# Question No. 1

- Let us assume that the two bodies are '$d$' distance apart from each other with masses $m_1$ and $m_2$ revolving with an angular frequency '$\omega$' about their Center of Mass.

$$R_{\text{com}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$R_{\text{com}} = \frac{m_2 d}{m_1 + m_2}$$

- Now, the gravitational force acting on mass $m_1$ is equal to the centripetal force acting on it.

$$\frac{G m_1 m_2}{d^2} = m_1 x_{\text{com}} \omega^2 \qquad [Reduced\ mass\ \mu = \frac{m_1 \cdot m_2}{m_1 + m_2}]$$

$$\frac{G m_2}{d^2} = \frac{m_2 d}{m_1 + m_2} \omega^2 \qquad [Total\ mass\ M = m_1 + m_2]$$

$$\omega = \sqrt{\frac{G(m_1 + m_2)}{d^3}}$$

$$\omega = \sqrt{\frac{GM}{d^3}}$$

- Total energy of the system = K.E + P.E
  For gravitational mechanics, we know

$$K.E = \frac{1}{2} m r \omega^2$$

$$P.E = -\frac{G m_1 m_2}{d}$$

So,

$$T.E = \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 - \frac{G m_1 m_2}{d}$$

$$r_1 = r_{\text{com}} = \frac{m_2 d}{m_1 + m_2}$$

$$r_2 = d - r_{\text{com}} = d - \frac{m_2 d}{m_1 + m_2} = \frac{m_1 d}{m_1 + m_2}$$

$$T.E = \frac{1}{2} m_1 \omega^2 \left[ \frac{m_2^2 d^2}{(m_1 + m_2)^2} \right] + \frac{1}{2} m_2 \omega^2 \left[ \frac{m_1^2 d^2}{(m_1 + m_2)^2} \right] - \frac{G m_1 m_2}{d}$$

$$T.E = \frac{1}{2}\frac{m_1 m_2}{(m_1+m_2)^{\cancel{2}}}\omega^2 d^2 \left[\cancel{m_1+m_2}\right] - \frac{Gm_1 m_2 \times M}{d \times M}$$

$$T.E = \frac{1}{2}\frac{\omega^2 \mu d^2 \times d}{d} - \frac{GM\mu}{d} \qquad [we\ know\ \omega^2 = \frac{GM}{d^3}]$$

$$T.E = \frac{1}{2}\frac{GM\mu}{d} - \frac{GM\mu}{d}$$

$$T.E = -\frac{1}{2}\frac{GM\mu}{d}$$

# Question No. 2

- An approach is to assume that the power radiated depends on $G$, $c$ and some time derivative of a mass multipole. It is straightforward to show that the first time-derivative of the monopole and dipole moments is zero because of conservation of mass, linear momentum and angular momentum. One is left then with using the quadrupole moment as the lowest viable order.

- Since power is a scalar and must be $\geq 0$, then the time derivative of the quadrupole moment must be squared and thus we can say

  $$P \sim G^a c^b \left(\frac{\partial^2 Q}{\partial t^2}\right)^2.$$

  Letting $Q$ have dimension of $ML^2$, then dimensional analysis yields $a = 1$, $b = -5$ and $c = 3$.

  In a binary, we have

  $Q \sim Ma^2 \sin 2\omega t$ (twice the frequency of the binary because the quadrupole moment flips twice per orbital period).

  Differentiate this three times, square it:

  $$Q^2 \sim M^2 a^4 \omega^3$$

  and then note that

  $$a^4 \sim \left(\frac{GM}{\omega^2}\right)^{\frac{4}{3}} \quad \text{(from Kepler's third law and you get)}$$

  $$P \sim \frac{G}{c^5}\left(\frac{GM}{\omega^2}\right)^{\frac{4}{3}} M\omega^{10/3} = \frac{32}{5}\frac{c^5}{G}\left(\frac{GM\omega}{c^3}\right)^{\frac{10}{3}}.$$

  $$P = \frac{32}{5}\frac{G^4\mu^2 M^3}{C^5 d^5}$$

- A binary orbit causes the binary system's geometry to change through 180 degrees and also causes the distance between each body of the binary system and the observer to change through 180 degrees causing a gravitational wave frequency of two times the orbital frequency).

