One-to-Two-Player Lifting for Mildly Growing Memory

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Abstract

We investigate so-called "one-to-two-player lifting" theorems for infinite-duration two-player games on graphs with zero-sum objectives. These theorems are concerned with questions of the following form. If that much memory is sufficient to play optimally in one-player games, then how much memory is needed to play optimally in two-player games? In 2005, Gimbert and Zielonka (CONCUR 2005) have shown that if no memory is needed in the one-player games, then the same holds for the two-player games. Building upon their work, Bouyer et al. (CONCUR 2020) have shown that if some constant amount of memory (independent of the size of a game graph) is sufficient in the one-player games, then exactly the same constant is sufficient in the two-player games. They also provide an example in which every one-player game requires only a finite amount of memory (now this amount depends on the size of a game) while some two-player game requires infinite memory.

Our main result states the following. If the memory grows just a bit slower (in the one-player games) than in the example of Bouyer et al., then in every two-player game it is sufficient to have finite memory. Thus, our work identifies the exact barrier for the one-to-two-player lifting theorems in a context of finite-memory strategies.

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1 Introduction

We extend a line of works establishing so-called "one-to-two-player lifting" theorems for different classes of strategies in deterministic two-player infinite-duration games on (finite) graphs. These games are famously known for their profound applications in logic, ranging from decidability of logical theories [11, 13] to controller synthesis and model checking [7, 12]. In this regard, one of the most important questions for these games is how much *memory* players need to play optimally (depending, of course, on their objectives). Qualitatively, one can ask for which objectives it is sufficient to have *finite* memory, or even *no* memory at all. One-to-two-player lifting theorems are quite useful in studying this kind of questions. Broadly speaking, they have the following form: as long as that much memory is needed in one-player games, at most that much memory is needed in two-player games. In a way, they reduce a reasoning about two-player games to just a reasoning about graphs (as in one-player games we are basically asked to find a path in a given graph maximizing/minimizing a certain objective).

Currently, all known one-to-two-player lifting theorems are only for zero-sum games. The first one of these theorems was obtained by Gimbert and Zielonka in [8]. Informally, their theorem states that as long as no memory is needed to play optimally in the one-player games, the same holds for the two-player games. Bouyer et al. [2] obtained the following generalization of this result. If only a constant (independent of the size of a graph) amount of memory is sufficient to play optimally in all one-player games, then exactly the same amount of memory is sufficient for the two-player games. It should be noted that analogous results were obtained for *stochastic* graph games [9, 3] (in this paper we only consider deterministic ones).

What if the memory is still finite for every specific one-player game but grows as games become larger and larger? Does it imply that every two-player game as well requires only a finite amount of memory? In [2] Bouyer et al. provide an example disproving it. This constitutes a barrier of applying one-to-two-player lifting theorems to a number of important objectives for which memory is finite but grows rather quickly with the size of a graph [5].

The aim of this work is to understand where exactly this barrier lie. It turns out that the example of Bouyer et al. is on the edge of it. More specifically, we observe that in their example it is sufficient to have linearly many states of the memory (in the number of nodes of a graph) to play optimally in the one-player games. So even this rather modest amount of memory may cause infinite memory in the two-player games. Our main result shows the sharpness of this example. Namely, we show that if it is sufficient to have a sublinear number of states of the memory in the one-player games, then every two-player game requires only a finite amount of memory. In fact, this result holds even when we have sublinearity not for all possible sizes of a game graph but for an arbitrary infinite increasing sequence of the sizes. Previous one-to-two-player lifting theorems follow from our result as special cases. In addition, our exposition includes a simple self-contained proof of the "memory-less" lifting theorem of Gimbert and Zielonka.

Technically, our results are subject to similar restrictions as the results of Bouyer et al. Namely, we consider only memory related to the objectives of the players. In turn, a graph-related memory falls out of our framework. Still, we make a step forward by allowing memory to grow with the size of a graph.

In order to proceed to further details, in the next section we introduce games on graphs formally. We return to our results in Section 3.

2 Games on Graphs

Notation. We denote the set of positive integer numbers by $\mathbb{Z}^+$. Given a set A, by A^* and A^ω we denote the sets of finite and, respectively, infinite sequences of elements of A. The length of a sequence $x \in A^* \cup A^\omega$ is denoted by $|x|$. We write $A = B \sqcup C$ for three sets A, B, C if $A = B \cup C$ and $B \cap C = \varnothing$. Function composition is denoted by $\circ$.

2.1 Arenas

Following previous papers [8, 9, 2, 3], we call graphs on which our games are played *arenas*. We start with some notation regarding arenas. First, fix an arbitrary finite set C. We will refer to the elements of C as to *colors*. Informally, an arena is just a directed graph with edges colored by elements of C and with nodes partitioned into two sets.

▶ **Definition 1.** *A tuple* $\mathcal{A} = \langle V, V_{\mathrm{Max}}, V_{\mathrm{Min}}, E, \mathsf{source}, \mathsf{target}, \mathsf{col}\rangle$*, where*
- $V, V_{\mathrm{Max}}, V_{\mathrm{Min}}, E$ *are four finite sets with* $V = V_{\mathrm{Max}} \sqcup V_{\mathrm{Min}}$*;*
- $\mathsf{source}, \mathsf{target}, \mathsf{col}$ *are functions of the form*

$$\mathsf{source}\colon E \to V, \qquad \mathsf{target}\colon E \to V, \qquad \mathsf{col}\colon E \to C;$$

is called an ***arena*** *if for every* $v \in V$ *there exists* $e \in E$ *with* $v = \mathsf{source}(e)$*.*

Elements of V will be called ***nodes*** *of $\mathcal{A}$ and elements of E will be called* ***edges*** *of $\mathcal{A}$. Nodes from V_{Max} will be called nodes of Max and nodes from V_{Min} will be called nodes of Min. The out-degree of a node $v \in V$ is $|\{e \in E \mid \mathsf{source}(e) = v\}|$. An arena is called* ***one-player*** *if either all nodes of Max have out-degree 1 or all nodes of Min have out-degree 1.*

Fix an arena $\mathcal{A} = \langle V, V_{\mathrm{Max}}, V_{\mathrm{Min}}, E, \mathsf{source}, \mathsf{target}, \mathsf{col}\rangle$. We extend the function col (which determines the coloring of the edges) to arbitrary sequences of edges by setting:

$$\mathsf{col}(e_1 e_2 e_3 \ldots) = \mathsf{col}(e_1)\mathsf{col}(e_2)\mathsf{col}(e_3)\ldots \qquad \text{for } e_1, e_2, e_3, \ldots \in E.$$

A non-empty sequence of edges $h = e_1 e_2 e_3 \ldots \in E^* \cup E^\omega$ is called a **path** if for every $1 \le n < |h|$ we have $\mathsf{target}(e_n) = \mathsf{source}(e_{n+1})$. We define $\mathsf{source}(h) = \mathsf{source}(e_1)$. When h is finite, we define $\mathsf{target}(h) = \mathsf{target}(e_{|h|})$. In addition, for every $v \in V$ we consider a 0-length path λ_v identified with v, for which we set $\mathsf{source}(\lambda_v) = \mathsf{target}(\lambda_v) = v$. For every $v \in V$ we define $\mathsf{col}(\lambda_v)$ as the empty string.

2.2 Infinite-duration games on arenas

An arena $\mathcal{A} = \langle V, V_{\mathrm{Max}}, V_{\mathrm{Min}}, E, \mathsf{source}, \mathsf{target}, \mathsf{col}\rangle$ induces an infinite-duration two-player game in the following way. First, we call players of this game Max and Min. Informally, Max and Min interact by gradually constructing a longer and longer path in $\mathcal{A}$. In each turn one of the players extends a current path by some edge from its endpoint. Which of the two players is the one to move is determined by whether this endpoint belongs to V_{Max} or to V_{Min}.

Formally, positions in the game are finite paths in $\mathcal{A}$. By definition, $\mathsf{target}(h) \in V_{\mathrm{Max}}$ for a finite path h means that Max is the one to move in the position h; respectively, $\mathsf{target}(h) \in V_{\mathrm{Min}}$ means that Min is the one to move in the position h. A set of moves available in a position h is the set $\{e \in E \mid \mathsf{source}(e) = \mathsf{target}(h)\}$. Making a move $e \in E$ in a position $h = e_1 e_2 \ldots e_{|h|}$ brings to a position $he = e_1 e_2 \ldots e_{|h|} e$.

We stress that no position is designated as the initial one. We assume that the game can start in any position of the form $\lambda_v, v \in V$, at our choice.

This mechanics naturally induces a notion of strategies. Namely, a strategy of Max is a function

$$\sigma \colon \{h \mid h \text{ is a finite path in } \mathcal{A} \text{ with } \mathsf{target}(h) \in V_{\mathrm{Max}}\} \to E$$

such that for every h from the domain of σ we have $\mathsf{source}(\sigma(h)) = \mathsf{target}(h)$. Respectively, a strategy of Min is a function

$$\tau \colon \{h \mid h \text{ is a finite path in } \mathcal{A} \text{ with } \mathsf{target}(h) \in V_{\mathrm{Min}}\} \to E$$

such that for every h from the domain of τ we have $\mathsf{source}(\tau(h)) = \mathsf{target}(h)$.

Observe that if $\mathcal{A}$ is one-player, then one of the players has exactly one strategy. For technical consistency we assume that even when one of the players owns all the nodes of $\mathcal{A}$, the other player still has one "empty" strategy.

A strategy induces a set of positions *consistent* with it (those that can be reached in a play against this strategy). Formally, a finite path $h = e_1 e_2 \ldots e_{|h|}$ is consistent with a strategy σ of Max if the following conditions hold:

- $\mathsf{source}(h) \in V_{\mathrm{Max}} \implies \sigma(\lambda_{\mathsf{source}(h)}) = e_1$;
- for every $1 \le i < |h|$ we have $\mathsf{target}(e_1 e_2 \ldots e_i) \in V_{\mathrm{Max}} \implies \sigma(e_1 e_2 \ldots e_i) = e_{i+1}$.

Consistency with the strategies of Min is defined similarly. Further, the notion of consistency can be extended to infinite paths. Namely, given a strategy, an infinite path is consistent with it if all finite prefixes of this path are.

For $v \in V$ and for a strategy $\mathcal{S}$ of one of the players $\mathsf{Cons}(v, \mathcal{S})$ denotes the set of all *infinite* paths that are consistent with $\mathcal{S}$ and whose source is v. For any strategy σ of Max, strategy τ of Min and $v \in V$ there is a unique path in the intersection $\mathsf{Cons}(v, \sigma) \cap \mathsf{Cons}(v, \tau)$. We denote this path by $h(v, \sigma, \tau)$ and call it *the play* of σ and τ from v.

