# ENPM 667 Control Systems For Robotics

# FINAL PROJECT

## Controller For Crane Model


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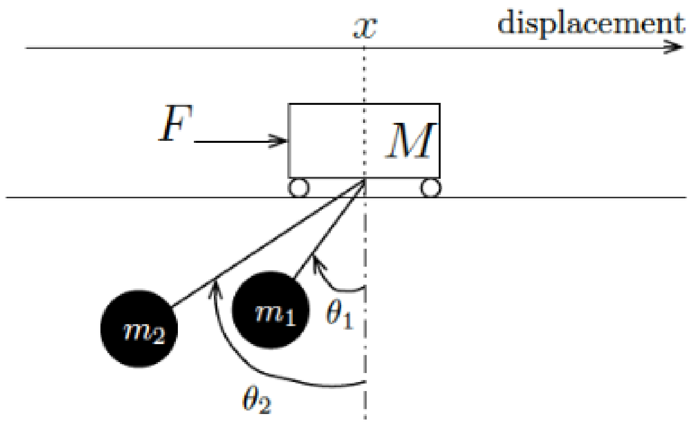


# 1 Part A

Obtain the Equations of Motion and Nonlinear State-Space Representation

Let us assume the following variables to represent the given system

- $M$: Mass of the cart
- $F$: Force applied to the cart
- $m_1$: Mass of load one
- $m_2$: Mass of load two
- $l_1$: Length of cable one to mass one
- $l_2$: Length of cable two to mass two
- $x$: Displacement of cart in positive x direction

**Lagrange equation's**

We know that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

and

$$L = PE - KE$$

where $PE$ is the Potential energy and $KE$ is the Kinetic energy of the system.

**Position of mass $m_1$ and $m_2$**

$$x_1 = x - l_1 \sin(\theta_1)$$

$$x_2 = x - l_2 \sin(\theta_1)$$

$$y_1 = -l_1 \cos(\theta_1)$$

$$y_2 = -l_2 \cos(\theta_2)$$

### 1.0.1 Velocity of mass $m_1$ and $m_2$

$$\dot{x}_1 = \dot{x} - l_1 \cos(\theta_1)\dot{\theta_1}$$

$$\dot{y}_1 = l_1 \sin(\theta_1)\dot{\theta_1}$$

$$\dot{x}_2 = \dot{x} - l_2 \cos(\theta_2)\dot{\theta_2}$$

$$\dot{y}_2 = l_2 \sin(\theta_2)\dot{\theta_2}$$

### 1.0.2 Magnitude of Velocity Vector

The magnitude of velocity is given by the following equation

$$v_1^2 = \dot{x_1}^2 + \dot{y_1}^2$$

Substituting the value of $\dot{x_1}$ and $\dot{y_1}$, we get

$$v_1^2 = (\dot{x} - l_1 \cos(\theta_1)\dot{\theta_1})^2 + (l_1 \sin(\theta_1)\dot{\theta_1})^2$$

$$v_1^2 = \dot{x}^2 + l_1^2 \cos^2(\theta_1)\dot{\theta_1}^2 - 2\dot{x_1} l_1 \cos(\theta_1)\dot{\theta_1} + l_1^2 \sin^2(\theta_1)\dot{\theta_1}^2$$

$$v_1^2 = \dot{x}^2 - 2\dot{x_1} l_1 \cos(\theta_1)\dot{\theta_1} + l_1\dot{\theta_1}(\sin^2(\theta_1) + cos^2(\theta_1)$$

$$v_1^2 = \dot{x}^2 + l_1^2\dot{\theta_1} - 2\dot{x_1} l_1 \cos(\theta_1)\dot{\theta_1}$$

Similarly for $v_2$

$$v_2^2 = \dot{x_2}^2 + \dot{y_2}^2$$

Substituting the value of $\dot{x_2}$ and $\dot{y_2}$, we get

$$v_2^2 = (\dot{x} - l_2 \cos(\theta_2)\dot{\theta_2})^2 + (l_2 \sin(\theta_2)\dot{\theta_2})^2$$

$$v_2^2 = \dot{x}^2 + l_2^2 \cos^2(\theta_2)\dot{\theta_2}^2 - 2\dot{x_2} l_2 \cos(\theta_2)\dot{\theta_2} + l_2^2 \sin^2(\theta_2)\dot{\theta_2}^2$$

$$v_2^2 = \dot{x}^2 - 2\dot{x_2} l_2 \cos(\theta_2)\dot{\theta_2} + l_2\dot{\theta_2}(\sin^2(\theta_2) + cos^2(\theta_2)$$

$$v_2^2 = \dot{x}^2 + l_2\dot{\theta_2} - 2\dot{x_2} l_2 \cos(\theta_2)\dot{\theta_2}$$

### 1.0.3 Computing the Kinetic Energy of the entire system

$$KE = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

Substituting the values of $v1$ and $v2$

$$KE = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m_1(\dot{x}^2 + l_1\dot{\theta}_1 - 2\dot{x}_1l_1\cos(\theta_1)\dot{\theta}_1)^2 + \frac{1}{2}m_2(\dot{x}^2 + l_2\dot{\theta}_2 - 2\dot{x}_2l_2\cos(\theta_2)\dot{\theta}_2)$$

$$KE = \frac{1}{2}\dot{X}(m_1 + m_2 + M) + \frac{1}{2}m_1l_1\dot{(\theta_1)}^2 + \frac{1}{2}m_2l_2\dot{(\theta_2)}^2$$

### 1.0.4 Computing the Potential Energy for the system

$$PE_1 \text{ for } m_1 = -m_1gl_1\cos(\theta_1)$$

$$PE_2 \text{ for } m_2 = -m_2gl_2\cos(\theta_2)$$

$$Total\text{PE} = \text{PE}_1 + PE_2 = -m_1gl_1\cos(\theta_1) - m_2gl_2\cos(\theta_2)$$

Now we will calculate L

$$L = KE - PE$$

$$L = \frac{1}{2}\dot{X}(m_1 + m_2 + M) + \frac{1}{2}m_1l_1\dot{\theta_1}^2 - m_1l_1\cos\theta_1\dot{\theta}_1\dot{x} - m_2l_2\cos\theta_2\dot{\theta}_2\dot{x} + m_1g_1\cos\theta_1 + m_2g_2\cos\theta_2$$

Now we will calculate $\frac{\partial L}{\partial X}$
Since there is no X term in the equation of L
Therefore

$$\frac{\partial L}{\partial X} = 0$$

Now,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{X}} = -m_1l_1\cos(\dot{\theta}_1)\dot{\theta}_1 - m_2l_2\cos(\dot{\theta}_2)\dot{\theta}_2 + \ddot{x}(m_1 + m_2 + M)$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{X}}) = -m_1l_1(-\sin(\dot{\theta}_1)\dot{\theta_1}^2 + \ddot{\theta}_1\cos(\theta_1)) - m_2l_2(-\sin(\dot{\theta}_2)\dot{\theta_2}^2 + \ddot{\theta}_2\cos(\theta_2)) + \ddot{x}(m_1 + m_2 + M) = F$$

Thus,

$$\frac{d}{dt}((\frac{\partial L}{\partial \dot{X}}) - \frac{\partial L}{\partial X} = -m_1 l_1(\ddot{\theta_1}\cos(\theta_1) + \dot{\theta_1}^2\sin(\theta_1)) - m_2 l_2(\ddot{\theta_2}\cos(\theta_2) + \dot{\theta_2}^2\sin(\theta_2))$$

$$\frac{\partial L}{\partial \dot{\theta_1}} = m_1 l_1^2 \dot{\theta}_1 - m_1 l_1 \cos(\theta_1)\dot{x}$$

$$\frac{\partial L}{\partial \dot{\theta_1}} = m_1 l_1 \sin(\theta_1) m_1 \dot{x}\dot{\theta_1} - m_1 g l_1 \sin(\theta_1)$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta_1}}) = m_1 l_1^2 \ddot{\theta_1} - m_1 l_1(\cos(\theta_1)\ddot{x} - \sin(\theta_1)\dot{x}\dot{\theta_1})$$

$$\frac{\partial L}{\partial \theta_2} = m_2 l_2 \dot{\theta}_2 \dot{x}\sin(\theta_2) - m_2 l_2 g \sin(\theta_2)$$

$$\frac{\partial L}{\partial \dot{\theta_2}} = m_2 l_2^2 \dot{\theta}_2 - m_1 l_2 \cos(\theta_2)\dot{x}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta_2}}) = m_2 l_2^2 \ddot{\theta_2} - m_2 l_2(\cos(\theta_2)\ddot{x} - \sin(\theta_2)\dot{x}\dot{\theta_2})$$

Thus, we get

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta_1}}) - \frac{\partial L}{\partial \dot{\theta_1}} = m_1 l_1^2 \ddot{\theta_1} - m_1 l_1(\cos(\theta_1)\ddot{x} - \sin(\theta_1)\dot{x}\dot{\theta_1}) - m_1 l_1 \sin(\theta_1) m_1 \dot{x}\dot{\theta_1} - m_1 g l_1 \sin(\theta_1)$$

On equating the equation with zero we have:

$$m_1 l_1 \ddot{\theta_1} - m_1 l_1 \ddot{x}_l \cos(\theta_1) + m_1 g l_1 \sin(\theta_1) = 0$$

$$\begin{aligned}
\dot{\theta}_1^2 &\approx 0 \\
\dot{\theta}_2^2 &\approx 0 \\
\sin(\theta_1) &\approx \theta_1 \\
\sin(\theta_2) &\approx \theta_2 \\
\cos(\theta_1) &\approx 1 \\
\cos(\theta_2) &\approx 1
\end{aligned}$$

$$m_1 l_1 \ddot{\theta_1} - m_1 l_1 \ddot{x}_l + m_1 g l_1 \theta_1 = 0$$

Similarly, we have:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta_2}}) - \frac{\partial L}{\partial \theta_2} = m_2 l_1^2 \ddot{\theta}_2 - m_2 l_2 (\cos(\theta_2)\ddot{x} - \sin(\theta_2)\dot{x}\dot{\theta}_2) - m_2 l_2 \sin(\theta_2) m_2 \dot{x}\dot{\theta}_2 + m_2 g l_2 \sin(\theta_2)$$

$$m_2 l_2 \ddot{\theta}_2 - m_2 l_2 \ddot{x}_2 \cos(\theta_2) + m_2 g l_2 \sin(\theta_2) = 0$$

