group representation theory of $\mathcal{G}\ell(m, R)$, the general linear group. The theory leading up to this definition is, however, quite difficult from a technical point of view, and for a detailed discussion of the group theoretic construction of zonal polynomials the reader is referred to Farrell (1976) and the papers of James and Constantine cited above, particularly James (1961b). Rather than outline a course in group representation theory, here we will start from another definition for the zonal polynomials which may appear somewhat arbitrary but probably has more pedagogic value. It should be emphasized that the treatment here is intended as an introduction to zonal polynomials and related topics. This is particularly true in Sections 7.2.1 and 7.2.2, where a rather informal approach is apparent. [For yet another approach, see an interesting paper by Saw (1977).]

7.2. ZONAL POLYNOMIALS

7.2.1. *Definition and Construction*

The zonal polynomials of a matrix are defined in terms of partitions of positive integers. Let k be a positive integer; a partition κ of k is written as $\kappa=(k_1, k_2,\dots)$, where $\Sigma_i k_i = k$, with the convention unless otherwise stated, that $k_1 \geq k_2 \geq \cdots$, where $k_1, k_2,\dots$ are non-negative integers. We will order the partitions of k *lexicographically*; that is, if $\kappa=(k_1, k_2,\dots)$ and $\lambda=(l_1, l_2,\dots)$ are two partitions of k we will write $\kappa>\lambda$ if $k_i>l_i$ for the first index i for which the parts are unequal. For example, if $k=6$,

$$\kappa=(2,2,2)>\lambda=(2,2,1,1).$$

Now suppose that $\kappa=(k_1,\dots,k_m)$ and $\lambda=(l_1,\dots,l_m)$ are two partitions of k (some of the parts may be zero) and let $y_1,\dots,y_m$ be m variables. If $\kappa>\lambda$ we will say that the monomial $y_1^{k_1}\dots y_m^{k_m}$ is of *higher weight* than the monomial $y_1^{l_1}\dots y_m^{l_m}$.

We are now ready to define a zonal polynomial. Before doing so, recall from the discussion in Section 7.1 that what we would like is a generalization of the function $f_k(x)=x^k$, which satisfies the differential equation $x^2 f_k''(x)=k(k-1)x^k$. Bearing this in mind may help to make the following definition seem a little less arbitrary. It is based on papers by James in 1968 and 1973.

DEFINITION 7.2.1. Let Y be an $m\times m$ symmetric matrix with latent roots $y_1,\dots,y_m$ and let $\kappa=(k_1,\dots,k_m)$ be a partition of k into not more than m parts. The zonal polynomial of Y corresponding to κ, denoted by $C_\kappa(Y)$, is a symmetric, homogeneous polynomial of degree k in the latent roots $y_1,\ldots,y_m$ such that:

(i) The term of highest weight in $C_\kappa(Y)$ is $y_1^{k_1}\ldots y_m^{k_m}$; that is,

$$C_\kappa(Y)=d_\kappa y_1^{k_1}\ldots y_m^{k_m}+\text{terms of lower weight,} \tag{1}$$

where d_κ is a constant.

(ii) $C_\kappa(Y)$ is an *eigenfunction* of the differential operator Δ_Y given by

$$\Delta_Y=\sum_{i=1}^m y_i^2\frac{\partial^2}{\partial y_i^2}+\sum_{i=1}^m\sum_{\substack{j=1\\ j\neq i}}^m \frac{y_i^2}{y_i-y_j}\frac{\partial}{\partial y_i}. \tag{2}$$

(iii) As κ varies over all partitions of k the zonal polynomials have unit coefficients in the expansion of $(\operatorname{tr} Y)^k$; that is,

$$(\operatorname{tr} Y)^k=(y_1+\cdots+y_m)^k=\sum_\kappa C_\kappa(Y). \tag{3}$$

We will now comment on various aspects of this definition.

Remark 1. By a symmetric, homogeneous polynomial of degree k in $y_1,\ldots,y_m$ we mean a polynomial which is unchanged by a permutation of the subscripts and such that every term in the polynomial has degree k.

