Problem A Day Marathon

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1 Intro

This PDF is a compilation of all the problems in the Problem a Day Marathon. The Problems are sorted into categories and are ordered from easiest to most difficult. The questions range from classroom math to olympiad level proofs in each topic. Note that some questions have multiple topics (i.e. trigonometric inequalities, which are in the inequalities section), but are only in one section.

2 Algebra

1. Find solutions to $x + 5 + \frac{6}{x} = 0$.

2. If we have: $a^2 + b^2 = 1$ $c^2 + d^2 = 1$ $ac - bd = \frac{1}{2}$ Find $ad + bc$

3. Two numbers sum to 1337 and have a product 9001. Find the two numbers.

4. Find the sum of all the real and nonreal roots of $x^{2001} + (\frac{1}{2} - x)^{2001} = 0$

5. Express $(0^3 - 350)(1^3 - 349)(2^3 - 348)...(350^3 - 0)$ concisely as possible.

6. Let $a + \frac{1}{a} = 3$. Find $a^4 + \frac{1}{a^4}$

7. Let r and s be roots of $x^2 - 5x + 2 = 0$. Find $\frac{r^3-1}{r-1} + \frac{s^3-1}{s-1}$.

8. $f(x)$ is a second degree polynomial such that $x^2 - 8x + 17 \leq f(x) \leq 2x^2 - 16x + 33$. If $f(10) = 55$ then find $f(22)$.

9. If $\frac{4}{5}x = y$ and $x^y = y^x$, then $x - y$ can be expressed as a REDUCED common fraction $\frac{a}{b}$. Find the value of $a + b$.

10. Find $(1 + i)^{12345678987654321}$

11. Find $e^{\ln(x+1)+\ln(x^2+1)+\ln(x^4+1)+...+\ln(x^{128}+1)}$ fully simplified and without any parentheses in the final result.

12. Let $F_0 = 0$ and $F_1 = 1$ and for $n \geq 2$ we have $F_{n-2} + F_{n-1} = F_n$. Find $\sum_{k=0}^{\infty} \frac{F_k}{7^k}$

13. The sum of the real coefficients of the terms of the expanded form of $(1 + i)^{1337}$ can be expressed as 2^n where n is an integer. Find n.

14. Find the solutions to $x^9 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + 1 = 0$.

15. If $a, b, c > 0$, find the minimum possible value of $\frac{a}{2b} + \frac{b}{4c} + \frac{c}{8a}$.

16. Show that if a polynomial with real coefficients has the root $a + bi$ for $a, b \neq 0$, Then $a - bi$ is also a root.

17. For what value of k is the following function continuous at k? $f(x) = k$ if $x = 2$ $f(x) = \frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}$ if $x \neq 2$

18. Let a, b, c be the roots of $x^3 - x - 3 = 0$. Find $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}$

19. $x_1, x_2, x_3, ...$ is an arithmetic sequence. $y_1, y_2, y_3, ...$ is a geometric sequence. $z_1, z_2, z_3, ...$ is a sequence such that for all positive integers n, we have $x_n + y_n = z_n$. If $z_1 = 1, z_2 = 8, z_3 = 10, z_4 = 32$ then find z_5.

20. Solve $(x+10)(x+11)(x+12)(x+13) = 1$.

21. Find the solution(s) to $\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{a}{3}}}} = 0.5$

22. For a certain series, we have $S_1 = 1, S_2 = 2$, and for $n \geq 3$ we have $S_n = S_{n-2}(S_{n-1} - S_{n-2}) + 1$. Find all solutions, real and complex, to $(S_{61})x^4 + (S_9 - S_5)x^3 + (S_{123456789})x^2 + (S_3)x + S_{72} = 0$

23. Find the sum of the real solutions to $5+(3+(2+(1+(1+(1000+x)^2)^2)^2)^2)^2 = 535829$.

24. Define $f(x)$ as: $\frac{x}{2}$ if x is even; $\frac{x+1023}{2}$ if x is odd.
Let k be the smallest possible value of x such that $f(f(f(f(f(x))))) = x$.
Let $p(x)$ be sum of the digits of x and the proper factors of x ($p(12) = 1+2+3+4+6+1+2 = 19$). Let $p(f(p(k))) = n$.
Find $\sqrt{\frac{n}{k+3} + 7}$

25. Find $\sum_{k=1}^{100} i^{(k^2)}$ where $i = \sqrt{-1}$.

26. Let $f(x) = (x-1) + (x-2)^2 + (x-3)^3 + ... + (x-9)^9 + (x-10)^{10}$. Find the sum of the roots of $f(x)$

27. Find an equation with rational coefficients and a root $\sqrt{3+\sqrt{7}}$.

28. Simplify $\frac{\sqrt{\sqrt{10}-3}+\sqrt{\sqrt{10}+3}}{\sqrt{\sqrt{10}+1}}$.

29. Prove that if $x+y+z = xyz$, then $\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = (\frac{2x}{1-x^2})(\frac{2y}{1-y^2})(\frac{2z}{1-z^2})$

30. The roots of $x^2 + ax + b$ are $r_1 \geq r_2$. Let $f(q) = x^2 + r_1(x) + r_2$ where q is a quadratic. Find the solutions to $f(f(q)) = q$ or prove there are no solutions.

31. Find all, real and/or complex, solutions to $a + \frac{a+8b}{a^2+b^2} = 2$ and $b + \frac{8a-b}{a^2+b^2} = 0$.

32. Find a formula in terms of n for a_n is $a_1 = 2, a_2 = 1$, and $a_n = 2a_{n-1} - 5a_{n-2}$.

33. Find the remainder when $x^{12345678} + x^{12345676} + x^{12345674} + ... + x^4 + x^2 + 12345678$ is divided by $x^2 - 1$.

