# Moving into the Frequency Domain

Frequency domains can be obtained through the transformation from one (**Time** or **Spatial**) domain to the other (**Frequency**) via

- **Discrete Cosine Transform (DCT)**— Heart of **JPEG** and **MPEG Video**, (alt.) MPEG Audio. (**New**)
- **Fourier Transform (FT)** — **MPEG Audio** (**Recap From CM0268**)

**Note**: We mention some image (and video) example in this section as DCT (in particular) but also the FT is commonly applied to filter multimedia data.

# 1D Example

Lets consider a 1D (e.g. Audio) example to see what the different domains mean:

Consider a complicated sound such as the noise of a car horn. We can describe this sound in two related ways:

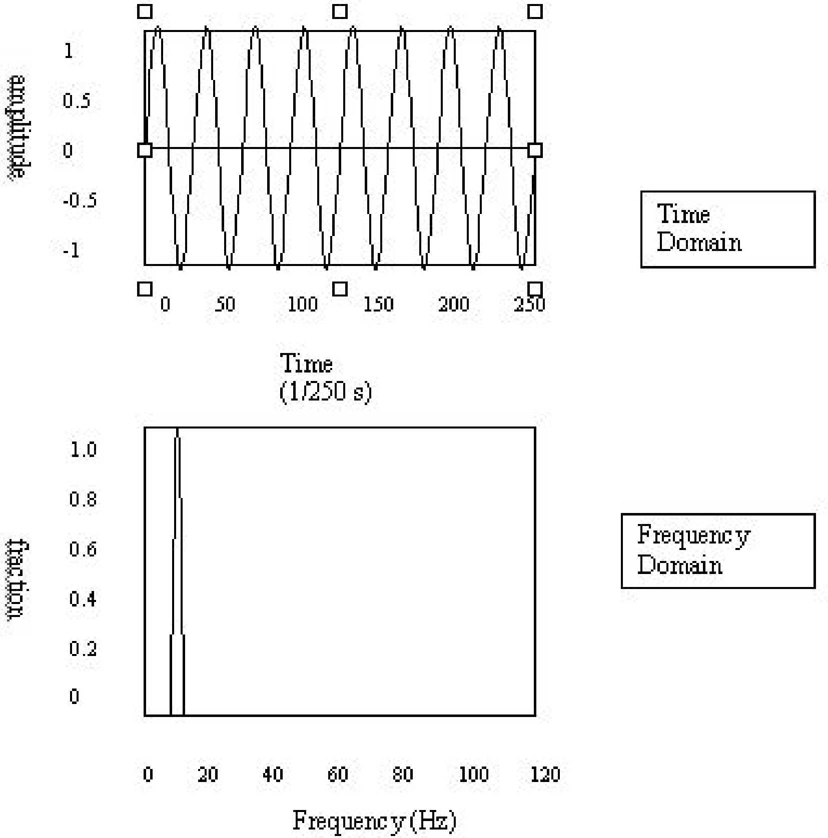
- Sample the amplitude of the sound many times a second, which gives an approximation to the sound as a function of time.
- Analyse the sound in terms of the pitches of the notes, or frequencies, which make the sound up, recording the amplitude of each frequency.

# An 8 Hz Sine Wave

In the example (next slide):

- A signal that consists of a sinusoidal wave at 8 Hz.
- 8 Hz means that wave is completing 8 cycles in 1 second
- The **frequency** of that wave (8 Hz).
- From the frequency domain we can see that the composition of our signal is
  - one wave (one peak) occurring with a frequency of 8 Hz
  - with a magnitude/fraction of 1.0 i.e. it is the whole signal.

# An 8 Hz Sine Wave (Cont.)



# 2D Image Example

Now images are no more complex really:

- Brightness along a line can be recorded as a set of values measured at equally spaced distances apart,
- **Or** equivalently, at a set of spatial frequency values.
- Each of these frequency values is a **frequency component**.
- An image is a 2D array of pixel measurements.
- We form a 2D grid of spatial frequencies.
- A given frequency component now specifies what contribution is made by data which is changing with specified $x$ and $y$ direction spatial frequencies.

# What do frequencies mean in an image?

- Large values at **high** frequency components then the data is changing rapidly on a short distance scale.

  *e.g.* a page of text

- Large **low** frequency components then the large scale features of the picture are more important.

  *e.g.* a single fairly simple object which occupies most of the image.

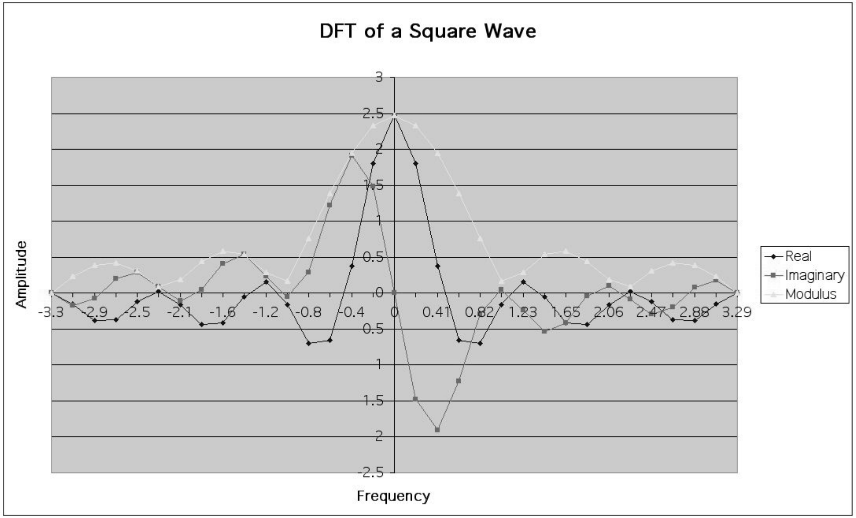
# The Road to Compression

How do we achieve compression?

- Low pass filter — ignore high frequency noise components
  - Only store lower frequency components
- High Pass Filter — Spot Gradual Changes
  - If changes to low Eye does not respond so ignore?

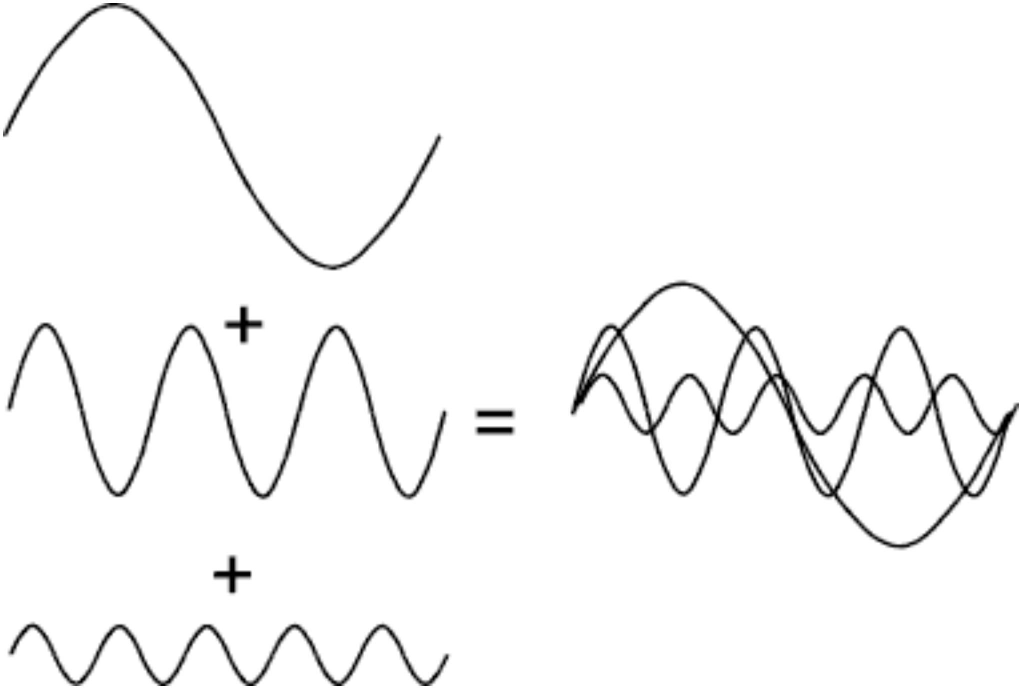
# Visualising Frequency Domain Transforms