2.3 Payoff functions and equilibria

We consider only zero-sum games; correspondingly, in our framework objectives of the players are always given by a *payoff function*. A payoff function is any function of the form $\varphi \colon C^\omega \to \mathcal{W}$, where $(\mathcal{W}, \leq)$ is a linearly ordered set. Informally, the aim of Max is to play in a way which maximizes the payoff function (with respect to the ordering of $\mathcal{W}$) while the aim of Min is the opposite one. Technically, to get the value of the payoff function on a play (which is an infinite path in the underlying arena) we first apply the function col to this play; this gives us an infinite sequence of colors; in conclusion we apply φ to this sequence.

Previous papers [8, 9, 2, 3] do not introduce payoff functions but rather consider preorders induced by them on C^ω; they refer to this preorders as to *preference relations*. These two approaches are equivalent.

Any payoff function in a standard way induces a notion of an *equilibrium* of two strategies of the players (with respect to this payoff function). Let us first introduce a notion of an *optimal response*. Namely, take $T \subseteq V$, a strategy σ of Max and a strategy τ of Min. We say that σ is a **T-wise optimal response** to τ if for all $v \in T$ and for all $h \in \mathsf{Cons}(v, \tau)$ we have:

$$\varphi \circ \mathsf{col}\big(h(v, \sigma, \tau)\big) \geq \varphi \circ \mathsf{col}(h)$$

(the inequality here, of course, is with respect to the ordering of $\mathcal{W}$). Similarly, we call τ a **T-wise optimal response** to σ if for all $v \in T$ and for all $h \in \mathsf{Cons}(v, \sigma)$ we have:

$$\varphi \circ \mathsf{col}\big(h(v, \sigma, \tau)\big) \leq \varphi \circ \mathsf{col}(h).$$

Next, we call a pair (σ, τ) a **T-wise equilibrium** if σ and τ are T-wise optimal responses to each other. By a standard game-theoretic argument the set of T-wise equilibria is always a Cartesian product. Strategies belonging to some T-wise equilibrium will be called **T-wise optimal**. When $T = V$ is the whole set of nodes, instead of “V-wise” we simply say “uniform”.

Recall that in the one-player arenas one of the players has exactly one strategy. Thus, the set of equilibria there is determined by the set of the optimal responses of the other player.

2.4 Restricted classes of strategies

Consider the game induced by an arena $\mathcal{A} = \langle V, V_{\mathrm{Max}}, V_{\mathrm{Min}}, E, \mathsf{source}, \mathsf{target}, \mathsf{col}\rangle$. In general, as there are infinitely many positions, strategies in this game do not have a finite description. We define some important classes of strategies that do have one. Each of these classes is given by a certain class of finite-valued functions on the set of positions.

Positional strategies. A strategy $\mathcal{S}$ of one of the players is called **positional** if for any two positions h_1, h_2 from its domain we have $\mathsf{target}(h_1) = \mathsf{target}(h_2) \implies \mathcal{S}(h_1) = \mathcal{S}(h_2)$. In other words, $\mathcal{S}(h)$ depends solely on $\mathsf{target}(h)$, which is just one of the finitely many nodes in the arena. In this regard, it is quite convenient to consider positional strategies as functions on the set of nodes of the corresponding players (rather than on the set of the positions of this player). I.e., positional strategies of Max can be identified with functions of the form:

$$\sigma\colon V_{\mathrm{Max}} \to E \text{ s.t. } \mathsf{source}(\sigma(v)) = v \text{ for all } v \in V_{\mathrm{Max}},$$

and positional strategies of Min can be identified with functions of the form:

$$\tau\colon V_{\mathrm{Min}} \to E \text{ s.t. } \mathsf{source}(\tau(v)) = v \text{ for all } v \in V_{\mathrm{Min}}.$$

Let us fix some notation regarding positional strategies. First, every edge $e \in E$ is a path (of length 1) and hence also a position in the game induced by $\mathcal{A}$. If $\mathcal{S}$ is a positional strategy of one of the players, we let $E_{\mathcal{S}}$ be the set of edges that are consistent with $\mathcal{S}$. Observe the following feature of positional strategies: the set of paths (positions) that are consistent with a positional strategy $\mathcal{S}$ is exactly the set of paths that consist only of edges from $E_{\mathcal{S}}$.

Given a positional strategy $\mathcal{S}$ of one of the players, by $\mathcal{A}_{\mathcal{S}}$ we denote the following arena:

$$\mathcal{A}_{\mathcal{S}} = \langle V, V_{\mathrm{Max}}, V_{\mathrm{Min}}, E_{\mathcal{S}}, \mathsf{source}, \mathsf{target}, \mathsf{col}\rangle.$$

I.e., $\mathcal{A}_{\mathcal{S}}$ is obtained from $\mathcal{A}$ by deleting all edges that are inconsistent with $\mathcal{S}$. Observe that the arena $\mathcal{A}_{\mathcal{S}}$ is one-player; each node of the player who plays $\mathcal{S}$ has exactly one out-going edge in $\mathcal{A}_{\mathcal{S}}$.

We call an equilibrium consisting of two positional strategies a "positional equilibrium".

Finite-state strategies. A finite-state strategy $\mathcal{S}$ is specified by a finite automaton whose input alphabet is the set of edges. If $\mathcal{S}$ is a finite-state strategy and h is a position, then $\mathcal{S}(h)$ is determined by $\mathsf{target}(h)$ and by a state into which the underlying automaton of $\mathcal{S}$ comes after reading h (note that h is a sequence of edges, i.e., a word over the input alphabet of our automaton).

Formally, consider any deterministic finite automaton $\mathcal{U} = \langle Q, q_{init}, \delta\colon Q \times E \to Q\rangle$ whose input alphabet is the set E. Here Q is the set of states of $\mathcal{U}$, a state $q_{init} \in Q$ is a designated initial state and δ is the transition function of $\mathcal{U}$. In a usual way for any $q \in Q$ we extend $\delta(q, \cdot)$ to finite sequences of edges. Now, a strategy $\mathcal{S}$ of one of the players is a $\mathcal{U}$-strategy if for any two positions h_1, h_2 from the domain of $\mathcal{S}$ it holds that

$$\big[\mathsf{target}(h_1) = \mathsf{target}(h_2) \text{ and } \delta(q_{init}, h_1) = \delta(q_{init}, h_2)\big] \implies \mathcal{S}(h_1) = \mathcal{S}(h_2).$$

A strategy which is a $\mathcal{U}$-strategy for some DFA $\mathcal{U}$ as above is a **finite-state strategy**.

For a DFA $\mathcal{U}$ with s states we interpret a $\mathcal{U}$-strategy as a strategy with s *states of memory*. We do not count the amount of memory needed to store $\mathsf{target}(h)$; in a sense, $\mathsf{target}(h)$ is given for free as it defines the set of available moves. In this regard, positional strategy are strategies with no memory as they store nothing besides $\mathsf{target}(h)$.

Arena-independent finite-state strategies. Among finite-state strategies we highlight those whose underlying automaton reads only the colors of edges. For technical reasons, we define them separately through a weaker class of automata whose input alphabet is the set of colors rather than the set of edges. Following Bouyer et al., we call this kind of automata *memory skeletons*.

Formally, a memory skeleton is a deterministic finite automaton $\mathcal{M} = \langle M, m_{init} \in M, \delta\colon M \times C \to M\rangle$ whose input alphabet is the set C. Here M is the set of states of $\mathcal{M}$, a state $m_{init} \in M$ is a designated initial state, and δ is the transition function of $\mathcal{M}$. Given $m \in M$, we extend $\delta(m, \cdot)$ to finite sequences of elements of C in a standard way. Now, a strategy $\mathcal{S}$ of one of the players is called an $\mathcal{M}$-strategy if for any two positions h_1 and h_2 from the domain of $\mathcal{S}$ it holds that

$$\big[\mathsf{target}(h_1) = \mathsf{target}(h_2) \text{ and } \delta(m_{init}, \mathsf{col}(h_1)) = \delta(m_{init}, \mathsf{col}(h_2))\big] \implies \mathcal{S}(h_1) = \mathcal{S}(h_2).$$

An important technical remark is that for any memory skeleton $\mathcal{M}$ the notion of an "$\mathcal{M}$-strategy" is arena-independent, i.e., it is well-defined in any arena. It is also instructive to note that positional strategies are exactly $\mathcal{M}_{triv}$-strategies, where $\mathcal{M}_{triv}$ is a memory skeleton with a single state.

An equilibrium consisting of two $\mathcal{M}$-strategies will simply be called an "$\mathcal{M}$-strategy equilibrium".

3 Overview of the Paper

3.1 Formal statements of the results

For an arbitrary payoff function φ we introduce two functions $s^1_\varphi, s^2_\varphi \colon \mathbb{Z}^+ \to \mathbb{Z}^+ \cup \{+\infty\}$. We define $s^1_\varphi(n)$ as minimal $s \in \mathbb{Z}^+$ for which there exists a memory skeleton $\mathcal{M}$ with s states such that all one-player arenas with at most n nodes have a uniform $\mathcal{M}$-strategy equilibrium (with respect to φ). If there is no such $\mathcal{M}$ at all, we set $s^1_\varphi(n) = +\infty$. The quantity $s^2_\varphi(n)$ is defined similarly, but now there must be a uniform $\mathcal{M}$-strategy equilibrium in all arenas with at most n nodes, not only in the one-player ones.

Informally, the functions s^1_φ, s^2_φ measure how much memory is needed to play optimally in the one-player and in the two-player games respectively (depending on the number of nodes). In general, we are interested in the relationship of s^1_φ and s^2_φ. More specifically, we aim to understand what conditions on the function s^1_φ ensure that $s^2_\varphi(n) \neq +\infty$ for all $n \in \mathbb{Z}^+$. Two important remarks regarding definitions of these functions have to be made. First, they are defined through memory skeletons and not through arbitrary finite-state strategies. In other words, we only consider memory which is related to the payoff function (and not to the graph's structure). The same assumption was made in the work of Bouyer et al. [2]. Dealing with the graph-related memory seems to be out of reach for the current technique. Second, we assume uniformity of the memory over arenas of a given size. This is because in the definition of $s^1_\varphi(n), s^2_\varphi(n)$ we are looking for a *single* memory skeleton $\mathcal{M}$ with which one can play optimally in *all* arenas with at most n nodes. Thus, in our setting the structure of the memory depends only on an upper bound on the number of nodes of an arena. To illustrate this with an example, assume that the colors of the edges are integral numbers. Then storing their sum along a play until its absolute value exceeds some threshold t falls into our framework, provided that t is a function of the number of nodes. On the other hand, storing this sum modulo, say, the number of edges falls out of our framework.

