Module 4

Signal Representation and Baseband Processing

Lesson 15

Orthogonality

After reading this lesson, you will learn about

- ***Basic concept of orthogonality and orthonormality;***
- ***Strum - Lion;***
- ***Slope overload distortion;***
- ***Granular Noise;***
- ***Condition for avoiding slope overloading;***

The Issue of Orthogonality

Let $f_m(x)$ and $f_n(x)$ be two real valued functions defined over the interval $a \le x \le b$. If the product $[f_m(x) \times f_n(x)]$ exists over the interval, the two functions are called orthogonal to each other in the interval $a \le x \le b$ when the following condition holds:

$$\int_b^a f_m(x) f_n(x) dx = 0 , \qquad m \neq n \qquad 4.15.1$$

A set of real valued functions $f_1(x)$, $f_2(x)\ldots f_N(x)$ is called an orthogonal set over an interval $a \le x \le b$ if

(i) all the functions exist in that interval and

(ii) all distinct pairs of the functions are orthogonal to each other over the interval, i.e.

$$\int_b^a f_i(x) f_j(x) dx = 0 \ , \ i = 1,2,\ldots; \qquad j = 1,2,\ldots \text{ and } i \neq j \qquad 4.15.2$$

The norm $\| f_m(x) \|$ of the function $f_m(x)$ is defined as,

$$\| f_m(x) \| = \sqrt{\int_b^a f_m^2(x) dx} \qquad 4.15.3$$

An orthogonal set of functions $f_1(x)$, $f_2(x)\ldots$ $f_N(x)$ is called an orthonormal set if,

$$\int_b^a f_m(x) . f_n(x) = \begin{cases} 0, m \neq n \\ 1, m = n \end{cases} \qquad 4.15.4$$

An orthonormal set can be obtained from a corresponding orthogonal set of functions by dividing each function by its norm. Now, let us consider a set of real functions $f_1(x)$, $f_2(x)\ldots$ $f_N(x)$ such that, for some non-negative weight function w(x) over the interval $a \le x \le b$

$$\int_b^a f_m(x) . f_n(x) . w(x) dx = 0 , \qquad m \neq n \qquad 4.15.5$$

Do f_i -s form an orthogonal set? We say that the f_i-s form an orthogonal set with respect to the weight function w(x) over the interval $a \le x \le b$ by defining the norm as,

$$\| f_m(x) \| = \sqrt{\int_b^a f_m^2(x).w(x)dx} . \qquad 4.15.6$$

The set of f_i-s is orthonormal with respect to w(x) if the norm of each function is 1.
The above extension of the idea of orthogonal set makes perfect sense. To see this, let

$$g_m(x) = \sqrt{w(x)}\, f_m(x), \text{ where w(x) is a non-negative function.} \qquad 4.15.7$$

It is now easy to verify that,

$$\int_b^a f_m(x).f_n(x).w(x)dx = \int_b^a g_m(x).g_n(x)dx = 0 . \qquad 4.15.8$$

This implies that if we have orthogonal f_i-s over $a \le x \le b$, with respect to a non-negative weight function w(x), then we can form an usual orthogonal set of f_i –s over the same interval $a \le x \le b$ by using the substitution,

$$g_m = \sqrt{w(x)}\, f_m(x)$$

Alternatively, an orthogonal set of g_i-s can be used to get an orthogonal set of f_i-s with respect to a specific non-negative weight function w(x) over $a \le x \le b$ by the following substitution (provided $\sqrt{w(x)} \ne 0$, $a \le x \le b$):

$$f_m(x) = \frac{g_m(x)}{\sqrt{w(x)}} . \qquad 4.15.9$$

A real orthogonal set can be generated by using the concepts of Strum-Liouville (S-L) equation. The S-L problem is a boundary value problem in the form of a second order differential equation with boundary conditions. The differential equation is of the following form:

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx} \right] + \left[q(x) + \lambda.\omega(x) \right] y = 0, \text{ for } a \le x \le b; \qquad 4.15.10$$

It satisfies the following boundary conditions:

i) $c_2 \dfrac{dy}{dx} + c_1 y = 0$; at x = a;

ii) $d_2 \dfrac{dy}{dx} + d_1 y = 0$; at x = b;

Here c_1, c_2, d_1 and d_2 are real constants such that at least one of c_1 and c_2 is non zero and at least one of d_1 and d_2 is non zero.