For example, if $m=2$ and $k=3$,

$$y_1^3+y_2^3+10y_1^2y_2+10y_1y_2^2$$

is a symmetric, homogeneous polynomial of degree 3 in y_1 and y_2.

Remark 2. The zonal polynomial $C_\kappa(Y)$ is a function only of the latent roots $y_1,\ldots,y_m$ of Y and so could be written, for example, as $C_\kappa(y_1,\ldots,y_m)$. However, for many purposes it is more convenient to use the matrix notation of the definition; see, for example, Theorem 7.2.4 later.

Remark 3. By saying that $C_\kappa(Y)$ is an eigenfunction of the differential operator Δ_Y given by (2) we mean that

$$\Delta_Y C_\kappa(Y)=\alpha C_\kappa(Y),$$

where α is a constant which does not depend on $y_1,\ldots,y_m$ (but which can depend on κ) and which is called the eigenvalue of Δ_Y corresponding to $C_\kappa(Y)$. This constant will be found in Theorem 7.2.2.

Remark 4. It has yet to be established that Definition 7.2.1 is not vacuous and that indeed there exists a unique polynomial in $y_1,\ldots,y_m$ satisfying all the conditions of this definition. Basically what happens is that condition (i), along with the condition that $C_\kappa(Y)$ is a symmetric, homogeneous polynomial of degree k, establishes what types of terms appear in $C_\kappa(Y)$. The differential equation for $C_\kappa(Y)$ provided by (ii) and Theorem 7.2.2 below then gives recurrence relations between the coefficients of these terms which determine $C_\kappa(Y)$ uniquely up to some normalizing constant. The normalization is provided by condition (iii), and this is the only role this condition plays. At this point it should be stated that no *general* formula for zonal polynomials is known; however, the above description provides a general *algorithm* for their calculation. We will illustrate the steps involved with concrete examples later. Before doing so, let us find the eigenvalue implicit in condition (ii).

THEOREM 7.2.2. The zonal polynomial $C_\kappa(Y)$ corresponding to the partition $\kappa=(k_1,\ldots,k_m)$ of k satisfies the partial differential equation

$$\Delta_Y C_\kappa(Y)=\left[\rho_\kappa+k(m-1)\right]C_\kappa(Y), \tag{4}$$

where Δ_Y is given by (2) and

$$\rho_\kappa=\sum_{i=1}^{m}k_i(k_i-i). \tag{5}$$

[Hence the eigenvalue α in Remark 3 is $\alpha=\rho_\kappa+k(m-1)$.]

Proof. By conditions (i) and (ii) it suffices to show that

$$\Delta_Y y_1^{k_1}\cdots y_m^{k_m}=\left[\rho_\kappa+k(m-1)\right]y_1^{k_1}\cdots y_m^{k_m}+\text{terms of lower weight.}$$

By straightforward differentiation it is seen that

$$\Delta_Y y_1^{k_1}\cdots y_m^{k_m}=y_1^{k_1}\cdots y_m^{k_m}\left[\sum_{i=1}^{m}k_i(k_i-1)+\sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m}\frac{y_ik_i}{y_i-y_j}\right]$$

$$=y_1^{k_1}\cdots y_m^{k_m}\left[\sum_{i=1}^{m}k_i^2-k+\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\left(\frac{y_ik_i}{y_i-y_j}+\frac{y_jk_j}{y_j-y_i}\right)\right]$$

Since

$$\frac{y_ik_i}{y_i-y_j}=k_i+\frac{y_jk_i}{y_i-y_j}$$

it follows that

$$\Delta_Y y_1^{k_1}\dots y_m^{k_m}=y_1^{k_1}\dots y_m^{k_m}\left\{\sum_{i=1}^{m}k_i^2-k+\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\left[k_i+\frac{y_j}{y_j-y_i}(k_j-k_i)\right]\right\}$$

$$=y_1^{k_1}\dots y_m^{k_m}\left[\sum_{i=1}^{m}k_i^2-k+\sum_{i=1}^{m-1}k_i(m-i)\right]+\text{terms of lower weight.}$$

Noting that

$$\sum_{i=1}^{m-1}k_i(m-i)=\sum_{i=1}^{m}k_i(m-i)=km-\sum_{i=1}^{m}ik_i$$

we then have

$$\Delta_Y y_1^{k_1}\dots y_m^{k_m}=y_1^{k_1}\dots y_m^{k_m}\left[\sum_{i=1}^{m}k_i(k_i-i)+k(m-1)\right]+\text{terms of lower weight,}$$

and the proof is complete.

