Hypergeometric Functions with Matrix Argument

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Hypergeometric function and Pochhammer symbol

Definition

$$
{}_sF_t\left(\begin{matrix} a_1,\ldots,a_s \\ b_1,\ldots,b_t \end{matrix} : z\right) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_s)_n}{(b_1)_n \cdots (b_t)_n} \cdot \frac{z^n}{n!},
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where $(a)_k = a(a+1)\cdots(a+k-1)$.

Examples

- ${}_2F_1\left(\begin{smallmatrix} a,b \\ c \end{smallmatrix} : z\right)$ is the Gaussian hypergeometric function s. t.

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z(1-z)\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + (c-(a+b+1)z)\frac{\mathrm{d}w}{\mathrm{d}z} - abw = 0.
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- $\log(1+z) = z\,{}_2F_1\left(\begin{smallmatrix} 1,1 \\ 2 \end{smallmatrix} : -z\right)$

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Given a $m \times m$ symmetric matrix Y,

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where,

- $\mathcal{P}_n$ is the set of all partitions of n;
- for $p = (p_1, \ldots, p_l) \in \mathcal{P}_n$, $(a)_p = \prod_{i=1}^{l} \left(a - \frac{i-1}{2}\right)_{p_i}$;
- $\mathcal{C}_p(Y)$ is ($\mathcal{C}$*-normalization of*) zonal polynomial, which is homogeneous, symmetric, polynomial of degree $n = |p|$, in the eigenvalues of Y.

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Zonal Polynomials

Definition

Zonal polynomial $\mathcal{Y}_p$ is defined by a vector form

$$\mathcal{Y} = \left(\mathcal{Y}_{(n)}, \mathcal{Y}_{(n-1,1)}, \ldots, \mathcal{Y}_{(1^n)}\right)^T = \Xi\mathcal{U},$$

- $\mathcal{U} = \left(\mathcal{U}_{(n)}, \ldots, \mathcal{U}_{(1^n)}\right)^T$ and $\mathcal{U}_p := u_1^{p_1-p_2} u_2^{p_2-p_3} \cdots u_l^{p_l}$, where $u_r := \sum_{i_1<\cdots<i_r} x_{i_1} \cdots x_{i_r}$; $\deg \mathcal{U}_p = (p_1 - p_2) + \cdots + l p_l = n$
$$\mathcal{U}_p(A) = \mathcal{U}_p(\alpha_1, \ldots, \alpha_k), \text{ where } \alpha_i\text{'s are eigenvalues.}$$
- Ξ is a matrix: nonsingular, upper triangular, such that $\Xi T_\nu = \Lambda_\nu \Xi$. $\dim \Xi = p(n) = |\mathcal{P}_n|$.

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Subindex ν involves random matrix $W \sim \mathcal{W}(I_k, \nu)$

1. $\lambda_{\nu p} := 2^n \prod_{i=1}^{l} \left(\frac{\nu+1-i}{2}\right)_{p_i}$ (Recall $(a)_p = \prod_{i=1}^{l} \left(a - \frac{i-1}{2}\right)_{p_i}$) Moments of χ^2
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$$V_n := \{f : f \text{ is homogeneous, symmetric, of degree } n \text{ or } f \equiv 0\}$$

(e.g. $u_n := \sum_{i_1 < \cdots < i_n} x_{i_n} \cdots x_{i_n}$ and $\mathcal{U}_p$) and $\tau_\nu : V_n \to V_n$ be (linear) transfom such that

$$\tau_\nu(\mathcal{U}_p)(A) = \mathbb{E}[\mathcal{U}_p(AW)].$$

A lemma guarantees

$$\tau_\nu(\mathcal{U}) = T_\nu \mathcal{U}$$

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Wishart distribution

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- $Z_1, \ldots, Z_k \sim \mathcal{N}(0,1)$ are independent, then $Q := Z_1 + \cdots + Z_k \sim \chi_k^2$;
- $X_{\nu \times m}$ such that each row is independently drawn from a m-variate normal distribution, $(x_i^1, \ldots, x_i^m) \sim \mathcal{N}_m(0, V) \Rightarrow S = X^T X \sim W_m(V, \nu)$ and ν is called the degree of freedom.

Remark

- Ξ is uniquely determined up to constant multiplication on each row.
- $\mathcal{C}_p(I_k) = \mathcal{C}_p\left(\underbrace{1, \ldots, 1}_{k}\right) = 1.$

Wishart distribution

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- $Z_1, \ldots, Z_k \sim \mathcal{N}(0,1)$ are independent, then $Q := Z_1 + \cdots + Z_k \sim \chi_k^2$;
- $X_{\nu \times m}$ such that each row is independently drawn from a m-variate normal distribution, $\left(x_i^1, \ldots, x_i^m\right) \sim \mathcal{N}_m(0, V) \Rightarrow S = X^T X \sim W_m(V, \nu)$ and ν is called the degree of freedom.

