Continuous Positional Payoffs

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Abstract

What payoffs are positionally determined for deterministic two-player antagonistic games on finite directed graphs? In this paper we study this question for payoffs that are continuous. The main reason why continuous positionally determined payoffs are interesting is that they include the multi-discounted payoffs.

We show that for continuous payoffs positional determinacy is equivalent to a simple property called prefix-monotonicity. We provide three proofs of it, using three major techniques of establishing positional determinacy – inductive technique, fixed point technique and strategy improvement technique. A combination of these approaches provides us with better understanding of the structure of continuous positionally determined payoffs as well as with some algorithmic results.

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1 Introduction

We study games of the following kind. A game takes place on a finite directed graph. There is a token, initially located in one of the nodes. Before each turn there is exactly one node containing the token. In each turn one of the two antagonistic players called Max and Min chooses an edge starting in a node containing the token. As a result the token moves to the endpoint of this edge, and then the next turn starts. To determine who makes a move in a turn we are given in advance a partition of the nodes into two sets. If the token is in a node from the first set, then Max makes a move, otherwise Min.

Players make infinitely many moves, and this yields an infinite trajectory of the token. Technically, we assume that each node of the graph has at least one out-going edge so that there is always at least one available move. To introduce competitiveness, we should somehow compare the trajectories of the token with each other. For that we first fix some finite set A and label the edges of the game graph by elements of A. We also fix a *payoff* φ which is a function from the set of infinite sequences of elements of A to $\mathbb{R}$. Each possible infinite trajectory of the token is then mapped to a real number called *the reward* of this trajectory as follows: we form an infinite sequence of elements of A by taking the labels of edges along the trajectory, and apply φ to this sequence. The larger the reward is the more Max is happy; on the contrary, Min wants to minimize the reward.

For both of the players we are interested in indicating *an optimal strategy*, i.e., an optimal instruction of how to play in all possible developments of the games. To point out among all the strategies the optimal ones we first introduce a notion of a *guarantee of a strategy*. The guarantee of a Max's strategy σ is the infimum of the payoff over all infinite trajectories, consistent with the strategy. The reward of a play against σ cannot be smaller than its guarantee, but can be arbitrarily close to it. Now, a strategy of Max is called optimal if its guarantee is maximal over all Max's strategies. Similarly, the guarantee of a Min's strategy is the supremum of the payoff over all infinite trajectories, consistent with this Min's strategy. Min's strategies minimizing the guarantee are called optimal.

Observe that the guarantee of any Min's strategy is at least as large as the guarantee of any Max's strategy. A pair (σ, τ) of a Max's strategy σ and a Min's strategy τ is called an *equilibrium* if the guarantee of σ *equals* the guarantee of τ. Both strategies appearing in an equilibrium must be optimal – one proves the optimality of the other. In this paper we only study the so-called *determined* payoffs – payoffs for which all games on finite directed graphs with this payoff have an equilibrium.

For general determined payoffs an optimal strategy might be rather complicated (since the game is infinite, it might even have no finite description). For what determined payoffs both players always have a "simple" optimal strategy? A word "simple" can be understood in different ways [2], and this leads to different classes of determined payoffs. Among these classes we study one for which "simple" is understood in, perhaps, the strongest sense possible. Namely, we study a class of *positionally determined* payoffs.

For a positionally determined payoff all game graphs must have a pair of *positional* strategies which is an equilibrium no matter in which node the game starts. Now, a positional strategy is a strategy which totally ignores the previous trajectory of the token[1] and only looks at its current location. Formally, a positional strategy of Max maps each Max's node to an edge which starts in this node (i.e., to a single edge which Max will use whenever this node contains the token). Min's positional strategies are defined similarly.

A lot of works are devoted to concrete positionally determined payoffs that are of particular interest in other areas of computer science. Classical examples of such payoffs are parity payoffs, mean payoffs and (multi-)discounted payoffs [5, 20, 19, 22]. Their applications range from logic, verification and finite automata theory [6, 12] to decision-making [21, 23] and algorithm design [3].

Along with this specialized research, in [9, 10] Gimbert and Zielonka undertook a thorough study of positionally determined payoffs in general. In [9] they showed that all the so-called *fairly mixing* payoffs are positionally determined. They also demonstrated that virtually all classical positionally determined payoffs are fairly mixing. Next, in [10] they established a property of payoffs which is *equivalent* to positional determinacy. Despite being rather technical, this property has a remarkable feature: if a payoff does not satisfy it, then this payoff violates positional determinacy in some *one-player* game graph (where one of the players owns all the nodes). As Gimbert and Zielonka indicate, this means that to establish positional determinacy of a payoff it is enough to do so only for one-player game graphs.

One could try to gain more understanding about positionally determined payoffs that satisfy certain additional requirements. Of course, this is interesting only if there are practically important positionally determined payoffs that satisfy these requirements. One such requirement studied in the literature is called *prefix-independence* [4, 8]. A payoff is prefix-independent if it is invariant under throwing away any finite prefix from an infinite sequence of edge labels. For instance, the parity and the mean payoffs are prefix-independent.

In [9] Gimbert and Zielonka briefly mention another interesting additional requirement, namely, *continuity.* They observe that the multi-discounted payoffs are continuous (they utilize this in showing that the multi-discounted payoffs are fairly mixing). In this paper we study continuous positionally determined payoffs in more detail. Continuity of a payoff, loosely speaking, means that its range converges to just a single point as more and more initial characters of an infinite sequence of edge labels are getting fixed. This contrasts with prefix-independent payoffs (such as the parity and the mean payoffs), for which any initial finite segment is irrelevant. Thus, continuity serves as a natural property which separates

[1] In particular, a node in which the game has started.

the multi-discounted payoffs from the other classical positionally determined payoffs. This is our main motivation to study continuous positionally determined payoffs in general, besides the general importance of the notion of continuity.

We show that for continuous payoff positional determinacy is equivalent to a simple property which we call *prefix-monotonicity.* Loosely speaking, prefix-monotonicity means the result of a comparison of the payoff on two infinite sequences of labels does not change after appending or deleting the same finite prefix. In fact, we prove this result in three different ways, using three major techniques of establishing positional determinacy:

- *An inductive argument.* Here we use a sufficient condition of Gimbert and Zielonka [9], which is proved by induction on the number of edges of a game graph. This type of argument goes back to a paper of Ehrenfeucht and Mycielski [5], where they provide an inductive proof of the positional determinacy of the Mean Payoff Games.
- *A fixed point argument.* Then we give a proof which uses a fixed point approach due to Shapley [22]. Shapley's technique is a standard way of establishing positional determinacy of Discounted Games. In this argument one derives positional determinacy from the existence of a solution to a certain system of equations (sometimes called *Bellman's equations*). In turn, to establish the existence of a solution one uses Banach's fixed point theorem.
- *A strategy improvement argument.* For Discounted Games the existence of a solution to Bellman's equations can also be proved by *strategy improvement.* This technique goes back to Howard [16]; for its thorough treatment (as well as for its applications to other payoffs) we refer the reader to [7]. We generalize it to arbitrary continuous positionally determined payoffs.

The simplest way to obtain our main result is via the inductive argument (at the cost of appealing without a proof to the results of Gimbert and Zielonka). We provide two other proofs for the following reasons.

First, they have applications (and it is unclear how to get these applications within the framework of the inductive approach). The fixed point approach provides a precise understanding of what do continuous positionally determined payoffs look like in general. We use this to answer a question of Gimbert [8] regarding positional determinacy in more general *stochastic* games. In turn, the strategy improvement approach has algorithmic consequences. More specifically, we show that a problem of finding a pair of optimal positional strategies is solvable in randomized subexponential time for any continuous positionally determined payoff.

Second, as far as we know, these two approaches were never used in such an abstract setting before. Thus, we believe that our paper makes a useful addition to these approaches from a technical viewpoint. For example, the main problem for the fixed point approach is to identify a metric with which one can carry out the same "contracting argument" as in the case of multi-discounted payoffs. To solve it, we obtain a result of independent interest about compositions of continuous functions. As for the strategy improvement approach, our main contribution is a generalization of such well-established tools as "modified costs" and "potential transformation lemma" [15, Lemma 3.6].

Organization of the paper. In Section 2 we formalize the concepts discussed in the introduction. Then in Sections 3–6 we expose our results in more detail. In Section 7 we indicate some possible future directions. All proofs are deferred to Appendix.

2 Preliminaries

We denote the function composition by $\circ$.

Sets and sequences. For two sets A and B by A^B we denote the set of all functions from B to A (sometime we will interpret A^B as the set of vectors consisting of elements of A and with coordinates indexed by elements of B). We write $C = A \sqcup B$ for three sets A, B, C if A and B are disjoint and $C = A \cup B$.