Before formulating our main result, we state the lifting theorems of Gimbert and Zielonka and Bouyer et al.

▶ **Theorem 2** ([8, 2]). *For any payoff function the following holds. Assume that all one-player arenas have a uniform positional equilibrium. Then all arenas have a uniform positional equilibrium.*

More generally, for any memory skeleton $\mathcal{M}$ the following holds. Assume that all one-player arenas have a uniform $\mathcal{M}$-strategy equilibrium. Then all arenas have a uniform $\mathcal{M}$-strategy equilibrium.

In terms of the functions s^1_φ and s^2_φ Theorem 2 states the following: if $C \in \mathbb{Z}^+$ is such that

$s^1_\varphi(n) \le C$ for all $n \in \mathbb{Z}^+$, then $s^2_\varphi(n) \le C$ for all $n \in \mathbb{Z}^+$. We generalize this in the following way.

▶ **Theorem 3** (Main result)**.** *For any payoff function φ the following holds. Assume that*

$$s^1_\varphi(n) \neq +\infty \text{ for all } n \in \mathbb{Z}^+ \text{ and } \inf_{n\in\mathbb{Z}^+} \frac{s^1_\varphi(n)}{n} = 0.$$

Then

$$s^2_\varphi(n) \neq +\infty \text{ and } s^2_\varphi(n) \le s^1_\varphi\left(\min\left\{m \in \mathbb{Z}^+ \mid \frac{s^1_\varphi(m)}{m+1} \le \frac{1}{2n}\right\}\right)$$

for all $n \in \mathbb{Z}^+$.

Notice that the condition $\inf_{n\in\mathbb{Z}^+} s^1_\varphi(n)/n = 0$ ensures that the minimum from our upper bound on $s^2_\varphi(n)$ is over a non-empty set. When the function s^1_φ is bounded, we obtain exactly the same relationship between s^1_φ and s^2_φ as stated in Theorem 2. It is also instructive to give an example of how s^1_φ and s^2_φ can be related with each other when s^1_φ is unbounded. For instance, for every $\gamma \in (0,1)$ Theorem 3 states that $s^1_\varphi(n) = O(n^\gamma)$ as $n \to \infty \implies s^2_\varphi(n) = O(n^{\gamma/(1-\gamma)})$ as $n \to \infty$. Notice that now there is a gap between s^1_φ and s^2_φ (and the closer γ to 1, the bigger is the gap).

In the next proposition we show that $s^2_\varphi(k)$ can be infinite for some fixed k already when $s^1_\varphi(n) = O(n)$ as $n \to \infty$. This shows the sharpness of the condition $\inf_{n\in\mathbb{Z}^+} s^1_\varphi(n)/n = 0$ from Theorem 3 (equivalently, this condition states that $s^1_\varphi(n_k) = o(n_k)$ as $k \to \infty$, for some strictly increasing sequence $\{n_k\}_{k\in\mathbb{Z}^+}$).

▶ **Proposition 4.** *There exists a payoff function ψ for which it holds that $s^1_\psi(n) \le 2n+2$ for every $n \in \mathbb{Z}^+$ while $s^2_\psi(2) = +\infty$. Moreover, there is a non-one-player arena with 2 nodes such that from one of its nodes Max and Min have no equilibrium of two finite-state strategies with respect to ψ.*

In fact, as discussed in the introduction, for this proposition we take a payoff considered in [2]. Our contribution here is a linear upper bound on s^1_φ for this payoff.

At this point one can ask whether Theorem 3 covers a strictly larger class of payoffs than the previous lifting theorems. In other words, is there a payoff which satisfies the assumption of Theorem 3 but not of Theorem 2 (regarding one-player arenas)? We answer positively in the following proposition.

▶ **Proposition 5.** *There exists a payoff function φ such that $s^1_\varphi(n) \neq +\infty$ for all $n \in \mathbb{Z}^+$, $\inf_{n\in\mathbb{Z}^+} \frac{s^1_\varphi(n)}{n} = 0$ and $\lim_{n\to\infty} s^1_\varphi(n) = +\infty$.*

3.2 Overview of the techniques

Our main result (Theorem 3) follows from the following technical result.

▶ **Theorem 6.** *Let $\mathcal{M}$ be a memory skeleton such that all one-player arenas with at most $2n \cdot |\mathcal{M}| - 1$ nodes have a uniform $\mathcal{M}$-strategy equilibrium. Then all arenas with at most n nodes have a uniform $\mathcal{M}$-strategy equilibrium.*

Derivation of Theorem 3 from Theorem 6. By definition, there must be a memory skeleton $\mathcal{M}$ with $s_1(m)$ states such that all one-player arenas with at most m nodes have a uniform $\mathcal{M}$-strategy equilibrium. By Theorem 6, provided that $2n \cdot |\mathcal{M}| - 1 = 2n \cdot s_1(m) - 1 \le m$, all arenas with at most n nodes have a uniform $\mathcal{M}$-strategy equilibrium. In other words, $s_1(m)$ is an upper bound on $s_2(n)$ as long as $2ns_1(m) - 1 \le m$. Equivalently, $s_2(n)$ is at most $s_1(m)$ for the minimal m satisfying $\frac{s_1(m)}{m+1} \le \frac{1}{2n}$, exactly as stated in Theorem 3. Again, the condition $\inf_{m\in\mathbb{Z}^+} s_1(m)/m = 0$ ensures that for every n there exists m with $\frac{s_1(m)}{m+1} \le \frac{1}{2n}$. ◀

Let us now give an overview of our proof of Theorem 6 (as well as of its connections to previous works). First, it should be noted that the previous one-to-two-player-lifting theorems for *deterministic* games [8, 2] were proved indirectly. More specifically, these papers first establish conditions on a payoff function that are equivalent to the conclusion of the corresponding one-to-two-player-lifting theorem (through rather involved concepts of *monotonicity* and *selectivity*). Then it is shown that as long as these conditions are not satisfied, then the assumption of the corresponding one-to-two-player-lifting theorem is not satisfied as well. A more direct approach was used for the counterparts of these theorems for *stochastic* games [9, 3]. We also use a direct approach.

We perform the proof through a reduction to a statement about positional strategies. Correspondingly, the argument has two independent parts: a proof of the reduction and a proof of the "positional statement". The "reduction part" of our argument is rather similar to those from [3]. The main novelty of our argument is contained in the second part, where the "positional statement" is proved. Though we are using an inductive technique due to Gimbert and Zielonka (as all papers in this line of work do), we require a new insight into it.

The key element of the both parts of the argument is a notion of $\mathcal{M}$*-trivial* arenas (similar notions were used in [10, 3]). Informally, these are arenas where $\mathcal{M}$-strategies degenerate to positional strategies. In other words, in these arenas one can reason only about positional strategies. We will reduce Theorem 6 to the following lemma about the $\mathcal{M}$-trivial arenas; correspondingly, this lemma mentions only positional strategies.

▶ **Lemma 7** (Main lemma, stated very informal). *If all one-player $\mathcal{M}$-trivial arenas with at most $2N - 1$ nodes have a positional equilibrium, then all $\mathcal{M}$-trivial arenas with at most N nodes have one.*

(We do not specify here with respect to which sets of nodes equilibria in this lemma are. This requires diving into further technical details while the key steps of the argument can be highlighted without it.)

To derive Theorem 6 from Lemma 7, we apply the lemma to $N = n \cdot |\mathcal{M}|$. Since all one-player arenas with at most $2N - 1$ nodes have an $\mathcal{M}$-strategy equilibrium, all $\mathcal{M}$-trivial one-player arenas with this many nodes have a positional equilibrium (simply because in these arenas $\mathcal{M}$-strategy equilibria degenerate to positional equilibria). This allows us to conclude that all $\mathcal{M}$-trivial arenas (not necessarily one-player) with at most N nodes have a positional equilibrium. To finish the argument it remains to refer to a standard *product arena* construction (sometimes called a *product game* construction, see, for instance, [1, Chapter 2]). Through this construction one can reduce an existence of an $\mathcal{M}$-strategy equilibrium in a given arena to an existence of a positional equilibrium in some $\mathcal{M}$-trivial arena with $|\mathcal{M}|$ times more nodes. Thus, we get the conclusion of Theorem 6 for all arenas with at most $N/|\mathcal{M}| = n$ nodes, as required.

It is rather important that we only lose a factor of 2 in the size of the arenas in Lemma 7. The additional factor of $|\mathcal{M}|$ is caused later by the product arena argument.

Let us now discuss the "positional part" of the argument. We rely on the technique of Gimbert and Zielonka [8]. Recall that in their paper it is established that as long as all one-player arenas have a uniform positional equilibrium, the same holds for all arenas.

However, in our framework we inevitably have to deal with arenas of bounded size. To this end, we highlight the following feature of the technique of Gimbert and Zielonka. Namely, to establish a uniform positional equilibrium in a *specific* arena $\mathcal{A}$ it is not required that *all* one-player arenas have one. Instead, we just need this for some *finite* family of one-player arenas; it turns out that all these one-player arenas have at most twice as many nodes as $\mathcal{A}$.

As far as we know, this has not been indicated in the literature earlier. There are close statements ([3, Lemma 24], [9, Theorem 9]) but they are not applicable in our setting. This is because the have the following form: if $\mathcal{F}$ is a family of arenas which is closed under certain operations, then having a positional equilibrium in all one-player arenas from $\mathcal{F}$ implies having one in all arenas from $\mathcal{F}$. The problem is that one of the operations with respect to which $\mathcal{F}$ has to be closed *increases* the size of the arenas (namely, one called the *split* in [9, 3]). Thus, $\mathcal{F}$ must have arenas with arbitrarily many nodes while in Lemma 7 we have to deal with arenas of bounded size.

3.3 Concluding remarks

Unfortunately, the barrier for the one-to-two-player lifting theorems which we encountered in this paper is quite low. A number of objectives that became popular in recent years require by far more states than in Theorem 3 (let alone the graph-related memory). This includes multi-dimensional energy games, mean-payoff parity games, etc. [4, 5, 6]. Still, our paper makes a step towards inventing a technique for treating these objectives in a unified manner.

Let us conclude by indicating the plausibility of extending our results to stochastic games.

Organization of the paper. Our central technical result (Theorem 6) is proved in Section 4. Our exposition includes a self-contained proof of the following special case of Theorem 6 (see Subsection 4.2).