- Any function (signal) can be decomposed into purely sinusoidal components (sine waves of different size/shape)
- When added together make up our original signal.
- **Fourier transform** is the tool that performs such an operation



# Summing Sine Waves

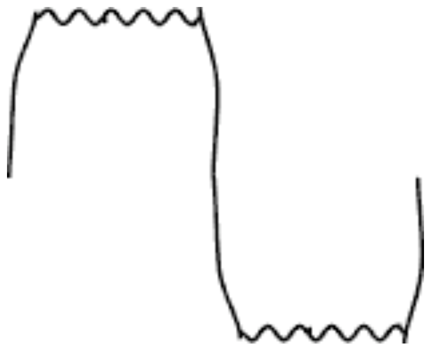
Digital signals are composite signals made up of many sinusoidal frequencies



# Summing Sine Waves to give a Square(ish) Wave

We can take the previous example a step further:

- A 200Hz digital signal (square(ish) wave) may be a composed of 200, 600, 1000, 1400, 1800, 2200, 2600, 3000, 3400 and 3800 sinusoidal signals which sum to give:



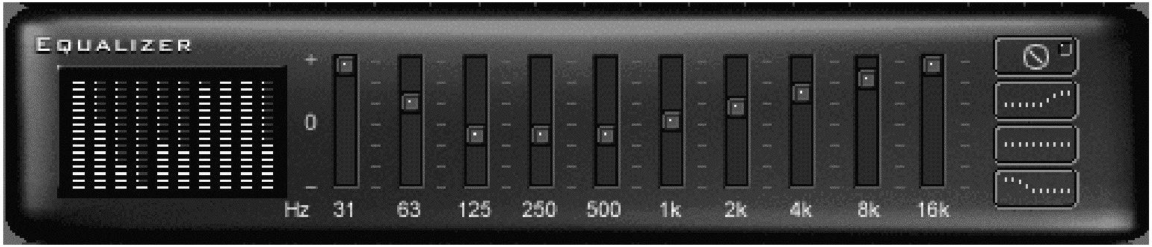
# So What Does All This Mean?

Transforming a signal into the frequency domain allows us

- **To see what sine waves make up our underlying signal**
- **E.g.**
  - One part sinusoidal wave at 50 Hz and
  - Second part sinusoidal wave at 200 Hz.
- More complex signals will give more complex graphs but the idea is exactly the same.
- Filtering now involves **attenuating** or **removing** certain frequencies — easily performed.
- The graph of the frequency domain is called the frequency spectrum — **more soon**

# Visualising Frequency Domain: Think Graphic Equaliser

An easy way to visualise what is happening is to think of a graphic equaliser on a stereo system (or some software audio players, *e.g. iTunes*).



# Fourier Theory

The tool which converts a spatial (real space) description of audio/image data into one in terms of its frequency components is called the **Fourier transform**

The new version is usually referred to as the **Fourier space description** of the data.
We then essentially process the data:

- *E.g.* for **filtering** basically this means attenuating or setting certain frequencies to zero

We then need to convert data back to real audio/imagery to use in our applications.

The corresponding **inverse** transformation which turns a Fourier space description back into a real space one is called the **inverse Fourier transform**.

# Fourier Transform

## 1D Case (*e.g.* Audio Signal)

Considering a continuous function $f(x)$ of a single variable $x$ representing distance.

The Fourier transform of that function is denoted $F(u)$, where $u$ represents spatial frequency is defined by

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x u}\, dx.$$

**Note**: In general $F(u)$ will be a complex quantity *even though* the original data is purely **real**.

The meaning of this is that not only is the magnitude of each frequency present important, but that its phase relationship is too.

# Inverse 1D Fourier Transform

The inverse Fourier transform for regenerating $f(x)$ from $F(u)$ is given by

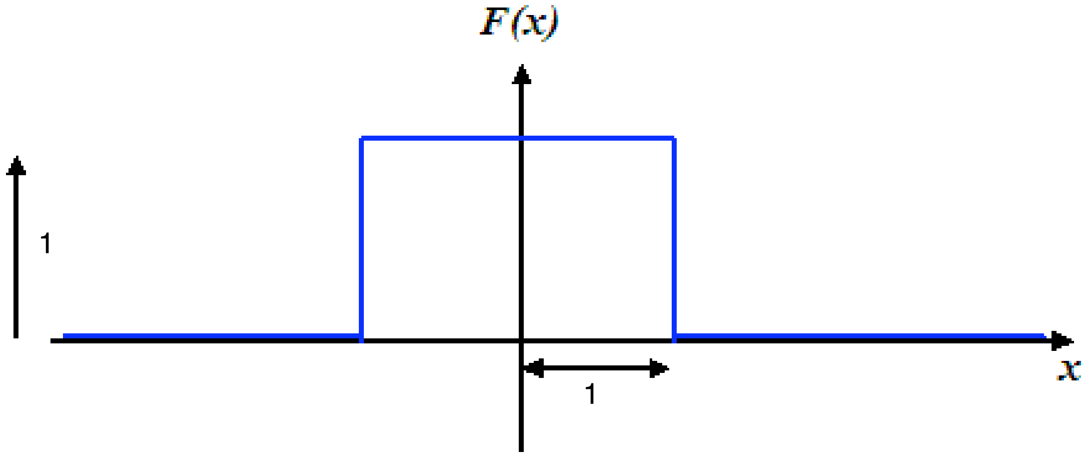
$$f(x) = \int_{-\infty}^{\infty} F(u) e^{2\pi i x u} \, du,$$

which is rather similar, except that the**exponential term has the opposite sign.** – not negative

# Example Fourier Transform

Let's see how we compute a Fourier Transform: consider a particular function $f(x)$ defined as

$$f(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{otherwise,} \end{cases}$$



So its Fourier transform is:

$$
\begin{aligned}
F(u) &= \int_{-\infty}^{\infty} f(x) e^{-2\pi i x u}\, dx \\
&= \int_{-1}^{1} 1 \times e^{-2\pi i x u}\, dx \\
&= \frac{-1}{2\pi i u}(e^{2\pi i u} - e^{-2\pi i u}) \\
&= \frac{\sin 2\pi u}{\pi u}.
\end{aligned}
$$

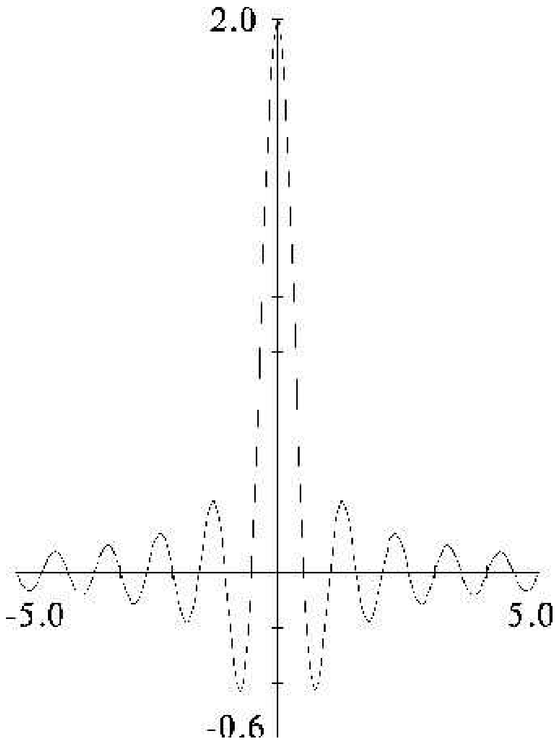
In this case $F(u)$ is purely real, which is a consequence of the original data being symmetric in $x$ and $-x$.

A graph of $F(u)$ is shown overleaf.

This function is often referred to as the Sinc function.

# The Sync Function

The Fourier transform of a top hat function:



# 2D Case (*e.g.* Image data)

If $f(x,y)$ is a function, for example the brightness in an image, its Fourier transform is given by

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-2\pi i(xu+yv)}\, dx\, dy,$$

and the inverse transform, as might be expected, is

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) e^{2\pi i(xu+yv)}\, du\, dv.$$

# But All Our Audio and Image data are Digitised!!

Thus, we need a *discrete* formulation of the Fourier transform:

- Which takes such regularly spaced data values, and
- Returns the value of the Fourier transform for a set of values in frequency space which are equally spaced.