For a set A by A^* we denote the set of all finite sequences of elements of A and by A^ω we denote the set of all infinite sequences of elements of A. For $w \in A^*$ we let $|w|$ be the length of w. For $\alpha \in A^\omega$ we let $|\alpha| = \infty$.

For $u \in A^*$ and $v \in A^* \cup A^\omega$ we let uv denote the concatenation of u and v. We call $u \in A^*$ a prefix of $v \in A^* \cup A^\omega$ if for some $w \in A^* \cup A^\omega$ we have $u = vw$. For $w \in A^*$ by wA^ω we denote the set $\{w\alpha \mid \alpha \in A^\omega\}$. Alternatively, wA^ω is the set of all $\beta \in A^\omega$ such that w is a prefix of β.

For $u \in A^*$ and $k \in \mathbb{N}$ we use a notation

$$u^k = \underbrace{uu \ldots u}_{k \text{ times}}.$$

In turn, we let $u^\omega \in A^\omega$ be a unique element of A^ω such that u^k is a prefix of u^ω for every $k \in \mathbb{N}$. We call $\alpha \in A^\omega$ ultimately periodic if α is a concatenation of u and v^ω for some $u, v \in A^*$.

Graphs notation. By a finite directed graph G we mean a pair $G = (V, E)$ of two finite sets V and E equipped with two functions $\mathsf{source}, \mathsf{target} \colon E \to V$. Elements of V are called nodes of G and elements of E are called edges of G.

The out-degree of a node $a \in V$ is $|\{e \in E \mid \mathsf{source}(e) = a\}|$. A node $a \in V$ is called a sink if its out-degree is 0. We call a graph G sinkless if there are no sinks in G.

A path in G is a non-empty (finite or infinite) sequence of edges of G with a property that $\mathsf{target}(e) = \mathsf{source}(e')$ for any two subsequent edges e and e' from the sequence. For a path p we define $\mathsf{source}(p) = \mathsf{source}(e)$, where e is the first edge of p. For a finite path p we define $\mathsf{target}(p) = \mathsf{target}(e')$, where e' is the last edge of p.

For technical convenience we also consider 0-length paths. Each 0-length path is associated with some node of G (so that there are $|V|$ different 0-length paths). For a 0-length path p, associated with $a \in V$, we define $\mathsf{source}(p) = \mathsf{target}(p) = a$.

When we write pq for two paths p and q we mean the concatenation of p and q (viewed as sequences of edges). Of course, this is well-defined only if p is finite. Note that pq is not necessarily a path. Namely, pq is a path if and only if $\mathsf{target}(p) = \mathsf{source}(q)$.

2.1 Deterministic infinite duration games on finite directed graphs

Mechanics of the game. A *game graph* is a sinkless finite directed graph $G = (V, E)$, equipped with two sets V_{Max} and V_{Min} such that $V = V_{\text{Max}} \sqcup V_{\text{Min}}$.

A game graph $G = (V = V_{\text{Max}} \sqcup V_{\text{Min}}, E)$ induces a so-called *infinite duration game* (IDG for short) on G. The game is always between two players called Max and Min. Positions of the game are finite paths in G (informally, these are possible finite trajectories of the token). We call a finite path p a Max's (a Min's) position if $\mathsf{target}(p) \in V_{\text{Max}}$ (if $\mathsf{target}(p) \in V_{\text{Min}}$). Max makes moves in Max's positions and Min makes moves in Min's positions.

The set of moves available at a position p is the set $\{e \in E \mid \mathsf{source}(e) = \mathsf{target}(p)\}$. A move e from a position p leads to a position pe.

A Max's strategy σ in a game graph G is a mapping assigning to every Max's position p a move available at p. Similarly, a Min's strategy τ in a game graph G is a mapping assigning to every Min's position p a move available at p.

Let $\mathcal{P} = e_1e_2e_3\dots$ be an infinite path in G. We say that $\mathcal{P}$ is *consistent* with a Max's strategy σ if the following conditions hold:

- if $s = \mathsf{source}(\mathcal{P}) \in V_{\mathrm{Max}}$, then $\sigma(s) = e_1$;
- for every $i \geq 1$ it holds that $\mathsf{target}(e_1e_2\dots e_i) \in V_{\mathrm{Max}} \implies e_{i+1} = \sigma(e_1e_2\dots e_i)$.

For $a \in V$ and for a Max's strategy σ we let $\mathsf{Cons}(a, \sigma)$ be a set of all infinite paths in G that start in a and are consistent with σ. We use similar terminology and notation for strategies of Min.

Given a Max's strategy σ, a Min's strategy τ and $a \in V$, we let *the play of σ and τ from a* be a unique element of the intersection $\mathsf{Cons}(a, \sigma) \cap \mathsf{Cons}(a, \tau)$. The play of σ and τ from a is denoted by $\mathcal{P}_a^{\sigma,\tau}$.

Positional strategies. A Max's strategy σ in a game graph $G = (V = V_{\mathrm{Max}} \sqcup V_{\mathrm{Min}}, E)$ is called *positional* if $\sigma(p) = \sigma(q)$ for all finite paths p and q in G with $\mathsf{target}(p) = \mathsf{target}(q) \in V_{\mathrm{Max}}$. Clearly, a Max's positional strategy σ can be represented as a mapping $\sigma \colon V_{\mathrm{Max}} \to E$ satisfying $\mathsf{source}(\sigma(u)) = u$ for all $u \in V_{\mathrm{Max}}$. We define Min's positional strategies analogously.

We call an edge $e \in E$ *consistent* with a Max's positional strategy σ if either $\mathsf{source}(e) \in V_{\mathrm{Min}}$ or $\mathsf{source}(e) \in V_{\mathrm{Max}}, e = \sigma(\mathsf{source}(e))$. We denote the set of edges that are consistent with σ by E^σ. If τ is a Min's positional strategy, then we say that an edge $e \in E$ is consistent with τ if either $\mathsf{source}(e) \in V_{\mathrm{Max}}$ or $\mathsf{source}(e) \in V_{\mathrm{Min}}, e = \tau(\mathsf{source}(e))$. The set of edges that are consistent with a Min's positional strategy τ is denoted by E_τ.

Labels and payoffs. Let A be a finite set. A game graph $G = (V = V_{\mathrm{Max}} \sqcup V_{\mathrm{Min}}, E)$ equipped with a function $\mathsf{lab} \colon E \to A$ is called an *A-labeled game graph.* If $p = e_1e_2e_3\dots$ is a (finite or infinite) path in an A-labeled game graph $G = (V = V_{\mathrm{Max}} \sqcup V_{\mathrm{Min}}, E), \mathsf{lab} \colon E \to A$, we define $\mathsf{lab}(p) = \mathsf{lab}(e_1)\mathsf{lab}(e_2)\mathsf{lab}(e_3)\dots \in A^* \cup A^\omega$. A *payoff* is a bounded function from A^ω to $\mathbb{R}$. Some papers allow A to be infinite and consider only infinite sequences that contain finitely many elements of A (as any game graph contains only finitely many labels). So basically we just have to deal with finite subsets of A, and this can be done with our approach.

Guarantees, optimal strategies and equilibria. Let A be a finite set, $\varphi \colon A^\omega \to \mathbb{R}$ be a payoff and $G = (V = V_{\mathrm{Max}} \sqcup V_{\mathrm{Min}}, E), \mathsf{lab} \colon E \to A$ be an A-labeled game graph. Take a Max's strategy σ in G. *The guarantee* of σ in a node $a \in V$ is the following quantity:

$$Guarantee[\sigma](a) = \inf \varphi \circ \mathsf{lab}\big(\mathsf{Cons}(a, \sigma)\big).$$

Similarly, if τ is a Min's strategy in G, then the guarantee of τ in a node $a \in V$ is the following quantity:

$$Guarantee[\tau](a) = \sup \varphi \circ \mathsf{lab}\big(\mathsf{Cons}(a, \tau)\big).$$

Observe that for any Max's strategy σ, for any Min's strategy τ and for any $a \in V$ we have:

$$Guarantee[\sigma](a) \leq \varphi \circ \mathsf{lab}\big(\mathcal{P}_a^{\sigma,\tau}\big) \leq Guarantee[\tau](a).$$

A Max's strategy σ is called *optimal* if $Guarantee[\sigma](a) \geq Guarantee[\sigma'](a)$ for any $a \in V$ and for any Max's strategy σ'. Similarly, A Min's strategy τ is called *optimal* if $Guarantee[\tau](a) \leq Guarantee[\tau'](a)$ for any $a \in V$ and for any Min's strategy τ'.