▶ **Lemma 8.** *Assume that all one-player arenas with at most $2N-1$ nodes have a uniform positional equilibrium. Then all arenas with at most N nodes have a uniform positional equilibrium.*

This is exactly what Theorem 6 states for a memory skeleton $\mathcal{M}$ consisting of 1 state. Observe also that this lemma implies the "memory-less" lifting theorem of Gimbert and Zielonka. Thus, our exposition provides a self-contained direct proof of their result.

Propositions 4 and 5 are proved in, respectively, Sections 5 and 6.

4 Proof of Theorem 6

4.1 Reduction to a statement about positional strategies

In this subsection we reduce Theorem 6 to a statement about positional strategies (namely, to Lemma 13 below). First we need a classical concept of *product arenas*.

▶ **Definition 9** (Product arenas). *Let $\mathcal{M} = \langle M, m_{init}, \delta\colon M \times C \to M\rangle$ be a memory skeleton and $\mathcal{A} = \langle V, V_{\mathrm{Max}}, V_{\mathrm{Min}}, E, \mathsf{source}, \mathsf{target}, \mathsf{col}\rangle$ be an arena. Then $\mathcal{M} \times \mathcal{A}$ stands for an arena, where*

- *the set of nodes is $M \times V$;*
- *the set of Max's nodes is $M \times V_{\mathrm{Max}}$;*
- *the set of Min's nodes is $M \times V_{\mathrm{Min}}$;*
- *the set of edges is $M \times E$;*
- *the source function is defined as follows: $\mathsf{source}((m, e)) = (m, \mathsf{source}(e))$;*

- *the target function is defined as follows:* $\mathsf{target}((m,e)) = \big(\delta(m, \mathsf{col}(e)), \mathsf{target}(e)\big)$*;*
- *the coloring function is defined as follows:* $\mathsf{col}((m,e)) = \mathsf{col}(e)$*.*

One can reduce a reasoning about $\mathcal{M}$-strategies in an arena $\mathcal{A}$ to a reasoning about positional strategies in the arena $\mathcal{M} \times \mathcal{A}$. More specifically, the following claim holds.

▶ **Observation 10.** *Let $\mathcal{M} = \langle M, m_{init}, \delta\colon M \times C \to M\rangle$ be a memory skeleton and $\mathcal{A} = \langle V, V_{\mathrm{Max}}, V_{\mathrm{Min}}, E, \mathsf{source}, \mathsf{target}, \mathsf{col}\rangle$ be an arena. Then for every $S \subseteq V$ the following holds: if $\mathcal{M} \times \mathcal{A}$ has an $(\{m_{init}\} \times S)$-wise positional equilibrium, then $\mathcal{A}$ has an S-wise $\mathcal{M}$-strategy equilibrium.*

This fact is rather standard, we provide a proof of it for completeness.

Proof of Observation 10. Let $(\widehat{\Sigma}, \widehat{T})$ be an $(\{m_{init}\} \times S)$-wise positional equilibrium in $\mathcal{M} \times \mathcal{A}$ (here $\widehat{\Sigma}$ is a strategy of Max and $\widehat{T}$ is a strategy of Min). Define an $\mathcal{M}$-strategy Σ of Max in $\mathcal{A}$ as follows. To determine $\Sigma(h)$ for a position h with $\mathsf{target}(h) = v \in V_{\mathrm{Max}}$ and $\delta(m_{init}, \mathsf{col}(h)) = m$, we consider a move which $\widehat{\Sigma}$ makes in a node (m, v). Assume that this move is a pair (m, e) (we must have $\mathsf{source}(e) = v$). Then we set $\Sigma(h) = e$. We define an $\mathcal{M}$-strategy T of Min in $\mathcal{A}$ similarly through the strategy $\widehat{T}$. We claim that (Σ, T) is an S-wise equilibrium in $\mathcal{A}$.

Assume for contradiction that for some $v \in S$ either Σ is not an optimal response to T or T is not an optimal response to Σ in v. We consider only the first option, the second one can be treated similarly. Then some infinite path $h \in \mathsf{Cons}(v, T)$ is better from the Max's perspective than $h(v, \Sigma, T)$ (the play of Σ and T from v), i.e.,

$$\varphi \circ \mathsf{col}(h) > \varphi \circ \mathsf{col}\big(h(v, \Sigma, T)\big). \tag{1}$$

For $n \in \mathbb{Z}^+$ let e'_n denote the nth edge of h and e_n denote the nth edge of $h(v, \Sigma, T)$. For each of these two sequences of edges define a sequence of states into which $\mathcal{M}$ comes while reading colors of these edges (assuming $\mathcal{M}$ is initially in m_{init}):

$$m'_1 = m_{init}, \qquad m'_{n+1} = \delta(m'_n, \mathsf{col}(e'_n)) \text{ for every } n \in \mathbb{Z}^+, \tag{2}$$

$$m_1 = m_{init}, \qquad m_{n+1} = \delta(m_n, \mathsf{col}(e_n)) \text{ for every } n \in \mathbb{Z}^+. \tag{3}$$

It is easy to see that the sequence $(m_1, e_1)(m_2, e_2)(m_3, e_3)\ldots$ is the play of $\widehat{\Sigma}$ and $\widehat{T}$ from (m_{init}, v). For example, let us show its consistency with $\widehat{\Sigma}$. We have to show that for every $n \in \mathbb{Z}^+$ such that $\mathsf{source}((m_n, e_n)) = (m_n, \mathsf{source}(e_n))$ is a node of Max we have $\widehat{\Sigma}((m_n, \mathsf{source}(e_n))) = (m_n, e_n)$. By definition of Σ it is sufficient to show that $e_n = \Sigma(h)$ for a position h in $\mathcal{A}$ with $\mathsf{target}(h) = \mathsf{source}(e_n)$ and $\delta(m_{init}, \mathsf{col}(h)) = m_n$. It is easy to see that we have this for a position $h = e_1 \ldots e_{n-1}$ if $n > 1$ and for $h = \lambda_v$ if $n = 1$. Indeed, we have $\Sigma(h) = e_n$ and $\mathsf{target}(h) = \mathsf{source}(e_n)$ because $e_1 e_2 \ldots e_n$ is a prefix of the play of Σ and T from v. Now, we have $\delta(m_{init}, \mathsf{col}(h)) = m_n$ because of (3).

Consistency of $(m_1, e_1)(m_2, e_2)(m_3, e_3)\ldots$ with $\widehat{T}$ can be shown similarly. Moreover, by the same argument the sequence $(m'_1, e'_1)(m'_2, e'_2)(m'_3, e'_3)\ldots$ is also consistent with $\widehat{T}$, due to (2). Now, since $\widehat{\Sigma}$ is an optimal response to $\widehat{T}$ in (m_{init}, v), we have that $(m_1, e_1)(m_2, e_2)(m_3, e_3)\ldots$ is at least as good as $(m'_1, e'_1)(m'_2, e'_2)(m'_3, e'_3)\ldots$ from the Max's perspective. However, by definition the first sequence is colored exactly as $h(v, \Sigma, T)$ and the second one exactly as h. This is a contradiction with (1). ◀

Next we introduce one more concept which we need for the reduction, namely, one of *$\mathcal{M}$-triviality.*

▶ **Definition 11.** *Let $\mathcal{M} = \langle M, m_{init}, \delta\colon M \times C \to M\rangle$ be a memory skeleton. A pair $(\mathcal{A}, f)$ of an arena $\mathcal{A} = \langle V, V_{\text{Max}}, V_{\text{Min}}, E, \mathsf{source}, \mathsf{target}, \mathsf{col}\rangle$ and a function $f\colon V \to M$ is called **$\mathcal{M}$-trivial** if for every $e \in E$ it holds that $\delta\big(f(\mathsf{source}(e)), \mathsf{col}(e)\big) = f(\mathsf{target}(e))$.*

Informally, f is a mapping from $\mathcal{A}$ to the transition graph of $\mathcal{M}$ which takes into account the colors of the edges. Of course, there are arenas that belong to no $\mathcal{M}$-trivial pair. We observe that $\mathcal{M}$-strategies, in a sense, degenerate to positional ones in $\mathcal{M}$-trivial pairs.

▶ **Observation 12.** *Let $\mathcal{M} = \langle M, m_{init}, \delta\colon M \times C \to M\rangle$ be a memory skeleton. Then for every $\mathcal{M}$-trivial pair $(\mathcal{A}, f)$ the following holds: if $\mathcal{A}$ has a uniform $\mathcal{M}$-strategy equilibrium, then $\mathcal{A}$ has an $f^{-1}(m_{init})$-wise positional equilibrium.*

Proof. Note that for any finite path h in $\mathcal{A}$ we have:

$$\delta(f(\mathsf{source}(h)), \mathsf{col}(h)) = f(\mathsf{target}(h)).$$

Indeed, this holds by definition as long as h is a single edge; for longer h this can be easily proved by induction on $|h|$.

To show the observation, we simply show that any $\mathcal{M}$-strategy coincides with some positional one for all the plays that start in the nodes of $f^{-1}(m_{init})$. Indeed, a move of an $\mathcal{M}$-strategy in a position h depends solely on $\mathsf{target}(h)$ and $\delta(m_{init}, \mathsf{col}(h))$. However, $\delta(m_{init}, \mathsf{col}(h)) = \delta\big(f(\mathsf{source}(h)), \mathsf{col}(h)\big) = f(\mathsf{target}(h))$ for all h with $\mathsf{source}(h) \in f^{-1}(m_{init})$. In other words, for all such h a move of an $\mathcal{M}$-strategy in h is a function only of $\mathsf{target}(h)$, as required. ◀

We are ready to formulate a statement about positional strategies to which we reduce Theorem 6.

▶ **Lemma 13.** *Let $\mathcal{M} = \langle M, m_{init}, \delta\colon M \times C \to M\rangle$ be a memory skeleton. Assume that for every $\mathcal{M}$-trivial pair $(\mathcal{A}, f)$ such that $\mathcal{A}$ is one-player and has at most $2N - 1$ nodes there exists an $f^{-1}(m_{init})$-wise positional equilibrium in $\mathcal{A}$.*

Then for every $\mathcal{M}$-trivial pair $(\mathcal{A}, f)$ such that $\mathcal{A}$ has at most N nodes there exists an $f^{-1}(m_{init})$-wise positional equilibrium in $\mathcal{A}$.