34. Find $\sum_{n=1}^{100} n(101-n) + \sum_{n=0}^{99} n(100-n)$.

35. Find $\sum_{n=0}^{10}\sum_{k=0}^{n} i^{k^{10+k^2}}$.

36. Let a and b be real with $a \neq 1$ such that $\frac{b^2}{1-a} = a+1$. Let $c = 2ab$ and $d = (a-b)(a+b)$. Prove that the hypotenuse of a triangle with legs c and d has length 1.

37. There is a sequence of complex numbers z such than $z_1 = 0$ and $z_{n+1} = z_n^2 + i$ for $n \geq 1$. Find $|z_{111}|$.

38. Find $(2+n)(2+n^2)(2+n^3)(2+n^4)(2+n^5)(2+n^6)$ if n is a $7th$ root of unity.

39. If $f(x) = \frac{1}{x} \forall x \in 1, 2, 3, 4, ..., 2007$ and f is a 2006 degree polynomial, then find $f(2008)$.

40. If $x^3 - 12x^2 + ax - 64$ has real, nonnegative roots, find a.

41. How many positive real solutions are there to $\sqrt{x} = |x^4 - 1|$.

42. Find x such that $\sqrt[3]{13x+37} - \sqrt[3]{13x-37} = \sqrt[3]{2}$.

43. If $\frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + ... = \frac{\pi^2}{6}$, then find $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

44. Find the largest real z such that there are real x, y that satisfy:
$x + y + z = 5$
$xy + yz + xz = 3$

45. If r is a root of $x^2 - x + 7$, then find $r^3 + 6r + \pi$

46. Find the exact solution of $3^{3x+2} = 5(2^{5x})$.

47. Show that if $a + b + c = 0$ then $3abc = a^3 + b^3 + c^3$.

48. Find $\sqrt{1 + \frac{1}{1^2} + \frac{1}{2^2}} + \sqrt{1 + \frac{1}{2^2} + \frac{1}{3^2}} + ... + \sqrt{1 + \frac{1}{1999^2} + \frac{1}{2000^2}}$.

49. Let $P(x)$ be a fourth degree polynomial such that: $P(2) = P(-2) = P(-3) = -1$ and $P(1) = P(-1) = 1$ Find $P(0)$

50. Solve all complex numbers such that $z^4 + 4z^2 + 6 = z$.

51. Let $n(x)$ be a 7th degree polynomial such that $n(\frac{2}{m}) = m$ for $m = 1, 2, 3, 4, ..., 8$. Find $n(\frac{1}{9})$

52. If $x + \frac{1}{x} = 5$, then find $\displaystyle\sum_{n=1}^{8} x^n + \frac{1}{x^n}$

53. If $a + b + c = 0, |a| = |b| = |c| = 1$ find $a^2 + b^2 + c^2$. $(a, b, c \in C)$

54. Let r, s, t be roots of $8x^3+1001x+2008 = 0$. Find $(r+s)^3+(r+t)^3+(s+t)^3$.

55. Find all functions $f(x)$ such that $f(a + b) - f(a - b) = 4ab$ for all real a, b

56. Find $\frac{(10^4+324)(22^4+324)(34^4+324)(46^4+324)(58^4+324)}{(4^4+324)(16^4+324)(28^4+324)(40^4+324)(52^4+324)}$ without a calculator.

57. Find all integers (a, b) with $a \leq 1000$ and $b \leq 1000$ such that $a+\sqrt{b}+\frac{1}{a+\sqrt{b}}$ is an integer.

58. Find $\sum_{n=1}^{\infty} \frac{(7n+32)*3^n}{n(n+2)*4^n}$.

59. Find all functions such that for all n we have $f(1 - x) = f(x) + 1 - 2x$.

60. Let $x > 0, x \neq 1$. Find the solution following these to: $x^{(x\sqrt{x})} = (x\sqrt{x})^x$.

61. Let $a_1 = a_2 = 1$ and $a_{n+2} = \frac{a_{n+1}+1}{a_n}$ for $n \geq 3$. Compute a_t if $t = 1998^5$.

62. The solutions to $x^{60} - x + 1 = 0$ are $r_1, r_2, ..., r_{60}$. Find $r_1^{60} r_2^{60} r_3^{60} ... r_{60}^{60}$.

63. Let C_n be the number of paths in the coordinate plane from $(0, 0)$ to (n, n) in $2n$ moves without going over the line drawn from $(0, 0)$ to (n, n). A move takes a point 1 unit to the right or upwards. Find a formula in terms of n for C_n.

64. Find $\lfloor x \rfloor$ if $x = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + ... + \frac{1}{\sqrt{1000000}}$.

3 Geometry

1. There is a triangle $\triangle RST$ with $RS = 13$, $ST = 14$, and $RT = 15$. M is the midpoint of ST. Find the length RM.

2. A regular hexagon is inscribed in a circle, which is inscribed in an equilateral triangle. If the area of the hexagon is 2, find the area of the equilateral triangle.

3. Find the radius of the circle with the equation $y^2 + 2x^2 + 4x + 6y = 5$.

4. Let $\triangle ABC$ be an equilateral triangle. Prove that for any point z in the triangle, the sums of the distances from z to the sides of the triangle is a constant. Find this constant in terms of the sides of the triangle.

5. A circle is inscribed in a semicircle such that it is tangent to the diameter and arc of the semicircle. If the area of the semicircle is 4, find the largest possible area of the circle.

6. Find the area of a regular octagon $ABCDEFGH$ if $AD = 4$

7. 4 spheres of radius 1 are all tangent to each other such that their centers form a regular tetrahedron. Find the radius of the smallest sphere that contains them all.

8. Indecisive Bob starts on point A in the coordinate plane at $(0, 0)$. He then travels $\frac{3}{5}$ of the way to point B at $(0, 1)$. He then travels $\frac{2}{5}$ of the way from where he is to point A. Then he again travels $\frac{3}{5}$ of the way from where he is to point B. He continues these two steps infinitely. Eventually the path converges to two points, C at (0, c) and D at $(0, d)$. Find $c + d$.

