , and the virtual mass,  $m_{\text{virtual}}$ . The balloon is a zero-pressure balloon (no pressure difference between the atmosphere and the balloon lifting gas).

As far back as 1850, Stokes [6] used the virtual mass concept to predict the motion of accelerating bodies in a fluid medium, and all models of balloon behavior in the literature use the virtual mass concept. In short, the virtual mass accounts for the mass of the surrounding environment that is dragged with the balloon as it ascends and descends through the atmosphere. In effect, the virtual mass reduces system acceleration. The virtual mass is given by

$$m_{\text{virtual}} = C_m \rho_{\text{atm}} V_{\text{gas}} \quad (4)$$

with

$$\begin{aligned} C_m &= \text{coefficient of virtual mass (typically 0.5, Iversen and Balent [3]),} \\ \rho_{\text{atm}} &= \text{atmospheric density [kg/m}^3\text{], and} \\ V_{\text{gas}} &= \text{balloon volume [m}^3\text{].} \end{aligned}$$

For purposes of the equations of motion, we define the effective system mass  $M$  as

$$M = m_{\text{sys}} + m_{\text{virtual}} \quad (5a)$$



$$m_{\text{sys}} = m_{\text{envelope}} + m_{\text{gas}} + m_{\text{gondola}}. \quad (5b)$$

The net force on the system is composed of gravitational, buoyancy, and drag forces:

$$\mathbf{F}_{\text{tot}} = \mathbf{F}_{\text{grav}} + \mathbf{F}_{\text{buoy}} + \mathbf{F}_{\text{drag}}. \quad (6)$$

Note that the pressure of the balloon gas is assumed to be equal to the atmospheric pressure at all times (zero-pressure balloon). Then finally the equation of motion for the balloon system is based on Equation 3 combined with Equations 5- 6:

$$\mathbf{F}_{\text{tot}} = M \mathbf{a}_{\text{tot}}. \quad (7)$$

### 2.3 Balloon System Energy

The energy equation for the balloon system is based on Equation 2:

$$\begin{aligned} & \frac{d}{dt} [(mu)_{\text{gas}} + (mu)_{\text{virtual}} + (mCT)_{\text{gondola}} + (mCT)_{\text{envelope}}] + \frac{d}{dt} \left[ \frac{MV^2}{2} \right] \\ &= \frac{dm_{\text{gas}}}{dt} \left[ h_{\text{gas}} + \frac{||\mathbf{V} - \mathbf{V}_{\text{discharge}}||^2}{2} \right] + \frac{dm_{\text{virtual}}}{dt} \left[ h_{\text{virtual}} + \frac{||\mathbf{V} - \mathbf{V}_{\text{discharge}}||^2}{2} \right] \\ & \quad - P \frac{d}{dt} [\mathcal{V}_{\text{gas}} + \mathcal{V}_{\text{virtual}}] + \mathbf{F}_{\text{tot}} \cdot \mathbf{V}, \end{aligned} \quad (8)$$

where  $\mathbf{V}_{\text{discharge}}$  is the velocity vector for the leaking gas and the virtual mass leaving the control volume, and  $C$  is the specific heat of a solid (gondola or envelope). The vector  $\mathbf{V}_{\text{discharge}}$  is taken relative to the control volume, whereas the total velocity vector for the system,  $\mathbf{V}$ , is taken relative to the planet surface. The vector  $\mathbf{V}_{\text{discharge}}$  is the integrated average of the leak velocity that occurs over the entire balloon (in the case of the lifting gas) or control volume surface (in the case of the virtual mass). Thus,  $\mathbf{V}_{\text{discharge}}$  is identically zero if the leak rate is uniform over the entire surface of the balloon.

With the assumption that  $V_{\text{discharge}}$  is extremely small ( $V_{\text{discharge}} \approx 0$ ), Equation 8 can be reduced to

$$\begin{aligned} & \frac{d}{dt} [(mu)_{\text{gas}} + (mu)_{\text{virtual}} + (mCT)_{\text{gondola}} + (mCT)_{\text{envelope}}] + \frac{d}{dt} \left[ \frac{MV^2}{2} \right] \\ &= \frac{dm_{\text{gas}}}{dt} \left[ h_{\text{gas}} + \frac{V^2}{2} \right] + \frac{dm_{\text{virtual}}}{dt} \left[ h_{\text{virtual}} + \frac{V^2}{2} \right] - P \frac{d}{dt} [\mathcal{V}_{\text{gas}} + \mathcal{V}_{\text{virtual}}] + \mathbf{F}_{\text{tot}} \cdot \mathbf{V}, \end{aligned} \quad (9)$$

We note that the kinetic energy term in Equation 9 can be expanded to

$$\frac{d}{dt} \left[ \frac{MV^2}{2} \right] = \frac{V^2}{2} \frac{dm_{\text{virtual}}}{dt} + \frac{V^2}{2} \frac{dm_{\text{gas}}}{dt} + M \mathbf{a}_{\text{tot}} \cdot \mathbf{V} \quad (10)$$

Note that Equation 10 explicitly states that the virtual mass  $m_{\text{virtual}}$  and the balloon gas mass  $m_{\text{gas}}$  may change with time. Substitution of Equations 7 and 10 into Equation 9 allows simplification of the energy equation. This step amounts to subtracting the mechanical energy terms from the total energy equation. The result is

$$\begin{aligned} \frac{d}{dt} [(mu)_{\text{gas}} + (mu)_{\text{virtual}} + (mCT)_{\text{gondola}} + (mCT)_{\text{envelope}}] \\ = q + \frac{dm_{\text{gas}}}{dt} [u_{\text{gas}}] + \frac{dm_{\text{virtual}}}{dt} [u_{\text{virtual}}] + \frac{d}{dt} [p_{\text{gas}} V_{\text{gas}} + p_{\text{virtual}} V_{\text{virtual}}], \end{aligned} \quad (11)$$

where  $q$  represents the heat transfer (both convection and radiation) from the environment to the control volume.

### 2.3.1 Energy Equations for Components of Balloon System

Equation 11 is difficult to solve in practice. To facilitate the solution, we realize that the energy equation can be applied not only to the entire system, but also to the components of the system. When applying the energy equation to all masses in the control volume, it is important to include heat transfer among components of the system. We begin with the energy equation for the virtual mass:

$$\begin{aligned} \frac{d}{dt} [(mu)_{\text{virtual}}] \\ = q_{\text{environment} \rightarrow \text{virtual}} + q_{\text{gondola} \rightarrow \text{virtual}} + q_{\text{envelope} \rightarrow \text{virtual}} + \frac{dm_{\text{virtual}}}{dt} h_{\text{virtual}} - P \frac{dV_{\text{virtual}}}{dt}. \end{aligned} \quad (12)$$

Employing ideal gas law relations for the virtual mass gives

$$\frac{d}{dt} m_{\text{virtual}} = C_m \frac{MW_{\text{atm}}}{MW_{\text{gas}}} \frac{T_{\text{gas}}}{T_{\text{virtual}}} \frac{dm_{\text{gas}}}{dt} \quad (13)$$

with  $MW$  representing molecular weight. Then from the definition of virtual mass, one can write:

$$\frac{dV_{\text{virtual}}}{dt} = C_m \frac{dV_{\text{gas}}}{dt}. \quad (14)$$

Assuming that  $q_{\text{environment} \rightarrow \text{virtual}}$  dominates all other terms on the right-hand side of Equation 12, we obtain

$$T_{\text{virtual}} = T_{\text{environment}}, \quad (15)$$

and we treat the virtual mass as part of the environment to simplify the heat transfer analysis.

Both the gondola and balloon envelope are assumed to have constant mass  $m$  and specific heat  $C$ . For the gondola we obtain

$$(mC)_{\text{gondola}} \frac{dT_{\text{gondola}}}{dt} = q_{\text{environment} \rightarrow \text{gondola}} + q_{\text{envelope} \rightarrow \text{gondola}}. \quad (16)$$

Similarly, the equation for the envelope becomes

$$(mC)_{\text{envelope}} \frac{dT_{\text{envelope}}}{dt} = q_{\text{environment} \rightarrow \text{envelope}} + q_{\text{gondola} \rightarrow \text{envelope}} + q_{\text{gas} \rightarrow \text{envelope}}. \quad (17)$$

Finally, the equation for the balloon gas becomes

$$\frac{d}{dt}[(mu)_{\text{gas}}] = q_{\text{environment} \rightarrow \text{gas}} + q_{\text{envelope} \rightarrow \text{gas}} + h_t \frac{dm_{\text{gas}}}{dt} - P \frac{dV_{\text{gas}}}{dt}, \quad (18)$$

Note that all the environment heat transfer terms in Equations 12- 18 should include both convective and radiative effects (solar and infrared).

