Module
4
Signal Representation and Baseband Processing

Lesson 18

Response of Linear System to Random Processes

After reading this lesson, you will learn about

- ***Modeling of thermal noise and power spectral density;***
- ***Time domain analysis of a linear filter for random input;***
- ***Representation of narrow-band Gaussian noise;***
- ***Low-pass equivalent components of narrow-band noise;***
- ***Band-pass Gaussian noise and its spectral density;***

A noise waveform is a sample function of a random process. Thermal noise is expected to manifest in a communication receiver for an infinite time and hence theoretically noise may have infinite energy. Thermal noise is typically modeled as a power signal. Usually, some statistical properties of thermal noise, such as its mean, variance, auto – correlation function and power spectrum are of interest.

Thermal noise is further modeled as a wide-sense-stationary (WSS) stochastic process. That is, if n(t) is a sample function of noise, a) the sample mean of $n(t_1)$, i.e. n(t) at $t = t_1$, is independent of the choice of sampling instant 't_1' and b) the correlation of two random samples, $n(t_1)$ and $n(t_2)$ depends only on the interval / delay (t_2-t_1), i.e., $E[n(t_1)n(t_2)] = R_n(t_2 - t_1) = R_n(\tau)$

The auto-correlation function (ACF), $R_x(\tau)$ of a WSS process, x(t) is defined as: $ACF = R_x(\tau) = E[x(t)x(t+\tau)]$. $R_x(\tau)$ indicates the extent to which two random variables separated in time by 'τ' vary with each other. Note that, $R_x(0) = E[x(t)x(t)] = \overline{x^2}$, the mean square of x (t).

Power Spectral Density (psd)

a. Specifies distribution of power of the random process over frequency 'f'. If $S_x(f)$ is the two-sided psd of x(t), the power in a small frequency band Δf at f_1 is $[S_x(f_1). \Delta f]$;

b. psd $S_x(f)$ of thermal noise is a real, positive even function of frequency. The power in a band f_1 to f_2 is:

$$\int_{-f_2}^{-f_1} S_x(f)df + \int_{f_1}^{f_2} S_x(f)df = 2\int_{f_1}^{f_2} S_x(f)df \; ; \quad \text{in Volt}^2\text{/Hz}$$

For a deterministic waveform, the psd and ACF form a Fourier transform pair. The concept is extended to random processes and we may write for thermal noise process,

$$S_x(f) = F\left[R_x(\tau)\right] = \int_{-\alpha}^{\alpha} R_x(\tau)e^{-j\varpi\tau}d\tau \qquad 4.18.1$$

and $$R_x(\tau) = F^{-1}\left[S_x(f)\right] = \int_{-\alpha}^{\alpha} S_x(\tau)e^{j\varpi\tau}df$$

Now, as noted in Lesson #17, the psd for white noise is constant:

$$S_n(f) = \frac{N_0}{2} \qquad 4.18.2$$

Hence, the ACF for such noise process is,

$$R_n(\tau) = F^{-1}\left\{\frac{N_0}{2}\right\} = \frac{N_0}{2}.\delta(\tau) \qquad 4.18.3$$

As we know, the signal, carrying information, occupies a specific frequency band and it is sufficient to consider the effect of noise, which manifests within this frequency band. So, it is useful to study the features of 'band-limited noise'. For a base band additive white Gaussian noise (AWGN) channel of bandwidth 'W' Hz,

$$\begin{aligned} S_n(f) &= \frac{N_0}{2}\ , \ |f| < W \\ &= 0, \quad \text{Elsewhere} \end{aligned} \qquad 4.18.4$$

Simple calculation now shows that, the auto-correlation function for this base band noise is:

$$R_n(\tau) = WN_0 \sin c(2W\tau) \qquad 4.18.5$$

For pass-band thermal noise of bandwidth 'B' around a centre frequency f_c, the results can be extended:

$$\begin{aligned} S_n(f) &= \frac{N_0}{2}\ , \ |f - f_c| < \frac{B}{2} \\ &= 0 \quad , \quad \text{Otherwise} \end{aligned} \qquad 4.18.6$$

The ACF now is given by, $R_n(\tau) = BN_0(\sin cB\tau).\cos 2\pi f_c\tau$ 4.18.7

In many situations, it is necessary to analyze the characteristics of a noise process at the output of a linear system, which transforms some excitation given at its input. This is important because, the system being linear in nature, obeys the principle of superposition and if we excite the system with a noise process and analyze the response noise process, we can use this knowledge for multiple situations. For example, a specific case of interest may be to analyze the output of a linear filter when a noisy received signal is fed to it. For simplicity, we discuss about response of linear systems which are time-invariant. Though such analysis is more elegant when carried out in the frequency domain, we start with a time-domain analysis to provide some insight.