A pair (σ, τ) of a Max's strategy σ and a Min's strategy τ is called an *equilibrium* if $Guarantee[\sigma](a) = Guarantee[\tau](a)$ for every $a \in V$. It is easy to see that any strategy appearing in an equilibrium is optimal. On the other hand, if at least one equilibrium exists, then the following holds: Cartesian product of the set of the optimal strategies of Max and the set of the optimal strategies of Min is exactly the set of equilibria. We say that φ is *determined* if in every A-labeled game graph there exists an equilibrium (with respect to φ).

Positionally determined payoffs. Let A be a finite set and $\varphi\colon A^\omega \to \mathbb{R}$ be a payoff. We call φ *positionally determined* if all A-labeled game graphs have (with respect to φ) an equilibrium consisting of two positional strategies.

▶ **Proposition 1.** *If A is a finite set, $\varphi\colon A^\omega \to \mathbb{R}$ is a positionally determined payoff and $g\colon \varphi(A^\omega) \to \mathbb{R}$ is a non-decreasing[2] function, then $g \circ \varphi$ is a positionally determined payoff.*

Proof. See Appendix A. ◀

2.2 Continuous payoffs

For a finite set A, we consider the set A^ω as a topological space. Namely, we take the discrete topology on A and the corresponding product topology on A^ω. In this product topology open sets are sets of the form

$$\mathcal{S} = \bigcup_{u \in S} uA^\omega,$$

where $S \subseteq A^*$. When we say that a payoff $\varphi\colon A^\omega \to \mathbb{R}$ is *continuous* we always mean continuity with respect to this product topology (and with respect to the standard topology on $\mathbb{R}$). The following proposition gives a convenient way to establish continuity of payoffs.

▶ **Proposition 2.** *Let A be a finite set. A payoff $\varphi\colon A^\omega \to \mathbb{R}$ is continuous if and only if for any $\alpha \in A^\omega$ and for any infinite sequence $\{\beta_n\}_{n=1}^\infty$ of elements of A^ω the following holds. If for all $n \geq 1$ the sequences α and β_n coincide in the first n elements, then $\lim\limits_{n\to\infty} \varphi(\beta_n)$ exists and equals $\varphi(\alpha)$.*

Proof. See Appendix B. ◀

For a finite A by Tychonoff's theorem the space A^ω is compact (because any finite set A with the discrete topology is compact). This has the following consequence which is important for this paper: if $\varphi\colon A^\omega \to \mathbb{R}$ is a continuous payoff, then $\varphi(A^\omega)$ is a compact subset of $\mathbb{R}$.

2.3 MDPs

This subsection concerns stochastic games, but we deal with them only in Theorem 26. So for the rest of our results one can skip this subsection.

In fact, we will need only one-player stochastic games, also known as Markov Decision Processes (MDPs). For definiteness, we assume player Max. Let us start with an informal discussion of MDPs.

[2] Throughout the paper we call a function $f\colon S \to \mathbb{R}, S \subseteq \mathbb{R}$ non-decreasing if for all $x, y \in S$ with $x \leq y$ we have $f(x) \leq f(y)$.

As before there is some finite set of possible locations of the token (in MDPs they are usually called *states* rather than nodes). Each state is equipped with a set of *actions* available in this state. An action is just a probability distribution over the set of nodes. In each turn Max chooses an action available at the state with the token. The next location of the token is then sampled according to the corresponding distribution. In a special case when all actions have support of size 1 we get deterministic one-player games.

Fixing a Max's strategy induces a probability distribution over the infinite trajectories of the token. An optimal strategy is one which maximizes the *expected reward* of the trajectory (with respect to its induced distribution). To compute the cost of a trajectory we first fix a *labeling* of our MDP so that each trajectory gets an infinite sequence of labels. As before, the reward is then the value of a payoff on this sequence of labels.

Similarly to deterministic games, a positional strategy is a strategy which never chooses two different actions in the same state (in this context positional strategies are called sometimes *pure stationary* strategies). A payoff for which in every MDP Max has an optimal positional strategy is called *positionally determined in MDPs*.

We now proceed to technical details (we follow a formalization of Gimbert [8]). Let A be a finite set. By $\mathfrak{S}_A^{bor}$ we mean σ-algebra of Borel subsets of A^ω (with respect to the product topology from the previous subsection). By $\Delta(S)$ we denote the set of all probability distributions over a finite set S.

▶ **Definition 3.** *An MDP is a pair $\mathcal{M} = (S, B)$ of two finite sets S and B equipped with two functions* $\mathsf{source}\colon B \to S$, $\mathrm{Dist}\colon B \to \Delta(S)$ *such that for any $s \in S$ there exists $b \in B$ with $s = \mathsf{source}(b)$. Elements of S are called states of $\mathcal{M}$ and elements of B are called actions of $\mathcal{M}$.*

Let $\mathcal{M} = (S, B)$ be an MDP. We call pairs $(b, s) \in B \times S$ *transitions* of $\mathcal{M}$. A non-epmty (finite of infinite) sequence of transitions $h = (b_1, s_1)(b_2, s_2)(b_3, s_3)\ldots$ is called *a history* if for every $1 \le i < |h|$ we have $s_i = \mathsf{source}(b_{i+1})$. For a history $h = (b_1, s_1)(b_2, s_2)(b_3, s_3)\ldots$ we define $\mathsf{source}(h) = \mathsf{source}(b_1)$. For finite h we define $\mathsf{target}(h) = s_{|h|}$.

For consistency, we also consider $|S|$ histories of length 0, each identified with some state. For a 0-length history h identified with $s \in S$ we write $\mathsf{source}(h) = \mathsf{target}(h) = s$.

▶ **Definition 4.** ***A strategy*** *in an MDP $\mathcal{M} = (S, B)$ is a mapping which to every finite history h assigns an action $b \in B$ such that* $\mathsf{target}(h) = \mathsf{source}(b)$.

Let $\mathcal{M} = (S, B)$ be an MDP, $s \in S$ be a state of $\mathcal{M}$ and σ be a strategy in $\mathcal{M}$. Denote by $T = B \times S$ the set of transitions of $\mathcal{M}$. Let us define a probability measure $\mathcal{P}_s^\sigma$ on $\mathfrak{S}_T^{bor}$. For that consider the following random process generating a history (a sequence of transitions) in $\mathcal{M}$:

Algorithm 1 Generating $\mathcal{P}_s^\sigma$.

```
initialization:
h is a history in M, h ← s;
b ∈ B;
u ∈ S;
while true do
    b ← σ(h);
    sample u ∈ S at random according to Dist(b) (and independently of the previous
      samples);
    append (b, u) to h;
end
```

We let $\mathcal{P}_s^\sigma$ be a probability measure on $\mathfrak{S}_T^{bor}$ such that for any $h \in T^*$ we have that $\mathcal{P}_s^\sigma(hT^\omega)$ equals the probability of the appearance of h in Algorithm 1. Existence of such $\mathcal{P}_s^\sigma$ needs some justification. A standard consequence of the Caratheodory's extension theorem is that for any $p\colon T^* \to [0,1]$ with

- $p(\text{empty string}) = 1$;
- $p(h) = \sum\limits_{t\in T} p(ht)$ for all $h \in T^*$

there exists a unique probability measure P on $\mathfrak{S}_T^{bor}$ with $P(hT^\omega) = p(h)$ for all $h \in T^*$. We apply this to p which to $h \in T^*$ assigns the probability of the appearance of h in Algorithm 1.

Let A be a finite set. An *A-labeled MDP* is an MDP $\mathcal{M} = (S, B)$ equipped with a function lab from the set T of transitions of $\mathcal{M}$ to A. Similarly to deterministic games we set $\mathsf{lab}(t_1t_2t_3\ldots) = \mathsf{lab}(t_1)\mathsf{lab}(t_2)\mathsf{lab}(t_3)\ldots$ for $t_1t_2t_3\ldots \in T^* \cup T^\omega$.

