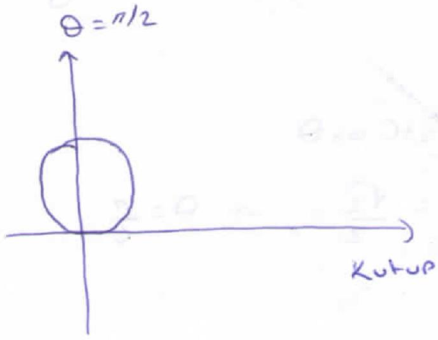
Pınar Albayrak
4. Uygulama

(*) $r=\sqrt{2}\sin\theta$ ile sınırlı bölgenin alanı?

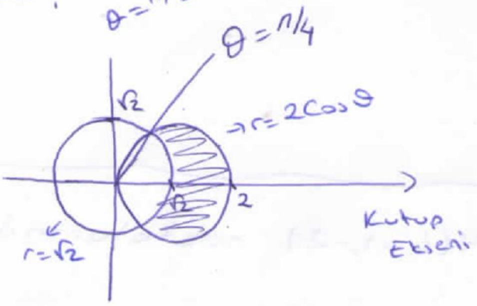


$$\frac{A}{2}=\frac{1}{2}\int_0^{\pi/2}(\sqrt{2}\sin\theta)^2\,d\theta$$

$$=\frac{1}{2}\int_0^{\pi/2}2\sin^2\theta\,d\theta$$

$$=\int_0^{\pi/2}\frac{1-\cos 2\theta}{2}\,d\theta=\frac{\pi}{4}\qquad \boxed{A=\frac{\pi}{2}}$$

(*) a) $r=2\cos\theta$ eğrisinin içinde $r=\sqrt{2}$ nin dışında kalan alan?



$$2\cos\theta=\sqrt{2}\rightarrow\boxed{\theta=\frac{\pi}{4}}$$

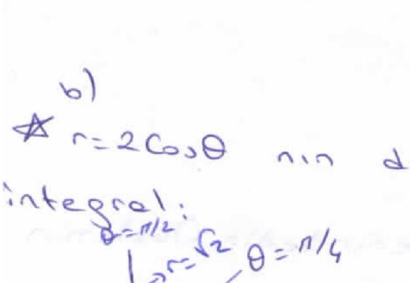
$$\frac{A}{2}=\frac{1}{2}\int_0^{\pi/4}(2\cos\theta)^2\,d\theta-\frac{1}{2}\int_0^{\pi/4}(\sqrt{2})^2\,d\theta$$

$$=\frac{1}{2}\left[\int_0^{\pi/4}(4\cos^2\theta-2)\,d\theta\right]$$

$$=\frac{1}{2}\int_0^{\pi/4}2\cos 2\theta\,d\theta=\left.\frac{\sin 2\theta}{2}\right|_0^{\pi/4}=\frac{1}{2}$$

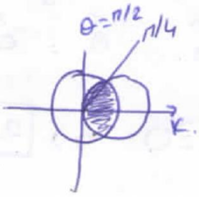
$$\boxed{A=1}$$

b) ☆ $r=2\cos\theta$ nın dışında, $r=\sqrt{2}$ nin içinde kalan alanı veren integral:



$$\frac{A}{2}=\frac{1}{2}\int_{\pi/4}^{\pi}(\sqrt{2})^2\,d\theta-\frac{1}{2}\int_{\pi/4}^{\pi/2}(2\cos\theta)^2\,d\theta$$

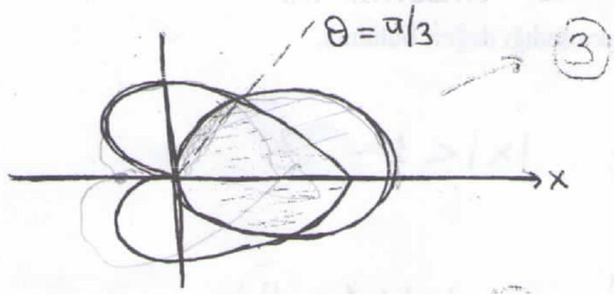
c) ☆ Ortak Alanı veren integral:



$$\frac{A}{2}=\frac{1}{2}\int_0^{\pi/4}(\sqrt{2})^2\,d\theta+\frac{1}{2}\int_{\pi/4}^{\pi/2}(2\cos\theta)^2\,d\theta$$