Time-domain analysis for random input to a linear filter

Let us consider a linear lowpass filter whose impulse response is h(t) and let us excite the filter with white Gaussian noise. The input being a random process, it is not so important to get only an expression for the filter output y(t). It is statistically more significant to obtain expressions for the mean, variance, ACF and other parameters of the output signal. Now, in general, if x(t) indicates the input to a linear system, the mean of the output y(t) is, $\overline{y} = \overline{x}\int_0^{\alpha} h(t)dt$, where $\overline{x}$ is the mean of the input process. The mean square value of the output is: $\overline{y^2} = \int_0^{\alpha}\int_0^{\alpha} R_x(\lambda_2 - \lambda_1)h(\lambda_1)h(\lambda_2)d\lambda_2 d\lambda_1$

When the input is white noise, we know

$$R_n(\tau) = \frac{N_0}{2}\delta(\tau) \ , \quad \text{and} \quad \overline{y} = 0 \qquad 4.18.8$$

So, the mean square of the output noise process is:

$$\overline{y^2} = \int_0^{\alpha}\int_0^{\alpha} \frac{N_0}{2}\delta(\lambda_2 - \lambda_1)h(\lambda_1)h(\lambda_2)d\lambda_2 d\lambda_1$$

$$= \frac{N_0}{2}\int_0^{\alpha} h^2(\lambda)d\lambda \qquad 4.18.9$$

Ex 4.18.1: Let us consider a single-stage passive R-C lowpass filter whose impulse response is well known:

$h(t) = \alpha e^{-\alpha t}u(t)$, where $\alpha = \frac{1}{\text{R.C}}$. The 3 dB cutoff frequency of the filter is $f_{cutoff} = \frac{1}{2\pi RC}$. It is straight forward to see that the average of noise at the output of this low-pass filter is zero: $\overline{y} = \overline{x}\int_0^{\alpha} \alpha e^{-\alpha t}d\lambda = \overline{n} = 0$

Further, $\overline{y^2} = \frac{N_0}{2}\int_0^{\alpha} \alpha^2 e^{-2\alpha\lambda}d\lambda = \frac{\alpha N_0}{4} = \frac{N_0}{4RC} = \frac{\pi}{2} \times N_0 \times f_{cutoff}$

We observe that $\overline{y^2}$, the noise power at the output of the filter, is proportional to the filter BW.

Auto-correlation function (ACF)

In general, the autocorrelation of a random process at the output of a linear two-port network is:

$$R_y(\tau) = \int_0^{\alpha}\int_0^{\alpha} R_x(\lambda_2 - \lambda_1 - \tau)h(\lambda_1).h(\lambda_2)d\lambda_2 d\lambda_1 \qquad 4.18.10$$

Specifically, for white noise,

$$R_y(\tau) = \int_0^{\alpha}\int_0^{\alpha} \frac{N_0}{2}\delta(\lambda_2 - \lambda_1 - \tau)h(\lambda_1).h(\lambda_2)d\lambda_2 d\lambda_1$$

$$= \frac{N_0}{2}\int_0^{\alpha} h(\lambda_1).h(\lambda_1+\tau)d\lambda_1 \qquad 4.18.11$$

Considering the RC lowpass filter of *Ex #4.18.1,* we see,

$$R_y(\tau) = \frac{N_0}{2}\int_0^{\alpha} \alpha e^{-\alpha\lambda}\alpha e^{-\alpha(\lambda+\tau)}d\lambda$$

$$= \frac{\alpha N_0}{4}e^{-\alpha\tau} \ , \tau \geq 0$$

As, R(τ) is an even function , R(τ) = R(- τ) and hence,

$$R_y(\tau) = \frac{\alpha N_0}{4}e^{-\alpha|\tau|} \qquad 4.18.12$$

This is an exponentially decaying function of τ with a peak value of $\frac{\alpha N_o}{4}$ at τ = 0.

