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Formulas for Zonal Polynomials

H. B. KUSHNER AND MORRIS MEISNER

Rockland Research Institute

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New integral and differential formulas for zonal polynomials are proved. As illustrations, zonal polynomials corresponding to partitions of two parts are computed. A method is presented, based on a certain partial differential operator, for expressing an orthogonally invariant polynomial as a linear combination of zonals. Zonal polynomials are expressed as linear combinations of well-known symmetric polynomials.

1. INTRODUCTION

Truly explicit formulas for zonal polynomials $C_\tau(V)$ of a real, symmetric $k \times k$ matrix V are unknown, except when $k \leqslant 2$; or when $k \geqslant 3$, and the number of parts of the partition τ is one (James [2]). Indeed, James [4] wondered whether explicit, usable formulas would soon be found. Recently, however, Kates [5] and the authors have independently obtained new integral and differential formulas for zonal polynomials. One of the integral formulas is, in principle, a computable formula for zonal polynomials. More precisely, using one method of evaluating the integral formula, the problem of calculating the coefficients of zonal polynomials is reduced to the problem of calculating the coefficients of a simpler, explicitly given polynomial. Using a second method of evaluating the integral formula, zonal polynomials can be represented as linear combinations of other symmetric polynomials which are defined by an elementary generating function. As illustrative examples of these methods, representations of zonal polynomials $C_\tau(V)$ are derived when the number of parts of τ is two. Differential formulas for zonals are found by introducing, for each partition τ with less than or equal to k parts, an explicitly given partial differential operator L_τ. Each zonal polynomial is then obtained by permitting L_τ to act on an elementary function. The L_τ

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operator may be used to expand a symmetric function as a linear combination of zonals. A general expression for zonal polynomials is given. However, the evaluation of this expression involves tedious calculation.

2. Integral Formulas for Zonal Polynomials; Coefficients of Zonal Polynomials

James [3] gave two integral representations of zonal polynomials, although neither integral is easily evaluated. Kates [5] gave another integral representation, discovered independently by the authors, which is, in principle, evaluable. In this section, we generalize James' and Kates' results and outline two methods of evaluating Kates' integral. We base the discussion on the concepts of [6, 7].

Let $u(V)$ be an EP polynomial, that is, a polynomial satisfying the eigenfunction property

$$(E_n u)(\Sigma) = \frac{C_{n,k}}{|\Sigma|^{n/2}} \int u(V)\,|V|^{(1/2)(n-k-1)}\, e^{-(1/2)\operatorname{tr} V\Sigma^{-1}}\, dV = \lambda_{n,k} u(\Sigma)$$

where $C_{n,k}^{-1} = 2^{(1/2)kn}\pi^{(1/4)k(k-1)} \prod_{j=1}^{k} \Gamma((n-j+1)/2)$. Then for any $k \times k$ matrix X, the function of V defined by $u(X'VX)$ is also an EP polynomial and so is the function

$$f(V) = \int_{X \in A} u(X'VX)\, g(X)\, dX \tag{2.1}$$

In (2.1) we assume that the integral, over the space of integration A, exists for proper choice of the function g. Moreover, if g satisfies the relation

$$g(HX) = g(X)$$

for $H \in O(k)$, the group of $k \times k$ orthogonal matrices, then the EP function $f(V)$ is symmetric or "orthogonally invariant," i.e.,

$$f(H'VH) = f(V) \qquad \text{for} \quad H \in O(k)$$

In particular, integral (2.1) exists when $A = O(k)$, $g \equiv 1$,

$$u(V) = \Phi(T'VT)$$

where

$$\Phi(V) = v_{11}^{m_1}(v_{11}v_{22} - v_{12}^2)^{m_2} \cdots |V|^{m_k} \tag{2.2}$$

is the "prototype" EP polynomial [6, 7], and $T \in \mathrm{Gl}(k)$, the group of invertible $k \times k$ matrices. For fixed T, Eq. (2.1) becomes

$$f(V) = \int_{O(k)} \Phi(T'H'VHT)\, dH \tag{2.3}$$

Since the function $f(V)$ (we suppress the dependence of f on T) is an orthogonally invariant EP polynomial, it must be a constant multiple of a zonal polynomial. Indeed, $\Phi \in V_{r,\tau}$, the space of EP polynomials, of degree r, corresponding to the partition $\tau = (t_i)$, $1 \leqslant i \leqslant k$, where

$$t_i = \sum_{p=i}^{k} m_p \qquad \text{and} \qquad r = \sum_{p=1}^{k} p m_p = \sum_{i=1}^{k} t_i.$$

