Module 2

Random Processes

Lesson

8

Stochastic Processes

After reading this lesson, you will learn about

- ***Stochastic Processes***
- ***Statistical Average***
- ***Wide sense stationary process***
- ***Complex valued stochastic process***
- ***Power Density Spectrum of a stochastic process***

- Many natural and man-made phenomena are random and are functions of time. e.g., i) Fluctuation of air temperature or pressure, ii) Thermal noise that are generated due to Brownian motion of carriers in conductors and semiconductors and iii) Speech signal.

- All these are examples of Stochastic Processes.

- An instantaneous sample value of a stochastic process is a random variable.

- A stochastic process, some time denoted by X(t), may be defined as an ensemble of (time) sample functions.

  Ex: Consider several identical sources each of which is generating thermal noise.

- Let us define $X_{ti}=X(t=t_i)$, i=1,2,….., n (n is arbitrarily chosen) as the random variables obtained by sampling the stochastic process X(t) for any set of time instants $t_1<t_2<t_3 \ldots <t_n$. So, these 'n' random variables can be characterized by their joint pdf, i.e. $p(x_{t1}, x_{t2}, x_{t3}, \ldots, x_{tn})$.

- Now, let us consider another set of random variables $X(t_i+\tau)$, generated from same stochastic process X(t) with an arbitrary time shift $'\tau'$

  $$X_{t_1+\tau} = X(t=t_1+\tau), \ldots\ldots X_{t_i+\tau} = X(t=t_i+\tau) \ldots\ldots$$

  and with a joint pdf $p(x_{t_1+\tau}, x_{t_2+\tau}, \ldots\ldots, x_{t_n+\tau})$.

  In general, these two joint pdf-s may not be same. However, if they are same for all τ and any 'n', then the stochastic process X(t) is said to be stationary in the strict sense. i.e.,

  $$p(x_{t1}, x_{t2}, \ldots\ldots, x_{tn}) = p(x_{t_1+\tau}, x_{t_2+\tau}, \ldots\ldots, x_{t_n+\tau}) \text{ for all } '\tau' \text{ and all 'n'}.$$

  Strict sense stationarity means that the statistics of a stochastic process is invariant to any translation of the time axis. Finding a quick example of a physical random process which is stationary in the strict sense may not be an easy task.

- If the joint pdf's are different for any $'\tau'$ or 'n' the stochastic process is non-stationary.

Statistical Averages

- Like random variables, we can define statistical averages or ensemble averages for a stochastic process.

  Let X(t) be a random process and $X_{ti} = X(t = t_i)$
  Then, the n-th moment of X_{ti} is:

  $$E[X_{ti}^{\ n}] = \int_{-\infty}^{+\infty} x_{ti}^{\ n}.p(x_{ti})dx_{ti} \qquad 2.8.1$$

- For a stationary process, $p(x_{ti+\tau}) = p(x_{ti})$ for all τ and hence the n-th moment is independent of time.

- Let us consider two random variables $X_{ti} = X(t_i), i = 1, 2$
  The correlation between X_{t1} & X_{t2} is measured by their joint moment:

  $$E[X_{t1}X_{t2}] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_{t1}.x_{t2}\, p(x_{t1}, x_{t2})dx_{t1}dx_{t2} \qquad 2.8.2$$

  This joint moment is also known as the autocorrelation function, $\Phi(t_1,t_2)$ of the stochastic process X(t). In general, $\Phi(t_1,t_2)$ is dependent on t_1 and t_2.

- However, for a strict-sense stationary process,

  $$p(x_{t1},x_{t2}) = p(x_{t1}+T, x_{t2}+T),$$

  For any T and this means that autocorrelation function $\Phi(t_1,t_2)$ for a stationary process is dependent on $(t_2 - t_1)$, rather than on t_2 and t_1,
  i.e.,

  $$E[X_{t1}X_{t2}] = \Phi(t_1, t_2) = \Phi(t_2 - t_1) = \Phi(\tau), \quad \text{say where } \tau = t_2 - t_1 \qquad 2.8.3$$

  It may be seen that, $\Phi(-\tau) = \Phi(\tau)$ and hence, $\Phi(\tau)$ is an even function.

- Note that hardly any physical stochastic process can be described as stationary in the strict sense, while many of such processes obey the following conditions:
  a) Mean value, i.e., E[X] of the process is independent of time and,
  b) $\Phi(t_1,t_2) = \Phi(t_1 - t_2)$ over the time domain of interest.

  Such stochastic processes are known as *wide sense stationary*. This is a less stringent condition on stationarity.

- The auto-covariance function of a stochastic process is closely related to the autocorrelation function:

  $$\begin{aligned}\mu(t_1,t_2) &= E\{[X_{t1} - m(t_1)][X_{t2} - m(t_2)]\} \\ &= \Phi(t_1 - t_2) - m(t_1).m(t_2)\end{aligned} \qquad 2.8.4$$

  Here, $m(t_1) = E[\ X_{t1}]$. For a wide sense stationary process,

$\mu(t_1 - t_2) = \mu(t_2 - t_1) = \mu(\tau) = \Phi(\tau) - m^2$ 2.8.5

- It is interesting to note that a Gaussian process is completely specified by its mean and auto covariance functions and hence, if a Gaussian process is wide-sense stationary, it is also strict-sense stationary.

- We will mean wide-sense stationary process when we discuss about a stationary stochastic process.