On linearizing we have:

$$m_1 l_1 \ddot{\theta}_1 - m_1 l_1 \ddot{x}_l + m_1 g l_1 \theta_1 = 0$$

$$m_2 l_2 \ddot{\theta}_2 - m_2 l_2 \ddot{x}_2 + m_2 g l_2 \theta_2 = 0$$

$$F = \ddot{x}(M + m_1 + m_2) - (m_1 l_1 \ddot{\theta}_1 \cos(\theta_1) + m_2 l_2 \ddot{\theta}_2 \cos(\theta_2) - m_1 l_1 \dot{\theta}_1^2 \sin(\theta_1) - m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2))$$

hence can be derived as:

$$\ddot{\theta_1} = \frac{\cos(\theta_1)\ddot{x} - g\sin(\theta_1)}{l_1}$$

$$\ddot{\theta_2} = \frac{\cos(\theta_2)\ddot{x} - g\sin(\theta_2)}{l_2}$$

$$\ddot{x} = \frac{m_1 g \sin(\theta_1)\cos(\theta_2) + m_2 g \sin(\theta_2)\cos(\theta_2) - m_1 l_1 \dot{\theta}_1^2 \sin(\theta_1) - m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2) + F}{M + m_1 + m_2 - m_1 \cos^2(\theta_1) - m_2 \cos^2(\theta_2)}$$

$$\ddot{x} = \frac{F - m_1(g\sin(\theta_1)\cos\theta_1 + l_1 \dot{\theta}_1^2 \sin(\theta_1)) - m_2(g\sin(\theta_2)\cos(\theta_2) + l_2 \dot{\theta}_2^2 \sin(\theta_2))}{M + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2)}$$

Substituting $\ddot{x}$ in $\ddot{\theta_1}$ and $\ddot{\theta_2}$

$$\ddot{\theta_1} = \frac{\cos\theta_1}{l_1}(\frac{F - m_1 g(\sin\theta_1 \cos\theta_1 + l_1 \sin\theta_1 \dot{\theta_1}^2) - m_2(g\sin\theta_2\cos\theta_2 + l\sin\theta_2\dot{\theta}^2)}{M + m_1 \sin{\theta_1}^2 + m_2 \sin\theta^2}) - g\frac{\sin\theta_1}{l_1}$$

$$\ddot{\theta_2} = \frac{\cos\theta_2}{l_2}(\frac{F - m_1 g(\sin\theta_1 \cos\theta_1 + l_1 \sin\theta_1 \dot{\theta_1}^2) - m_2(g\sin\theta_2\cos\theta_2 + l\sin\theta_2\dot{\theta}^2)}{M + m_1 \sin^2(\theta_1) + m_2^2(\sin\theta)}) - g\frac{\sin\theta_2}{l_2}$$

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$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \frac{F - m_1(g\sin(\theta_1)\cos\theta_1 + l_1 \dot{\theta}_1^2 \sin(\theta_1)) - m_2(g\sin(\theta_2)\cos(\theta_2) + l_2 \dot{\theta}_2^2 \sin(\theta_2))}{M + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2)} \\ \dot{\theta_1} \\ \frac{\cos\theta_1}{l_1}(\frac{F - m_1 g(\sin\theta_1 \cos\theta_1 + l_1 \sin\theta_1 \dot{\theta_1}^2) - m_2(g\sin\theta_2\cos\theta_2 + l\sin\theta_2\dot{\theta}^2)}{M + m_1 \sin{\theta_1}^2 + m_2 \sin\theta^2}) - g\frac{\sin\theta_1}{l_1} \\ \dot{\theta_2} \\ \frac{\cos\theta_2}{l_2}(\frac{F - m_1 g(\sin\theta_1 \cos\theta_1 + l_1 \sin\theta_1 \dot{\theta_1}^2) - m_2(g\sin\theta_2\cos\theta_2 + l\sin\theta_2\dot{\theta}^2)}{M + m_1 \sin^2(\theta_1) + m_2^2(\sin\theta)}) - g\frac{\sin\theta_2}{l_2} \end{bmatrix}$$

# 2 Part B

Obtain the linearized system around the equilibrium point specified by $x = 0$ and $\theta_1 = \theta_2 = 0$. Write the state-space representation of the linearized system.

From the above linearized equation, we have the following state-space representation

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-gm_1}{M} & 0 & \frac{-gm_2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-g(M+m_1)}{Ml_2} & 0 & \frac{-gm_1}{Ml_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-gm_1}{Ml_2} & 0 & \frac{-g(M+m_2)}{Ml_2} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml_1} \\ 0 \\ \frac{1}{Ml_2} \end{bmatrix} F$$

$$\text{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-gm_1}{M} & 0 & \frac{-gm_2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-g(M+m_1)}{Ml_2} & 0 & \frac{-gm_1}{Ml_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-gm_1}{Ml_2} & 0 & \frac{-g(M+m_2)}{Ml_2} & 0 \end{bmatrix}$$

$$\text{B} = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml_1} \\ 0 \\ \frac{1}{Ml_2} \end{bmatrix}$$

# 3 Part C

Obtain conditions on M, m1 , m2 , l1 , l2 for which the linearized system is controllable.

The Controllability matrix,

The matrices $A$ and $B$ obtained above are independent of time, indicating that the system is a time-invariant (LTI) system. The controllability of an LTI system is determined by examining the controllability matrix, denoted as $C$, which has dimensions $n \times nm$. For controllability, the rank of matrix $C$ must satisfy the full-rank condition. Therefore, the rank of the controllability matrix, denoted as rank($C$), should be equal to $n$. It is expressed as follows:

$$\text{rank}(C) = \text{rank}\begin{bmatrix} B & AB & A^2B & A^3B & A^4B & A^5B \end{bmatrix} = n$$

The controllability matrix received is as follows:

Column 1 of Controllability Matrix:

```
>> C
Controllability matrix Mc:
[       0,      1/M,                                          0,            - (g*m1)/(L1*M^2) - (g*m2)/(L2*M^2),
[     1/M,        0,            - (g*m1)/(L1*M^2) - (g*m2)/(L2*M^2),                                          0,
[       0, 1/(L1*M),                                          0, - (M*g + g*m1)/(L1^2*M^2) - (g*m2)/(L1*L2*M^2),
[1/(L1*M),        0, - (M*g + g*m1)/(L1^2*M^2) - (g*m2)/(L1*L2*M^2),                                          0,
[       0, 1/(L2*M),                                          0, - (M*g + g*m2)/(L2^2*M^2) - (g*m1)/(L1*L2*M^2),
[1/(L2*M),        0, - (M*g + g*m2)/(L2^2*M^2) - (g*m1)/(L1*L2*M^2),                                          0,
```

Column 2 of Controllability Matrix:

```
                                                                                                                                    0,
            ((g*m1*(M*g + g*m1))/(L1*M^2) + (g^2*m1*m2)/(L2*M^2))/(L1*M) + ((g*m2*(M*g + g*m2))/(L2*M^2) + (g^2*m1*m2)/(L1*M^2))/(L2*M),
                                                                                                                                    0,
((g*m2*(M*g + g*m1))/(L1^2*M^2) + (g*m2*(M*g + g*m2))/(L1*L2*M^2))/(L2*M) + ((M*g + g*m1)^2/(L1^2*M^2) + (g^2*m1*m2)/(L1*L2*M^2))/(L1*M),
                                                                                                                                    0,
((g*m1*(M*g + g*m2))/(L2^2*M^2) + (g*m1*(M*g + g*m1))/(L1*L2*M^2))/(L1*M) + ((M*g + g*m2)^2/(L2^2*M^2) + (g^2*m1*m2)/(L1*L2*M^2))/(L2*M),
```

Column 3 of Controllability Matrix:

```
            ((g*m1*(M*g + g*m1))/(L1*M^2) + (g^2*m1*m2)/(L2*M^2))/(L1*M) + ((g*m2*(M*g + g*m2))/(L2*M^2) + (g^2*m1*m2)/(L1*M^2))/(L2*M)]
                                                                                                                                    0]
((g*m2*(M*g + g*m1))/(L1^2*M^2) + (g*m2*(M*g + g*m2))/(L1*L2*M^2))/(L2*M) + ((M*g + g*m1)^2/(L1^2*M^2) + (g^2*m1*m2)/(L1*L2*M^2))/(L1*M)]
                                                                                                                                    0]
((g*m1*(M*g + g*m2))/(L2^2*M^2) + (g*m1*(M*g + g*m1))/(L1*L2*M^2))/(L1*M) + ((M*g + g*m2)^2/(L2^2*M^2) + (g^2*m1*m2)/(L1*L2*M^2))/(L2*M)]
                                                                                                                                    0]
```

For the controllability matrix provided above to attain full rank, it is imperative that its determinant remains non-zero, denoted as $\det(C) \neq 0$, where $\det(C)$ is expressed as:

$$\det(C) = \frac{-g^6(\ell_1^2 - \ell_2^2)}{M^6(l_1^6 - l_2^6)} \neq 0$$

This condition holds true as long as $\ell_{12} - \ell_{22} \neq 0$ or equivalently $\ell_1 \neq \ell_2$. Hence, the system exhibits controllability solely when the lengths of the crane cables are not equal for $\quad \mathrm{m}_1 > 0, \quad m_2 > 0, \quad M > 0$

# 4 Part D

For the following state space representation

$$\mathrm{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-gm_1}{M} & 0 & \frac{-gm_2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-g(M+m_1)}{Ml_2} & 0 & \frac{-gm_1}{Ml_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-gm_1}{Ml_2} & 0 & \frac{-g(M+m_2)}{Ml_2} & 0 \end{bmatrix}$$

$$
\mathrm{B} = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml_1} \\ 0 \\ \frac{1}{Ml_2} \end{bmatrix}
$$

For the given initial conditions.

$$
[x(t), \dot{x}(t), \theta_1(t), \dot{\theta}_1(t), \theta_2(t), \dot{\theta}_2(t)] = [0; 0; 45°; 0; 45°; 0]
$$

And for M=1000kg, m1=100kg, m2=100kg, l1=20m, l2=10m and substituting this in the matrix and B we get:

$$
\mathrm{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.981 & 0 & -0.981 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -0.5395 & 0 & -0.0491 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -0.0981 & 0 & -1.0791 & 0 \end{bmatrix}
$$

$$
\mathrm{B} = \begin{bmatrix} 0 \\ 0.001 \\ 0 \\ 5 * e^{-5} \\ 0 \\ 0.0001 \end{bmatrix}
$$

Let us check the controllability of the system using the condition for controllability:

$$
\text{rank}(C) = \text{rank} \begin{bmatrix} B & AB & A^2B & A^3B & A^4B & A^5B \end{bmatrix} = n
$$

Controllablity can be checked from the following function in MATLAB

```
%To Check for controllabilty
r=rank([B A*B (A^2)*B (A^3)*B (A^4)*B (A^5)*B])
```

Rank of this matrix is 6=n, hence controllable
To drive an LQR controller the following cost function is considered:

$$
J(U) = \sum_{\tau=0}^{N-1} x_\tau^T Q x_\tau + u_\tau^T R u_\tau
$$

Where Q is used to penalize bad performance, with the increase in value, and R is used to penalize effort. For the following system, the following Q and R functions are considered.

