Module
2
Random Processes

Lesson
5

Introduction to Random Variables

After reading this lesson, you will learn about

- ***Definition of a random variable***
- ***Properties of cumulative distribution function (cdf)***
- ***Properties of probability density function (pdf)***
- ***Joint Distribution***

- A random variable (RV) is a real number x(s) assigned to every outcome 's' of an experiment. An experiment may have a finite or an infinite number of outcomes. Ex: i) Voltage of a random noise source, ii) gain points in a game of chance.
- An RV is also a function whose domain is the set 'S' of outcomes of the experiment.
- Domain of a function: A 'function' y(t) is a rule of correspondence between values of 't' and 'y'. All possible values of the independent variable 't' form a set 'T', known as the 'domain of y(t)'.
- The values of the dependent variable y(t) form a set Y on the y-axis and the set Y is called the range of the function y(t).
- The rule of correspondence between 't' and 'y' may be a table, a curve, a formula or any other precise form of description.
- Events: Any subset of S, the set of valid outcomes, may define an event. The subset of outcomes usually depends on the nature of interest / investigation in the experiment. Each event, however, must have a notion of probability associated with it, represented by a non-negative real number less than or equal to unity.

A more formal definition of a Random Variable (RV)

A random variable 'X' is a process of assigning a (real) number x(s) to every outcome s of a statistical experiment. The resulting function must satisfy the following two conditions:

1. The set $\{X \le a\}$ is an event for every 'a'.
2. The probabilities of the events $\{X = +\infty\}$ and $\{X = -\infty\}$ equal zero, i.e., $P\{X = +\infty\} = \{X = -\infty\} = 0$.

A complex random variable is defined as, $\mathbf{z} = x + jy$ where x and y are real random variables.

Cumulative Distribution Function [cdf]

The cdf of a random variable X is the function $F_x(a) = P\{X \le a\}$, for $-\infty < a < \infty$. Here, note that $\{X \le a\}$ denotes an event and 'a' is a real number. The subscript 'x' is added to F() to remind that 'X' is the random variable. The cdf $F_x(a)$ is a positive number which depends on 'a'. We may also use the simpler notation F (x) to indicate the cdf $F_x(a)$ when there is no confusion.

A complex RV **z** = x + jy has no cdf in the ordinary sense because, an inequality of the type $(x + jy) \le (a + jb)$ has no meaning. Hence, the statistics of a complex RV (like **z**) are specified in terms of the joint statistics of the random variables x & y.

Some fundamental properties of cdf $F_x(a)$

For a small and positive ε, let, $F_x(a^+) = \lim_{\varepsilon \to 0} F_x(a+\varepsilon)$

i.e., $F_x(a^+)$ is the cdf of X when the number 'a' is approached from the right hand side (RHS) of the real number axis.

Similarly, let, for a small and positive ε, $F_x(a^-) = \lim_{\varepsilon \to 0} F_x(a-\varepsilon)$

i.e., $F_x(a^-)$ is the cdf of X when the number 'a' is approached from the left hand side (LHS) of the real number axis.

Property #1 $F_x(+\infty) = 1$ and $F_x(-\infty) = 0$

Hint: $F_x(+\infty) = P\{X \le \infty\} = P\{S\} = 1$ and $F_x(-\infty) = P\{X \le -\infty\} = 0$

Property #2 The cdf $F_x(a)$ is a non-decreasing function of 'a', i.e.,

if $a_1 < a_2$, $F_x(a_1) \le F_x(a_2)$

Hint: An event $\{x \le a_1\}$ is a subset of the event $\{x \le a_2\}$ because, if $x(s) \le a_1$ for some 's', then $x(s) \le a_2$ too.

Hence, $P\{x \le a_1\} \le P\{x \le a_2\}$

From properties #1 and #2, it is easy to see that $F_x(a)$ increases from 0 to 1 as 'a' increases from $-\infty$ to $+\infty$. The particular value of $a = a_m$ such that, $F_x(a_m) = 0.5$ is called the *median* of the RV 'X'.