9. Bobby's family is into hexagons. Their "tree room" is in the shape of a hexagonal prism with side lengths 8 feet and height 12 feet (the floor and ceiling are regular hexagons). Bobby's family is getting a Christmas tree in the shape of a cone such that the base of the cone is a circle. If the base of the chrismas tree must rest completely on the floor, find the volume of the cone with the greatest possible volume that fits in the room.

10. The graph of $\frac{x^2}{4} + \frac{y^2}{9} = 1$ is rotated around the origin an angle of $\frac{\pi}{4}$ radians. If the new equation can be written in the form $ax^2 + bxy + cy^2 + d = 72$ then find a.
. Find the mximum perimeter of a rectangle inscribed in a semicircle with radius 1.

11. What is the greatest possible area of a square inscribed in a semicircle inscribed in an equilateral triangle inscribed in a square with side length 240?

express your answer in simplest radical form.

12. Compute the greatest possible perimeter of a rectangle inscribed in a reegular hexagon inscribed in a circle inscribed in an equilateral triangle inscribed in a square with side length 180.

13. Find the area of the largest equilateral triangle that can be inscribed in a hexagon with area 120.

14. Find the greatest perimeter of a right isoceles triangle inscribed in an equilateral triangle inscribed in a circl with radius 120.

15. A circle is inscribed in a triangle with lengths $3, 4, 5$. Then, a triangle with side length ratios $3 - 4 - 5$ is inscribed in the circle. Find the area of this small triangle.

16. A semicircle is inscribed in a right isoceles triangle maximally. Find the ratio of the area of the semicircle to the area of the right isoceles triangle.

17. The altitude, median, and angle bisector of $\angle A$ hit BC at d, e, f respectively. If $BD = DE = EF = FC$, then find $\angle A$.

18. Prove that if D is on BC such that $\angle BAD = \angle DAC$, then $\frac{AB}{BD} = \frac{AC}{CD}$. This is the angle bisector theorem.

19. Find the number of intersections of the following curves on the coordinate plane: $y = -100, y = -99, ..., y = 99, y = 100$ $x = -100, x = -99, ..., x = 99, x = 100$. The circles centered at $(0, 0)$ with radii $\frac{1}{\pi}, 1 + \frac{1}{\pi}, 2 + \frac{1}{\pi}, ..., 99 + \frac{1}{\pi}$

20. What is the smallest natural k such that a cube with edge k can be dissected into 2008 cubes, each of integer length?

21. The shortest altitude of a triangle is 3. What is the smallest possible area of the triangle?

22. There is a regular hexagon with area $\frac{21}{2}$. Another hexagon is made by connecting the midpoints of this hexagon. Then, a hexagon is formed by connecting the midpoints of [b]this[/b] hexagon. This continues infinitely. Find the total area of all the hexagons

23. Let ABC be a triangle with side lengths a, b, c. we have $2a + 3b = 4c$. Also, ABC is a right triangle! Find the triplet(s) (a, b, c).

24. If the $3D$ plane is colored 4 different colors, show that there must exist two points that are the same color and have distance 1.

25. What is the radius of the largest sphere that can fit entirely in a cone (tangent to the cone's base and lateral surface) of diameter 16 and height 15?

26. There is a square with side length 2. Prove that if 10 points are placed somewhere on the square, then at least 2 points exist with distance less than 1.

27. Given regular dodecahedron $ABCDEFGHIJKLMNOPQRST$ with area 1000 find the area of the points closest to vertex A.

28. In triangle ABC, we have $\angle A = 60°, \angle B = 45°$. T is on BC such that $AT = 24$ is the angle bisector of $\angle BAC$. Find the area of the triangle.

29. Find the smallest possible area of a triangle with integer side lengths and one angle $120°$

30. In triangle ABC, we have $AB = c, AC = b, BC = a$. Prove that if R is the circumradius and $[x]$ is the area of x, then $[ABC] = \frac{abc}{4R}$.

31. Diagonals AC and BD of regular heptagon $ABCDEFG$ meet at X. Prove that $AX + AB = AD$.

32. Show that if the plane is colored 3 distinct colors, then there exist two point such that their distance is 1 and they are the same color.

33. There is an equilateral triangle ABC inscribed in a circle. Prove that for any point P on $\overset{\frown}{AB}$, we have $PA + PB = PC$.

34. Three circles, each radius 1, are externally tangent to each other. Find the largest possible area of a circle with one vertex on each circle.

35. $ABCDE$ is a regular pentagon and P is a point on its circumcircle between A and E. Prove: $PA + PC + PE = PB + PD$

36. Triangle ABC is isoceles with $AB = AC$ and $BC = 1$. D is on AB such that $AD = 1$ and $\angle DAC = \frac{\pi}{7}$. Find CD.

37. Prove that the area of a cyclic quadrilateral with sides a, b, c, d and $s = \frac{a+b+c+d}{2}$ is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$

4 Trigonometry

1. Find solutions to $\sin x + \cos x - \sin x \cos x = -1$

2. If $\cot\theta + \csc\theta = 3$, find $\csc\theta - \cot\theta$

3. Given that $\sin 2\theta = \frac{40}{41}$ find $\cos^8\theta - \sin^8\theta$

4. Prove that if $< W+ < x+ < y+ < z = 180°$, then $\sin w \sin x + \sin y \sin z = \sin(w+z)\sin(x+y)$

5. Find solutions to $\sin^5\theta + \cos^7\theta = 1$.

6. Find the least k such that $kx = \sin x$ has 5 real solutions.

7. Find $2(\sin 54° - \sin 18°)$ without using calculators, computers, etc

8. If $\sec\theta + \tan\theta = \frac{22}{7}$, then find the value of $\csc\theta + \cot\theta$.

9. If $\sin\theta + \cos\theta + \tan\theta + \sec\theta + \csc\theta + \cot\theta = 7$, Then find $(\sin\theta + \cos\theta)^2$

10. Find $(\cos 20°)(\cos 40°)(\cos 80°)$

11. Find $\frac{\tan^2\frac{\pi}{11} - sin^2\frac{\pi}{11}}{\tan^2\frac{\pi}{11}\sin^2\frac{\pi}{11}}$

12. Find $(\sin 1°)(\sin 3°)(\sin 5°)(\sin 7°)...(\sin 177°)(\sin 179°)$

13. Solve $123\cos\theta + 321\sin\theta = 111$ for $\tan\theta$.

14. x is a positive integer in the interval $[0, 360]$, such that $\sin x°, \cos x°, \cot x°$, and $\tan x°$ take exactly 3 distinct, defined values.Find the sum of all possible values of x.