Derivation of Theorem 6 from Lemma 13. Let $\mathcal{A} = \langle V, V_{\text{Max}}, V_{\text{Min}}, E, \mathsf{source}, \mathsf{target}, \mathsf{col}\rangle$ be an arena with at most n nodes. Our goal is to show that $\mathcal{A}$ has a uniform $\mathcal{M}$-strategy equilibrium. By Observation 10 it is sufficient to show that the arena $\mathcal{M} \times \mathcal{A}$ has an $\{m_{init}\} \times V$-wise positional equilibrium. Observe that a pair $(\mathcal{M} \times \mathcal{A}, f)$, where

$$f\colon M \times V \to M, \qquad f((m, v)) = m,$$

is an $\mathcal{M}$-trivial pair, by definition of $\mathcal{M} \times \mathcal{A}$. Since $\{m_{init}\} \times V = f^{-1}(m_{init})$, we only have to show that $\mathcal{M} \times \mathcal{A}$ has an $f^{-1}(m_{init})$-wise positional equilibrium. Since $\mathcal{M} \times \mathcal{A}$ has at most $|\mathcal{M}| \cdot n$ nodes, it remains to explain why the assumption of Lemma 13 holds for $N = |\mathcal{M}| \cdot n$.

By the assumption of Theorem 6 all one-player arenas with at most $2|\mathcal{M}| \cdot n - 1 = 2N - 1$ nodes have a uniform $\mathcal{M}$-strategy equilibrium. In particular, this applies to any one-player arena $\mathcal{A}'$ with at most $2N - 1$ nodes belonging to some $\mathcal{M}$-trivial pair $(\mathcal{A}', f)$. By Observation 12 this means that all such $\mathcal{A}'$ have a $f^{-1}(m_{init})$-wise positional equilibrium, as required. ◀

In the next two subsections we prove Lemma 13. We first prove it in a special case when $\mathcal{M}$ is a memory skeleton with just 1 state.

4.2 Lemma 13 for a single-state memory skeleton

For a memory skeleton $\mathcal{M}$ with just 1 state the notion of $\mathcal{M}$-triviality is vacuous. Indeed, all that we can do is to label all nodes of an arena by the unique state of $\mathcal{M}$, and this always gives an $\mathcal{M}$-trivial pair. So in this special case Lemma 13 takes the following form.

▶ **Lemma 14** (Special case of Lemma 13, restatement of Lemma 8)**.** *Assume that all one-player arenas with at most $2N-1$ nodes have a uniform positional equilibrium. Then all arenas with at most N nodes have a uniform positional equilibrium.*

Proof. The proof is by induction on the number of edges of an arena. More precisely, we are proving by induction on m the following claim: for every m every arena with m edges and at most N nodes has a uniform positional equilibrium.

The induction base ($m = 1$) is trivial (any arena with one edge is one-player and has exactly one node, so we can just refer to the assumption of the lemma). We proceed to the induction step. Take an arena $\mathcal{A} = \langle V, V_{\text{Max}}, V_{\text{Min}}, E, \mathsf{source}, \mathsf{target}, \mathsf{col}\rangle$ with at most N nodes and assume that all arenas with at most N nodes and with fewer edges than $\mathcal{A}$ have a uniform positional equilibrium. We prove the same for $\mathcal{A}$. Since the set of uniform equilibria is a Cartesian product, it is enough to establish the following two claims:

- ***(a)*** in $\mathcal{A}$ there exists a uniform equilibrium including a positional strategy of Max;
- ***(b)*** in $\mathcal{A}$ there exists a uniform equilibrium including a positional strategy of Min.

We only show ***(a)***, a proof of ***(b)*** is similar.

We may assume that $\mathcal{A}$ is not one-player (otherwise we are done due to the assumptions of the lemma). Hence there exists a node $w \in V_{\text{Max}}$ with out-degree at least 2. Partition the set $E(w) = \{e \in E \mid \mathsf{source}(e) = w\}$ into two non-empty disjoint subsets $E_1(w)$ and $E_2(w)$. Define two new arenas $\mathcal{A}_1$ and $\mathcal{A}_2$. The arena $\mathcal{A}_1$ is obtained from $\mathcal{A}$ by deleting edges from the set $E_2(w)$. Similarly, the arena $\mathcal{A}_2$ is obtained from $\mathcal{A}$ by deleting edges from the set $E_1(w)$. So in $\mathcal{A}_i$ for $i = 1, 2$ the set of edges with the source in w is $E_i(w)$.

Both $\mathcal{A}_1$ and $\mathcal{A}_2$ have fewer edges than $\mathcal{A}$. So both these arenas have a uniform positional equilibrium. Let (σ_i, τ_i) be a uniform positional equilibrium in $\mathcal{A}_i$ for $i = 1, 2$. We will first define two auxiliary strategies τ_{12} and τ_{21} of Min; then we will show that either (σ_1, τ_{12}) or (σ_2, τ_{21}) is a uniform equilibrium in $\mathcal{A}$. After that ***(a)*** will be proved.

Strategies τ_{12} and τ_{21} will not be positional. In a sense, they are combinations of τ_1 and τ_2. In both strategies Min has a counter I which can only take two values, 1 and 2. The counter I indicates to Min which of the strategies τ_1 and τ_2 to use. I.e., whenever Min should make a move from a node $v \in V_{\text{Min}}$, he uses an edge $\tau_I(v)$. The value of I changes each time in the node w Max uses an edge not from a set $E_I(w)$. It only remains to specify the initial value of I. There are two ways to do this, one will give us strategy τ_{12}, and the other will give τ_{21}. More specifically, in τ_{12} the initial value of I is 1 and in τ_{21} the initial value of I is 2.

It is not hard to see that τ_{12} is a uniformly optimal response to σ_1 and τ_{21} is a uniformly optimal response to σ_2. For instance, let us show this for τ_{12} and σ_1. By definition, τ_1 is a uniformly optimal response to σ_1 in the arena $\mathcal{A}_1$, and hence also in the arena $\mathcal{A}$ (because any play against σ takes place inside $\mathcal{A}_1$). It remains to notice that τ_{12} plays exactly as τ_1 against σ_1. Indeed, σ_1 never uses edges from $E_2(w)$, so the counter I always equals 1 against σ_1.

It remains to show that either σ_1 is a uniformly optimal response to τ_{12} or σ_2 is a uniformly optimal response to τ_{21} (in the arena $\mathcal{A}$). We derive it from the assumption of the lemma applied to an auxiliary one-player arena $\mathcal{B}$ with at most $2N-1$ nodes.

Namely, we define $\mathcal{B}$ as follows. Recall that in our notation $(\mathcal{A}_1)_{\tau_1}$ and $(\mathcal{A}_2)_{\tau_2}$ stand for two arenas obtained from, respectively, $\mathcal{A}_1$ and $\mathcal{A}_2$ by throwing away edges that are inconsistent with, respectively, τ_1 and τ_2. Consider an arena consisting of two "independent" parts one of which coincides with $(\mathcal{A}_1)_{\tau_1}$ and the other with $(\mathcal{A}_2)_{\tau_2}$ ("independent" means that there are no edges between the parts). From each part take a node corresponding to the node w. Then merge these two nodes into a single one. The resulting arena with $2|V| - 1 \leq 2N - 1$ nodes will be $\mathcal{B}$ (see Figure 1).

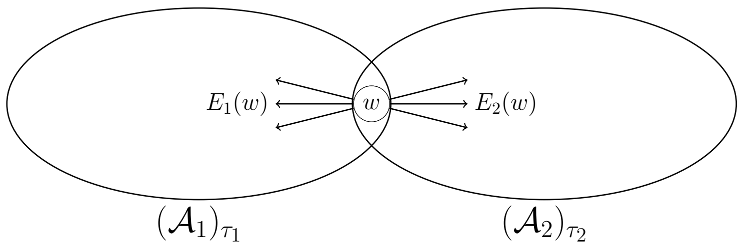

Figure 1 Arena $\mathcal{B}$.

For each node of $\mathcal{A}$ there are two "copies" of it in $\mathcal{B}$ – one from $(\mathcal{A}_1)_{\tau_1}$ and the other from $(\mathcal{A}_2)_{\tau_2}$. We will call copies of the first kind *left copies* and copies of the second kind *right copies*. Note that the left and the right copy of w is the same node in $\mathcal{B}$. Any other node of $\mathcal{A}$ has two distinct copies. Now, by the *prototype* of a node v' of $\mathcal{B}$ we mean a node v of $\mathcal{A}$ of which v' is a copy.

Note that in $\mathcal{B}$ all nodes of Min have out-degree 1 (because they do so inside $(\mathcal{A}_1)_{\tau_1}$ and $(\mathcal{A}_2)_{\tau_2}$, and the only node of $\mathcal{B}$ which was obtained by merging two nodes is a node of Max). Thus, $\mathcal{B}$ is a one-player arena.

An important feature of $\mathcal{B}$ is that it can "emulate" any play against τ_{12} and τ_{21} in $\mathcal{A}$. Formally,

▶ **Lemma 15.** *For any infinite path h in $\mathcal{A}$ which is consistent with τ_{12} there exists an infinite path h' in $\mathcal{B}$ with $\mathsf{col}(h') = \mathsf{col}(h)$ and with the source in the left copy of $\mathsf{source}(h)$.*

Similarly, for any infinite path h in $\mathcal{A}$ which is consistent with τ_{21} there exists an infinite path h' in $\mathcal{B}$ with $\mathsf{col}(h') = \mathsf{col}(h)$ and with the source in the right copy of $\mathsf{source}(h)$.

Proof. We only give an argument for τ_{12}, the argument for τ_{21} is similar. We construct h' from the left copy of $\mathsf{source}(h)$ by always moving in the same "local direction" as h. There will be no problem with that for the nodes of Max because they have the same set of out-going edges in $\mathcal{B}$ as their prototypes have in $\mathcal{A}$. Now, for the nodes of Min we should be more accurate. The path h is consistent with τ_{12}, so from the nodes of Min it applies either τ_1 or τ_2. Now, in $\mathcal{B}$ strategy τ_1 is available only in the left ellipse of Figure 1, and τ_2 is available only in the right ellipse. So each time h wants to apply τ_1, the path h' should be in the left ellipse. Similarly, each time h wants to apply τ_2, the path h' should be in the right ellipse. Initially, until its counter changes, τ_{12} applies τ_1, and correspondingly h' starts in the left ellipse. Now, each time τ_{12} switches to τ_2, it does so because Max used an edge from $E_2(w)$ in w. Correspondingly, h' enters the right ellipse at this moment. Similarly, whenever τ_{12} switches back to τ_1, the path h' returns to the left ellipse. ◀

Note that in $\mathcal{B}$ Min has exactly one strategy. We denote it by T. The arena $\mathcal{B}$ is one-player and has at most $2N - 1$ nodes, so by the assumption of the lemma there is a uniform positional equilibrium $(\widehat{\Sigma}, T)$ in it. We claim the following:

- if $\widehat{\Sigma}$ applies an edge from $E_1(w)$ in w, then σ_1 is a uniformly optimal response to τ_{12} in $\mathcal{A}$;
- if $\widehat{\Sigma}$ applies an edge from $E_2(w)$ in w, then σ_2 is a uniformly optimal response to τ_{21} in $\mathcal{A}$.

