Non regular tilings review

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1 Introduction

There are many patterns in nature. They can be seen everywhere: from honeycomb to animal print to patterns on leaves. Artists often draw inspiration from all of these naturally occurring patterns to create their own. Looking more closely, we can find patterns at a smaller scale in cell structure and in crystal structures and their atom arrangements called latices. In 3D crystals, we can observe many different kinds of 3D lattices such as triclinic, monoclinic, and orthorhombic lattices. In 2D lattices, we can observe five different Bravais lattices (distinct lattice types), including the oblique lattice, which can describe atomic lattice planes of crystals (for example, NaCl) [1]. From all of this, we see that the plane can be filled with different geometric figures in multiple ways. In mathematics, those geometric figures can be called tiles and the filled space, without overlaps or gaps between the tiles, can be called a tiling [2, 3]. With common shapes such as squares, hexagons, and equilateral triangles, we can build regular, periodic tilings. Regular tilings have been studied for over a century and well known[4]. These tilings are characterized by translational symmetry: the tiling pattern maps on itself under a translation. However, there are tilings that lack translation symmetry while still being non-random. Such tilings are called non-regular.

Non-regular tilings range from being non-periodic, having a different rotational symmetry, to being made up of non-regular shapes. These tilings can make up lattices with 5- , 8 - , 12- fold symmetries: such a rotational symmetries where the tiling will not change if you rotate the tiling by 360/n (where n could be 5, 8, or 12). An example of a non-regular tiling is the famous Penrose tiling: it is featured in many different artworks and studied thoroughly over the past years[5]. In this overview, I will focus on non-regular tilings with their mathematical properties as well as the properties and applications of quasicrystals: crystals that have a non-regular lattice structure.

2 Non regular tilings

There are many different types of non-regular tiling. One of the simplest non regular tilings is one where the tiles consist of the same shape. Such tiles are called rep-tiles.

The tiling that consists of rep-tiles is a tiling that consists only of one shape. Rep-tiles are fractals: they are polygons that consist of smaller versions of themselves, as well as having the ability to form together to form larger versions of themselves[6]. The tilings created by them consist purely of themselves and different transformations of themselves [7]. In fact, rep-tiles can be regular tilings (for example, an equilateral triangle); however, when their shapes are not the three common polygons (square, hexagon, and equilateral triangle), they become non-regular tilings. The interesting thing is that, when repeated indefinitely, the rep-tile tilings can become aperiodic[8]. This usually happens when the translation transformation isn't the only transformation used between each tile of irregular shape[7]. Such tiles are usually found in different art forms.

There are many more purely aperiodic tilings such as octagonal, pinwheel, random, and Penrose tilings [9]. Penrose tilings consist of two tiles: kites and darts. They have an interesting property: if you repeat your tiling indefinitely, a "golden mean" (the ratio between darts and kites) emerges equal to 1.618 [10]. What is interesting about all of the tilings listed above is that they exhibit quasi-symmetry - a symmetry through partial rotation[10]. This symmetry can be found in many places. For example, it was very prominent in art, and even Easter's schedule follows a quasicrystal like lattice [11]. However, most importantly of all, this symmetry is most prominent in science, in special crystals called quasicrystals - crystals that defy normal symmetry.

3 Quasi crystals

Quasiperiodic crystals (more commonly known as quasicrystals) are crystals that exhibit quasi symmetry. 2D quasicrystals specifically can exhibit 5- , 10- , 8- or even 12-fold symmetries [12]. The first quasi crystal that was discovered was that of Al_4Mn, however most quasi crystals were discovered to be intermetallic systems (crystals of metal alloys): specifically those of simple metals (of which the most commonly occurring in the crystals would be aluminum) and different transition metals [13]. Some examples of quasicrystalline alloys include GaMgZn, AlCuLi, AlCuFe, and AlCuRu[14]. Most quasicrystals divided into one of two groups: metastable and stable crystals. Metastable crystals (those in which the crystal arrangement does not use the lowest amount of energy) occur as crystals developed in laboratories[15]. However, some quasicrystals are naturally occurring/stable. Examples include AlCuFe, AlPdMn, and AlLiCu [14].

Quasi crystals have very unique properties due to their structure. They are of brittle nature because of their aperiodic lattices: the aperiodic nature of this crystal does not allow them to easily bend or slip since there are not many options for displacement, causing brittleness of the material [13]. Their strength can be reinforced, though, using hydrogen due to hydrogen's ability to fill the "holes" in the structure [16]. They do not have good static (electrical) conductivity or optical conductivity (a current is not induced when they are exposed to an oscillating electric field, specifically that of optical frequencies). That is also due to their aperiodic structure because of which the mobility of electrons is limited. However, there are a few exceptions in the form of superconductive graphene quasicrystals that allow electricity to pass without resistance[17]. For the same reason (limited mobility of electrons), the thermal conductivities of quasicrystals are also on the reduced side. They do not have great adhesion to different surfaces, and they have a lower wetting than most other metals. That can be explained by their low surface energy, which reduces their bonds with other materials and elements. Due to the low surface energy, quasicrystals also have a smaller coefficient of friction. Their low surface energy also affects their corrosiveness and oxidation: both have no effect on the crystals [13].

Due to their unique nature, quasicrystals have been investigated for many uses. Their low optical conductivity made them good candidates for solar light absorbers. Due to their poor thermal conductivity, they have been suggested for heat insulation. Their non-reactivity has been a point for the innovation of uses of quasicrystals as anti oxidation or corrosion protectors, and their potential in improving non-stick surfaces (where apparently a quasicrystal nonstick pan almost was launched, but was canceled due to some tests that the owner did not go through) [13]. They also been researched for potential uses in microelectronics and aerospace. Due to the property of quasicrystals, where hydrogen reinforces their strength, they have also been researched for hydrogen storage. Since hydrogen reinforces the strength of the quasicrystals, they are a potential option for hydrogen storage [13]. When dissolved in a solution, quasi crystals have also shown properties of very efficient catalysts, making them a great option for further development [18].

4 Conclusion

Mathematically, the various non-regular tilings are complex combinatorial problems. Their study is related to "group theory", of which I do not have a deep understanding yet. In solid-state physics, non-regular tilings occur in materials with unique physical properties such as quasicrystals. As material science and engineering advance, it will be possible to find more of such materials in nature or synthesize them in a way that is useful for application.

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