Matsci 406 A2

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1

To derive the weak form for $u(x)$, we start with the strong form given in the problem:

$$\frac{d}{dx}\left(AE\,\frac{du}{dx}\right) + b = 0 \quad \text{on} \quad 0 < x < 6 \tag{1}$$

$$u(0) = 0\,\mathrm{m} \tag{2}$$

$$\sigma(6) = \left(E\,\frac{du}{dx}\right)_{x=6} = -4\times 10^{6}\,\mathrm{N/m^2} = t_\ell \tag{3}$$

Multiply (1) and (3) by $w(x)$, and integrate (1) over the domain:

$$\int_0^\ell w\left[\frac{d}{dx}\left(AE\,\frac{du}{dx}\right) + b\right]dx = 0 \tag{4}$$

$$wA\left[\left(E\,\frac{du}{dx}\right) - t_\ell\right]_{x=6} = 0 \tag{5}$$

Next, we manipulate equation (4). Letting $f = AE\,\frac{du}{dx}$, equation (4) becomes:

$$\int_0^\ell w\left(\frac{df}{dx} + b\right)dx = 0 \quad \text{, with} \tag{6}$$

$$\begin{aligned}\int_0^\ell w\,\frac{df}{dx}dx &= wf\Big|_0^\ell - \int_0^\ell f\,\frac{dw}{dx}dx \\ &= wAE\,\frac{du}{dx}\Big|_0^\ell - \int_0^\ell \frac{dw}{dx}AE\,\frac{du}{dx}dx\end{aligned} \tag{7}$$

$$\implies \quad wAE\,\frac{du}{dx}\Big|_0^\ell - \int_0^\ell \frac{dw}{dx}AE\,\frac{du}{dx}dx + \int_0^\ell wb\,dx = 0 \tag{8}$$

Since $\sigma = E\,\frac{du}{dx}$, we get

$$wA\sigma|_{x=\ell} - wA\sigma|_{x=0} - \int_0^\ell \frac{dw}{dx}AE\,\frac{du}{dx}dx + \int_0^\ell wb\,dx = 0 \tag{9}$$

From equation (2), it follows that $w(0) = 0$, and the first term in (9) disappears:

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\begin{aligned}
& w A\sigma|_{x=0} - \int_0^\ell \frac{dw}{dx} AE \, \frac{du}{dx} dx + \int_0^\ell wb \, dx = 0 && (10) \\
\Longrightarrow \quad & \int_0^\ell \frac{dw}{dx} AE \, \frac{du}{dx} dx = (w A t_\ell) + \int_0^\ell wb \, dx && (11)
\end{aligned}
$$

Thus, we arrive at the statement of the weak form:

Find $u(x)$ among smooth functions satisfying $u(0) = 0$ such that

$$
\int_0^6 \frac{dw}{dx} AE \, \frac{du}{dx} dx = w(x{=}6) \, A t_\ell + \int_0^6 wb \, dx \qquad \forall w \text{ with } w(0) = 0 \tag{12}
$$