1FC3 Theorem List

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Contents

1 Equivalence

(3.1) Axiom, Associativity of $\equiv$: $((p \equiv q) \equiv r) \equiv (p \equiv (q \equiv r))$
(3.2) Axiom, Symmetry of $\equiv$: $p \equiv q \equiv q \equiv p$
(3.3) Axiom, Identity of $\equiv$: $\mathit{true} \equiv q \equiv q$
(3.4) true
(3.5) Reflexivity of $\equiv$: $p \equiv p$
(3.7) Metatheorem: Any two theorems are equivalent.

2 Negation

(3.8) Axiom, Definition of false: $\mathit{false} \equiv \neg \mathit{true}$
(3.9) Axiom, Distributivity of $\neg$ over $\equiv$: $\neg(p \equiv q) \equiv (\neg p \equiv q)$
(3.10) Axiom, Definition of $\not\equiv$: $(p \not\equiv q) \equiv \neg(p \equiv q)$
(3.11) $\neg p \equiv q \equiv p \equiv \neg q$
(3.12) Double negation: $\neg\neg p \equiv p$
(3.13) Negation of false: $\neg \mathit{false} \equiv \mathit{true}$
(3.14) $(p \not\equiv q) \equiv \neg p \equiv q$
(3.15) $\neg p \equiv p \equiv \mathit{false}$
(3.16) Symmetry of $\not\equiv$: $(p \not\equiv q) \equiv (q \not\equiv p)$
(3.17) Associativity of $\not\equiv$: $((p \not\equiv q) \not\equiv r) \equiv (p \not\equiv (q \not\equiv r))$
(3.18) Mutual associativity: $((p \not\equiv q) \equiv r) \equiv (p \not\equiv (q \equiv r))$
(3.19) Mutual interchangeability: $p \not\equiv q \equiv r \equiv p \equiv q \not\equiv r$

3 Disjunction

(3.24) Axiom, Symmetry of $\vee$: $p \vee q \equiv q \vee p$
(3.25) Axiom, Associativity of $\vee$: $(p \vee q) \vee r \equiv p \vee (q \vee r)$
(3.26) Axiom, Idempotency of $\vee$: $p \vee p \equiv p$
(3.27) Axiom, Distributivity of $\vee$ over $\equiv$: $p \vee (q \equiv r) \equiv p \vee q \equiv p \vee r$
(3.28) Axiom, Excluded Middle: $p \vee \neg p$
(3.29) Zero of $\vee$: $p \vee \mathit{true} \equiv \mathit{true}$
(3.30) Identity of $\vee$: $p \vee \mathit{false} \equiv p$
(3.31) Distributivity of $\vee$ over $\vee$: $p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$
(3.32) $p \vee q \equiv p \vee \neg q \equiv p$

4 Conjunction

(3.35) Axiom, Golden rule: $p \wedge q \equiv p \equiv q \equiv p \vee q$
(3.36) Symmetry of $\wedge$: $p \wedge q \equiv q \wedge p$
(3.37) Associativity of $\wedge$: $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
(3.38) Idempotency of $\wedge$: $p \wedge p \equiv p$
(3.39) Identity of $\wedge$: $p \wedge \mathit{true} \equiv p$
(3.40) Zero of $\wedge$: $p \wedge \mathit{false} \equiv \mathit{false}$
(3.41) Distributivity of $\wedge$ over $\wedge$: $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge (p \wedge r)$
(3.42) Contradiction: $p \wedge \neg p \equiv \mathit{false}$
(3.43) Absorption:
- **(a)** $p \wedge (p \vee q) \equiv p$
- **(b)** $p \vee (p \wedge q) \equiv p$

(3.44) Absorption:
- **(a)** $p \wedge (\neg p \vee q) \equiv p \wedge q$
- **(b)** $p \vee (\neg p \wedge q) \equiv p \vee q$

(3.45) Distributivity of $\vee$ over $\wedge$: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
(3.45) Distributivity of $\wedge$ over $\vee$: $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
(3.47) De Morgan:

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

(3.48): $p \wedge q \equiv p \wedge \neg q \equiv \neg p$

(3.49): $p \wedge (q \equiv r) \equiv p \wedge q \equiv p \wedge r \equiv p$

(3.50): $p \wedge (q \equiv p) \equiv p \wedge q$

(3.51) Replacement: $(p \equiv q) \wedge (r \equiv p) \equiv (p \equiv q) \wedge (r \equiv q)$