15 +1 cos 180

4. a) $r = 3\cos\theta$ ve $r = 1+\cos\theta$ eğrilerinin içinde kalan ortak bölgenin alanını veren integrali yazınız. (İntegral hesaplanmayacak.)



a)

$$\frac{A}{2} = \frac{1}{2}\int_0^{\pi/3} (1+\cos\theta)^2\, d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2} (3.\cos\theta)^2\, d\theta$$

(2) (2) (2) (2) (2) (2)

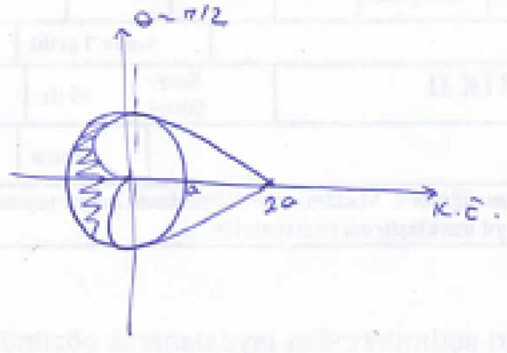
b) Kardiyoid içi, çember dışı

$$\frac{A}{2} = \frac{1}{2}\int_{\pi/3}^{\pi} (1+\cos\theta)^2\, d\theta - \frac{1}{2}\int_{\pi/3}^{\pi/2} (3\cos\theta)^2\, d\theta$$

c) Çember içi kar. dışı

$$\frac{A}{2} = \frac{1}{2}\int_0^{\pi/3} (3\cos\theta)^2 - (1+\cos\theta)^2\, d\theta$$

2) $a > 0$ olmak üzere $r = a(1+\cos\theta)$ kardiyoidinin dışında, $r = a$ çemberinin içinde kalan bölgenin alanını hesaplayınız. (Şekil çiziniz)



$$\frac{A}{2} = \int_{\pi/2}^{\pi} a^2 - (a + a\cos\theta)^2\, d\theta = \int_{\pi/2}^{\pi} \Big(-2a^2\cos\theta - a^2 \underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}}\Big)\, d\theta$$

$$= -2a^2\sin\theta - a^2\frac{\theta}{2} - \frac{a^2}{4}\sin 2\theta \Big|_{\pi/2}^{\pi}$$

$$= -\frac{a^2\pi}{2} - \left(-2a^2 - \frac{a^2\pi}{4}\right) = -\frac{a^2\pi}{2} + 2a^2 + \frac{a^2\pi}{4}$$

(*) $\begin{cases} x = 8\cos t + 8t\sin t \\ y = 8\sin t - 8t\cos t \\ 0 \le t \le \frac{\pi}{2} \end{cases}$ parametrizasyonu ile verilen eğrinin uzunluğu?

$$S = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = -8\cancel{\sin t} + 8\cancel{\sin t} + 8t\cos t \Rightarrow \left(\frac{dx}{dt}\right)^2 = 64t^2\cos^2 t$$

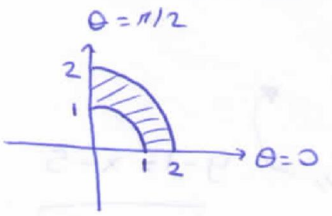
$$\frac{dy}{dt} = 8\cancel{\cos t} - \cancel{8\cos t} + 8t\sin t \Rightarrow \left(\frac{dy}{dt}\right)^2 = 64t^2\sin^2 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{64t^2} = 8t$$

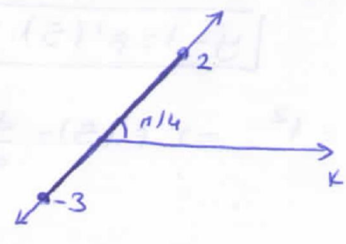
$$S = \int_0^{\pi/2} 8t\, dt = 4t^2 \Big|_0^{\pi/2} = \underline{\underline{\pi^2}}$$

(*) Kutupsal koordinatları aşağıdaki şartları sağlayan noktalar kümesinin grafiğini çiziniz.

