Module
5
Carrier Modulation

Lesson 27

Performance of BPSK and QPSK in AWGN Channel

After reading this lesson, you will learn about

- ***Bit Error Rate (BER) calculation for BPSK;***
- ***Error Performance of coherent QPSK;***
- ***Approx BER for QPSK;***
- ***Performance Requirements;***

We introduced the principles of Maximum Likelihood Decision and the basic concepts of correlation receiver structure for AWGN channel in Lesson #19, Module #4. During the discussion, we also derived a general expression for the related likelihood function for use in the design of a receiver. The concept of likelihood function plays a key role in the assessment of error performance of correlation receivers. For ease of reference, we reproduce the expression (Eq.4.19.14) below with usual meaning of symbols and notations:

$$f_{\bar{r}}\left(\bar{r}\middle|m_i\right) = \left(\pi N_0\right)^{-\frac{N}{2}}.\exp\left[-\frac{1}{N_0}\sum_{j=1}^{N}\left(r_j - s_{ij}\right)^2\right] \quad i = 1,2,\ldots,M. \qquad 5.27.1$$

Bit Error Rate (BER) calculation for BPSK

We consider ordinary BPSK modulation with optimum demodulation by a correlation receiver as discussed in Lesson #24. **Fig. 5.27.1** shows the familiar signal constellation for the scheme including an arbitrarily chosen received signal point, denoted by vector $\bar{r}$.

The two time limited signals are $s_1(t) = \sqrt{E_b}.\varphi_1(t)$ and $s_2(t) = -\sqrt{E_b}.\varphi_1(t)$, while the basis function, as assumed earlier in Lesson #24 is $\varphi_1(t) = \sqrt{\frac{2}{T_b}}.\cos 2\pi f_c t$; $0 \le t < T_b$. Further, $s_{11} = \sqrt{E_b}$ and $s_{21} = -\sqrt{E_b}$. The two signals are shown in **Fig.5.27.1** as two points $\bar{s}_1$ and $\bar{s}_2$. The discontinuous vertical line dividing the signal space in two halves identifies the two decision zones Z_1 and Z_2. Further, the received vector $\bar{r}$ is the vector sum of a signal point ($\bar{s}_1$ or $\bar{s}_2$) and a noise vector (say, $\bar{w}$) as, in the time domain, r(t) = s(t) + n(t). Upon receiving $\bar{r}$, an optimum receiver makes the best decision about whether the corresponding transmitted signal was $s_1(t)$ or $s_2(t)$.

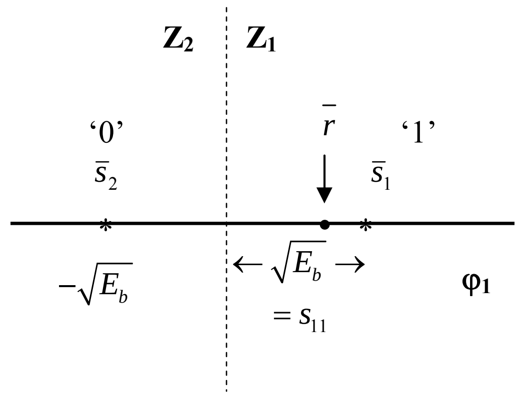

Fig.5.27.1 *Signal constellation for BPSK showing an arbitrary received vector* $\bar{r}$

Now, with reference to **Fig. 5.27.1**, we observe that, an error occurs if

a) $s_1(t)$ is transmitted while $\bar{r}$ is in Z_2 or
b) $s_2(t)$ is transmitted while $\bar{r}$ is in Z_1.

Further, if 'r' denotes the output of the correlator of the BPSK demodulator, we know the decision zone in which $\bar{r}$ lies from the following criteria:

a) $\bar{r}$ lies in Z_1 if $r = \int_0^{T_b} r(t)\varphi_1(t)dt > 0$
b) $\bar{r}$ lies in Z_2 if $r \le 0$

Now, from **Eq. 5.27.1**, we can construct an expression for a Likelihood Function:

$f_r\left(\bar{r}\middle| s_2(t)\right) = f_r(r(t)$/message '0' was transmitted)