(3.52) Definition of $\equiv$: $p \equiv q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

(3.53) Definition of $\not\equiv$ (Exclusive or): $p \not\equiv q \equiv (\neg p \wedge q) \vee (p \wedge \neg q)$

5 Implication

(3.57) Axiom, Definition of Implication: $p \Rightarrow q \equiv p \vee q \equiv q$

(3.58) Axiom, Consequence: $p \Leftarrow q \equiv q \Rightarrow p$

(3.59) (Alternative) Definition of Implication: $p \Rightarrow q \equiv \neg p \vee q$

(3.60) (Dual) Definition of Implication: $p \Rightarrow q \equiv p \wedge q \equiv p$

(3.61) Contrapositive: $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$

Ex. 3.45 $p \Rightarrow q \equiv \neg p \vee \neg q \equiv \neg p$

Ex. 3.46 $p \Rightarrow q \equiv \neg p \wedge \neg q \equiv \neg q$

(3.62): $p \Rightarrow (q \equiv r) \equiv p \wedge q \equiv p \wedge r$

(3.63) Distributivity of $\Rightarrow$ over $\equiv$: $p \Rightarrow (q \equiv r) \equiv p \Rightarrow q \equiv p \Rightarrow r$

(3.64) Self-distributivity of $\Rightarrow$: $p \Rightarrow (q \Rightarrow r) \equiv (p \Rightarrow q) \Rightarrow (p \Rightarrow r)$

(3.65) Shunting: $p \wedge q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r)$

(3.66) $p \wedge (p \Rightarrow q) \equiv p \wedge q$

(3.67) $p \wedge (q \Rightarrow p) \equiv p$

(3.68) $p \vee (p \Rightarrow q) \equiv \mathit{true}$

(3.69) $p \vee (q \Rightarrow p) \equiv q \Rightarrow p$

(3.70) $p \vee q \Rightarrow p \wedge q \equiv p \equiv q$

(3.71) Reflexivity of $\Rightarrow$: $p \Rightarrow p \equiv \mathit{true}$

(3.72) Right zero of $\Rightarrow$: $p \Rightarrow \mathit{true} \equiv \mathit{true}$

(3.73) Left identity of $\Rightarrow$: $\mathit{true} \Rightarrow p \equiv p$

(3.74) $p \Rightarrow \mathit{false} \equiv \neg p$

(3.75) $\mathit{false} \Rightarrow p \equiv \mathit{true}$

(3.76) Weakening/strengthening:

- **(a)** $p \Rightarrow p \vee q$
- **(b)** $p \wedge q \Rightarrow p$
- **(c)** $p \wedge q \Rightarrow p \vee q$
- **(d)** $p \vee (q \wedge r) \Rightarrow p \vee q$
- **(e)** $p \wedge q \Rightarrow p \wedge (p \vee r)$

(3.77) Modus ponens: $p \wedge (p \Rightarrow q) \Rightarrow q$

(3.78) Case analysis: $(p \Rightarrow r) \wedge (q \Rightarrow r) \equiv (p \vee q \Rightarrow r)$

(3.79) Case analysis: $(p \Rightarrow r) \wedge (\neg p \Rightarrow r) \equiv r$

(3.80) Mutual implication: $(p \Rightarrow q) \wedge (q \Rightarrow p) \equiv p \equiv q$

(3.80b) Reflexivity wrt. Equivalence: $(p \equiv q) \Rightarrow (p \Rightarrow q)$

(3.81) Antisymmetry: $(p \Rightarrow q) \wedge (q \Rightarrow p) \Rightarrow p \equiv q$

(3.82) Transitivity:

- **(a)** $(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$
- **(b)** $(p \equiv q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$
- **(c)** $(p \Rightarrow q) \wedge (q \equiv r) \Rightarrow (p \Rightarrow r)$

Ex. 6.3.A $p \Rightarrow (q \wedge r) \equiv (p \Rightarrow q) \wedge (p \Rightarrow r)$

Ex. 6.3.B $(p \wedge q) \vee (\neg p \wedge r) \equiv (p \Rightarrow q) \wedge (\neg p \Rightarrow r)$

Ex. 6.3.C $(p \wedge q) \vee (\neg p \wedge r) \equiv (\neg p \vee r) \wedge (p \vee r)$