### 2.3.2 Simplification of the Balloon Gas Energy Equation

At this point we can simplify the energy equation for the balloon gas. First, we both add and subtract the term  $V_{\text{gas}} \frac{dP}{dt}$  to the right side of Equation 18:

$$\frac{d}{dt}[(mu)_{\text{gas}}] = q_{\text{environment} \rightarrow \text{gas}} + q_{\text{envelope} \rightarrow \text{gas}} + h_t \frac{dm_{\text{gas}}}{dt} - P \frac{dV_{\text{gas}}}{dt} + V_{\text{gas}} \frac{dP}{dt} + V_{\text{gas}} \frac{dP}{dt} \quad (19)$$

Next, we group terms to get

$$\frac{d}{dt}[(mu)_{\text{gas}}] = q_{\text{environment} \rightarrow \text{gas}} + q_{\text{envelope} \rightarrow \text{gas}} + h_t \frac{dm_{\text{gas}}}{dt} - \frac{d}{dt}[PV_{\text{gas}}] + V_{\text{gas}} \frac{dP}{dt} \quad (20)$$

Using the definition of enthalpy ( $h = u + Pv$ ) and specific volume ( $v = \frac{V}{m}$ ) we obtain

$$\frac{d}{dt}[(mh)_{\text{gas}}] = q_{\text{environment} \rightarrow \text{gas}} + q_{\text{envelope} \rightarrow \text{gas}} + h_t \frac{dm_{\text{gas}}}{dt} + V_{\text{gas}} \frac{dP}{dt} \quad (21)$$

The pressure change with time  $\frac{dP}{dt}$  depends on: (a) bulk atmospheric changes and (b) the velocity of the balloon through the atmosphere. Let the density of the atmosphere at a given altitude  $\tilde{z}$  and time  $t$  be given by  $\rho_{\text{atm}}(\tilde{z}, t)$ , let the pressure at zero altitude be given by  $P_{\text{atm}}^0(t)$ , and let the gravitational constant  $g$  be independent of altitude and time. Then the pressure of the atmosphere at altitude  $\tilde{z}$  and time  $t$  is given by

$$P_{\text{atm}}(\tilde{z}, t) = P_{\text{atm}}^0(t) - g \int_0^{\tilde{z}} \rho_{\text{atm}}(\zeta, t) d\zeta. \quad (22)$$

The balloon pressure  $P(t)$  satisfies  $P(t) = P_{\text{atm}}(z(t), t)$  for a zero-pressure balloon at altitude  $z(t)$ . Then differentiation of Equation 22 gives

$$\begin{aligned} \frac{dP(t)}{dt} &= \frac{dP_{\text{atm}}(z(t), t)}{dt} \\ &= \frac{dP_{\text{atm}}^0(t)}{dt} - g \int_0^{z(t)} \frac{\partial \rho_{\text{atm}}(\zeta, t)}{\partial t} d\zeta - g \rho_{\text{atm}}(z(t), t) \frac{dz(t)}{dt}. \end{aligned} \quad (23)$$

Finally we assume that both the atmospheric density profile and the pressure at zero altitude are independent of  $t$  (i.e.,  $\rho_{\text{atm}}(\tilde{z}, t) \equiv \rho_{\text{atm}}(\tilde{z})$ ,  $P_{\text{atm}}^0(t) \equiv P_{\text{atm}}^0$ ) Then Equation 23 simplifies to:

$$\frac{dP(t)}{dt} = -g \rho_{\text{atm}}(z(t)) \frac{dz(t)}{dt}. \quad (24)$$

Thus, from Equation 21 the energy equation for the balloon gas becomes

$$\frac{d}{dt}[(mh)_{\text{gas}}] = q_{\text{environment} \rightarrow \text{gas}} + q_{\text{envelope} \rightarrow \text{gas}} + h_t \frac{dm_{\text{gas}}}{dt} - g \rho_{\text{atm}}(z) V \frac{dz}{dt}. \quad (25)$$

Equation 25 is identical to the energy equation for balloon gas used by Wu and Jones [7]. It is also equivalent to the energy equation used by Carlson [1].

### 2.3.3 Summary of Assumptions

The balloon gas energy equation (Equation 25) contains the following assumptions:

- o The surface average velocity of the leaking gas is small relative to the velocity of the balloon system,
- o The atmospheric density profile does not vary with time,
- o Gravitational acceleration does not vary with time or location,
- o The virtual mass can be treated as part of the surroundings in the heat transfer analysis,
- o The balloon acts as a rigid body for purposes of calculating bulk motion, and
- o The balloon is a zero-pressure balloon.

### 2.3.4 A Generalized Form for the Energy Equation

To simplify the presentation of the subsystem energy equations, it is useful to group all heat transfer terms into a single variable. Let  $q$  be the net heat transfer into the subsystem. Thus, we can re-write Equation 25 as

$$\frac{d}{dt}[(mh)_{\text{gas}}] = q_{\text{gas}} + h_t \frac{dm_{\text{gas}}}{dt} - g\rho_{\text{atm}}(z)V \frac{dz}{dt}. \quad (26)$$

The underlying energy conservation principle used to derive Equation 25 applies to a generic component of the system which can be characterized by thermodynamic state variables. Thus, Equation 26 can be written more generally:

$$\frac{d}{dt}[(mh)] = q + h_t \frac{dm}{dt} - g\rho_{\text{atm}}(z)V \frac{dz}{dt}. \quad (27)$$

Next we introduce a composite forcing term  $F$  to include all energy flow terms independent of mass exchange

$$F = q - g\rho_{\text{atm}}(z)V \frac{dz}{dt} \quad (28)$$

and the term  $L$ , the mass leak rate. Furthermore, the equations can be cast in terms of the *total enthalpy*

$$H = mh. \quad (29)$$

Thus, we have the general form of the energy and mass equations:

$$\frac{d}{dt}H = F + Lh \quad (30a)$$

$$\frac{d}{dt}m = L. \quad (30b)$$

### 3 Heat Transfer

Heat transfer between the balloon system and its surroundings is a major driving force for balloon behavior. In the case of an oscillatory aerobot, both heat transfer between the primary balloon and the environment and heat transfer between the secondary balloon and the environment are important. The following list gives the various heat transfer paths that must be analyzed and the corresponding values of  $q$  necessary for implementation of Equation 30. Details on the heat transfer models can be found in Wu [7].

$q_{pg}$  Heat flow into the Primary Balloon Gas: The primary gas is modeled as a uniform-temperature sphere having diffuse, gray radiation characteristics with solar absorptivity  $\alpha_{pg}$  and IR emissivity  $\epsilon_{pg}$ . It is important to consider radiation between the gas and the sun, the ground, and the sky, as the film may be semi-transparent. The primary gas may or may not participate in the radiative exchange depending on the gas characteristics and gas purity. Convection between the balloon film and the gas must also be considered.

$q_{pf}$  Heat Flow into the Primary Balloon Film: The film is modeled as a single node of uniform-temperature having diffuse, gray radiative characteristics with solar absorptivity  $\alpha_{pf}$  and IR emissivity  $\epsilon_{pf}$ . The values of  $\alpha_{pf}$  and  $\epsilon_{pf}$  must account for film semi-transparency if required. Consideration of radiative exchange with the sun, the sky, and the gas and convective exchange with the lifting gas and the atmosphere is necessary.

$q_{sg}$  Heat Flow into the Secondary Gas: The secondary gas is modeled as a uniform temperature sphere having diffuse, gray radiation characteristics with solar absorptivity  $\alpha_{sg}$  and IR emissivity  $\epsilon_{sg}$ . Radiative exchange between the secondary gas and the sun, the ground, the sky and the secondary film, as the film may be semi-transparent. The secondary gas may or may not participate in the radiation depending on its properties. Convection between the secondary gas and the secondary film must also be considered.

$q_{sf}$  Heat Flow into the Secondary Film: The secondary film is modeled as two flat plates for purposes of convective heat transfer. Convection between the secondary film and both the secondary gas and atmosphere must be included. For radiation calculations, the secondary film is modeled as a sphere with equivalent volume. Radiation between the secondary film and the sun, the ground, the sky, and the secondary gas must be considered.

$q_{sl}$  Heat Flow into the Secondary Liquid: The secondary liquid resides in the heat exchanger which is modeled as a vertical flat plate. It is assumed that the heat exchanger wall temperature is equal to the temperature of the secondary liquid. Radiative heat exchange between the secondary liquid and the sun, the sky, and the ground must be considered. Convection between the heat exchanger walls and the atmosphere must be included.