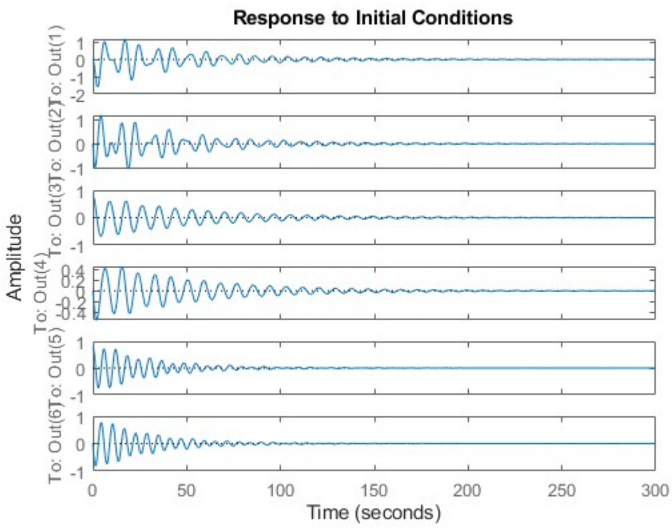
$$Q = \begin{bmatrix} 5000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1000 \end{bmatrix}$$

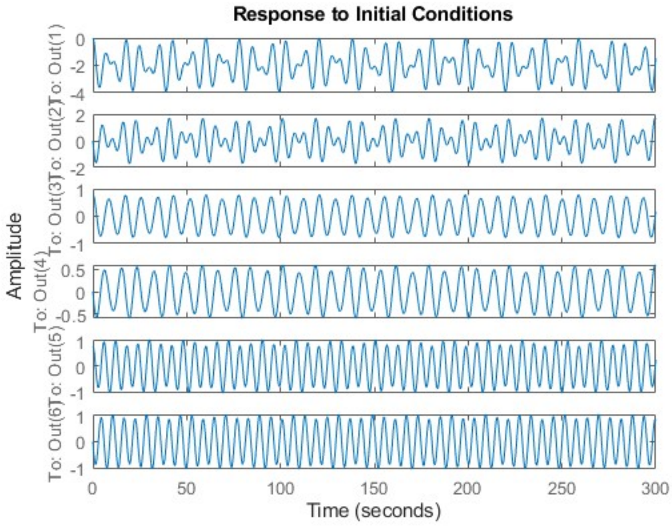
$$R = 0.02$$

For the LQR controller, MATLAB provides a function to derive K, S, and P for the given system. It can be used as shown below

```
%[K, S, P] = lqr(A, B, Q, R)
```

Following, are the plots for the initial response of the state with and without an LQR controller.






To check for the stability of the system, eigen values of the state need to be considered. To check if the state is stable with the LQR controller, consider the eigne values of the closed loop system. For the following system using MATLAB, the eigen values are determined as shown in the code above and have the following values.
The eigenvalues of the closed-loop $A$ matrix are:

$$\begin{aligned}
&-0.4597 + 0.4766i \\
&-0.4597 - 0.4766i \\
&-0.0347 + 1.0317i \\
&-0.0347 - 1.0317i \\
&-0.0188 + 0.7173i \\
&-0.0188 - 0.7173i
\end{aligned}$$

Since all the real part of the eigen values are negative, the system is stable

# 5 Part E

Suppose that you can select the following output vectors:

$$x(t), (\theta_1(t), \theta_2(t)), (x(t), \theta_2(t)) or (x(t), \theta_1(t), \theta_2(t))$$

Determine for which output vectors the linearized system is observable.

The following output vectors are considered:

- $x(t)$

- $\theta_1(t)$
- $\theta_2(t)$
- $(x(t), \theta_2(t))$
- $(x(t), \theta_1(t), \theta_2(t))$

For vector $x(t)$:

$$c = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For vectors $\theta_1(t), \theta_2(t)$:

$$c = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

For vector $x(t), \theta_2(t)$:

$$c = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

For vectors $x(t), \theta_1(t), \theta_2(t)$:

$$c = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Using the following Code, the observability has been calculated:

```
syms m1 g m2 M L1 L2
A = [0 1 0 0 0 0; 0 0 -m1*g/M 0 -m2*g/M 0;
0 0 0 1 0 0; 0 0 -((M*g)+(m1*g))/(M*L1) 0 -g*m2/(M*L1) 0;
0 0 0 0 0 1; 0 0 -m1*g/(M*L2) 0 -((M*g)+(m2*g))/(M*L2) 0];
B = [0; 1/M; 0; 1/(L1*M); 0; 1/(L2*M)];
n = 6;
Mc = sym(zeros(size(A, 1), n*size(B, 2)));
for i = 1:n
Mc(:, (i-1)*size(B, 2)+1:i*size(B, 2)) = A^(i-1) * B;
end
disp('Controllability matrix Mc:');
disp(Mc);
% Rank calculation
```

```
Rank = rank([B A*B (A^2)*B (A^3)*B (A^4)*B (A^5)*B]);
disp('Rank:');
disp(Rank);
% Determinant calculation
d = det([B A*B (A^2)*B (A^3)*B (A^4)*B (A^5)*B]);
disp('Determinant:');
disp(d);
```

Output

```
>> E_1
     6     4     6     6
```

From the code, we get the rank for respective output vector as 6,4,6,6. The output vector ( 1 (t); 2 (t)) is not observable.

# 6 Part F

Obtain your "best" Luenberger observer for each one of the output vectors for which the system is observable and simulate its response to initial conditions and unit step input. The simulation should be done for the observer applied to both the linearized system and the original nonlinear system.

The Luenberger Observer Equation is as follows:

$$\dot{\hat{\mathbf{X}}}(t) = \mathbf{AX}(t) + \mathbf{BkU}_k(t) + \mathbf{L}(\mathbf{Y}(t) - \mathbf{CX}(t)) \tag{1}$$

$\hat{\mathbf{x}}(t)$ is the state estimator, $\mathbf{L}$ is the observer gain matrix, $\mathbf{Y}(t) - \mathbf{C}\hat{\mathbf{x}}(t)$ is the correction term, and $\hat{\mathbf{x}}(0) = \mathbf{0}$. The estimation error $\mathbf{X}_e(t) = \mathbf{X}(t) - \hat{\mathbf{X}}(t)$ has the following state-space representation:

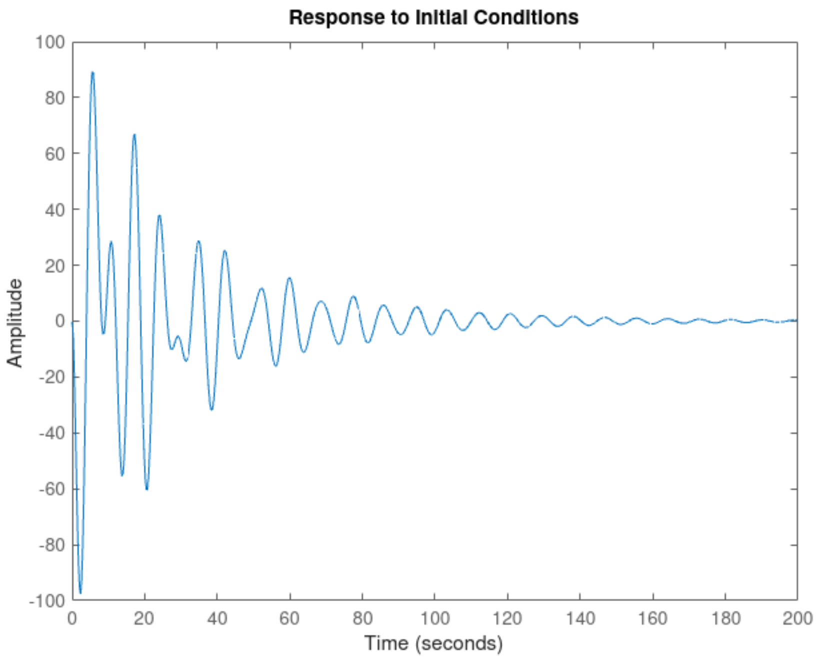
$$\dot{\mathbf{X}}_e(t) = \dot{\mathbf{X}}(t) - \dot{\hat{\mathbf{X}}}(t); \tag{2}$$

$$\dot{\mathbf{X}}_e(t) = \mathbf{AX}_e(t) - \mathbf{L}(\mathbf{Y}(t) - \mathbf{C}\hat{\mathbf{x}}(t)) + \mathbf{B}_d\mathbf{U}_d(t), \tag{3}$$

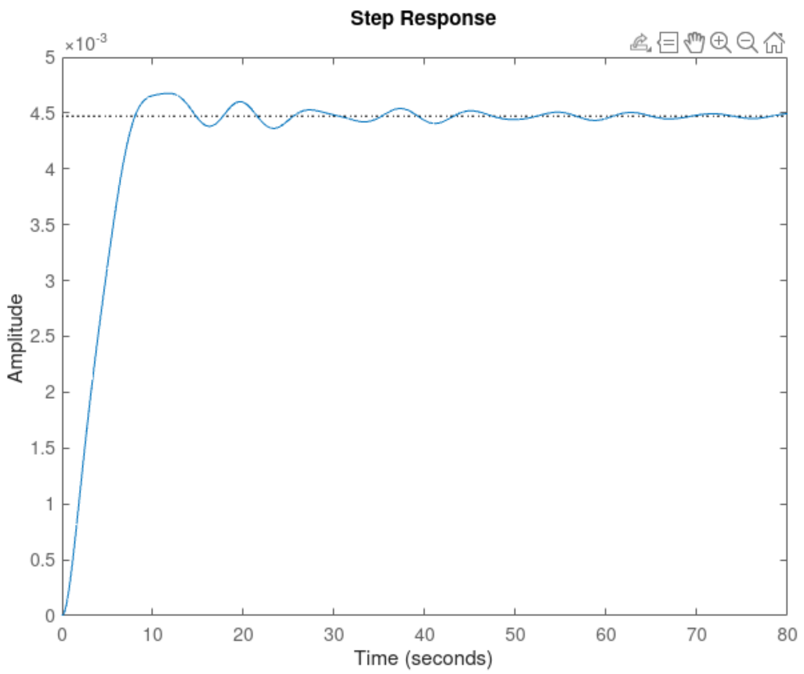
where we assume $\mathbf{D} = \mathbf{0}$, $\mathbf{Y}(t) = \mathbf{Cx}(t)$. Therefore, the equation can be written as

$$\dot{\mathbf{X}}_e(t) = (\mathbf{A} - \mathbf{LC})\mathbf{X}_e(t) + \mathbf{B}_d\mathbf{U}_d(t). \tag{4}$$