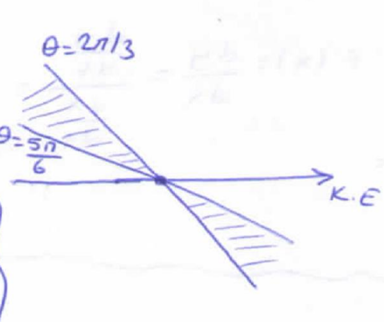
a) $1 \le r \le 2$ ve $0 \le \theta \le \pi/2$



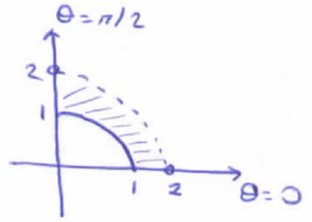
b) $-3 \le r \le 2$ ve $\theta = \frac{\pi}{4}$



c) $\frac{2\pi}{3} \le \theta \le \frac{5\pi}{6}$



d) $1 \le r < 2$, $0 \le \theta \le \frac{\pi}{2}$



S.3 a) $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$ serisinin toplamını bulunuz.(13p)

$$\frac{2n+1}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

$$S_n = \sum_{k=1}^{n} \frac{2k+1}{k^2(k+1)^2} = \sum_{k=1}^{n} \left(\frac{1}{k^2} - \frac{1}{(k+1)^2} \right)$$

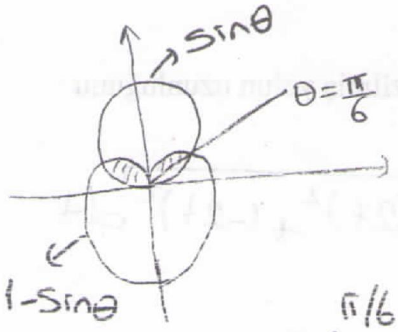
$$= \left(1 - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \dots + \left(\frac{1}{(n-1)^2} - \frac{1}{n^2}\right) + \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right)$$

$$S_n = 1 - \frac{1}{(n+1)^2}$$

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = \lim_{n \to \infty} \left(1 - \frac{1}{(n+1)^2}\right) = 1 \,//$$

b) $\rho = 1 - \sin\theta$ kardiyoidi ve $\rho = \sin\theta$ çemberinin her ikisinin de içinde kalan bölgenin alanını bulunuz .(12p)

$$1 - \sin\theta = \sin\theta \implies \theta = \frac{\pi}{6}, \ \frac{5\pi}{6}$$

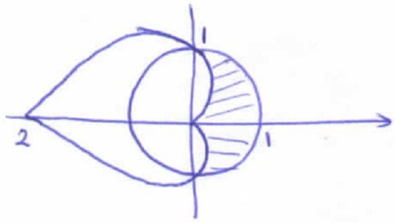


$$\frac{A}{2} = \frac{1}{2}\int_0^{\pi/6} (\sin\theta)^2 \, d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2} (1 - \sin\theta)^2 \, d\theta$$

$$A = \int_0^{\pi/6} \frac{1 - \cos 2\theta}{2} \, d\theta + \int_{\pi/6}^{\pi/2} \frac{1}{2}(3 - 4\sin\theta - \cos 2\theta) \, d\theta$$

$$A = \left(\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right) + \left(\frac{\pi}{2} - \frac{7\sqrt{3}}{8}\right) = \frac{7\pi}{12} - \sqrt{3} \ br^2$$