Ex. 6.3.D $\neg p \vee (p \Rightarrow q) \equiv p \Rightarrow q$

Ex. 6.3.E $p \vee (\neg p \Rightarrow q) \equiv \neg p \Rightarrow q$

(4.1) $p \Rightarrow (q \Rightarrow p)$

(4.2) Monotonicity of $\vee$: $(p \Rightarrow q) \Rightarrow (p \vee r \Rightarrow q \vee r)$

(4.3) Monotonicity of $\wedge$: $(p \Rightarrow q) \Rightarrow p \wedge r \Rightarrow q \wedge r$

(4.4) (Extended) Deduction Theorem: Suppose adding $P_1, \ldots, P_n$ as axioms to propositional logic **E**, with the variables of the Pi considered to be constants, allows Q to be proved. Then $P_1 \wedge \ldots \wedge P_n \Rightarrow Q$ is a theorem.

(4.6) $(p \vee q \vee r) \wedge (p \Rightarrow s) \wedge (q \Rightarrow s) \wedge (r \Rightarrow s) \Rightarrow s$

6 Substitution (Leibniz as an axiom)

(3.84a) $(e = f) \wedge E[z := e] \equiv (e = f) \wedge E[z := f]$
(3.84b) $(e = f) \Rightarrow E[z := e] \equiv (e = f) \Rightarrow E[z := f]$
(3.84c) $q \wedge (e = f) \Rightarrow E[z := e] \equiv q \wedge (e = f) \Rightarrow E[z := f]$
(3.85a) Replace by ***true*****:** $p \Rightarrow E[z := p] \equiv p \Rightarrow E[z := true]$
(3.85b) Replace by ***true*****:** $q \wedge p \Rightarrow E[z := p] \equiv \wedge p \Rightarrow E[z := true]$
(3.86a) Replace by ***false*****:** $E[z := p] \Rightarrow p \equiv E[z := false] \Rightarrow p$
(3.86b) Replace by ***false*****:** $E[z := p] \Rightarrow p \vee q \equiv E[z := false] \Rightarrow p \vee q$
(3.87) Replace by ***true*****:** $p \wedge E[z := p] \equiv p \wedge E[z := true]$
(3.88) Replace by ***false*****:** $p \vee E[z := p] \equiv p \vee E[z := false]$
(3.89) Shannon: $E[z := p] \equiv (p \wedge E[z := true]) \vee (\neg p \wedge E[z := false])$
(4.5) Metatheorem, Case analysis (Shannon):
If $E[z := true]$ and $E[z := false]$ are theorems, then so is $E[z := p]$

7 Quantification

7.1 General laws of quantification

(8.11) Substitution: Provided $\neg occurs(`y', `x, F')$,
$(\star x \mid R \bullet P)[x := F] = (\star x \mid R[x := F] \bullet P[x := F]$
(8.13) Axiom, Empty range (where u is the identity of $\star$): $(\star x \mid false \bullet P) = u$
(8.14) Axiom, One-point rule: Provided $\neg occurs(`x', `E')$,
$(\star x \mid x = E \bullet P) = P[x := E]$
(8.15) Axiom, (Quantification) Distributivity: Provided each quantification is defined,
$(\star x \mid R \bullet P) \star (\star x \mid R \bullet Q) = (\star x \mid R \bullet P \star Q)$
(8.16) Axiom, Range split: Provided $R \wedge S = false$ and each quantification is defined,
$(\star x \mid R \vee S \bullet P) = (\star x \mid R \bullet P) \star (\star x \mid S \bullet P)$
(8.17) Axiom, Range split: Provided each quantification is defined,
$(\star x \mid R \vee S \bullet P) \star (\star x \mid R \wedge S \bullet P) = (\star x \mid R \bullet P) \star (\star x \mid S \bullet P)$
(8.18) Range split for idempotent $\star$: Provided each quantification is defined,
$(\star x \mid R \vee S \bullet P) = (\star x \mid R \bullet P) \star (\star x \mid S \bullet P)$
(8.19) Axiom, Interchange of dummies: Provided $\neg occurs(`y', `R`)$ and $\neg occurs(`x', `S`)$, and each quantification is defined,
$(\star x \mid R \bullet (\star y \mid S \bullet P)) = (\star y \mid R \bullet (\star x \mid S \bullet P))$
(8.20) Axiom, Nesting: Provided $\neg occurs(`y', `R`)$,
$(\star x, y \mid R \wedge S \bullet P) = (\star x \mid R \bullet (\star y \mid S \bullet P))$
(8.20a) Axiom, Dummy list permutation: $(\star x, y \mid R \bullet P) = (\star y, x \mid R \bullet P)$
(8.21) Axiom, Dummy renaming (α conversion): Provided $\neg occurs(`y', `R, P`)$,
$(\star x \mid R \bullet P) = (\star y \mid R[x := y] \bullet P[x := y])$
(8.22) Change of dummy: Provided f has an inverse and $\neg occurs(`y', `x, R, P`)$,
$(\star x \mid R \bullet P) = (\star y \mid R[x := f.y] \bullet P[x := f.y])$
(8.23) Theorem Split off term: For $n : \mathbb{N}$ and dummies $i : \mathbb{N}$,
$(\star x \mid 0 \leq i < n+1 \bullet P) = (\star x \mid 0 \leq i < n \bullet P) \star P[i := n]$
$(\star x \mid 0 \leq i < n+1 \bullet P) = P[i := 0] \star (\star i \mid 0 < i < n+1 \bullet P)$

