Jahangirnagar University

Department of Statistics and Data Science

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COURSE TITLE: Time Series Analysis and Forecasting

Assignment On

" **Impact of Transformation in Modeling and Forecasting with ARIMA**"

Submitted By	Submitted To
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Impact of Transformation in Modeling and Forecasting with ARIMA

As my ID is 20231**040**, 40%%6=4; AirPassengers dataset has been used for Time Series Analysis.

Dataset Description

The AirPassengers dataset in R is a built-in dataset containing monthly international airline passenger numbers from 1949 to 1960 (144 data points). Here are its main features:

- Data Type: Time Series - The data represents observations over a fixed time interval (monthly) for 11 years.
- Variable: Passengers - This single variable represents the number of international airline passengers for each month.
- Format: The data is typically loaded as a numeric vector named "AirPassengers".

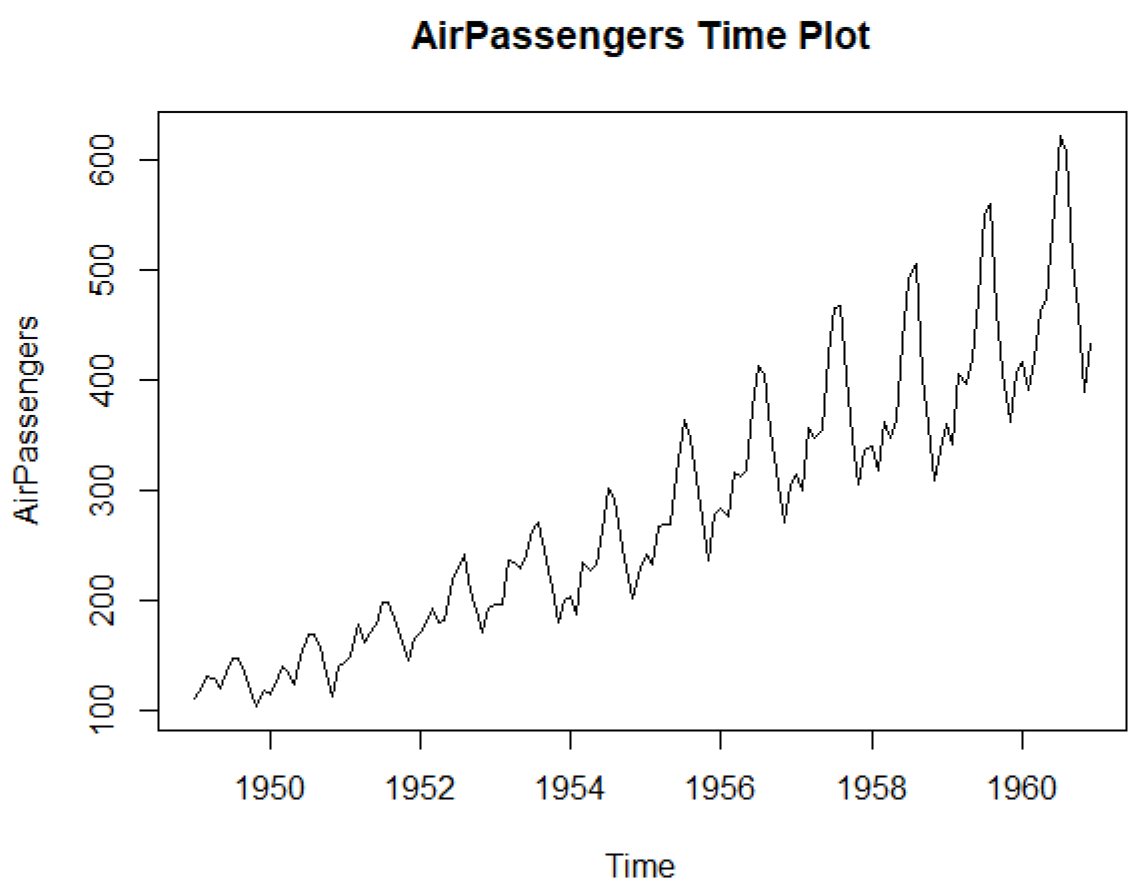


Figure 1: Time plot for Air Passengers data

From the time plot in Fig. 1 it can be seen that,

i. **Seasonality:** The dataset exhibits clear seasonality, specifically multiplicative seasonality, with recurring patterns that occur at regular intervals.

ii. **Trend:** There's an underlying upward trend in passenger numbers over time, indicating an overall increase in air travel during this period.

iii. **Stationarity:** The data might not be stationary due to the mean, and variance is not constant over time. While the data exhibits a clear trend and seasonality, it may require appropriate differencing to achieve stationarity.

iv. **Missing Values:** The dataset does not contain any missing values, making it suitable for analysis and modeling.

v. **Potential Outliers:** While there are no significant outliers visible in the dataset, it's essential to perform exploratory data analysis and diagnostic checks to identify any potential outliers or anomalies that may need to be addressed.

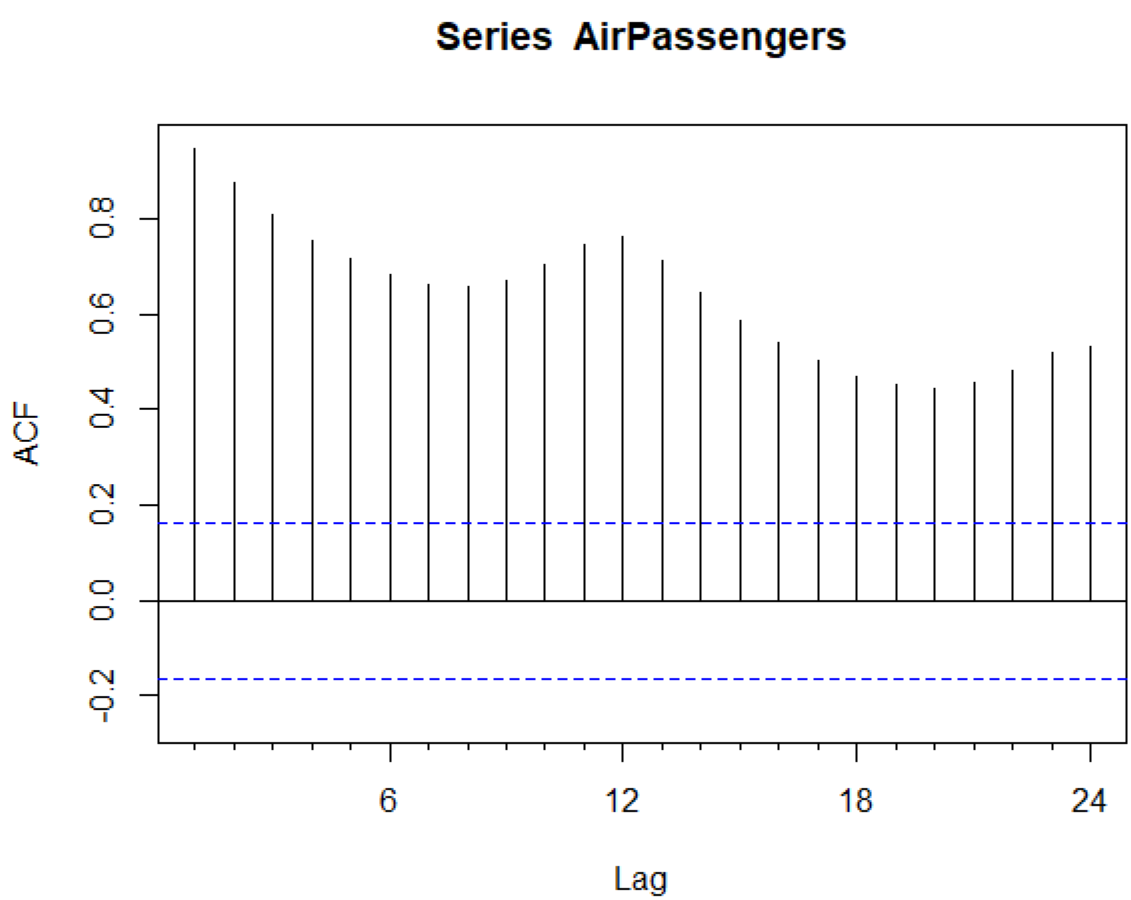


Figure 2: ACF plot for Airpassengers dataset

From the ACF plot in Fig. 2 it can be seen that,

i. **Trend**: The dataset contains trends, as the autocorrelations for small lags are large and positive.
ii. **Seasonality**: The presence of seasonality is evident in the dataset, as indicated by larger autocorrelations at the seasonal lags. The dataset is recorded monthly with a frequency of 12, the autocorrelation at lag 12 or 24 is slightly higher than at previous lags.
iii. **Stationarity**: The autocorrelations for the lags have surpassed the threshold for the 95% confidence interval, it suggests the possibility that the data might be non-stationary.
iv. **Altering Series**: Since the autocorrelation function (ACF) plot does not exhibit any negative values for any lags, it suggests that the time series data is not undergoing any significant alterations.
v. **Outliers**: Since there are no abnormal spikes observed for any lags, it can be concluded that the dataset does not contain any outliers.