We only show the first claim, the proof of the second one is analogous. Consider a restriction of $\widehat{\Sigma}$ to the left ellipse of $\mathcal{B}$. This defines a positional strategy $\widehat{\sigma}$ of Max in $\mathcal{A}$. Note that in each node of $\mathcal{A}$ the strategy σ_1 is at least as good against τ_{12} as $\widehat{\sigma}$. Indeed, $\sigma_1(w), \widehat{\sigma}(w) \in E_1(w)$. Hence $\sigma_1, \widehat{\sigma}$ are strategies in the arena $\mathcal{A}_1$, where σ_1 is a uniformly optimal response to τ_1. It remains to notice that τ_{12} plays exactly as τ_1 against σ_1 and $\widehat{\sigma}$ since these two strategies of Max never use edges from $E_2(w)$.

Therefore, it is enough to show that $\widehat{\sigma}$ is a uniformly optimal response to τ_{12} in $\mathcal{A}$. Take any node $v \in V$ and any play h against τ_{12} from v. Our goal is to show that the play of $\widehat{\sigma}$ and τ_{12} from v is at least as good from the Max's perspective as h. Now, by Lemma 15 some infinite path h' from the left copy of v is colored exactly as h. On the other hand, the play of $\widehat{\Sigma}$ and T from the left copy of v is at least as good for Max as h' (and hence as h). This is because h' is consistent with T (as there are simply no other strategies of Min in $\mathcal{B}$) and because $(\widehat{\Sigma}, T)$ is an equilibrium. It remains to note that the play of $\widehat{\Sigma}$ and T from the left copy of v is colored exactly as the play of $\widehat{\sigma}$ and τ_{12} from v. Indeed, as we have already observed, τ_{12} plays exactly as τ_1 against $\widehat{\sigma}$. On the other hand, the play of $\widehat{\Sigma}$ and T can never leave the left ellipse as $\widehat{\Sigma}$ points to the left in w. Moreover, restrictions of these strategies to the left ellipse coincide with $\widehat{\sigma}$ and τ_1; for $\widehat{\Sigma}$ this is just by definition and for T this is because the left ellipse coincides with the arena $(\mathcal{A}_1)_{\tau_1}$ ◀

4.3 Proof of Lemma 13 in general case

We are now proving by induction on m the following claim: for every m and for every $\mathcal{M}$-trivial pair $(\mathcal{A}, f)$ such that $\mathcal{A}$ has m edges and at most N nodes there exists an $f^{-1}(m_{init})$-wise positional equilibrium in $\mathcal{A}$.

Induction base ($m = 1$) again requires no argument, and we proceed to the induction step. Consider any $\mathcal{M}$-trivial pair $(\mathcal{A}, f)$, where $\mathcal{A} = \langle V, V_{\mathrm{Max}}, V_{\mathrm{Min}}, E, \mathsf{source}, \mathsf{target}, \mathsf{col}\rangle$ has at most N nodes. Our goal is to show that $\mathcal{A}$ has an $f^{-1}(m_{init})$-wise positional equilibrium, provided that an analogous claim is already proved for all $\mathcal{M}$-trivial pairs $(\mathcal{A}', f')$ in which $\mathcal{A}'$ has at most N nodes and fewer edges than $\mathcal{A}$. Since the set of $f^{-1}(m_{init})$-wise equilibria is a Cartesian product, it is enough to establish the following two claims:

- ***(a)*** in $\mathcal{A}$ there exists an $f^{-1}(m_{init})$-wise equilibrium including a positional strategy of Max;
- ***(b)*** in $\mathcal{A}$ there exists an $f^{-1}(m_{init})$-wise equilibrium including a positional strategy of Min.

We only show ***(a)***, a proof of ***(b)*** is similar. As before, we may assume that $\mathcal{A}$ is not one-player so that there exists a node $w \in V_{\mathrm{Max}}$ with out-degree at least 2. We partition the set of its out-going edges into two disjoint non-empty sets $E_1(w)$ and $E_2(w)$. Then we define arenas $\mathcal{A}_1$ and $\mathcal{A}_2$ exactly as in the proof of Lemma 14. Since $(\mathcal{A}, f)$ is an $\mathcal{M}$-trivial pair, then so are pairs $(\mathcal{A}_1, f)$ and $(\mathcal{A}_2, f)$. Indeed, $\mathcal{A}_1$ and $\mathcal{A}_2$ were obtained by simply throwing away some edges of $\mathcal{A}$. The remaining edges satisfy the definition of $\mathcal{M}$-triviality with respect to f just because they do so inside $\mathcal{A}$.

Note that $\mathcal{A}_1$ and $\mathcal{A}_2$ both have fewer edges than $\mathcal{A}$ and at most as many nodes. So by the induction hypotheses both these arenas have an $f^{-1}(m_{init})$-wise positional equilibrium. Let (σ_1, τ_1) be an $f^{-1}(m_{init})$-wise positional equilibrium in $\mathcal{A}_1$ and (σ_2, τ_2) be an $f^{-1}(m_{init})$-wise positional equilibrium in $\mathcal{A}_2$. Next, we define two auxiliary strategies τ_{12} and τ_{21} of

Min exactly as in the proof of Lemma 14. Our goal is to show that either (σ_1, τ_{12}) is an $f^{-1}(m_{init})$-wise equilibrium in $\mathcal{A}$ or (σ_2, τ_{21}) is an $f^{-1}(m_{init})$-wise equilibrium in $\mathcal{A}$.

Similarly to the proof of Lemma 14 we have that τ_{12} is an $f^{-1}(m_{init})$-wise optimal response to σ_1 and τ_{21} is an $f^{-1}(m_{init})$-wise optimal response to σ_2. The main challenge is to show the opposite for at least one of the pairs (σ_1, τ_{12}) and (σ_2, τ_{21}).

For that we define a one-player arena $\mathcal{B}$ exactly as in the proof of Lemma 14 (see Figure 1). It has $2|V| - 1 \leq 2N - 1$ nodes. We will apply the assumption of Lemma 13 to $\mathcal{B}$. More precisely, this will be done for some $\mathcal{M}$-trivial pair which includes $\mathcal{B}$. For that we define the following mapping g from the set of nodes of $\mathcal{B}$ to the set of states of $\mathcal{M}$. Namely, if v' is a node of $\mathcal{B}$, we set $g(v') = f(v)$, where v is the prototype of v'. Observe that $(\mathcal{B}, g)$ is an $\mathcal{M}$-trivial pair. Indeed any edge of $\mathcal{B}$ is between two nodes whose prototypes are connected in $\mathcal{A}$ by an edge of the same color. Thus, by the assumption of Lemma 13, the arena $\mathcal{B}$ has a $g^{-1}(m_{init})$-wise positional equilibrium $(\widehat{\Sigma}, T)$ (as before, in $\mathcal{B}$ there are no strategies of Min other than T). It is sufficient to establish the following two claims:

- if $\widehat{\Sigma}$ applies an edge from $E_1(w)$ in w, then σ_1 is an $f^{-1}(m_{init})$-wise optimal response to τ_{12} in $\mathcal{A}$;
- if $\widehat{\Sigma}$ applies an edge from $E_2(w)$ in w, then σ_2 is an $f^{-1}(m_{init})$-wise optimal response to τ_{21} in $\mathcal{A}$.

A key observation here is that $g^{-1}(m_{init})$ is the union of the left copies of the nodes from $f^{-1}(m_{init})$ and the right copies of the nodes of $f^{-1}(m_{init})$. In fact, for a proof of the first claim we only need a fact that $g^{-1}(m_{init})$ includes all the left copies of the nodes from $f^{-1}(m_{init})$. Correspondingly, only the right copies of $f^{-1}(m_{init})$ are relevant for a proof of the second claim.

We only show the first claim, the second one can be proved similarly. As before, the argument is carried out through a positional strategy $\widehat{\sigma}$ of Max in $\mathcal{A}$ obtained by restricting $\widehat{\Sigma}$ to the left ellipse. First we observe that in any node from $f^{-1}(m_{init})$ the strategy σ_1 is at least as good against τ_{12} as $\widehat{\sigma}$. Indeed, both σ_1 and $\widehat{\sigma}$ are strategies in $\mathcal{A}_1$ whereas σ_1 is an $f^{-1}(m_{init})$-wise optimal response to τ_1 in $\mathcal{A}_1$ by definition. On the other hand, τ_{12} plays against σ_1 and $\widehat{\sigma}$ exactly as τ_1.

It remains to show that $\widehat{\sigma}$ is an optimal response to τ_{12} in any node from $f^{-1}(m_{init})$. This can be done by exactly the same argument as in the last paragraph of the proof of Lemma 14. A difference is that now we have a weaker assumption about $\widehat{\Sigma}$; namely, we only know that $\widehat{\Sigma}$ is optimal in the nodes from $g^{-1}(m_{init})$ (while before it was optimal everywhere in $\mathcal{B}$). Correspondingly, we are proving a weaker statement. Namely, instead of proving that $\widehat{\sigma}$ is an optimal response to τ_{12} everywhere in $\mathcal{A}$ we are only proving this for all $v \in f^{-1}(m_{init})$. It can be checked that in the argument for a specific v we only require optimality of $\widehat{\Sigma}$ in the left copy of v; so if $v \in f^{-1}(m_{init})$, then its left copy is in $g^{-1}(m_{init})$ so that the argument still works.

5 Proof of Proposition 4

Let the set of colors be $C = \{-1, 1\}$. Define $\psi \colon C^\omega \to \{0, 1\}$ as follows:

$$\psi(c_1 c_2 c_3 \ldots) = 1 \iff \text{either } \Big(\lim_{n\to\infty} \sum_{i=1}^{n} c_i = +\infty\Big) \text{ or } \Big(\sum_{i=1}^{n} c_i = 0 \text{ for infinitely many } n\Big).$$

We assume the standard ordering on $\{0, 1\} = \psi(C^\omega)$ so that 1 is interpreted as victory of Max and 0 is interpreted as victory of Min. It will be more convenient to refer to the elements of C as to *weights* rather than as to colors. Correspondingly, by the weight of a path we will mean the sum of the weights of its edges. Further, we will call a path *positive* if its weight is positive. We define negative and zero paths similarly (in fact, we will apply this terminology only to cycles).