From our previous discussion,

$$f_r\left(\bar{r}\middle| s_2(t)\right) = f_r\left(\bar{r}\middle|'0'\right)$$

$$= \frac{1}{\sqrt{\pi N_0}}.\exp\left[-\frac{1}{N_0}\left(r - s_{21}\right)^2\right]$$

$$= \frac{1}{\sqrt{\pi N_0}}.\exp\left\{\frac{-\left[r - \left(-\sqrt{E_b}\right)\right]^2}{N_0}\right\}$$

$$= \frac{1}{\sqrt{\pi N_0}}.\exp\left[-\frac{1}{N_0}\left(r + \sqrt{E_b}\right)^2\right] \qquad 5.27.2$$

$\therefore$ The conditional Probability that the receiver decides in favour of '1' while '0' was transmitted $= \int_0^\infty f_r\left(r|0\right)dr = P_e\left(0\right)$, say.

$$\therefore P_e\left(0\right) = \frac{1}{\sqrt{\pi N_0}} \int_0^\infty \exp\left[-\frac{1}{N_0}\left(r+\sqrt{E_b}\right)^2\right]dr \qquad 5.27.3$$

Now, putting $\frac{1}{\sqrt{N_0}}\left(r+\sqrt{E_b}\right) = Z$, we get,

$$P_e\left(0\right) = \frac{1}{\sqrt{\pi}} \int_{\sqrt{E_b/N_0}}^{\infty} \exp\left(-Z^2\right)dz$$

$$= \frac{1}{2}\cdot\frac{2}{\sqrt{\pi}} \int_{\sqrt{\frac{E_b}{N_0}}}^{\infty} e^{-z^2}dz$$

$$= \frac{1}{2}.erfc\left(\sqrt{\frac{E_b}{N_o}}\right) = Q\left(\sqrt{\frac{2E_b}{N_o}}\right), \qquad \left[\because erf_c(u) = 2Q\left(\sqrt{2}\,u\right)\right] \qquad 5.27.4$$

Fig. 5.27.2 shows the profiles for the error function erf(x) and the complementary error function erfc(x)