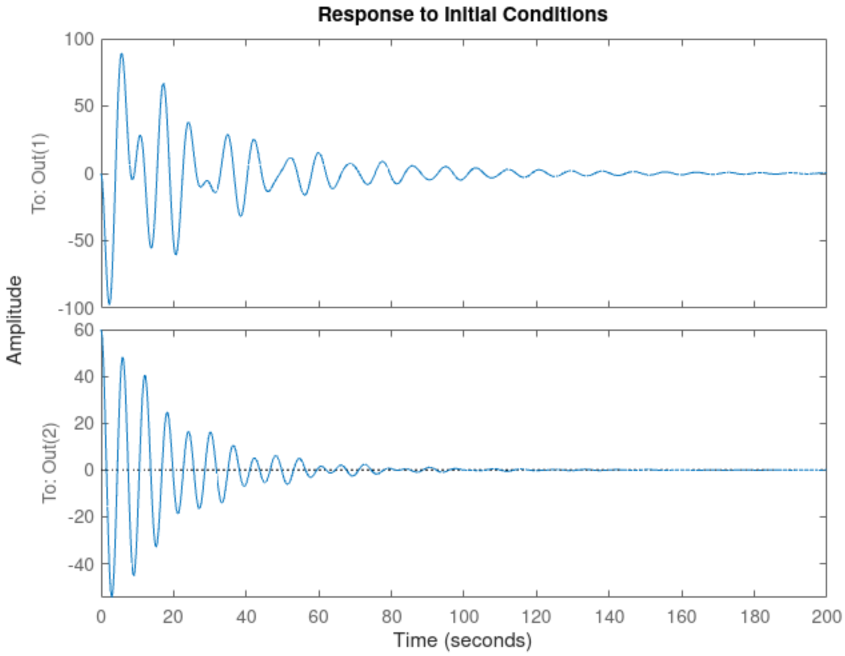
Linear Observer

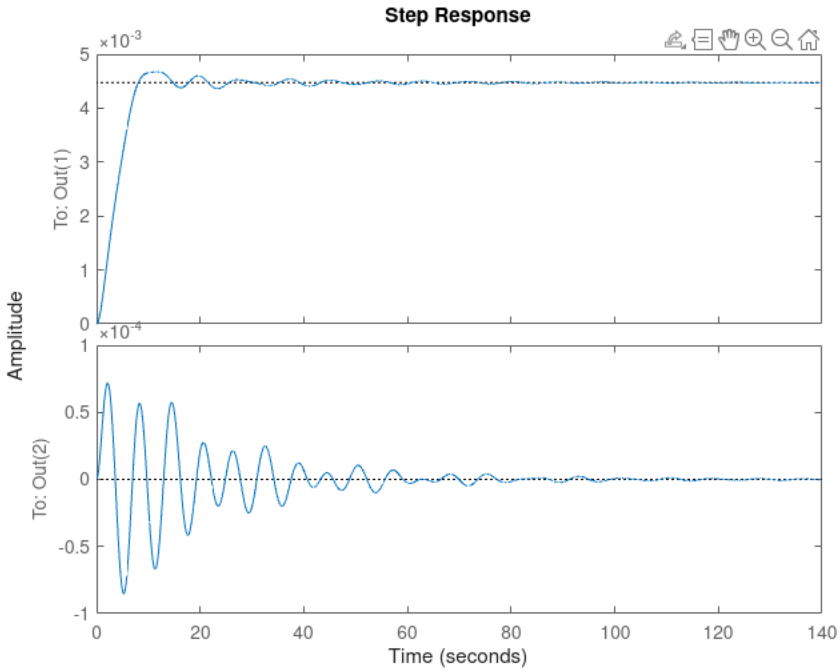


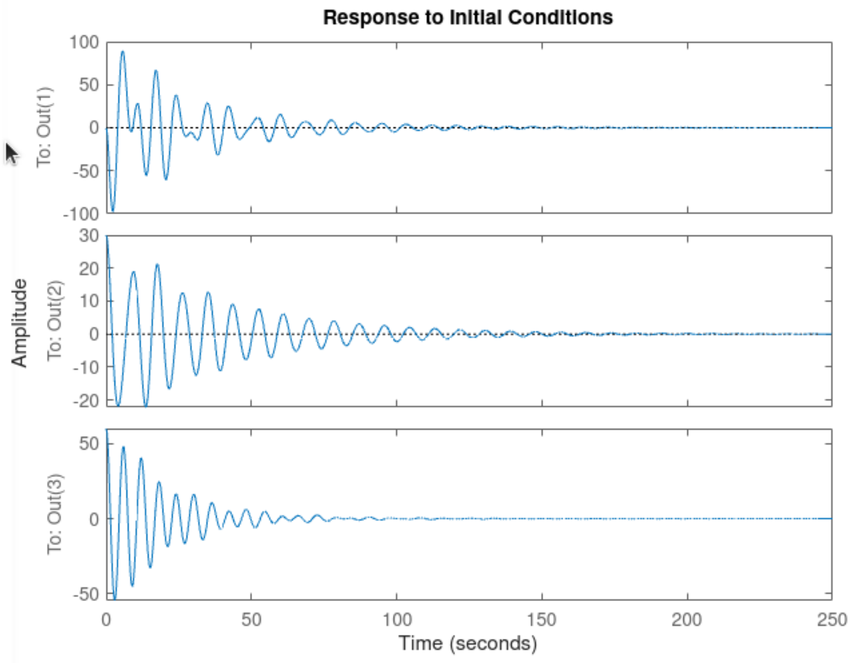
Response to Initial Conditions
Amplitude
Time (seconds)

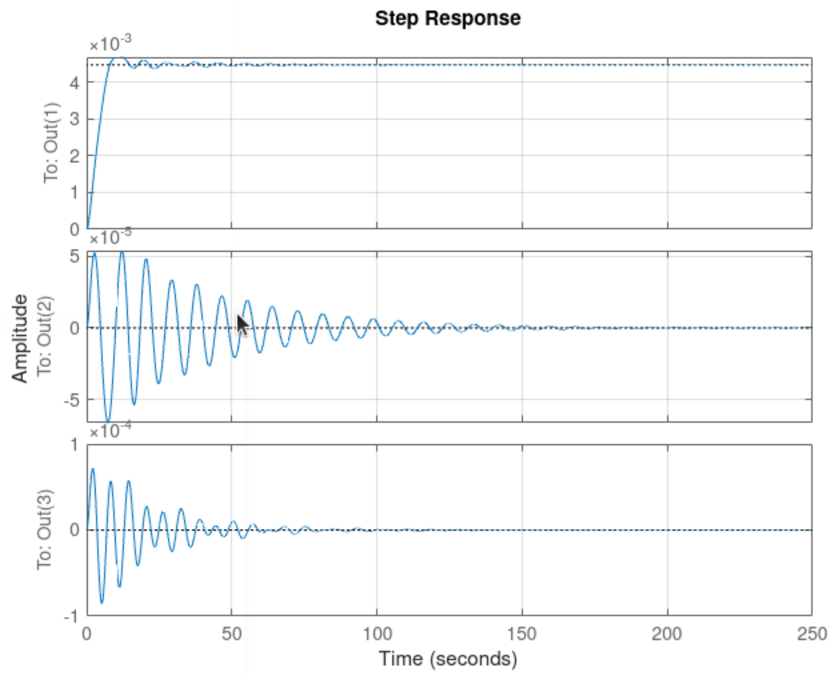



Step Response
×10^-3
Amplitude
Time (seconds)

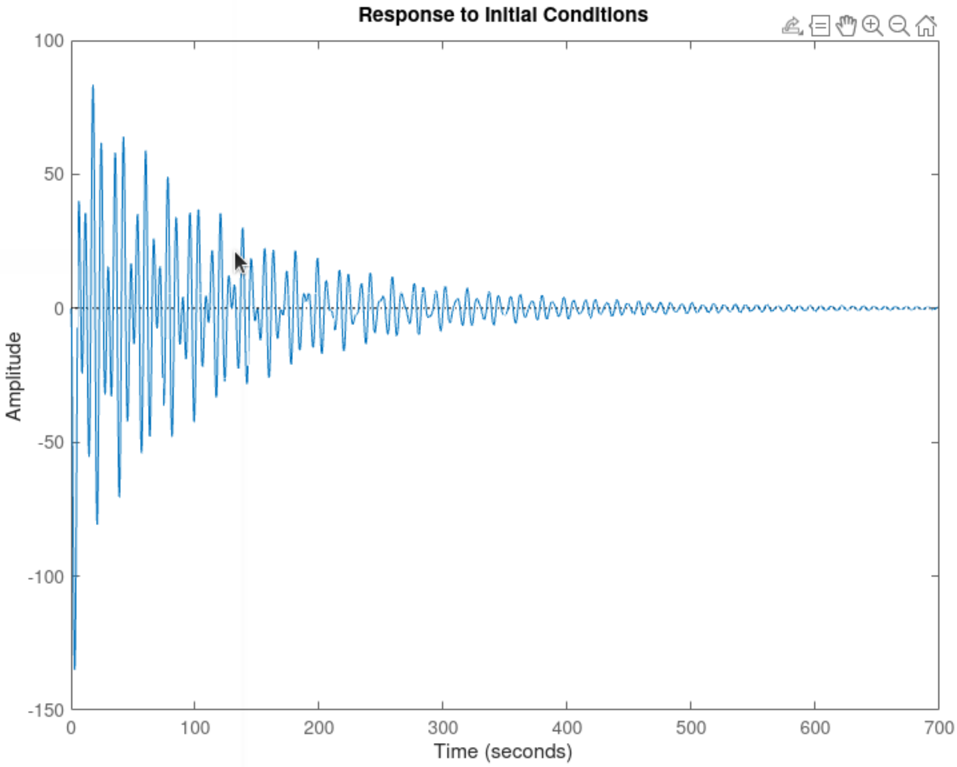

Response to Initial Conditions
Amplitude
To: Out(1)
To: Out(2)
100
50
0
-50
-100
60
40
20
0
-20
-40
0
20
40
60
80
100
120
140
160
180
200
Time (seconds)