In the sequel, we shall denote Φ more fully by Φ_τ. Returning to (2.3), we now have, for some function $B(T)$,

$$f(V) = B(T)\, C_\tau(V)$$

where $C_\tau(V)$ is the zonal polynomial corresponding to the partition τ, i.e., the unique orthogonally invariant polynomial belonging to $V_{r,\tau}$. Substituting $V = I$ in (2.3) we conclude that $\Phi(T'T) = f(I) = B(T)\, C_\tau(I)$ so that

$$f(V) = \int_{O(k)} \Phi_\tau(T'H'VHT)\, dH = \Phi_\tau(T'T) \frac{C_\tau(V)}{C_\tau(I)} = \Phi_\tau(T'T)\, C_\tau^*(V) \tag{2.4}$$

where $C_\tau^*(V) = C_\tau(V)/C_\tau(I)$.

For $T = I$, (2.4) is the integral representation of James [3].

We vary the procedure to obtain an evaluable integral for $C_\tau(V)$. Let $A = E_{k^2}$, Euclidean space of dimension k^2, i.e., the space of $k \times k$ matrices. For $X \in E_{k^2}$, Σ a $k \times k$ positive definite matrix, and $n \geqslant k$, define a function satisfying $g(HX) = g(X)$ by $g(X) = |X'X|^{(1/2)(n-k)} \exp(-\frac{1}{2} \operatorname{tr} \Sigma^{-1} X'X)$. Let $u(V) = \Phi_\tau(V)$, the EP polynomial given in (2.2) which satisfies $E_n \Phi_\tau = \lambda_{n,\tau} \Phi_\tau$ where $\lambda_{n,\tau} = 2^r (n/2)_\tau$ and $(a)_\tau = \prod_{i=1}^{k} (a - (i-1)/2)_{t_i}$ is the generalized hypergeometric symbol. Equation (2.1) now becomes

$$f(V) = \int_{E_{k^2}} \Phi_\tau(X'VX)\, |X'X|^{(1/2)(n-k)}\, e^{-(1/2) \operatorname{tr} \Sigma^{-1} X'X}\, dX \tag{2.5}$$

As before, for some function $b(\Sigma)$,

$$f(V) = b(\Sigma)\, C_\tau(V).$$

Setting $V=I$ in (2.5), we obtain

$$f(I)=C_\tau(I)\,b(\Sigma)=\int \Phi_\tau(X'X)\,|X'X|^{(1/2)(n-k)}\,e^{-(1/2)\operatorname{tr}\Sigma^{-1}X'X}\,dX \quad (2.6)$$

As in Herz [1], for almost every X, the change of variables, $X=HS^{1/2}$, where S is a $k\times k$ positive definite matrix and $H\in O(k)$, gives

$$dX=G_k\,|S|^{-1/2}\,dS\,dH$$

where $G_k=\pi^{(1/2)k^2}/(\pi^{(1/4)k(k-1)}\prod_{i=0}^{k-1}\Gamma((k-i)/2))$ and dH is normalized Haar measure on $O(k)$. From (2.6) we thus calculate

$$\begin{aligned} f(I) &= G_k\int \Phi_\tau(S)\,|S|^{(n-k-1)/2}\,e^{-(1/2)\operatorname{tr}\Sigma^{-1}S}dS\,dH \\ &= G_k\int_{S>0} \Phi_\tau(S)\,|S|^{(n-k-1)/2}e^{-(1/2)\operatorname{tr}\Sigma^{-1}S}\,dS \\ &= G_k(C_{n,k})^{-1}\,|\Sigma|^{n/2}\,(E_n\Phi_\tau)(\Sigma) \end{aligned}$$