15. I am riding my bicycle along the flat road at 20 miles per hour when suddenly it rides over a spot of red paint, leaving a mark on my tire. My tire is 2 feet in diameter. If t is the number of [b]seconds[/b] since I rolled over the spot, find an equation for the distance between the spot on my tire and the spot on the ground in [b]feet[/b]in terms of t. (remember the tire's rotation!!!)

16. Find the number of solutions to $(\sin 2\theta)(\sin 42\theta)(\sin 1337\theta) = 0$ in the interval $(0, \pi]$

17. Evaluate $\frac{\sin 40° + \sin 80°}{\sin 110°}$

18. Prove that $\displaystyle\sum_{k=1}^{n} \arctan(\frac{1}{2k^2}) = \arctan(\frac{n}{n+1}.$

19. Find the product of $\sum_{n=1}^{50}(\tan^{-1}(\frac{1}{n}) + \cot(\frac{1}{n}) + \tan(\frac{1}{n}) + \cot^{-1}(\frac{1}{n}))$ and $(\cos\frac{\pi}{7})(\cos\frac{2\pi}{7})(\cos\frac{4\pi}{7})$

20. Find $\cos(\frac{2\pi}{7})\cos(\frac{\pi}{7})\cos(\frac{4\pi}{7})$

21. Prove $\arctan\frac{1}{3} + \arctan\frac{1}{2} + \arctan\frac{1}{1} = \frac{\pi}{2}$.

22. Find $\sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{5\pi}{8} + \sin^4\frac{7\pi}{8}$

23. Prove $\arctan 1 + \arctan 2 + \arctan 3 = \pi$.

24. Let $AC > AB$ is triangle ABC with right angle A. The incircle of $\triangle ABC$ hits BC at W. Show that $2CW - 2WB = 2R(\frac{b\sin A - c\sin(B+C)}{R}$ where R is the circumradius.

25. Show that $\tan 50° + \tan 60° + \tan 70° = \tan 80°$.

26. Show that: $\tan 10° + \tan 70° - \tan 50° = \sqrt{3}$.

27. Find $(\sin\frac{\pi}{5})(\sin\frac{2\pi}{5})(\sin\frac{3\pi}{5})(\sin\frac{4\pi}{5})$

28. Find $\sin^4(\frac{\pi}{8}) + \sin^4(\frac{3\pi}{8}) + \sin^4(\frac{5\pi}{8}) + \sin^4(\frac{7\pi}{8})$

29. Prove $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ in triangle ABC.

30. Show that in triangle ABC, we have $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$

31. Prove that $sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + ... + \sin(\alpha+10\beta) = \frac{\sin(\alpha+5\beta)\sin(\frac{11\beta}{2})}{\sin\frac{\beta}{2}}$.

32. Let S be a point in triangle ABC such that $\angle BAS = \angle CBS = \angle ACS = k$. Prove $\cot k = \cot a + \cot b + \cot c$

33. Find the positive integer n if $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{n} = \frac{\pi}{4}$

34. Let a, b, c be the sides of $\triangle ABC$ such that each side is opposite its corresponding angle. Show that:
$ab = \frac{(\sin A\sin B)(a(a+b\cos(A+B))+b(b+a\cos(A+B))}{(1-\cos C)(1+\cos C)}$.

35. Show that is A, B, C are the sides of a triangle, then $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1$.

36. In triangle ABC, prove that $\cos A + \cos B + \cos C = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} + 1$.

37. Prove that $\prod_{k=1}^{6} \sin \frac{k\pi}{7} = \frac{7}{64}$.

38. Find the angles of a triangle with lengths $2, 2, \sqrt{6} - \sqrt{2}$.

39. Show that in non-right triangle ABC, we have $\cot A + \cot B + \cot C = \frac{a^2+b^2+c^2}{4[ABC]}$

40. If $\cos\alpha + \cos\beta = 1$, prove $\frac{\cos\beta\cos(\frac{\alpha}{2})}{\cos(\beta-\frac{\alpha}{2})} + \frac{\cos\alpha\cos(\frac{\beta}{2})}{\cos(\alpha-\frac{\beta}{2})} = 1$.

41. Prove $2(\cos\frac{4\pi}{19} + \cos\frac{6\pi}{19} + \cos\frac{10\pi}{19})$ is a solution to $\sqrt{4+\sqrt{4+\sqrt{4-x}}} = x$

42. Let $0^\circ < \theta < 90^\circ$ Find solutions to: $\sin(80^\circ - \theta) = 2\cos 40\theta \sin\theta$.

5 Calculus

1. Find $\sum_{k=1}^{1000} k(i^k)$

2. If $18e^{\frac{i\pi}{6}}$ is expressed in the simplified form $a\sqrt{b}+ci$, find the solutions to $ax^2 - abx + abc = 0$ not necessarily real.

3. Bobby is writing his Christmas list. If he writes n items on the list, the chances of getting each item is $\frac{1}{2n}$. For example, If he writes 3 items, he will get each item with a probability of $\frac{1}{6}$. As he begins to write infinitely many gifts on his list (Bobby is quite greedy), what does the probability of him getting no gifts converge to?