Before proceeding further it is worth pointing out explicitly two consequences of Definition 7.2.1. The first is that if $m=1$, condition (iii) becomes $y^k=C_{(k)}(Y)$ so that the zonal polynomials of a matrix variable are analogous to powers of a single variable. The second consequence is that if β is a constant then the fact that $C_\kappa(Y)$ is homogeneous of degree k implies that $C_\kappa(\beta Y)=\beta^k C_\kappa(Y)$.

We will now illustrate how Definition 7.2.1 can be used to construct an algorithm for calculating zonal polynomials by using it to find explicit formulas corresponding to the values $k=1$, 2, and 3. We will express these zonal polynomials in terms of the *monomial symmetric functions*. If $\kappa=(k_1,\dots,k_m)$, the monomial symmetric function of $y_1,\dots,y_m$ corresponding to κ is defined as

$$M_\kappa(Y)=\sum\dots\sum y_{i_1}^{k_1}y_{i_2}^{k_2}\dots y_{i_p}^{k_p},$$

where p is the number of nonzero parts in the partition κ and the summation is over the distinct permutations $(i_1,\dots,i_p)$ of p different integers from the integers $1,\dots,m$. Hence

$$M_\kappa(Y)=y_1^{k_1}\dots y_m^{k_m}+\text{symmetric terms.}$$

Thus, for example,

$$M_{(1)}(Y)=y_1+\cdots+y_m,$$

$$M_{(2)}(Y)=y_1^2+\cdots+y_m^2,$$

$$M_{(1,1)}(Y)=\sum_{i<j}^{m} y_i y_j,$$

and so on.

$\underline{k=1}$: When $k=1$ there is only one partition $\kappa=(1)$ so, by condition (iii), $C_{(1)}(Y)=\operatorname{tr} Y=y_1+\cdots+y_m=M_{(1)}(Y)$.

$\underline{k=2}$: When $k=2$ there are two zonal polynomials corresponding to the partitions (2) and (1, 1) of the integer 2. Using condition (i) and the fact that the zonal polynomials are symmetric and homogeneous of degree 2 we have

$$\begin{aligned}C_{(2)}(Y)&=d_{(2)}y_1^2+\text{terms of lower weight}\\&=d_{(2)}y_1^2+\beta y_1y_2+\text{symmetric terms}\\&=d_{(2)}(y_1^2+\cdots+y_m^2)+\beta(y_1y_2+\cdots+y_{m-1}y_m)\\&=d_{(2)}M_{(2)}(Y)+\beta M_{(1,1)}(Y)\end{aligned}$$

for some constant β, and

$$\begin{aligned}C_{(1,1)}(Y)&=d_{(1,1)}y_1y_2+\text{terms of lower weight}\\&=d_{(1,1)}(y_1y_2+\cdots+y_{m-1}y_m)\\&=d_{(1,1)}M_{(1,1)}(Y).\end{aligned}$$

By condition (iii) we have

$$\begin{aligned}(\operatorname{tr} Y)^2&\equiv M_{(2)}(Y)+2M_{(1,1)}(Y)=C_{(2)}(Y)+C_{(1,1)}(Y)\\&=d_{(2)}M_{(2)}(Y)+(\beta+d_{(1,1)})M_{(1,1)}(Y),\end{aligned}$$

and equating coefficients of $M_{(2)}(Y)$ and $M_{(1,1)}(Y)$ on both sides shows that

$$d_{(2)}=1,\qquad d_{(1,1)}=2-\beta,$$

so that

$$C_{(2)}(Y)=M_{(2)}(Y)+\beta M_{(1,1)}(Y) \tag{6}$$

and

$$C_{(1,1)}(Y)=(2-\beta)M_{(1,1)}(Y).$$

The constant β is now found using the differential equation for $C_{(2)}(Y)$. Since $\rho_{(2)}=2(2-1)=2$, Theorem 7.2.2 shows that $C_{(2)}(Y)$ satisfies the partial differential equation