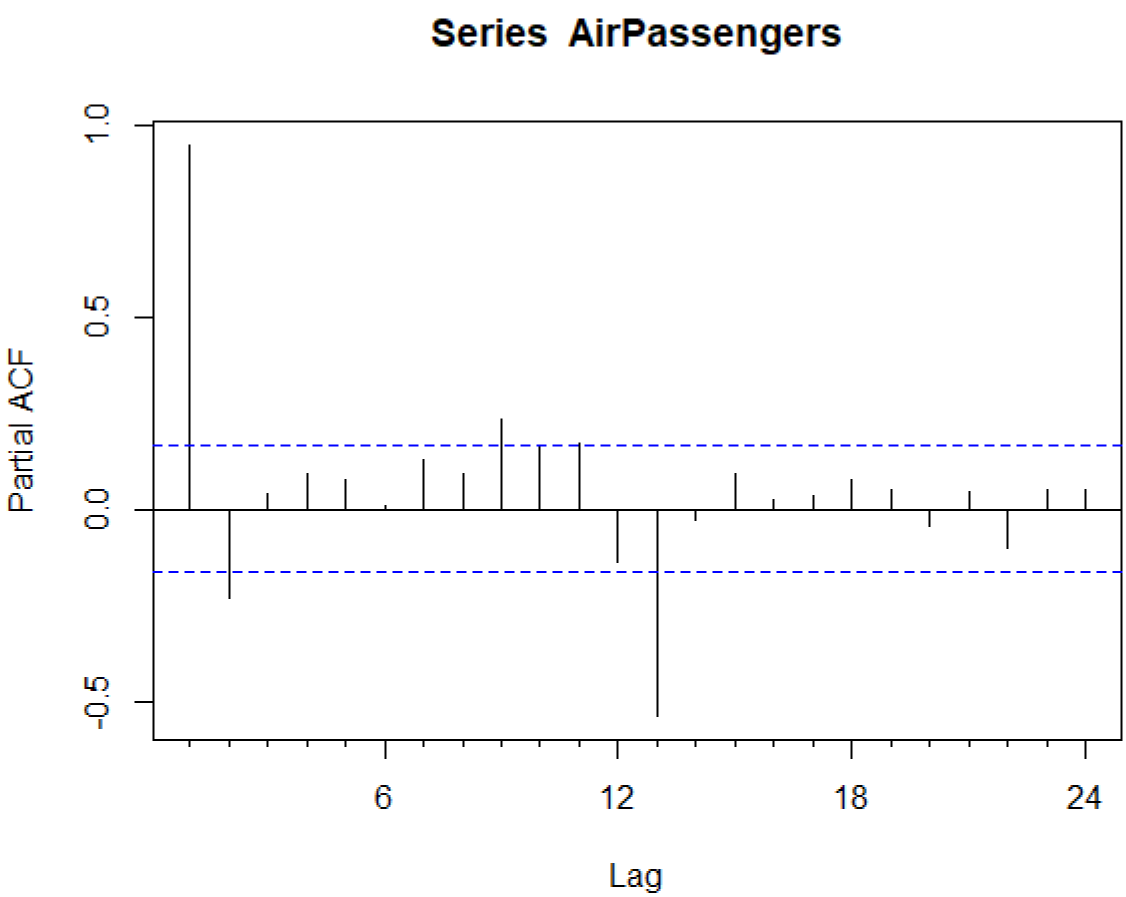


Figure 3: Partial ACF plot for air passenger dataset

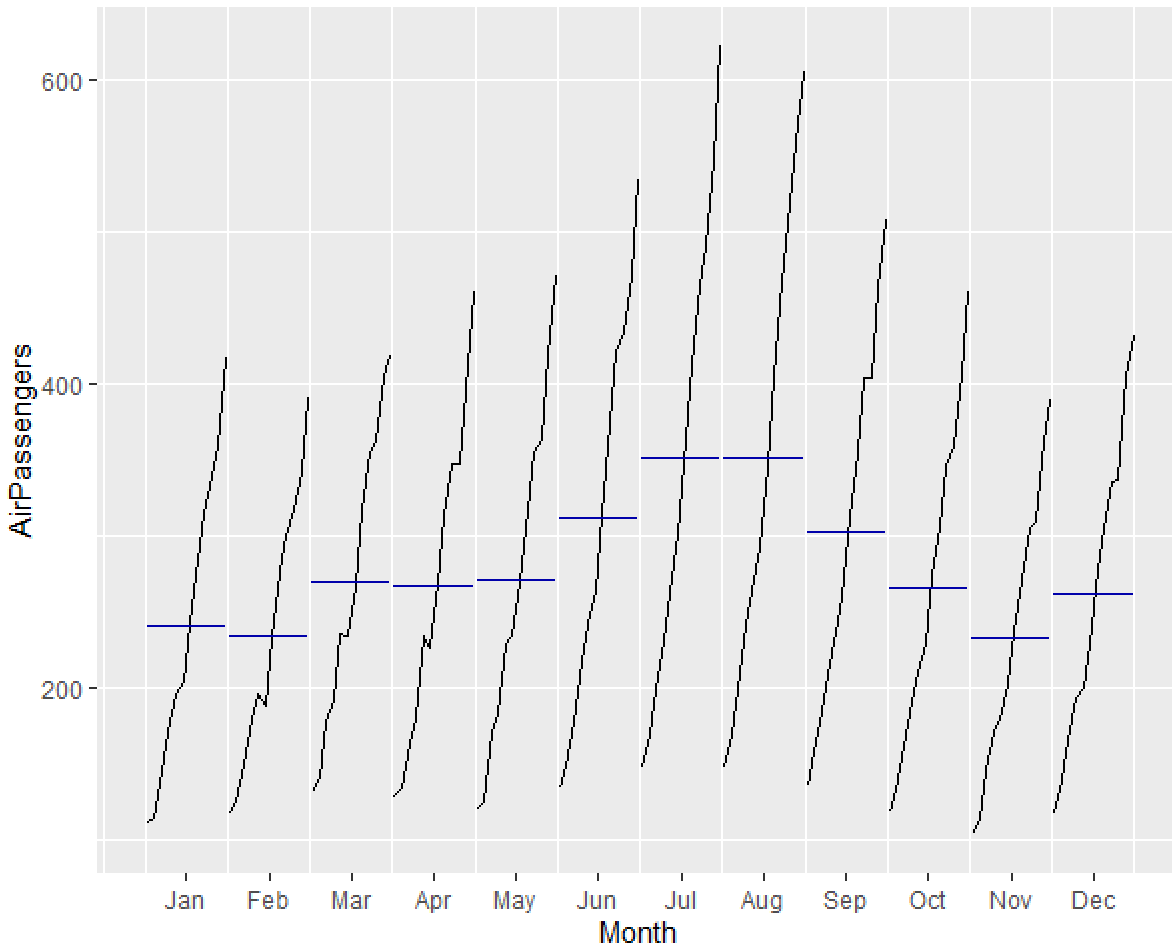


Figure 4: Seasonal subseries plot of air passenger dataset

Fig. 4 depicts that; the air passenger data exhibits a clear seasonal pattern with higher passenger numbers in peak travel months (Jul-Aug) and lower numbers in off-peak months.

Dataset Spliting:

For a 70-30% split, the AirPassengers dataset, consisting of 144 observations, allocates the first 101 instances to the training data, while the remaining instances are assigned to the test data.

```
Code:
traindata=ts(AirPassengers[1:101],frequency=12,start=c(1949,1))
testdata=ts(AirPassengers[102:144],frequency=12,start=c(1957,6))
```

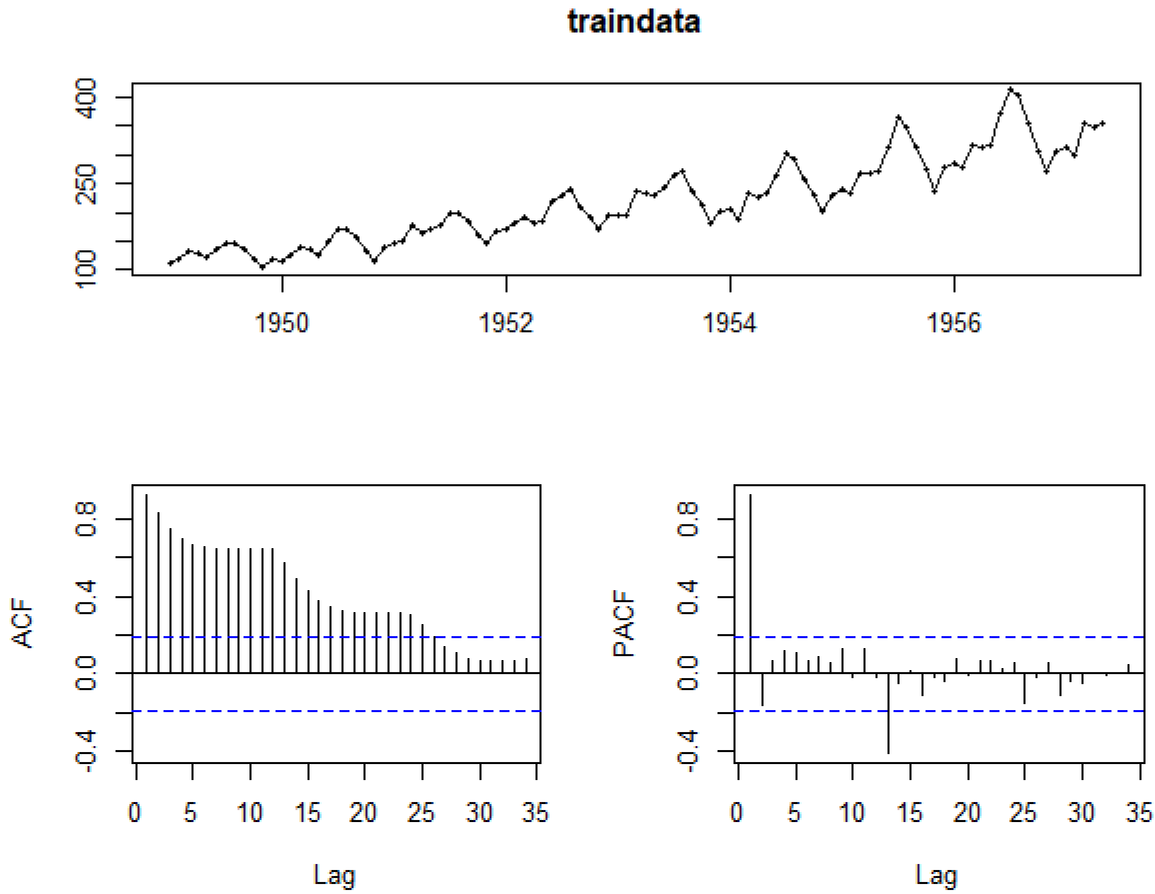


Figure 5: Time plot, ACF, and PACF plot for train data

Stationarity Test:

While the time plot may suggest non-stationarity in the data, conducting stationarity tests can strengthen the robustness of the conclusion.