4. Find the smallest possible area of a triangle with integer side lengths and one angle 120°

5. Find the area of a leaf of the rose given by the function $r = 3\sin 2\theta$

6. Find $\lim\limits_{x\to 1} \dfrac{x^{2011}-1}{x^{2010}-1}$

7. Find the slope of the line tangent to $x\sin(x+y) = y\cos(x-y)$ at the point $(0, \frac{\pi}{2})$.

8. Find $\frac{d}{dx}\tan^{-1} x$. Your answer should not include any trigonometric functions.

9. Find a non-constant function $f(x)$ that satisfies $\frac{d}{dx}f(x)(\frac{d}{dx}(\frac{d}{dx}f(x)))(\frac{d}{dx}(\frac{d}{dx}(\frac{d}{dx}f(x)))) = f(x)$ or prove that none exist.

10. The graph $r = 1 + 2\sin\theta$ has two closed areas. Find the area of the larger of these.

11. Find $\int \frac{1}{x^2-4x+3}dx$.

12. Find $\lim\limits_{x\to 1} \dfrac{x^50 - 2x^42 + x^23}{8x^2 - 23x + 15}$.

13. Find the derivative of $\sqrt[3]{1+\sqrt{x+\sqrt{e^x}}}$

14. Find $\int \frac{1}{x^2+x+1}dx$

15. There is an equilateral triangle. Joe chooses a random point on its base, and draws a line upwards perpendicular to the base. This divides the triangle into two regions, the smaller of which he colors blue. What is the expected amount of blue paint needed if the triangle has a side length 1?

6 Number Theory

1. Find the sum of the 7 digit palindromes.

2. Only parentheses, brackets, and $+x/-$ can be used to make 24 out of $1, 5, 5, 5$. No combining the numbers (1,5 can't be 15).

3.There is a pyramid of numbers which has a bottom row consisting of 11, the next of 10, and so on, until the top row has a single number. The bottom row consists only of 673s and 198s. Each number other than the bottom row is the sum of the two numbers below. for example, if the bottom row read $673, 198, 198, 673, 673...$ then the next row would read $871, 396, 871, 1346, ...$ and so on. How many original numberings of the bottom row result in a top number divisible by 3?

4. Let k be the smallest of 6 consecutive integers with sum S. Find the smallest possible value for k if S has 3 distinct prime factors.

5. Let a be the sum of the divisors of 100^{100}.
Let b be the number of positive integer solutions to $a+b+c$ given that $a, b, c \leq 42$.
Let c be the number of roots of $x^{100} - 1 = 0$ that aren't roots of $x^{60)-1=0}$
Find the sum of the squares of the roots of $cx^2 + bx + a = 0$ in exponential form.
(100 day anniversary problem)

6. If $\lfloor x \rfloor$ is equal to the greatest integer less than or equal to x, Then find solutions to $\lfloor \frac{x}{2} \rfloor + \lfloor \frac{x}{3} \rfloor + \lfloor \frac{x}{4} \rfloor + \lfloor \frac{x}{5} \rfloor + \lfloor \frac{x}{6} \rfloor = 100$.

7. Find the smallest number with the same numbers of 1s an 0s in their binary representation as 97 has, and is equivalent to $6 \mod 16$.

8. Show that $1337^5 - 1337$ is divisible by 30.

9. In the decade $101 - 110$, there are 4 prime numbers. Find the smallest decade with 0 primes. (Note: a decade starts with a number with last digit 1).

. Two numbers a and b are [i]pairwise awesome[/i] if/only if the sum of the proper divisors of a is b and the sum of the proper divisors of b is a. Note: the proper divisors are the factors of the number other than itself. For example, the first such pair is $220-284$, as $1+2+4+5+10+11+20+22+44+55+110 = 284$ and $1 + 2 + 4 + 71 + 142 = 220$. Find the second such pair.

10. A number is called [i]magical[/i] if it's proper divisors add to itself. For example: $6 = 1 + 2 + 3$ and $28 = 1 + 2 + 4 + 7 + 14$. Find: a) The next two [i]magical[/i] numbers b) The form for all [i]magical[/i] numbers (all magical numbers can be expressed in this form)

11. Three planets orbit the same sun. Planet A has an orbital period of 174 years. Planet B has an orbital period of 406 years. Planet C has an orbital period of 609 years. All orbitals are perfect circles, and the orbital period is the time it takes to travel the orbital once. Right now all the planets are colinear. Find how many years will pass before they are next collinear.

12.Find the smallest number consisting entirely of 1's divisible by 19.

13. Bobby's brother, Billy, is also making his Christmas list. If Billy puts n gifts on his list, then the whole list will cost $\sum_{k=1}^{n} k^2$ dollars. For example, 1 gift costs $1^2 = 1$ dollar, while 4 gifts would cost $1^2 + 2^2 + 3^2 + 4^2 = 30$ dollars. Billy is a spoiled child, and his parents are willing to spend up to $1,000,000$ dollars on his list. Find the greatest number of presents Billy can put on his list to satisfy this $1,000,000$ dollar cap.

14. Bobby's family consists of 9 children and 4 adults. They each received exactly one present. The adult's presents are distinguuishable from the children's presents, but otherwise they are indistinguishable (someone forgot to label them). Each child randomly takes a child present, and eacch adult randomly takes an adult present. Find the probability that nobody gets the present meant for them. (Example. A,B,C have presents a, b, c. A gets b, B gets c, C gets a.)

15. The number of factors of 2010^{2011} is the same as the number of factors in 2^k for some k. k Find the remainder when k is divided by 2012.

16. Note that the sum of the divisors of 99 is $1 + 3 + 9 + 11 + 33 + 99 = 156$. Find the least and greatest positive integers whose divisor sum is 99, or prove that there are none.

17. What is the first integer n such that n^{n+1} has at least 1500 factors?

18. What is the expected amount of pairs of consecutive tosses that come up heads if a coin is flipped 10 times? For example, $HHTHHHTTTH$ has 3 pairs.