Let $\varphi\colon A^\omega \to \mathbb{R}$ be a bounded measurable payoff (with respect to $\mathfrak{S}_A^{bor}$). It is easy to see that a composition $\varphi \circ \mathsf{lab}\colon T^\omega \to \mathbb{R}$ is also measurable (now with respect to $\mathfrak{S}_T^{bor}$) and bounded. In particular, $\varphi \circ \mathsf{lab}$ is integrable with respect to a probability measure $\mathcal{P}_s^\sigma$, for any $s \in S$ and for any strategy σ in $\mathcal{M}$. Thus we can consider the following quantity:

$$\mathbb{E}_{x\sim\mathcal{P}_s^\sigma}\varphi \circ \mathsf{lab}(x).$$

For the sake of readability we will use the following notation for this quantity:

$$\mathbb{E}\varphi \circ \mathsf{lab}\big(\mathcal{P}_s^\sigma\big).$$

▶ **Definition 5.** *Let A be a finite set, $\varphi\colon A^\omega \to \mathbb{R}$ be a bounded measurable payoff and $\mathcal{M} = (S, B), \mathsf{lab}\colon B \times S \to A$ be an A-labeled MDP. Next, let σ be a strategy in $\mathcal{M}$. We say that σ is **optimal** if for every strategy σ' in $\mathcal{M}$ and for every $s \in S$ we have:*

$$\mathbb{E}\varphi \circ \mathsf{lab}\big(\mathcal{P}_s^\sigma\big) \geq \mathbb{E}\varphi \circ \mathsf{lab}\big(\mathcal{P}_s^{\sigma'}\big).$$

A strategy σ in an MDP $\mathcal{M} = (S, B)$ is called *positional* if $\sigma(h_1) = \sigma(h_2)$ for any two histories in $\mathcal{M}$ with $\mathsf{target}(h_1) = \mathsf{target}(h_2)$.

▶ **Definition 6.** *Let A be a finite set and $\varphi\colon A^\omega \to \mathbb{R}$ be a continuous payoff. We say that φ is **positionally determined in MDPs** if for every A-labeled MPD $\mathcal{M} = (S, B), \mathsf{lab}\colon B \times S \to A$ there exists an optimal positional strategy for φ.*

Clearly, any continuous payoff is bounded (because A^ω is compact) and measurable, so we can apply these definitions to continuous payoffs without any clauses.

3 Statement of the Main Result and Preliminary Discussion

Our main result establishes a simple property which is equivalent to positional determinacy for continuous payoffs.

▶ **Definition 7.** *Let A be a finite set. A payoff $\varphi\colon A^\omega \to \mathbb{R}$ is called **prefix-monotone** if there are no $u, v \in A^*$, $\beta, \gamma \in A^\omega$ such that $\varphi(u\beta) > \varphi(u\gamma)$ and $\varphi(v\beta) < \varphi(v\gamma)$.*

(One can note that prefix-independence trivially implies prefix-monotonicity. On the other hand, no prefix-independent payoff which takes at least 2 values is continuous.)

▶ **Theorem 8.** *Let A be a finite set and $\varphi\colon A^\omega \to \mathbb{R}$ be a continuous payoff. Then φ is positionally determined if and only if φ is prefix-monotone.*

The fact that any continuous positionally determined payoff must be prefix-monotone[3] is proved in Appendix C. Three different proofs of the "if" part of Theorem 8 are given in, respectively, Sections 4, 5 and 6. Before going into the proofs, let us discuss the notions of continuity and prefix-monotonicity by means of the multi-discounted payoffs.

▶ **Definition 9.** *A payoff $\varphi\colon A^\omega \to \mathbb{R}$ for a finite set A is* ***multi-discounted*** *if there are functions $\lambda\colon A \to [0,1)$ and $w\colon A \to \mathbb{R}$ such that*

$$\varphi(a_1a_2a_3\ldots) = \sum_{n=1}^{\infty} \lambda(a_1)\cdot\ldots\cdot\lambda(a_{n-1})\cdot w(a_n) \tag{1}$$

for all $a_1a_2a_3\ldots \in A^\omega$.

A few technical remarks: since the set A is finite, the coefficients $\lambda(a)$ are bounded away from 1 uniformly over $a \in A$. This ensures that the series (1) converges. In fact, this means that a tail of this series converges to 0 uniformly over $a_1a_2a_3\ldots \in A^\omega$. Thus, the multi-discounted payoffs are continuous. As the multi-discounted payoffs are positionally determined, by Theorem 8 they also must be prefix-monotone. Of course, prefix-monotonicity of the multi-discounted payoffs can be established without Theorem 8. Indeed, from (1) it is easy to derive that $\varphi(a\beta) - \varphi(a\gamma) = \lambda(a)\cdot(\varphi(\beta) - \varphi(\gamma))$ for all $a \in A, \beta, \gamma \in A^\omega$. Due to the condition $\lambda(a) \geq 0$, we have that $\varphi(a\beta) > \varphi(a\gamma)$ implies that $\varphi(\beta) > \varphi(\gamma)$. Moreover, the same holds if we append more than one character to β and γ. Hence it is impossible to simultaneously have $\varphi(u\beta) > \varphi(u\gamma)$ and $\varphi(v\beta) < \varphi(v\gamma)$ for $u, v \in A^*$, as required in the definition of prefix-monotonicity.

At this point one can ask whether there *exists* a payoff which is continuous and prefix-monotone but not multi-discounted (or, in other words, whether Theorem 8 is vacuous or not). The answer is that such payoffs exist in abundance, but constructing one is actually not that easy. Here we only give a high-level idea of the construction. A key insight is that (1) can be viewed not only as a series but also as a limit of compositions of linear functions. Namely, observe that (1) can be re-written as follows:

$$\varphi(a_1a_2a_3\ldots) = \lim_{n\to\infty} f_{a_1}\circ f_{a_2}\circ\ldots\circ f_{a_n}(0),$$

where $f_a = \lambda(a)x + w(a)$ for $a \in A$. One can obtain an example which we are looking for through the same expression with non-linear f_a. To ensure prefix-monotonicity, these functions have to be non-decreasing; to ensure convergence and continuity, these functions have to be "contracting". We return to this question in more detail in Subsection 5.1.

4 Inductive Argument

Here we show that any continuous prefix-monotone payoff is positionally determined using a sufficient condition of Gimbert and Zielonka [9, Theorem 1], which, in turn, is proved by an inductive argument. As Gimbert and Zielonka indicate [9, Lemma 2], their sufficient condition takes the following form for continuous payoffs.

[3] Here it is crucial that in our definition of positional determinacy we require that some positional strategy is optimal for all the nodes. Allowing each starting node to have its own optimal positional strategy gives us a weaker, "non-uniform" version of positional determinacy. It is not clear whether non-uniform positional determinacy implies prefix-monotonicity. At the same time, we are not even aware of a payoff which is positional only "non-uniformly".

▶ **Proposition 10.** *Let A be a finite set. Any continuous payoff $\varphi\colon A^\omega \to \mathbb{R}$, satisfying the following two conditions:*

- ***(a)*** *for all $u \in A^*$ and $\alpha, \beta \in A^\omega$ we have that $\varphi(\alpha) \le \varphi(\beta) \implies \varphi(u\alpha) \le \varphi(u\beta)$;*
- ***(b)*** *for all non-empty $u \in A^*$ and for all $\alpha \in A^\omega$ we have that*

$$\min\{\varphi(u^\omega), \varphi(\alpha)\} \le \varphi(u\alpha) \le \max\{\varphi(u^\omega), \varphi(\alpha)\};$$

is positionally determined.

We observe that one can get rid of the condition ***(b)*** in this Proposition.

▶ **Proposition 11.** *For continuous payoffs the condition* ***(a)*** *of Proposition 10 implies the condition* ***(b)*** *of Proposition 10.*

Proof. See Appendix D. ◀

So to establish positional determinacy of a continuous payoff it is enough to demonstrate that this payoff satisfies the condition ***(a)*** of Proposition 10. Let us now reformulate this condition using the following definition.

▶ **Definition 12.** *Let A be a finite set. A payoff $\varphi\colon A^\omega \to \mathbb{R}$ is called* ***shift-deterministic*** *if for all $a \in A, \beta, \gamma \in A^\omega$ we have $\varphi(\beta) = \varphi(\gamma) \implies \varphi(a\beta) = \varphi(a\gamma)$.*

▶ **Observation 13.** *Let A be a finite set. A payoff $\varphi\colon A^\omega \to \mathbb{R}$ satisfies the condition* ***(a)*** *of Proposition 10 if and only if φ is prefix-monotone and shift-deterministic.*

The above discussion gives the following sufficient condition for positional determinacy.

▶ **Proposition 14.** *Let A be a finite set. Any continuous prefix-monotone shift-deterministic payoff $\varphi\colon A^\omega \to \mathbb{R}$ is positionally determined.*

Still, some argument is needed for continuous prefix-monotone payoffs that are not shift-deterministic. To tie up loose ends we prove the following:

▶ **Proposition 15.** *Let A be a finite set and let $\varphi\colon A^\omega \to \mathbb{R}$ be a continuous prefix-monotone payoff. Then $\varphi = g \circ \psi$ for some continuous prefix-monotone shift-deterministic payoff $\psi\colon A^\omega \to \mathbb{R}$ and for some continuous[4] non-decreasing function $g\colon \psi(A^\omega) \to \mathbb{R}$.*

Proof. See Appendix E. ◀

Due to Proposition 1 this finishes our first proof of Theorem 8. In fact, we do not need continuity of g here, but it will be useful later.