Remark

- Ξ is uniquely determined up to constant multiplication on each row.
- $\mathcal{C}_p(I_k) = \mathcal{C}_p\left(\underbrace{1, \ldots, 1}_{k}\right) = 1.$

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${}_1F_1\left(a; c; Y\right)$

Integral Rerpesentation

When $\Re\left(a\right) > \frac{m+1}{2}$ and $\Re\left(c-a\right) > \frac{m+1}{2}$,

$$
\begin{aligned}
{}_1F_1\left(a; c; Y\right) & \left(= \sum_{n=0}^{\infty} \sum_{p \in \mathcal{P}_n} \frac{\left(a\right)_p}{\left(c\right)_p} \cdot \frac{\mathcal{C}_p\left(Y\right)}{n!}\right) \\
& = \frac{\Gamma_m\left(c\right)}{\Gamma_m\left(a\right)\Gamma_m\left(c-a\right)} \int_{0<X<I_m} e^{\mathrm{tr}(XY)} \left(\det X\right)^{a-\frac{m+1}{2}} \left(\det\left(I_m - X\right)\right)^{c-a-\frac{m+1}{2}} \mathrm{d}X,
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where

- $0 < X < I_m$ means both X and $I_m - X$ are positive definite;
- $\mathrm{d}X = \prod_{i \le j} dx_{ij}$ is the Lebesgue measure of the upper triangular entries of X;
- $\Gamma_m\left(a\right) := \pi^{\frac{m(m-1)}{4}} \prod_{i=1}^{m} \Gamma\left(a - \frac{i-1}{2}\right)$

Recall that

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{}_1F_1\left(a; c; z\right) = \frac{\Gamma\left(c\right)}{\Gamma\left(a\right)\Gamma\left(c-a\right)} \int_0^1 e^{zt} t^{a-1} \left(1-t\right)^{c-a-1} dt, \text{ when } \Re\left(c\right) > \Re\left(a\right) > 0.
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${}_1F_1\left(a;c;Y\right)$

Kummer relation

$$e^{-\operatorname{tr}Y}\,{}_1F_1\left(a;c;Y\right)={}_1F_1\left(c-a;c;-Y\right)$$

Recall that

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Theorem

[Differential Equation] Let $F\left(Y\right)={}_1F_1\left(a;c;Y\right)$ for $Y=\operatorname{diag}\left(y_1,\ldots,y_m\right)$, then F is the unique solution to

$$\left(y_i\partial_i^2+\left(c-\frac{m-1}{2}-y_i+\frac{1}{2}\sum_{j=1,j\neq i}^{m}\frac{y_i}{y_i-y_j}\right)\partial_i-\frac{1}{2}\sum_{j=1,j\neq i}^{m}\frac{y_j}{y_i-y_j}\partial_j-a\right)F=0,\ \ i=1,\ldots,m$$

subject to that$F$ is symmetric in $y_1,\ldots,y_m$ and F is analytic at $Y=0$ with $F\left(0\right)=1$.

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${}_1F_1\left(a; c; Y\right)$

Remark

Restrict to the open region $\mathcal{X} := \left\{ y \in \mathbb{C}^m : \prod_{i=1}^{m} y_i \prod_{i \neq j} (y_i - y_j) \neq 0 \right\}$.

Note that $\frac{y_i}{y_i - y_j} = 1 + \frac{y_j}{y_i - y_j}$.

$$g_i := y_i \partial_i^2 + (c - y_i)\,\partial_i + \frac{1}{2} \sum_{j=1, j\neq i}^{m} \frac{y_i}{y_i - y_j} (\partial_i - \partial_j) - a \Rightarrow g_i F = 0.$$

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$\mathcal{C}_p(Y)$

- If $A_{n\times n}$ is symmetric, then

$$(\operatorname{tr} A)^n = \sum_{p\in\mathcal{P}_n} \mathcal{C}_p(A).$$

$$\Rightarrow {}_0F_0(;A) = \sum_{n=0}^{\infty}\sum_{p\in\mathcal{P}_n}\frac{\mathcal{C}_p(A)}{n!} = \sum_{n=0}^{\infty}\frac{(\operatorname{tr} A)^n}{n!} = e^{\operatorname{tr} A}.$$

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Thank you