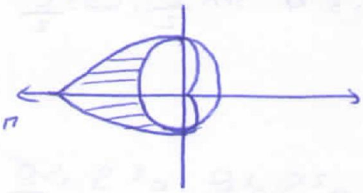
(*) $r=1$ çemberinin içinde, $r=1-\cos\theta$ kardiyoidinin dışında kalan bölgenin alanını veren integral?



$$\frac{A}{2}=\int_0^{\pi/2}\frac{1}{2}\,d\theta-\frac{1}{2}\int_0^{\pi/2}(1-\cos\theta)^2\,d\theta$$

$$A=\int_0^{\pi/2}\left(1-(1-\cos\theta)^2\right)d\theta$$

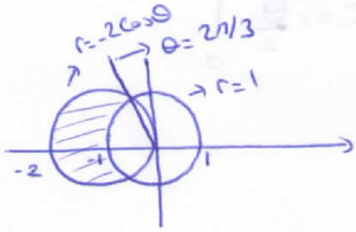
b) çemberin dışı, kardiyoidin içi:



$$\frac{A}{2}=\int_{\pi/2}^{\pi}\frac{1}{2}\cdot(1-\cos\theta)^2\,d\theta-\int_{\pi/2}^{\pi}\frac{1}{2}\,d\theta$$

$$A=\int_{\pi/2}^{\pi}\left((1-\cos\theta)^2-1\right)d\theta$$

(*) $r=-2\cos\theta$ çemberinin içinde, $r=1$ çemberinin dışında kalan alan?



$$-2\cos\theta=1 \Rightarrow \theta=\frac{2\pi}{3}$$

$$\frac{A}{2}=\frac{1}{2}\int_{2\pi/3}^{\pi}(-2\cos\theta)^2\,d\theta-\frac{1}{2}\int_{2\pi/3}^{\pi}1^2\,d\theta$$

$$A=\int_{2\pi/3}^{\pi}(4\overbrace{\cos^2\theta}^{\frac{1+\cos2\theta}{2}}-1)\,d\theta=\int_{\frac{2\pi}{3}}^{\pi}(1+2\cos2\theta)\,d\theta$$

$$=\theta+\sin2\theta\Big|_{2\pi/3}^{\pi}=\pi-\frac{2\pi}{3}-\sin\frac{\pi}{3}=\frac{\pi}{3}-\frac{\sqrt{3}}{2}$$

(*) $x = 4\sin t \quad y = 2\cos t$ eğrisinin $t = \frac{\pi}{4}$ deki teğeti ?

$$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2\sin t}{4\cos t}\Bigg|_{t=\pi/4} = -\frac{1}{2}$$

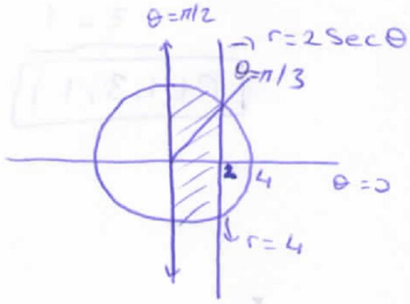
$$t = \frac{\pi}{4} \Rightarrow x = 2\sqrt{2} \quad y = \sqrt{2}$$

Teğet Denklemi $\Rightarrow$ $y - y_0 = m(x - x_0)$

$$y - \sqrt{2} = -\frac{1}{2}(x - 2\sqrt{2}) \Rightarrow \boxed{y = -\frac{1}{2}x + 2\sqrt{2}}$$

(*) $r = 4$, $\theta = \frac{\pi}{2}$, $r = 2\sec\theta$ arasında kalan bölgenin alanını veren integral?

$$r = 2\sec\theta = \frac{2}{\cos\theta} \Rightarrow \underbrace{r\cos\theta}_{x} = 2 \Rightarrow \boxed{x = 2} \text{ doğrusu}$$



$$2\sec\theta = 4 \Rightarrow \sec\theta = 2 \Rightarrow \cos\theta = \frac{1}{2}$$

$$\theta = \pi/3$$

$$\frac{A}{2} = \frac{1}{2}\int_0^{\pi/3} (2\sec\theta)^2 d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2} (4)^2 d\theta$$