Now $E_n\Phi_\tau=\lambda_{n,\tau}\Phi_\tau$. Therefore,

$$f(I)=\lambda_{n,\tau}G_k(C_{n,k})^{-1}\,|\Sigma|^{n/2}\,\Phi_\tau(\Sigma)$$

and

$$\begin{aligned} f(V) &= \lambda_{n,\tau}G_k(C_{n,k})^{-1}\,|\Sigma|^{n/2}\,\Phi_\tau(\Sigma)\,C^*_\tau(V) \\ &= \int \Phi_\tau(X'VX)\,|X'X|^{(1/2)(n-k)}e^{-(1/2)\operatorname{tr}\Sigma^{-1}X'X}\,dX \qquad (2.7) \end{aligned}$$

We now obtain Kates' integral as a special case of (2.7).

Setting $\Sigma=I$ and $n=k$ in (2.7) and noting $G_k(C_{k,k})^{-1}=(2\pi)^{k^2/2}$, we get

$$\lambda_{k,\tau}(2\pi)^{k^2/2}\,C^*_\tau(V)=\int \Phi_\tau(X'VX)\,e^{-(1/2)\operatorname{tr}X'X}\,dX \quad (2.8)$$

which becomes, after the transformation $V^{1/2}X\to X$,

$$\lambda_{k,\tau}(2\pi)^{k^2/2}\,|V|^{k/2}\,C^*_\tau(V)=\int \Phi_\tau(X'X)\,e^{-(1/2)\operatorname{tr}X'V^{-1}X}\,dX \quad (2.9)$$

In (2.9) we set $V=\Lambda$, the diagonal matrix of the eigenvalues of V, to obtain

$$\lambda_{k,\tau}(2\pi)^{k^2/2}\,|\Lambda|^{k/2}\,C^*_\tau(\Lambda)=\int \Phi_\tau(X'X)\,e^{-(1/2)\operatorname{tr}X'\Lambda^{-1}X}\,dX \quad (2.10)$$

Note that when the partition τ has number of parts g less than k, then $\Phi_\tau(X'X)$ depends only on the first g column vectors of X, and (2.10) generalizes to

$$\lambda_{k,\tau}(2\pi)^{kg/2}\,|\Lambda|^{g/2}\,C_\tau^*(\Lambda)=\int \Phi_\tau(X'X)\,e^{-(1/2)\operatorname{tr}X'\Lambda^{-1}X}\,dX \tag{2.11}$$

where X is a $k\times g$ matrix and the integration is over kg-dimensional Euclidean space.

Equation (2.11) can be concisely expressed as

$$\lambda_{k,\tau}C_\tau^*(\Lambda)=E[\Phi_\tau(X'X)] \tag{2.12}$$

where E is the expectation operator of the distribution of the $k\times g$ sample matrix X, each of whose columns is independently distributed as $N(0,\Lambda)$. When $m_i=0$, $2\leqslant i\leqslant k$, i.e., when $\tau=(m_1)$, Eq. (2.12) was obtained by Rubens [9] and, for general τ, by Kates [5].

Another consequence of (2.7) is the equation

$$E[\Phi_\tau(Y'X'XY)]=\lambda_{k,\tau}^2\,C_\tau^*(\Lambda_1)\,C_\tau^*(\Lambda_2)$$

where E is the expectation operator of the two independent sample matrices, $X\sim N(0,\Lambda_1)$ and $Y\sim N(0,\Lambda_2)$, which may be useful in the evaluation of hypergeometric functions of two matrix variables, James [2]. We also observe that, if in Eq. (2.10) the change of variables $X=HS^{1/2}$ is used, followed by integration with respect to dS, the resulting formula will be of a form dual to that of James [3]. In (2.10), and (2.11) the integration is performed not on the orthogonal group but on Euclidean space, a domain over which integrals are easier to compute. Indeed, formula (2.11) yields an explicit expression for $C_\tau(V)$, when an explicit expression is known for $\Phi_\tau(X'X)$ as a polynomial in x_{pq}, $1\leqslant p\leqslant k$, $1\leqslant q\leqslant g$, the kg entries of X. For example, suppose that

$$\Phi_\tau(X'X)=\sum a_\tau(j_{11},j_{12}\cdots j_{kg})\,x_{11}^{j_{11}}x_{12}^{j_{12}}\cdots x_{kg}^{j_{kg}} \tag{2.13}$$

