Module 4

Signal Representation and Baseband Processing

Lesson

20

Matched Filter

After reading this lesson, you will learn about

- ***Principle of matched filter (MF);***
- ***Properties of a matched filter;***
- ***SNR maximization and minimization of average symbol error probability;***
- ***Schwartz's Inequality;***

Certain structural modification and simplifications of the correlation receiver are possible by observing that,

(a) All orthonormal basis functions $\varphi_j - s$ are defined between $0 \le t \le T_b$ and they are zero outside this range .

(b) Analog multiplication, which is not always very simple and accurate to implement, of the received signal r(t) with time limited basis functions may be replaced by some filtering operation.

Let, $h_j(t)$ represent the impulse response of a linear filter to which r(t) is applied.

Then, the filter output $y_j(t)$ may be expressed as:

$$y_j(t) = \int_{\infty}^{\infty} r(\tau)\, h_j(t-\tau) d\tau \qquad 4.20.1$$

Now, let, $h_j(t) = \varphi_j(T-t)$, a time reversed and time-shifted version of $\varphi_j(t)$.

$$\text{Now, } y_j(t) = \int_{-\infty}^{\infty} r(\tau).\varphi_j[T-(t-\tau)]d\tau$$

$$= \int_{-\infty}^{\infty} r(\tau).\varphi_j(T+\tau-t)d\tau \qquad 4.20.2$$

If we sample this output at t = T,

$$y_j(T) = \int_{-\infty}^{\infty} (\tau).\varphi_j(\tau)d\tau \qquad 4.20.3$$

Let us recall that $\varphi_j(t)$ is zero outside the interval $0 \le t \le T$. Using this, the above equation may be expressed as,

$$y_j(T) = \int_0^T r(\tau)\varphi_j(\tau)d\tau$$

From our discursion on correlation receiver, we recognize that,

$$r_j = \int_0^T r(\tau)\varphi_j(\tau)d\tau = y_j(\tau) \qquad 4.20.4$$

The important expression of (Eq.4.20.4) tells us that the j – th correlation output can equivalently be obtained by using a filter with $h_j(t) = \varphi_j(T-t)$ and sampling its output at t = T.

The filter is said to be matched to the orthonormal basis function $\varphi_j(t)$ and the alternation receiver structure is known as a matched filter receiver. The detector part of the matched filter receiver is shown in [**Fig.4.20.1**].