Augmented Dickey-Fuller (ADF) Test:

```
Augmented Dickey-Fuller Test

data:  traindata
Dickey-Fuller = -4.6441, Lag order = 4, p-value = 0.01
alternative hypothesis: stationary
```

Since the p-value is less than 0.05 (at a 95% confidence level), we reject the null hypothesis, indicating that the training data is stationary.

Kwiatkowski-Phillips-Schmidt-Shin (KPSS) Test:

```
KPSS Test for Level Stationarity

data:  traindata
```

```
KPSS Level = 1.9493, Truncation lag parameter = 4, p-value = 0.01
```

Since the p-value is less than 0.05 (at a 95% confidence level), we reject the null hypothesis, indicating that the training data is non-stationary.

As the tests do not provide definitive evidence regarding the stationarity of the data, differencing can be applied to achieve stationarity.

In our analysis, one differencing operation has been performed to remove trends and non-seasonal patterns from the data. Additionally, one lag has been incorporated to address and mitigate seasonality effects in the dataset.

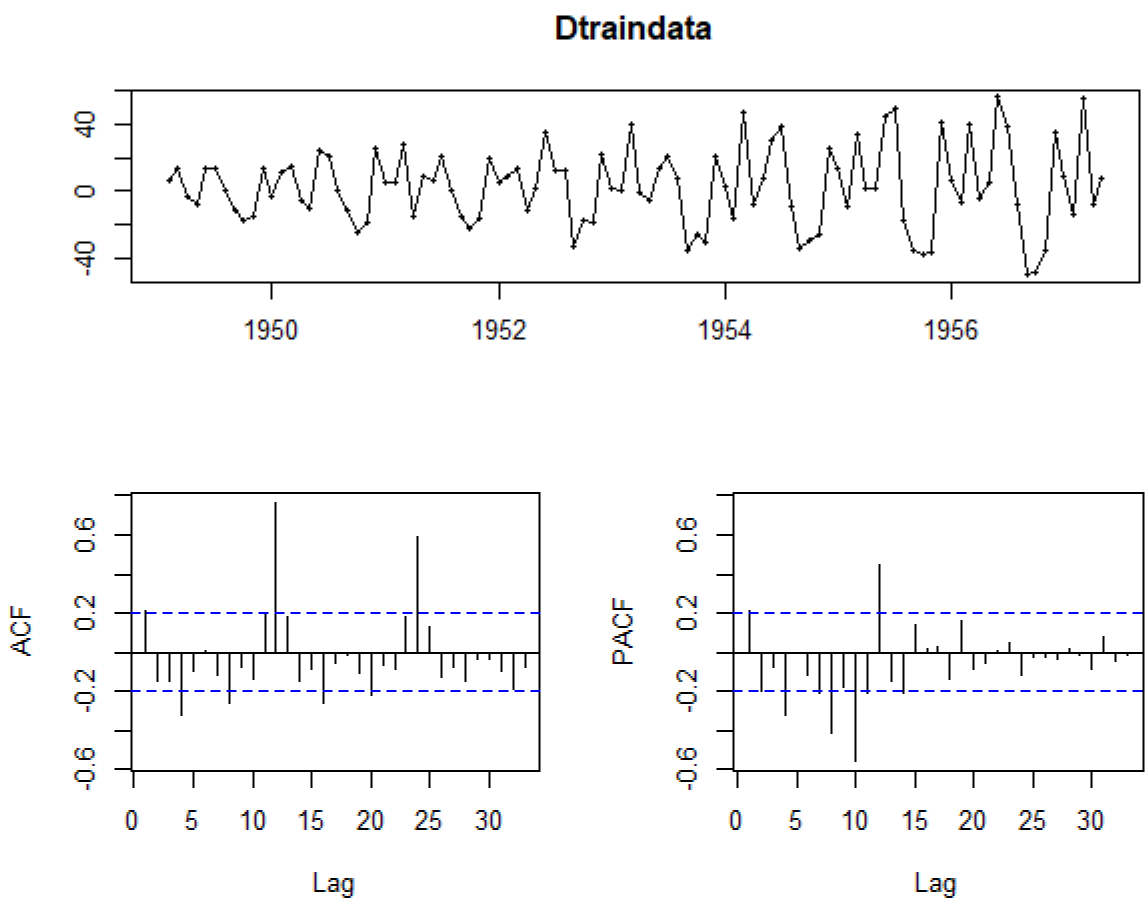


Figure 6: Time plot, ACF, and PACF plot for train data with one seasonal and one non-seasonal difference

ADF Test:

```
Augmented Dickey-Fuller Test

data:  Dtraindata
Dickey-Fuller = -5.5737, Lag order = 4, p-value = 0.01
alternative hypothesis: stationary
```

p-value < alpha (0.05); we reject the null hypothesis, indicating that the training data is stationary.

KPSS Test:

```
KPSS Test for Level Stationarity

data:  Dtraindata
KPSS Level = 0.022411, Truncation lag parameter = 4, p-value = 0.1
```

p-value > alpha (0.05); we do not reject the null hypothesis, indicating that the training data is stationary.

ARIMA Model

ARIMA stands for Auto Regressive Integrated Moving Average. It's a statistical method used for analyzing and forecasting time series data. ARIMA models are described using three different factors which are described below shortly.

- Autoregressive (p): This refers to how much a future value depends on past values in the series.
- Integrated (d): This indicates how many times the data needs to be differenced (essentially taking the difference between subsequent data points) to make it stationary, which means the statistical properties don't change over time.
- Moving Average (q): This considers the average of past forecast errors (the difference between predicted and actual values) to improve the accuracy of future forecasts.

The characteristics equation for an ARIMA (Autoregressive Integrated Moving Average) and SARIMA (seasonal ARIMA) model, commonly denoted as **ARIMA (p, d, q) (P, D, Q) [m]**. In simpler terms ARIMA (p, d, q) is like the base model for forecasting and (P, D, Q) [m] adds a seasonal twist to the model, accounting for recurring patterns in the data. The breakdown is given below:

- p is the order of the autoregressive part.
- d is the degree of differencing (the number of times the data have had past values subtracted).
- q is the order of the moving average part.
- P is the seasonal order of the autoregressive part.
- D is the seasonal degree of differencing.
- Q is the seasonal order of the moving average part.
- m is the number of periods in each season.

The **mathematical expression** is:

$$\left(1-\varphi_1 B^1-\varphi_2 B^2-\cdots-\varphi_p B^p\right)\left(1-\Phi_1 B^{1m}-\Phi_2 B^{2m}-\cdots-\Phi_P B^{Pm}\right)(1-B)^d\left(1-B^m\right)^D X_t=\left(1+\theta_1 B^1+\theta_2 B^2+\cdots+\theta_q B^q\right)\left(1+\Theta_1 B^{1m}+\Theta_2 B^{2m}+\cdots+\Theta_Q B^{Qm}\right),$$
$$\boldsymbol{where}\ \{Z_t\}\sim WN(0,\sigma^2) \quad (1)$$

Where:

- B is the backward shift operator.
- $\varphi_1, \varphi_2, \ldots\ldots, \varphi_p$ are the autoregressive coefficients.
- $\Phi_1, \Phi_2, \ldots\ldots, \Phi_P$ are the seasonal autoregressive coefficients.
- $\theta_1, \theta_2, \ldots\ldots, \theta_q$ are the moving average coefficients.
- $\Theta_1, \Theta_2, \ldots\ldots, \Theta_Q$ are the seasonal moving average coefficients.

This equation characterizes the behavior of the ARIMA model in terms of its autoregressive and moving average components, both for the non-seasonal and seasonal parts.

Model 1

From Fig. 6, it is evident that in the PACF plot, the first two spikes exceed the 95% threshold, and in the ACF plot, one spike surpasses the 95% threshold. Additionally, regarding seasonal characteristics, one spike in the PACF plot and two spikes in the ACF plot exceed the 95% threshold for a yearly (12-month) interval. Consequently, based on these observations, the ARIMA model is suggested to be **ARIMA (2,1,1) (1,1,2) [12].**

```
Series: traindata
ARIMA(2,1,1)(1,1,2)[12]

Coefficients:
          ar1      ar2     ma1    sar1     sma1    sma2
      -0.7848  -0.1104  0.5132  0.9982  -1.4603  0.4864
s.e.   0.9183   0.2985  0.9057  0.0118   0.2447  0.2072

sigma^2 = 71.02:  log likelihood = -319.1
AIC=652.21   AICc=653.61   BIC=669.55
```

From equation 1, the **mathematical expression** for the model 1 written as,

$$(1 + 0.7848\,\mathrm{B}^1 + 0.1104\,\mathrm{B}^2)(1 - 0.9982\,\mathrm{B}^{12})(1 - \mathrm{B})^1(1 - \mathrm{B}^{12})^1 X_t = (1 + 0.5132\,\mathrm{B}^1)(1 - 1.4603\,\mathrm{B}^{12} + 0.4864\,\mathrm{B}^{24})\ \boldsymbol{where}\ \{Z_t\} \sim WN(0,\ \sigma^2 = 71.02) \quad (2)$$

<u>Tests for the residual diagnostics checking:</u>

```
Null hypothesis: Residuals are iid noise.
Test                        Distribution Statistic   p-value
Ljung-Box Q                Q ~ chisq(20)     17.92    0.5929
McLeod-Li Q                Q ~ chisq(20)     32.05    0.0428 *
Turning points T      (T-66)/4.2 ~ N(0,1)       63     0.475
Diff signs S          (S-50)/2.9 ~ N(0,1)       47    0.3035
Rank P            (P-2525)/170.4 ~ N(0,1)     2353    0.3128
```