Then, since $\operatorname{tr}(X'\Lambda^{-1}X)=\sum_{p=1}^k\lambda_p^{-1}\sum_{q=1}^g x_{pq}^2$, we find, after integrating (2.11), that

$$\left(\frac{k}{2}\right)_\tau \pi^{kg/2}C_\tau^*(\Lambda)=\sum a_\tau(j_{11},j_{12},\ldots,j_{kg})\prod_{p,q}\Gamma\left(\frac{j_{pq}+1}{2}\right)\lambda_1^{j_1/2}\cdots\lambda_k^{j_k/2} \tag{2.14}$$

where $j_p=\sum_{q=1}^g j_{pq}$. The sum in (2.14) runs only over those $k\times g$ matrices $J=(j_{pq})$ for which each j_{pq}, occurring in the sum (2.13) is even, and in the product $\prod_{p,q}$, p goes from 1 to k and q goes from 1 to g. Introduce the notation

$$X^J = \prod_{p,q} x_{pq}^{j_{pq}}$$

$$\Gamma\left(\frac{J+1}{2}\right) = \prod_{p,q} \Gamma\left(\frac{j_{pq}+1}{2}\right)$$

and $y^j = (y_1^{j_1}, y_2^{j_2}, \ldots, y_k^{j_k})$, where $X = (x_{pq})$, $J = (j_{pq})$ are $k \times g$ matrices, $y = (y_p)$ is any k-dimensional vector and $j = (j_p)$ is a k-dimensional "integral vector," i.e., a vector whose components are non-negative integers. We can now express (2.13)–(2.14) as follows.

If $\Phi_\tau(X'X) = \sum_J a_\tau(J)\, X^J$ then

$$\left(\frac{k}{2}\right)_\tau \pi^{(kg)/2} C_\tau^*(\Lambda) = \sum_{J \text{ even}} a_\tau(J)\, \Gamma\left(\frac{J+1}{2}\right) \lambda^{j/2} \tag{2.15}$$

where "J even" indicates that $J = (j_{pq})$ is a $k \times g$ matrix of even entries; j is a k dimensional vector whose pth component is the pth row sum of J, i.e.,

$$j_p = \sum_{q=1}^{g} j_{pq};$$

and λ is a k-dimensional vector consisting of the eigenvalues of Λ.

We now outline a second way to evaluate (2.12). In this method, zonal polynomials are expanded as linear combinations of other symmetric polynomials, defined below. Let x_p denote pth column vector of the $k \times g$ matrix X. Then the (p, q)th entry of $X'X$ is (x_p, x_q), the inner product of the vectors x_p and x_q. From

$$E[e^{(1/2)\operatorname{tr} SX'X}] = |I - S \times \Lambda|^{-1/2}$$

where S is a $g \times g$ positive definite matrix, E is the expectation operator appearing in (2.12), and $S \times \Lambda$ denotes the Kronecker product of S and Λ, we have

$$E[(x_1, x_1)^{j_{11}} (x_1, x_2)^{j_{12}} \cdots (x_g, x_g)^{j_{gg}}] = 2^{\operatorname{tr} J} \prod_{p \geqslant q}^{g} (j_{pq}!)\, \Psi_J(\Lambda)$$

where, for each symmetric $g \times g$ matrix $J = (j_{pq})$ of non-negative integers, $\Psi_J(\Lambda)$ is the symmetric polynomial defined by the generating function

$$|I - S \times \Lambda|^{-1/2} = \sum_J \Psi_J(\Lambda) \prod_{p \geqslant q}^{g} s_{pq}^{j_{pq}} \tag{2.16}$$