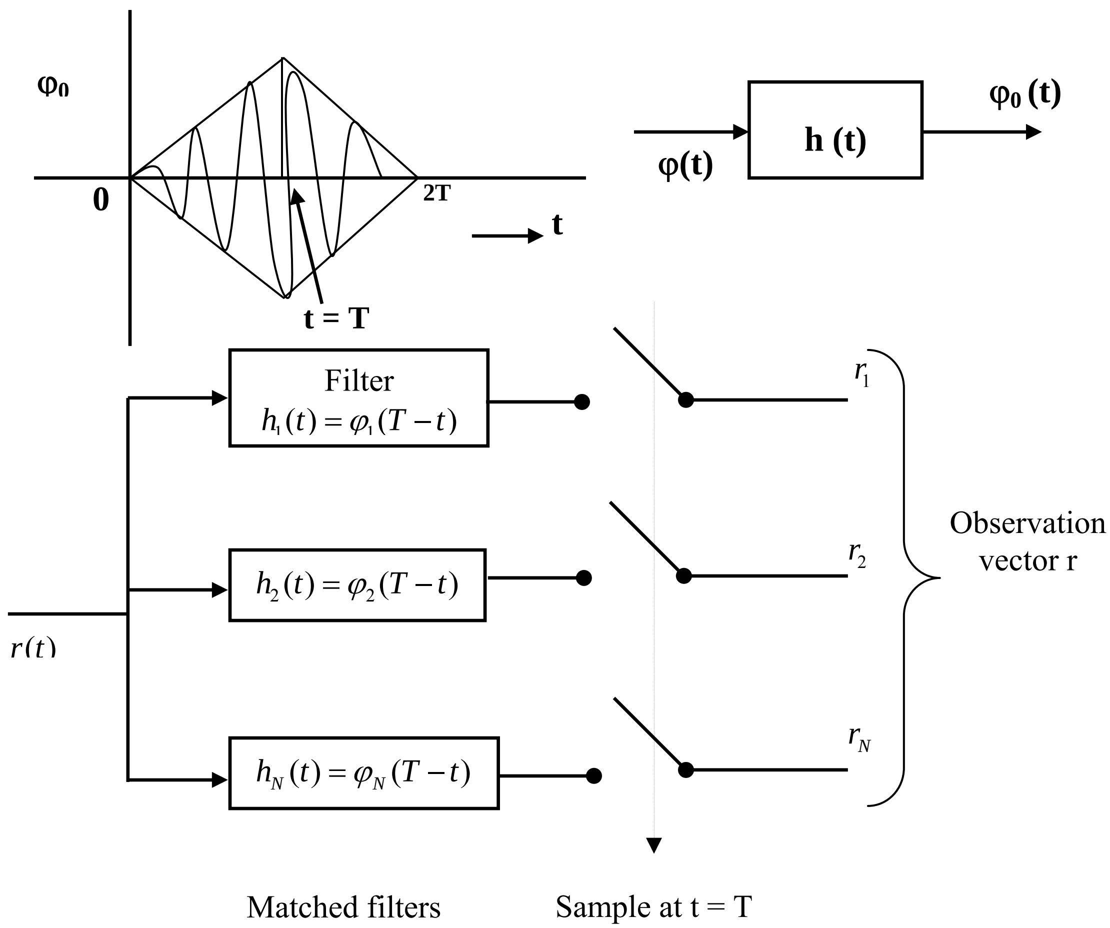

Fig. 4.20.1: *The block diagram of a matched filter bank that is equivalent to a Correlation Detector*

A physically realizable matched filter is to be causal and $h_j(t)=0$ *for* $t<0$. Note that if $\varphi_j(t)$ is zero outside $0 \le t \le T$, $h_j(t)=\varphi_j(T-t)$ is a causal impulse response.

Properties of a Matched Filter

We note that a filter which is matched to a known signal $\varphi(t), 0 \le t \le T$, is characterized by an impulse response h(t) which is a time reversed and delayed version of $\varphi(t)$ i.e.

$$h(t)=\varphi(T-t) \qquad 4.20.5$$

In the frequency domain, the matched filter is characterized (without mach explanation at this point), by a transfer function, which is, except for a delay factor, the complex conjugate of the F.T. of $\varphi(t)$, i.e.

$$\mathrm{H}(f) = \Phi^*(f)\exp(-j2\pi fT) \qquad 4.20.6$$

Property (1) : The spectrum of the output signal of a matched filter with the matched signal as input is, except for a time delay factor, proportional to the energy spectral density of the input signal.

Let, $\Phi_0(f)$ denote the F.T. of the filter of output $\varphi_0(t)$. Then,

$$\begin{aligned}\Phi_0(f) &= \mathrm{H}(f)\Phi(f)\\ &= \Phi^*(f)\Phi(f)\exp^{(-j2\pi fT)}\\ &= \underbrace{|\Phi(f)|^2}_{\substack{\text{Energy spectral}\\ \text{density of}\varphi(\text{t})}}\exp^{(-j2\pi fT)}\end{aligned} \qquad 4.20.7$$

Property (2): The output signal of a matched filter is proportional to a shifted version of the autocorrelation function of the in the input signal to which the filter is matched.

This property follows from Property (1). As the auto-correlation function and the energy spectral density form F.T. pair, by taking IFT of (Eq.4.20.7), we may write,

$$\varphi_0(t) = R_\varphi(t-T) \qquad 4.20.8$$

Where $R_\varphi(\tau)$ is the act of $\varphi(t)$ for $'lag\ \tau'$. Note that at t = T,

$$R_\varphi(0) = \varphi_0(t) = \text{Energy of } \varphi(t). \qquad 4.20.9$$

Property (3): The output SNR of a matched filter depends only on the ratio of the signal energy to the psd of the white noise at the filter input.

Let us consider a filter matched to the input signal $\varphi(t)$.

From property (2), we see that the maximum value of $\varphi_0(t)$ at t = T is $\varphi_0(t-T) = E$.

Now, it may be shown that the average noise power at the output of the matched filter is given by, $E[n^2(t)] = \frac{N_0}{2}\int_{-\infty}^{\infty}|\varphi(f)|^2\,df = \frac{N_0}{2}.E$ 4.20.10

The maximum signal power $= |\varphi_0(T)|^2 = E^2$.

Hence, $$(SNR)_{\max} = \frac{E^2}{\frac{N_0}{2}E} = \frac{2E}{N_0} \qquad 4.20.11$$

Note that SNR in the above expression is a dimensionless quantity.

This is a very significant result as we see that the $SNR_{\max}$ depends on E and N_0 but not on the shape of $\varphi(t)$. This means a freedom to the designer to select specific pulse shape to optimize other design requirement (the most usual requirement being the spectrum or, equivalently, the transmission bandwidth) while ensuring same SNR.