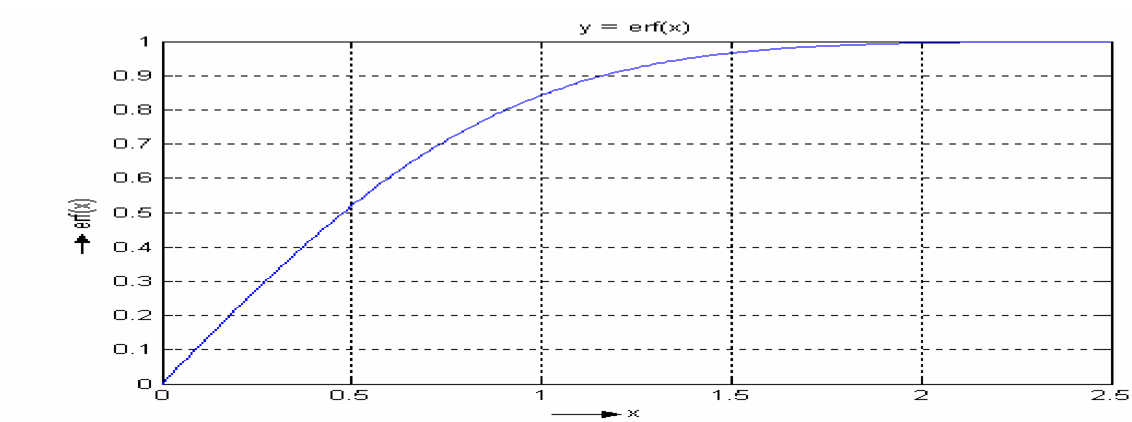

Fig.5.27.2(a) *The 'error function' y = erf(x)*

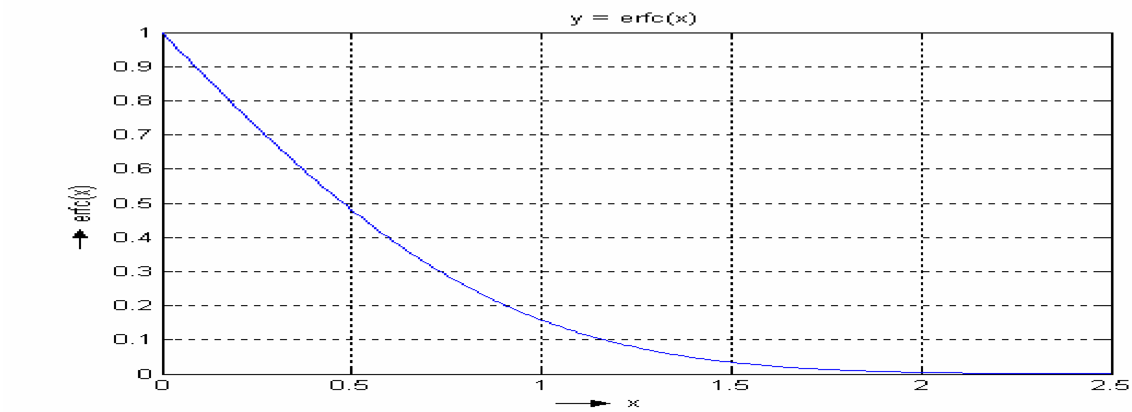

Fig.5.27.2(b) *The 'complementary error function' y = erfc(x)*

Following a similar approach as above, we can determine the probability of error when '1' is transmitted from the modulator, i.e. $P_e(1)$ as,

$$P_e(1) = \frac{1}{2}.erfc\left(\sqrt{\frac{E_b}{N_o}}\right) \qquad 5.27.5$$

Now, as we have assumed earlier, the '0'-s and '1'-s are equally likely to occur at the input of the modulator and hence, the average probability of a received bit being decided erroneously (P_e) is,

$$P_e = \frac{1}{2}.P_e(0) + \frac{1}{2}P_e(1) = \frac{1}{2}.erfc\left(\sqrt{\frac{E_b}{N_o}}\right) \qquad 5.27.6$$

We can easily recognize that P_e is the BER, or equivalently the SER(Symbol error rate) for the optimum BPSK modulator. This is the best possible error performance any BPSK modulator-demodulator can achieve in presence of AWGN. **Fig. 5.27.3** depicts the above relationship. This figure demands some attention as it is often used as a benchmark for comparing error performance of other carrier modulation schemes. Careful observation reveals that about 9.6 dB of E_b/N_o is necessary to achieve a BER of 10^{-05} while an E_b/N_o of 8.4 dB implies an achievable BER of 10^{-04}.