This is done quite naturally by replacing the integral by a summation, to give the *discrete Fourier transform* or DFT for short.

In 1D it is convenient now to assume that $x$ goes up in steps of $1$, and that there are $N$ samples, at values of $x$ from $0$ to $N-1$.

# 1D Discrete Fourier transform

So the DFT takes the form

$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-2\pi i x u/N},$$

while the inverse DFT is

$$f(x) = \sum_{x=0}^{N-1} F(u) e^{2\pi i x u/N}.$$

**NOTE:** Minor changes from the continuous case are a factor of $1/N$ in the exponential terms, and also the factor $1/N$ in front of the forward transform which does not appear in the inverse transform.

# 2D Discrete Fourier transform

The 2D DFT works is similar. So for an $N \times M$ grid in $x$ and $y$ we have

$$F(u,v) = \frac{1}{NM} \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} f(x,y) e^{-2\pi i (xu/N + yv/M)},$$

and

$$f(x,y) = \sum_{u=0}^{N-1} \sum_{v=0}^{M-1} F(u,v) e^{2\pi i (xu/N + yv/M)}.$$

# Balancing the 2D DFT

Often $N = M$, and it is then it is more convenient to redefine $F(u, v)$ by multiplying it by a factor of $N$, so that the forward and inverse transforms are more symmetrical:

$$F(u, v) = \frac{1}{N} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x, y) e^{-2\pi i(xu+yv)/N},$$

and

$$f(x, y) = \frac{1}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} F(u, v) e^{2\pi i(xu+yv)/N}.$$

# Visualising the Fourier Transform

Back to Fourier Transform for Moment:

- It's useful to visualise the Fourier Transform
- Standard tools
- Easily plotted in MATLAB

# The Magnitude Spectrum of Fourier Transform

Recall that the Fourier Transform of even our **real** audio/image data is always **complex**.

- How can we visualise a complex data array?

**Compute the absolute value of the complex data**:

$$|F(k)| = \sqrt{F_R^2(k) + F_I^2(k)} \text{ for } k = 0, 1, \ldots, N-1$$

where $F_R(k)$ is the real part and $F_I(k)$ of the $N$ sampled Fourier Transform, $F(k)$.

This is called the **magnitude spectrum** of the Fourier Transform

**Easy in MATLAB**: `Sp = abs(fft(X,N))/N;`
(Normalised form)

# The Phase Spectrum of the Fourier Transform

The Fourier Transform also represent phase, the **phase spectrum** is given by:

$$\varphi = \arctan \frac{F_I(k)}{F_R(k)} \text{ for } k = 0, 1, \ldots, N-1$$

# Relating a Sample Point to a Frequency Point

When plotting graphs of *FT Spectra* and doing other FT processing we may wish to plot the $x$-axis in Hz (Frequency) rather than sample point number $k = 0, 1, \ldots, N-1$

There is a simple relation between the two:
The sample points go in steps $k = 0, 1, \ldots, N-1$
For a given sample point $k$ the frequency relating to this is given by:

$$f_k = k\frac{f_s}{N}$$

where $f_s$ is the *sampling frequency* and $N$ the number of samples.
Thus we have equidistant frequency steps of $\frac{f_s}{N}$ ranging from $0$ Hz to $\frac{N-1}{N}f_s$ Hz

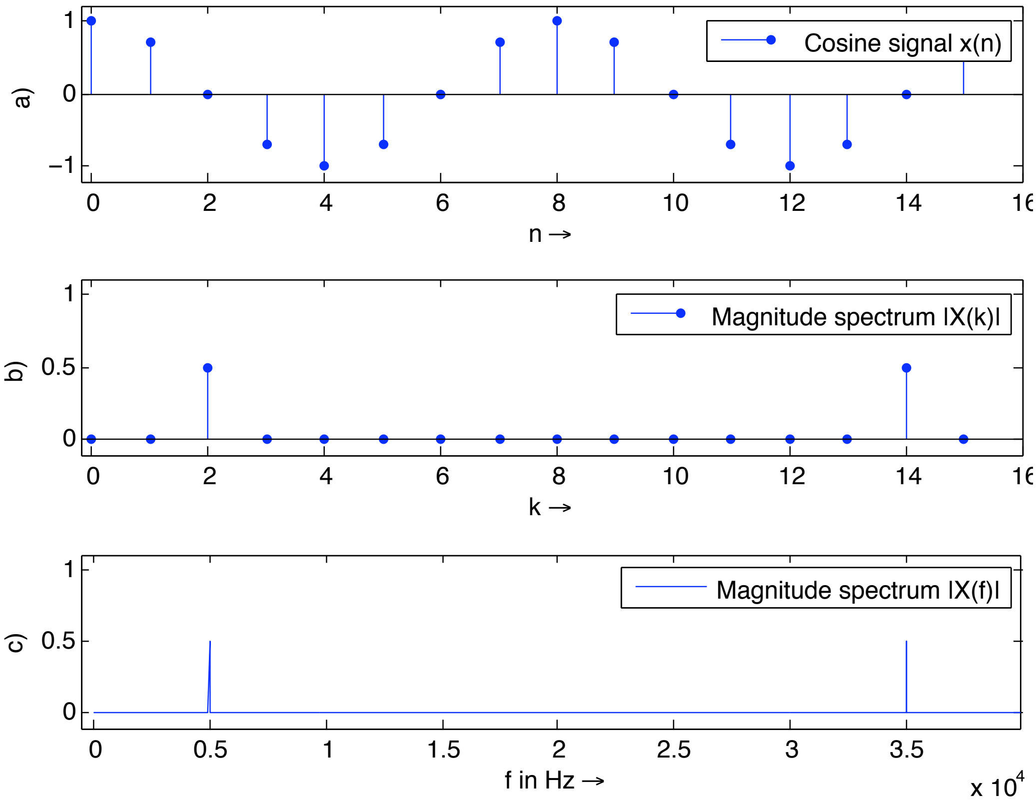
# MATLAB Fourier Frequency Spectra Example

The following code (fourierspectraeg.m):

```
N=16;
x=cos(2*pi*2*(0:1:N-1)/N)';

figure(1)
subplot(3,1,1);stem(0:N-1,x,'.');
axis([-0.2 N -1.2 1.2]);
legend('Cosine signal x(n)');
ylabel('a)');
xlabel('n \rightarrow');

X=abs(fft(x,N))/N;
subplot(3,1,2);stem(0:N-1,X,'.');
axis([-0.2 N -0.1 1.1]);
legend('Magnitude spectrum |X(k)|');
ylabel('b)');
xlabel('k \rightarrow')

N=1024;
x=cos(2*pi*(2*1024/16)*(0:1:N-1)/N)';

FS=40000;
f=((0:N-1)/N)*FS;
X=abs(fft(x,N))/N;
subplot(3,1,3);plot(f,X);
axis([-0.2*44100/16 max(f) -0.1 1.1]);
legend('Magnitude spectrum |X(f)|');
ylabel('c)');
xlabel('f in Hz \rightarrow')

figure(2)
subplot(3,1,1);
plot(f,20*log10(X./(0.5)));
axis([-0.2*44100/16 max(f) ...
-45 20]);
legend('Magnitude spectrum |X(f)| ...
in dB');
ylabel('|X(f)| in dB \rightarrow');
xlabel('f in Hz \rightarrow')
```

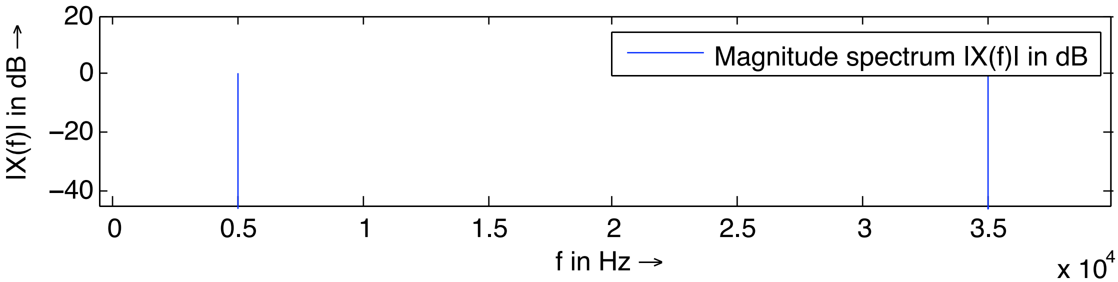
# MATLAB Fourier Frequency Spectra Example (Cont.)

