Module
2
Random Processes

Lesson

6

Functions of Random Variables

After reading this lesson, you will learn about

- ***cdf of function of a random variable.***
- ***Formula for determining the pdf of a random variable.***

Let, X be a random variable and $g(a)$ is a function of a real variable a. Then, the expression $y = g(x)$ leads to a new random variable Y with the following connotation:

Let 's' indicate an outcome of a random experiment, as introduced earlier in Lesson #5. For a given 's', x(s) is a real number and $g[x(s)]$ is another real number specified in terms of x(s) and g(a). This new number is the value $y(s) = g[x(s)]$, which is assigned to the random variable Y. In brief, Y = g (X) indicates this functional relationship between the random variables X and Y.

The cdf $F_y(b)$ of the new random variable Y, so formed, is the probability of the event $\{y \le b\}$, consisting of all outcomes 's' such that $y(s) = g[x(s)] \le b$.
This means,

$$F_y(b) = P\{y \le b\} = P\{g(s) \le b \qquad 2.6.1$$

For a specific b, there may be multiple values of 'a' for which $g(a) \le b$. Let us assume that all these values of 'a' for which $g(a) \le b$, form a set on the a-axis and let us denote this set as I_y. This set is known as the point set.

So, $g[x(s)] \le b$ if x(s) is a number in the set I_y, i.e. $F_y(b) = P\{x \in Iy\}$ 2.6.2

Now, g(a) must have the following properties so that g(x) is a random variable :

a) The domain of g(a) must include the range of the random variable X.
b) For every b such that $g(a) \le b$, the set I_y must consist of the union and intersection of a countable number of intervals since then only $\{y \le b\}$ is an event.
c) The events $\{g(x) = \pm\infty\}$ must have zero probability.

Cumulative Distribution Function [cdf] of g(x)

We wish to express the cdf $F_y(b)$ of the new random variable Y where y = g(x) in term of the cdf $F_x(a)$ of the random variable X and the function g(a). To do this, we determine the set I_y on the a-axis so that $g(a) \le b$ and also the probability that the random variable X is in this set.

Let us assume that $F_x(a)$ is continuous and consider a few examples to illustrate the point.

Example #2.6.1

Let, y = g(x) = c.x + d, where c and d are constants [This is an equation of a straight line].

To find $F_y(b)$, we have to find the values of 'a' such that, c.a + d ≤ b.

$$For\ c > 0:\ ca + d \le b\ means\ \ a \le \frac{b-d}{c}$$

So, $F_y(b) = P\left\{x \le \frac{b-d}{c}\right\} = F_x\left(\frac{b-d}{c}\right)$

While, for c < 0, $ca + d \le b\ means\ a \ge \frac{b-d}{c}$ and so

$$F_y(b) = P\left\{x \ge \frac{b-d}{c}\right\} = 1 - F_x\left(\frac{b-d}{c}\right)$$

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Example #2.6.2

Let, y = g(x) = x^2

It is easy to see that, for b < 0, $F_y(b) = 0$

However, for b ≥ 0 $a^2 \le b$ for $-\sqrt{b} \le a \le \sqrt{b}$ and hence,

$$F_y(b) = P\left\{-\sqrt{b} \le x \le \sqrt{b}\right\} \quad = F_x\left(\sqrt{b}\right) - F_x\left(\sqrt{b}\right)$$

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Example #2.6.3

Let us consider the following function g(a):

$$g(a) = \begin{Bmatrix} a + c, a < -c \\ 0, -c \le a \le c \\ a - c, a > c \end{Bmatrix}$$

It is a good idea to sketch g(a) versus 'a' to gain a closer look at the function.

Note that, $F_y(b)$ is discontinuous at $b = g(a) = 0$ by the amount $F_x(c) - F_x(-c)$

Further,

$$for\ b \ge 0, \quad P\{y \le b\} = P\{x \le b + c\} = F_x(b + c)$$

$$\&\ for\ b < 0, \quad P\{y \le b\} = P\{x \le b - c\} = F_x(b - c)$$

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Example #2.6.4

While we will discuss more about linear and non-linear quantizers in the next Module, let us consider the simple transfer characteristics of a linear quantizer here:

Let, $g(a) = n.s, (n - a)s < a \le ns$ where 's' is a constant, indicating a fixed step size and 'n' is an integer, representing the n-th quantization level.

Then for $y = g(x)$, the random variable Y takes values

$$b_n = ns\ with$$

$$P\{y = ns\} = P\{(n-1)s < x \le ns\} = F_x(ns) - F_x(ns - s)$$

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Example #2.6.5

Let, $g(a) = \begin{cases} a+c, a \geq 0 \\ a-c, a < 0 \end{cases}$, where 'c' is a constant. Plot g(a) versus 'a' and see that g(a) is discontinuous at a = 0, with $g(0^-) = -c$ and $g(0^+) = +c$. This implies that, $F_Y(b) = F_X(0)$, for $|b| \leq c$.

Further, for $b \geq c,\ g(a) \leq b \quad for\ a \leq b-c; \quad hence,\ F_y(b) = F_x(b-c)$

$-c \leq b \leq c,\ g(a) \leq b \quad for\ a \leq c; \quad hence,\ F_y(b) = F_x(0)$

$b \leq -c,\ g(a) \leq b \quad for\ a \leq b+c; \quad hence,\ F_y(b) = F_x(b+c)$

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An important step while dealing with functions of random variables is to find the point set I_y and thereby the cdf $F_Y(Y)$ when the functions g(x) and $F_X(X)$ are known. In terms of probability, it is equivalent to finding the values of the random variable X such that, $F_Y(y) = P\{Y \leq y\} = P\{X \in I_y\}$. We now briefly discuss about a concise and convenient relationship for determination of the pdf of Y, i.e $f_Y(Y)$.