5 Fixed point argument

Here we present a way of establishing positional determinacy of continuous prefix-monotone shift-deterministic payoffs (Proposition 14) via a fixed point argument. Together with Proposition 15 this constitutes our second proof of Theorem 8.

Obviously, for any shift-deterministic payoff $\varphi\colon A^\omega \to \mathbb{R}$ and for any $a \in A$ there is a unique function $\mathrm{sh}[a,\varphi]\colon \varphi(A^\omega) \to \varphi(A^\omega)$ such that $\mathrm{sh}[a,\varphi]\big(\varphi(\beta)\big) = \varphi(a\beta)$ for all $\beta \in A^\omega$.

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[4] Throughout the paper we call a function $f\colon S \to \mathbb{R}$, $S \subseteq \mathbb{R}^n$ continuous f is continuous with respect to a restriction of the standard topology of $\mathbb{R}^n$ to S.

▶ **Observation 16.** *A shift-deterministic payoff* $\varphi\colon A^\omega \to \mathbb{R}$ *is prefix-monotone if and only if* $\mathrm{sh}[a,\varphi]$ *is non-decreasing for every* $a \in A$.

We use this notation to introduce the so-called *Bellman's equations*, playing a key role in our fixed point argument.

▶ **Definition 17.** *Let A be a finite set, $\varphi\colon A^\omega \to \mathbb{R}$ be a shift-deterministic payoff and $G = (V = V_{\mathrm{Max}} \sqcup V_{\mathrm{Min}}, E), \mathsf{lab}\colon E \to A$ be an A-labeled game graph.*

*The following equations in $\mathbf{x} \in \varphi(A^\omega)^V$ are called **Bellman's equations** for φ in G:*

$$\mathbf{x}_u = \max_{e \in E, \mathsf{source}(e) = u} \mathrm{sh}[\mathsf{lab}(e), \varphi](\mathbf{x}_{\mathsf{target}(e)}), \qquad \text{for } u \in V_{\mathrm{Max}}, \tag{2}$$

$$\mathbf{x}_u = \min_{e \in E, \mathsf{source}(e) = u} \mathrm{sh}[\mathsf{lab}(e), \varphi](\mathbf{x}_{\mathsf{target}(e)}), \qquad \text{for } u \in V_{\mathrm{Min}}. \tag{3}$$

The most important step of our argument is to show the existence of a solution to Bellman's equations.

▶ **Proposition 18.** *For any finite set A, for any continuous prefix-monotone shift-deterministic payoff $\varphi\colon A^\omega \to \mathbb{R}$ and for any A-labeled game graph G there exists a solution to Bellman's equations for φ in G.*

This proposition requires some additional work, and we first discuss how to derive positional determinacy of continuous prefix-monotone shift-deterministic payoffs from it. Assume that we are give a solution $\mathbf{x}$ to (2–3). How can one extract an equilibrium of positional strategies from it? For that we take any pair of positional strategies that use only $\mathbf{x}$-*tight* edges. Now, an edge e is $\mathbf{x}$-tight if $\mathbf{x}_{\mathsf{source}(e)} = \mathrm{sh}[a,\varphi](\mathbf{x}_{\mathsf{target}(e)})$. Note that each node must contain an out-going $\mathbf{x}$-tight edge (this will be any edge on which the maximum/minimum in (2–3) is attained for this node). So clearly each player has at least one positional strategy which only uses $\mathbf{x}$-tight edges. It remains to show that for continuous prefix-monotone shift-deterministic φ any two such strategies of the players form an equilibrium.

▶ **Lemma 19.** *If A is a finite set, $\varphi\colon A^\omega \to \mathbb{R}$ is a continuous prefix-monotone shift-deterministic payoff, and $\mathbf{x} \in \varphi(A^\omega)^V$ is a solution to (2–3) for an A-labeled game graph $G = (V = V_{\mathrm{Max}} \sqcup V_{\mathrm{Min}}, E), \mathsf{lab}\colon E \to A$, then the following holds. Let σ^* be a positional strategy of Max and τ^* be a positional strategy of Min such that $\sigma^*(V_{\mathrm{Max}})$ and $\tau^*(V_{\mathrm{Min}})$ consist only of $\mathbf{x}$-tight edges. Then (σ^*, τ^*) is an equilibrium in G.*

Proof. See Appendix F. ◀

We now proceed to details of our proof of Proposition 18. Consider a function $T\colon \varphi(A^\omega)^V \to \varphi(A^\omega)^V$, mapping $\mathbf{x} \in \varphi(A^\omega)^V$ to the vector of the right-hand sides of (2–3). We should argue that T has a fixed point. For that we will construct a continuous metric $D\colon \varphi(A^\omega)^V \times \varphi(A^\omega)^V \to [0, +\infty)$ with respect to which T is *contracting*. More precisely, $D(T\mathbf{x}, T\mathbf{y})$ will always be smaller than $D(\mathbf{x}, \mathbf{y})$ as long as $\mathbf{x}$ and $\mathbf{y}$ are distinct. Due to the compactness of the domain of T this will prove that T has a fixed point.

Now, to construct such D we show that for continuous shift-deterministic φ there must be a continuous metric $d\colon \varphi(A^\omega) \times \varphi(A^\omega) \to [0, +\infty)$ such that for all $a \in A$ the function $\mathrm{sh}[a, \varphi]$ is d-contracting. Once we have such d, we let $D(\mathbf{x}, \mathbf{y})$ be the maximum of $d(\mathbf{x}_a, \mathbf{y}_a)$ over $a \in V$. Checking that T is contracting with respect to such D will be rather straightforward (technically, we will need an additional property of d which can be derived from the prefix-monotonicity of φ).

The main technical challenge is to prove the existence of d. We do so via the following general fact about compositions of continuous functions.

▶ **Theorem 20.** *Let $K \subseteq \mathbb{R}$ be a compact set, $m \geq 1$ be a natural number and $f_1, \ldots, f_m \colon K \to K$ be m continuous functions. Then the following two conditions are equivalent:*

- ***(a)*** *for any $a_1 a_2 a_3 \ldots \in \{1, 2, \ldots, m\}^\omega$ we have $\lim_{n\to\infty} \mathsf{diam}\big(f_{a_1} \circ f_{a_2} \circ \ldots \circ f_{a_n}(K)\big) = 0$ (by $\mathsf{diam}(S)$ for $S \subseteq \mathbb{R}$ we mean $\sup_{x,y\in S} |x - y|$);*
- ***(b)*** *there exists a continuous metric $d\colon K \times K \to [0, +\infty)$ such that $f_1, f_2, \ldots, f_m$ are all d-contracting (a function $h\colon K \to K$ is called d-contracting if for all $x, y \in K$ with $x \neq y$ we have $d(h(x), h(y)) < d(x, y)$).*

*If $f_1, \ldots, f_m$ are non-decreasing, then one can strengthen item **(b)** by demanding that d satisfies the following property: for all $x, y, s, t \in K$ with $x \leq s \leq t \leq y$ we have $d(s, t) \leq d(x, y)$.*

Proof. See Appendix G. ◀

Then we accurately check that this theorem is applicable to $\mathrm{sh}[a, \varphi]$ for continuous shift-deterministic φ.

▶ **Proposition 21.** *Let A be a finite set and $\varphi\colon A^\omega \to \mathbb{R}$ be a continuous shift-deterministic payoff. Then the functions $\mathrm{sh}[a, \varphi], a \in A$ are continuous and satisfy the condition **(a)** of Theorem 20 for $K = \varphi(A^\omega)$.*

Proof. See Appendix H. ◀

A formal derivation of Proposition 18 from Theorem 20 and Proposition 21 can be found in Appendix I.

5.1 Applications of the fixed point technique

Theorem 20 additionally provides an exhaustive method of generating continuous positionally determined payoffs.

