1FC3 Theorem List

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Contents

1 Equivalence

(3.1) Axiom, Associativity of $\equiv$: $((p \equiv q) \equiv r) \equiv (p \equiv (q \equiv r))$
(3.2) Axiom, Symmetry of $\equiv$: $p \equiv q \equiv q \equiv p$
(3.3) Axiom, Identity of $\equiv$: $\mathit{true} \equiv q \equiv q$
(3.4) true
(3.5) Reflexivity of $\equiv$: $p \equiv p$
(3.7) Metatheorem: Any two theorems are equivalent.

2 Negation

(3.8) Axiom, Definition of false: $\mathit{false} \equiv \neg \mathit{true}$
(3.9) Axiom, Distributivity of $\neg$ over $\equiv$: $\neg(p \equiv q) \equiv (\neg p \equiv q)$
(3.10) Axiom, Definition of $\not\equiv$: $(p \not\equiv q) \equiv \neg(p \equiv q)$
(3.11) $\neg p \equiv q \equiv p \equiv \neg q$
(3.12) Double negation: $\neg\neg p \equiv p$
(3.13) Negation of false: $\neg \mathit{false} \equiv \mathit{true}$
(3.14) $(p \not\equiv q) \equiv \neg p \equiv q$
(3.15) $\neg p \equiv p \equiv \mathit{false}$
(3.16) Symmetry of $\not\equiv$: $(p \not\equiv q) \equiv (q \not\equiv p)$
(3.17) Associativity of $\not\equiv$: $((p \not\equiv q) \not\equiv r) \equiv (p \not\equiv (q \not\equiv r))$
(3.18) Mutual associativity: $((p \not\equiv q) \equiv r) \equiv (p \not\equiv (q \equiv r))$
(3.19) Mutual interchangeability: $p \not\equiv q \equiv r \equiv p \equiv q \not\equiv r$

3 Disjunction

(3.24) Axiom, Symmetry of $\vee$: $p \vee q \equiv q \vee p$
(3.25) Axiom, Associativity of $\vee$: $(p \vee q) \vee r \equiv p \vee (q \vee r)$
(3.26) Axiom, Idempotency of $\vee$: $p \vee p \equiv p$
(3.27) Axiom, Distributivity of $\vee$ over $\equiv$: $p \vee (q \equiv r) \equiv p \vee q \equiv p \vee r$
(3.28) Axiom, Excluded Middle: $p \vee \neg p$
(3.29) Zero of $\vee$: $p \vee \mathit{true} \equiv \mathit{true}$
(3.30) Identity of $\vee$: $p \vee \mathit{false} \equiv p$
(3.31) Distributivity of $\vee$ over $\vee$: $p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$
(3.32) $p \vee q \equiv p \vee \neg q \equiv p$

4 Conjunction

(3.35) Axiom, Golden Rule: $p \wedge q \equiv p \equiv q \equiv p \vee q$
(3.36) Symmetry of $\wedge$: $p \wedge q \equiv q \wedge p$
(3.37) Associativity of $\wedge$: $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
(3.38) Idempotency of $\wedge$: $p \wedge p \equiv p$
(3.39) Identity of $\wedge$: $p \wedge \mathit{true} \equiv p$
(3.40) Zero of $\wedge$: $p \wedge \mathit{false} \equiv \mathit{false}$
(3.41) Distributivity of $\wedge$ over $\wedge$: $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge (p \wedge r)$
(3.42) Contradiction: $p \wedge \neg p \equiv \mathit{false}$
(3.43) Absorption:
$p \wedge (p \vee q) \equiv p$
$p \vee (p \wedge q) \equiv p$
(3.44) Absorption:
$p \wedge (\neg p \vee q) \equiv p \wedge q$
$p \vee (\neg p \wedge q) \equiv p \vee q$
(3.45) Distributivity of $\vee$ over $\wedge$: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
(3.45) Distributivity of $\wedge$ over $\vee$: $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
(3.47) De Morgan:
$\neg(p \wedge q) \equiv \neg p \vee \neg q$

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

(3.48) Axiom, Golden Rule:
(3.49) Axiom, Golden Rule:
(3.50) Axiom, Golden Rule:
(3.51) Replacement: $(p \equiv q) \wedge (r \equiv p) \equiv (p \equiv q) \wedge (r \equiv q)$
(3.52) Definition of $\equiv$: $p \equiv q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
(3.53) Definition of $\not\equiv$ (Exclusive or): $p \not\equiv q \equiv (\neg p \wedge q) \vee (p \wedge \neg q)$

5 Implication

(3.57) Axiom, Definition of Implication: $p \Rightarrow q \equiv p \vee q \equiv q$
(3.58) Axiom, Consequence: $p \Leftarrow q \equiv q \Rightarrow p$
(3.59) (Alternative) Definition of Implication: $p \Rightarrow q \equiv \neg p \vee q$
(3.60) (Dual) Definition of Implication: $p \Rightarrow q \equiv p \wedge q \equiv p$
(3.61) Contrapositive: $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$
Ex. 3.45 $p \Rightarrow q \equiv \neg p \vee \neg q \equiv \neg p$
Ex. 3.46 $p \Rightarrow q \equiv \neg p \wedge \neg q \equiv \neg q$
(3.62): $p \Rightarrow (q \equiv r) \equiv p \wedge q \equiv p \wedge r$
(3.63) Distributivity of $\Rightarrow$ over $\equiv$: $p \Rightarrow (q \equiv r) \equiv p \Rightarrow q \equiv p \Rightarrow r$
(3.64) Self-distributivity of $\Rightarrow$: $p \Rightarrow (q \Rightarrow r) \equiv (p \Rightarrow q) \Rightarrow (p \Rightarrow r)$
(3.65) Shunting: $p \wedge q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r)$
(3.66) $p \wedge (p \Rightarrow q) \equiv p \wedge q$
(3.67) $p \wedge (q \Rightarrow p) \equiv p$
(3.68) $p \vee (p \Rightarrow q) \equiv true$
(3.69) $p \vee (q \Rightarrow p) \equiv q \Rightarrow p$
(3.70) $p \vee q \Rightarrow p \wedge q \equiv p \equiv q$
(3.71) Reflexivity of $\Rightarrow$: $p \Rightarrow p \equiv true$
(3.72) Right zero of $\Rightarrow$: $p \Rightarrow true \equiv true$
(3.73) Left identity of $\Rightarrow$: $true \Rightarrow p \equiv p$
(3.74) $p \Rightarrow false \equiv \neg p$
(3.75) $false \Rightarrow p \equiv true$
(3.76) Weakening/strengthening:

- **(a)** $p \Rightarrow p \vee q$
- **(b)** $p \wedge q \Rightarrow p$
- **(c)** $p \wedge q \Rightarrow p \vee q$
- **(d)** $p \vee (q \wedge r) \Rightarrow p \vee q$
- **(e)** $p \wedge q \Rightarrow p \wedge (p \vee r)$

(3.77) Modus ponens: $p \wedge (p \Rightarrow q) \Rightarrow q$
(3.78) Case analysis: $(p \Rightarrow r) \wedge (q \Rightarrow r) \equiv (p \vee q \Rightarrow r)$
(3.79) Case analysis: $(p \Rightarrow r) \wedge (\neg p \Rightarrow r) \equiv r$
(3.80) Mutual implication: $(p \Rightarrow q) \wedge (q \Rightarrow p) \equiv p \equiv q$
(3.80b) Reflexivity wrt. Equivalence: $(p \equiv q) \Rightarrow (p \Rightarrow q)$
(3.81) Antisymmetry: $(p \Rightarrow q) \wedge (q \Rightarrow p) \Rightarrow p \equiv q$
(3.82) Transitivity:

- **(a)** $(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$
- **(b)** $(p \equiv q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$
- **(c)** $(p \Rightarrow q) \wedge (q \equiv r) \Rightarrow (p \Rightarrow r)$

Ex. 6.3.A $p \Rightarrow (q \wedge r) \equiv (p \Rightarrow q) \wedge (p \Rightarrow r)$
Ex. 6.3.B $(p \wedge q) \vee (\neg p \wedge r) \equiv (p \Rightarrow q) \wedge (\neg p \Rightarrow r)$
Ex. 6.3.C $(p \wedge q) \vee (\neg p \wedge r) \equiv (\neg p \vee r) \wedge (p \vee r)$
Ex. 6.3.D $\neg p \vee (p \Rightarrow q) \equiv p \Rightarrow q$
Ex. 6.3.E $p \vee (\neg p \Rightarrow q) \equiv \neg p \Rightarrow q$
(4.1) $p \Rightarrow (q \Rightarrow p)$
(4.2) Monotonicity of $\vee$: $(p \Rightarrow q) \Rightarrow (p \vee r \Rightarrow q \vee r)$
(4.3) Monotonicity of $\wedge$: $(p \Rightarrow q) \Rightarrow p \wedge r \Rightarrow q \wedge r$
(4.4) (Extended) Deduction Theorem: Suppose adding $P_1, \ldots, P_n$ as axioms to propositional

logic **E**, with the variables of the Pi considered to be constants, allows Q to be proved. Then $P_1 \wedge \ldots \wedge P_n \Rightarrow Q$ is a theorem.

(4.6) $(p \vee q \vee r) \wedge (p \Rightarrow s) \wedge (q \Rightarrow s) \wedge (r \Rightarrow s) \Rightarrow s$

6 Substitution

(3.84a) $(e = f) \wedge E[z := e] \equiv (e = f) \wedge E[z := f]$

(3.84b) $(e = f) \Rightarrow E[z := e] \equiv (e = f) \Rightarrow E[z := f]$

(3.84c) $q \wedge (e = f) \Rightarrow E[z := e] \equiv q \wedge (e = f) \Rightarrow E[z := f]$

(3.85a) Replace by *true*: $p \Rightarrow E[z := p] \equiv p \Rightarrow E[z := \mathit{true}]$

(3.85b) Replace by *true*: $q \wedge p \Rightarrow E[z := p] \equiv \wedge p \Rightarrow E[z := \mathit{true}]$

(3.86a) Replace by *false*: $E[z := p] \Rightarrow p \equiv E[z := \mathit{false}] \Rightarrow p$

(3.86b) Replace by *false*: $E[z := p] \Rightarrow p \vee q \equiv E[z := \mathit{false}] \Rightarrow p \vee q$

(3.87) Replace by *true*: $p \wedge E[z := p] \equiv p \wedge E[z := \mathit{true}]$

(3.88) Replace by *false*: $p \vee E[z := p] \equiv p \vee E[z := \mathit{false}]$

(3.89) Shannon: $E[z := p] \equiv (p \wedge E[z := \mathit{true}]) \vee (\neg p \wedge E[z := \mathit{false}])$

(4.5) Metatheorem Case Analysis (Shannon):
If $E[z := \mathit{true}]$ and $E[z := \mathit{false}]$ are theorems, then so is $E[z := p]$

7 Quantification

(8.11) Substitution: Provided $\neg \mathit{occurs}(‘y’, ‘x, F’)$,
$(\star y \mid R \bullet P)[x := F] = (\star y \mid R[x := F] \bullet P[x := F])$