Here and in the subsequent sections, we use the following subscript convention:

- pg: primary gas
- pf: primary film
- sg: secondary gas
- sf: secondary film
- sl: secondary liquid

## 4 Energy Conservation Equations for the Aerobot System

The results of the previous two sections can be applied immediatly to express the equations which model the gas in the primary balloon, which remains always in a gaseous form. Thus, Equation 30 becomes:

$$\frac{d}{dt}H_{pg} = F_{pg} + L_{pg}h_{pg} \quad (31a)$$

$$\frac{d}{dt}m_{pg} = L_{pg} \quad (31b)$$

where

$$h_{pg} = \frac{H_{pg}}{m_{pg}} \quad (32a)$$

$$F_{pg} = q_{pg} - g\rho_{\text{atm}}(z)V_{pg}\frac{dz}{dt}. \quad (32b)$$

Both the primary and secondary balloons have films which act as a barrier to the outside environment :

$$\frac{d}{dt}H_{pf} = F_{pf} \quad (33a)$$

$$\frac{d}{dt}H_{sf} = F_{sf} \quad (33b)$$

where

$$F_{pf} = q_{pf} \quad (34a)$$

$$F_{sf} = q_{sf}. \quad (34b)$$

$$H_{pf} = m_{pf}C_{pf}T_{pf} \quad (34c)$$

$$H_{sf} = m_{sf}C_{sf}T_{sf}. \quad (34d)$$

The next section presents a detailed discussion of the modeling of the secondary balloon system, which consists of a liquid and a gaseous component.

## 5 Dual-Phase Models for the Secondary Balloon System

### 5.1 Thermodynamic Inputs

Unlike the primary balloon, the secondary balloon system consists of both a liquid and a gaseous component.<sup>2</sup> Because the internal flow of mass and total enthalpy between these components must be modeled to capture the dynamics of the phase changes, state variables must be introduced for

<sup>2</sup>The secondary *ballon* system comprises both the secondary ballon (condenser) and heat exchanger (evaporator) as illustrated in Figure 1

the mass and total enthalpy of the secondary liquid and the secondary gaseous system:  $m_{sl}$ ,  $m_{sg}$ ,  $H_{sl}$ ,  $H_{sg}$ .

As in Equation 28, let  $F_{sl}$  and  $F_{sg}$  denote all energy flow terms independent of mass flow. Then let  $L_{sl}$  and  $L_{sg}$  denote mass leakage terms which are independent of the internal phase changes within the secondary balloon. That is, these mass flow terms do not contain the effects of condensation and evaporation which are essential to the modeling of the secondary balloon system as in Equation 30.

The remainder of this section completes the model by using principles of energy and mass conservation to characterize the effects of condensation and evaporation within the secondary system. More precisely, under thermodynamic conditions determined by other state variables, a given amount of secondary mass  $m_s$  and secondary total enthalpy  $H$ , will uniquely determine a separation into phases:

$$m_s = m_{sl} + m_{sg} \quad (35a)$$

$$H_s = H_{sl} + H_{sg} \quad (35b)$$

This principle is first derived generally, that is, without reference to a particular application and then applied to the secondary balloon system. The separation into phases will depend crucially on the system's specific saturation enthalpy for a liquid,  $\hat{h}^l$ , and specific saturation enthalpy for a gas,  $\hat{h}^g$ . For future reference the time derivatives of these quantities can be represented in terms of other state variables and the saturation values of temperature and pressure,  $\hat{T}$ ,  $\hat{P}$ :

$$\dot{\hat{h}}^l = \frac{d}{dt} \hat{h}^l = \frac{d\hat{h}^l}{d\hat{T}} \frac{d\hat{T}}{dt} + \frac{d\hat{h}^l}{d\hat{P}} \frac{d\hat{P}}{dt} \quad (36a)$$

$$\dot{\hat{h}}^g = \frac{d}{dt} \hat{h}^g = \frac{d\hat{h}^g}{d\hat{T}} \frac{d\hat{T}}{dt} + \frac{d\hat{h}^g}{d\hat{P}} \frac{d\hat{P}}{dt}. \quad (36b)$$

Here the functional forms of  $\hat{h}^l(\hat{T})$ ,  $\hat{h}^g(\hat{T})$ , and  $\hat{T}(\hat{P})$  are presumed to be known for the given substance, and  $\frac{d\hat{P}}{dt}$  is interpreted through Equation 24 for the zero-pressure aerobot application.

## 5.2 Single-System Model

Let the state of a mixed-phase system be characterized by mass  $m$  and total enthalpy  $H$ , with corresponding specific enthalpy  $h = H/m$ . In addition, the given thermodynamic state determines values for the specific saturation enthalpy for a liquid,  $\hat{h}^l$ , and the specific saturation enthalpy for a gas,  $\hat{h}^g$ . Then the *Enthalpy-Fraction Function* gives a relative separation into phases (i.e., quality):

$$\Phi[h; \hat{h}^l, \hat{h}^g] = \begin{cases} 0, & \text{if } h < \hat{h}^l \\ (h - \hat{h}^l)/(\hat{h}^g - \hat{h}^l), & \text{if } \hat{h}^l < h < \hat{h}^g \\ 1, & \text{if } \hat{h}^g < h. \end{cases} \quad (37)$$

The decomposition into liquid and gaseous components is uniquely determined in terms of the functions  $\Phi[h; \hat{h}^l, \hat{h}^g]$ ,  $\min[h, \hat{h}^l]$  and  $\max[h, \hat{h}^g]$  as:

$$m^L[m, H; \hat{h}^l, \hat{h}^g] = (1 - \Phi[H/m; \hat{h}^l, \hat{h}^g])m \quad (38a)$$

$$m^G[m, H; \hat{h}^l, \hat{h}^g] = (\Phi[H/m; \hat{h}^l, \hat{h}^g])m \quad (38b)$$

$$H^L[m, H; \hat{h}^l, \hat{h}^g] = m^L[m, H; \hat{h}^l, \hat{h}^g] \min[H/m, \hat{h}^l] \quad (38c)$$

$$H^G[m, H; \hat{h}^l, \hat{h}^g] = m^G[m, H; \hat{h}^l, \hat{h}^g] \max[H/m, \hat{h}^g], \quad (38d)$$

where the superscripts "L" and "G" are used here and elsewhere to denote liquid and gaseous components of a dual-phase system. One can easily verify that this decomposition preserves both mass and total enthalpy:

$$m = m^L[m, H; \hat{h}^l, \hat{h}^g] + m^G[m, H; \hat{h}^l, \hat{h}^g] \quad (39a)$$

$$H = H^L[m, H; \hat{h}^l, \hat{h}^g] + H^G[m, H; \hat{h}^l, \hat{h}^g]. \quad (39b)$$

For notational convenience the parametric dependance of  $\Phi$ ,  $m^L$ , and  $m^G$  on  $\hat{h}^g$  and  $\hat{h}^l$  will generally be omitted (i.e.,  $\Phi[h] = \Phi[h; \hat{h}^l, \hat{h}^g]$ ).

The inputs to the system are characterized by the differential quantities

$$\frac{dH}{dt} = F + L(H/m) \quad \frac{dm}{dt} = L \quad (40a)$$

$$\frac{d\hat{h}^l}{dt} = \hat{p}^l \quad \frac{d\hat{h}^g}{dt} = \hat{p}^g \quad (40b)$$

and also for  $h = H/m$ :

$$\frac{dh}{dt} = (1/m) \frac{dH}{dt} - (H/m^2) \frac{dm}{dt} = (F/m). \quad (41)$$

This makes it possible to determine differential changes in the liquid and and gaseous components of the system:

$$\frac{dm^L}{dt} = \Psi_m^L[m, H; \hat{h}^l, \hat{h}^g; F, L, \hat{p}^l, \hat{p}^g] \quad (42a)$$

$$\frac{dm^G}{dt} = \Psi_m^G[m, H; \hat{h}^l, \hat{h}^g; F, L, \hat{p}^l, \hat{p}^g] \quad (42b)$$

$$(42c)$$

$$\frac{dH^L}{dt} = \Psi_H^L[m, H; \hat{h}^l, \hat{h}^g; F, L, \hat{p}^l, \hat{p}^g] \quad (42d)$$

$$\frac{dH^G}{dt} = \Psi_H^G[m, H; \hat{h}^l, \hat{h}^g; F, L, \hat{p}^l, \hat{p}^g] \quad (42e)$$

where

$$\Psi_m^L[m, H; \hat{h}^l, \hat{h}^g; F, L, \hat{p}^l, \hat{p}^g] = \quad (43a)$$

$$-\frac{\partial}{\partial h} \Phi[H/m] F + (1 - \Phi[H/m]) L - \frac{\partial}{\partial \hat{h}^l} \Phi[H/m] m \hat{p}^l - \frac{\partial}{\partial \hat{h}^g} \Phi[H/m] m \hat{p}^g$$

$$\Psi_m^G[m, H; \hat{h}^l, \hat{h}^g; F, L, \hat{p}^l, \hat{p}^g] = \quad (43b)$$

$$\frac{\partial}{\partial h} \Phi[H/m] F + \Phi[H/m] L + \frac{\partial}{\partial \hat{h}^l} \Phi[H/m] m \hat{p}^l + \frac{\partial}{\partial \hat{h}^g} \Phi[H/m] m \hat{p}^g$$