As $R_y(\tau)$ and the power spectrum $S_y(\omega)$ are Fourier transform pair, we see that the power spectrum of the output noise is:

$$S_y(\omega) = \frac{N_0}{2}\left(\frac{\alpha^2}{\omega^2+\alpha^2}\right) \qquad 4.18.13$$

Representation of Narrow-band Gaussian Noise

Representation and analysis of narrow pass band noise is of fundamental importance in developing insight into various carrier-modulated digital modulation schemes which are discussed in Module #5. The following discussion is specifically relevant for narrowband digital transmission schemes.

Let, x(t) denote a zero-mean Gaussian noise process band-limited to ± B/2 around centre frequency 'f_c'. There are several ways of analyzing such narrowband noise process and we choose an easy-to-visualize approach, which somewhat approximate. To start with, we consider a sample of a noise process over a finite time interval and apply Fourier series expansion while stretching the time interval to ∞. If x(t) is observed over an interval $-\frac{T}{2} \leq t \leq \frac{T}{2}$, we may write

$$x(t) = \sum_{n=1}^{\alpha}(x_{cn}\cos nw_0t + x_{sn}\sin w_0t) \qquad \text{where,} \quad w_0 = \frac{2\pi}{T}$$

and $x_{cn} = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos nw_0 t dt$, n = 1,2,……

and $x_{sn} = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin nw_0 t dt$, n = 1,2,…..

4.18.14

It can be shown that x_{cn} and x_{sn} are Gaussian random variables.
The centre frequency 'f_c' can now be brought in by the following substitution:

$$nw_0 = (nw_0 - w_c) + w_c$$

$$\therefore \quad x(t) = \sum_{n=1}^{\alpha} \{x_{cn} \cos[(nw_0 - w_c)t + w_c t] + x_{sn} \sin[(nw_0 - w_c)t + w_c t]\}$$

$$\equiv x_c(t)\cos w_c t - x_s(t)\sin w_c t \; ; \qquad 4.18.15$$

where, $x_c(t) = \sum_{n=1}^{\alpha} x_{cn} \cos(nw_0 - w_c)t + x_{sn} \sin(nw_0 - w_c)t$ 4.18.16

and $x_s(t) = \sum_{n=1}^{\alpha} x_{cn} \cos(nw_0 - w_c)t + x_{sn} \sin(nw_0 - w_c)t$ 4.18.17

Another elegant and equivalent expression for x(t) is:

$$x(t) = x_c(t)\cos w_c t - x_s(t)\sin w_c t$$

$$= r(t)\cos[w_c t + \Phi(t)] \qquad 4.18.18$$

where, $r(t) = \sqrt{x_c{}^2(t) + x_s{}^2(t)}$;

and $\Phi(t) = \tan^{-1}\left[\dfrac{x_s(t)}{x_c(t)}\right]$; $0 \le \Phi(t) < 2\pi$

It is easy to recognize that, $x_c(t) = r(t)\cos\Phi(t)$ and $x_s(t) = r(t)\sin\Phi(t)$

Low pass equivalent components of narrow band noise

Let, x_{ct} and x_{st} represent samples of $x_c(t)$ and $x_s(t)$. These are Gaussian distributed random variables with zero mean as the original noise process has zero mean.

$$\therefore E[x_{ct}] = E[x_{st}] = 0 \qquad 4.18.19$$

Now, an expression for variance of x_{ct}, by definition, looks like the following:

$$E\left[x_{ct}{}^2\right] = E\left[\sum_{n=1}^{\alpha}\sum_{m=1}^{\alpha} \begin{array}{l}[x_{cn}\cos(nw_0 - w_c)t + x_{sn}\sin(nw_0 - w_c)t] \times \\ [x_{cm}\cos(mw_0 - w_c)t + x_{sm}\sin(mw_0 - w_c)t]\end{array}\right] \qquad 4.18.20$$