Property (4): The matched-filtering operation may be separated into two matching condition: namely, spectral phase matching that produces the desired output peak at t = T and spectral amplitude matching that gives the peak value its optimum SNR.

$$\Phi(f) = |\Phi(f)| \exp[j\theta(f)] \qquad 4.20.12$$

The filter is said to be matched to the signal $\varphi(t)$ in spectral phase if the transfer function of the filter follows:

$$\mathrm{H}(f) = |\mathrm{H}(f)| \exp[-j\theta(f) - j2\pi fT] \qquad 4.20.13$$

Here $|\mathrm{H}(f)|$ is real non-negative and 'T' is a positive constant.

The output of such a filter is,

$$\varphi_0{}'(t) = \int_{-\infty}^{\infty} \mathrm{H}(f).\Phi(f), \exp(j2\pi ft)df$$

$$= \int_{-\infty}^{\infty} |\mathrm{H}(f)||\Phi(f)|.\exp[j2\pi f(t-T)df$$

Note that, $|\mathrm{H}(t)||\varphi(t)|$ is real and non-negative. Spectral phase matching ensures that all spectral components of $\varphi_0{}'(t)$ add constructively at t = T and thus cause maximum value of the output:

$$\varphi_0{}'(T) = \int_{-\infty}^{\infty} |\Phi(f)||\mathrm{H}(f)| df. \geq \varphi_0{}'(t) \qquad 4.20.14$$

For spectral amplitude matching, we choose the amplitude response $|\mathrm{H}(f)|$ of the filter to shape the output for best SNR at t = T by using $|\mathrm{H}(f)| = |\Phi(f)|$. The standard matched filter achieves both these features.

Maximization of output Signal –to-Noise Raito:

Let, h(t) be the impulse response of a linear filter and $x(t) = \varphi(t) + \omega(t), 0 \leq t \leq T$: is the input to the filter where $\varphi(t)$ is a known signal and $\omega(t)$ is an additive white noise sample function with zero mean and psd of $(N_0/2)$ Watt/Hz. Let, $\varphi(t)$ be one of the orthonormal basis functions. As the filter is linear, its output can be expressed as, $y(t) = \varphi_0(t) + n(t)$, where $\varphi_0(t)$ is the output due to the signal component $\varphi(t)$ and n(t) is the output due to the noise component $\omega(t)$.[**Fig. 4.20.2**].

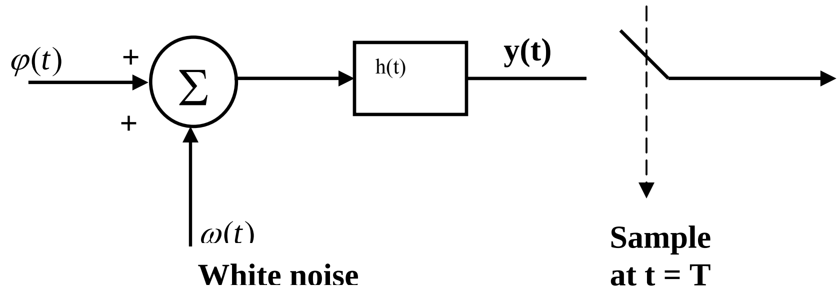

Fig. 4.20.2: *A matched filter is fed with a noisy basis function to which it is matched*

We can now re-frame the requirement of minimum probability of error (or maximum likelihood detection) as: The filter should make power of $\varphi_0(t)$ considerably greater (in fact, as large as possible) compared to the power of n(t) at t = T. That is, the filter should maximize the output signal-to-noise power ratio $[(SNR)_0]$

$$\triangleq \left[\frac{|\varphi_0(T)|^2}{E[n^2(t)]}\right]\max$$

The following discussion shows that the SNR is indeed maximized when h(t) is matched to the known input signal $\varphi(t)$.

Let, $\Phi(f)$: F.T. of known signal $\varphi(t)$

$\mathrm{H}(f)$: Transfer function of the linear filter.

$$\therefore \Phi_0(f) = \mathrm{H}(f)\Phi(f)$$

and $$\Phi_0(t) = \int_{-\infty}^{\infty} \mathrm{H}(f)\Phi(f)\exp(j2\pi ft)df \qquad 4.20.15$$

The filter output is sampled at t = T. Now,

$$|\varphi_0(T)|^2 = \left|\int_{-\infty}^{\infty} \mathrm{H}(f)\Phi(f)\exp(j2\pi fT)df\right|^2 \qquad 4.20.16$$