7.2 Universal Quantification

(9.2) Axiom, Trading: $(\forall x \mid R \bullet P) \equiv (\forall x \mid\bullet R \Rightarrow P)$
(9.3) Trading:
(a) $(\forall x \mid R \bullet P) \equiv (\forall x \mid\bullet \neg R \vee P)$
(b) $(\forall x \mid R \bullet P) \equiv (\forall x \mid\bullet R \wedge P \equiv R)$
(c) $(\forall x \mid R \bullet P) \equiv (\forall x \mid\bullet R \vee P \equiv P)$
(9.4) Trading:
(a) $(\forall x \mid Q \wedge R \bullet P) \equiv (\forall x \mid Q \bullet R \Rightarrow P)$

(b) $(\forall x \mid Q \wedge R \bullet P) \equiv (\forall x \mid Q \bullet \neg R \vee P)$
(c) $(\forall x \mid Q \wedge R \bullet P) \equiv (\forall x \mid Q \bullet R \wedge P \equiv R)$
(d) $(\forall x \mid Q \wedge R \bullet P) \equiv (\forall x \mid Q \bullet R \vee P \equiv P)$

(9.4.1) Universal double trading: $(\forall x \mid R \bullet P) \equiv (\forall x \mid P \bullet R)$

(9.5) Axiom, Distributivity of $\vee$ over $\forall$: Provided $\neg \mathit{occurs}(\text{‘}x\text{’}, \text{‘}P\text{‘})$,
$P \vee (\forall x \mid R \bullet Q) \equiv (\forall x \mid R \bullet P \vee Q)$

(9.6) Provided $\neg \mathit{occurs}(\text{‘}x\text{’}, \text{‘}P\text{‘})$, $(\forall x \mid R \bullet P) \equiv P(\forall x \mid\bullet \neg R)$

(9.7) Axiom, Distributivity of $\wedge$ over $\forall$: Provided $\neg \mathit{occurs}(\text{‘}x\text{’}, \text{‘}P\text{‘})$,
$\neg(\forall \mid\bullet \neg R) \Rightarrow ((\forall x \mid R \bullet P \wedge Q) \equiv P \wedge (\forall x \mid R \bullet Q))$

(9.8 $(\forall x \mid R \bullet \mathit{true}) \equiv \mathit{true}$

(9.9) $(\forall x \mid R \bullet P \equiv Q) \Rightarrow ((\forall x \mid R \bullet P) \equiv (\forall x \mid R \bullet Q))$

(9.10) Range weakening/strengthening: $(\forall x \mid R \vee Q \bullet P) \Rightarrow (\forall x \mid R \bullet P)$

(9.11) Body weakening/strengthening: $(\forall x \mid R \bullet P \wedge Q) \Rightarrow (\forall x \mid R \bullet P)$

(9.12) Monotonicity of $\forall$: $(\forall x \mid R \bullet Q \Rightarrow P) \Rightarrow ((\forall x \mid R \bullet Q) \Rightarrow (\forall x \mid R \bullet P))$

(9.12a) Range-antitonicity of $\forall$: $(\forall x \mid\bullet Q \Rightarrow R) \Rightarrow ((\forall x \mid R \bullet P) \Rightarrow (\forall x \mid Q \bullet P))$

(9.13) Instantiation: $(\forall x \mid\bullet P) \Rightarrow P[x := E]$

(9.16) Metatheorem: P is a theorem iff $(\forall x \mid\bullet P)$ is a theorem.