The above code produces the following:



# Magnitude Spectrum in dB

**Note**: It is common to plot both spectra magnitude (also frequency ranges not show here) on a dB/log scale:
(Last Plot in above code)



# Time-Frequency Representation: Spectrogram

It is often useful to look at the frequency distribution over a short-time:

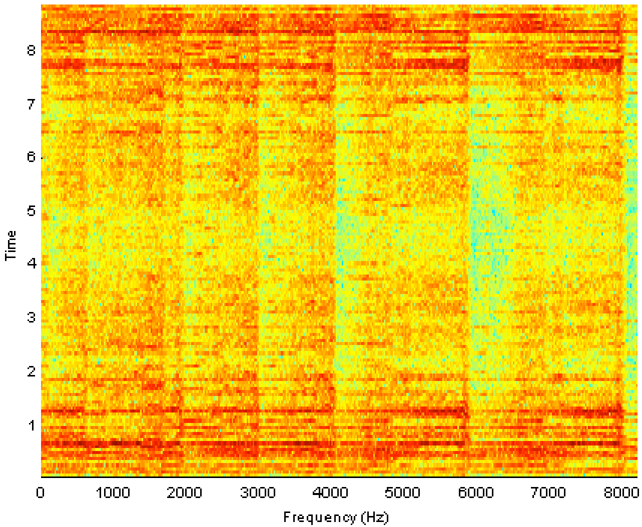
- Split signal into $N$ segments
- Do a **windowed** Fourier Transform
  - Window needed to reduce *leakage* effect of doing a short sample FFT.
  - Apply a Blackman, Hamming or Hanning Window
- MATLAB function does the job: `Spectrogram` — see `help spectrogram`
- See also MATLAB's `specgramdemo`

# MATLAB Example

The code:

```
load('handel')
[N M] = size(y);
figure(1)
spectrogram(fft(y,N),512,20,1024,Fs);
```

Produces the following:



# The Discrete Cosine Transform (DCT)

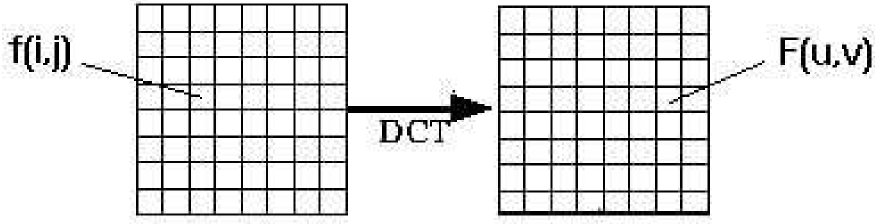
## Relationship between DCT and FFT

DCT (Discrete Cosine Transform) is actually a *cut-down* version of the Fourier Transform or the Fast Fourier Transform (FFT):

- Only the **real** part of FFT
- Computationally simpler than FFT
- DCT — Effective for Multimedia Compression
- DCT **MUCH** more commonly used (than FFT) in Multimedia Image/Video Compression — **more later**
- Cheap MPEG Audio Variant — **more later**

# Applying The DCT

- Similar to the discrete Fourier transform:
  - it transforms a signal or image from the spatial domain to the frequency domain
  - DCT can approximate lines well with fewer coefficients




- Helps separate the image into parts (or spectral sub-bands) of differing importance (with respect to the image's visual quality).

# 1D DCT

For $N$ data items 1D DCT is defined by:

$$F(u) = \left(\frac{2}{N}\right)^{\frac{1}{2}} \sum_{i=0}^{N-1} \Lambda(i).cos\left[\frac{\pi.u}{2.N}(2i+1)\right] f(i)$$

and the corresponding **inverse** 1D DCT transform is simple $F^{-1}(u)$, i.e.:

$$\begin{aligned} f(i) &= F^{-1}(u) \\ &= \left(\frac{2}{N}\right)^{\frac{1}{2}} \sum_{u=0}^{N-1} \Lambda(i).cos\left[\frac{\pi.u}{2.N}(2i+1)\right] F(u) \end{aligned}$$

where

$$\Lambda(\xi) = \begin{cases} \frac{1}{\sqrt{2}} & \text{for} \xi = 0 \\ 1 & \text{otherwise} \end{cases}$$

# 2D DCT

For a 2D $N$ by $M$ image 2D DCT is defined :

$$
\begin{aligned}
F(u,v) &= \left(\frac{2}{N}\right)^{\frac{1}{2}} \left(\frac{2}{M}\right)^{\frac{1}{2}} \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} \Lambda(i).\Lambda(j). \\
&\quad cos\left[\frac{\pi.u}{2.N}(2i+1)\right] cos\left[\frac{\pi.v}{2.M}(2j+1)\right].f(i,j)
\end{aligned}
$$

and the corresponding **inverse** 2D DCT transform is simple $F^{-1}(u,v)$, i.e.:

$$
\begin{aligned}
f(i,j) &= F^{-1}(u,v) \\
&= \left(\frac{2}{N}\right)^{\frac{1}{2}} \sum_{u=0}^{N-1} \sum_{v=0}^{M-1} \Lambda(u).\Lambda(u). \\
&\quad cos\left[\frac{\pi.u}{2.N}(2i+1)\right].cos\left[\frac{\pi.v}{2.M}(2j+1)\right].F(u,v)
\end{aligned}
$$

where

$$
\Lambda(\xi) = \begin{cases} \frac{1}{\sqrt{2}} & \text{for} \xi = 0 \\ 1 & \text{otherwise} \end{cases}
$$

# Performing DCT Computations

The basic operation of the DCT is as follows:

- The input image is N by M;
- f(i,j) is the intensity of the pixel in row i and column j;
- F(u,v) is the DCT coefficient in row $u_i$ and column $v_j$ of the DCT matrix.
- For JPEG image (and MPEG video), the DCT input is usually an 8 by 8 (or 16 by 16) array of integers.
  This array contains each image window's respective colour band pixel levels;

# Compression with DCT

- For most images, much of the signal energy lies at low frequencies;
  - These appear in the upper left corner of the DCT.
- Compression is achieved since the lower right values represent higher frequencies, and are often small
  - Small enough to be neglected with little visible distortion.

# Computational Issues (1)

- Image is partitioned into 8 x 8 regions — The DCT input is an 8 x 8 array of integers.
- An 8 point DCT would be:

$$
\begin{aligned}
F(u,v) &= \frac{1}{4}\sum_{i,j}\Lambda(i).\Lambda(j).cos\left[\frac{\pi.u}{16}(2i+1)\right].\\
&\quad cos\left[\frac{\pi.v}{16}(2j+1)\right]f(i,j)
\end{aligned}
$$

where

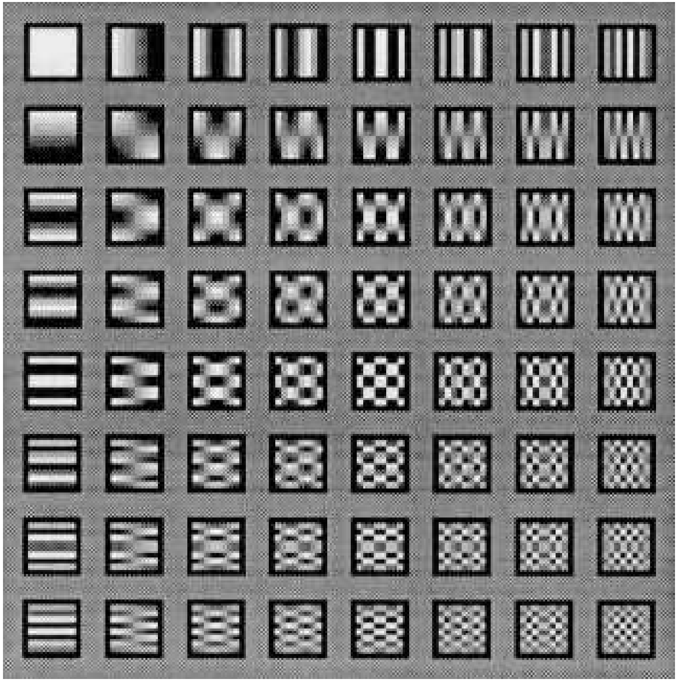
$$
\Lambda(\xi) = \begin{cases} \frac{1}{\sqrt{2}} & \text{for} \xi = 0 \\ 1 & \text{otherwise} \end{cases}
$$

- The output array of DCT coefficients contains integers; these can range from -1024 to 1023.