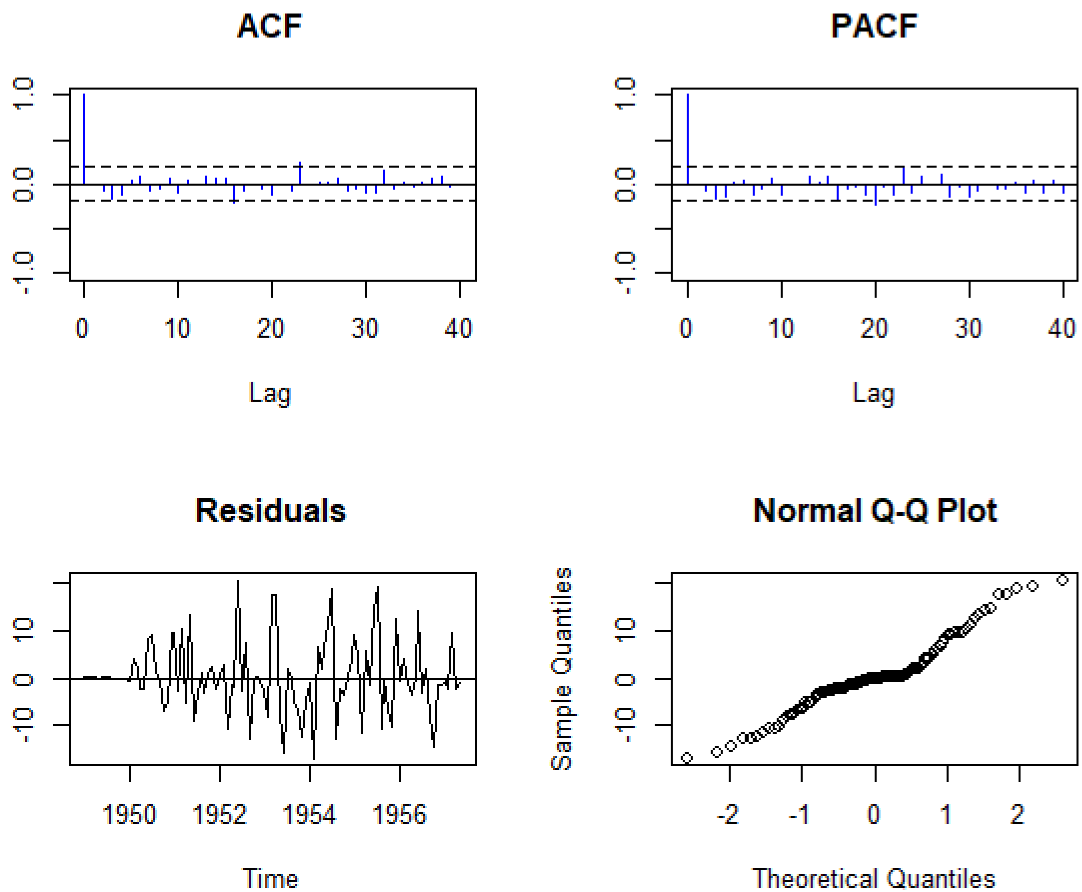


Figure 7: Residuals normality checking for model 1.

From Fig. 7, the q-q plot of the residuals for model 1 indicates a nearly normal distribution. Furthermore, based on the results of the statistical tests, it can be inferred that the residuals are independently and identically distributed (iid), although one statistical test suggests otherwise, the other four tests confirm the iid assumption. Autocorrelation is notably significant at a lag of 23, as evidenced by the ACF plot. Moreover, at a lag of 20, the spike surpassed the 95% confidence interval boundary.

Cubic Root Transformation

Based on the time plot of the training data, it appears that there are upward (increasing) trends present. Therefore, to stabilize the variance, we can consider applying a cubic root transformation to the data.

The transformation is defined as:

$$y_{transformed} = \sqrt[3]{y} \tag{3}$$

And the reverse transformation is defined as:

$$y = \left(y_{transformed}\right)^3 \tag{4}$$

where, y is the original observation and $y_{transformed}$ is the transformed observation.

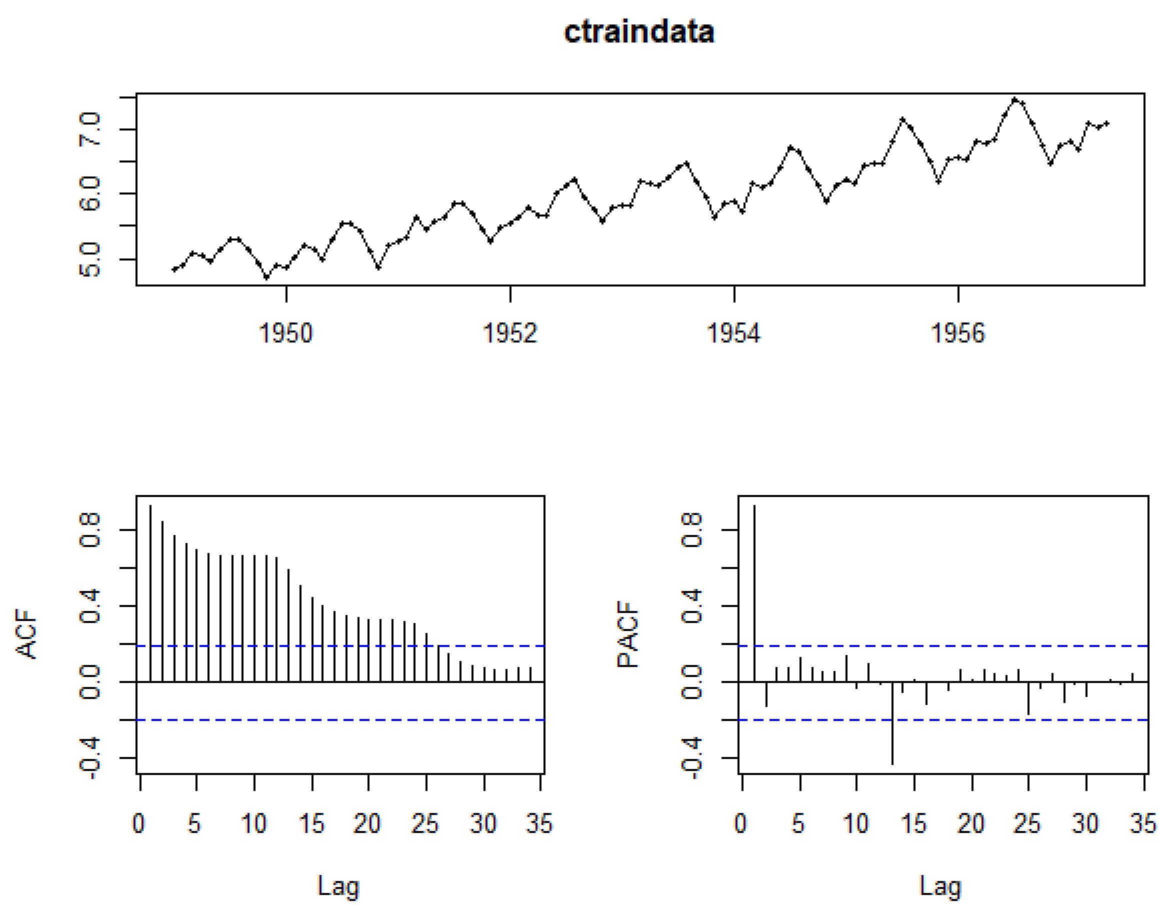
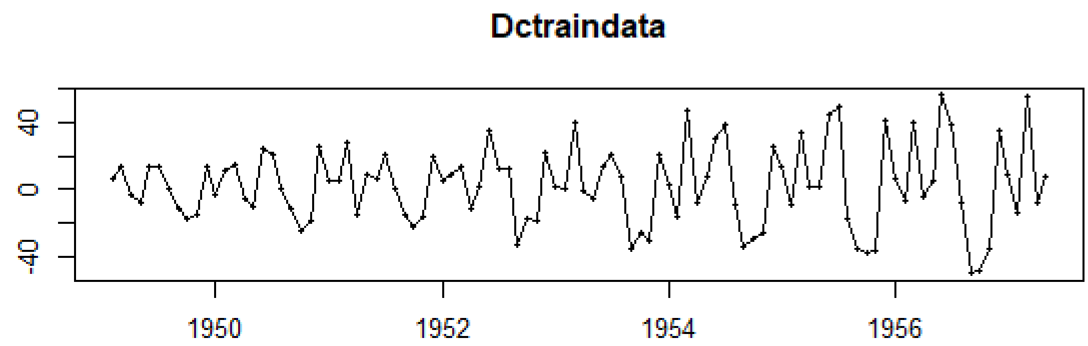


Figure 8: Time plot, ACF and PACF plot for cubic root transformation of train data

The cubic transformed data exhibits a high representation of seasonal differences. Consequently, the incorporation of one lag has been undertaken to eliminate seasonal patterns from the data.