▶ **Theorem 22.** *Let m be a natural number. The set of continuous positionally determined payoffs from[5] $\{1, 2, \ldots, m\}^\omega$ to $\mathbb{R}$ coincides with the set of φ that can be obtained in the following 5 steps.*

- ***Step 1.*** *Take a compact set $K \subseteq \mathbb{R}$.*
- ***Step 2.*** *Take a continuous metric $d\colon K \times K \to [0, +\infty)$.*
- ***Step 3.*** *Take m non-decreasing d-contracting functions $f_1, f_2, \ldots, f_m\colon K \to K$ (they will automatically be continuous due to continuity of d).*
- ***Step 4.*** *Define $\psi\colon \{1, \ldots, m\}^\omega \to K$ so that*

$$\{\psi(a_1 a_2 a_3 \ldots)\} = \bigcap_{n=1}^{\infty} f_{a_1} \circ f_{a_2} \circ \ldots \circ f_{a_n}(K)$$

  for every [6] $a_1 a_2 a_3 \ldots \in \{1, 2, \ldots, m\}^\omega$.
- ***Step 5.*** *Choose a continuous non-decreasing function $g\colon \psi(\{1, 2, \ldots, m\}^\omega) \to \mathbb{R}$ and set $\varphi = g \circ \psi$.*

Proof. See Appendix J. ◀

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[5] Of course, in this theorem a set of labels can be any finite set, we let it be $\{1, 2, \ldots, m\}$ for some $m \in \mathbb{N}$ just to simplify the notation.

[6] Note that this intersection always consists of a single point due to Cantor's intersection theorem and item ***(a)*** of Theorem 20. This will also be $\lim_{n\to\infty} f_{a_1} \circ f_{a_2} \circ \ldots \circ f_{a_n}(x)$ for any $x \in K$.

▶ Remark 23. Recall that we did not use continuity of g from Proposition 15 in the inductive argument. It becomes important for Theorem 22 – otherwise we could not argue that all continuous positionally payoffs can be obtained in these 5 steps.

We get the multi-discounted payoffs when the functions $f_1, f_2, \ldots, f_m$ are linear, each with the slope from $[0,1)$. In this case they will be contracting with respect to a standard metric $d(x,y) = |x - y|$. We get the whole set of continuous positionally determined payoffs by relaxing the multi-discounted payoffs in the following three regards: ***(a)*** functions $f_1, f_2, \ldots, f_m$ do not have to be linear; ***(b)*** d can be an arbitrary continuous metric; ***(c)*** any continuous non-decreasing function g can be applied to a payoff.

We use Theorem 22 to construct a continuous positionally determined payoff which does not "reduce" to the multi-discounted ones, in a sense of the following definition.

▶ **Definition 24.** *Let A be a finite set, $\varphi, \psi \colon A^\omega \to \mathbb{R}$ be two payoffs, and G be an A-labeled game graph. We say that φ **positionally reduces** to ψ **inside** G if any pair of positional strategies in G which is an equilibrium for ψ is also an equilibrium for φ.*

This definition has an algorithmic motivation. Namely, note that finding a positional equilibrium for ψ in G is at least as hard as for φ, provided that φ reduces to ψ inside G. There are classical reductions from Parity to Mean Payoff games [17] and from Mean Payoff to Discounted games [24] that work in exactly this way. See also [11] for a reduction from *Priority* Mean Payoff games to Multi-Discounted games. As far as we know, our next proposition provides the first example of a positionally determined payoff which does not reduce to the multi-discounted ones in this sense.

▶ **Proposition 25.** *There exist a finite set A, a continuous positionally determined payoff $\varphi \colon A^\omega \to \mathbb{R}$ and an A-labeled game graph G such that there exists no multi-discounted payoff to which φ reduces inside G.*

Proof. See Appendix K. ◀

Proposition 25 means, in particular, that there exists a continuous positionally determined payoff which differs from all the multi-discounted ones (as was stated in Section 3). This fact alone can be used to disprove a conjecture of Gimbert [8]. Namely, Gimbert conjectured the following: "Any payoff function which is positional for the class of non-stochastic one-player games is positional for the class of Markov decision processes". To show that this is not the case, we establish the following theorem.

▶ **Theorem 26.** *Any continuous payoff which is positionally determined in MDPs is a multi-discounted payoff.*

Proof. See Appendix L. ◀

6 Strategy improvement argument

Here we establish the existence of a solution to Bellman's equations (Proposition 18) via the *strategy improvement*. This will yield our third proof of Theorem 8. We start with an observation that a vector of guarantees of a positional strategy always gives a solution[7] to a restriction of Bellman's equations to edges that are consistent with this strategy.

[7] Bellman's equations involve the functions $\mathrm{sh}[a, \varphi]$ for $a \in A$, and these functions are defined on $\varphi(A^\omega)$. So formally we should argue that the guarantees of any strategy belong to $\varphi(A^\omega)$. Indeed, for continuous φ the set $\varphi(A^\omega)$ is compact and hence is closed, and all guarantees are the infimums/supremums of some subsets of $\varphi(A^\omega)$.

▶ **Lemma 27.** *Let A be a finite set, $\varphi\colon A^\omega \to \mathbb{R}$ be a continuous prefix-monotone shift-deterministic payoff and $G = (V = V_{\text{Max}} \sqcup V_{\text{Min}}, E), \mathsf{lab}\colon E \to A$ be an A-labeled game graph. Then for every positional strategy σ of Max in G we have:*

$$Guarantee[\sigma](u) = \mathrm{sh}[\mathsf{lab}(\sigma(u)), \varphi]\big(Guarantee[\sigma]\big(\mathsf{target}(\sigma(u))\big)\big) \text{ for } u \in V_{\text{Max}},$$
$$Guarantee[\sigma](u) = \min_{e \in E, \mathsf{source}(e) = u} \mathrm{sh}[\mathsf{lab}(e), \varphi]\big(Guarantee[\sigma](\mathsf{target}(e))\big) \text{ for } u \in V_{\text{Min}}.$$

Proof. See Appendix M. ◀

Next, take a positional strategy σ of Max. If the vector $\{Guarantee[\sigma](u)\}_{u\in V}$ happens to be a solution to the Bellman's equations, then we are done. Otherwise by Lemma 27 there must exist an edge $e \in E$ with $\mathsf{source}(e) \in V_{\text{Max}}$ such that $Guarantee[\sigma](\mathsf{source}(e)) < \mathrm{sh}[\mathsf{lab}(e), \varphi]\big(Guarantee[\sigma](\mathsf{target}(e))\big)$. We call edges satisfying this property *σ-violating*. We show that *switching* σ to any σ-violating edge gives us a positional strategy which *improves* σ.

▶ **Lemma 28.** *Let A be a finite set, $\varphi\colon A^\omega \to \mathbb{R}$ be a continuous prefix-monotone shift-deterministic payoff and $G = (V = V_{\text{Max}} \sqcup V_{\text{Min}}, E), \mathsf{lab}\colon E \to A$ be an A-labeled game graph. Next, let σ be a positional strategy of Max in G. Assume that the vector $Guarantee[\sigma] = \{Guarantee[\sigma](u)\}_{u\in V}$ does not satisfy (2–3) and let $e' \in E$ be any σ-violating edge. Define a positional strategy σ' of Max as follows:*

$$\sigma'(u) = \begin{cases} e' & u = \mathsf{source}(e'), \\ \sigma(u) & \textit{otherwise.} \end{cases}$$

Then $\sum\limits_{u\in V} Guarantee[\sigma'](u) > \sum\limits_{u \in V} Guarantee[\sigma](u)$.

Proof. See Appendix N. ◀

By this lemma, a Max's positional strategy σ^* maximizing the quantity $\sum_{u\in V} Guarantee[\sigma](u)$ (over positional strategies σ of Max) gives a solution to (2–3). Such σ^* exists just because there are only finitely many positional strategies of Max. This finishes our strategy improvement proof of Proposition 18. Let us note that the same argument can be carried out with positional strategies of Min (via analogues of Lemma 27 and Lemma 28 for Min).

6.1 Applications of the strategy improvement technique

In this subsection we discuss implications of our strategy improvement argument to the *strategy synthesis problem*. Strategy synthesis for a positionally determined payoff φ is an algorithmic problem of finding an equilibrium (with respect to φ) of two positional strategies for a given game graph. It is classical that strategy synthesis for classical positionally determined payoffs admits a randomized algorithm which is subexponential in the number of nodes [14, 1]. We obtain the same subexponential bound for all continuous positionally determined payoffs. From a technical viewpoint, we just observe that a technique which was used for classical positionally determined payoffs is applicable in a more general setting. Specifically, we use a framework of recursively local-global functions due to Björklund and Vorobyov [1].

Let us start with an observation that for continuous positionally determined shift-deterministic payoffs a non-optimal positional strategy can always be improved by changing it just in a single node.