19. How many 9 digit palindromes are there not divisible by 4?

20. Let $T_0 = 0$ and $T_n = 1 + 2 + 3 + ... + n$. Show that $8T_n + 1$ is always a perfect square.

21. Find the least positive cube to end in the digits 888

22. Find the largest n such that $n^3 + 100$ is divisible by $n + 10$.

23. Let $e(n)$ be the expected value of the number picked randomly from the numbers $1, 4, 9, ..., n^2$. Fidn the first n such that $e(n) \geq 1000$.

24. Denote $b(n)$ as the "backwards factorization" of a number. This is formulated by flipping the bases with the exponents. For example, $b(289) = 2^{17}$ and $b(75) = 2^5 1^3$. It doesn't matter if the bases of this new factorization are prime: $b(81) = 4^3$. Let $p(n)$ be the product of the digits of n. Find the smallest positive n such that $p(n) + b(n) = n$.

25. Find the remainder when 3^{2007} is divided by 2007

26. I am thinking of a 22 digit number. There are exactly 3 distinct digits in it. If my number is w, what is the largest single-digit number that must divide $w^{1001} - w^{999}$.

27. Which of the following cannot be expressed as $a^3 + b^3$ for integers a and b?
a) 700056
b) 707713
c) 7077915
d) 7000639
e) 7077283

28. In the game Tower of Hanoi, 8 rings of size 1, 2, 3, 4, 5, 6, 7, 8 are all stacked on the leftmost peg of three pegs. They are currently stacked 1 on top of 2 on top of 3 on top of... on top of 8. Only the top rings can be moves, and only to either empty pegs or pegs containing only sizes of rings greater than it. Find the least number of moves possible to get all 8 rings onto the rightmost peg.

29. Good ol' num theo classic: Let $s(x)$ denote the sum of the digits of x. Find $s(s(s(4444^{4444})))$.

30. 90 is special because its the $5th$ smallest positive integer with 12 factors. What is the $5th$ smallest integer with 90 factors?

31. Simplify the product $\prod_{k=2}^{100}(\frac{F_k}{F_{k-1}} - \frac{F_k}{F_{k+1}})$, where F_n is the nth number in the Fibonacci sequence that begins $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...$

32. Prove that for every nonnegative integer a, there is at least one nonnegative integer solution (x, y) to $x^2 = y^2 + a^3$.

33. Let $x + \frac{1}{x}$ be an integer. Prove that all $x^n + \frac{1}{x^n}$ are integers for positive integer n.

34. Let $n(x)$ be the number of digits before the decimal place of x. Find $n(n(2^{2011}))$

35. Show that $2^{2^n} + 2^{2^{n-1}} + 1$ can be factored into at least n prime factors.

36. For your allowance, your parents give you a deal; they give you 8 two-dollar bills and 2 ten-dollar bills. You can distribute the bills in 2 hats any way you like. Then, with your eyes closed and after them shuffling the hats, you randomly pick a hat, then randomly pick a bill inside of it. This bill represents your allowance. What is the optimal distribution that gives the greatest expected value of your allowance?

37. Find a formula for F_n in terms of n.

38. Prove that $\displaystyle\sum_{k=0}^{n}(2k+1)^2 = \binom{2n+3}{3}$ without any induction.

39. We take a series $0^n, 1^n, 2^n, 3^n, ...$ and "difference it" until we get a constant. For example, take the series of cubes:
$0, 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, ...$
We take the difference of consecutive terms:
$1 - 0 = 1$
$8 - 1 = 7$
$27 - 8 = 19$
$64 - 27 = 37$
$125 - 64 = 61$
$216 - 125 = 91$
$343 - 216 = 127$
Now we take the difference of those terms:
$7 - 1 = 6$
$19 - 7 = 12$
$37 = 19 = 18$
$61 - 37 = 24$
$91 - 61 = 30$
$127 - 91 = 36$ And again:
$12 - 6 = 6$
$18 - 12 = 6$
$24 - 18 = 6$
$30 - 24 = 6$
$36 - 30 = 6$
So 6 is our constant. We reached 6 in three steps.
a) prove that for integer $n \geq 1$, we have the constant is reached in n steps
b) prove that the constant is $n!$ for $n \geq 1$.

40. Let $L_1 = 1$, $L_2 = 3$, and for $n \geq 3$ let $L_n = L_{n-1} + L_{n-2}$. Prove that:
a) $L_{2n} = (L_n)^2 - 2(-1)^n$

b) $L_{n-1}L_{n+1} = (L_n)^2 - 5(-1)^n$

41. Bob has a number x that has 5 digits. If a 1 is placed in front of x, the value is $\frac{1}{3}$ the value if 1 is placed after x instead. Find the number of minutes in $7x + 1$ seconds, and call this s. Find the number of factors in s

42. Find the number of nonnegative integer solutions to $a + b + c = 50$, where a is a multiple of 3, and b and c are odd.

43. Show that the only perfect square factorials of nonnegative integers are 0! and 1!

44. Let $R_n = \sqrt{\frac{1^2+2^2+3^2+4^2+...+n^2}{n}}$. Find the smallest integer $n > 1$ such that R_n is an integer.

45. Find the $5th$ digit from the end of $5^{5^{5^{5^5}}}$

46. Let a great number be a number made of only 123456789 in thaat order and plus signs. For example, $1+2+3+4+5+6+7+8+9 = 45$ is great, and so is $1 + 23456789 = 23456790$. Find the gcd of all great numbers.

47. I will now show a cool way to multiply positive integers: take the two numbers. but one in the half column and the other in the double column. For example, take 19 and 14.
19—14 Now, double the number in the double column, and half the number in the half column. Fractions are ugly, so get rid of them. Repeat this until you get to 1.
19—14
9—28
4—56
2—112
1—224
Now, even numbers are totally out of style, so cross out all rows with an even number in the left column:
19—14
9—28
4—56 scratch out
2—112 scratch out
1—224
Finally, add up the remaining numbers in the right column: $224 + 28 + 14 = 266 = 19 * 14$
This works every time, as long as you carefully follow these rules. Don't believe me? Try it yourself! Here's a simple example:
5—7
2—14 scratch out

1—28
Note that $28 + 7 = 35 = 5 * 7$
Now to your job: Prove that this cool method always works.

