

Lin JIU

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Austrian Academy of Sciences (ÖAW)

Linz, Austria

April 5, 2017

Definitions



Definition

$$_sF_t\left(\frac{a_1,\ldots,a_s}{b_1,\ldots,b_t}:z\right):=\sum_{n=0}^{\infty}\frac{\left(a_1\right)_n\cdots\left(a_s\right)_n}{\left(b_1\right)_n\cdots\left(b_t\right)_n}\cdot\frac{z^n}{n!},$$

where $(a)_k = a(a+1)\cdots(a+k-1)$.

Examples

 \blacksquare ${}_{2}F_{1}\left({}_{c}^{a,b}:z\right)$ is the Gaussian hypergeometric function s. t.

$$z\left(1-z\right)\frac{\mathrm{d}^{2}w}{\mathrm{d}z^{2}}+\left(c-\left(a+b+1\right)z\right)\frac{\mathrm{d}w}{\mathrm{d}z}-abw=0.$$

 $\log (1+z) = z_2 F_1 \begin{pmatrix} 1,1 \\ 2 \end{pmatrix} : -z)$



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where,

 $\blacksquare \mathcal{P}_n$ is the set of all partitions of n;

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 $\mathbb{C}_p(Y)$ is (*C-normalization of*) zonal polynomial, which is homogeneous, symmetric, polynomial of degree n = |p|, in the eigenvalues of Y.



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Zonal polynomial \mathcal{Y}_{p} is defined by a vector form

$$\mathcal{Y} = (\mathcal{Y}_{(n)}, \mathcal{Y}_{(n-1,1)}, \dots, \mathcal{Y}_{(1^n)})^T = \Xi \mathcal{U},$$

$$\mathcal{U}_{p}(A) = \mathcal{U}_{p}(\alpha_{1}, \dots, \alpha_{k})$$
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- $\blacksquare \ \lambda_{\nu p} := 2^{p} \prod_{i=1}^{l} \left(\frac{\nu+1-i}{2}\right)_{p_{i}} (\operatorname{Recall}\ (a)_{p} = \prod_{i=1}^{l} \left(a \frac{i-1}{2}\right)_{p_{i}}) \operatorname{Moments} \ \operatorname{of}\ \chi^{2}$
- $\Lambda_{\nu} = \operatorname{diag}(\lambda_{\nu p} : p \in \mathcal{P}_n)$
- \mathcal{T}_{ν} is an upper triangular matrix with diagonal Λ_{ν} . Let

$$V_n := \{f : f \text{ is homogeneous, symmetric, of degree } n \text{ or } f \equiv 0\}$$

(e.g.
$$u_n:=\sum\limits_{i_1<\dots< i_n}x_{i_n}\dots x_{i_n}$$
 and \mathcal{U}_p) and $au_{
u}:V_n o V_n$ be (linear) transfom such

that

$$\tau_{\nu}\left(\mathcal{U}_{p}\right)\left(A\right) = \mathbb{E}\left[\mathcal{U}_{p}\left(AW\right)\right]$$

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- \blacksquare $Z_1,\ldots,Z_k\sim\mathcal{N}\left(0,1\right)$ are independent, then $Q:=Z_1+\cdots+Z_k\sim\chi_k^2$;
- $X_{
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Remark

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Integral Rerpesentation

When $\Re(a) > \frac{m+1}{2}$ and $\Re(c-a) > \frac{m+1}{2}$,

$${}_{1}F_{1}\left(a;c;Y\right)\left(=\sum_{n=0}^{\infty}\sum_{\rho\in\mathcal{P}_{n}}\frac{\left(a\right)_{\rho}}{\left(c\right)_{\rho}}\cdot\frac{\mathcal{C}_{\rho}\left(Y\right)}{n!}\right)$$

$$=\frac{\Gamma_{m}\left(c\right)}{\Gamma_{m}\left(a\right)\Gamma_{m}\left(c-a\right)}\int_{0<\mathcal{X}<\mathcal{I}_{m}}e^{\operatorname{tr}\left(XY\right)}\left(\det X\right)^{\beta-\frac{m+1}{2}}\left(\det\left(\mathcal{I}_{m}-X\right)\right)^{c-\beta-\frac{m+1}{2}}dX$$

where

- $0 < X < I_m$ means both X and $I_m X$ are positive definite;
- $\mathbf{d}X = \prod_{i < i} dx_{ij}$ is the Lebesgue measure of the upper triangular entries of X_i

Recall that

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Definitions



Integral Rerpesentation

When $\Re(a) > \frac{m+1}{2}$ and $\Re(c-a) > \frac{m+1}{2}$,

$${}_{1}F_{1}(a;c;Y)\left(=\sum_{n=0}^{\infty}\sum_{p\in\mathcal{P}_{n}}\frac{(a)_{p}}{(c)_{p}}\cdot\frac{\mathcal{C}_{p}(Y)}{n!}\right)$$

$$=\frac{\Gamma_{m}(c)}{\Gamma_{m}(a)\Gamma_{m}(c-a)}\int_{0< X< I_{m}}e^{tr(XY)}\left(\det X\right)^{a-\frac{m+1}{2}}\left(\det\left(I_{m}-X\right)\right)^{c-a-\frac{m+1}{2}}dX,$$

where

- $0 < X < I_m$ means both X and $I_m X$ are positive definite;
- $\mathbf{d}X = \prod_{i \leq i} dx_{ij}$ is the Lebesgue measure of the upper triangular entries of X;

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Kummer relation

$$e^{-\operatorname{tr} Y} {}_{1}F_{1}(a; c; Y) = {}_{1}F_{1}(c - a; c; - Y)$$

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Theorem

[Differential Equation] Let $F(Y) = {}_1F_1(a; c; Y)$ for $Y = \text{diag}(y_1, \dots, y_m)$, then F is the unique solution to

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subject to that F is symmetric in v_1, \ldots, v_m and F is analytic at Y = 0 with F(0) = 1



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Remark

Restrict to the open region
$$\mathcal{X}:=\left\{y\in\mathbb{C}^m:\prod\limits_{i=1}^my_i\prod\limits_{i\neq i}\left(y_i-y_j\right)
eq 0\right\}$$

$$g_i := y_i \partial_i^2 + (c - y_i) \partial_i + \frac{1}{2} \sum_{j=1, j \neq i}^m \frac{y_i}{y_i - y_j} (\partial_i - \partial_j) - a \Rightarrow g_i F = 0$$



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If $A_{n \times n}$ is symmetric, then

$$(\operatorname{tr} A)^n = \sum_{p \in \mathcal{P}_n} C_p(A).$$

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End

Thank you