## Computational Issues (2)

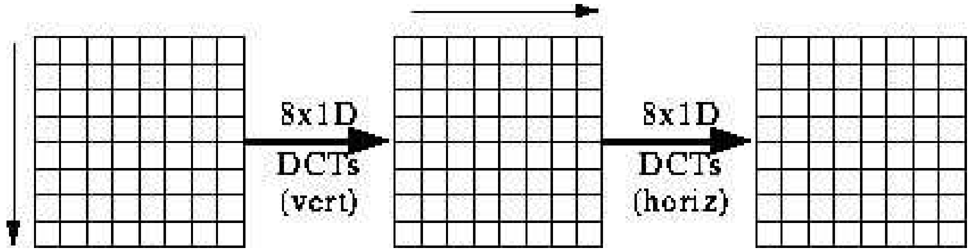
- Computationally easier to implement and more efficient to regard the DCT as a set of **basis functions**
  - Given a known input array size (8 x 8) can be precomputed and stored.
  - Computing values for a convolution mask (8 x 8 window) that get applied
    - Sum values $x$ pixel the window overlap with image apply window across all rows/columns of image
  - The values as simply calculated from DCT formula.

# Computational Issues (3)
# Visualisation of DCT basis functions



## Computational Issues (4)

- Factoring reduces problem to a series of 1D DCTs (No need to apply 2D form directly):
  - apply 1D DCT (Vertically) to Columns
  - apply 1D DCT (Horizontally) to resultant Vertical DCT above.
  - or alternatively Horizontal to Vertical.



## Computational Issues (5)

- The equations are given by:

$$G(i,v) = \frac{1}{2}\sum_i \Lambda(u).cos\left[\frac{\pi.v}{16}(2j+1)\right] f(i,j)$$

$$F(u,v) = \frac{1}{2}\sum_i \Lambda(u).cos\left[\frac{\pi.u}{16}(2i+1)\right] G(i,v)$$

- Most software implementations use fixed point arithmetic. Some fast implementations approximate coefficients so all multiplies are shifts and adds.

# Filtering in the Frequency Domain

FT and DCT methods pretty similar:

- DCT has less data overheads — no complex array part
- FT captures more frequency 'fidelity' (e.g. Phase).

## Low Pass Filter

*Example: Frequencies above the Nyquist Limit,*
*Noise*:

- The idea with noise smoothing is to reduce various spurious effects of a local nature in the image, caused perhaps by
  - noise in the acquisition system,
  - arising as a result of transmission of the data, for example from a space probe utilising a low-power transmitter.

# Frequency Space Smoothing Methods

**Noise = High Frequencies**:

- In audio data many spurious peaks in over a short timescale.
- In an image means there are many rapid transitions (over a short distance) in intensity from high to low and back again or vice versa, as faulty pixels are encountered.
- **Not all high frequency data noise though!**

Therefore noise will contribute heavily to the high frequency components of the image when it is considered in Fourier space.

Thus if we reduce the high frequency components — **Low-Pass Filter**, we should reduce the amount of noise in the data.

# (Low-pass) Filtering in the Fourier Space

We thus create a new version of the image in Fourier space by computing

$$G(u, v) = H(u, v)F(u, v)$$

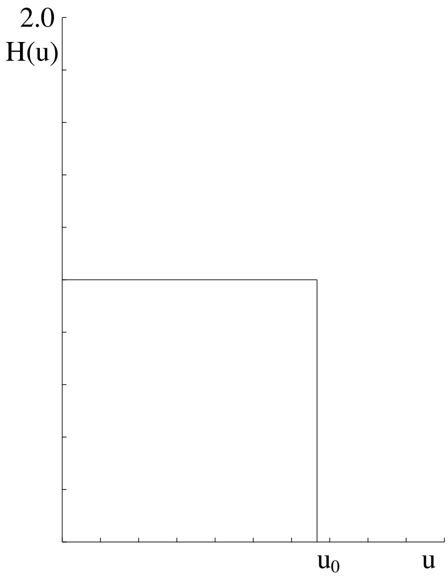
where:

- $F(u, v)$ is the Fourier transform of the original image,
- $H(u, v)$ is a filter function, designed to reduce high frequencies, and
- $G(u, v)$ is the **Fourier transform of the improved image**.
- Inverse Fourier transform $G(u, v)$ to get $g(x, y)$ our **improved image**

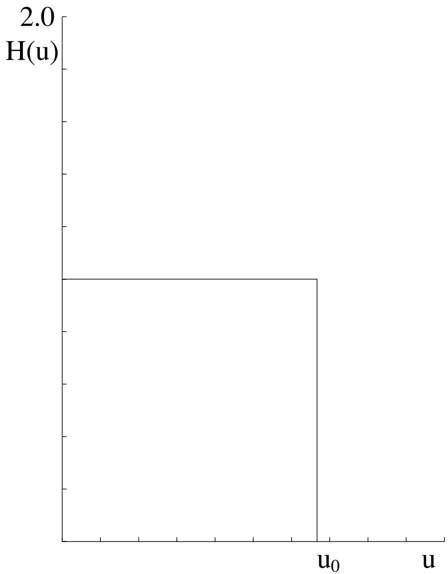
**Note:** Discrete Cosine Transform approach identical, sub. FT with DCT

# Ideal Low-Pass Filter

The simplest sort of filter to use is an *ideal low-pass filter*, which in one dimension appears as :



# Ideal Low-Pass Filter (Cont.)




This is a top hat function which is 1 for $u$ between $0$ and $u_0$, the *cut-off frequency*, and zero elsewhere.

- So All frequency space space information above $u_0$ is thrown away, and all information below $u_0$ is kept.
- A **very simple** computational process.

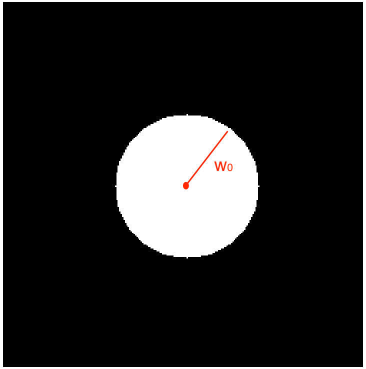
# Ideal 2D Low-Pass Filter

The two dimensional analogue of this is the function

$$H(u,v) = \begin{cases} 1 & \text{if } \sqrt{u^2+v^2} \leq w_0 \\ 0 & \text{otherwise,} \end{cases}$$

where $w_0$ is now the cut-off frequency.

Thus, all frequencies inside a radius $w_0$ are kept, and all others discarded.



# Not So Ideal Low-Pass Filter?

The problem with this filter is that as well as the noise:

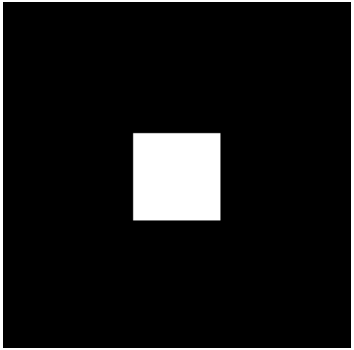
- In audio: plenty of other high frequency content
- In Images: edges (places of rapid transition from light to dark) also significantly contribute to the high frequency components.

Thus an ideal low-pass filter will tend to *blur* the data:

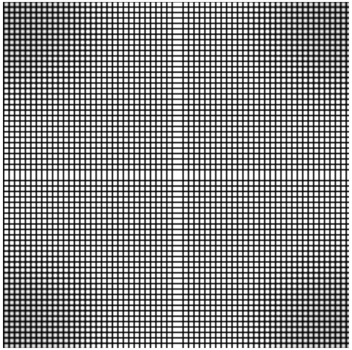
- High audio frequencies become muffled
- Edges in images become blurred.