$$\Psi_H^L[m, H; \hat{h}^l, \hat{h}^g; F, L, \hat{p}^l, \hat{p}^g] = \quad (43c)$$

$$\begin{aligned} & \Psi_m^L[m, H; \hat{h}^l, \hat{h}^g; F, L, \hat{p}^l, \hat{p}^g] \min[H/m, \hat{h}^l] \\ & + (1 - \Phi[H/m]) \left( \frac{\partial}{\partial h} \min[H/m, \hat{h}^l] F + \frac{\partial}{\partial \hat{h}^l} \min[H/m, \hat{h}^l] m \hat{p}^l \right) \end{aligned}$$

$$\Psi_H^G[m, H; \hat{h}^l, \hat{h}^g; F, L, \hat{p}^l, \hat{p}^g] = \quad (43d)$$

$$\begin{aligned} & \Psi_m^G[m, H; \hat{h}^l, \hat{h}^g; F, L, \hat{p}^l, \hat{p}^g] \max[H/m, \hat{h}^g] \\ & + \Phi[H/m] \left( \frac{\partial}{\partial h} \max[H/m, \hat{h}^g] F + \frac{\partial}{\partial \hat{h}^g} \max[H/m, \hat{h}^g] m \hat{p}^g \right). \end{aligned}$$

Here  $\Phi[h; \hat{h}^l, \hat{h}^g]$ ,  $\min[h, \hat{h}^l]$ , and  $\max[h, \hat{h}^g]$ , as well as their relevant partial derivatives, can be easily expressed in terms of the *Heaviside* function

$$\tilde{H}[x] = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0. \end{cases} \quad (44)$$

We have:

$$\Phi[h; \hat{h}^l, \hat{h}^g] = \frac{1}{\hat{h}^g - \hat{h}^l} \int_0^h (\tilde{H}[h - \hat{h}^l] - \tilde{H}[h - \hat{h}^g]) dh \quad (45a)$$

$$\min[h, \hat{h}^l] = \int_0^h (\tilde{H}[h] - \tilde{H}[h - \hat{h}^l]) dh \quad (45b)$$

$$\max[h, \hat{h}^g] = \hat{h}^g + \int_0^h \tilde{H}[h - \hat{h}^g] dh \quad (45c)$$

$$\frac{\partial}{\partial h} \Phi[h; \hat{h}^l, \hat{h}^g] = \frac{1}{\hat{h}^g - \hat{h}^l} (\tilde{H}[h - \hat{h}^l] - \tilde{H}[h - \hat{h}^g]) \quad (45d)$$

$$\begin{aligned} \frac{\partial}{\partial \hat{h}^l} \Phi[h; \hat{h}^l, \hat{h}^g] &= \frac{1}{(\hat{h}^g - \hat{h}^l)^2} \int_0^h (\tilde{H}[h - \hat{h}^l] - \tilde{H}[h - \hat{h}^g]) dh + \frac{-1}{\hat{h}^g - \hat{h}^l} \tilde{H}[h - \hat{h}^l] \\ &- \frac{1}{\hat{h}^g - \hat{h}^l} \Phi[h; \hat{h}^l, \hat{h}^g] + \frac{-1}{\hat{h}^g - \hat{h}^l} H[h - \hat{h}^l] \end{aligned} \quad (45e)$$

$$\begin{aligned} \frac{\partial}{\partial \hat{h}^g} \Phi[h; \hat{h}^l, \hat{h}^g] &= \frac{1}{(\hat{h}^g - \hat{h}^l)^2} \int_0^h (\tilde{H}[h - \hat{h}^l] - \tilde{H}[h - \hat{h}^g]) dh + \frac{-1}{\hat{h}^g - \hat{h}^l} \tilde{H}[h - \hat{h}^g] \\ &- \frac{-1}{\hat{h}^g - \hat{h}^l} \Phi[h; \hat{h}^l, \hat{h}^g] + \frac{-1}{\hat{h}^g - \hat{h}^l} H[h - \hat{h}^g] \end{aligned} \quad (45f)$$

$$\frac{\partial}{\partial h} \min[h, \hat{h}^l] = 1 - \tilde{H}[h - \hat{h}^l] \quad (45g)$$

$$\frac{\partial}{\partial \hat{h}^l} \min[h, \hat{h}^l] = H[h - \hat{h}^l] \quad (45h)$$

$$\frac{\partial}{\partial h} \max[h, \hat{h}^g] = \tilde{H}[h - \hat{h}^g] \quad (45i)$$

$$\frac{\partial}{\partial \hat{h}^g} \max[h, \hat{h}^g] = 1 - \tilde{H}[h - \hat{h}^g]. \quad (45j)$$

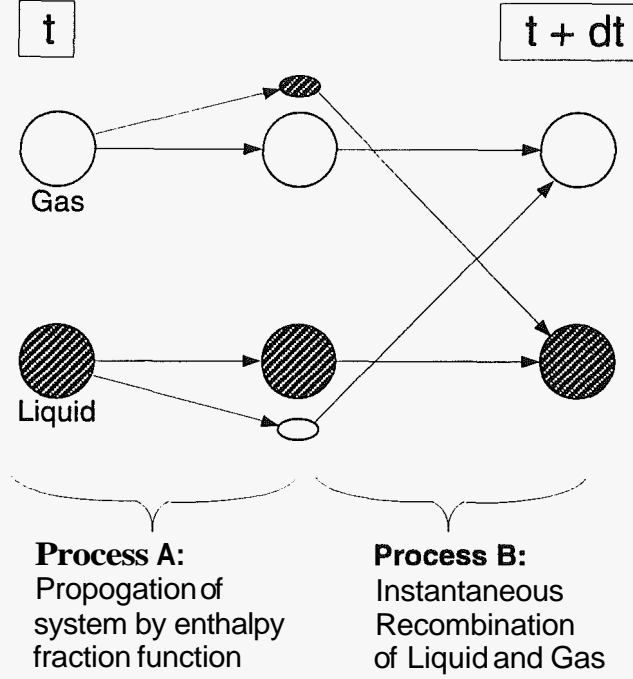


Figure 4: Dual-Phase Model with No Phase Holdup

### 5.3 Model with No Phase Holdup

The previous section applies strictly to a single thermodynamic system with a combination of phases. These results can be used to derive a system of two coupled thermodynamic subsystems where each subsystem can have different thermodynamic properties (e.g., different temperatures).

Here we develop the subsystem models to allow transitions between the liquid and gaseous phases (i.e., condensation and evaporation) but to require each subsystem remain a pure phase (i.e., purely liquid or purely gas). Let the liquid subsystem be denoted by subscript “1” and the gaseous subsystem be denoted by subscript “2”. Then, as illustrated in Figure 4, condensation and evaporation are combined with an instantaneous recombination of phases so that each subsystem remains strictly of one phase.

Since Equation 42 allows one to separate phases on a differential level for each subsystem, the integrity of the phases can be maintained simply by a differential recombination of liquid and gaseous components. We have:

$$\frac{dm_1}{dt} = \Psi_m^L[m_1, H_1; \hat{h}_1^l, \hat{h}_1^g; F_1, L_1, \hat{p}_1^l, \hat{p}_1^g] + \Psi_m^L[m_2, H_2; \hat{h}_2^l, \hat{h}_2^g; F_2, L_2, \hat{p}_2^l, \hat{p}_2^g] \quad (46a)$$

$$\frac{dm_2}{dt} = \Psi_m^G[m_1, H_1; \hat{h}_1^l, \hat{h}_1^g; F_1, L_1, \hat{p}_1^l, \hat{p}_1^g] + \Psi_m^G[m_2, H_2; \hat{h}_2^l, \hat{h}_2^g; F_2, L_2, \hat{p}_2^l, \hat{p}_2^g] \quad (46b)$$

$$\frac{dH_1}{dt} = \Psi_H^L[m_1, H_1; \hat{h}_1^l, \hat{h}_1^g; F_1, L_1, \hat{p}_1^l, \hat{p}_1^g] + \Psi_H^L[m_2, H_2; \hat{h}_2^l, \hat{h}_2^g; F_2, L_2, \hat{p}_2^l, \hat{p}_2^g] \quad (46c)$$

$$\frac{dH_2}{dt} = \Psi_H^G[m_1, H_1; \hat{h}_1^l, \hat{h}_1^g; F_1, L_1, \hat{p}_1^l, \hat{p}_1^g] + \Psi_H^G[m_2, H_2; \hat{h}_2^l, \hat{h}_2^g; F_2, L_2, \hat{p}_2^l, \hat{p}_2^g]. \quad (46d)$$