The solution y = 0 is a trivial solution. All other solutions of the above equation subject to specific boundary conditions are known as characteristic functions or eigen-functions of the S-L problem. The values of the parameter 'λ' for the non trivial solutions are known as characteristic values or eigen values. A very important property of the eigen-functions is that they are orthogonal.

Orthogonality Theorem:

Let the functions p(x), q(x) and ω(x) in the S-L equation (4.15.10) be real valued and continuous in the interval $a \le x \le b$. Let $y_m(x)$ and $y_n(x)$ be eigen functions of the S-L problem corresponding to distinct eigenvalues λ_m and λ_n respectively. Then, $y_m(x)$ and $y_n(x)$ are orthogonal over $a \le x \le b$ with respect to the weight function w(x).

Further, if p(x = a) = 0, then the boundary condition (i) may be omitted and if p(x = b) = 0, then boundary condition (ii) may be omitted from the problem. If p(x = a) = p(x = b), then the boundary condition can be simplified as,

$$y(a) = y(b) \text{ and } \frac{dy}{dx}\Big|_{x=a} = y'(a) = y'(b) = \frac{dy}{dx}\Big|_{x=b}$$

Another useful feature is that, the eigenvalues in the S-L problem, which in general may be complex based on the forms of p(x), q(x) and w(x), are real valued when the weight function ω(x) is positive in the interval $a \le x \le b$ or always negative in the interval $a \le x \le b$

Examples of orthogonal sets:

Ex#1: We know that, for integer 'm' and 'n',

$$\int_{-\pi}^{\pi} \cos mx.\cos nx dx = \begin{cases} 0, & m \ne n \\ \pi, & m = n \end{cases} \qquad \text{E4.15.1}$$

$$\int_{-\pi}^{\pi} \sin mx.\sin nx dx = \begin{cases} 0, & m \ne n \\ \pi, & m = n \end{cases} \qquad \text{E4.15.2}$$

and
$$\int_{-\pi}^{\pi} \cos mx.\sin nx dx = 0 \qquad \text{E4.15.3}$$

Let us consider equation E4.15.1 and rewrite it as:

$$\int_{-1/2f}^{1/2f} (\cos 2\pi mft).(\cos 2\pi nft)dt = \begin{cases} 0, & m \ne n \\ \pi, & m = n \end{cases} \qquad \text{E4.15.4}$$

by substituting x = 2πft = ω t and dx = 2πfdt = ω dt

Note that the functions 'cosmx' and 'cosnx' are orthogonal over the range 2π of the independent variable x and its integral multiple, i.e. M. 2π, in general, where 'M' is an integer. This implies that equation (E4.15.4) is orthogonal in terms of the independent variable 't' over the fundamental range $\frac{1}{f}$ and, in general, over M$\frac{1}{f}$ = M T_0, where 'T_0' indicates the fundamental time interval over which cos2πmft and cos2πnft are orthogonal to each other. Now 'm' and 'n' can have a minimum difference '1' if

$$\int_{-T_0}^{T_0} (\cos 2\pi mft).(\cos 2\pi nft)dt = 0 \qquad \text{E4.15.5}$$

i.e., mf –nf = f = $\frac{1}{T_0}$

So, if two cosine signals have a frequency difference 'f', then we may say,

$$\int_{-1/2f}^{1/2f} \cos 2\pi(f_c + \frac{f}{2})t.\cos 2\pi(f_c - \frac{f}{2})t.dt = 0 \qquad \text{E4.15.6}$$

Re-writing equation (E4.15.6)

$$\int_{-T_0}^{T_0} \cos 2\pi(f_c + \frac{f}{2})t.\cos 2\pi(f_c - \frac{f}{2})t.dt = 0 \qquad \text{where, } T_0 = \frac{1}{f}$$

Looking back at equation E4.15.5, we may write a general form for equation (E4.15.6):

$$\int_{-T_0/2}^{T_0/2} \cos 2\pi(f_c + p\frac{f}{2})t.\cos 2\pi(f_c - p\frac{f}{2})t.dt = 0 \qquad \text{E 4.15.7}$$

where mf = (n+p)f and 'p' is an integer.