However, after some manipulation, the above expression can be put in the following form:

$$E\left[x_{ct}{}^2\right]$$

$$= \left\{\sum_{n=1}^{\alpha}\sum_{m=1}^{\alpha} \begin{array}{l}[\overline{x_{cn}.x_{cm}}\cos(nw_0 - w_c)t.\cos(mw_0 - w_c)t + \overline{x_{cn}.x_{sm}}\cos(nw_0 - w_c)t. \\ \sin(mw_0 - w_c)t + \ldots + \overline{x_{sn}.x_{cm}}\sin(nw_0 - w_c)t.\cos(mw_0 - w_c)t \\ + \overline{x_{sn}.x_{sm}}\sin(nw_0 - w_c)t.\sin(mw_0 - w_c)t]\end{array}\right\}$$

4.18.21

Here,

$$\overline{x_{cn}x_{cm}} = E\left[\left(\frac{2}{T}\int_{-T/2}^{T/2} x(t).\cos nw_0 t dt\right).\left(\frac{2}{T}\int_{-T/2}^{T/2} x(t).\cos mw_0 t dt\right)\right]$$

$$= \frac{4}{T^2}\int_{-T/2}^{T/2}\int_{-T/2}^{T/2} \overline{x(t_1)x(t_2)}\cos nw_0 t_1.\cos mw_0 t_2 dt_1 dt_2$$

$$= \frac{4}{T^2}\int_{-T/2}^{T/2}\int_{-T/2}^{T/2} R_x(t_2 - t_1)\cos nw_0 t_1.\cos mw_0 t_2 dt_1 dt_2 \qquad 4.18.22$$

$R_x(t_2\text{-}t_1)$ in the above expression is the auto-correlation of the noise process x(t). Now, putting $t_2 - t_1 = u$ and $\frac{t_1}{T} = v$, we get,

$$\overline{x_{cn}x_{cm}} = \frac{4}{T}\int_{-1/2}^{1/2} \cos nw_0 Tv \left\{ \int_{-T(\frac{1}{2}+v)}^{T(\frac{1}{2}+v)} R_x(u).\cos mw_0(u + v.T)du \right\} dv$$

$$= \frac{4}{T}\int_{-1/2}^{1/2} \cos 2\pi nv \left[\cos 2\pi mv \int_{-T(\frac{1}{2}+v)}^{T(\frac{1}{2}+v)} R_x(u).\cos\frac{2\pi mu}{T}du \right.$$

$$\left. - \sin 2\pi mv \int_{-T(\frac{1}{2}+v)}^{T(\frac{1}{2}+v)} R_x(u).\sin\frac{2\pi mu}{T}du \right] dv \qquad 4.18.23$$

Now for $T \to \infty$ and putting $f_m = \frac{m}{T}$, where f_m tends to 0 for all 'm', the inner integrands are:

$$\lim_{T\to\alpha} \int_{-T(\frac{1}{2}+v)}^{T(\frac{1}{2}+v)} R_x(u).\cos 2\pi f_m u du = S_x(f_m) \text{ , say}$$

$$\text{and } \lim_{T\to\alpha} \int_{-T(\frac{1}{2}+v)}^{T(\frac{1}{2}+v)} R_x(u).\sin 2\pi f_m u du = 0$$

Let us choose to write, $\lim_{T\to\alpha} f_m \left(= \frac{m}{T}\right) = f \to 0$

Now from Eq. 4.18.23, we get a cleaner expression in the limit:

$$\lim_{T\to\alpha} T\overline{x_{cn}x_{cm}} = 4S_x(f)\int_{-1/2}^{1/2} \cos 2\pi nv \cos 2\pi mv dv$$
$$= 2S_x(f), \quad \mathrm{m = n}$$
$$= 0, \quad \mathrm{m \neq n} \qquad 4.18.24$$

Following similar procedure as outlined above, it can be shown that,

$$\lim_{T\to\alpha} T\overline{x_{sn}x_{sm}} = 2S_x(f), \quad \mathrm{m = n}$$
$$= 0, \quad \mathrm{m \neq n} \qquad 4.18.25$$

and $\lim_{T\to\alpha} T\overline{x_{cn}x_{sm}} = 0$, all m, n 4.18.26

Eq. 4.18.24 – 26 establish that the coefficients x_c-s and x_s-s are uncorrelated as T approaches ∞.