48. Let $x = pn + 1$ where p is a prime, and $\gcd(p, n) = 1$. Let r be the largest possible integer such that p^r divides $x^{(p^2)} - 1$. Find r.

49. Prove that (1)(3)(5)...(1993) + (2)(4)(6)...(1994) is divisible by 1995.

50. Find the last two digits of $98765432103^{79808182838485}$.

51. Find the last three digits of $2003^{2002^{2001}}$

52. 20 pairwise distinct positive integer are each less than 70. Prove that among their positive differences there are 4 equal numbers.

53. Prove that $\sum_{k=0}^{m} \binom{n}{k} = \binom{n-k}{m-k} = 2^m \binom{n}{m}$ for $m < n$.

54. Let S_n denote the number of subsets of $1, 2, 3, ..., n$ without two consecutive integers in them. Find S_n in terms of F_n, where $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

55. Let $z(n, k)$ be the number of ways to partition Then set $1, 2, 3, ..., n$ into k non-empty disjoint subsets. Prove that $z(n, k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$.

56. Prove that any natural number can be represented by the sum of 4 nonnegative perfect squares.

7 Counting and Probability

1. How may ways can I make a strictly increasing alphabetical 4-letter sequence?

2. Bob has a $2 * 4$ rectangle. He randomly picks a side of the rectangle. Then, he randomly picks a point inside the rectangle. What is the probability that the triangle formed by this is acute? Express as common fraction.

3. How many ways can I choose 15 distinct letters out of the alphabet without choosing every letter of either "NEW", "$YEAR$" or "$FLAMINGO$"?

4. How many ways can 5 X's, 7 Y's, 4 Z's be arranged so there are at least three showings of ZX? One such to consider is $ZXZXYYYZXYYYYXXZ$, but $ZXZXZYXYYYZYYXXY$ would not work.

5. How many ways can the numbers $1, 2, 3, 4, 5$ be rearranged such that no number k is in the kth position? For example, 25413 and 51234 work, but 15432 and 31542 do not.

6. Bobby got a board game for Christmas, which he is playing with his brother Billy. Bobby wins each game with a probability $\frac{2}{3}$. If Billy loses 5 games in a row, he storms off in tears, and refuses to play again. If Billy wins 3 games in a row, Bobby storms off in tears and refuses to play again. What is the probability of Bobby and Billy playing 6 gamess straight, without either storming off (even at the end)?

7. Al and Bob are taking two random true/false tests. They both didn't study, so they guess on every question. Al's is 11 questions long, while Bob's is 7. Find the probability that Bob got at least as mant questions right as Al.

8. Bobby has magnetic letters on his fridge that spell "$MERRYCHRISTMAS$". Bobby has an infinite supply of each letter. At random, he takes a letter fromm his letter bin and sticks it on the fridge. What is the expected number of letters he puts up before he has all of the letters of "$NEWYEAR$ on the fridge (in any order, not necessarily next to each other)?

9. How many ways can you make permutations of "applepi2000" if there must be at least one pair of double letters or numbers? examples:
0palei20pp0
000ppp2alie
pla00eip2p0
applepi2000
are all fine, but
aplp02p0ie0
is not.

10. Al and Bob both write out the numbers $ONE, TWO, THREE, FOUR, FIVE$. Al randomly picks a letter from these 19 letters and Bob randomly picks one of the five words, then randomly picks a letter from that word. What is the probability that they both picked a vowel? Express your answer as a common fraction. All random pickings have an equal probability.

11. Two real numbers r and s are chosen at random from 0 to 1. If we know that $|r - s| < \frac{1}{4}$, then what is the probability that $r < \frac{1}{2} < s$

12. Three points A, B, C are randomly chosen on a square, with uniform distribution everywhere. . Then, C is reflected over AB to get D. What is the probability that D lies inside the square? Express as common fraction.

13. Ed, Ned, and Fred are invited to a party. Ed will come with a 40% chance. If Ed comes, them Ned will come with a 80% chance. If Ed doesn't come, then Ned will come with a 30% chance. If Ned comes, then Fred will come with a 60% chance. If Ned doesn't come, then Fred will come with a 75% chance. If you see Fred at the party, what is the chance that exactly one of Ed and Ned are also there? Express as a common fraction.

14. Particle Boy is in space. He travels from (x, y, z) to either $(x + 1, y, z)$, $(x, y + 1, z)$, or $(x, y, z + 1)$ in the 3D coordinate plane. How many ways can he go from the origin to $(2, 4, 6)$ without passing through $(1, 1, 1)$ or $(2, 2, 2)$ in exactly 12 steps?

15. There are 32 letters in the language of Spoonish. 7 different letters of these 32 letters are randomly picked. What is the probability that exactly 6 of the letters picked are "sacred", If 10 of the original 32 were sacred?

16. In how many ways can the integers from 1 to 36 inclusive be ordered such that no two multiples of 6 are adjacent?

17. x and y, not necessarily distinct, are chosen randomly from the set $1, 2, 3, ..., 99$ with each (x, y) having equal probability. Given that $x + y$ is even, what is the probability that the sum of the digits of y is less than 10?

18. My watch becomes $(-1)^n n^2$ seconds slow the nth hour after I get it. So after: 1 hour- 1 minute fast. 2 hours- $-1+4 = 3$ minutes slow 3 hours- $-3+9 = 6$ minutes fast etc. If it is noon January 1st, at what hour on which day will my watch become at least 1 hour off? include am/pm.

19. A coin is flipped 20 times. Given that exactly 14 heads appeared, find the probability that no two consecutive tosses were both tails.

20. Bob has a cube. In how many ways can he color the 6 sides of the cube with 100 colors if every two adjacent faces are different colors?

21. If 6 points are randomly placed on a circle with circumference 2, what is the probability they can be fully covered with an arc length 1?