Let, $S_N(f)$: Power spectral density of noise at the output of the linear filter. So,

$$S_N(f) = \frac{N_0}{2}.|\mathrm{H}(f)|^2 \qquad 4.20.17$$

Now, the average noise power at the output of the filter

$$= E[n^2(t)] = \int_{-\infty}^{\infty} S_N(f)df$$

$$= \frac{N_0}{2} \int_{-\infty}^{\infty} |\mathrm{H}(f)|^2 \, df \qquad 4.20.18$$

Form Eq. 4.20.16 and 4.20.18, we can write an expression of the output SNR as:

$$(SNR)_0 = \frac{|\varphi^2(T)|^2}{E[n^2(t)]} = \frac{\left| \int_{-\infty}^{\infty} \mathrm{H}(f).\varphi(f)\exp(j2\pi fT)df \right|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |\mathrm{H}(f)|^2 \, df} \qquad 4.20.19$$

Our aim now is to find a suitable form of H(f) such that $(\mathrm{SNR})_0$ is maximized. We use Schwarz's inequality for the purpose.

Schwarz's Inequality

Let $\overline{x}(t)$ and $\overline{y}(t)$ denote any pair of complex-valued signals with finite energy, i.e. $\int_{-\infty}^{\infty} |\overline{x}(t)|^2 \, dt < \infty$ & $\int_{-\infty}^{\infty} |\overline{y}(t)|^2 \, dt < \infty$. Schwarz's Inequality states that,

$$\left| \int_{-\infty}^{\infty} \overline{x}(t)\overline{y}(t)dt \right|^2 \leq \int_{-\infty}^{\infty} |\overline{x}(t)|^2 \, dt . \int_{-\infty}^{\infty} |\overline{y}(t)|^2 \, dt. \qquad 4.20.20$$

The equality holds if and only if $\overline{y}(t) = k.\overline{x}^*(t)$, where 'k' is a scalar constant. This implies, $\overline{y}(t)\overline{x}(t) = k.\overline{x}(t)\overline{x^*}(t) \rightarrow$ a real quantity.

Now, applying Schwarz's inequality on the numerator of (Eq.4.20.19), we may write,

$$\left| \int_{-\infty}^{\infty} \mathrm{H}(f)\Phi(f)\exp(j2\pi fT)df \right|^2 \leq \int_{-\infty}^{\infty} |\mathrm{H}(f)|^2 \, df \int_{-\infty}^{\infty} |\Phi(f)|^2 \, df \qquad 4.20.21$$

Using inequality (4.20.21), equation (4.20.19) may be expressed as,

$$(SNR)_0 \leq \frac{2}{N_0} \int_{-\infty}^{\infty} |\varphi(f)|^2 \, df \qquad 4.20.22$$

Now, from Schwarz's inequality, the SNR is maximum i.e. the equality holds, when

$\mathrm{H}_{opt}(f) = \Phi^*(f).\exp(-j2\pi fT)$. [Assuming k = 1, a scalar)

We see, $h_{opt}(t) = \int_{-\infty}^{\infty} \Phi^*(f)\exp[-j2\pi(T-t)]df$ 4.20.23

Now, $\varphi(t)$ is a real valued signal and hence,

$$\Phi^*(f) = \Phi(-f) \qquad 4.20.24$$

Using Eq. 4.20.24 we see,

$$h_{opt}(t) = \int_{-\infty}^{\infty} \Phi(-f)\exp[-j2\pi)(T-t)]df \ = \varphi(T-t)$$

$$\therefore h_{opt}(t) = \Phi(T-t)$$

This relation is the same as we obtained previously for a matched filter receiver. So, we can infer that, *SNR maximization is an operation, which is equivalent to minimization of average symbol error (P_e) for an AWGN Channel.*

Example #4.20.1: Let us consider a sinusoid, defined below as the basis function:

$$\varphi(t) = \begin{cases} \sqrt{\dfrac{2}{T}}\cos w_c t, \ 0 \le t \le T \\ 0, \quad elsewhere. \end{cases}$$

$$h_{opt.}(t) = \varphi(T-t) = \varphi(t). \qquad h(t) = \varphi(T-t) = \varphi(t)$$

$$\varphi_0(t) = \begin{cases} \dfrac{t}{T}\cos w_c t. & 0 \le t \le T \\ \left(2-\dfrac{t}{T}\right)\cos w_c t. & T \le t \le 2T \\ 0 & \text{else.} \end{cases}$$

Problems

Q4.20.1) Under what conditions matched filter may be considered equivalent to an optimum correlation receiver?

Q4.20.2) Is a matched filter equivalent to an optimum correlation receiver if sampling is not possible at the right instants of time?

Q4.20.3) Explain the significance of the fact that a matched filter ensures maximum output signal-to-noise ratio.