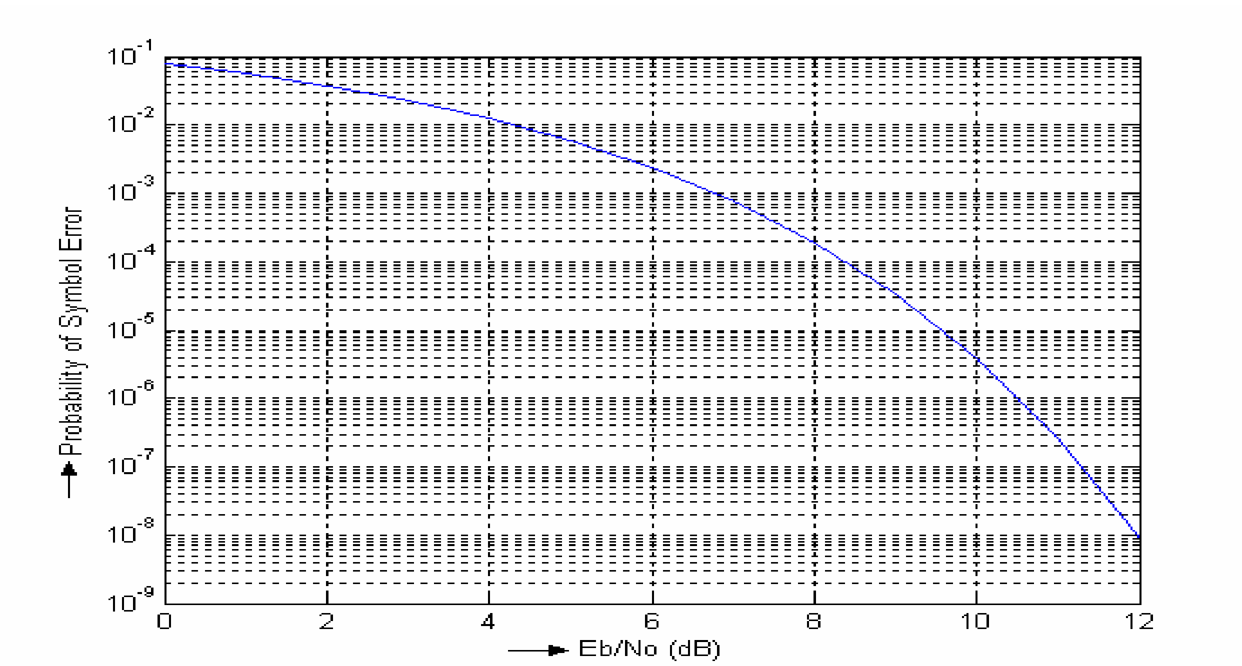

Fig.5.27.3 *Optimum error performance of a Maximum Likelihood BPSK demodulator in presence of AWGN*

Error Performance of coherent QPSK

Fig.5.27.4, drawn following an earlier **Fig.5.25.1**, shows the QPSK signal constellation along with the four decision zones. As we have noted earlier, all the four signal points are equidistant from the origin. The dotted lines divide the signal space in four segments.

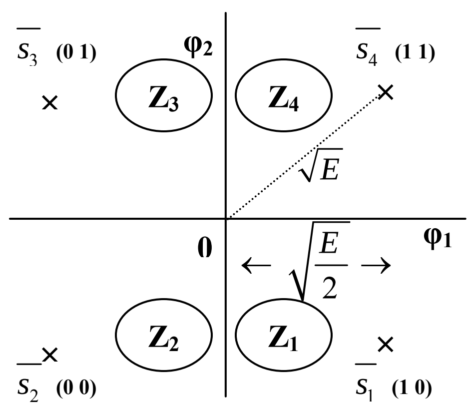

Fig.5.27.4 *QPSK signal constellation showing the four-decision zone*

To recapitulate, a time-limited QPSK modulated signal is expressed as,

$$s_i(t) = \sqrt{\frac{2E}{T}}.\cos\left[(2i-1)\pi/4\right]\cos w_c t - \sqrt{\frac{2E}{T}}.\sin\left[(2i-1)\frac{\pi}{4}\right]\sin w_c t\,,\ 1 \le i \le 4 \qquad 5.27.7$$

The corresponding signal at the input of a QPSK receiver is r(t) = $s_i(t)$ + w(t), $0 \le t \le T$, where 'w(t)' is the noise sample function and 'T' is the duration of one symbol. Following our discussion on correlation receiver, we observe that the received vector $\bar{r}$, at the output of a bank of I-path and Q-path correlators, has two components:

$$r_1 = \int_0^T r(t)\varphi_1(t)\,dt = \sqrt{E}.\cos\left[(2i-1)\frac{\pi}{4}\right] + w_1$$

and $$r_2 = \int_0^T r(t)\varphi_2(t)\,dt = -\sqrt{E}\sin\left[(2i-1)\frac{\pi}{4}\right] + w_2 \qquad 5.27.8$$

Note that if $r_1>0$, it implies that the received vector is either in decision zone Z_1 or in decision zone Z_4. Similarly, if $r_2 > 0$, it implies that the received vector is either in decision zone Z_3 or in decision zone Z_4.

We have explained earlier in Lesson #19, Module #4 that w_1 and w_2 are independent, identically distributed (iid) Gaussian random variables with zero mean and variance $= \frac{N_0}{2}$. Further, r_1 and r_2 are also sample values of independent Gaussian random variables with means $\sqrt{E}\cos\left[(2i-1)\frac{\pi}{4}\right]$ and $-\sqrt{E}\sin\left[(2i-1)\frac{\pi}{4}\right]$ respectively and with same variance $\frac{N_0}{2}$.

Let us now assume that $s_4(t)$ is transmitted and that we have received $\bar{r}$. For a change, we will first compute the probability of correct decision when a symbol is transmitted.

Let, $P_{c_{s4(t)}}$ = Probability of correct decision when $s_4(t)$ is transmitted.