Following similar observations on equation E4.15.2, one can say,

$$\int_{-T_0/2}^{T_0/2} \sin 2\pi(f_c + p\frac{f}{2})t.\sin 2\pi(f_c - p\frac{f}{2})t.dt = 0 \qquad \text{E4.15.8}$$

Equation E4.15.3 may also be expressed as,

$$\int_{-T_0/2}^{T_0/2} \cos 2\pi(f_c + p\frac{f}{2})t.\sin 2\pi(f_c - p\frac{f}{2})t.dt$$

$$= \int_{-T_0/2}^{T_0/2} \sin 2\pi(f_c + p\frac{f}{2})t.\cos 2\pi(f_c - p\frac{f}{2})t.dt = 0 \qquad \text{E4.15.9}$$

Let us define $s_1 = \cos 2\pi\left(f_c + p\frac{f}{2}\right)t$, $s_2 = \cos 2\pi\left(f_c - p\frac{f}{2}\right)t$, $s_3 = \sin 2\pi\left(f_c + p\frac{f}{2}\right)t$ and $s_4 = \sin 2\pi\left(f_c - p\frac{f}{2}\right)t$. Can we use the above observations on orthogonality to distinguish among 's_i-s' over a decision interval of $T_5 = T_0 = \frac{1}{f}$?

■

Ex#2: $x_1(t)$ = 1.0 for $0 \le t \le T/2$ and zero elsewhere, ■
$x_2(t)$ = 1.0 for $T/2 \le t \le T$ and zero elsewhere, ■

Ex#3: $x_1(t) = 1.0$ for $0 \le t \le T/2$ and $x_1(t) = -1.0$ for $T/2 < t \le T$, while $x_2(t) = -1.0$ for $0 \le t \le T$ ■

Importance of the concepts of Orthogonality in Digital Communications

a. In the design and selection of information bearing pulses, orthogonality over a symbol duration may be used to advantage for deriving efficient symbol-by-symbol demodulation scheme.
b. Performance analysis of several modulation demodulation schemes can be carried out if the information-bearing signal waveforms are known to be orthogonal to each other.
c. The concepts of orthogonality can be used to advantage in the design and selection of single and multiple carriers for modulation, transmission and reception.

Orthogonality in a complex domain

Let, $z_1(t) = x_1(t) + jy_1(t)$ and $z_2(t) = x_2(t) + jy_2(t)$

Now, $x_1(t) = \dfrac{z_1(t) + z_1^*(t)}{2}$ and $x_2(t) = \dfrac{z_2(t) + z_2^*(t)}{2}$

If x_1 and x_2 are orthogonal to each other over $a \le t \le b$,

$$\int_a^b x_1(t).x_2(t)\,dt = 0$$

i.e., $$\int_a^b \left[z_1(t) + z_1^*(t)\right]\left[z_2(t) + z_2^*(t)\right] dt = 0$$

or, $$\int_a^b \left[z_1(t).z_2(t) + z_1(t).z_2^*(t) + z_1^*(t).z_2(t) + z_1^*(t).z_2^*(t)\right] dt = 0$$

Let us consider a complex function

$$z_1(t) = x(t) + jy(t), \quad a \le t \le b$$
$$= r(t)\left[\cos\Phi(t) + j\sin\Phi(t)\right]$$

where, $r(t) = \left|\tilde{z}(t)\right|$, a non – negative function of 't'.

$$\therefore x(t) = r(t)\cos\Phi(t) \quad \text{and} \quad y(t) = r(t)\sin\Phi(t)$$