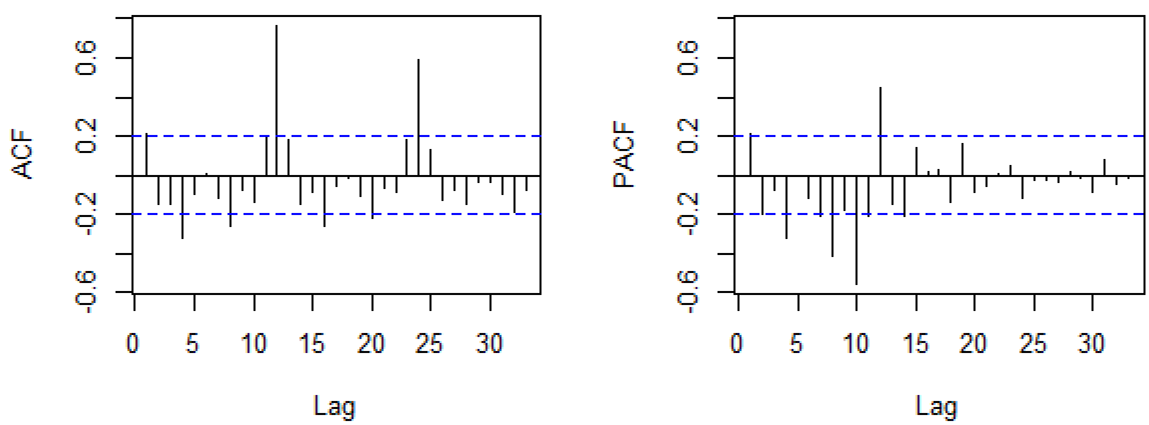


Figure 9: Time plot, ACF, and PACF plot for cubic transformed train data with one seasonal difference

ADF Test:

```
Augmented Dickey-Fuller Test

data:  Dctraindata
Dickey-Fuller = -5.5737, Lag order = 4, p-value = 0.01
alternative hypothesis: stationary
```

p-value < alpha (0.05); we reject the null hypothesis, indicating that the training data is stationary.

KPSS Test:

```
KPSS Test for Level Stationarity

data:  Dctraindata
KPSS Level = 0.022411, Truncation lag parameter = 4, p-value = 0.1
```

p-value > alpha (0.05); we do not reject the null hypothesis, indicating that the training data is stationary.

Model 2

From Fig. 9, it is evident that in the PACF plot, the first two spikes slightly exceed the 95% threshold, and in the ACF plot, one spike surpasses the 95% threshold. Additionally, regarding seasonal characteristics, one spike in the PACF plot and two spikes in the ACF plot exceed the 95% threshold for a yearly (12-month) interval. Consequently, based on these observations, the ARIMA model is suggested to be **ARIMA (2,0,1) (0,1,2) [12].**

```
Series: ctraindata
ARIMA(2,0,1)(0,1,2)[12]

Coefficients:
         ar1      ar2      ma1     sma1    sma2
      1.3113  -0.3141  -0.6357  -0.5225  0.0659
s.e.  0.3683   0.3648   0.3078   0.1153  0.1211

sigma^2 = 0.006235:  log likelihood = 99.26
AIC=-186.51   AICc=-185.49   BIC=-171.58
```

From equation 1, the **mathematical expression** for the model 2 written as,

$$(1 - 1.3113\, B^1 + 0.3141\, B^2)(1 - B^{12})^1 X_t = (1 - 0.6357\, B^1)(1 - 0.5225\, B^{12} + 0.0659\, B^{24})\ \boldsymbol{where}\ \{Z_t\} \sim WN(0, \sigma^2 = 0.006235) \quad (5)$$

<u>Tests for the residual diagnostics checking:</u>

```
Null hypothesis: Residuals are iid noise.
Test                        Distribution Statistic   p-value
Ljung-Box Q                Q ~ chisq(20)     16.43    0.6893
McLeod-Li Q                Q ~ chisq(20)     21.71    0.3562
Turning points T     (T-66)/4.2 ~ N(0,1)        62    0.3408
Diff signs S         (S-50)/2.9 ~ N(0,1)        53    0.3035
Rank P           (P-2525)/170.4 ~ N(0,1)      2365    0.3478
```

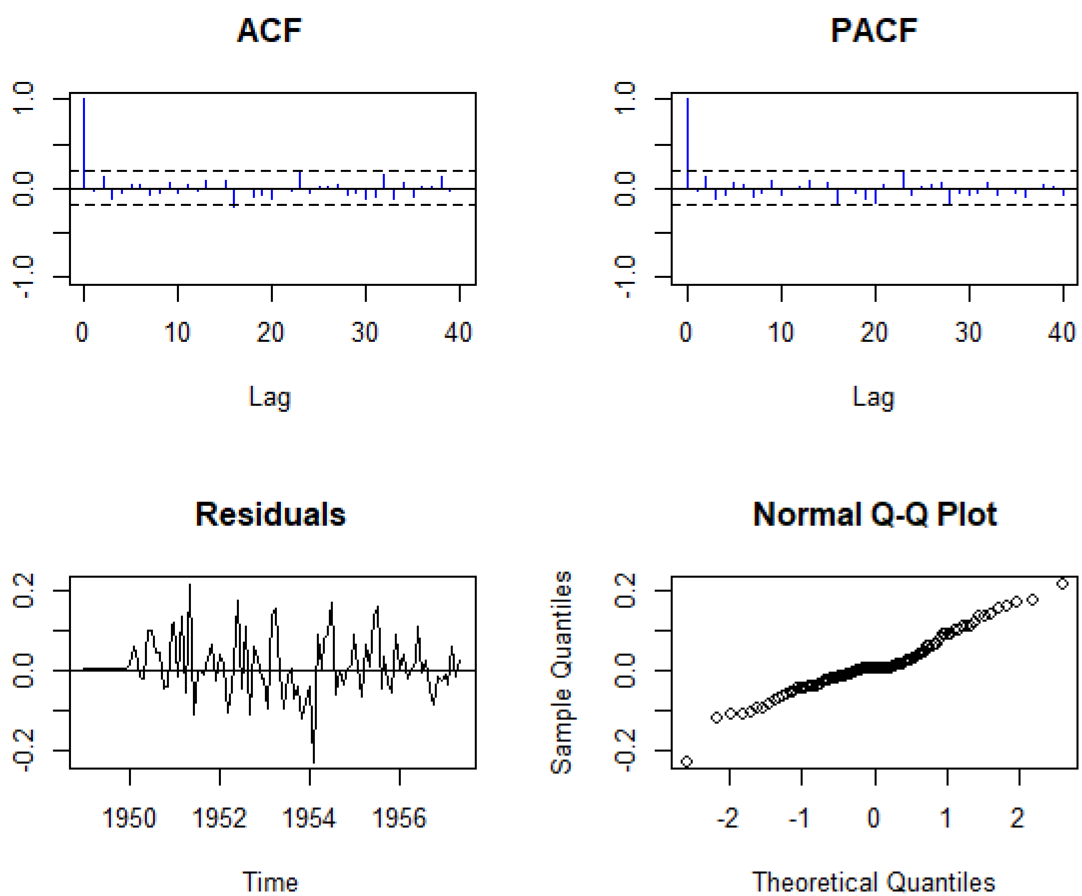


Figure 10: Residuals normality checking for model 2.

The q-q plot of the residuals for model 2 indicates a nearly normal distribution. Furthermore, based on the results of the statistical tests, it can be inferred that the residuals are independently and identically distributed (iid). The ACF and PACF plots should not show significant autocorrelation or partial autocorrelation at any lag, indicating that the residuals are independent and random.

Box-Cox Transformation

Box-Cox transformation is a method used to stabilize the variance and improve the normality of a dataset. It involves applying a power transformation to the data, expressed as:

$$y^{\lambda} = \begin{cases} \frac{y^{\lambda}-1}{\lambda} & if\ \lambda \neq 0 \\ \log(y) & if\ \lambda = 0 \end{cases} \tag{6}$$

Where:

- y is the original data.
- λ is the transformation parameter.
- If λ = 0, the transformation becomes a logarithmic transformation.
- If λ ≠ 0, the transformation adjusts the data according to the specified power.

The optimal value of λ is often chosen to maximize the log-likelihood function or minimize other criteria, such as the coefficient of variation or the variance of the transformed data.

Box-Cox transformation is useful when the data violates the assumptions of constant variance or normality required by many statistical models. By transforming the data to meet these assumptions, it can improve the performance and interpretability of the subsequent analysis or modeling.

The general rule for **reversing** a Box-Cox transformation applied to a forecasted value y^{λ} is as follows:

$$y = \begin{cases} \frac{(\lambda * forecasted\ value + 1)^{\frac{1}{\lambda}}}{\lambda} & if\ \lambda \neq 0 \\ e^{forecasted\ value} & if\ \lambda = 0 \end{cases} \tag{7}$$

Where:

- y is the forecasted value in the original scale.
- λ is the lambda value used in the Box-Cox transformation.
- If λ = 0, the reverse transformation involves raising the forecasted value to the power of $\frac{1}{\lambda}$ then subtracting 1 and dividing by λ.
- If λ ≠ 0, the reverse transformation involves taking the exponential of the forecasted value.

This reverse transformation rule helps to bring the forecasted values back to the original scale after applying a Box-Cox transformation. It's important to use the correct lambda value and apply the appropriate transformation based on the lambda value being zero or non-zero.