Averages for joint stochastic processes

Let X(t) & Y(t) denote any two stochastic processes and $X_{ti} = X(t_i)$, i=1,2,….n and $Y_{t'_j} = Y(t'_j)$, j=1,2,…………,m are the random variables at $t_1 > t_2 > t_3 \ldots > t_n$ and $t'_1 > t'_2 > \ldots\ldots > t'_m$ [n and m are arbitrary integers].

The two processes are statistically described by their joint pdf:
$p(x_{t1}, x_{t2}, \ldots\ldots x_{tn}, yt'_1, yt'_2, \ldots\ldots yt'_m)$

Now, the cross-correlation function of X(t) and Y(t) is defined as their joint moment:

$$\Phi_{xy}(t_1, t_2) = E(X_{t1}Y_{t2}) = \int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} x_{t1}y_{t2}\Phi(x_{t1}, y_{t2})dx_{t1}dy_{t2} \qquad 2.8.6$$

The cross-covariance is defined as,

$$\mu_{xy}(t_1, t_2) = \Phi_{xy}(t_1, t_2) - m_x(t_1).m_y(t_2) \qquad 2.8.7$$

Here, $m_x(t_1)$ = E[X_{t1}] and $m_y(t_1)$ = E[Y_{t2}]. If X and Y are individually and jointly stationary, we have,

$$\Phi_{xy}(t_1, t_2) = \Phi_{xy}(t_1 - t_2)$$

& 2.8.8

$$\mu_{xy}(t_1, t_2) = \mu_{xy}(t_1 - t_2)$$

In particular, $\Phi_{xy}(-\tau) = E(X_{t1}Y_{t1+\tau}) = E(X_{t'_1-\tau}.Y_{t'_1}) = \Phi_{yx}(\tau)$

Now, two stochastic processes are said to be statistically independent if and only if,

$$p(x_{t1}, x_{t2}, \ldots\ldots x_{tn}, y_{t'1}, y_{t'2} \ldots\ldots, y_{t'm})$$
$$= p(x_{t1}, x_{t2}, \ldots\ldots t_n).p(y_{t'1}, y_{t'2} \ldots\ldots, y_{t'm}) \qquad 2.8.9$$

for all t_i, t'_j, n and m

Two stochastic processes are said to be uncorrelated if,

$\Phi_{xy}(t_1, t_2) = E[X_{t1}].E[Y_{t2}]$. This implies, $\mu_{xy}(t_1, t_2) = 0$ 2.8.10

i.e., *the processes have zero cross-covariance.*

Complex valued stochastic process

Let, Z(t) = X(t) + jY(t), where X(t) and Y(t) are individual stochastic processes.
One can now define joint pdf of Z_{ti} = Z(t =i), i=1,2,…………,n, as
$p(x_{t1}, x_{t2}, \ldots\ldots x_{tn}, y_{t1}, y_{t2}, \ldots\ldots y_{tn})$

Now the auto correlation function $\tilde{\Phi}_{zz}(t_1, t_2)$ of the complex process z(t) is defined as,

$$\tilde{\Phi}_{zz}(t_1, t_2) = \frac{1}{2}.E[Z_{t1}.Z^*_{t2}] = \frac{1}{2}.E[(x_{t1} + jy_{t1})(x_{t2} + jy_{t2})]$$

$$= \frac{1}{2}.[\Phi_{xx}(t_1, t_2) + \Phi_{yy}(t_1, t_2) + j\{\Phi_{yx}(t_1, t_2) - \Phi_{xy}(t_1, t_2)\}] \qquad 2.8.11$$

Here, $\Phi_{xx}(t_1, t_2)$: Auto-correlation of X;
$\Phi_{yy}(t_1, t_2)$: Auto-correlation of Y;
$\Phi_{yx}(t_1, t_2)$ and $\Phi_{xy}(t_1, t_2)$ are cross correlations of X&Y.

For a stationary $\tilde{z}(t)$, $\Phi_{zz}(t_1, t_2) = \Phi_{zz}(\tau)$ and $\Phi^*_{zz}(t_1, t_2) = \Phi^*_{zz}(\tau) = \Phi_{zz}(-\tau)$.
2.8.12

Power Density Spectrum of a Stochastic Process

A stationary stochastic process is an infinite energy signal and hence its Fourier Transform does not exist. The spectral characteristic of a stochastic process is obtained by computing the Fourier Transform of the auto correlation function.

That is, the distribution of power with frequency is described as:

$$\Phi(f) = \int_{-\alpha}^{\alpha} \Phi(\tau) e^{-j2\pi f\tau} d\tau \qquad 2.8.13$$

The Inverse Fourier Transform relationship is:

$$\Phi(\tau) = \int_{-\infty}^{\infty} \Phi(f) e^{j2\pi f\tau} df \qquad 2.8.14$$

Note that, $\Phi(0) = \int_{-\alpha}^{\alpha} \Phi(f) df = E(X_1^2) \geq 0$

$\therefore \Phi(0)$ represents the average power of the stochastic signal, which is the area under the $\Phi(f)$ curve.
Hence, $\Phi(f)$ is called the power density spectrum of the stochastic process.

If X(t) is real, then $\Phi(\tau)$ is real and even.
Hence $\Phi(f)$ is real and even.

Problems

Q2.8.1) Define and explain with an example a “wide sense stationary stochastic process”.

Q2.8.2) What is the condition for two stochastic processes to be uncorrelated?

Q2.8.3) How to verify whether two stochastic processes are statistically independent of each other?