The lower the cut-off frequency is made, the more pronounced this effect becomes in *useful data content*

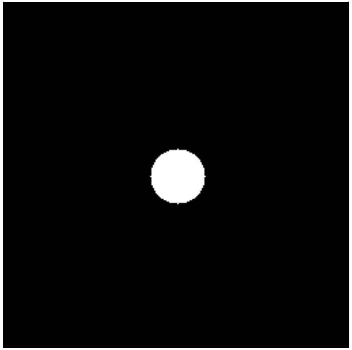
# Ideal Low Pass Filter Example 1



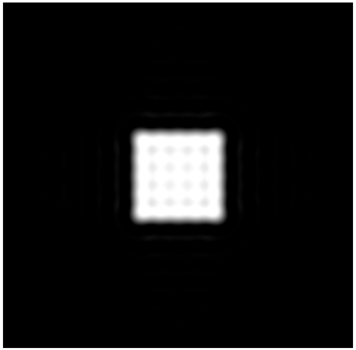
(a) Input Image



(b) Image Spectra



(c) Ideal Low Pass Filter



(d) Filtered Image

# Ideal Low-Pass Filter Example 1 MATLAB Code

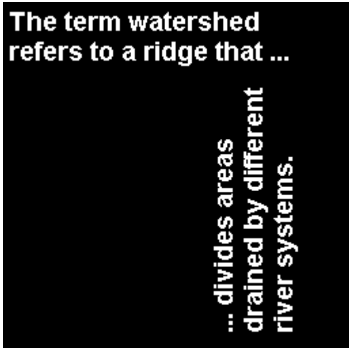
low pass.m:

```
% Create a white box on a black background image
M = 256; N = 256;
image = zeros(M,N)
box = ones(64,64);
%box at centre
image(97:160,97:160) = box;

% Show Image

figure(1);
imshow(image);

% compute fft and display its spectra

F=fft2(double(image));
figure(2);
imshow(abs(fftshift(F)));
```

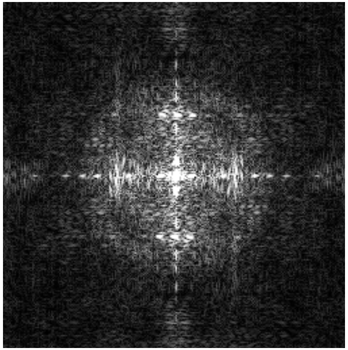
# Ideal Low-Pass Filter Example 1 MATLAB Code (Cont.)

```
%compute Ideal Low Pass Filter
u0 = 20; % set cut off frequency

u=0:(M-1);
v=0:(N-1);
idx=find(u>M/2);
u(idx)=u(idx)-M;
idy=find(v>N/2);
v(idy)=v(idy)-N;
[V,U]=meshgrid(v,u);
D=sqrt(U.^2+V.^2);
H=double(D<=u0);

% display
figure(3);
imshow(fftshift(H));

% Apply filter and do inverse FFT
G=H.*F;
g=real(ifft2(double(G)));

% Show Result
figure(4);
imshow(g);
```

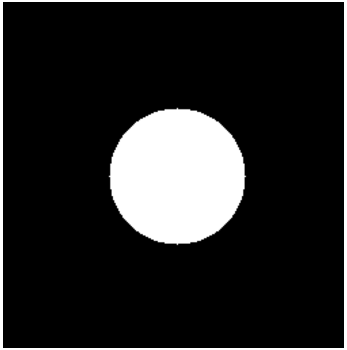
# Ideal Low-Pass Filter Example 2




(a) Input Image



(b) Image Spectra



(c) Ideal Low-Pass Filter


(d) Filtered Image



Back
Close

# Ideal Low-Pass Filter Example 2 MATLAB Code

lowpass2.m:

```
% read in MATLAB demo text image
image = imread('text.png');
[M N] = size(image)


% Show Image

figure(1);
imshow(image);

% compute fft and display its spectra

F=fft2(double(image));
figure(2);
imshow(abs(fftshift(F))/256);
```

# Ideal Low-Pass Filter Example 2 MATLAB Code (Cont.)

```
%compute Ideal Low Pass Filter
u0 = 50; % set cut off frequency

u=0:(M-1);
v=0:(N-1);
idx=find(u>M/2);
u(idx)=u(idx)-M;
idy=find(v>N/2);
v(idy)=v(idy)-N;
[V,U]=meshgrid(v,u);
D=sqrt(U.^2+V.^2);
H=double(D<=u0);

% display
figure(3);
imshow(fftshift(H));

% Apply filter and do inverse FFT
G=H.*F;
g=real(ifft2(double(G)));

% Show Result
figure(4);
imshow(g);
```

# Low-Pass Butterworth Filter

Another filter sometimes used is the *Butterworth low pass filter*.

In the 2D case, $H(u, v)$ takes the form

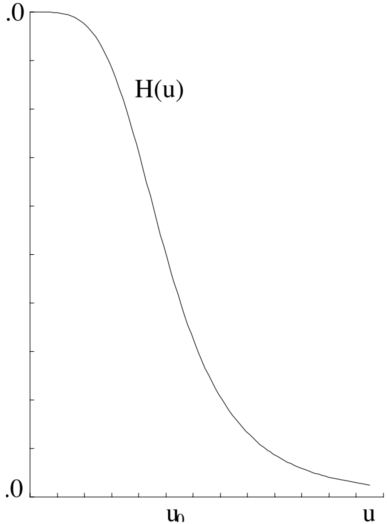
$$H(u, v) = \frac{1}{1 + [(u^2 + v^2)/w_0^2]^n},$$

where $n$ is called the **order** of the filter.

## Low-Pass Butterworth Filter (Cont.)



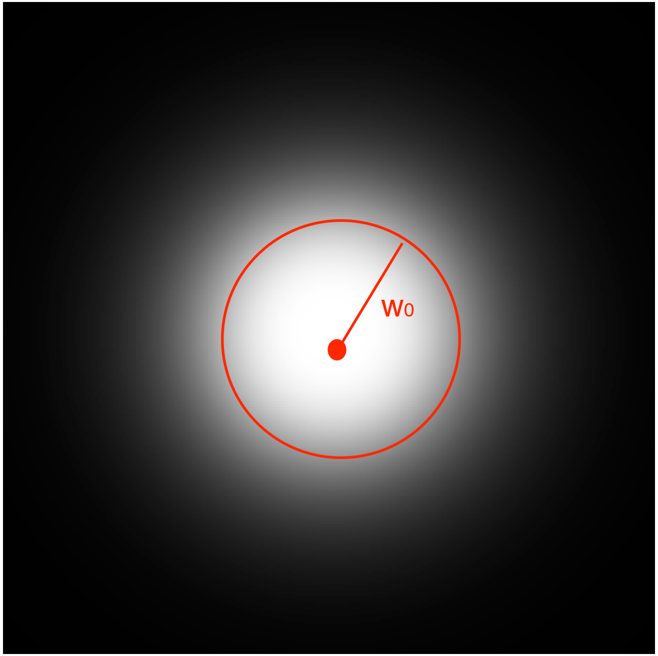
This keeps some of the high frequency information, as illustrated by the second order one dimensional Butterworth filter:



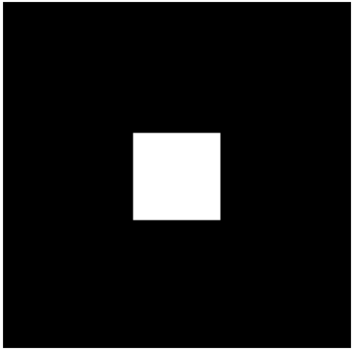

Consequently reduces the blurring.