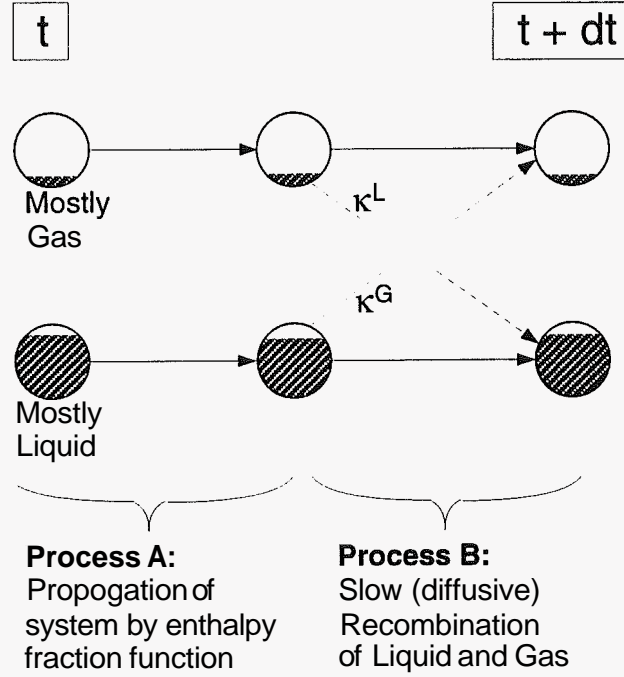


Figure 5: Dual-Phase Model with Phase Holdup

## 5.4 Model with Phase Holdup

At a cost of increasing the number of states, one can **also** model the latency of the recombination of phases so that within each subsystem some mixing of phases is inevitable. However, the recombination process is designed to be sufficiently *fast* so that one phase predominates in each subsystem.

Let the *mostly liquid* subsystem be denoted by subscript “1” and the *mostly gaseous* subsystem be denoted by subscript “2”. First assume the two systems are decoupled so that each subsystem can be independently propagated as in Equation 42. Next, introduce the coupling parameters  $\kappa^L$ ,  $\kappa^G$ , which respectively model the diffusive recombination of the liquid and gaseous components. Then, as illustrated in Figure 5, condensation and evaporation are combined with a slower recombination of phases so that neither subsystem remains strictly of one phase. We have:

$$\frac{dm_1^L}{dt} = \Psi_m^L[m_1, H_1; \hat{h}_1^l, \hat{h}_1^g; F_1, L_1, \hat{p}_1^l, \hat{p}_1^g] + \kappa^L m_2^L \quad (47a)$$

$$\frac{dm_1^G}{dt} = \Psi_m^G[m_1, H_1; \hat{h}_1^l, \hat{h}_1^g; F_1, L_1, \hat{p}_1^l, \hat{p}_1^g] - \kappa^G m_1^G \quad (47b)$$

$$\frac{dH_1^L}{dt} = \Psi_H^L[m_1, H_1; \hat{h}_1^l, \hat{h}_1^g; F_1, L_1, \hat{p}_1^l, \hat{p}_1^g] + \kappa^L H_2^L \quad (47c)$$

$$\frac{dH_1^G}{dt} = \Psi_H^G[m_1, H_1; \hat{h}_1^l, \hat{h}_1^g; F_1, L_1, \hat{p}_1^l, \hat{p}_1^g] - \kappa^G H_1^G \quad (47d)$$

$$\frac{dm_2^L}{dt} = \Psi_m^L[m_2, H_2; \hat{h}_2^l, \hat{h}_2^g; F_2, L_2, \hat{p}_2^l, \hat{p}_2^g] - \kappa^L m_2^L \quad (47e)$$

$$\frac{dm_2^G}{dt} = \Psi_m^G[m_2, H_2; \hat{h}_2^l, \hat{h}_2^g; F_2, L_2, \hat{p}_2^l, \hat{p}_2^g] + \kappa^G m_2^G \quad (47f)$$

$$\frac{dH_2^L}{dt} = \Psi_H^L[m_2, H_2; \hat{h}_2^l, \hat{h}_2^g; F_2, L_2, \hat{p}_2^l, \hat{p}_2^g] - \kappa^L H_2^L \quad (47g)$$

$$\frac{dH_2^G}{dt} = \Psi_H^G[m_2, H_2; \hat{h}_2^l, \hat{h}_2^g; F_2, L_2, \hat{p}_2^l, \hat{p}_2^g] + \kappa^G H_2^G. \quad (47h)$$

By choosing the coupling parameters,  $\kappa^L$ ,  $\kappa^G$ , sufficiently large, one can ensure that diffusive recombination process is sufficiently fast so that a single phase predominates in each subsystem. In fact, one can show by elementary techniques that as  $\kappa \rightarrow \infty$ , where  $\kappa = \min[\kappa^L, \kappa^G]$ , the solutions to the model with phase holdup converge to solutions of the model with no phase holdup. Details are provided in Appendix A.

## 5.5 Summary of Dual-Phase Model

The preceeding sections have presented options for modeling the secondary balloon with and without latency in the recombination of phases. In addition, a consistency in these models has been demonstrated. For the remainder of this paper we focus on the dual-phase model with no phase holdup. Then Equation 46 can be re-written with explicit reference to the “secondary liquid” and “secondary gaseous” subsystems:

$$\frac{dm_{sl}}{dt} = \Psi_m^L[m_{sl}, H_{sl}; \hat{h}_{sl}^l, \hat{h}_{sl}^g; F_{sl}, L_{sl}, \hat{p}_{sl}^l, \hat{p}_{sl}^g] \quad (48a)$$

$$\begin{aligned} & + \Psi_m^L[m_{sg}, H_{sg}; \hat{h}_{sg}^l, \hat{h}_{sg}^g; F_{sg}, L_{sg}, \hat{p}_{sg}^l, \hat{p}_{sg}^g] \\ \frac{dm_{sg}}{dt} & = \Psi_m^G[m_{sl}, H_{sl}; \hat{h}_{sl}^l, \hat{h}_{sl}^g; F_{sl}, L_{sl}, \hat{p}_{sl}^l, \hat{p}_{sl}^g] \quad (48b) \\ & + \Psi_m^G[m_{sg}, H_{sg}; \hat{h}_{sg}^l, \hat{h}_{sg}^g; F_{sg}, L_{sg}, \hat{p}_{sg}^l, \hat{p}_{sg}^g] \end{aligned}$$

$$\frac{dH_{sl}}{dt} = \Psi_H^L[m_{sl}, H_{sl}; \hat{h}_{sl}^l, \hat{h}_{sl}^g; F_{sl}, L_{sl}, \hat{p}_{sl}^l, \hat{p}_{sl}^g] \quad (48c)$$

$$\begin{aligned} & + \Psi_H^L[m_{sg}, H_{sg}; \hat{h}_{sg}^l, \hat{h}_{sg}^g; F_{sg}, L_{sg}, \hat{p}_{sg}^l, \hat{p}_{sg}^g] \\ \frac{dH_{sg}}{dt} & = \Psi_H^G[m_{sl}, H_{sl}; \hat{h}_{sl}^l, \hat{h}_{sl}^g; F_{sl}, L_{sl}, \hat{p}_{sl}^l, \hat{p}_{sl}^g] \quad (48d) \\ & + \Psi_H^G[m_{sg}, H_{sg}; \hat{h}_{sg}^l, \hat{h}_{sg}^g; F_{sg}, L_{sg}, \hat{p}_{sg}^l, \hat{p}_{sg}^g]. \end{aligned}$$

Here the input functions,  $F_{sl}$ ,  $L_{sl}$ ,  $\hat{p}_{sl}^l$ ,  $\hat{p}_{sl}^g$ ,  $F_{sg}$ ,  $L_{sg}$ ,  $\hat{p}_{sg}^l$ , and  $\hat{p}_{sg}^g$ , are defined as in Section 5.1 for both subsystems.