$$\Delta_Y C_{(2)}(Y)=2mC_{(2)}(Y), \tag{7}$$

where Δ_Y is the differential operator given by (2). It is easily verified that

$$\Delta_Y M_{(2)}(Y)=2mM_{(2)}(Y)+2M_{(1,1)}(Y)$$

and

$$\Delta_Y M_{(1,1)}(Y)=(2m-3)M_{(1,1)}(Y),$$

and hence substitution of (6) in (7) yields

$$2mM_{(2)}(Y)+2M_{(1,1)}(Y)+\beta(2m-3)M_{(1,1)}(Y)$$
$$=2mM_{(2)}(Y)+2m\beta M_{(1,1)}(Y).$$

Equating coefficients of $M_{(1,1)}(Y)$ on both sides then gives $\beta=2/3$. Hence the two zonal polynomials corresponding to $k=2$ are

$$C_{(2)}(Y)=M_{(2)}(Y)+\frac{2}{3}M_{(1,1)}(Y)$$

and

$$C_{(1,1)}(Y)=\frac{4}{3}M_{(1,1)}(Y).$$

$\underline{k=3}$: When $k=3$ there are three zonal polynomials corresponding to the partitions (3), (2, 1), and (1, 1, 1); we will indicate how these can be evaluated, leaving the details as an exercise. Conditions (i) and (iii) of Definition 7.2.1, together with the symmetric homogeneous nature of the zonal polynomials, are sufficient to show that

$$C_{(3)}(Y)=M_{(3)}(Y)+\beta M_{(2,1)}(Y)+\gamma M_{(1,1,1)}(Y), \tag{8}$$

$$C_{(2,1)}(Y)=(3-\beta)M_{(2,1)}(Y)+\delta M_{(1,1,1)}(Y),$$

and

$$C_{(1,1,1)}(Y)=(6-\gamma-\delta)M_{(1,1,1)}(Y),$$

for some constants β,γ, and δ. The partial differential equation for $C_{(3)}(Y)$ then determines β and γ. Once β is known, the partial differential equation for $C_{(2,1)}(Y)$ determines δ. To demonstrate this, we need the effect of the operator Δ_Y on the monomial symmetric functions $M_{(3)}(Y)$, $M_{(2,1)}(Y)$, and $M_{(1,1,1)}(Y)$. It can be readily verified that

$$\Delta_Y M_{(3)}(Y)=3(m+1)M_{(3)}(Y)+3M_{(2,1)}(Y), \tag{9}$$

$$\Delta_Y M_{(2,1)}(Y)=(3m-2)M_{(2,1)}(Y)+6M_{(1,1,1)}(Y),$$

and

$$\Delta_Y M_{(1,1,1)}(Y)=3(m-2)M_{(1,1,1)}(Y).$$

Since $\rho_{(3)}=3(3-1)=6$, Theorem 7.2.2 shows that $C_{(3)}(Y)$ satisfies the partial differential equation

$$\Delta_Y C_{(3)}(Y)=3(m+1)C_{(3)}(Y). \tag{10}$$

Substituting for $C_{(3)}(Y)$ from (8) in (10), using the differential relations (9), and equating coefficients of $M_{(2,1)}(Y)$ and $M_{(1,1,1)}(Y)$ on both sides then gives $\beta=3/5$ and $\gamma=2/5$. Since $\rho_{(2,1)}=2(2-1)+1(1-2)=1$, the partial differential equation given by Theorem 7.2.2 for $C_{(2,1)}(Y)$ is

$$\Delta_Y C_{(2,1)}(Y)=(3m-2)C_{(2,1)}(Y). \tag{11}$$

Substituting for $C_{(2,1)}(Y)$ from (8), with $\beta=3/5$, in (11), using the differential relations (9), and equating coefficients of $M_{(1,1,1)}(Y)$ on both sides then gives $\delta=18/5$. Hence the three zonal polynomials of degree 3 are