Expressing $\Phi_\tau(X'X)$ as a polynomial in the inner products (x_p, x_q) $1 \leqslant p, q \leqslant g$ now leads to the result: If $\Phi_\tau(X'X) = \sum_M b_\tau(M) \prod_{p \geqslant q}^{g} (x_p, x_q)^{m_{pq}}$ where $M = (m_{pq})$ is a symmetric matrix, then

$$\lambda_{k,\tau} C_\tau^*(\Lambda) = \sum_M b_\tau(M)\, 2^{\operatorname{tr} M} \prod_{p \geqslant q}^{g} (m_{pq}!)\, \Psi_M(\Lambda) \tag{2.17}$$

where $\Psi_M(\Lambda)$ is defined in (2.16).

3. Illustrations

We now illustrate results (2.15) and (2.17) by giving two formulas for zonal polynomials corresponding to two-part partitions, $\tau = (t_1, t_2)$. In the first formula, $C_\tau^*(V)$ is expressed as a polynomial in the eigenvalues of V; in the second formula, $C_\tau^*(V)$ is expressed as a bilinear combination of zonal polynomials corresponding to one part.

Since

$$C_\rho^*(V) = |V|^l\, C_\tau^*(V), \qquad \rho = (t_1 + l, t_2 + l, l),\ \tau = (t_1, t_2)$$

the most general zonal polynomial can be obtained, when $k = 3$, from the formulas proved in this section.

To apply formula (2.15), with $t_1 = m_1 + m_2$, $t_2 = m_2$, expand $\Phi_\tau(V) = v_{11}{}^{m_1}(v_{11}v_{22} - v_{12}^2)^{m_2}$ by the binomial theorem and substitute $V = X'X$:

$$\Phi_\tau(X'X) = \sum_{i=0}^{m_2} (-)^{m_2 - i} \binom{m_2}{i} (x_1, x_1)^{i+m_1} (x_2, x_2)^i (x_1, x_2)^{2(m_2 - i)} \tag{3.1}$$

where x_1, x_2 are the two k-dimensional column vectors of X. Again expand the power of each inner product (x_p, x_q) by the multinomial theorem

$$\Phi_\tau(X'X) = \sum (-)^{m_2 - i} \binom{m_2}{i} M(u)\, M(v)\, M(w)\, x_1^{2u+w} x_2^{2v+w}$$

$$|u| = m_1 + i$$

$$|v| = i$$

$$|w| = 2(m_2 - i)$$

Here, if $s = (s_1, s_2 \cdots s_n)$ is a vector with integral components, we use the notation

$$M(s) = \frac{(s_1 + s_2 + \cdots + s_n)!}{s_1!\, s_2! \cdots s_n!},$$

the multinomial coefficient, and

$$|s| = s_1 + s_2 + \cdots + s_n$$

Now only even vectors $w = 2s$ contribute to the computation of the zonal polynomial. From (2.15), and the above equation, we have

$$\begin{aligned}\left(\frac{k}{2}\right)_\tau \pi^k C_\tau^*(\Lambda) = \sum_{s,u,v} (-)^{|s|} \binom{t_2}{|s|} M(u)\, M(v)\, M(2s) \\ \times \Gamma(u + s + \tfrac{1}{2})\, \Gamma(v + s + \tfrac{1}{2})\, \lambda^{u+v+2s} \\ |s| + |v| = t_2 \\ |u| - |v| = t_1 - t_2\end{aligned} \tag{3.2}$$

where s, u, and v are k-dimensional integral vectors, and λ is a k-dimensional vector consisting of the eigenvalues of Λ.