## 6 Summary of Equations for the Aerobot System

The thermodynamic equations of the previous sections must be coupled with dynamical equations to control the aerobot motion. Let  $x$ ,  $y$ , and  $z$  be the Cartesian coordinates of the aerobot where  $z$  represents the altitude. First we model the motion in the  $x$  and  $y$  directions as dependent only on a prescribed wind velocity field so that the balloon simply travels at the prevailing wind speed in the horizontal plane:

$$\frac{d}{dt}x = w_x(x, y, z, t) \quad (49a)$$

$$\frac{d}{dt}y = w_y(x, y, z, t). \quad (49b)$$

The motion in the  $z$  direction reflects a complex interaction between the thermodynamic variations and the forces on the aerobot. Substitution of Equations 5a and 6 into Equation 7 yields:

$$(m_{sys} + m_{virtual})\mathbf{a}_{tot} = \mathbf{F}_{grav} + \mathbf{F}_{buoy} + \mathbf{F}_{drag}. \quad (50)$$

Then, for the implementation here, this equation is re-written as:

$$(m_{sys} + C_m \rho_{atm} \mathcal{V}) \frac{d^2 z}{dt^2} = g(\rho_{atm} \mathcal{V} - m_{sys}) - \frac{1}{2} \rho_{atm} C_D \left( \frac{dz}{dt} + w_z \right) \left| \frac{dz}{dt} + w_z \right| \dot{A}_p. \quad (51)$$

The inertial factor on the left-hand side reflects not only the total mass of the system,  $m_{sys}$ , but also the effect of displaced air mass. Here  $C_m$  is the coefficient of virtual mass,  $\rho_{atm}$  is the density of air, and  $\mathcal{V}$  is the total volume displaced by the aerobot. This factor also arises on the right-hand side of the equation multiplied by  $g$ , the gravitational constant, to represent the force of buoyancy which acts on the system. The other term on the right-hand side models aerodynamic drag. Here  $C_D$  is the coefficient of aerodynamic drag,  $\dot{A}_p$  is the effective cross-sectional area, and  $w_z$  models the wind updraft analogously to the terms in Equation 49 [7]. From Equation 51 one can also directly characterize the vertical acceleration:

$$\frac{d^2 z}{dt^2} = \mathcal{A}_z = \frac{\left( g(\rho_{atm} \mathcal{V} - m_{sys}) - \frac{1}{2} \rho_{atm} C_D \left( \frac{dz}{dt} + w_z \right) \left| \frac{dz}{dt} + w_z \right| \dot{A}_p \right)}{(m_{sys} + C_m \rho_{atm} \mathcal{V})}. \quad (52)$$

Then finally the combined set of equations to model the aerobot system can be written in terms of a state equation

$$\frac{d\mathcal{X}}{dt} = \mathcal{F}(\mathcal{X}), \quad (53)$$

where the state is defined as:

$$\mathcal{X} = (x, y, z, \dot{z}, m_{pg}, m_{sl}, m_{sg}, H_{pf}, H_{pg}, H_{sf}, H_{sl}, H_{sg})^T. \quad (54)$$

Here  $x$ ,  $y$ , and  $z$  are the Cartesian coordinates of the aerobot where  $z$  represents the altitude and  $\dot{z}$  represents the velocity in this direction. The values  $m_{pg}$ ,  $m_{sl}$ , and  $m_{sg}$  respectively represent the

masses of the primary gas, the secondary liquid and the secondary gas. And the values  $H_{pf}$ ,  $H_{pg}$ ,  $H_{sf}$ ,  $H_{sl}$ , and  $H$ , respectively represent the total enthalpies of the primary film, primary gas, the secondary film, the secondary liquid and the secondary gas. The relevant differential equations can be summarized from the previous sections:

$$\frac{d}{dt}x = w_x(x, y, z, t) \quad (55a)$$

$$\frac{d}{dt}y = w_y(x, y, z, t) \quad (55b)$$

$$\frac{d}{dt}z = \dot{z} \quad (55c)$$

$$\frac{d}{dt}\dot{z} = \mathcal{A}_z(\mathcal{X}) \quad (55d)$$

$$\frac{d}{dt}m_{pg} = L_{pg} \quad (55e)$$

$$\begin{aligned} \frac{d}{dt}m_{sl} = & \Psi_m^L[m_{sl}, H_{sl}; \hat{h}_{sl}^l, \hat{h}_{sl}^g; F_{sl}, L_{sl}, \hat{p}_{sl}^l, \hat{p}_{sl}^g] \\ & + \Psi_m^L[m_{sg}, H_{sg}; \hat{h}_{sg}^l, \hat{h}_{sg}^g; F_{sg}, L_{sg}, \hat{p}_{sg}^l, \hat{p}_{sg}^g] \end{aligned} \quad (55f)$$

$$\begin{aligned} \frac{d}{dt}m_{sg} = & \Psi_m^G[m_{sl}, H_{sl}; \hat{h}_{sl}^l, \hat{h}_{sl}^g; F_{sl}, L_{sl}, \hat{p}_{sl}^l, \hat{p}_{sl}^g] \\ & + \Psi_m^G[m_{sg}, H_{sg}; \hat{h}_{sg}^l, \hat{h}_{sg}^g; F_{sg}, L_{sg}, \hat{p}_{sg}^l, \hat{p}_{sg}^g] \end{aligned} \quad (55g)$$

$$\frac{d}{dt}H_{pf} = F_{pf} \quad (55h)$$

$$\frac{d}{dt}H_{pg} = F_{pg} \quad (55i)$$

$$\frac{d}{dt}H_{sf} = F_{sf} \quad (55j)$$

$$\begin{aligned} \frac{d}{dt}H_{sl} = & \Psi_H^L[m_{sl}, H_{sl}; \hat{h}_{sl}^l, \hat{h}_{sl}^g; F_{sl}, L_{sl}, \hat{p}_{sl}^l, \hat{p}_{sl}^g] \\ & + \Psi_H^L[m_{sg}, H_{sg}; \hat{h}_{sg}^l, \hat{h}_{sg}^g; F_{sg}, L_{sg}, \hat{p}_{sg}^l, \hat{p}_{sg}^g] \end{aligned} \quad (55k)$$

$$\begin{aligned} \frac{d}{dt}H_{sg} = & \Psi_H^G[m_{sl}, H_{sl}; \hat{h}_{sl}^l, \hat{h}_{sl}^g; F_{sl}, L_{sl}, \hat{p}_{sl}^l, \hat{p}_{sl}^g] \\ & + \Psi_H^G[m_{sg}, H_{sg}; \hat{h}_{sg}^l, \hat{h}_{sg}^g; F_{sg}, L_{sg}, \hat{p}_{sg}^l, \hat{p}_{sg}^g]. \end{aligned} \quad (55l)$$

Here Equations 55a and 55b are defined as in Equation 49 while Equations 55c and 55d are defined as in Equation 52. Equations 55e and 55i for the primary balloon mass and total enthalpy come from Equation 31. Likewise, Equations 55h and 55j for the total enthalpies of the primary and secondary films come from Equation 33. And also Equations 55f, 55g, 55k, and 55l for the

masses and total enthalpies of the secondary liquid and the secondary gas come from Equation 48.

## 7 Simulation Results

An obvious part of developing a new model is to test it with simulations. Before describing that work for the new phase-change model, it is necessary to describe a little of the simulation history leading up to this point.

After the concept was developed to use a phase-change fluid to control the vertical motion of exploratory balloons, our team took the basic balloon dynamics described in Sections 2-4 and extended it with a discrete approximation to the phase-change thermodynamics [7]. This early simulation system was implemented in a personal computer spreadsheet program. This environment was excellent for the rapid changes and improvements that were necessary during the early development, but simulations were slow. It is not unusual to simulate balloon flights of ten to twenty-four hours with a time step of one to two seconds. As personal computer hardware improved over recent years, more powerful machines speeded up these simulations, but even current systems take roughly an hour to run interesting simulations.

Numerous design studies, preparations for ground-based flights, and post-flight analysis of flight data all require extensive simulations. Therefore our team re-implemented the simulation in C. The same algorithm as in the spreadsheet version was implemented, with its deficiencies, in a C program which runs many times faster than the spreadsheet version (several seconds instead of approximately an hour).

When the new phase-change differential equations were derived, our team implemented them in the the C simulation program. Various simulations were run and compared to previous simulations. Before showing simulation results, it is worthwhile describing the scenario being simulated. As part of the process of improving our understanding of oscillatory balloon behavior, we have run a series of earth-based experiments called Altitude Control Experiment (ALICE) [5]. In preparation for our next ALICE flight, we have been running simulations based on the previous C program.

The simulated flight starts at JPL in Pasadena, California, and uses weather data for late May of 1996. The main balloon is filled with approximately 0.520 kg of Helium and the secondary system is filled with a nearly azeotropic mixture of two commercial refrigerants which we call “Mix 2” for historical reasons.

A number of mixtures were evaluated with the end goal of finding a mixture substitute for R114 that was non-toxic, non-flammable, low-ozone-depleting, and of minimal molecular weight (to produce larger delta buoyancy changes for any given mixture mass). The preferred final mixture has come to be known as Mix 2, which consists of a mixture of 35% by weight of R124 (2-chloro-1,1,1,2-tetrafluorethane or  $\text{CHClFCF}_3$ ) and 65% by weight of R142b (chlorodifluorethane or  $\text{CH}_3\text{CClF}_2$ ). Both of these fluids are non-toxic and low-ozone-depleting. R142b is slightly flammable and has been mixed with R124 in such proportions that the mixture itself is always non-flammable for any given air/fuel ratio. The normal boiling point (NBP) of R124 is -12.00 C, and that of R142b is -9.78 C, while the mixture NBP is -10.6 C. The NBP is significantly lower than that of the single fluid, R114, at 3.61 C, and thus the equilibrium altitude for Mix 2 is several km higher than that for R114. The molecular weight for Mix 2 is 110.7, compared to 171 for R114, and thus Mix 2 can create delta buoyancy changes that are 54% larger than R114, for any given mass of buoyancy fluid. An atmospheric model was constructed using data from upper atmosphere sounding balloons

released from nearby weather stations. With reference to that atmospheric model, the equilibrium altitude for Mix 2 is approximately 8620 meters.