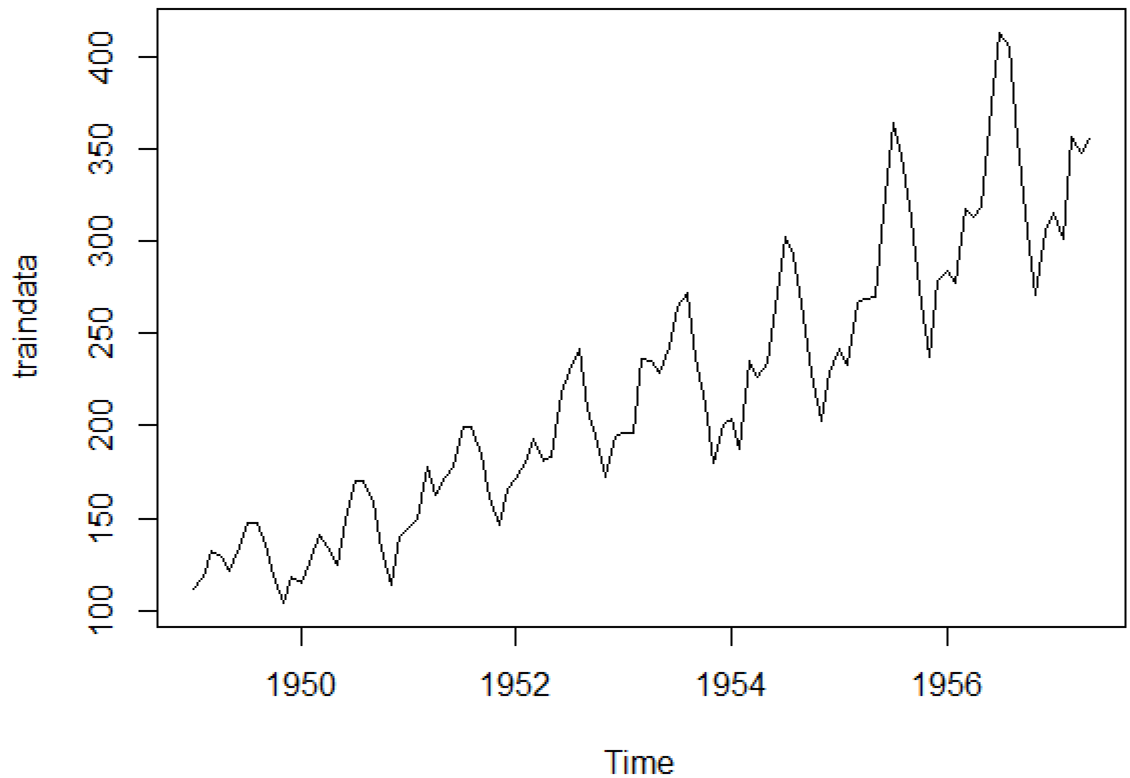


Figure 11: Box-Cox transformed Time plot for train data.

In Fig. 11, the representation of the box-cox transformed training data is observed. Notably, the visual depiction unveils an unmistakable upward trend present within the dataset.

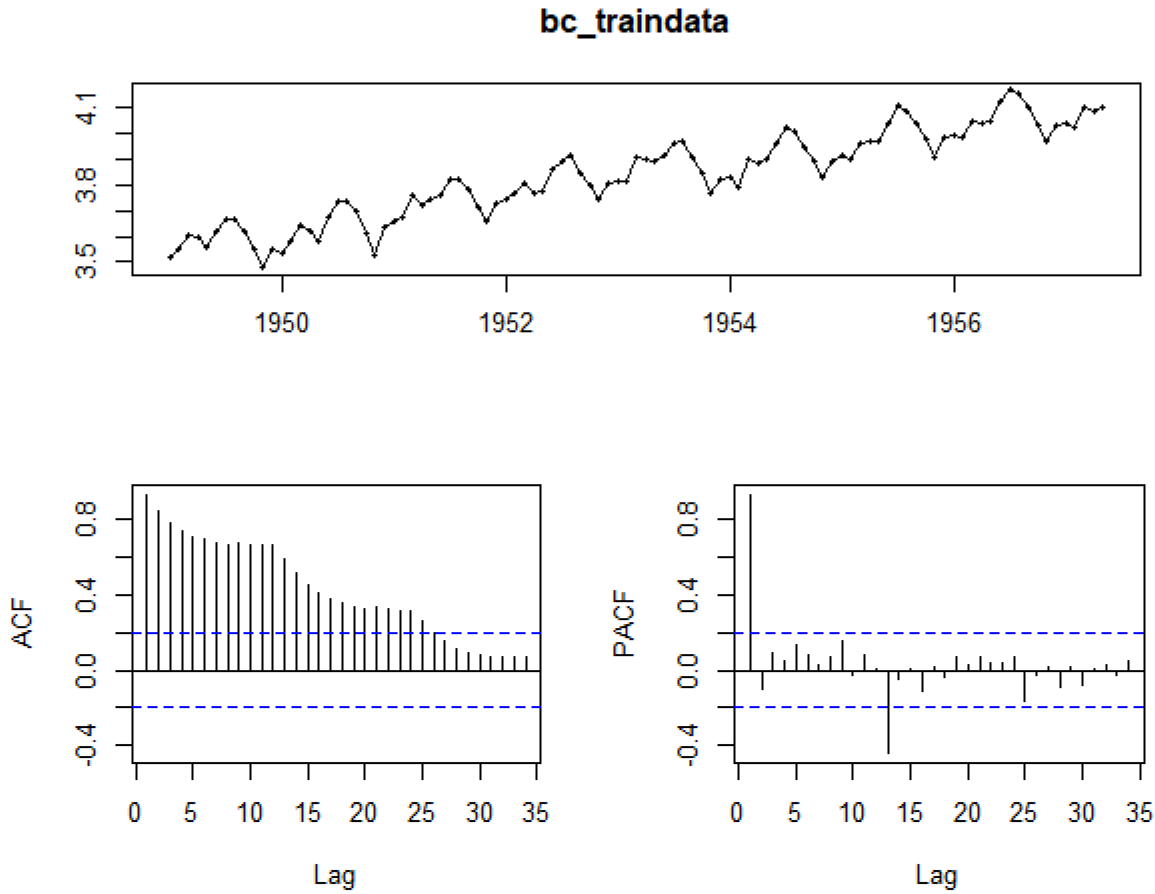


Figure 12: Time plot, ACF, and PACF plot for box-cox transformed train data.

In Fig. 12, the ACF and PACF plots are accompanied by the time plot for the box-cox transformed training data. It is observed from the figure that the data exhibit seasonality and trend, indicating non-stationarity. Consequently, to address these observed patterns, consideration may be given to applying either seasonal or non-seasonal differencing techniques to eliminate trend or seasonality, thereby rendering the data stationarity prior to fitting an ARIMA model.

.

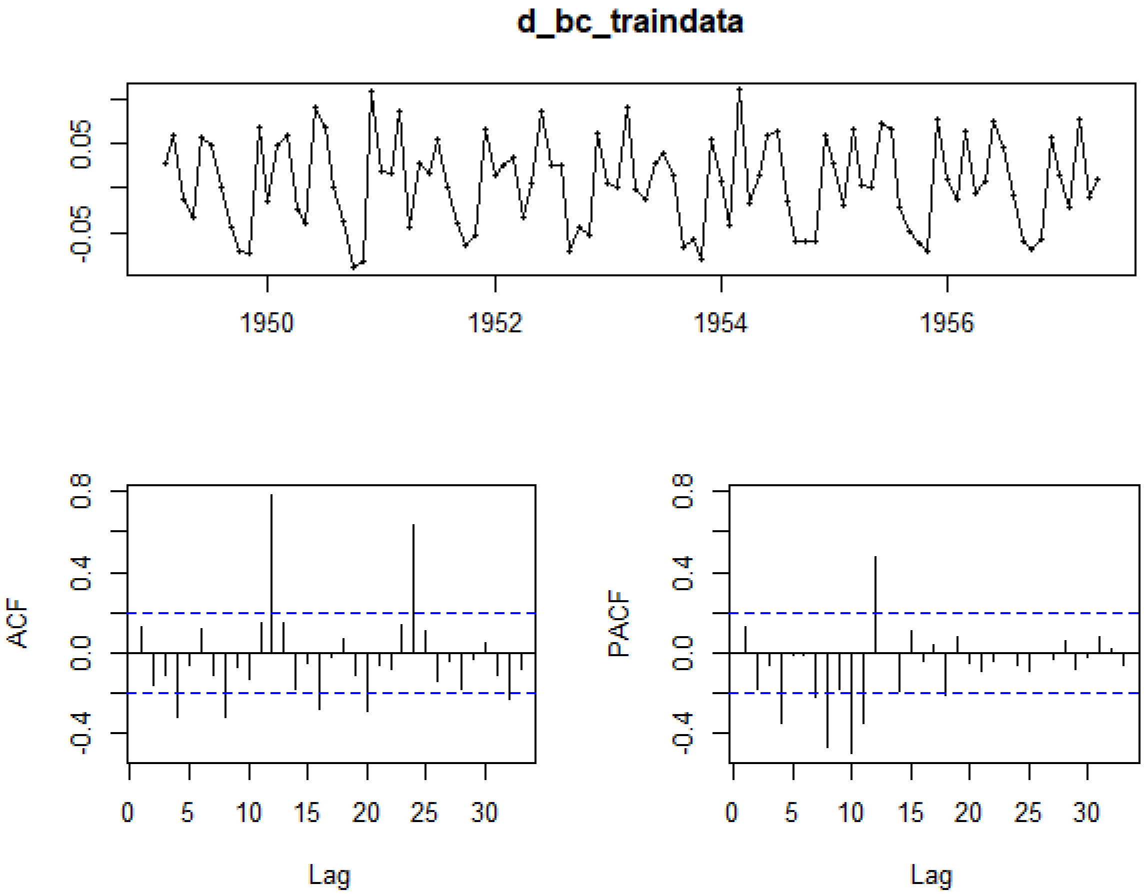


Figure 13: Time plot, ACF, and PACF plot for box-cox transformed train data with one seasonal difference.