Property #3 If $F_x(a_0) = 0$ then, $F_x(a) = 0$ for every $a \le a_0$

Hint: As $F_x(-\infty) = 0$ and $F_x(a_1) \le F_x(a_2)$ when $a_1 < a_2$.

Property #4 $P\{x > a\} = 1 - F_x(a)$

Hint: The events $\{x \le a\}$ *and* $\{x \ge a\}$ are mutually exclusive and

$\{x \le a\} \cup \{x > a\} = S$

Property #5 The function $F_x(a)$ is continuous from the right, i.e.,

$F_x(a^+) = F_x(a)$

Hint: Observe that, $P\{x \le a + \in\} \to F_x(a)$ as $\in \to 0$

Because, $P\{x \le a + \in\} \to F_x(a + \in)$ and $F_x^{'}(a + \in) \to F_x(a^+)$

Also observe that the set $\{x \le a + \in\}$ tends to the set $\{x \le a\}$ *as* $\in \to 0$

Property #6 $P\{a_1 < x \le a_2\} = F_x(a_2) - F_x(a_1)$

Hint: The events $\{x \le a_1\}$ *and* $\{a_1 < x \le a_2\}$ are mutually exclusive. Further, $\{ x \le a_2 \} = \{ x \le a_1 \} + \{ a_1 < x \le a_2 \}$ and hence, $P\{ x \le a_2 \} = P\{ x \le a_1 \} + P\{ a_1 < x \le a_2 \}$.

Property #7 $P\{x = a\} = F_x(a) - F_x(a^-)$

Hint: Put $a_1 = a - \in$ *and* $a_2 = a$ in Property #6.

Property #8 $P\{a_1 \le x \le a_2\} = F_x(a_2) - F_x(a_1^-)$

Hint: Note that, $\{a_1 \le x \le a_2\} = \{a_1 < x \le a_2\} + \{x = a_1\}$

Now use Property #6 and Property #7.

- **Continuous RV** A random variable X is of continuous type if its cdf $F_x(a)$ is continuous. In this case, $F_x(a^-) = F_x(a)$ and hence, $P\{x = a\} = 0$ for every 'a'. [Property #7]

- **Discrete RV** A random variable X is of discrete type if $F_x(a)$ is a staircase function. If 'a_i' indicates the points of discontinuity in $F_x(a)$, we have,
  $F_x(a_i) - F_x(a_i^-)$
  $= P\{x = a_i\} = p_i$
  In this case, the statistics of X are determined in terms of a_i-s and p_i-s.
- **Mixed RV** A random variable X is of mixed type if $F_x(a)$ is discontinuous but not a staircase.

Probability Density Function (pdf) / Frequency Function / Density Function

- The derivative $f_x(a) = \dfrac{dF_x(a)}{da}$ is called the probability density function (pdf) / density function / frequency function of the random variable X.
  For a discrete random variable X taking values a_i-s with probabilities p_i,
  $f_x(a) = \sum_i p_i \delta(a - a_i)$, where $p_i = P\{x = a_i\}$ and $\delta(x)$ is the impulse function.

Properties of Probability Density Function (pdf)

- From the monotonicity of $F_x(a)$, $f_x(a) \ge 0$
- Integrating the basic expression of $f_x(a) = \dfrac{dF_x(a)}{da}$ from $-\infty$ to 'a' and recalling that $F_x(-\infty) = 0$, we get,

$$F_x(a) = \int_{-a}^{a} f_x(\tau) d\tau \qquad 2.5.1$$

- $\int_{-a}^{a} f_x(a)da = 1$ 2.5.2

- $F_X(a_2) - F_X(a_1) = \int_{a_1}^{a_2} f_X(\tau)d\tau$ or $P\{a_1 < x \le a_2\} = \int_{a_1}^{a_2} f_X(\tau)d\tau$ 2.5.3

  Note, if X is a continuous RV, $P\{a_1 < x \le a_2\} = P\{a_1 \le x \le a_2\}$. However, if $F_x(a)$ is discontinuous at, say a_2, the integration in 2.5.3 must include the impulse at x = a_2.