# Low-Pass Butterworth Filter (Cont.)

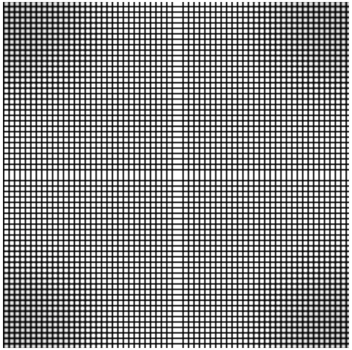
The 2D second order Butterworth filter looks like this:



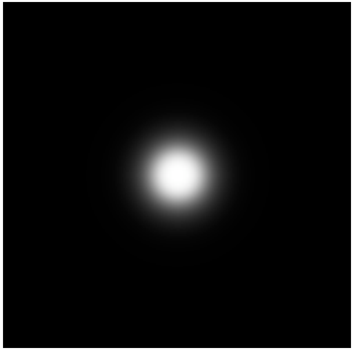
# Butterworth Low Pass Filter Example 1



(a) Input Image



(b) Image Spectra



(c) Butterworth Low-Pass Filter

(d) Filtered Image

# Butterworth Low-Pass Filter Example 1 (Cont.)

Comparison of Ideal and Butterworth Low Pass Filter:

Ideal Low-Pass

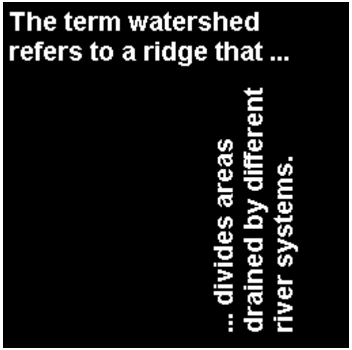
Butterworth Low Pass

# Butterworth Low-Pass Filter Example 1 MATLAB Code

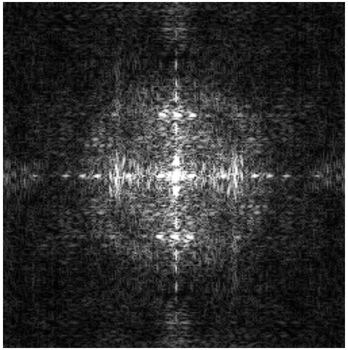
butterworth.m:

```
% Load Image and Compute FFT as in Ideal Low Pass Filter
% Example 1
.......
% Compute Butterworth Low Pass Filter
u0 = 20; % set cut off frequency

u=0:(M-1);
v=0:(N-1);
idx=find(u>M/2);
u(idx)=u(idx)-M;
idy=find(v>N/2);
v(idy)=v(idy)-N;
[V,U]=meshgrid(v,u);

for i = 1: M
    for j = 1:N
      %Apply a 2nd order Butterworth
      UVw = double((U(i,j)*U(i,j) + V(i,j)*V(i,j))/(u0*u0));
      H(i,j) = 1/(1 + UVw*UVw);
    end
end
% Display Filter and Filtered Image as before
```

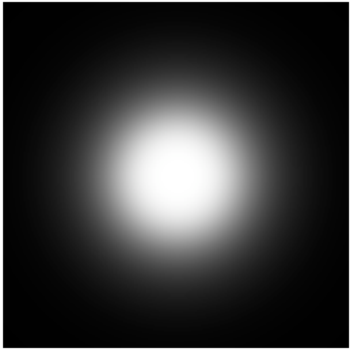
# Butterworth Low-Pass Butterworth Filter Example 2



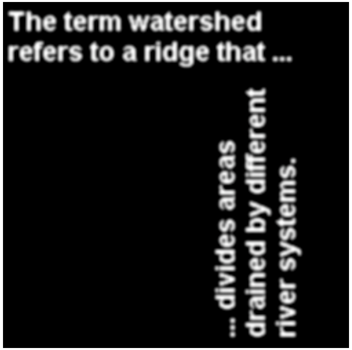

(a) Input Image



(b) Image Spectra



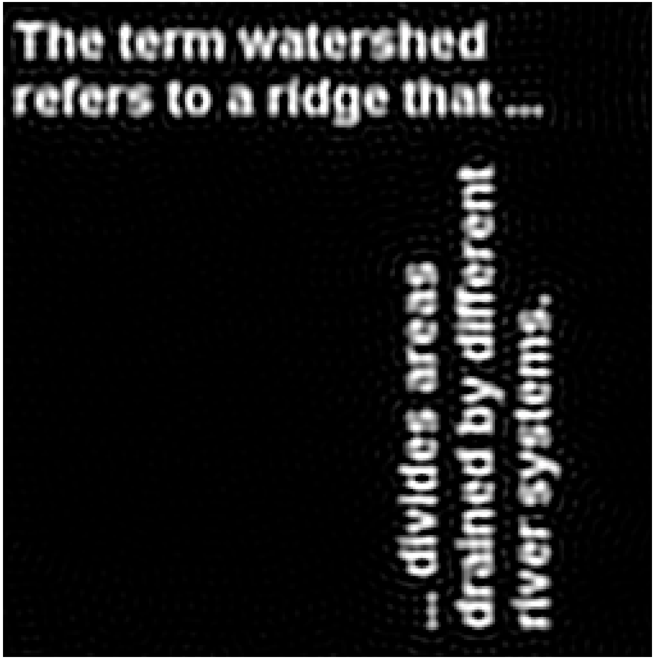
(c) Butterworth Low-Pass Filter



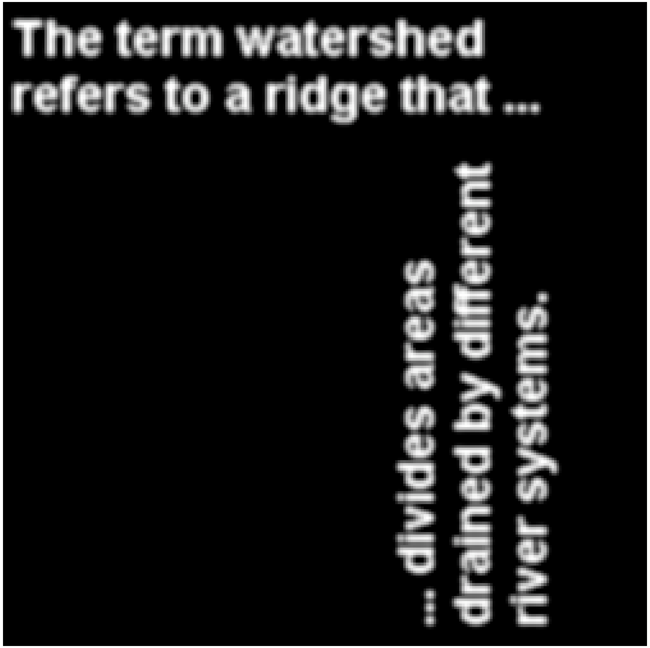

(d) Filtered Image

# Butterworth Low-Pass Filter Example 2 (Cont.)

Comparison of Ideal and Butterworth Low-Pass Filter:




Ideal Low Pass




Butterworth Low Pass

# Butterworth Low Pass Filter Example 2 MATLAB Code

butterworth2.m:

```
% Load Image and Compute FFT as in Ideal Low Pass Filter
% Example 2
.......
% Compute Butterworth Low Pass Filter
u0 = 50; % set cut off frequency

u=0:(M-1);
v=0:(N-1);
idx=find(u>M/2);
u(idx)=u(idx)-M;
idy=find(v>N/2);
v(idy)=v(idy)-N;
[V,U]=meshgrid(v,u);

for i = 1: M
    for j = 1:N
      %Apply a 2nd order Butterworth
      UVw = double((U(i,j)*U(i,j) + V(i,j)*V(i,j))/(u0*u0));
      H(i,j) = 1/(1 + UVw*UVw);
    end
end
% Display Filter and Filtered Image as before
```

# Other Filters

**High-Pass Filters** — opposite of low-pass, select high frequencies, attenuate those **below** $u_0$

**Band-pass** — allow frequencies in a range $u_0 \ldots u_1$ attenuate those outside this range

**Band-reject** — opposite of band-pass, attenuate frequencies within $u_0 \ldots u_1$ **select** those **outside** this range

**Notch** — attenuate frequencies in a narrow bandwidth around cut-off frequency, $u_0$

**Resonator** — amplify frequencies in a narrow bandwidth around cut-off frequency, $u_0$

Other filters exist that are a combination of the above

# Convolution

Several important audio and optical effects can be described in terms of convolutions.

- In fact the above Fourier filtering is applying convolutions of low pass filter where the equations are Fourier Transforms of real space equivalents.
- deblurring — high pass filtering
- reverb — see CM0268.

# 1D Convolution

Let us examine the concepts using 1D continuous functions.

The convolution of two functions $f(x)$ and $g(x)$, written $f(x) * g(x)$, is defined by the integral

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(\alpha) g(x - \alpha)\, d\alpha.$$

# 1D Convolution Example

For example, let us take two top hat functions of the type described earlier.