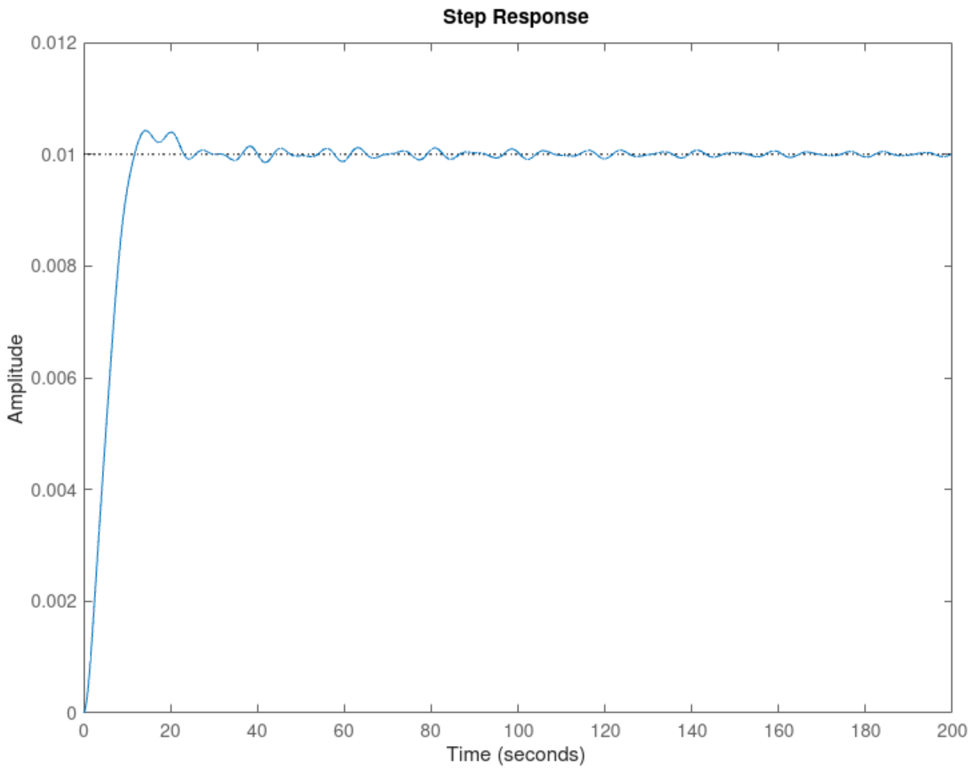

Step Response
Amplitude
To: Out(1)
To: Out(2)
×10⁻³
5
4
3
2
1
0
×10⁻⁴
1
0.5
0
-0.5
-1
0
20
40
60
80
100
120
140
Time (seconds)

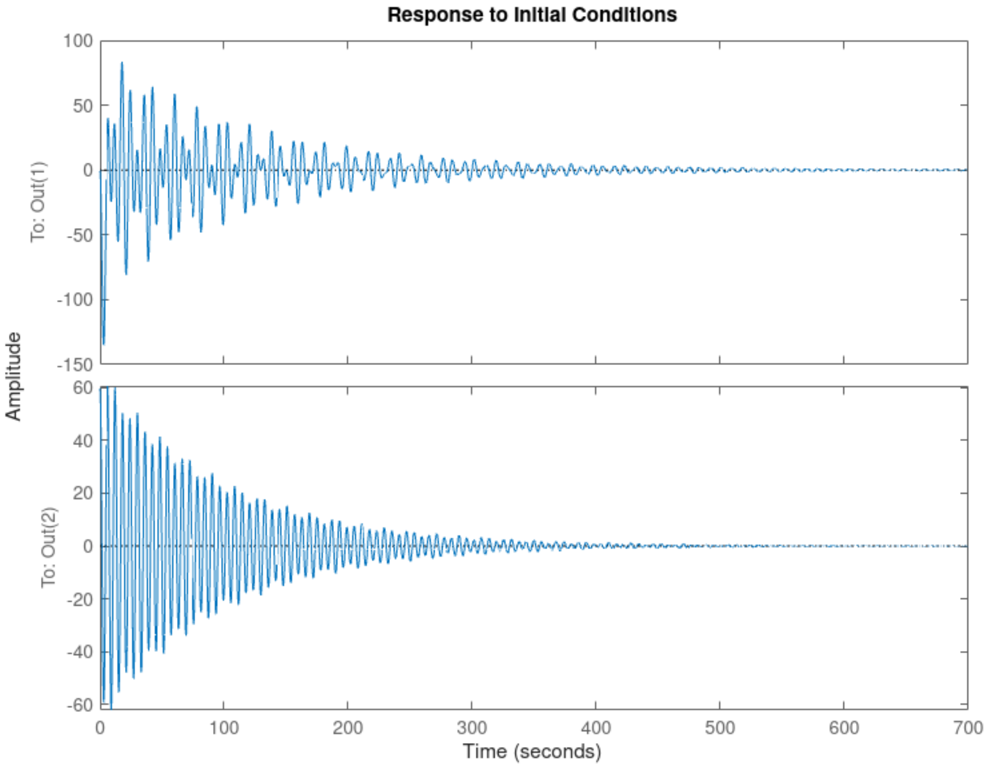


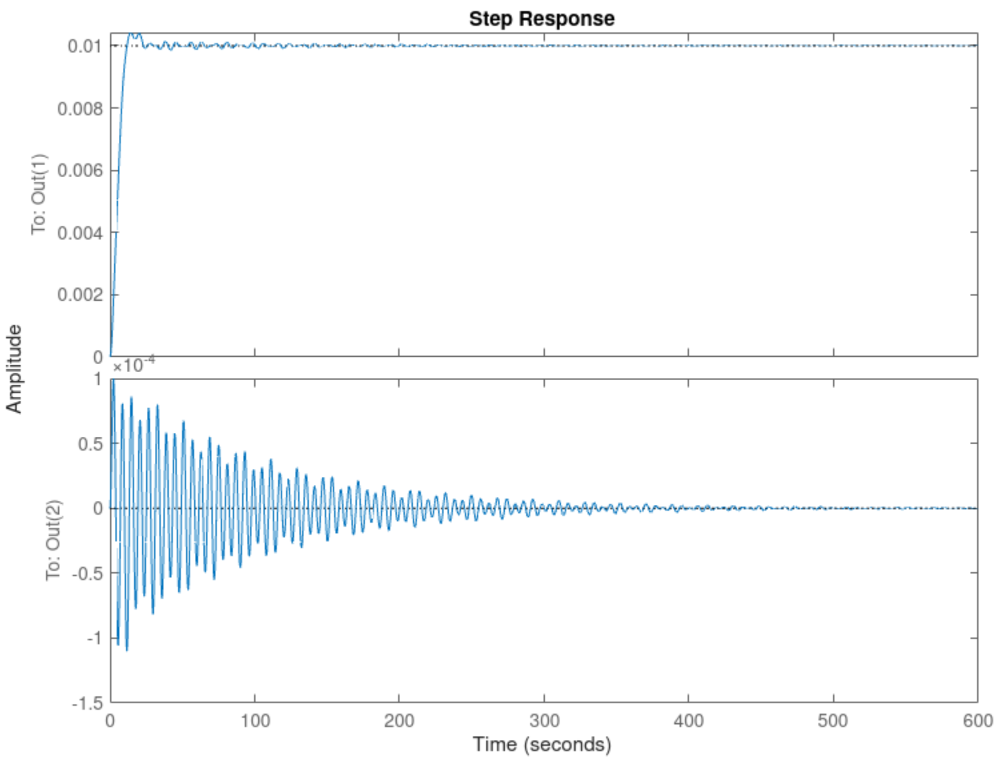

Non-Linear Observer


Response to Initial Conditions
100
50
0
-50
-100
-150
Amplitude
0
100
200
300
400
500
600
700
Time (seconds)

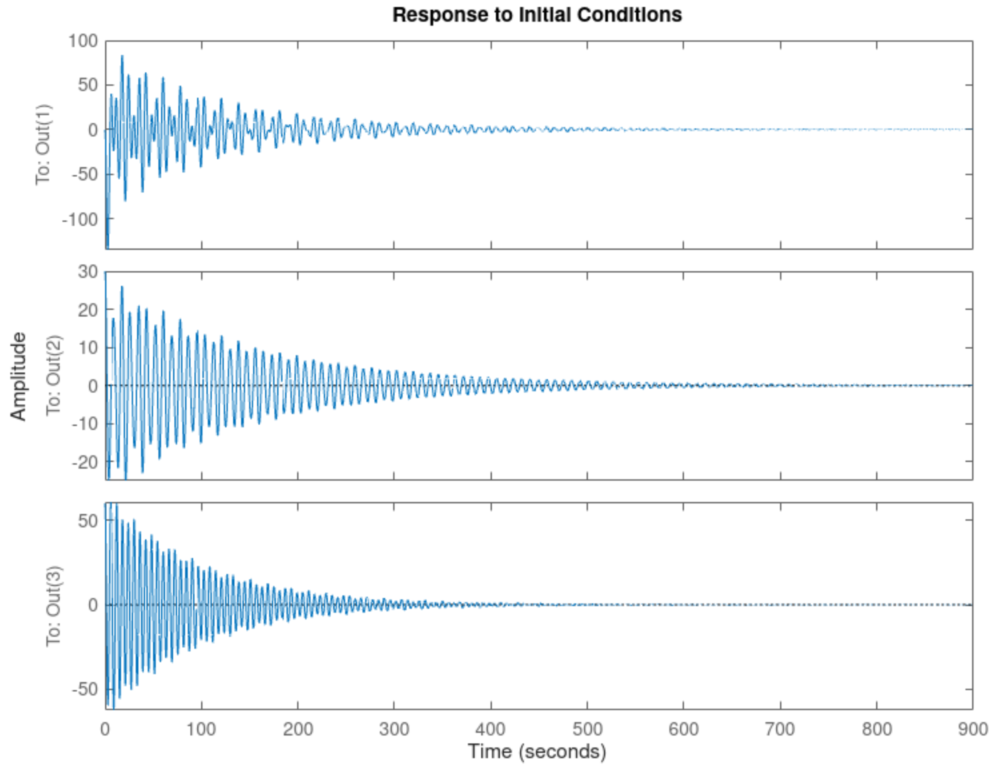

Step Response
0.012
0.01
0.008
0.006
0.004
0.002
0
Amplitude
0
20
40
60
80
100
120
140
160
180
200
Time (seconds)