Given $n \in \mathbb{Z}^+$, define a memory skeleton $\mathcal{M}_n$ which stores the current sum of the weights until its absolute value exceeds n (in this case it comes into a special "invalid" state and stays in it forever). We will denote its normal states by integers from $-n$ to n, and we will denote its invalid state by $\perp$. The number of states of $\mathcal{M}_n$ is $2n+2$. The rest of the proof is organized as follows:

- in Subsection 5.1 we show that all one-player arenas with at most n nodes have a uniform $\mathcal{M}_n$-strategy equilibrium (with respect to ψ);
- in Subsection 5.2 we construct a 2-node arena and a node v in it for which there exists no $\{v\}$-wise finite-state equilibrium (with respect to ψ).

5.1 One-player arenas

Consider a one-player arena $\mathcal{A} = \langle V, V_{\text{Max}}, V_{\text{Min}}, E, \mathsf{source}, \mathsf{target}, \mathsf{col}\rangle$ with at most n nodes. By definition, either all nodes of Min have out-degree 1 or all nodes of Max have out-degree 1. We will consider these two cases separately. In both cases we will deal with a product arena $\mathcal{M}_n \times \mathcal{A}$ (see Definition 9). Recall that the nodes of $\mathcal{M}_n \times \mathcal{A}$ are pairs of the form (state of $\mathcal{M}_n$, node of $\mathcal{A}$) so that it will be convenient to use the following notation for these nodes:

$$(-n, v), \ldots (-1, v), (0, v), (1, v), \ldots, (n, v), (\perp, v), \qquad v \in V.$$

Case 1: all nodes of Min have out-degree 1. Let V_+ be the set of nodes of $\mathcal{A}$ from where one can reach a positive cycle. By a standard reasoning, inside V_+ Max has a positional strategy which guarantees that the sum of the weights goes to $+\infty$. This strategy is winning for Max with respect to ψ as well.

Obviously, there are no edges from $V \setminus V^+$ to V^+. Thus, for the rest of the argument we may only deal with a restriction of $\mathcal{A}$ to $V \setminus V^+$. In other words, we may assume WLOG that V^+ is empty so that all cycles of $\mathcal{A}$ are non-positive. In particular, this implies that the weight of any path is at most n (as it can be decomposed into a simple path and a union of cycles).

Now the sum of the weights cannot go to $+\infty$ so that Max can only win by making this sum equal to 0 infinitely many times. Observe also that Max looses as long as the sum of the weights becomes smaller than $-n$. Indeed, in this case it is impossible to make it non-negative again (for that we would need a path of weight bigger than n).

These considerations show that our winning condition for Max is now equivalent to the following one: Max wins if the sum of the weights equals 0 infinitely often but never exceeds n in the absolute value. Equivalently (in terms of the memory skeleton $\mathcal{M}_n$) Max wins if $\mathcal{M}_n$ comes into state 0 infinitely often but never comes into the invalid state. We can further simplify our winning condition by recalling $\mathcal{M}_n$ stays in the invalid state forever once this state is reached. So we can just forget about the requirement of avoiding the invalid state; as long as $\mathcal{M}_n$ comes into state 0 infinitely often, we automatically have that it never comes into the invalid state.

In terms of the product arena $\mathcal{M}_n \times \mathcal{A}$ this just means that Max wins from $w \in V$ if and only there exists an infinite path from $(0, w)$ in $\mathcal{M}_n \times \mathcal{A}$ which visits some node of the form $(0, v), v \in V$ infinitely often. In terms of the arena $\mathcal{M}_n \times \mathcal{A}$ this winning condition is just a parity game with 2 priorities (a bigger one labels all the nodes of the form $(0, v)$

and a smaller one labels all the other nodes). By positional determinacy of parity games some positional strategy σ of Max in $\mathcal{M}_n \times \mathcal{A}$ is winning for him wherever he has a winning strategy. Similarly to the proof of Observation 10, strategy σ defines an $\mathcal{M}_n$-strategy Σ of Max in $\mathcal{A}$ which is winning wherever Max has a winning strategy. This strategy Σ (together with a unique strategy of Min) forms a uniform $\mathcal{M}_n$-strategy equilibrium in $\mathcal{A}$.

Case 2: all nodes of Max have out-degree 1. Similarly to Case 1 we may assume WLOG that all cycles of $\mathcal{A}$ are non-negative. Indeed, in all nodes from where one can reach a negative cycle Min can win by making the sum of the weights going to $-\infty$; moreover, he has a single positional strategy which does this for all these nodes. There are no edges to these nodes from the remaining ones; so we can restrict our arena to the set of nodes from where no negative cycle is reachable.

By definition, Min wins if and only if the sequence of the running sums of the weights satisfies the following two conditions:

- ***(a)*** infinitely many of its elements are smaller than some constant C;
- ***(b)*** it has only finitely many 0's.

Let us show that the condition ***(b)*** can be replaced by the following one:

- ***(c)*** $\mathcal{M}_n$ comes into state 0 only finitely many times on our sequence of weights.

For the ***(b)*** $\Longrightarrow$ ***(c)*** direction observe that $\mathcal{M}_n$ can be in state 0 only if indeed the current sum of the weights is 0. Hence if the sum was 0 only finitely many times, then $\mathcal{M}_n$ was in state 0 only finitely many times as well. For the other direction, however, a more subtle argument is need. This is because the sum of weights can be 0 while $\mathcal{M}_n$ is in state $\perp$ (this may happen if previously the sum exceeded n in the absolute value). So ***(c)*** $\Longrightarrow$ ***(b)*** may be false only if the sum was 0 infinitely many times after $\mathcal{M}_n$ came into state $\perp$. However, due to our assumptions about $\mathcal{A}$ the sum *never* equals 0 once $\mathcal{M}_n$ came into $\perp$. Namely, recall that in $\mathcal{A}$ all cycles are non-negative. Hence (by the same argument as in Case 1) there is no path of weight smaller than $-n$. So $\mathcal{M}_n$ can come into $\perp$ only if the sum exceeded n. But if this happened, the sum will never be 0 again (for that we would need a path of weight smaller than $-n$).

So Min wins if and only if both the conditions ***(a)*** and ***(c)*** are satisfied. In terms of the arena $\mathcal{M}_n \times \mathcal{A}$ the condition ***(c)*** just means that the nodes of the form $(0, v)$ should be visited only finitely many times. Now, to finish the argument it is sufficient to show that in $\mathcal{M}_n \times \mathcal{A}$ some positional strategy τ of Min wins wherever Min has a winning strategy. Indeed, then by turning τ into corresponding $\mathcal{M}_n$-strategy T in $\mathcal{A}$ (as in Case 1) we obtain a uniform $\mathcal{M}_n$-strategy equilibrium.

Call a cycle of $\mathcal{M}_n \times \mathcal{A}$ *good* if its weight is 0 and it contains no node of the form $(0, v)$. Call all the other cycles of $\mathcal{M}_n \times \mathcal{A}$ *bad*.

First, observe that if only bad cycles are reachable from a node of $\mathcal{M}_n \times \mathcal{A}$, then Min looses in this node. Indeed, consider any infinite path h which starts at such a node. There exists a simple cycle C such that h passes through it infinitely many times. This cycle must be bad, so it either contains a node of the form $(0, v)$ or its weight is strictly positive. If the first option holds, then Max wins on h because of visiting a node of the form $(0, v)$ infinitely many times. If the second option holds, then Max wins on h because the sum of the weights goes to $+\infty$. Indeed, any prefix of h which passes through C at least k times has weight at least $k - n$ (each pass through C contributes at least 1, and the rest of the prefix is a path whose weight is at least $-n$). Since k goes to $+\infty$, so does the sum of the weights.

Second, we claim that Min wins in all nodes from where one can reach a good cycle; moreover, there is a single positional strategy τ of Min which wins in all these nodes. Denote the set of these nodes of $\mathcal{M}_n \times \mathcal{A}$ by G. Since all cycles are non-negative, any good cycle contains a simple good subcycle. Hence in G there exists a set S of disjoint good simple cycles such that any node of G has a path to a cycle from S. Consider the following strategy τ. If a node belongs to one of the cycles from S, then move along this cycle. Otherwise, move by the shortest path to a cycle from S. Clearly, from any node of G the strategy τ first reaches a cycle from S and then stays on it forever. Since all cycles of S of are zero, this means that the sum of the weights is bounded from above; moreover, these cycles do not contain nodes of the form $(0, v)$, so we will see these nodes only finitely many times.

5.2 Example with no finite-state equilibrium

This part of the argument can also be found in [2, Section 3.4]. We reproduce it for completeness.

We claim that in the arena from Figure 2 there exists a node from where Max and Min have no equilibrium of two finite-state strategies.

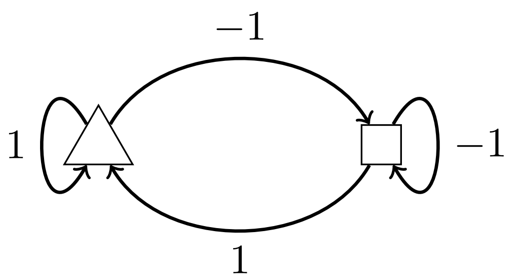

Figure 2 The square is owned by Max and the triangle is owned by Min.

More specifically, we show that if the game starts in the square, then Max has a winning strategy but no finite-state one. Namely, the following strategy of Max is winning: if the current sum of the weights is positive, use a -1 loop, otherwise go to the triangle. This strategy guarantees that whenever we reach the square, the sum of the weights will become 0 once more. So provided that the square is visited infinitely many times, the sum of the weights equals 0 infinitely often. Now, it might be that from some moment Min stays in the triangle forever, but in this case the sum of the weights goes to $+\infty$, and hence Max also wins.

To show that Max has no finite-state winning strategy from the square, for every s we define a strategy of Min which wins against any finite-state strategy of Max with at most s states. Namely, this strategy of Min is as follows: if the current sum of the weights is smaller than $s + 2$, then use a 1 loop, otherwise go to the square.

Obviously, this strategy guaranties that the sum of the weights never exceeds $s+2$. Hence Max can win only by making the sum of weights equal to 0 infinitely many times. However, for that there must be infinitely many periods in which Max stays in the square for at least $s + 1$ moves. But if Max stays in the square for $s + 1$ moves, then it stays there forever after (during these $s + 1$ moves he was in the same state twice). This means that the sum of the weights goes to $-\infty$ and that Min wins.