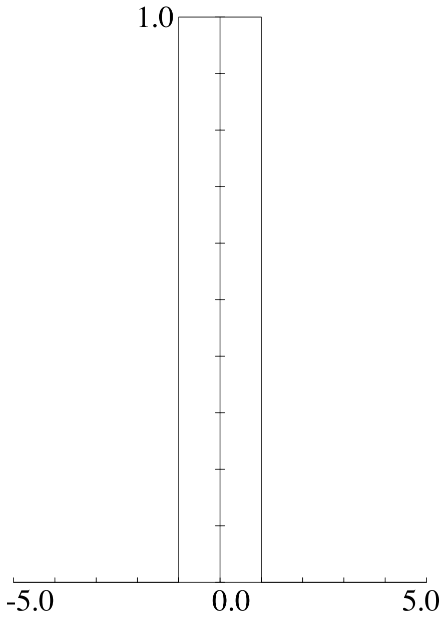
Let $f(\alpha)$ be the top hat function shown:

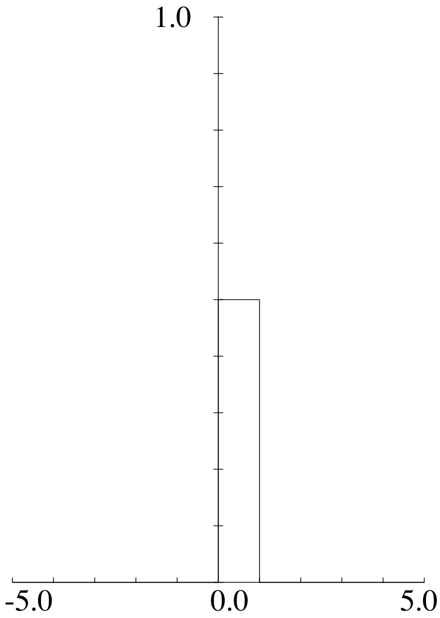
$$f(\alpha) = \begin{cases} 1 & \text{if } |\alpha| \leq 1 \\ 0 & \text{otherwise,} \end{cases}$$

and let $g(\alpha)$ be as shown in next slide, defined by

$$g(\alpha) = \begin{cases} 1/2 & \text{if } 0 \leq \alpha \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

# 1D Convolution Example (Cont.)

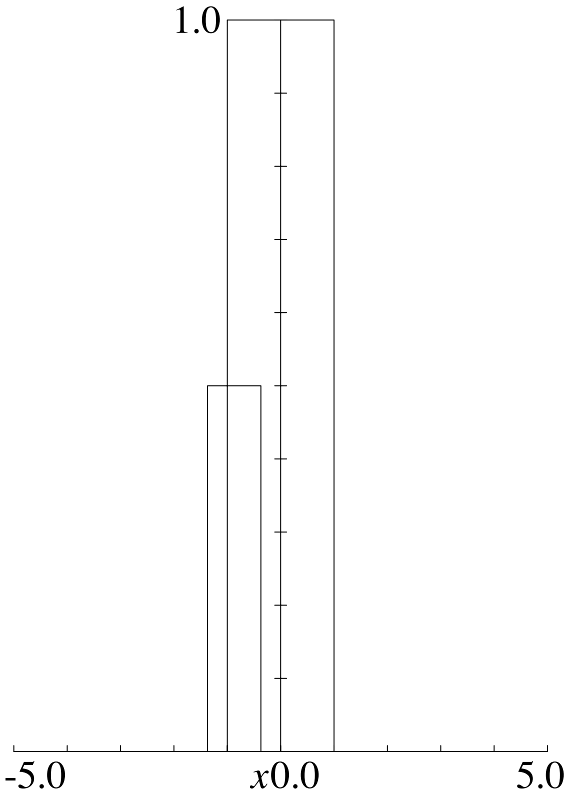



$$f(\alpha) = \begin{cases} 1 & \text{if } |\alpha| \leq 1 \\ 0 & \text{otherwise,} \end{cases} \qquad g(\alpha) = \begin{cases} 1/2 & \text{if } 0 \leq \alpha \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

# 1D Convolution Example (Cont.)

- $g(-\alpha)$ is the reflection of this function in the vertical axis,
- $g(x-\alpha)$ is the latter shifted to the right by a distance $x$.
- Thus for a given value of $x$, $f(\alpha)g(x-\alpha)$ integrated over all $\alpha$ is the area of overlap of these two top hats, as $f(\alpha)$ has unit height.
- An example is shown for $x$ in the range $-1 \le x \le 0$ opposite

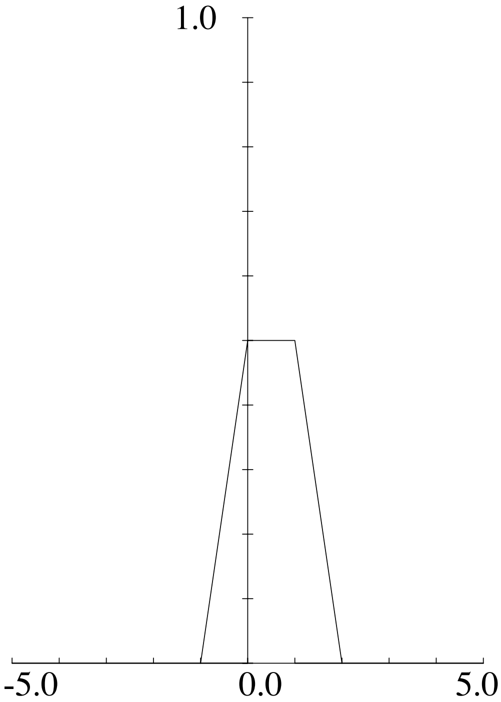


# 1D Convolution Example (cont.)

If we now consider $x$ moving from $-\infty$ to $+\infty$, we can see that

- For $x \leq -1$ or $x \geq 2$, there is no overlap;
- As $x$ goes from $-1$ to $0$ the area of overlap steadily increases from $0$ to $1/2$;
- As $x$ increases from $0$ to $1$, the overlap area remains at $1/2$;
- Finally as $x$ increases from $1$ to $2$, the overlap area steadily decreases again from $1/2$ to $0$.
- Thus the convolution of $f(x)$ and $g(x)$, $f(x) * g(x)$, in this case has the form shown on next slide

# 1D Convolution Example (cont.)




Result of $f(x) * g(x)$

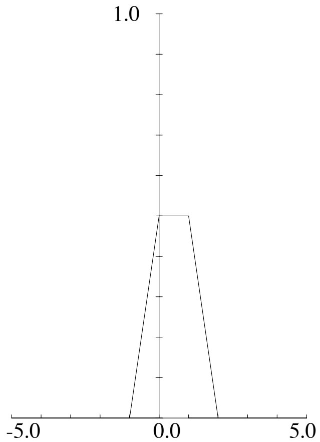



Back
Close

# 1D Convolution Example (cont.)

Mathematically the convolution is expressed by:

$$f(x) * g(x) = \begin{cases} (x+1)/2 & \text{if } -1 \leq x \leq 0 \\ 1/2 & \text{if } 0 \leq x \leq 1 \\ 1 - x/2 & \text{if } 1 \leq x \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$



# Fourier Transforms and Convolutions

One major reason that Fourier transforms are so important in image processing is the **convolution theorem** which states that:

*If $f(x)$ and $g(x)$ are two functions with Fourier transforms $F(u)$ and $G(u)$, then the Fourier transform of the convolution $f(x) * g(x)$ is simply the product of the Fourier transforms of the two functions, $F(u)G(u)$.*

## Recall our Low Pass Filter Example (MATLAB CODE)

```
% Apply filter
G=H.*F;
```

Where `F` was the Fourier transform of the image, `H` the filter

# Computing Convolutions with the Fourier Transform

*E.g.*:

- To apply some reverb to an audio signal, **example later**
- To compensate for a less than ideal image capture system:

To do this **fast convolution** we simply:

- Take the Fourier transform of the audio/imperfect image,
- Take the Fourier transform of the function describing the effect of the system,
- Multiply by the effect to apply effect to audio data
- To remove/compensate for effect: Divide by the effect to obtain the Fourier transform of the ideal image.
- Inverse Fourier transform to recover the new audio/ideal image.

This process is sometimes referred to as **deconvolution**.