From **Fig.5.27.4,** we can say that,

$P_{c_{s4(t)}}$ = Joint probability of the event that, $r_1 > 0$ and $r_2 > 0$

As $s_4(t)$ is transmitted,

Mean of $r_1 = \sqrt{E}\cos\left[7\frac{\pi}{4}\right] = \sqrt{\frac{E}{2}}$ and

Mean of $r_2 = -\sqrt{E}\sin\left[7\pi/4\right] = \sqrt{\frac{E}{2}}$

$$\therefore \quad Pc_{s_4(t)} = \int_0^\infty \frac{1}{\sqrt{\pi N_0}}.\exp\left[-\frac{\left(r_1-\sqrt{\frac{E}{2}}\right)^2}{N_0}\right]dr_1.\int_0^\infty \frac{1}{\sqrt{\pi N_0}}.\exp\left[-\frac{\left(r_2-\sqrt{\frac{E}{2}}\right)^2}{N_0}\right]dr_2 \qquad 5.27.9$$

As r_1 and r_2 are statistically independent, putting $\dfrac{r_j - \sqrt{\frac{E}{2}}}{\sqrt{N_0}} = Z$, j = 1, 2, we get,

$$Pc_{s_4(t)} = \left[\frac{1}{\sqrt{\pi}} \cdot \int_{-\sqrt{\frac{E}{2N_0}}}^{\infty} \exp\left(-Z^2\right) dz \right]^2 \qquad 5.27.10$$

Now, note that, $\dfrac{1}{\sqrt{\pi}} \int_{-a}^{\infty} e^{-x^2} dx = 1 - \dfrac{1}{2} erfc(a)$. 5.27.11

$$\therefore Pc_{s_4(t)} = \left[1 - \frac{1}{2} erfc\left(\sqrt{\frac{E}{2N_0}} \right) \right]^2$$

$$= 1 - erfc\left(-\sqrt{\frac{E}{2N_0}} \right) + \frac{1}{4} . erfc^2\left(\sqrt{\frac{E}{2N_0}} \right) \qquad 5.27.12$$

So, the probability of decision error in this case, say, $P_{e_{s4(t)}}$ is

$$= Pe_{s_4(t)} = 1 - Pc_{s_4(t)} = erfc\left(\sqrt{\frac{E}{2N_0}} \right) - \frac{1}{4} erfc^2\left(\sqrt{\frac{E}{2N_0}} \right) \qquad 5.27.13$$

Following similar argument as above, it can be shown that $P_{e_{s1(t)}} = P_{e_{s2(t)}} = P_{e_{s3(t)}} = P_{e_{s4(t)}}$.

Now, assuming all symbols as equally likely, the average probability of symbol error

$$= Pe = 4 \times \frac{1}{4} \left[erfc\left(\sqrt{\frac{E}{2N_0}} \right) - \frac{1}{4} erfc^2\left(\sqrt{\frac{E}{2N_0}} \right) \right] \qquad 5.27.14$$

A relook at **Fig.5.27.2(b)** reveals that the value of erfc(x) decreases fast with increase in its argument. This implies that, for moderate or large value of E_b/N_0, the second term on the R.H.S of **Eq.5.27.14** may be neglected to obtain a shorter expression for the average probability of symbol error, P_e:

$$P_e \cong erfc\left(\sqrt{\frac{E}{2N_o}} \right) = erfc\left(\sqrt{\frac{2.E_b}{2.N_o}} \right) = erfc\left(\sqrt{\frac{E_b}{N_o}} \right) \qquad 5.27.15$$

Fig.5.27.5 shows the average probabilities for symbol error for BPSK and QPSK. Note that for a given E_b/N_o, the average symbol error probability for QPSK is somewhat more compared to that of BPSK.