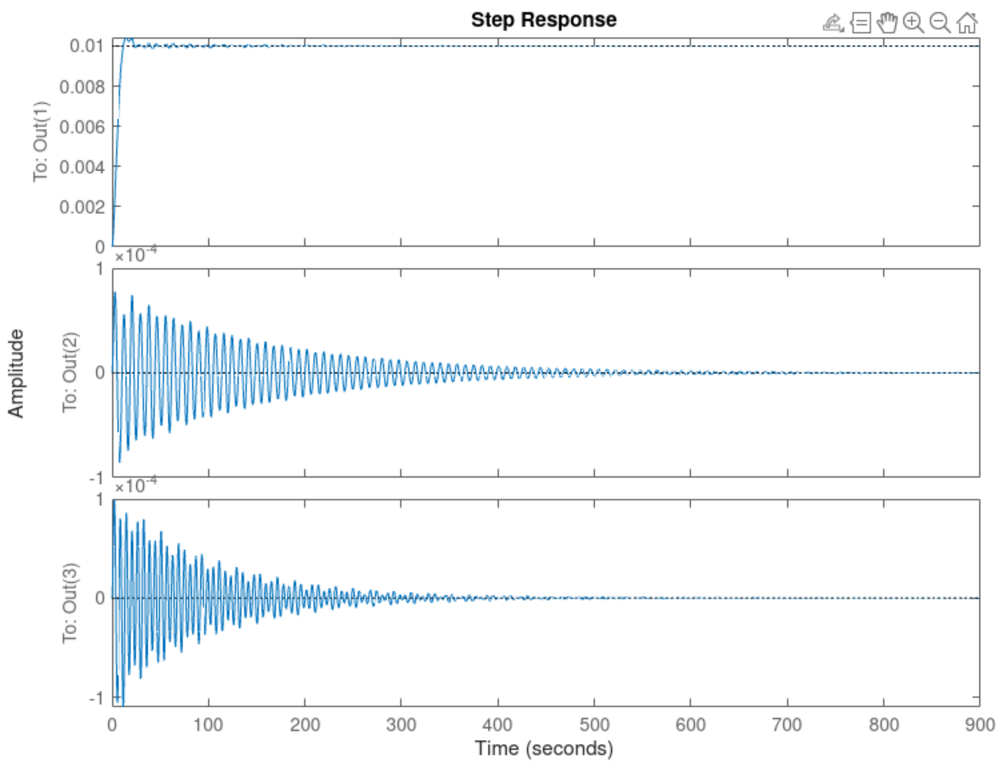


Response to Initial Conditions
Amplitude
To: Out(1)
To: Out(2)
Time (seconds)

Step Response
Amplitude
To: Out(1)
To: Out(2)
×10^-4
Time (seconds)


Response to Initial Conditions
Amplitude
To: Out(1)
To: Out(2)
To: Out(3)
Time (seconds)


Step Response
Amplitude
To: Out(1)
To: Out(2)
To: Out(3)
Time (seconds)

# 7 Part G

Design an output feedback controller for your choice of the ”smallest” output vector. Use the LQG method and apply the resulting output feedback controller to the original nonlinear system. Obtain your best design and illustrate its performance in simulation. How would you reconfigure your con-troller to asymptotically track a constant reference on x ? Will your design reject constant force disturbances applied on the cart ?

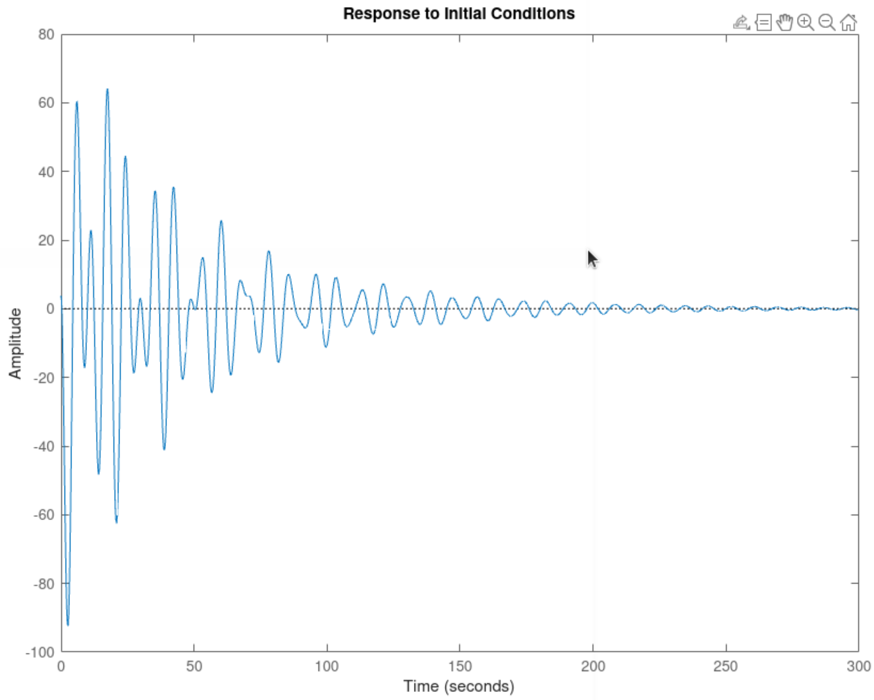
Linear LQG:

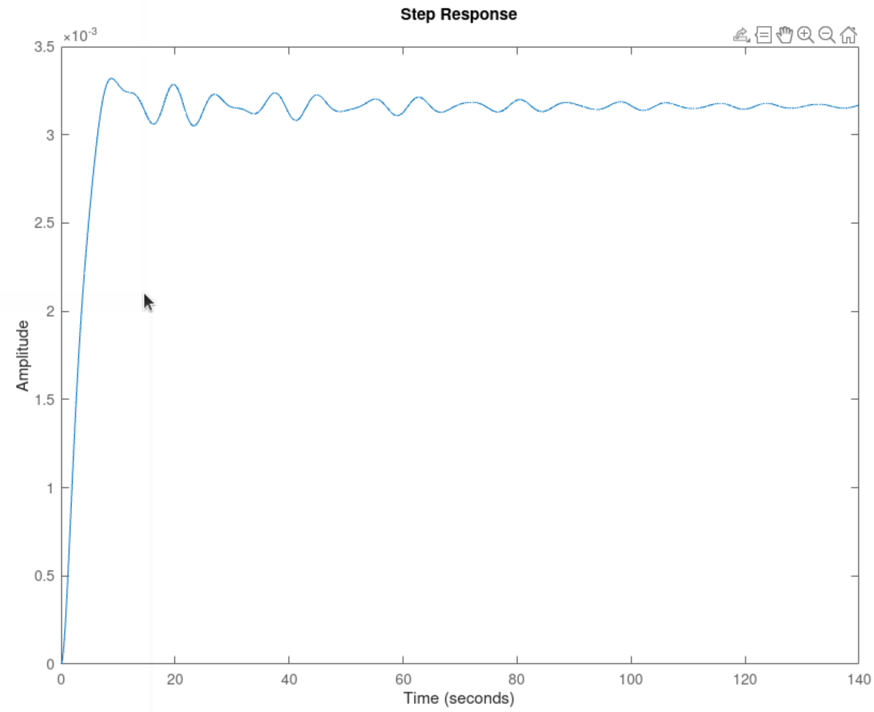
An LQG (Linear Quadratic Gaussian) Controller is an advanced control strategy used in the field of control systems engineering. It combines two major control methodologies: Linear Quadratic Regulator (LQR) and Kalman Filter. Here's a breakdown of its definition and key components:

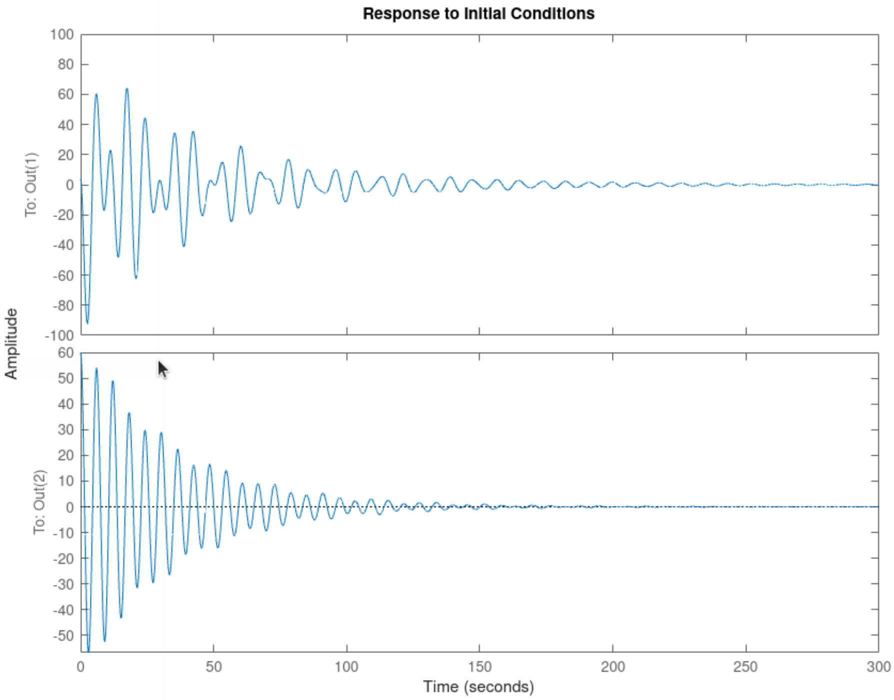
- Linear Quadratic Regulator (LQR): The LQR part of an LQG controller is designed to optimize the performance of a linear system. It does so by minimizing a quadratic cost function, which typically includes terms for both the state deviation (how far the system is from its desired state) and control effort (how much energy or effort is used to control the system). The goal is to achieve the best balance between performance (like tracking a set point) and effort (like energy consumption).