We now illustrate result (2.17) by expanding zonal polynomials as linear combinations of the symmetric polynomials $\Psi_J(\Lambda)$. According to (2.17) and (3.1), we must calculate $\Psi_{M_j}(\Lambda)$ where $M_j = \binom{j+m_1 \quad 2(m_2-j)}{2(m_2-j) \quad j}$. Since

$$|I - S \times \Lambda| = \prod_{j=1}^{2} |I - \mu_j \Lambda|$$

where μ_1 and μ_2 are the eigenvalues of $S = \binom{s_{11} s_{12}}{s_{12} s_{22}}$, we use (James [2])

$$|I - t\Lambda|^{-1/2} = \sum_{m=0}^{\infty} \frac{Z_{(m)}(\Lambda)\, t^m}{2^m m!}$$

to write

$$\begin{aligned}|I - S \times \Lambda|^{-1/2} &= \sum_{m,n=0}^{\infty} \frac{Z_{(m)}(\Lambda)\, Z_{(n)}(\Lambda)}{2^{m+n} m!\, n!} \mu_1^m \mu_2^n \\ &= \sum_{i,l=0}^{\infty} \frac{Z_{(i+l)}(\Lambda)\, Z_{(l)}(\Lambda)}{2^{i+2l} (i+l)!\, l!} P_{il}(S)\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}P_{0,l}(S) &= |S|^l \\ P_{i,l}(S) &= \operatorname{tr}(S^i)\, |S|^l, \qquad i \geqslant 1\end{aligned}$$

$P_{il}(S)$ can be expanded as a polynomial in s_{11}, s_{12}, s_{22}:

$$P_{il}(S) = \sum_{p+2n+q=i+2l} a_{il}(p, 2n, q)\, s_{11}^p s_{12}^{2n} s_{22}^q \tag{3.4}$$

where for $p \geqslant q$,

$$a_{il}(p, 2n, q) = \frac{i}{p!\, q!} \sum_{j=0}^{\min(l,q)} \binom{l}{j} (-)^{l-j} (-p)_j (-q)_j$$
$$\times (n+1-l+j)_{p-j-1} (n-l+j)_{q-j} \tag{3.5}$$

The derivation of (3.5) is omitted. Note that, in (3.4), $a_{il}(p, 2n, q) = a_{il}(q, 2n, p)$ and, in (3.5), if $p = q$, interpret $i(n+1-l+q)_{-1} = i((i/2)+1)_{-1}$ as equal to 1 or 2, according as $i = 0$ or $i > 0$. Setting $m_2 = t_2$, $m_1 = t_1 - t_2$ in M_j, we therefore obtain

$$\Psi_{M_j}(\Lambda) = \frac{1}{2^{t_1+t_2}} \sum_{i+2l=t_1+t_2} \frac{a_{il}(t_1 - t_2 + j, 2(t_2 - j), j)}{(i+l)!\, l!} Z_{(i+l)}(\Lambda)\, Z_{(l)}(\Lambda)$$

Finally, we have for $\tau = (t_1, t_2)$

$$2^{t_1+t_2} \left(\frac{k}{2}\right)_\tau C_\tau^*(\Lambda) = \sum_{j=0}^{t_2} (-)^{t_2-j} \binom{t_2}{j} 2^{2j+t_1-t_2}$$
$$\times (t_1 - t_2 + j)!\, j!(2(t_2 - j))!\, \Psi_{M_j}(\Lambda)$$

4. A Differential Formula for Zonals; Applications

In this section, we derive a differential formula for zonals. By means of this formula, zonal polynomials are obtained by partial differentiation of an elementary function of two matrix variables. Two interesting applications are noted. In one application, a method is presented which enables one to express an orthogonally invariant polynomial as a linear combination of zonal polynomials. In the second application, zonal polynomials are expressed as linear combinations of other well-known orthogonally invariant polynomials.

Consider the $k \times k$ matrix of partial differential operators

$$D = \begin{bmatrix} 2\dfrac{\partial}{\partial s_{11}} & \dfrac{\partial}{\partial s_{12}} & \dfrac{\partial}{\partial s_{1k}} \\ & \ddots & \\ & & 2\dfrac{\partial}{\partial s_{kk}} \end{bmatrix}$$

Let D_p denote the determinant of the $p \times p$ upper left corner, i.e.,

$$D_1 = 2 \frac{\partial}{\partial s_{11}}$$
$$D_2 = 4 \frac{\partial^2}{\partial s_{11}\,\partial s_{22}} - \frac{\partial^2}{\partial s_{12}^2}$$
$$\vdots$$
$$D_k = \det D$$

and let L_τ denote the operator

$$L_\tau = D_1^{m_1} D_2^{m_2} \cdots D_k^{m_k}$$

The relationship between the partition τ and the vector $m = (m_i)$ is as given in Section 2.