▶ **Proposition 29.** *Let A be a finite set and $\varphi\colon A^\omega \to \mathbb{R}$ be a continuous positionally determined shift-deterministic payoff. Then for any A-labeled game graph $G = (V = V_{\text{Max}} \sqcup V_{\text{Min}}, E), \mathsf{lab}\colon E \to A$ the following two conditions hold:*

- *if σ is a non-optimal positional strategy of Max in G, then in G there exists a Max's positional strategy σ' such that $|\{u \in V_{\text{Max}} \mid \sigma(u) \neq \sigma'(u)\}| = 1$ and $\sum_{u\in V} Guarantee[\sigma'](u) > \sum_{u\in V} Guarantee[\sigma](u)$;*
- *if τ is a non-optimal positional strategy of Min in G, then in G there exists a Min's positional strategy τ' such that $|\{u \in V_{\text{Min}} \mid \tau(u) \neq \tau'(u)\}| = 1$ and $\sum_{u\in V} Guarantee[\tau'](u) < \sum_{u\in V} Guarantee[\tau](u)$.*

Proof. See Appendix O. ◀

It is instructive to visualize this proposition by imagining the set of positional strategies of one of the players (say, Max) as a *hypercube*. Namely, in this hypercube there will be as many dimensions as there are nodes of Max. A coordinate corresponding to a node $u \in V_{\text{Max}}$ will take values in the set of edges that start at u. Obviously, vertices of such hypercube are in a one-to-one correspondence with positional strategies of Max. Let us call two vertices *neighbors* of each other if they differ in exactly one coordinate. Now, Proposition 29 means in this language the following: any vertex σ, maximizing $\sum_{u\in V} Guarantee[\sigma](u)$ over its neighbors, also maximizes this quantity over the *whole* hypercube.

So an optimization problem of maximizing $\sum_{u\in V} Guarantee[\sigma](u)$ (equivalently, finding an optimal positional strategy of Max) has the following remarkable feature: all its *local* maxima are also *global*. For positional strategies of Min the same holds for the minima. Optimization problems with this feature are in a focus of numerous works, starting from a classical area of convex optimization.

Observe that in our case this local-global property is *recursive*; i.e., it holds for any restriction to a *subcube* of our hypercube. Indeed, subcubes correspond to subgraphs of our initial game graph, and for any subgraph we still have Proposition 29. Björklund and Vorobyov [1] noticed that a similar phenomenon occurs for all classical positionally determined payoffs. In turn, they showed that any optimization problem on a hypercube with this recursive local-global property admits a randomized algorithm which is subexponential in the dimension of a hypercube. In our case this yields a randomized algorithm for the strategy synthesis problem which is subexponential in the number of nodes of a game graph.

Still, this only applies to continuous payoffs that are shift-deterministic (as we have Proposition 29 only for shift-deterministic payoffs). One more issue is that we did not specify how our payoffs are represented. We overcome these difficulties in the following result.

▶ **Theorem 30.** *Let A be a finite set and $\varphi\colon A^\omega \to \mathbb{R}$ be a continuous positionally determined payoff. Consider an oracle which for given $u, v, a, b \in A^*$ tells, whether there exists $w \in A^*$ such that $\varphi(wu(v)^\omega) > \varphi(wa(b)^\omega)$. There exists a randomized algorithm which with this oracle solves the strategy synthesis problem for φ in expected $e^{O\left(\log m + \sqrt{n \log m}\right)}$ time for game graphs with n nodes and m edges. In particular, every call to the oracle in the algorithm is for $u, v, a, b \in A^*$ that are of length $O(n)$, and the expected number of the calls is $e^{O\left(\log m + \sqrt{n \log m}\right)}$.*

Proof. See Appendix P. ◀

So to deal with the issue of representation we assume a suitable oracle access to φ. Still, the oracle from Theorem 30 might look unmotivated. Here it is instructive to recall that all continuous positionally determined φ must be prefix-monotone. For prefix-monotone

φ a formula $\exists w \in A^* \ \varphi(w\alpha) > \varphi(w\beta)$ defines a total preorder on A^ω, and our oracle just compares ultimately periodic sequences according to this preorder. In fact, it is easy to see that the formula $\exists w \in A^* \ \varphi(w\alpha) > \varphi(w\beta)$ defines a total preorder on A^ω if and *only if* φ is prefix-monotone. This indicates a fundamental role of this preorder for prefix-monotone φ and justifies a use of the corresponding oracle in Theorem 30. Let us note that $\left[\exists w \in A^* \ \varphi(w\alpha) > \varphi(w\beta)\right] \iff \varphi(\alpha) > \varphi(\beta)$ if φ is additionally shift-deterministic.

7 Discussion

As Gimbert and Zielonka show by their characterization of the class of positionally determined payoffs [10], positional determinacy can always be proved by an inductive argument. Does the same hold for two other techniques that we have considered in the paper – the fixed point technique and the strategy improvement technique? The answer is positive in the continuous case, so this suggests that the answer might also be positive at least in some other special cases, for instance, for prefix-independent payoffs. E.g., for the mean payoff, a major example of a prefix-independent positionally determined payoff, both the strategy improvement and the fixed point arguments are applicable [13, 18].

These questions are specifically interesting for the strategy improvement argument. In all instances that we know the strategy improvement argument immediately yields a subexponential-time algorithm for the strategy synthesis (as, e.g., we have shown for continuous positionally determined payoffs). So this resonates with a question of how hard strategy synthesis for a positionally determined payoff can be. Loosely speaking, do we have the same subexponential bound for all positionally determined payoffs?

Finally, is it possible to characterize positionally determined payoffs more explicitly (say, as in Theorem 22)? This question sounds more approachable in special cases, and a natural special case to start is again the prefix-independent case.

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A Proof of Proposition 1

Let (σ, τ) be an equilibrium for φ, where σ is a Max's strategy and τ is a Min's strategy. We claim that (σ, τ) is also an equilibrium for $g \circ \varphi$. If φ is positionally determined, then we can take σ, τ to be positional, and this implies positional determinacy of $g \circ \varphi$.

By a standard argument a pair (σ, τ) is an equilibrium if and only if for all $a \in V$ and for all $\mathcal{P} \in \mathsf{Cons}(a, \sigma)$, $\mathcal{Q} \in \mathsf{Cons}(a, \tau)$ we have:

$$\varphi \circ \mathsf{lab}(\mathcal{P}) \geq \varphi \circ \mathsf{lab}(\mathcal{P}_a^{\sigma,\tau}) \geq \varphi \circ \mathsf{lab}(\mathcal{Q}).$$

Since g is non-decreasing, it preserves these two inequalities for all $a, \mathcal{P}, \mathcal{Q}$.

B Proof of Proposition 2

First, assume that φ is continuous. Take any $\varepsilon > 0$. We have to show that for some n_0 it holds that $\varphi(\beta_n) \in (\varphi(\alpha) - \varepsilon, \varphi(\alpha) + \varepsilon)$ for all $n \geq n_0$. The set $\varphi^{-1}((\varphi(\alpha) - \varepsilon, \varphi(\alpha) + \varepsilon))$ must be open. So for some $S \subseteq A^*$ we have:

$$\varphi^{-1}((\varphi(\alpha) - \varepsilon, \varphi(\alpha) + \varepsilon)) = \bigcup_{u \in S} uA^\omega.$$

Since obviously $\alpha \in \varphi^{-1}((\varphi(\alpha)-\varepsilon, \varphi(\alpha)+\varepsilon))$, there exists $u \in S$ such that $\alpha \in uA^\omega$. Hence for $n \geq |u|$ we have $\beta_n \in uA^\omega \subseteq \varphi^{-1}((\varphi(\alpha)-\varepsilon, \varphi(\alpha)+\varepsilon))$, as required.

Let us now establish the other direction of the proposition. It is enough to show that for any $x, y \in \mathbb{R}$ with $x < y$ the set $\varphi^{-1}((x,y))$ is open. Take any $\alpha \in \varphi^{-1}((x,y))$. Let us show that there exists $n(\alpha)$ such that all $\beta \in A^\omega$ that coincide with α in the first $n(\alpha)$ elements belong to $\varphi^{-1}((x,y))$. Indeed, otherwise for any n there exists β_n, coinciding with α in the first n elements, such that $\beta_n \notin \varphi^{-1}((x,y))$. Now, the limit $\lim_{n\to\infty} \varphi(\beta_n)$ must exist and must be equal to $\varphi(\alpha)$. But $\varphi(\alpha) \in (x,y)$ and all $\varphi(\beta_n)$ are not in this interval, contradiction.

Now, for $\alpha \in \varphi^{-1}((x,y))$ let $u_\alpha \in A^{n(\alpha)}$ be the $n(\alpha)$-length prefix of α. Observe that

$$\varphi^{-1}((x,y)) = \bigcup_{\alpha \in \varphi^{-1}((x,y))} u_\alpha A^\omega.$$

So the set $\varphi^{-1}((x,y))$ is open, as required.