- For a continuous RV x, with $a_1 = a$ & $a_2 = a + \Delta a$, we may write from 2.5.3:
  $P\{a_1 \le x \le a_2\} \cong f_x(x).\Delta a$ for $\Delta a \to 0$ which means,

  $$f_x(a) = \lim_{\Delta a \to 0} \frac{P\{a \le x \le a + \Delta a\}}{\Delta a} \qquad 2.5.4$$

Eq. 2.5.4 shows that for $\Delta a \to 0$, the probability that the random variable X is in a small interval Δa is proportional to $f_x(a)$, i.e. $f_x(a) \propto P\{a \le x \le a + \Delta a\}$. When the interval Δa includes the point a = a_{mode} where $f_x(a)$ is maximum, a_{mode} is called the *mode* or *most likely value* of the random variable X.

Now, if f(x) denotes the pdf of a random variable X, then, $P\{ x_0 < X \le x_0 + dx \} = f(x_0)dx$, for vanishingly small dx.

If the random variable X takes values between - ∞ and + ∞, then its average (expectation or mean) value is: $E[X] = \lim_{N\to\infty} \sum_{i=1}^{N} x_i P\{X = x_i\} = \int_{-\infty}^{+\infty} xf(x)dx$

The mean-squared value of a random variable X is defined as:

$$E\left[X^2\right] = \lim_{N\to\infty} \sum_{i=1}^{N} x_i^2 P\{X = x_i\} = \int_{-\infty}^{+\infty} x^2 f(x)dx \qquad 2.5.5$$

The variance of a random variable X is its mean squared value about its mean, i.e.

$$\text{Variance of X} = E\left[\{X - E(X)\}^2\right] = \lim_{N\to\infty} \sum_{i=1}^{N} \{x_i - E(X)\}^2 P\{X = x_i\}$$

$$= \int_{-\infty}^{+\infty} \{x - E(X)\}^2 f(x)dx \qquad 2.5.6$$

The variance of a random variable is popularly denoted by σ^2. The positive square root of the variance is called the 'standard deviation' of the random variable (from its mean value).

In addition to the above basic statistical parameters, several 'moments' are also defined to provide detail description of a random experiment. These are the means or averages of positive integer powers of the random variable X. In general, the n-th moment of X is expressed as:

$$\alpha_n = E\left[X^n\right] = \lim_{N\to\infty}\sum_{i=1}^{N} x_i^n P\{X = x_i\} = \int_{-\infty}^{+\infty} x^n f(x)\,dx \qquad 2.5.7$$

It is easy to verify and good to remember that, $\alpha_0 = 1$, $\alpha_1 = E[X]$, $\alpha_2 = E[X^2]$, the mean squared value of X and, the variance $\sigma^2 = \alpha_2 - \alpha_1^2$

Another class of moments, known as 'central moments' of the distribution is obtained by subtracting the mean of X from its values and taking moments as defined above. That is, the n-th central moment of X is defined as:

$$\mu_n = E\left\{\left[X - E(X)\right]^n\right\} = \lim_{N\to\infty}\sum_{i=1}^{N}\left[x_i - E(X)\right]^n P(X = x_i)$$

$$= \int_{-\infty}^{+\infty}\left[x - E(X)\right]^n f(x)\,dx \qquad 2.5.8$$

This time, note that, $\mu_0 = 1$, $\mu_1 = 0$ and $\mu_2 = \sigma^2$.

Two or more random variables

Let X and Y are two random variables. We define a joint distribution function F(x, y) as the (joint) probability of the combined events { $X \le x$ } and { $Y \le y$ }, i.e.,

$$F(x, y) = \text{Prob.}[\, X \le x \text{ and } Y \le y \,] \qquad 2.5.9$$

Extending our earlier discussion over a joint event, it is easy to note that,

$$F(-\infty, y) = 0,\ F(x, -\infty) = 0 \text{ and } F(+\infty, +\infty) = 1 \qquad 2.5.10$$

The joint probability density function [joint pdf] can be defined over the joint probability space when the partial derivatives exist and are continuous:

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y} \qquad 2.5.11$$

This implies,

$$\int_{a_1}^{a_2}\int_{b_1}^{b_2} f(x,y)\,dxdy = F(a_2,b_2) - F(a_1,b_2) - \left[F(a_2,b_1) - F(a_1,b_1)\right]$$

$$= \text{Prob.}\left[a_1 < X \le a_2 \text{ and } b_1 < Y \le b_2\right] \qquad 2.5.12$$

For two or more random variables with defined joint distribution functions F (x, y) and joint pdf f (x , y), several second order moments can be defined from the following general expression:

$$\alpha_{ij} = E\left(x^i y^j\right) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^i y^j f(x,y)\,dydx, \qquad 2.5.13$$

The following second order moment α_{11} is called the correlation function of X and Y:

$$\alpha_{11} = E(X\ Y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\, f(x,y)\,dydx \qquad 2.5.14$$

The set of second order central moments can also be expressed in general as:

$$\mu_{ij} = E\left\{\left[X - E(X)\right]^i\right\} E\left\{\left[Y - E(Y)\right]^j\right\}$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left[x - E(X)\right]^i \left[Y - E(Y)\right]^j\ f(x,y)\,dydx, \qquad 2.5.15$$

The reader may try out some exercise at this point to verify the following:
$\mu_{20} = \alpha_{20} - \alpha_{10}^2$; $\mu_{02} = \alpha_{02} - \alpha_{01}^2$; and $\mu_{11} = \alpha_{11} - \alpha_{01}.\alpha_{10}$;

Now, let $F_X(x)$ denote the cdf of X and $F_Y(y)$ denote the cdf of Y. That is,
$F_X(x) = \text{Prob}\left[X \le x \text{ and } Y \le \infty\right]$ and $F_Y(y) = \text{Prob}\left[Y \le y \text{ and } X \le \infty\right]$.
It is straightforward to see that,

$$F_X(x) = \text{Prob}\left[X \le x \text{ and } Y \le \infty\right] = \int_{-\infty}^{x}\int_{-\infty}^{+\infty} f(x,y)\,dydx \qquad 2.5.16$$

$$F_Y(y) = \text{Prob}\left[\ X \le \infty \text{ and } Y \le y\right] = \int_{-\infty}^{+\infty}\int_{-\infty}^{y} f(x,y)\,dydx \qquad 2.5.17$$

If the first order probability density functions of X and Y are now denoted by $f_X(x)$ and $f_Y(y)$ respectively, by definition, $f_X(x) = \dfrac{dF_X(x)}{dx}$ and $f_Y(y) = \dfrac{dF_Y(y)}{dy}$.

Further,

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy \quad \text{and} \quad f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx \qquad 2.5.18$$

The above expressions highlight that individual probability density functions are contained within the joint probability density function.
If the two random variables X and Y are known to be independent of each other, we may further see that,

$$\text{Prob}\left[X \le x \text{ and } Y \le y\right] = \text{Prob}\left[\ X \le x\right]\cdot \text{Prob}\left[\ Y \le y\right]$$

and hence,

$$F(x,y) = F_X(x)\cdot F_Y(y),$$

$$f(x,y) = f_X(x)\cdot f_Y(y)\ ,$$

and also,

$$\alpha_{11} = E(X,Y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy \cdot f(x,y)\,dydx = E(X)E(Y) \qquad 2.5.19$$

Occasionally it is necessary to consider and analyze a joint distribution involving three or more random variables and one has to extend the above ideas carefully to obtain the desired statistics. For example, the joint distribution function F(x, y, z) for three random variables X, Y and Z, by definition is:

$$\mathrm{F}(x,y,z) = \text{Prob}\left[X \le x \text{ and } Y \le y \text{ and } \mathrm{Z} \le z\right] \qquad 2.5.20$$

And their joint probability density function [pdf] f(x, y, z) can be expressed in a general form as:

$$f(x,y,z) = \frac{\partial^3 F(x,y,z)}{\partial x \partial y \partial z} \qquad 2.5.21$$

Problems

Q2.5.1) Mention the two basic conditions that a random variable must satisfy?

Q2.5.2) Prove property #5 under section "Some fundamental properties of cdf $F_x(a)$.

Q2.5.3) What is a mixed RV? Explain with two examples.

Q2.5.4) Give an example where the concept of joint pdf may be useful.