Now, $$\int_a^b x(t).y(t)\,dt = \int_a^b r^2(t).\cos\Phi(t).\sin\Phi(t)\,dt$$

We know that cosθ & sinθ are orthogonal to each other over $-\pi \le \theta < \pi$, i.e.,

$$\int_{-\pi}^{\pi} \cos\theta.\sin\theta d\theta = 0$$

So, using a constant weight function w = r, which is non-negative, we may say

$$\int_{-\pi}^{\pi} r^2 \cos\theta.\sin\theta d\theta = 0$$

Now, x = r cos θ and y = r sin θ are also orthogonal over -π ≤ θ < π.
Now, let 'θ' be a continuous function of 't' over -π ≤ θ < π. And,

$\theta\big|_{t=a} = \theta_a = -\pi$ and $\theta\big|_{t=b} = \theta_b = \pi$

Assuming a linear relationship, let, $\theta(t) = 2\pi ft$

$\therefore d\theta(t) = 2\pi fdt$

Under these conditions, we see,

$$\int_a^b r^2(t).\cos\Phi(t).\sin\Phi(t)dt = \frac{1}{2\pi f}\int_{-\pi}^{\pi} r^2(t).\cos\Phi(t).\sin\Phi(t)d\Phi$$

$$= \frac{1}{2\pi f}\int_{-\pi}^{\pi} r^2.\cos\Phi(t).\sin\Phi(t)d\Phi = 0$$

i.e., x(t) and y(t) are orthogonal over the interval $-\frac{1}{2f} \le t \le \frac{1}{2f}$ or $\frac{T}{2} \le t \le \frac{T}{2}$

So, if $\tilde{z}(t) = x(t) + jy(t)$ represents a phasor in the complex plane rotating at a uniform frequency of 'f' , then x(t) and y(t) are orthogonal to each other over the interval $-\frac{1}{2f} \le t \le \frac{1}{2f}$ or, equivalently $-\frac{T}{2} \le t \le \frac{T}{2}$ where $T = \frac{1}{f}$. i.e., $\int_{-T/2}^{T/2} x(t).y(t)dt = 0$

Now, let us consider two complex functions:

$\tilde{z}_1(t) = x_1(t) + jy_1(t) = |\tilde{z}_1(t)|e^{j\Phi_1(t)}$

and $\tilde{z}_2(t) = x_2(t) + jy_2(t) = |\tilde{z}_2(t)|e^{j\Phi_2(t)}$

[x_1(t),y_1(t)] and [x_2(t),y_2(t)] are orthogonal pairs over the interval $-\frac{T}{2} \le t \le \frac{T}{2}$. So, $\tilde{z}_1(t)$ and $\tilde{z}_2(t)$ may be viewed as two phasors rotating with equal speed.

Now, two static phasors are orthogonal to each other if their dot or scalar product is zero, i.e.,

$\overline{A}.\overline{B} = |A||B|\cos\gamma = A_x.B_x + A_y.B_y = 0$, where 'γ is the angle between $\overline{A}$ *and* $\overline{B}$

In general, two complex functions $\tilde{z}_1(t)$ *and* $\tilde{z}_2(t)$ with finite energy are said to be orthogonal to each other over an interval a ≤ t ≤ b, if

$$\int_a^b \tilde{z}_1(t).\tilde{z}_2^{\,*}(t)dt = 0$$

Problems

Q4.15.1) Verify whether two signals are orthogonal over one time period of the signal with smallest frequency signal.

i) $X_1(t) = \text{Cos } 2\pi ft$ and $X_2(t) = \text{Sin } 2\pi ft$

ii) $X_1(t) = \text{Cos } 2\pi ft$ and $X_2(t) = \text{Cos }(2\pi ft + \frac{\pi}{3})$

iii) $X_1(t) = \text{Cos } 2\pi ft$ and $X_2(t) = \text{Cos }(4\pi ft + \frac{\pi}{4})$

iv) $X_1(t) = \text{Sin } 4\pi ft$ and $X_2(t) = -\text{ Cos }(\pi ft - \frac{\pi}{6})$