In Figure 6, two simulations from our preliminary studies are shown. One is the simulation using the previous version of the C simulation program (dubbed "old") and the other is the "new" simulation based on results presented in this paper. The lighter line is the previous C simulation

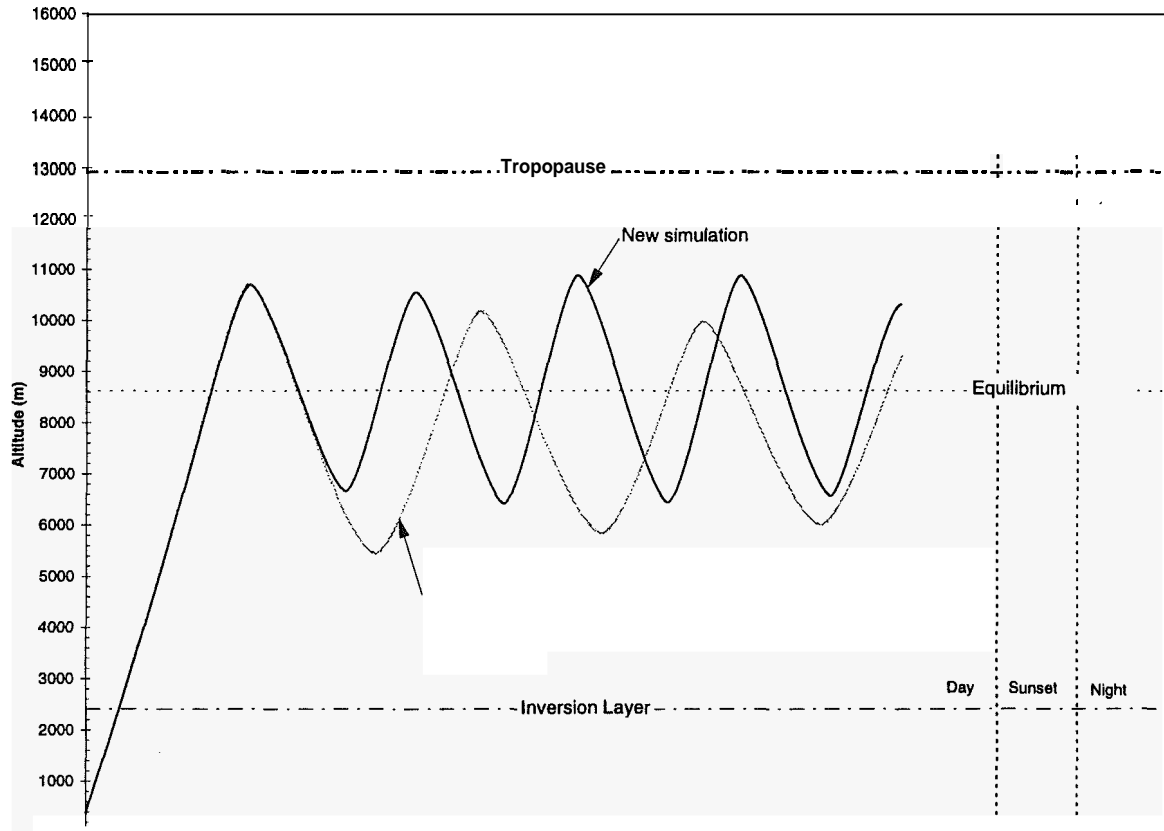


Figure 6: Aerobot Flight Simulation

and the dark line is the new simulation. Notice that there is some divergence at about 8000 seconds, when the system descends below the equilibrium altitude and the phase-change fluid starts vaporizing in the heat exchanger. We believe this divergence is due to inaccuracies of the previous model although more testing will be necessary to validate the new model fully. Notice that the oscillations of the old simulation are not centered about the equilibrium altitude whereas the oscillations of the new simulation are.

## 8 Conclusion

This paper presents a non-equilibrium, differential-equation model for a two-balloon aerobot system including a dual-phase model for phase change and mass transfer. This model provides a significant improvement over previously developed models of phase change and mass transfer for two-balloon



systems. The differential equation formulation and the modeling of phase mixing holdup will advance the accuracy and flexibility of the models used to design and analyze these systems.

## Appendix A Asymptotic Equivalence of Dual-Phase Models

Here an equivalence between the two dual-phase models is demonstrated in the limit as the recombination of the phases becomes arbitrarily fast in the model with phase holdup (Equation 47). That is, we are interested in the limiting case  $\kappa \rightarrow \infty$  where  $\kappa = \min[\kappa^L, \kappa^G]$ . The result essentially follows from estimates involving decaying exponential functions. For example, let  $f(t)$  be a continuous function with the bound  $|f(t)| < f_0$ . It then follows:

$$|\exp(-\kappa t) \int_0^t \exp(\kappa \tau) f(\tau) d\tau| < \frac{f_0}{\kappa} (1 - \exp(-\kappa t)). \quad (56)$$

Thus, terms of this form decay to zero as  $\kappa \rightarrow \infty$  with a rate denoted by  $\mathcal{O}(\frac{1}{\kappa})$ . The following technical assumption is required:

For sufficiently large  $\kappa = \min[\kappa^L, \kappa^G]$ , the System of Equations 47 has solutions which are bounded independently of  $\kappa^L, \kappa^G$ . For all solution values, the functions defined by Equation 43 are then bounded independently of  $\kappa^L, \kappa^G$ .

Consider the System of Equations 47. In the context of this analysis, one can consider all functions simply as time-dependent (e.g.,  $\Psi_m^L(t; 1) \equiv \Psi_m^L[m_1, H_1; \hat{h}_1^l, \hat{h}_1^g; F_1, L_1, \hat{p}_1^l, \hat{p}_1^g]$ ). The components which represent phase holdup are modeled by Equations 47b, 47d, 47e, and 47g. With the assumption of initial conditions at  $t = 0$ , these can be integrated to give:

$$m_1^G(t) = m_1^G(0) \exp(-\kappa^G t) + \exp(-\kappa^G t) \int_0^t \exp(\kappa^G \tau) \Psi_m^G(\tau; 1) d\tau \quad (57a)$$

$$H_1^G(t) = H_1^G(0) \exp(-\kappa^G t) + \exp(-\kappa^G t) \int_0^t \exp(\kappa^G \tau) \Psi_H^G(\tau; 1) d\tau \quad (57b)$$

$$m_2^L(t) = m_2^L(0) \exp(-\kappa^L t) + \exp(-\kappa^L t) \int_0^t \exp(\kappa^L \tau) \Psi_m^L(\tau; 2) d\tau \quad (57c)$$

$$H_2^L(t) = H_2^L(0) \exp(-\kappa^L t) + \exp(-\kappa^L t) \int_0^t \exp(\kappa^L \tau) \Psi_H^L(\tau; 2) d\tau. \quad (57d)$$

And also, by means of integration by parts, one can estimate the corresponding first integrals:

$$\kappa^G \int_0^t m_1^G(\tau) d\tau \quad (58a)$$

$$= \kappa^G \int_0^t m_2^L(0) \exp(-\kappa^G \tau) d\tau + \kappa^G \int_0^t \exp(-\kappa^G \tau) \int_0^\tau \exp(\kappa^G \sigma) \Psi_m^G(\sigma; 1) d\sigma d\tau$$

$$= m_1^G(0)(1 - \exp(-\kappa^G t)) + \int_0^t \Psi_m^G(\tau; 1) d\tau - \exp(-\kappa^G t) \int_0^t \exp(\kappa^G \tau) \Psi_m^G(\tau; 1) d\tau$$

$$\kappa^G \int_0^t H_1^G(\tau) d\tau \quad (58b)$$

$$\begin{aligned} &= \kappa^G \int_0^t H_1^G(0) \exp(-\kappa^G \tau) d\tau + \kappa^G \int_0^t \exp(-\kappa^G \tau) \int_0^\tau \exp(\kappa^G \sigma) \Psi_H^G(\sigma; 1) d\sigma d\tau \\ &= H_1^G(0)(1 - \exp(-\kappa^G t)) + \int_0^t \Psi_H^G(\tau; 1) d\tau - \exp(-\kappa^G t) \int_0^t \exp(\kappa^G \tau) \Psi_H^G(\tau; 1) d\tau \end{aligned}$$

$$\kappa^L \int_0^t m_2^L(\tau) d\tau = \quad (58c)$$

$$\begin{aligned} &= \kappa^L \int_0^t m_2^L(0) \exp(-\kappa^L \tau) d\tau + \kappa^L \int_0^t \exp(-\kappa^L \tau) \int_0^\tau \exp(\kappa^L \sigma) \Psi_m^L(\sigma; 2) d\sigma d\tau \\ &= m_2^L(0)(1 - \exp(-\kappa^L t)) + \int_0^t \Psi_m^L(\tau; 2) d\tau - \exp(-\kappa^L t) \int_0^t \exp(\kappa^L \tau) \Psi_m^L(\tau; 2) d\tau \end{aligned}$$

$$\kappa^L \int_0^t H_2^L(\tau) d\tau \quad (58d)$$

$$\begin{aligned} &= \kappa^L \int_0^t H_2^L(0) \exp(-\kappa^L \tau) d\tau + \kappa^L \int_0^t \exp(-\kappa^L \tau) \int_0^\tau \exp(\kappa^L \sigma) \Psi_H^L(\sigma; 2) d\sigma d\tau \\ &= H_2^L(0)(1 - \exp(-\kappa^L t)) + \int_0^t \Psi_H^L(\tau; 2) d\tau - \exp(-\kappa^L t) \int_0^t \exp(\kappa^L \tau) \Psi_H^L(\tau; 2) d\tau. \end{aligned}$$