---

e) $t, [-1,0]$ aralığında değişken, $x(t) = t^2, y(t) = 1 - t^2$ ile çizilmiş yolun uzunluğunu bulunuz.(10p)

$$L = \int_{-1}^{0} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt = \int_{-1}^{0} \sqrt{(2t)^2 + (-2t)^2}\, dt$$

$$= 2\sqrt{2}\int_{-1}^{0} |t|\, dt = -2\sqrt{2}\int_{-1}^{0} t\, dt = \sqrt{2} \text{ br}$$

$\circledast$ $r = 1+\cos\theta$ eğrisinin $\theta = \frac{5\pi}{4}$ deki teğet doğrusunun denklemini yazınız.

$$\left.\begin{aligned} x &= r\cos\theta = (1+\cos\theta)\cos\theta \\ y &= r\sin\theta = (1+\cos\theta).\sin\theta \end{aligned}\right] \theta = \frac{5\pi}{4} \Rightarrow x_0 = \left(1-\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) = \frac{1}{2} - \frac{\sqrt{2}}{2}$$

$$y_0 = \left(1-\frac{\sqrt{2}}{2}\right)\cdot\left(-\frac{\sqrt{2}}{2}\right) = \frac{1}{2} - \frac{\sqrt{2}}{2}$$

$$y - y_0 = f'(x_0).(x-x_0) \Rightarrow y - \left(\frac{1}{2}-\frac{\sqrt{2}}{2}\right) = \underbrace{f'\left(\frac{1}{2}-\frac{\sqrt{2}}{2}\right)}_{\text{bulmalıyız!}}.\left(x - \left(\frac{1}{2}-\frac{\sqrt{2}}{2}\right)\right)$$

$$f'(x) = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos\theta + \cos^2\theta - \sin^2\theta}{-\sin\theta - 2\cos\theta\sin\theta}$$

$$f'\left(\frac{1}{2}-\frac{\sqrt{2}}{2}\right) = \left.\frac{dy}{dx}\right|_{\theta=\frac{5\pi}{4}} = \frac{\overbrace{\cos\frac{5\pi}{4}}^{-\sqrt{2}/2} + \cancel{\cos^2\frac{5\pi}{4}} - \cancel{\sin^2\frac{5\pi}{4}}}{-\underbrace{\sin\frac{5\pi}{4}}_{-\frac{\sqrt{2}}{2}} - 2\underbrace{\cos\frac{5\pi}{4}}_{-\frac{\sqrt{2}}{2}}.\underbrace{\sin\frac{5\pi}{4}}_{-\frac{\sqrt{2}}{2}}} = \frac{-\frac{\sqrt{2}}{2}}{-1+\frac{\sqrt{2}}{2}} = \underline{1+\sqrt{2}}$$

$$\boxed{y - \left(\frac{1}{2}-\frac{\sqrt{2}}{2}\right) = (1+\sqrt{2})\left(x - \left(\frac{1}{2}-\frac{\sqrt{2}}{2}\right)\right)}$$ teğet doğrunun denklemi

---

$\circledast$ $\sum_{n=1}^{\infty} \frac{1}{n^3}.\sin\left(\frac{\pi}{n^2}\right)$ serisinin karakteri?

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ serisini seçelim. $p=2>1$ yakınsaktır

$$\lim_{n\to\infty} \frac{\overset{\to 0}{\frac{1}{n^3}}.\sin\left(\frac{\pi}{n^2}\right)}{\frac{1}{n^2}.\frac{n}{\pi}} \overset{\to 1}{} = 0 \Rightarrow$$

Limit Testine göre; $\sum_{n=1}^{\infty} \frac{1}{n^2}$ yakınsak olduğundan

$\sum_{n=1}^{\infty} \frac{1}{n^3}.\sin\left(\frac{\pi}{n^2}\right)$ de yakınsaktır.