6 Proof of Proposition 5

Let the set of colors be $C = \{0, 1\}$. Fix a set $T \subseteq \mathbb{Z}^+$. Define a payoff function $\varphi\colon \{0, 1\}^\omega \to \{0, 1\}$ by setting $\varphi(\alpha) = 1$ for $\alpha = \alpha_1\alpha_2\alpha_3 \ldots \in \{0, 1\}^\omega$ if and only if at least one of the

following two conditions holds:
- α contains only finitely many 0's;
- for some $t \in T$ a word

  $$0\underbrace{11\ldots1}_{t\text{ times}}0$$

  is a subword of α (here we call $x \in \{0,1\}^*$ a subword of α if $x = \alpha_n\alpha_{n+1}\ldots\alpha_{n+|x|-1}$ for some $n \in \mathbb{Z}^+$).

  Call T *sparse* if there are infinitely many $k \in T$ such that $l \notin T$ for all $k < l < k^4$.

▶ **Lemma 16.** *If $T \subseteq \mathbb{Z}^+$ is sparse, then $s^1_\varphi(n) \neq +\infty$ for all $n \in \mathbb{Z}^+$ and $\inf_{n\in\mathbb{Z}^+} \frac{s^1_\varphi(n)}{n} = 0$.*

Call T *isolated* if for all $m \in \mathbb{Z}^+$ there exists $k \in T, k > m$ such that $l \notin T$ for all $k - m < l < k + m$, $l \neq k$.

▶ **Lemma 17.** *If T is isolated, then $\lim\limits_{n\to\infty} s^1_\varphi(n) = +\infty$.*

Assuming Lemmas 16 and 17 are proved, it remains to construct a sparse isolated set. For instance, one can take $T = \{2^{4^n} \mid n \in \mathbb{Z}^+\}$.

6.1 Proof of lemma 16

Call $k \in T$ *good* if $l \notin T$ for all $k < l < k^4$. Since T is sparse, there are infinitely many good k in it. To prove the lemma it is sufficient to show that for every good k there exists a memory skeleton $\mathcal{M}_k$ with $k + 4$ states such that all one-player arenas with at most k^2 nodes have a uniform $\mathcal{M}_k$-strategy equilibrium. In fact, we will show this for all arenas with at most k^2 nodes, not only for the one-player ones.

Let us define $\mathcal{M}_k$. States of $\mathcal{M}_k$ will be denoted as follows:

$$I, F, q_0, q_1, \ldots, q_k, q_{>k}.$$

State I is the initial one. Our memory skeleton stays in it until it sees the first 0. Once it happened, $\mathcal{M}_k$ starts memorizing the number of 1's after the last 0 read so far, until this number exceeds k. So once we see the first 0, we come into q_0 (there were no 1's after this 0 yet). Next, if we see 1 in a state q_i for $0 \leq i < k$, then we come into q_{i+1}. In turn, if we see 1 in q_k, we come into $q_{>k}$ and stay in it as long as we see only 1's.

When a new 0 appears, this interrupts the previous sequence of consecutive 1's. Correspondingly, when $\mathcal{M}_k$ sees 0, in most of the cases it comes into q_0. However, in some cases it comes into state F in which it then stays forever. More specifically, if $\mathcal{M}_k$ sees 0 in a state q_i for $i \leq k, i \notin T$, or in $q_{>k}$, then it comes into q_0. In turn, if $\mathcal{M}_k$ sees 0 in a state q_i for $i \leq k, i \in T$, then it comes into F.

Take any arena $\mathcal{A} = \langle V, V_{\text{Max}}, V_{\text{Min}}, E, \mathsf{source}, \mathsf{target}, \mathsf{col}\rangle$ with at most k^2 nodes. Our goal is to show an existence of a uniform $\mathcal{M}_k$-strategy equilibrium in $\mathcal{A}$ (with respect to φ). By Observation 10 it is sufficient to establish an $\{I\} \times V$-wise positional equilibrium in a product arena $\mathcal{M}_k \times \mathcal{A}$ (again, with respect to φ).

Define an auxiliary payoff $\psi\colon \{0,1\}^\omega \to \{0,1\}$ by setting $\psi(\alpha) = 1$ for $\alpha \in \{0,1\}^\omega$ if and only if either α contains only finitely many 0's or a word

$$0\underbrace{11\ldots1}_{t\text{ times}}0$$

is a subword of α for some $t \in \{1, 2, \ldots, k\} \cap T$. Our argument consists of proving the following two claims:

- *Claim 1.* If (σ, τ) is an $\{I\} \times V$-wise positional equilibrium with respect to ψ, then (σ, τ) is also an $\{I\} \times V$-wise positional equilibrium with respect to φ. Here σ is a positional strategy of Max in $\mathcal{M}_k \times \mathcal{A}$ and τ is a positional strategy of Min in $\mathcal{M}_k \times \mathcal{A}$.
- *Claim 2.* There exists an $\{I\} \times V$-wise positional equilibrium in $\mathcal{M}_k \times \mathcal{A}$ with respect to ψ.

Proving Claim 1. It is sufficient to show that as long as σ (correspondingly, τ) is winning in a node $(I, w), w \in V$ with respect to ψ, then σ (correspondingly, τ) is winning in this node with respect to φ. For σ this is immediate because $\psi(\alpha) = 1 \implies \varphi(\alpha) = 1$ for every $\alpha \in \{0,1\}^\omega$. Now, let (I, w) be a node for which τ is winning with respect to ψ. Assume for contradiction that there exists an infinite path with the source in (I, w) which is consistent with τ and which is winning for Max with respect to φ. Since this path is loosing for Max with respect to ψ, the corresponding infinite sequence of colors must have a subword of the form

$$0\underbrace{11\ldots1}_{l \text{ times}}0$$

for some $l \in T \setminus \{1, 2, \ldots, k\}$. Since k is good, we must have $l \geq k^4$. This means that in $(\mathcal{M}_k \times \mathcal{A})_\tau$ there is a path which starts in (I, w) and contains k^4 consecutive edges colored by 1. Since in $\mathcal{M}_k \times \mathcal{A}$ there are at most $(k+4) \cdot k^2 < k^4$ nodes, this means that in $(\mathcal{M}_k \times \mathcal{A})_\tau$ one can reach from (I, w) a cycle colored only by 1's. Therefore there is a strategy of Max such that in its play again τ there are only finitely many edges colored by 0. Hence τ could not be winning in (I, w) with respect to ψ, contradiction.

Proving Claim 2. We will show that there is a parity game with 3 priorities on $\mathcal{M}_k \times \mathcal{A}$ such that for every $w \in V$ Max wins in this parity game from (I, w) if and only if he wins from (I, w) with respect to ψ. Once this claim is proved it remains to refer to the positional determinacy of parity games.

It is easy to see that $\mathcal{M}_k$ is in state F if and only if the current sequence of colors has a subword

$$0\underbrace{11\ldots1}_{i}0$$

for some $i \in T \cap \{1, 2, \ldots, k\}$. So Max wins with respect to ψ in the following two cases: ***(a)*** $\mathcal{M}_k$ ever comes into F ***(b)*** all but finitely many edges of a play are colored by 1. In terms of the arena $\mathcal{M}_k \times \mathcal{A}$ condition ***(a)*** means that a node of the form (F, v) is visited at least once. To put it differently, ***(a)*** means that some *edge* which starts in a node of the form (F, v) is passed at least once. Let us denote the set of these edges by E_2. Partition all the other edges of $\mathcal{M}_k \times \mathcal{A}$ into two sets E_0 and E_1 according to their color. So Max wins if either a play contains an edge from E_2 or it contains only finitely many edges of E_0. In fact, instead of requiring to have at least one edge from E_2 we may require to have infinitely many such edges (because once $\mathcal{M}_k$ came into F, it stays in it forever). This is equivalent to a parity game with 3 priorities in which edges from E_2 are labeled by the largest priority, edges from E_0 are labeled by the middle one and edges from E_1 are labeled by the smallest one.

6.2 Proof of lemma 17

For every m we construct a one-player arena $\mathcal{A}_m$ for which there exists no memory skeleton $\mathcal{M}$ with less than m states such that $\mathcal{A}_m$ has a uniform $\mathcal{M}$-strategy equilibrium. Clearly, this implies that $\lim\limits_{n \to \infty} s^1_\varphi(n) = +\infty$.

By definition of isolation, there exists $k \in T, k > m$ such that $l \notin T$ for all $k - m < l < k + m$, $l \neq k$. Let $\mathcal{A}_m$ be as on Figure 3. All its nodes are owned by Max. On the left it has m nodes. The ith one (from the top) has a single simple path to the central node; the colors along this simple path form a word 01^i (a zero followed by i ones). On the right $\mathcal{A}_m$ also has m nodes. For each $i \in \{1, 2, \ldots, m\}$ there is a single simple path from the central node to the ith node (from the top) on the right; colors along this path form a word 1^{k-i}. Finally, all the nodes on the right have a 0 loop.

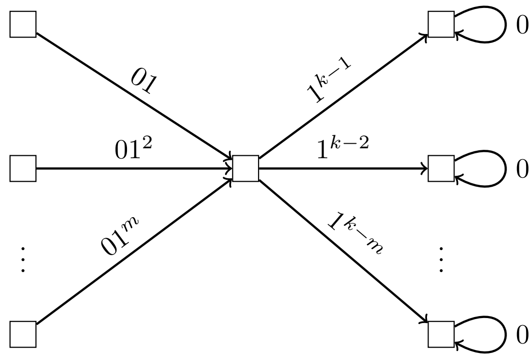

Figure 3 Arena $\mathcal{A}_m$.

Assume for contradiction that there exists a memory skeleton $\mathcal{M}$ with less than m states such that $\mathcal{A}_m$ has a uniform $\mathcal{M}$-strategy equilibrium.This means that some $\mathcal{M}$-strategy Σ of Max is winning for him in all the nodes where Max has a winning strategy. Observe that all the nodes on the left are winning for Max. Indeed, from the ith one Max should go (through the central node) to the ith node on the right. The resulting infinite sequence of colors will be 01^k0^ω, and this is winning for Max since $k \in T$. Note also that this is the only infinite path which is winning for Max from the ith node on the left. Indeed, any other path is colored by $01^{i+k-j}0^\omega$ for some $j \in \{1, 2, \ldots, m\}, j \neq i$. This is loosing for Max because $i + k - j \neq k$, $k - m < i + k - j < k + m$, and hence $i + k - j \notin T$.

Our $\mathcal{M}$-strategy Σ must be winning for all the nodes on the left. So if the game starts in the ith node on the left, then in the central node Σ must go to the ith node on the right. However, as there are less than m states in $\mathcal{M}$, there must be two distinct nodes on the left from which $\mathcal{M}$ comes into the same state upon reaching the central node. So $\mathcal{M}$ must make the same move from the central node no matter in which of these two nodes on the left the game started. This means that Σ will be loosing for at least one of these two nodes.

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