In accordance with Equation 39, the liquid and gaseous components of each subsystem can be combined to recover that subsystem's mass and total enthalpy. Thus, combining alternate lines of Equation 47 gives:

$$\frac{dm_1}{dt} = \Psi_m^L(t; 1) + \Psi_m^G(t; 1) + \kappa^L m_2^L - \kappa^G m_1^G \quad (59a)$$

$$\frac{dm_2}{dt} = \Psi_m^L(t; 2) + \Psi_m^G(t; 2) - \kappa^L m_2^L + \kappa^G m_1^G \quad (59b)$$

$$\frac{dH_1}{dt} = \Psi_H^L(t; 1) + \Psi_H^G(t; 1) + \kappa^L H_2^L - \kappa^G H_1^G \quad (59c)$$

$$\frac{dH_2}{dt} = \Psi_H^L(t; 2) + \Psi_H^G(t; 2) - \kappa^L H_2^L + \kappa^G H_1^G. \quad (59d)$$

Then, with initial conditions at  $t = 0$  in the decomposition given by Equation 39, integration of this system combined with Equation 58 gives:

$$m_1(t) = \left( m_1^L(0) + m_1^G(0) \right) + \int_0^t \left( \Psi_m^L(\tau; 1) + \Psi_m^G(\tau; 1) + \kappa^L m_2^L(\tau) - \kappa^G m_1^G(\tau) \right) d\tau \quad (60a)$$

$$\begin{aligned} &= \left( m_1^L(0) + m_2^L(0) \right) + \int_0^t \left( \Psi_m^L(\tau; 1) + \Psi_m^L(\tau; 2) \right) d\tau \\ &\quad - \left( m_2^L(0) + \int_0^t \exp(\kappa^L \tau) \Psi_m^L(\tau; 2) d\tau \right) \exp(-\kappa^L t) \\ &\quad + \left( m_1^G(0) + \int_0^t \exp(\kappa^G \tau) \Psi_m^G(\tau; 1) d\tau \right) \exp(-\kappa^G t) \end{aligned}$$

$$m_2(t) = \left( m_2^L(0) + m_2^G(0) \right) + \int_0^t \left( \Psi_m^L(\tau; 2) + \Psi_m^G(\tau) - \kappa^L m_2^L(\tau) + \kappa^G m_1^G(\tau) \right) d\tau \quad (60b)$$

$$\begin{aligned} &= \left( m_2^G(0) + m_1^G(0) \right) + \int_0^t \left( \Psi_m^G(\tau; 2) + \Psi_m^G(\tau; 1) \right) d\tau \\ &\quad + \left( m_2^L(0) + \int_0^t \exp(\kappa^L \tau) \Psi_m^L(\tau; 2) d\tau \right) \exp(-\kappa^L t) \\ &\quad - \left( m_1^G(0) + \int_0^t \exp(\kappa^G \tau) \Psi_m^G(\tau; 1) d\tau \right) \exp(-\kappa^G t) \end{aligned}$$

$$H_1(t) = \left( H_1^L(0) + H_1^G(0) \right) + \int_0^t \left( \Psi_H^L(\tau; 1) + \Psi_H^G(\tau; 1) + \kappa^L H_2^L(\tau) - \kappa^G H_1^G(\tau) \right) d\tau \quad (60c)$$

$$\begin{aligned} &= \left( H_1^L(0) + H_2^L(0) \right) + \int_0^t \left( \Psi_H^L(\tau; 1) + \Psi_H^L(\tau; 2) \right) d\tau \\ &\quad - \left( H_2^L(0) + \int_0^t \exp(\kappa^L \tau) \Psi_H^L(\tau; 2) d\tau \right) \exp(-\kappa^L t) \\ &\quad + \left( H_1^G(0) + \int_0^t \exp(\kappa^G \tau) \Psi_H^G(\tau; 1) d\tau \right) \exp(-\kappa^G t) \end{aligned}$$

$$H_2(t) = \left( H_2^L(0) + H_2^G(0) \right) + \int_0^t \left( \Psi_H^L(\tau; 2) + \Psi_H^G(\tau; 2) - \kappa^L H_2^L(\tau) + \kappa^G H_1^G(\tau) \right) d\tau \quad (60d)$$

$$\begin{aligned} &= \left( H_2^G(0) + H_1^G(0) \right) + \int_0^t \left( \Psi_H^G(\tau; 2) + \Psi_H^G(\tau; 1) \right) d\tau \\ &\quad + \left( H_2^L(0) + \int_0^t \exp(\kappa^L \tau) \Psi_H^L(\tau; 2) d\tau \right) \exp(-\kappa^L t) \\ &\quad - \left( H_1^G(0) + \int_0^t \exp(\kappa^G \tau) \Psi_H^G(\tau; 1) d\tau \right) \exp(-\kappa^G t). \end{aligned}$$

Then finally we eliminate all terms which vanish in the limit as  $\kappa \rightarrow \infty$  as in Equation 56. This results simply in an integrated form of the model with no phase holdup (Equation 46):

$$m_1(t) = \left( m_1^L(0) + m_2^L(0) \right) + \int_0^t \left( \Psi_m^L(\tau; 1) + \Psi_m^L(\tau; 2) \right) d\tau \quad (61a)$$

$$m_2(t) = \left( m_1^G(0) + m_2^G(0) \right) + \int_0^t \left( \Psi_m^G(\tau; 2) + \Psi_m^G(\tau; 1) \right) d\tau \quad (61b)$$

$$H_1(t) = \left( H_1^L(0) + H_2^L(0) \right) + \int_0^t \left( \Psi_H^L(\tau; 1) + \Psi_H^L(\tau; 2) \right) d\tau \quad (61c)$$

$$H_2(t) = \left( H_1^G(0) + H_2^G(0) \right) + \int_0^t \left( \Psi_H^G(\tau; 2) + \Psi_H^G(\tau; 1) \right) d\tau. \quad (61d)$$

The initial conditions for this final model with no phase holdup come from the phase decomposition of the *combined* initial state of the model with phase holdup.

## Acknowledgments

The work described in this paper was carried out by the Jet Propulsion Laboratory, California Institute of Technology, under a contract with the National Aeronautics and Space Administration.

## References

- [1] Carlson, L.A. and W.J. Horn. 1981. A Unified Thermal and Vertical Trajectory Model For the Prediction of High Altitude Balloon Performance. NASA-CR-156884 (TAMRF Rep. No. 4217-81-01), Texas A&M University.
- [2] Cutts, J.A., Nock, K.T., Jones, J.A., Rodriguez, G., and Balaram, J. 1996. Planetary Exploration by Robotic Aerovehicles. Journal of Autonomous Robots, January.
- [3] Iversen, H.W. and R. Balent. 1951. "A Correlating Modulus for Fluid Resistance in Accelerated Motion." Journal of Applied Physics. Volume 22, Number 3, pp. 324-328.
- [4] Jones, J.A., 1995, Reversible Fluid Balloon Altitude Control Concepts. 11th Lighter-Than-Air Systems Technology Conference, Clearwater Beach, FL, 16-18 May.
- [5] Nock, K.T., Aaron, K.M, Jones, J.A., McGee, D.P., Powell, G.E., Yavrouian, A.H., and Wu, J.J. 1995. Balloon Altitude Control Experiment (ALICE) Project. 11th Lighter-Than-Air Systems Technology Conference, Clearwater Beach, FL, 16-18 May.
- [6] Stokes, Sir G.G. 1850. Transactions of the Cambridge Philosophical Society. Vol. 9, pp. 8-106.
- [7] Wu, J-J and J.A. Jones. 1995. Performance Model for Reversible Fluid Balloons. 11th Lighter-Than-Air Systems Technology Conference, Clearwater Beach, FL, 16-18 May.