Formula for determining the pdf of Y, i.e., $f_Y(Y)$:

Let, X be a continuous random variable with pdf $f_x(X)$ and g(x) be a differentiable function of x. $[i.e.\ g'(x) \neq 0]$. We wish to establish a general expression for the pdf of Y = g(X).

Note that, an event $\{y < Y \leq y + dy\}$ can be written as a union of several disjoint elementary events $\{E_i\}$.

Let, the equation $y = g(x)$ have n real roots $x_1, x_2, \ldots\ldots, x_n$,

i.e. $y - g(x_i) = 0$, for i = 1, 2, … n.

Then, the disjoint events are of the forms:

$$E_i = \{x_i - |dx_i| < X < x_i\}, \quad if\ g'(x_i)\ is - ve$$

$$or\ E_i = \{x_i < X < x_i + |dx_i|\}, \quad if\ g'(x_i)\ is + ve$$

In either case, we can write (following the basic definition of pdf), that,

Pr. of an event = (pdf at x = xi).$|dx_i|$

So, for the above disjoint events $\{E_i\}$, we may, approximately write,

$P\{E_i\} =$ Probability of event $E_i = f_X(x_i)|dx_i|$

As we have considered the events E_i – s disjoint, we may now write that,

Prob.{ $y < Y \leq (y + dy)$ } = $f_Y(y).\ |dy|$

$= f_x(x_1).|dx_1| + f_x(x_2).|dx_2| + \ldots\ldots\ldots.+ f_x(x_n).|dx_n|$

$= \sum_{i=1}^{n} f_X(x_i).\ |dx_i|$

The above expression can equivalently be written as,

$$f_Y(y) = \sum_{i=1}^{n} f_X(x_i).\left|\frac{dx_i}{dy}\right|$$

$$= \sum_{i=1}^{n} f_X(x_i).\left|\frac{dy}{dx_i}\right|^{-1}$$

Let us note that, at the i-th root of y = g(x), $\frac{dy}{dx_i} = g^{'}(x_i).$ = value of the derivative of g(x) with respect to 'x' , evaluated at x = x_i.

Using the above convenient notation, we finally get,

$$f_Y(y) = \sum_{i=1}^{n} f_X(x_i) / \left|g^{'}(x_i)\right|, \qquad 2.6.3$$

Here, x_i is the i-th real root of y = g(x) and $g^{'}(x_i) \neq 0.$ If, for a given y, y = g(x) has no real root, then $f_Y(y) = 0$ as X being a random variable and 'x' being real, it can not take imaginary values with non-zero probability.

Let us take up a small example before concluding this lesson.

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Example #2.6.6

Let X be a random variable known to follow uniform distribution between -π and +π. So, the mean of X is 0 and its probability density function [pdf] is:

$$f_X(x) = \begin{cases} \frac{1}{2\pi}, -\pi < x \le \pi \\ 0, otherwise \end{cases}$$

Now consider a new random variable Y which is a function of X and the functional relationship is, Y = g(X) = sinX.

So, we can write, y = g(x) = sin x . Further, one can easily observe that, the pdf of Y exists for -1.0 ≤ y <1.0.

Let us first consider the interval 0 ≤ y < 1.0:

The roots of y – sin x = 0 for y >0 are, $x_1 = \sin^{-1}(y)$ and $x_2 = \pi - \sin^{-1}(y)$.

$$Further, \frac{dg(x)}{dx} = \cos x \quad while$$

$$\left.\frac{dg(x)}{dx}\right|_{x=x_1} = \cos\left(\sin^{-1} y\right) \quad and$$

$$\left.\frac{dg(x)}{dx}\right|_{x=x_2} = \cos\left(\pi - \sin^{-1} y\right)$$

$$= \cos\pi.\cos\left(\sin^{-1} y\right) + \sin\pi.\sin\left(\sin^{-1} y\right) \;= -\cos\left(\sin^{-1} y\right)$$

We see that,

$$\left|\frac{dg(x)}{dx}\right|_{x_1} \mp \left|\frac{dg(x)}{dx}\right|_{x_2} = \sqrt{1-y^2}$$

$$\therefore f_Y(y) = \frac{f_X(x_1)}{\left|g^{'}(x_1)\right|} + \frac{f_X(x_2)}{\left|g^{'}(x_2)\right|}$$

$$= \frac{f_X(\sin^{-1} y)}{\sqrt{1-y^2}} + \frac{f_X(\pi - \sin^{-1} y)}{\sqrt{1-y^2}}$$

$$= \frac{1}{2\pi} \times \frac{2}{\sqrt{1-y^2}} = \frac{1}{\pi}.\frac{1}{\sqrt{1-y^2}}, \qquad 0 \le y < 1$$

Following similar procedure for the range $-1 \le y < 0$, it can ultimately be shown that,

$$f_Y(y) = \begin{cases} \dfrac{1}{\pi}.\dfrac{1}{\sqrt{1-y^2}}, |y| < 1 \\ 0, \quad otherwise \end{cases}$$

Problems

Q2.6.1) Let, $y=2x^2 + 3x+1$. If pdf is x is $f_X(x)$, determine an expression for pdf of y.

Q2.6.2) Sketch the pdf of y of problem 2.6.1, if X has u form distribution between -1 and +1.