$$C_{(3)}(Y)=M_{(3)}(Y)+\frac{3}{5}M_{(2,1)}(Y)+\frac{2}{5}M_{(1,1,1)}(Y), \tag{12}$$

$$C_{(2,1)}(Y)=\frac{12}{5}M_{(2,1)}(Y)+\frac{18}{5}M_{(1,1,1)}(Y),$$

and

$$C_{(1,1,1)}(Y)=2M_{(1,1,1)}(Y).$$

In general, it should now be apparent that the differential equation for $C_\kappa(Y)$ gives rise to a recurrence relation between the coefficients of the monomial symmetric functions in $C_\kappa(Y)$; once the coefficient of the term of highest weight is given, the other coefficients are uniquely determined by the recurrence relation. We will *state* a general result, due to James (1968). Let κ be a partition of k; condition (i) of Definition 7.2.1 and the fact that the zonal polynomial $C_\kappa(Y)$ is symmetric and homogeneous of degree k show that $C_\kappa(Y)$ can be expressed in terms of the monomial symmetric functions as

$$C_\kappa(Y)=\sum_{\lambda\le\kappa} c_{\kappa,\lambda}M_\lambda(Y), \tag{13}$$

where the $c_{\kappa,\lambda}$ are constants and the summation is over all partitions λ of k with $\lambda\le\kappa$ (that is, λ is below or equal to κ in the lexicographical ordering). Substituting this expression (13) in the partial differential equation

$$\Delta_Y C_\kappa(Y)=\left[\rho_\kappa+k(m-1)\right]C_\kappa(Y)$$

and equating coefficients of like monomial symmetric functions on both sides leads to a recurrence relation for the coefficients, namely,

$$c_{\kappa,\lambda}=\sum_{\lambda<\mu\le\kappa}\frac{\left[(l_i+t)-(l_j-t)\right]}{\rho_\kappa-\rho_\lambda}c_{\kappa,\mu} \tag{14}$$

where $\lambda=(l_1,\dots,l_m)$ and $\mu=(l_1,\dots,l_i+t,\dots,l_j-t,\dots,l_m)$ for $t=1,\dots,l_j$ such that, when the parts of the partition μ are arranged in descending order, μ is above λ and below or equal to κ in the lexicographical ordering. The summation in (14) is over all such μ, including possibly, nondescending ones, and any empty sum is taken to be zero. This recurrence relation determines $C_\kappa(Y)$ uniquely once the coefficient of the term of highest weight is given. Using condition (iii) of Definition 7.2.1 it follows that for $\kappa=(k)$ the coefficient of the term of highest weight in $C_{(k)}(Y)$ is unity; that is, $c_{(k),(k)}=1$. This determines all the other coefficients $c_{(k),\lambda}$ in the expansion (13) of $C_{(k)}(Y)$ in terms of monomial symmetric functions. These determine, in turn, the coefficient of the term of highest weight in $C_{(k-1,1)}(Y)$, and once this is known, the recurrence relation gives all the other coefficients, and so on. The reader can readily verify that the general recurrence relation (14) gives the coefficients of the monomial symmetric functions found earlier in the expressions for the zonal polynomials of degree $k=1$, 2, and 3. We will look at one further example, namely, $k=4$. Here there are five zonal polynomials, corresponding to the partitions (4), (3,1), (2,2), (2,1,1), and (1,1,1,1). Consider the zonal polynomial $C_{(4)}(Y)$. Using (13) this can be written in terms of the monomial symmetric functions as

$$\begin{aligned} C_{(4)}(Y) = M_{(4)}(Y) &+ c_{(4),(3,1)}M_{(3,1)}(Y) + c_{(4),(2,2)}M_{(2,2)}(Y) \\ &+ c_{(4),(2,1,1)}M_{(2,1,1)}(Y) + c_{(4),(1,1,1,1)}M_{(1,1,1,1)}(Y), \end{aligned}$$

where we have used the fact that $c_{(4),(4)}=1$. Consider the coefficient $c_{(4),(3,1)}$. Putting $\kappa=(4)$, $\lambda=(3,1)$ in (14) and using $\rho_{(4)}=12$, $\rho_{(3,1)}=5$ gives

$$c_{(4),(3,1)} = \frac{4-0}{12-5} = \frac{4}{7}.$$

The coefficient $c_{(4),(2,2)}$ comes from the partitions (3,1) and (4) and, since $\rho_{(2,2)}=2$, it is

$$\begin{aligned} c_{(4),(2,2)} &= \frac{4-0}{12-2}c_{(4),(4)} + \frac{3-1}{12-2}c_{(4),(3,1)} \\ &= \frac{18}{35}. \end{aligned}$$