ADF Test:

```
Augmented Dickey-Fuller Test

data:  bc_traindata
Dickey-Fuller = -4.7192, Lag order = 4, p-value = 0.01
alternative hypothesis: stationary
```

KPSS Test:

```
KPSS Test for Level Stationarity

data:  bc_traindata
KPSS Level = 1.9941, Truncation lag parameter = 4, p-value = 0.01
```

Model 3

From Fig. 13, it is evident that in the ACF and PACF plot, the first three spikes don't exceed the 95% threshold, the fourth spike surpasses the 95% threshold. Additionally, regarding seasonal characteristics, one spike in the PACF plot and two spikes in the ACF plot exceed the 95% threshold for a yearly (12-month) interval. Due to the possibility of information being lost, the suggested order of nonseasonal autoregressive (AR) and moving average component (MA) is 0. Consequently, based on these observations, the ARIMA model is suggested to be **ARIMA (2,1,1) (1,1,2) [12].**

```
Series: bc_traindata
ARIMA(0,1,0)(1,1,2)[12]

Coefficients:
         sar1    sma1     sma2
      -0.7042  0.0651  -0.4909
s.e.   1.2061  1.2081   0.7691

sigma^2 = 0.0004279:  log likelihood = 214.73
AIC=-421.47   AICc=-420.99   BIC=-411.56
```

From equation 1, the **mathematical expression** for the model 3 written as,

$$(1 + 0.7042\mathrm{B}^{12})(1 - \mathrm{B})^1(1 - \mathrm{B}^{12})^1 X_t = (1 + 0.0651\mathrm{B}^{12} - 0.4909\mathrm{B}^{24})$$
$$\boldsymbol{where}\ \{Z_t\} \sim WN(0, \sigma^2 = 0.0004279) \quad (8)$$

<u>Tests for the residual diagnostics checking:</u>

```
Null hypothesis: Residuals are iid noise.
Test                        Distribution Statistic   p-value
Ljung-Box Q                Q ~ chisq(20)     29.89    0.0717
McLeod-Li Q                Q ~ chisq(20)     25.69    0.1762
Turning points T     (T-66)/4.2 ~ N(0,1)        61    0.2338
Diff signs S         (S-50)/2.9 ~ N(0,1)        44    0.0396 *
Rank P           (P-2525)/170.4 ~ N(0,1)      2398    0.4561
```

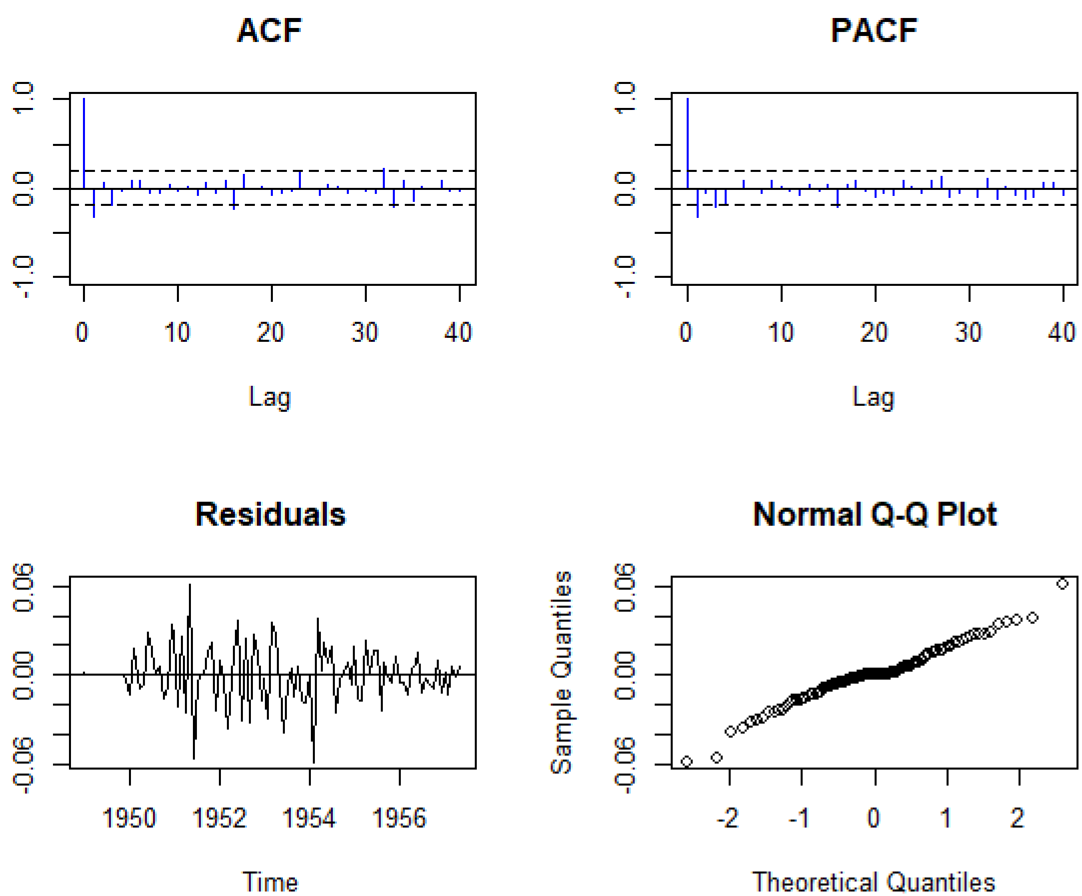


Figure 14: Residuals normality checking for model 3.

The q-q plot of the residuals for model 3 indicates a nearly normal distribution. Furthermore, based on the results of the statistical tests, it can be inferred that the residuals are independently and identically distributed (iid). Autocorrelation and partial autocorrelation are notably significant at a lag of 1, it suggests that model might be inadequately capturing the underlying patterns in the data.

Model 4: Auto ARIMA

The model is fitted using the auto.arima function from the forecast library in R.

```
Series: traindata
ARIMA(0,1,3)(1,1,0)[12]

Coefficients:
          ma1     ma2      ma3     sar1
      -0.2873  0.0780  -0.2652  -0.2036
s.e.   0.1038  0.1242   0.1116   0.1070

sigma^2 = 86.76:  log likelihood = -319.61
AIC=649.21   AICc=649.94   BIC=661.6
```

From equation 1, the **mathematical expression** for the model 4 written as,

$$(1+0.2036\mathrm{B}^{12})(1-\mathrm{B})^{1}(1-\mathrm{B}^{12})^{1}X_t = (1-0.2873\mathrm{B}^{1}+0.0780\mathrm{B}^{2}-\ 0.2652\mathrm{B}^{3})$$
$$\boldsymbol{where}\ \{Z_t\}\sim WN(0,\sigma^2=86.76) \quad (9)$$

Tests for the residual diagnostics checking:

```
Null hypothesis: Residuals are iid noise.
Test                        Distribution Statistic   p-value
Ljung-Box Q                  Q ~ chisq(20)     15.76    0.7315
McLeod-Li Q                  Q ~ chisq(20)     32.41    0.0391 *
Turning points T        (T-66)/4.2 ~ N(0,1)       65    0.8118
Diff signs S            (S-50)/2.9 ~ N(0,1)       49    0.7316
Rank P              (P-2525)/170.4 ~ N(0,1)     2329    0.2501
```

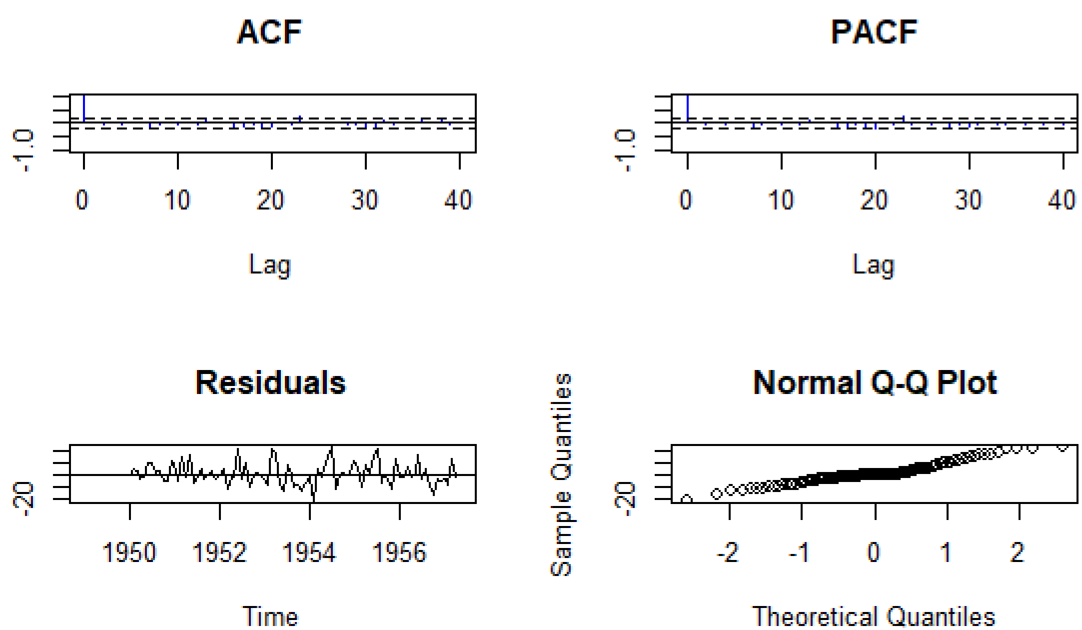


Figure 15: Residuals normality checking for model 4.

Model Evaluation

The selection of the best model among the four different ARIMA models is based on considering both forecasting error and information criteria.

Table I: *Comparison of Models Based on Information Criteria*

Model	AIC	AICc	BIC
M1	652.21	653.61	669.55
M2	-186.51	-185.49	-171.58
M3	-421.47	-420.99	-411.56
M4	649.21	649.94	661.6

The Table I presents the values of different information criteria (AIC, AICc, BIC) for four models (M1, M2, M3, M4). Models M2 and M3 are based on cubic root transformation and Box-Cox transformation, respectively.

To compare the models, lower values of AIC, AICc, and BIC indicate better model fit.

Observing the Table I, it's evident that Model 2 (M2), which is based on cubic root transformation, has considerably lower AIC, AICc, and BIC values compared to other models, indicating superior model fit. This suggests that the cubic root transformation is well-fitted with the dataset, leading to better forecasted results. Model 3 (M3), based on Box-Cox transformation, also shows lower AIC, AICc, and BIC values compared to M1 and M4, suggesting a relatively better fit than these models. However, between M2 and M3, M2 appears to have the best fit based on the information criteria.