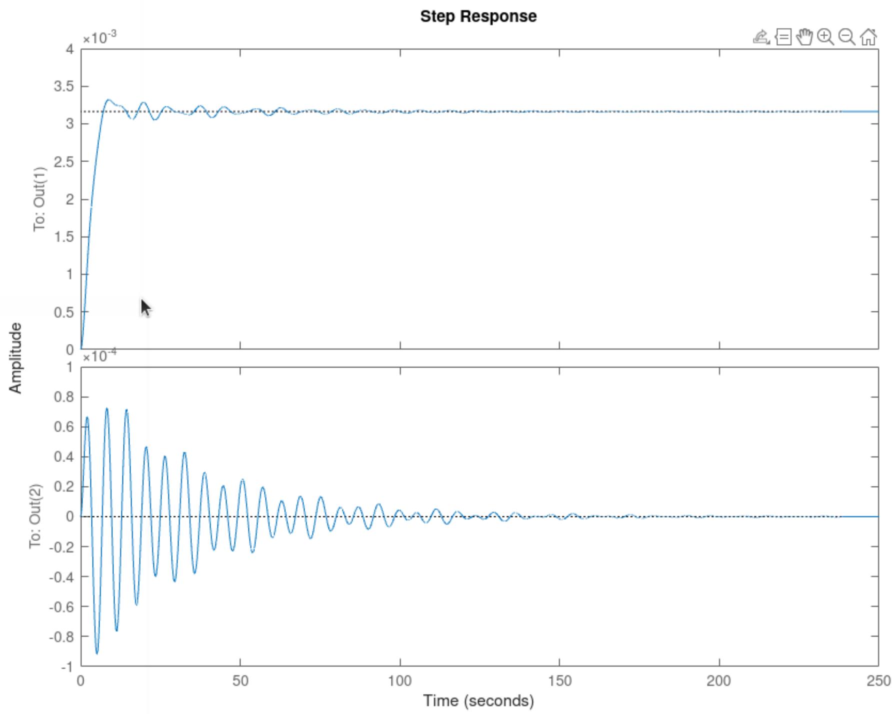
- Kalman Filter: The Kalman Filter component is used for estimating the internal states of the system. In many practical applications, not all states of a system can be measured directly. The Kalman Filter provides a way to estimate these unmeasurable states in an optimal manner, based on noisy measurements. It's particularly effective in systems with Gaussian noise.

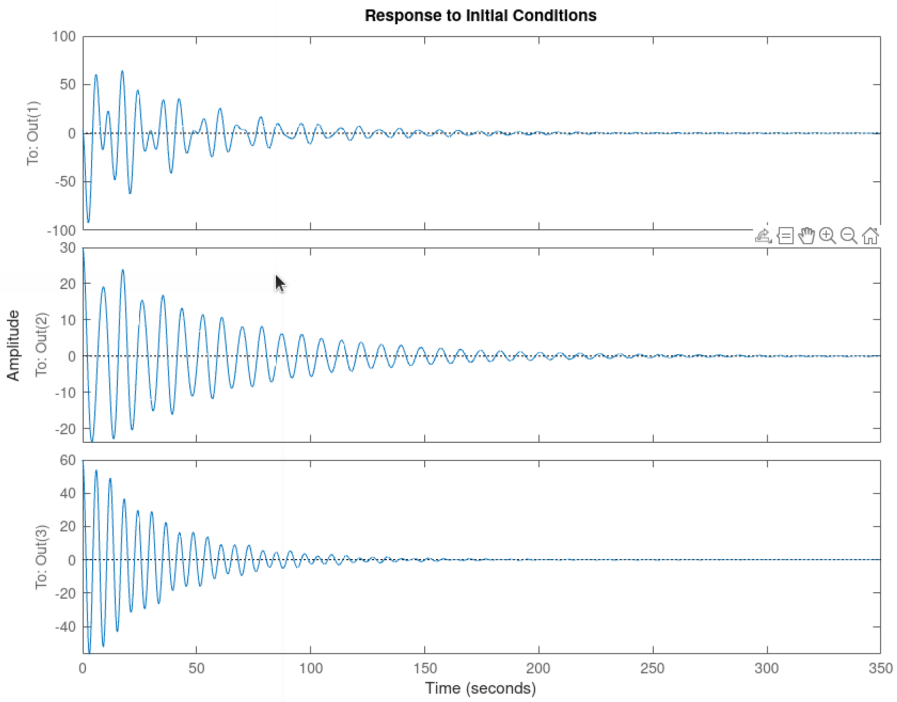
- Gaussian Noise Assumption: The ”G” in LQG stands for Gaussian, indicating that the controller is designed under the assumption that the system experiences noise or uncertainties that follow a Gaussian (normal) distribution. This assumption is crucial for the Kalman Filter's effectiveness in state estimation.

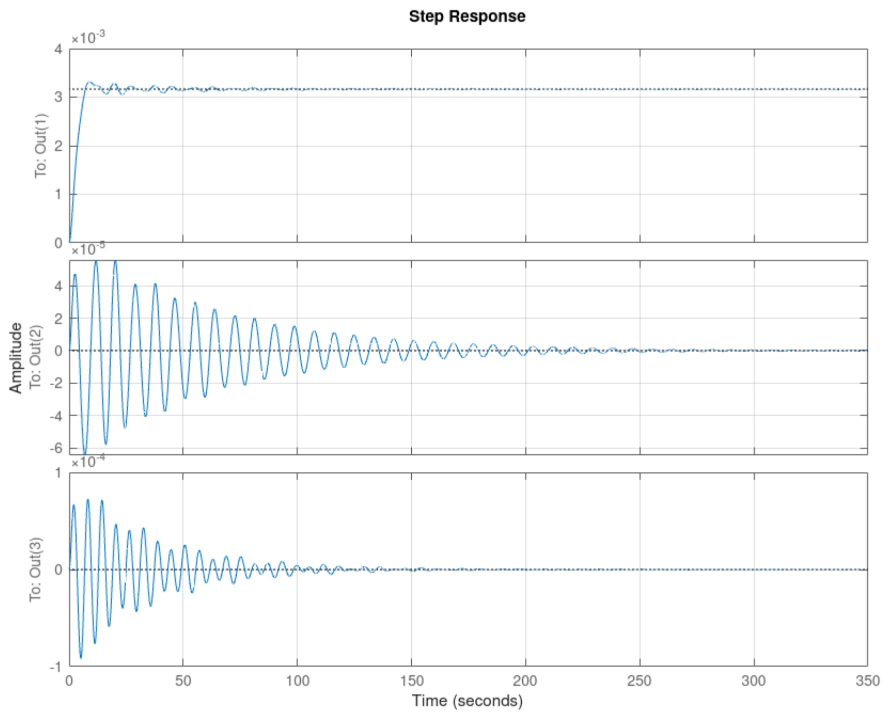

Response to Initial Conditions
Amplitude
Time (seconds)
80
60
40
20
0
-20
-40
-60
-80
-100
0
50
100
150
200
250
300


Step Response
×10^-3
Amplitude
Time (seconds)
3.5
3
2.5
2
1.5
1
0.5
0
0
20
40
60
80
100
120
140


Response to Initial Conditions
Amplitude
To: Out(1)
To: Out(2)
Time (seconds)

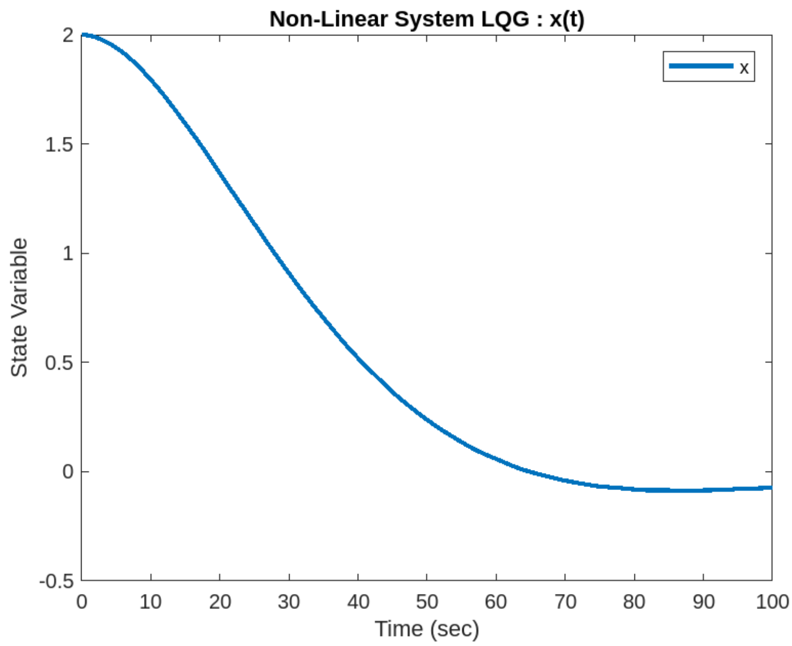

Step Response
Amplitude
To: Out(1)
To: Out(2)
Time (seconds)




For Non-Linear Part, utilising LQG = LQR + Kalman Filter

In our model, we have taken into consideration: $Q = 1$, $R = 0.1$, noise $B_d = 0.1$, and $V_d = 0.01$.




The depicted illustration illustrates the feedback loop of the Nonlinear System LQG. To align the feedback with the intended outcome, we will readjust our controller by supplying it with the suitable $x$ vector and fine-tuning the LQR controller accordingly. Indeed, our design enables the cart to endure recurrent force disturbances. As you escalate the disturbance noise in the Kalman filter, you will notice the formidable capability of the LQR controller to rapidly stabilize the cart position $x(t)$ within seconds.