7.3 Existential Quantification

(9.17) Axiom, Generalized De Morgan: $(\exists x \mid R \bullet P) \equiv \neg(\forall x \mid R \bullet \neg P)$

(9.18) Generalized De Morgan:
(a) $\neg(\exists x \mid R \bullet \neg P) \equiv (\forall x \mid R \bullet P)$
(b) $\neg(\exists x \mid R \bullet P) \equiv (\forall x \mid R \bullet \neg P)$
(c) $(\exists x \mid R \bullet \neg P) \equiv \neg(\forall x \mid R \bullet P)$

(9.19) Trading for $\exists$: $(\exists x \mid R \bullet P) \equiv (\exists x \mid\bullet R \wedge P)$

(9.20) Trading for $\exists$: $(\exists x \mid Q \wedge R \bullet P) \equiv (\exists x \mid Q \bullet R \wedge P)$

(9.20.1) Existential double trading: $(\exists x \mid R \bullet P) \equiv (\exists x \mid P \bullet R)$

(9.20.2) $(\exists x \mid\bullet R) \Rightarrow ((\forall x \mid R \bullet P) \Rightarrow (\exists x \mid R \bullet P))$

(9.21) Distributivity of $\wedge$ over $\exists$: Provided $\neg \mathit{occurs}(\text{‘}x\text{’}, \text{‘}P\text{‘})$,
$P \wedge (\exists x \mid R \bullet Q) \equiv (\exists x \mid R \bullet P \wedge Q)$

(9.22) Provided $\neg \mathit{occurs}(\text{‘}x\text{’}, \text{‘}P\text{‘})$, $(\exists x \mid R \bullet P) \equiv P \wedge (\exists x \mid\bullet R)$

(9.23) Distributivity of $\vee$ over $\exists$: Provided $\neg \mathit{occurs}(\text{‘}x\text{’}, \text{‘}P\text{‘})$,
$(\exists x \mid\bullet R) \Rightarrow ((\exists x \mid R \bullet P \vee Q) \equiv P \vee (\exists x \mid R \bullet Q))$

(9.24) $(\exists x \mid R \bullet \mathit{false}) \equiv \mathit{false}$

(9.25) Range weakening/strengthening: $(\exists x \mid R \bullet P) \Rightarrow (\exists x \mid Q \vee R \bullet P)$

(9.26) Body weakening/strengthening: $(\exists x \mid R \bullet P) \Rightarrow (\exists x \mid R \bullet P \vee Q)$

(9.27) (Body) Monotonicity of $\exists$: $(\exists x \mid R \bullet Q \Rightarrow P) \Rightarrow ((\exists x \mid R \bullet Q) \Rightarrow (\exists x \mid R \bullet P))$

(9.27) Range-Monotonicity of $\exists$: $(\exists x \mid\bullet Q \Rightarrow R) \Rightarrow ((\exists x \mid Q \bullet P) \Rightarrow (\exists x \mid R \bullet P))$

(9.28) $\exists$ Introduction: $P[x := p] \Rightarrow (\exists x \mid\bullet P)$

(9.29) Interchange of quantifications: Provided $\neg \mathit{occurs}(\text{‘}y\text{’}, \text{‘}R\text{‘}) \wedge \neg \mathit{occurs}(\text{‘}x\text{’}, \text{‘}Q\text{‘})$,
$(\exists x \mid R \bullet (\forall y \mid Q \bullet P)) \Rightarrow (\forall y \mid Q \bullet (\exists x \mid R \bullet P))$

(9.30) Metatheorem Witness: If $\neg \mathit{occurs}(\text{‘}\hat{x}\text{’}, \text{‘}P, Q, R\text{‘})$, then:
$\exists x \mid R \bullet P) \Rightarrow Q$ is a theorem iff $(R \wedge P)[x := \hat{x}] \Rightarrow Q$ is a theorem.

(9.30v) Metatheorem Witness (variant): If $\neg \mathit{occurs}(\text{‘}x\text{’}, \text{‘}Q\text{‘})$, then:
$\exists x \mid R \bullet P) \Rightarrow Q$ is a theorem iff $(R \wedge P) \Rightarrow Q$ is a theorem.

8 Sets

8.1 General set theorems

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8.2 Operations on sets

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8.3 Basic properties of union

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8.4 Basic properties of intersection

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8.5 Properties of combinations of union and intersection

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8.6 Properties of set difference

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8.7 Implication versus subset

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8.8 Subsets

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8.9 Powersets

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