⊛ $\{a_n\} = \left\{ n\left(1 - n \sin\frac{1}{n}\right) \right\}$ dizisinin yakınsaklığını araştırınız.

$$\lim_{n\to\infty} n\cdot\left(1 - n\sin\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\frac{1}{n} - \sin\frac{1}{n}}{\frac{1}{n^2}} = \lim_{k\to 0} \frac{k - \sin k}{k^2} \to \frac{0}{0} \text{ L'H}$$

$k = \frac{1}{n}$ olsun. $n\to\infty \Rightarrow k\to 0$

$$= \lim_{k\to 0} \frac{1-\cos k}{2k} \to \frac{0}{0} \text{ L'H}$$

$$= \lim_{k\to 0} \frac{\sin k}{2} = 0 \Rightarrow$$ Dizi yakınsak

---

⊛ $\{a_n\} = \left\{ (2 - e^{1/n})^n \right\}$ dizisinin yakınsaklığını araştırın.

☆ $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$ olduğundan $\lim_{n\to\infty} f(n) = 0 \Rightarrow \lim_{n\to\infty} (1 + f(n))^{1/f(n)} = e$ dir.

Bu formüle benzetirsek:

$$\lim_{n\to\infty} (2 - e^{1/n})^n = \lim_{n\to\infty} \left(1 + (1 - e^{1/n})\right)^n = \lim_{n\to\infty} \left(\underbrace{\left(1 + (\overbrace{1 - e^{1/n}}^{f(n)})\right)^{\frac{1}{\underset{f(n)}{1 - e^{1/n}}}}}_{e} \right)^{n\cdot(1 - e^{1/n})}$$

$$= e^{\lim_{n\to\infty} n\cdot(1 - e^{1/n})} = e^{\lim_{n\to\infty} \frac{1 - e^{1/n}}{\frac{1}{n}}} \to \frac{0}{0} \to \text{L'H}$$

$$= e^{\lim_{n\to\infty} \frac{\cancel{\frac{1}{n^2}}\cdot e^{1/n}}{-\cancel{\frac{1}{n^2}}}} \to -1$$

$$= e^{-1} = \frac{1}{e}$$

$$\sum_{n=0}^{\infty} \frac{2^{2n+1} - 3^n}{5^n} = ?$$

$$\sum_{n=0}^{\infty} \frac{2^{2n+1}}{5^n} - \sum_{n=0}^{\infty} \frac{3^n}{5^n} = \sum_{n=0}^{\infty} \frac{4^n \cdot 2}{5^n} - \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n$$

$$= \underbrace{\sum_{n=0}^{\infty} 2\cdot\left(\frac{4}{5}\right)^n}_{a=2 \quad r=\frac{4}{5}} - \underbrace{\sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n}_{a=1 \quad r=\frac{3}{5}} = \frac{2}{1-\frac{4}{5}} - \frac{1}{1-\frac{3}{5}}$$

$$= 10 - \frac{5}{2} = \frac{15}{2}$$

$\{a_n\} = \left\{\frac{\cos n}{n^2+1}\right\}$ dizisinin yakınsaklığını Sıkıştırma Teo. ile araştırınız.

$$\lim_{n\to\infty} \frac{\cos n}{n^2+1} = ?$$

$$-1 \le \cos n \le 1$$

$$\downarrow$$

$$\frac{-1}{n^2+1} \le \frac{\cos n}{n^2+1} \le \frac{1}{n^2+1}$$

$$\lim_{n\to\infty} -\frac{1}{n^2+1} = \lim_{n\to\infty} \frac{1}{n^2+1} = 0$$ olduğundan

Sıkıştırma Teo. göre $\lim_{n\to\infty} \frac{\cos n}{n^2+1} = 0$ dır.

Dolayısıyla dizi yakınsaktır.

Sevgili MAT2 öğrencileri,

Elimde başka soru kalmadı; soru yetiştiremediğim öğrencilere duyurulur ☺

Hepinize ilk vize sınavında başarılar dilerim...

Sevgiler,

Pınar Albayrak