Table II: *Forecasted Values using Four Different Models.*

Test data			Forecasts using the models			
Year	**Month**	**Number of Passenger**	**M1**	**M2**	**M3**	**M4**
1957	Jun	422	408.438	406.3424	410.7774	405.4472
1957	Jul	465	454.5084	452.1936	461.2436	446.9283
1957	Aug	467	442.8837	440.7139	454.4451	437.3362
1957	Sep	404	393.0226	392.6642	400.3686	390.3908
1957	Oct	347	346.0857	345.3688	348.2943	343.6308
1957	Nov	305	308.0693	305.8512	304.2891	308.2235
1957	Dec	336	346.3715	345.6836	349.3796	344.4453
1958	Jan	340	356.3876	354.8137	359.6842	352.8344
1958	Feb	318	341.0174	342.1236	349.617	340.2599
1958	Mar	362	395.3525	395.1437	411.8993	392.2053
1958	Apr	348	389.841	388.8638	403.247	385.0199
1958	May	363	397.8436	394.5107	407.6292	391.6126

1958	Jun	435	455.9673	451.1519	471.0296	443.1906
1958	Jul	491	505.3825	499.7434	530.7804	484.1664
1958	Aug	505	492.3528	487.533	522.5027	474.8985
1958	Sep	404	438.2197	435.2636	459.8993	427.3311
1958	Oct	359	388.1021	383.5899	399.3123	380.1149
1958	Nov	310	347.541	341.128	347.3309	344.7906
1958	Dec	337	388.2296	383.6054	400.213	380.7636
1959	Jan	360	399.2452	393.265	411.6313	389.2771
1959	Feb	342	381.5079	379.1941	400.1649	376.4123
1959	Mar	406	439.8053	436.543	472.3751	428.9797
1959	Apr	396	434.3331	429.3286	461.763	421.6284
1959	May	420	443.7564	435.4519	466.9933	428.3041
1959	Jun	472	506.2932	495.5913	542.624	479.6518
1959	Jul	548	559.2084	547.0458	612.2035	520.7305
1959	Aug	559	544.679	533.8741	602.7339	511.3966
1959	Sep	463	486.3399	478.1017	528.7241	463.9558
1959	Oct	407	433.0121	422.8575	457.6881	416.8326
1959	Nov	362	389.932	377.3391	397.6791	381.4913
1959	Dec	405	432.99	422.5522	459.04	417.515
1960	Jan	417	445.0111	432.6898	472.8725	426.0032
1960	Feb	391	424.9064	417.53	459.2509	413.1975
1960	Mar	419	487.1616	478.4182	544.1736	465.6383
1960	Apr	461	481.7271	470.5787	532.1454	458.3207
1960	May	472	492.5696	476.9167	538.1861	464.9795
1960	Jun	535	559.511	540.5483	625.8184	516.3741
1960	Jul	622	615.9202	594.8219	708.2126	557.4319
1960	Aug	606	599.8935	580.6988	696.8357	548.1114
1960	Sep	508	537.3562	521.4735	610.1848	500.6448
1960	Oct	461	480.8238	462.7052	526.9392	453.5026
1960	Nov	390	435.2294	414.1648	456.2004	418.1649
1960	Dec	432	480.6525	462.0555	528.2762	454.1782

Table II presents the forecasted values of four different models, all expressed in the original scale. Back-transformation is required for models M2 and M3 to obtain the data in the original scale. Equation 2 has been utilized for model M2, while equation 3 has been employed for model M3 to derive the forecasts in the original scale.

Table III: *Comparison of Models Based on Performance Metrics*

Model	MSE	RMSE	MAE	MPE	MAPE
M1	864.9422	29.4099	25.90495	-5.52267	6.483892
M2	602.6924	24.54979	21.05863	-3.77014	5.260441
M3	3420.827	58.48783	51.07934	-11.628	11.96692
M4	691.2021	26.29072	21.66817	-2.03255	5.190799

From Table III, the best model is typically determined by prioritizing the one with the lowest error metrics (MSE, RMSE, MAE, MPE, MAPE), as lower values indicate better performance. Among the provided metrics, Model 2 (M2) is observed to have the lowest MSE, RMSE, MAE, and MAPE compared to all other models. Although the MPE score for Model 2 is the second lowest, Model M2 still emerges as the best model based on all metrics.

Estimated Model 2 Expression

From equation 5, The forecasted value of the time series at time *t* for model 2 is expressed as

$$\boldsymbol{X_t} = 1.3113X_{t-1} + 0.3141X_{t-2} - X_{t-12} + 1.3113X_{t-13} - 0.3141X_{t-14} + Z_t - 0.6357Z_{t-1} - 0.5225Z_{t-12} + 0.3322Z_{t-13} + 0.0659Z_{t-24} - 0.04189Z_{t-25} \ , where\ \{Z_t\} \sim WN(0, \sigma^2 = 0.006235) \quad (10)$$

From equation 10, it can be summarized that,

- $X_{t-1}$and X_{t-13} have positive impacts on X_t.
- $X_{t-2}$and X_{t-14} have negative impacts on X_t.
- Lagged seasonal effects: X_{t-12} has a negative impact, while X_{t-24} has a positive impact on X_t
- Z_t represents the influence of the differenced series on X_t.
- Lagged differences: Z_{t-12}, and Z_{t-25} have negative impacts, while Z_{t-13} has a positive impact on X_t
- The model accounts for both non-seasonal and seasonal patterns in the time series and differenced series for forecasting.

M2 Visualization (Best Model)

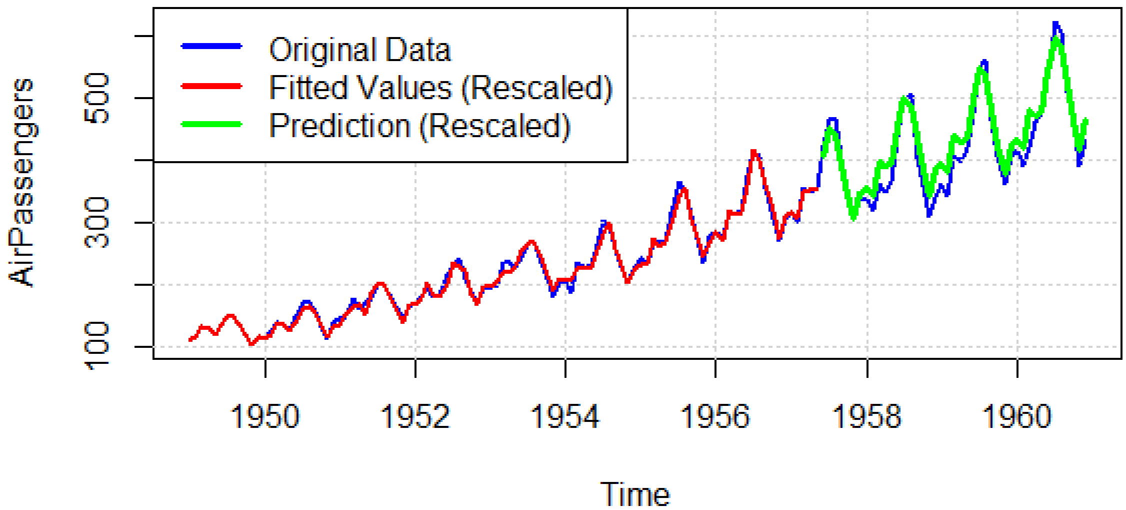


Figure 16: Outcomes of M2 (fitted values and predictions).

In Fig. 16, the original values, including both the training and testing data, are represented by blue lines. The fitted values for the training data are shown in red, while the forecasted values, corresponding to the length of the testing data, are depicted by green lines. It is observed that model 2 did not accurately predict the lower peak values evident in the test data.

Forecasting 10 points using M2

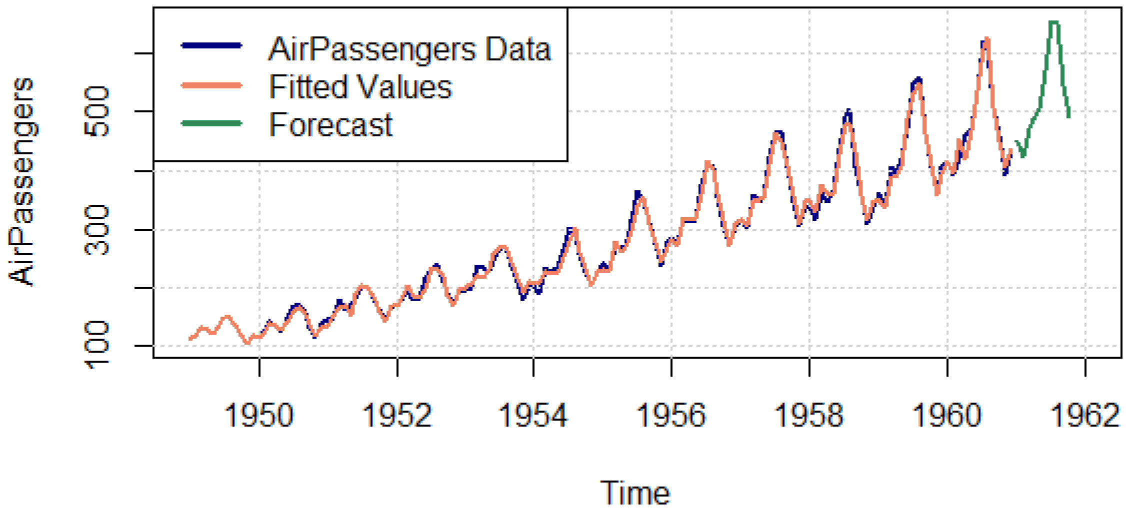


Figure 17: Time plot for original data, fitted values and forecast 10 points with M2.

In Fig. 17, the AirPassengers data is shown as a blue line, while the fitted values for Model 2 are displayed as a red line, and the 10-point forecast is represented by a green line. It's evident that Model 2 is accurately fitted to the original data.