An operator analogue of the prototype function Φ_τ, the operator L_τ, applied to the function

$$g(V, S) = \int e^{-(1/2)\operatorname{tr} X'X + (1/2)\operatorname{tr} SX'VX}\, dX \tag{4.1}$$

yields, after S is set to zero, the right and side of Eq. (2.8). Since $g(V, S) = (2\pi)^{k^2/2}\, |I - S \times V|^{-1/2}$, where $S \times V$ is the Kronecker product of matrices S and V, we thus have a differential formula for $C_\tau(V)$:

$$L_\tau\, |I - S \times V|^{-1/2}\, (S = 0) = \lambda_{k,\tau} C_\tau^*(V) \tag{4.2}$$

Differentiation is of course easier to contemplate than integration. However, computations based on formula (4.2) are very tedious. A different version of (4.2) is given by Kates [5].

To obtain a first consequence of (4.2), we recall the identity used by Saw [10]

$$|I - S \times V|^{-1/2} = \sum_\tau \frac{(k/2)_\tau\, C_\tau(V)\, C_\tau(S)}{r!\, C_\tau(I)}, \quad r = |\tau| = t_1 + t_2 + \cdots + t_k \tag{4.3}$$

Operating with L_ρ on both sides of (4.3), we obtain

$$\lambda_{k,\rho} C_\rho^*(V) = \sum_\tau \frac{(k/2)_\tau}{r!}\, b_{\rho\tau} C_\tau^*(V)$$

where $b_{\rho\tau} = (L_\rho C_\tau(S))_0$ is the result of L_ρ operating on $C_\tau(S)$ and then setting $S = 0$. We thus obtain the "orthogonality" property

$$(L_\rho C_\tau)_0 = b_{\rho\tau} = 2^r r!\, \delta_\rho^\tau, \qquad r = |\tau|$$

where

$$\delta_\rho^\tau = 1, \qquad \text{if} \quad \tau = \rho$$
$$= 0, \qquad \text{otherwise}$$

From this we see that the coefficients b_τ in the expansion of

$$p(V) = \sum b_\tau C_\tau(V)$$

where p is an orthogonally invariant polynomial, are given by

$$b_\tau = (L_\tau p)_0 / r!\, 2^r, \qquad r = |\tau|$$

Kates [5] gives an integration formula for b_τ. Richards [8] has recently defined a differential operator, ∂_κ, with the above "orthogonality" property. However, the operator ∂_κ, unlike the operator L_ρ, involves coefficients which are difficult to compute.

A second consequence of (4.2) is a representation of $C_\tau(V)$ as a polynomial in $\mathscr{E}_\rho(V) = (\operatorname{tr} V)^{\pi_1} (\operatorname{tr} V^2)^{\pi_2} \cdots (\operatorname{tr} V^r)^{\pi_r}$, where π_j is the multiplicity of j in the partition ρ of r. According to the identity [10],

$$|I - S \times V|^{-1/2} = \sum_\rho \frac{a(\rho)}{|\rho|!} \mathscr{E}_\rho(V) \mathscr{E}_\rho(S)$$

where

$$(a(\rho))^{-1} = \prod_{j=1}^{r} (2j)^{\pi_j} \pi_j!$$

and the partitions ρ are unrestricted as to the number of its parts. Operating on both sides of the above equation with L_τ, we find

$$|\tau|!\, \lambda_{k,\tau} C_\tau^*(V) = \sum_{|\rho| = |\tau|} a(\rho)(L_\tau \mathscr{E}_\rho)_0 \mathscr{E}_\rho(V)$$

The practicality of this representation hinges on whether the coefficients $(L_\tau \mathscr{E}_\rho)_0$ can be swiftly calculated. A different expression for the coefficients of $\mathscr{E}_\rho$ is given by Kates [5].

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