The coefficient $c_{(4),(2,1,1)}$ comes from the partitions (3,1,0), (3,0,1), and (2,2,0) and, since $\rho_{(2,1,1)}=-1$, it is

$$\begin{aligned} c_{(4),(2,1,1)} &= 2\cdot\frac{3-0}{12+1}c_{(4),(3,1)} + \frac{2-0}{12+1}c_{(4),(2,2)} \\ &= \frac{12}{35}. \end{aligned}$$

The coefficient $c_{(4),(1,1,1,1)}$ comes from the partitions (2,0,1,1), (2,1,0,1), (2,1,1,0), (1,2,0,1), (1,2,1,0), and (1,1,2,0) and, since $\rho_{(1,1,1,1)}=-6$, it is

$$c_{(4),(1,1,1,1)} = 6\cdot\frac{2-0}{12+6}c_{(4),(2,1,1)} = \frac{8}{35}.$$

Hence the zonal polynomial $C_{(4)}(Y)$ is

$$(15)\qquad C_{(4)}(Y)=M_{(4)}(Y)+\frac{4}{7}M_{(3,1)}(Y)+\frac{18}{35}M_{(2,2)}(Y)$$
$$+\frac{12}{35}M_{(2,1,1)}(Y)+\frac{8}{35}M_{(1,1,1,1)}(Y).$$

The next zonal polynomial $C_{(3,1)}(Y)$ can be written

$$(16)\quad C_{(3,1)}(Y)=c_{(3,1),(3,1)}M_{(3,1)}(Y)+c_{(3,1),(2,2)}M_{(2,2)}(Y)$$
$$+c_{(3,1),(2,1,1)}M_{(2,1,1)}(Y)+c_{(3,1),(1,1,1,1)}M_{(1,1,1,1)}(Y),$$

and condition (iii) of Definition 7.2.1, in conjunction with the expression (15) for $C_{(4)}(Y)$, shows that $c_{(3,1),(3,1)}=24/7$. The recurrence relation (14) then determines the other coefficients in (16); the remaining computations for $k=4$ are left as an exercise (see Problem 7.1).

Without delving deeply into the details we will give two properties of zonal polynomials which can be proved using the recurrence relation (14). They are consequences of the following lemma.

LEMMA 7.2.3. Let the coefficients $c_{\kappa,\lambda}$ be given by (13) and suppose that κ is a partition of k into p nonzero parts. If the partition λ of k has less than p nonzero parts and $\lambda<\kappa$ then $c_{\kappa,\lambda}=0$.

Rather than giving a tedious algebraic proof, we will illustrate the lemma with an example. The partition $\kappa=(4,1,1,1)$ of $k=7$ is followed in the lexicographical ordering by two partitions with less than four parts, namely, $(3,3,1)$ and $(3,2,2)$. Considering first $\lambda=(3,3,1)$, the recurrence relation (14) immediately shows that $c_{\kappa,\lambda}=0$ because there are no partitions μ satisfying $\lambda<\mu\leq\kappa$ [see the discussion following (14)]. Now taking $\lambda=(3,2,2)$, the coefficient $c_{\kappa,\lambda}$ comes from the partition $(3,3,1)$ so that

$$c_{\kappa,\lambda}=2\cdot\frac{3-1}{\rho_\kappa-\rho_\lambda}c_{\kappa,\mu},$$

where $\mu=(3,3,1)$, and it has just been established that $c_{\kappa,\mu}=0$.

The two aforementioned properties of zonal polynomials are given in the following corollary.

COROLLARY 7.2.4.