# Question No. 3

- From above we got the expression for the power radiated by gravitational waves as

$$P = \frac{32}{5}\frac{G^4}{C^5}\frac{\mu^2 M^3}{d^5}$$

$$d^3 = \frac{GM}{\omega^2}\ ,\ d = \left(\frac{GM}{\omega^2}\right)^{\frac{1}{3}}$$

$$P = \frac{32}{5}\frac{G^3\mu^2 M^2\omega^2}{C^5\cancel{GM}d^2}$$

$$P = \frac{32}{5}\frac{G^3\mu^2 M^2\omega^2}{C^5 d^2}$$

$$P = \frac{32}{5}\frac{G\mu^2\omega^2}{C^5 d^2} \times \left(\omega^4 d^6\right)$$

$$P = \frac{32}{5}\frac{G\mu^2\omega^6 d^4}{C^5}$$

$$P = \frac{32}{5}\frac{G\mu^2\omega^6}{C^5}\left(\frac{GM}{\omega^2}\right)^{\frac{4}{3}}$$

# Question No. 4

- In the adiabatic regime, the system has the time to adjust the orbit to compensate the energy lost in gravitational waves with a change in the orbital energy, in such a way that

$$\frac{\mathrm{d}E_{orb}}{\mathrm{d}t} + P = 0$$

$$E_{orb} = -\frac{GM\mu}{2d}$$

$$\frac{\mathrm{d}E_{orb}}{\mathrm{d}t} = \frac{GM\mu\dot{d}}{2d^2}$$

$$d^3 = \frac{GM}{\omega^2}$$

$$3d^2\dot{d} = -2GM\omega^{-3}\dot{\omega}$$

$$3d^3\dot{d} = -2GM\omega^{-3}\dot{\omega}d$$

$$3\frac{\cancel{GM}\dot{d}}{\omega^2} = -2\cancel{GM}\omega^{-3}\dot{\omega}d$$

$$\dot{d} = -\frac{2}{3}\frac{d\dot{\omega}}{\omega}$$

$$\frac{\mathrm{d}E_{orb}}{\mathrm{d}t} = \frac{GM\mu}{\cancel{2}d^{\cancel{2}}} \times -\frac{\cancel{2}}{3}\frac{\cancel{d}\dot{\omega}}{\omega}$$

$$= -\frac{GM\mu\dot{\omega}}{3d\omega}$$

$$\cancel{-}\frac{GM\mu\dot{\omega}}{3d\omega} = \cancel{-}p$$

$$\frac{\cancel{G}M\cancel{\mu}\dot{\omega}}{3d\omega} = \frac{32}{5}\frac{\cancel{G}\mu^{\cancel{2}}\omega^6 d^4}{C^5}$$

$$\dot{\omega} = \frac{96}{5}\frac{\mu\omega^7 d^5}{c^5 M} \quad [we\ know\ d^3 = \frac{GM}{\omega^2}]$$

$$(\dot{\omega})^3 = \left(\frac{96}{5}\right)^3 \frac{\mu^3\omega^{21}d^{15}}{c^{15}M^3}$$

$$(\dot{\omega})^3 = \left(\frac{96}{5}\right)^3 \frac{\mu^3\omega^{21}G^5M^5}{c^{15}M^3\omega^10}$$

$$M_c\,(\text{chirp mass}) = \left(\mu^3 M^2\right)^{\frac{1}{5}}$$

$$(\dot{\omega})^3 = \left(\frac{96}{5}\right)^3 \frac{\omega^{11}}{c^{15}}G^5{M_c}^5$$

$$M_c^5 = \left(\frac{5}{96}\right)^3 c^{15}\omega^{-11}G^{-5}\dot{\omega}^3$$

$$M_c = \frac{c^3}{G}\left(\left(\frac{5}{96}\right)^3 \omega^{-11}\dot{\omega}^3\right)^{\frac{1}{5}}$$

# Question No. 5

$$\omega_{orb} = 2\pi f_{orb}$$

$$\omega_{orb} = \pi f_{GW}$$

$$\dot{\omega}_{orb} = \pi \dot{f}_{GW}$$

$$M_c = \frac{c^3}{G}\left(\left(\frac{5}{96}\right)^3 \pi^{-8} f_{GW}\dot{f_{GW}}^3\right)^{\frac{1}{5}}$$