22. Call a set "magical" if no three consecutive integers are in it. How manny subsets of $1, 2, 3, ..., 12$ are magical?

23. How many subsets of $1, 2, 3, ..., 12$ have no 2 consecutive integers as elements?

24. How many subsets of $1, 2, 3, 4, 5, 6, 7, 8, 9$ don't contain exactly two of $2, 4, 6, 8$ and don't contain exactly one of $1, 2, 3$?

25. There is an interesting math competition, with k teams and k competitors on each team. Each team has k problems to solve. In each team, one person knows how to solve all k problems, one knows how to solve $k - 1$ of them, and so on down to a member that only knows how to solve 1 problem. Each team randomly assigns exactly one question to each competitor. What is the probability that every team gets every question right?

8 Inequalities

1. Prove $a^4 + 20 > 16a$ for all real a.

2. Find the smallest possible value of $\frac{12x^2 \sin^2 x + 3}{x \sin x}$

3. a, b are real numbers such that $a + b = 1 - ab$. Also, $0 < a < b < 1$. Show that $0 < \frac{b-a}{1+ab} < 1$.

4. Write a geometric proof for 2-variable AM-GM.

5. a, b, c, x, y, z are all positive. $a^2 + b^2 + c^2 = 784$, $x^2 + y^2 + z^2 = 49$ and $ax + by + cz = 196$. Find all, real and/or complex, solutions to $(\frac{a+b+c}{x+y+z})x^2 + (\frac{x+y+z}{a+b+c})x + ((a+x)^2 + (b+y)^2 + (c+z)^2) = 0$.

6. Show that $abc(a + b + c) \leq a^3b + b^3c + c^3a$ for positive a, b, c

7. For positive x, y, z find the minimum of $(1 + \frac{x}{2y})(1 + \frac{y}{2z})(1 + \frac{z}{2x})$.

8. Let $a, b, c, d > 0$ and $a+b+c+d = 4$. Prove $\displaystyle\sum_{cyc}^{4} \frac{a^2}{bc + d} \geq 4 - \frac{4ab + 4bc + 4cd + 4ad}{4 + ab + bc + cd + ad}$.

9. If $a \geq b > 1$ then find the greatest poss. value of $log_a \frac{a}{b} + \log_b \frac{b}{a}$

10. Prove that for positive real a, b, c, d we have $\frac{a^2+b^2+c^2}{a+b+c} + \frac{a^2+b^2+d^2}{a+b+d} + \frac{a^2+c^2+d^2}{a+c+d} + \frac{b^2+c^2+d^2}{b+c+d} \geq a + b + c + d$

11. If $a, b, c > 0$ and $a + b + c = 6$, prove that $ab + ac + bc \geq \frac{a^2bc + ab^2c + abc^2}{4}$

12. Let $p(x) = x^{24} + a_{23}x^{23} + ... + a_1x + a_0$. Prove that if all the roots of $p(x)$ are positive, then $a_{23}a_1 \geq 576a_0$.

13. Let the incircle of $\triangle ABC$ be tangent to AB at D and AC at E. Let the center of the incircle be I. Prove that $AI \geq DE$, and tell when equality holds.

14. Prove $1 < \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} < 2$ for positive a, b, c.

15. Prove that $\cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2} \geq 3\sqrt{3}$ for triangle ABC

16. Let $PQRS$ be a quadrilateral. Prove $\frac{(PQ)^2+(QR)^2+(RS)^2+(SP)^2+2((PQ)(RS)+(PS)(QR))}{4} \geq (PR)(QS)$.

17. Prove that if a, b, c are positive, then $\frac{a}{a + \sqrt{(a+b)(a+c)}} + \frac{b}{b + \sqrt{(b+a)(b+c)}} +$

$$\frac{c}{c+\sqrt{(a+c)(b+c)}} \leq 1.$$

18. Let there be 3 circles, pairwise externally tangent and tangent to a straight line. If the smallest circle has radius 1, and the other circles have radii a, b, prove:
$ab \geq 16$

19. Prove that is $x, y, z \in R^+$, then: $\displaystyle\sum_{cyc} \frac{(2x+y+z)^2}{2x^2+(y+z)^2} \leq 8$

20. Show that for all positive real a, b, c we have $\frac{1}{a^3+b^3+abc} + \frac{1}{b^3+c^3+abc} + \frac{1}{c^3+a^3+abc} \leq \frac{1}{abc}$.

21. Prove that for $x_1, x_2 \geq 0$ then we have $\sqrt{\frac{x_1^2+x_2^2}{2}} - \frac{x_1+x_2}{2} \geq \sqrt{x_1x_2} - \frac{1}{\frac{1}{x_1}+\frac{1}{x_2}}$

22. Prove that if a, b, c are the sides of ABC with each angle opposite its corresponding side, that: $\sin B(a+b+c) \geq 3b\sqrt[3]{\sin A \sin B \sin C}$.

23. Let ABC be an acute triangle. Prove that $\tan A \tan B \tan C \geq 3\sqrt{3}$.

24. Let R be the circumradius of the circle, and let a, b, c be the side lengths of triangle ABC. Prove that $\frac{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{3} \geq \frac{\sqrt[3]{abc}}{2R}$.

25. Prove $(a^5-a^2+3)(b^5-b^2+3)(c^5-c^2+3) \geq (a+b+c)^3$ for nonnegative a, b, c.

26. Let all a_i and b_i be nonnegative reals for $i \in 1, 2, 3, ..., n$.. Prove $\displaystyle\sum_{i,j=1}^{n} \min a_ia_j, b_ib_j \leq \sum_{i,j=1}^{n} \min a_ib_j, a_jb_i$.

9 Outro

In this project, some of the problems were borrowed from contests such as the AMC series, ARML, USC, etc. Also, some came from various AoPS books. Most of the problems, however, were made up.

Once again, I would like to thank all the competitors, especially aznluster, edisonchew240, Abe27342, SCP, smarthjain, soulspeedy, and raskvin for your help with the in-thread solutions.