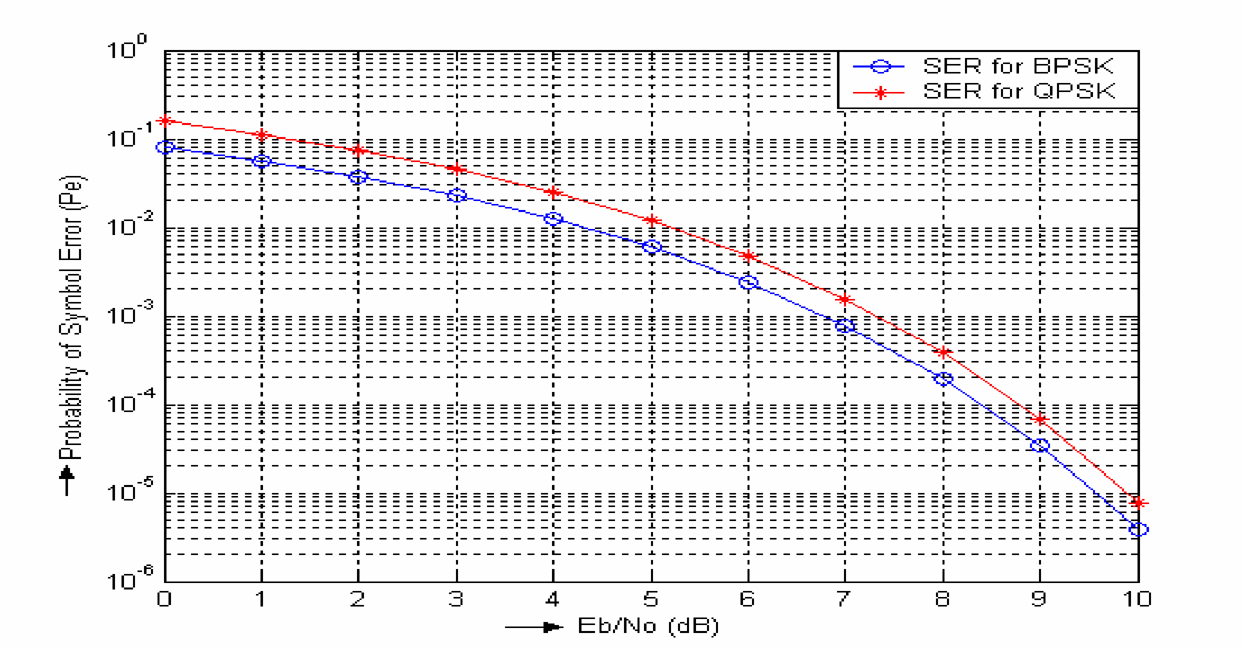

Fig.5.27.5 *SER for BPSK and QPSK*

Approx BER for QPSK:

When average bit error rate, BER, for QPSK is of interest, we often adopt the following approximate approach:

By definition, $Av.\,BER = \lim_{N_{Tot.}\to\infty} \frac{\text{No of erroneous bits}}{\text{Total No. of bits transmitted}\left(N_{Tot}\right)}$

Now, let us note that, one decision error about a QPSK symbol may cause an error in one bit or errors in both the bits that constitute a symbol. For example, with reference to **Fig.5.27.4**, if $s_4(t)$ is transmitted and it is wrongly detected as $s_1(t)$, information symbol (1,1) will be misinterpreted as (1,0) and this actually means that one bit is in error. On the contrary, if $s_4(t)$ is misinterpreted as $s_2(t)$, this will result in two erroneous bits. However, the probability of $s_4(t)$ being wrongly interpreted as $s_2(t)$ is much less compared to the probability that $s_4(t)$ is misinterpreted as $s_1(t)$ or $s_3(t)$. We have tacitly taken advantage of this observation while representing the information symbols in the signal space. See that two adjacent symbols differ in one bit position only. This scheme, which does not increase the complexity of the modem, is known as gray encoding. It ensures that one wrong decision on a symbol mostly results in a single erroneous bit. This observation is good especially at moderate or high E_b/N_o. Lastly, the total number of message bits transmitted over an observation duration is twice the number of transmitted symbols.

$$\therefore Av.\,BER = \frac{1}{2} \times \lim_{Ns \to \infty} \frac{\text{no. of erroneous symbols}}{\text{Total no. of symbols (Ns)}} = \frac{1}{2} \times Pe \cong \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right) \qquad 5.27.16$$

That is, the BER for QPSK modulation is almost the same as the BER that can be achieved using BPSK modulation. So, to summarize, for a given data rate and a given channel condition (E_b/N_o), QPSK as as good as BPSK in terms of error performance while it requires half the transmission bandwidth needed for BPSK modulation. This is very important in wireless communications and is a major reason why QPSK is widely favoured in digital satellite communications and other terrestrial systems.

Problems

Q5.27.1) Suppose, 1 million information bits are modulated by BPSK scheme & the available $\frac{E_b}{N_o}$ is 6.0 dB in the receiver.

Q5.27.2) Determine approximately how many bits will be erroneous at the output of the demodulator.

Q5.27.3) Find the same if QPSK modulator is used instead of BPSK.

Q5.27.4) Mention three situations which can be describe suitable using error function.