(i) If the $m\times m$ symmetric matrix Y has rank r, so that $y_{r+1}=\cdots=y_m=0$, and if κ is a partition of k into more than r parts, then $C_\kappa(Y)=0$.

(ii) If Y is a positive definite matrix ($Y>0$) then $C_\kappa(Y)>0$.

Proof. To prove (i), write $C_\kappa(Y)$ as

$$C_\kappa(Y)=\sum_{\lambda\leq\kappa} c_{\kappa,\lambda}M_\lambda(Y).$$

Now note that $M_\lambda(Y)=0$ if the number of nonzero parts in λ is greater than or equal to the number of nonzero parts in κ, while if the reverse is true then $c_{\kappa,\lambda}=0$ by Lemma 7.2.3. Part (ii) is proved by noting that the monomial symmetric functions are positive when $Y>0$, and the coefficients $c_{\kappa,\lambda}$ generated by the recurrence relation (14) are non-negative.

Zonal polynomials have so far been defined only for symmetric matrices. The definition can be extended: if Y is symmetric and X is positive definite then the latent roots of XY are the same as the latent roots of $X^{1/2}YX^{1/2}$ and we define $C_\kappa(XY)$ as

$$C_\kappa(XY)=C_\kappa(X^{1/2}YX^{1/2}). \tag{17}$$

As stated earlier there is no known general formula for zonal polynomials. Expressions are known for some special cases (see James, 1964, 1968). One of these special cases is when $Y=I_m$. Although we will not derive the result here, it is worth stating. If the partition κ of k has p nonzero parts, the value of the zonal polynomial at I_m is given by

$$C_\kappa(I_m)=2^{2k}k!(\tfrac{1}{2}m)_\kappa\frac{\prod_{i<j}^{p}(2k_i-2k_j-i+j)}{\prod_{i<i}^{p}(2k_i+p-i)!} \tag{18}$$

where

$$(\tfrac{1}{2}m)_\kappa=\prod_{i=1}^{p}\left(\tfrac{1}{2}(m-i+1)\right)_{k_i},$$

with $(a)_k=a(a+1)\ldots(a+k-1),(a)_0=1$. For a proof of this result the reader is referred to Constantine (1963). Although no general formula is known, the recurrence relation (14) enables the zonal polynomials to be computed quite readily. The coefficients $c_{\kappa,\lambda}$ of the monomial symmetric functions $M_\lambda(Y)$ in $C_\kappa(Y)$ obtained from (14) are given in Table 1 to $k=5$. They have been tabulated to $k=12$ by Parkhurst and James (1974) in terms of the sums of powers of the latent roots and in terms of the elementary

Table 1. Coefficients of monomial symmetric functions $M_\lambda(Y)$ in the zonal polynomial $C_\kappa(Y)$

$k=2$

κ \ λ	(2)	(1,1)
(2)	1	2/3
(1,1)	0	4/3

$k=3$

κ \ λ	(3)	(2,1)	(1,1,1)
(3)	1	3/5	2/5
(2,1)	0	12/5	18/5
(1,1,1)	0	0	2

$k=4$

κ \ λ	(4)	(3,1)	(2,2)	(2,1,1)	(1,1,1,1)
(4)	1	4/7	18/35	12/35	8/35
(3,1)	0	24/7	16/7	88/21	32/7
(2,2)	0	0	16/5	32/15	16/5
(2,1,1)	0	0	0	16/3	64/5
(1,1,1,1)	0	0	0	0	16/5

$k=5$

κ \ λ	(5)	(4,1)	(3,2)	(3,1,1)	(2,2,1)	(2,1,1,1)	(1,1,1,1,1)
(5)	1	5/9	10/21	20/63	2/7	4/21	8/63
(4,1)	0	40/9	8/3	46/9	4	14/3	40/9
(3,2)	0	0	48/7	32/7	176/21	64/7	80/7
(3,1,1)	0	0	0	10	20/3	130/7	200/7
(2,2,1)	0	0	0	0	32/3	16	32
(2,1,1,1)	0	0	0	0	0	80/7	800/21
(1,1,1,1,1)	0	0	0	0	0	0	16/3