# 8 Simulation

Video Recording for Part C, E, F and G: Google-Drive Link

Video Recording for Part D: Google-Drive Link Part: D

# 9 Appendix

CODE FOR PART(D).

```
syms M m1 m2 l1 l2 g
%%Units of measurements taken in metric system
M = 1000;
m1 = 100;
m2 = 100;
l1 = 20;
l2 = 10;
g = 9.81;

%% State space representation of the system as matrice A, B, C, D

A = [0 1 0 0 0 0; 0 0 -m1*g/M 0 -m2*g/M 0;
```

```
0 0 0 1 0 0; 0 0 -((M+m1)*g)/(M*l1) 0 -g*m2/(M*l1) 0;
0 0 0 0 0 1; 0 0 -m1*g/(M*l2) 0 -((M+m2)*g)/(M*l2) 0];
disp("martix A is:")
disp(A)
B = [0; 1/M; 0; 1/(l1*M); 0; 1/(l2*M)];
disp("martix B is:")
disp(B)
% THE C MATRIX
C = [1 0 0 0 0 0;
     0 1 0 0 0 0;
     0 0 1 0 0 0;
     0 0 0 1 0 0;
     0 0 0 0 1 0;
     0 0 0 0 0 1]
% THE D MATRIX
D = [0; 0; 0; 0; 0; 0]


% Taking intial conditions assumption
X_init = [0; 0; deg2rad(45); deg2rad(0); deg2rad(45); deg2rad(0)]

%Q and R values assumption
Q = [5000 0 0 0 0 0;
     0 5000 0 0 0 0;
     0 0 5000 0 0 0;
     0 0 0 5000 0 0;
     0 0 0 0 5000 0;
     0 0 0 0 0 5000]
R = 0.02


disp("The response of the system without the LQR Controller for initial states\n")

ss_rep = ss(A, B, C, D)
time=0:0.01:2
step(ss_rep,t)

figure("Name","Intial State without LQR controller")

initial(ss_rep, X_init,300)

disp("With LQR Controller \n")

[K, S, P] = lqr(A, B, Q, R)

ss_rep_closed = ss(A-(B*K), B, C, D)

% Plotting the change in response of the system with the addition of LQR
```

```
% Controller

figure("Name","Intial State with LQR controller")
initial(ss_rep_closed, X_init,300)

%Check for stablity
disp("The eigen values of closed loop A matrix is:")
disp(eig(A-B*K))
```

Part F

```
clc
clear

% Defining constants
cartMass = 1000;        % Mass of the cart (kg)
pendulum1Mass = 100;    % Mass of Pendulum 1 (kg)
pendulum2Mass = 100;    % Mass of Pendulum 2 (kg)
pendulum1Length = 20;   % Length of the string of Pendulum 1 (m)
pendulum2Length = 10;   % Length of the string of Pendulum 2 (m)
gravity = 9.81;         % Acceleration due to gravity (m/s^2)

% State-space matrices
A = [0 1 0 0 0 0;
     0 0 -(pendulum1Mass*gravity)/cartMass 0 -(pendulum2Mass*gravity)/cartMass 0;
     0 0 0 1 0 0;
     0 0 -((cartMass+pendulum1Mass)*gravity)/(cartMass*pendulum1Length
     ) 0 -(pendulum2Mass*gravity)/(cartMass*pendulum1Length) 0;
     0 0 0 0 0 1;
     0 0 -(pendulum1Mass*gravity)/(cartMass*pendulum2Length)
     0 -(gravity*(cartMass+pendulum2Mass))/(cartMass*pendulum2Length)
     0];
B = [0; 1/cartMass; 0; 1/(cartMass*pendulum1Length)
; 0; 1/(cartMass*pendulum2Length)];

% Output matrices for observable states
C_x = [1 0 0 0 0 0];                        % Output measurement for x component
C_x_theta2 = [1 0 0 0 1 0];                  % Output measurement for x and theta2
C_x_theta1_theta2 = [1 0 0 0 0 0; 0 0 1 0 0 0; 0 0 0 0 1 0]; % Output measurement for
D = 0;

% LQR design
Q = 100*eye(6);
R = 0.01;
initialState = [0, 0, 30, 0, 60, 0, 0, 0, 0, 0, 0, 0];
poles = -1:-1:-6;
K = lqr(A, B, Q, R);
```

```
% Observer design
L_x = place(A', C_x', poles)';
L_x_theta2 = place(A', C_x_theta2', poles)';
L_x_theta1_theta2 = place(A', C_x_theta1_theta2', poles)';

% Closed-loop systems
A_cl_x = [(A-B*K) B*K; zeros(size(A)) (A-L_x*C_x)];
B_cl = [B; zeros(size(B))];
C_cl_x = [C_x zeros(size(C_x))];
system_x = ss(A_cl_x, B_cl, C_cl_x, D);

A_cl_x_theta2 = [(A-B*K) B*K; zeros(size(A)) (A-L_x_theta2*C_x_theta2)];
C_cl_x_theta2 = [C_x_theta2 zeros(size(C_x_theta2))];
system_x_theta2 = ss(A_cl_x_theta2, B_cl, C_cl_x_theta2, D);

A_cl_x_theta1_theta2 =
[(A-B*K) B*K; zeros(size(A)) (A-L_x_theta1_theta2*C_x_theta1_theta2)];
C_cl_x_theta1_theta2 =
[C_x_theta1_theta2 zeros(size(C_x_theta1_theta2))];
system_x_theta1_theta2 = ss(A_cl_x_theta1_theta2, B_cl, C_cl_x_theta1_theta2, D);

% Simulation
figure, initial(system_x, initialState)
figure, step(system_x)
figure, initial(system_x_theta2, initialState)
figure, step(system_x_theta2)
figure, initial(system_x_theta1_theta2, initialState)
figure, step(system_x_theta1_theta2)
```

Part F(b)

```
M = 1000; % Cart mass
m1 = 100; % Pendulum 1 mass
m2 = 100; % Pendulum 2 mass
l1 = 20;  % Pendulum 1 length
l2 = 10;  % Pendulum 2 length
g = 9.81; % Gravity

% Initial conditions
x0 = [0, 0, 10*pi/180, 0, 20*pi/180, 0];
% [position, velocity, angle1, angular velocity1, angle2, angular velocity2]

% Time span for the simulation
tspan = [0 30]; % Simulate for 30 seconds

% Numerical integration using ode45
```

```
[t, x] = ode45(@(t, x)
nonlinearCartDoublePendulumDynamics(t, x, M, m1, m2, l1, l2, g),
tspan, x0);

% Plotting results
figure;

% Plotting cart position vs time
subplot(3,2,1);
plot(t, x(:,1));
title('Cart Position vs Time');
xlabel('Time (s)');
ylabel('Position (m)');

% Plotting cart velocity vs time
subplot(3,2,2);
plot(t, x(:,2));
title('Cart Velocity vs Time');
xlabel('Time (s)');
ylabel('Velocity (m/s)');

% Plotting pendulum 1 angle vs time
subplot(3,2,3);
plot(t, x(:,3) * 180/pi); % Converting radians to degrees
title('Pendulum 1 Angle vs Time');
xlabel('Time (s)');
ylabel('Angle (degrees)');

% Plotting pendulum 1 angular velocity vs time
subplot(3,2,4);
plot(t, x(:,4) * 180/pi); % Converting radians to degrees
title('Pendulum 1 Angular Velocity vs Time');
xlabel('Time (s)');
ylabel('Angular Velocity (degrees/s)');

% Plotting pendulum 2 angle vs time
subplot(3,2,5);
plot(t, x(:,5) * 180/pi); % Converting radians to degrees
title('Pendulum 2 Angle vs Time');
xlabel('Time (s)');
ylabel('Angle (degrees)');

% Plotting pendulum 2 angular velocity vs time
subplot(3,2,6);
plot(t, x(:,6) * 180/pi); % Converting radians to degrees
title('Pendulum 2 Angular Velocity vs Time');
xlabel('Time (s)');
ylabel('Angular Velocity (degrees/s)');
```

Part G(1)

```
    % Clearing workspace and command window
clear;
clc;

% System Parameters
cartMass = 1000;                % Mass of the cart (kg)
pendulumMass1 = 100;            % Mass of Pendulum 1 (kg)
pendulumMass2 = 100;            % Mass of Pendulum 2 (kg)
pendulumLength1 = 20;           % Length of Pendulum 1 (m)
pendulumLength2 = 10;           % Length of Pendulum 2 (m)
gravity = 9.81;                 % Gravity (m/s^2)

% State-Space Matrices
A = [0 1 0 0 0 0;
     0 0 -(pendulumMass1*gravity)/cartMass 0 -(pendulumMass2*gravity)
     /cartMass 0;
     0 0 0 1 0 0;
     0 0 -((cartMass+pendulumMass1)*gravity)
     /(cartMass*pendulumLength1) 0
     -(pendulumMass2*gravity)/
     (cartMass*pendulumLength1) 0;
     0 0 0 0 0 1;
     0 0 -(pendulumMass1*gravity)/
     (cartMass*pendulumLength2)
     0 -(gravity*(cartMass+pendulumMass2))/
     (cartMass*pendulumLength2)
     0];
B = [0; 1/cartMass; 0;
1/(cartMass*pendulumLength1); 0;
1/(cartMass*pendulumLength2)];

% LQR Design Matrices
Q = 100*eye(6);
R = 0.001;

% Output Matrices
C1 = [1 0 0 0 0 0];
C3 = [1 0 0 0 0 0; 0 0 0 0 1 0];
C4 = [1 0 0 0 0 0; 0 0 1 0 0 0; 0 0 0 0 1 0];
D = 0;

% Initial Conditions
initialState = [4; 0; 30; 0; 60; 0; 0; 0; 0; 0; 0; 0];

% Control and Observer Design
K = lqr(A, B, Q, R);
```

```
vd = 0.3*eye(6); % Process noise
vn = 1; % Measurement noise

% Kalman filter gains for different output matrices
K_pop1 = lqr(A', C1', vd, vn)';
K_pop3 = lqr(A', C3', vd, vn)';
K_pop4 = lqr(A', C4', vd, vn)';

% System Definitions
outputMatrices = {C1, C3, C4};
kalmanGains = {K_pop1, K_pop3, K_pop4};
systemTitles = {'System with C1', 'System with C3', 'System with C4'};

for i = 1:length(outputMatrices)
    sys = ss([(A-B*K) B*K; zeros(size(A)) (A-kalmanGains{i}*outputMatrices{i})], ...
             [B; zeros(size(B))], ...
             [outputMatrices{i} zeros(size(outputMatrices{i}))], D);

    figure;
    subplot(2,1,1);
    initial(sys, initialState);
    title(['Initial Response - ', systemTitles{i}]);

    subplot(2,1,2);
    step(sys);
    title(['Step Response - ', systemTitles{i}]);
    grid on;
end
```

Part G(2)

```
    % Clear workspace
clear all

%% Defining variables
syms m1 g m2 M L1 L2 x dx
m1 = 100;
m2 = 100;
M = 1000;
L1 = 20;
L2 = 10;
g = 9.81;
tspan = 0:0.1:100;
q0 = [2 0 deg2rad(0) 0 deg2rad(0) 0];


%% LQR Controller
Q = [1 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 0 0;
```

```
0 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 0 0];
R = 0.1;
[K, ~, ~] = lqr(A, B, Q, R);
sys = ss(A - B * K, B, c1, d);

%% Kalman Estimator Design
Bd = 0.1 * eye(6);  % Process Noise
Vn = 0.01;          % Measurement Noise
[Lue1, ~, ~] = lqe(A, Bd, c1, Bd, Vn * eye(3));
Ac1 = A - Lue1 * c1;
e_sys1 = ss(Ac1, [B, Lue1], c1, 0);

%% Non-linear Model LQG Response
[t, q1] = ode45(@(t, q) nonLinearObs1(t, q, -K * q, Lue1), tspan, q0);
figure();
plot(t, q1(:,1), 'LineWidth', 2);
ylabel('State Variable');
xlabel('Time (sec)');
title('Non-Linear System LQG for output vector: x(t)');
legend('x');
```