Now, referring back to Eq. 4.18.21, we can see that,

$$E\left[x_{ct}^2\right] = \lim_{T\to\alpha}\sum_{n=1}^{\alpha} x_{cn}^{\ 2}\left[\cos^2(nw_0 - w_c)t + \sin^2(nw_0 - w_c)t\right] \qquad 4.18.27$$

$$= \lim_{T\to\alpha}\sum_{n=1}^{\alpha} S_x(f_m)\left(\frac{2}{T}\right) = 2\int_0^{\alpha} S_x(f)df = \overline{x_t^{\ 2}} \qquad 4.18.28$$

Here $\overline{x_t^{\ 2}}$ denotes the mean square value of x(t).

Similarly, it can be shown that,

$$E\left[x_{st}^{\ 2}\right] = E\left[x_{ct}^{\ 2}\right] = \overline{x_t^{\ 2}} \qquad 4.18.29$$

Since, $\overline{x_{st}} = \overline{x_{ct}} = 0$, we finally get, $\sigma_{st}^{\ 2} = \sigma_{ct}^{\ 2} = \sigma_x^{\ 2}$, the variance of x(t).

It may also be shown that the covariance of x_{ct} and x_{st} approach 0 as T approaches ∞. Therefore, ultimately it can be shown that x_{ct} and x_{st} are statistically independent.

So, x_{ct} and x_{st} are uncorrelated Gaussian distributed random variables and they are statistically independent. They have zero mean and a variance equal to the variance of the original bandpass noise process**.** This is an important observation. 'x_{ct}' is called the in-phase component and 'x_{st}' is called the quadrature component of the noise process.

Spectral Density of In-phase and Quadrature Component of Bandpass Gaussian Noise

Following similar procedures as adopted for determining mean square values of $x_c(t)$ and $x_s(t)$, we can compute their auto-correlation and cross-correlation functions as below:

ACF of $x_{ct} = R_{x_c}(\tau) = 2\int_0^{\alpha} S_x(f).cos2\pi(f - f_c)\tau df$ 4.18.30

$R_{x_s}(\tau) =$ ACF of $x_{st} = 2\int_0^{\alpha} S_x(f).cos2\pi(f - f_c)\tau df = R_{xc}(\tau)$ 4.18.31

Cross Correlation Function (CCF) between x_{ct} and x_{st} is:

$$R_{x_c x_s}(\tau) = 2\int_0^{\alpha} S_x(f).\sin 2\pi(f - f_c)\tau df \qquad 4.18.32$$

and $R_{x_c x_s}(\tau) = -R_{x_c x_s}(\tau)$ 4.18.33

Eq.4.18.30 can be expressed in the following convenient manner:

$$R_{x_c}(\tau) = \int_0^{\alpha} S_x(f)\cos 2\pi(f - f_c)\tau df + \int_0^{-\alpha} S_x(-f)\cos 2\pi(-f - f_c)\tau(-df)$$

$$= \int_{-f_c}^{\alpha} S_x(f + f_c)\cos 2\pi f\tau df + \int_0^{f_c} S_x(f - f_c)\cos 2\pi f\tau df$$

$$= \int_{-f_c}^{f_c} [S_x(f + f_c) + S_x(f - f_c)]\cos 2\pi f\tau df \qquad 4.18.34$$

From this, using inverse Fourier Transform, one gets,

$$S_{x_c}(f) = S_x(f + f_c) + S_x(f - f_c) \qquad 4.18.35$$

Moreover, $S_{x_c}(f) = S_{x_s}(f) = S_x(f + f_c) + S_x(f - f_c)$ 4.18.36

Note that the power spectral density of $x_c(t)$ is the sum of the negative and positive frequency components of $S_x(f)$ after their translation to the origin.

The following steps summarize the method to construct psd of $S_{x_c}(f)$ or $S_{x_s}(f)$ from $S_x(f)$:

a. Displace the +ve frequency portion of the plots of $S_x(f)$ to the left by ‘f_c’.
b. Displace the –ve frequency portion of $S_x(f)$ to the right by ‘f_c’.
c. Add the two displaced plots.

If ‘f_c’ is not the centre frequency, the psd of $x_c(t)$ or $x_s(t)$ may be significantly different from what may be guessed intuitively.

Problems

Q4.18.1) Consider a pass band thermal noise of bandwidth 10 MHz around a center frequency of 900 MHz. Sketch the auto co-relation function of this pass band thermal noise normalized to its PSD.

Q4.18.2) Sketch the pdf of typical narrow band thermal noise.