C The "Only If" Part of Theorem 8

Assume that φ is not prefix-monotone. Then for some $u, v \in A^*$ and $\alpha, \beta \in A^\omega$ we have

$$\varphi(u\alpha) > \varphi(u\beta) \text{ and } \varphi(v\alpha) < \varphi(v\beta). \tag{4}$$

First, notice that by continuity of φ we may assume that α and β are ultimately periodic. Indeed, consider any two sequences $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ of ultimately periodic sequences from A^ω such that α_n and α (respectively, β_n and β) have the same prefix of length n. Then from continuity of φ we have:

$$\lim_{n\to\infty} \varphi(u\alpha_n) = \varphi(u\alpha), \qquad \lim_{n\to\infty} \varphi(v\alpha_n) = \varphi(v\alpha),$$
$$\lim_{n\to\infty} \varphi(u\beta_n) = \varphi(u\beta), \qquad \lim_{n\to\infty} \varphi(v\beta_n) = \varphi(v\beta).$$

So if u, v, α, β violate prefix-monotonicity, then so do u, v, α_n, β_n for some $n \in \mathbb{N}$.

Now, if α, β are ultimately periodic, then $\alpha = p(q)^\omega$ and $\beta = w(r)^\omega$ for some $p, q, w, r \in A^*$. Consider an A-labeled game graph from Figure 1 (all nodes there are owned by Max).

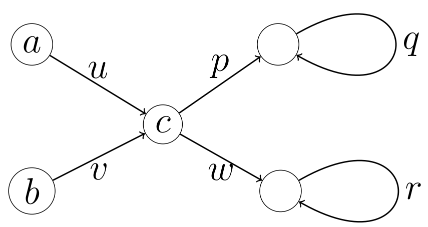

Figure 1 A game graph where φ is not positionally determined.

In this game graph there are two positional strategies of Max, one which from c goes by p and the other which goes from c by w. The first one is not optimal when the game starts in b, and the second one is not optimal when the game starts in a (because of (4)). So φ is not positionally determined in this game graph.

D Proof of Proposition 11

We only show that $\varphi(u\alpha) \le \max\{\varphi(u^\omega), \varphi(\alpha)\}$, the other inequality can be proved similarly. If $\varphi(u\alpha) \le \varphi(\alpha)$, then we are done. Assume now that $\varphi(u\alpha) > \varphi(\alpha)$. By repeatedly applying ***(a)*** we obtain $\varphi(u^{i+1}\alpha) \ge \varphi(u^i\alpha)$ for every $i \in \mathbb{N}$. In particular, for every $i \ge 1$ we get that $\varphi(u^i\alpha) \ge \varphi(u\alpha)$. By continuity of φ the limit of $\varphi(u^i\alpha)$ as $i \to \infty$ exists and equals $\varphi(u^\omega)$. Hence $\varphi(u^\omega) \ge \varphi(u\alpha)$.

E Proof of Proposition 15

Define a payoff $\psi\colon A^\omega \to \mathbb{R}$ as follows:

$$\psi(\gamma) = \sum_{w \in A^*} \left(\frac{1}{|A|+1}\right)^{|w|} \varphi(w\gamma), \qquad \gamma \in A^\omega. \tag{5}$$

First, why is ψ well-defined, i.e., why does this series converge? Since A^ω is compact, so is $\varphi(A^\omega) \subseteq \mathbb{R}$, because φ is continuous. Hence $\varphi(A^\omega) \subseteq [-W, W]$ for some $W > 0$ and (5) is bounded by the following absolutely converging series:

$$\sum_{w \in A^*} W \cdot \left(\frac{1}{|A|+1}\right)^{|w|}.$$

We shall show that ψ is continuous, prefix-monotone and shift-deterministic, and that $\varphi = g \circ \psi$ for some continuous non-decreasing $g\colon \psi(A^\omega) \to \mathbb{R}$.

Why is ψ continuous? Consider any $\alpha \in A^\omega$ and any infinite sequence $\{\beta_n\}_{n\in\mathbb{N}}$ of elements of A^ω such that for all n the sequences α and β_n coincide in the first n elements. We have to show that $\psi(\beta_n)$ converges to $\psi(\alpha)$ as $n \to \infty$. By definition:

$$\psi(\beta_n) = \sum_{w \in A^*} \left(\frac{1}{|A|+1}\right)^{|w|} \varphi(w\beta_n), \qquad \psi(\alpha) = \sum_{w \in A^*} \left(\frac{1}{|A|+1}\right)^{|w|} \varphi(w\alpha).$$

The first series, as we have seen, is bounded uniformly (in n) by an absolutely converging series. So it remains to note that the first series converges to the second one term-wise, by continuity of φ.

Why is ψ prefix-monotone? Let $\alpha, \beta \in A^\omega$. We have to show that either $\psi(u\alpha) \ge \psi(u\beta)$ for all $u \in A^*$ or $\psi(u\alpha) \le \psi(u\beta)$ for all $u \in A^*$.

Since φ is prefix-monotone, then either $\varphi(w\alpha) \ge \varphi(w\beta)$ for all $w \in A^*$ or $\varphi(w\alpha) \le \varphi(w\beta)$ for all $w \in A^*$. Up to swapping α and β we may assume that $\varphi(w\alpha) \ge \varphi(w\beta)$ for all $w \in A^*$. Then for any $u \in A^*$ the difference

$$\psi(u\alpha) - \psi(u\beta) = \sum_{w \in A^*} \left(\frac{1}{|A|+1}\right)^{|w|} \left[\varphi(wu\alpha) - \varphi(wu\beta)\right]$$

consists of non-negative terms. Hence $\psi(u\alpha) \ge \psi(u\beta)$ for all $u \in A^*$, as required.

Why is ψ shift-deterministic? Take any $a \in A$ and $\beta, \gamma \in A^\omega$ with $\psi(\beta) = \psi(\gamma)$. We have to show that $\psi(a\beta) = \psi(a\gamma)$. Indeed, assume that

$$0 = \psi(\beta) - \psi(\gamma) = \sum_{w \in A^*} \left(\frac{1}{|A|+1}\right)^{|w|} \left[\varphi(w\beta) - \varphi(w\gamma)\right].$$

If this series contains a non-zero term, then it must contain a positive term and a negative term. But this contradicts prefix-monotonicity of φ. So all the terms in this series must be 0. The same then must hold for a series:

$$\psi(a\beta) - \psi(a\gamma) = \sum_{w \in A^*} \left(\frac{1}{|A|+1}\right)^{|w|} [\varphi(wa\beta) - \varphi(wa\gamma)]$$

(all the terms in this series also appear in the series for $\psi(\beta) - \psi(\gamma)$). So we must have $\psi(a\beta) = \psi(a\gamma)$.

Why $\varphi = g \circ \psi$ for some continuous non-decreasing $g\colon \psi(A^\omega) \to \mathbb{R}$? Let us first show that

$$\varphi(\alpha) > \varphi(\beta) \implies \psi(\alpha) > \psi(\beta) \text{ for all } \alpha, \beta \in A^\omega. \tag{6}$$

Indeed, if $\varphi(\alpha) > \varphi(\beta)$, then we also have $\varphi(w\alpha) \geq \varphi(w\beta)$ for every $w \in A^*$, by prefix-monotonicity of φ. Now, by definition,

$$\psi(\alpha) - \psi(\beta) = \sum_{w \in A^*} \left(\frac{1}{|A|+1}\right)^{|w|} [\varphi(w\alpha) - \varphi(w\beta)].$$

All the terms in this series are non-negative, and the term corresponding to the empty w is strictly positive. So we have $\psi(\alpha) > \psi(\beta)$, as required.

Now, let us demonstrate that (6) implies that $\varphi = g \circ \psi$ for some non-decreasing $g\colon \psi(A^\omega) \to \mathbb{R}$. Namely, define g as follows. For $x \in \psi(A^\omega)$ take an arbitrary $\gamma \in \psi^{-1}(x)$ and set $g(x) = \varphi(\gamma)$. First, why do we have $\varphi = g \circ \psi$? By definition, $g(\psi(\alpha)) = \varphi(\gamma)$ for some $\gamma \in A^\omega$ with $\psi(\alpha) = \psi(\gamma)$. By (6) we also have $\varphi(\alpha) = \varphi(\beta)$, so $g(\psi(\alpha)) = \varphi(\gamma) = \varphi(\alpha)$, as required. Now, why is g non-decreasing? I.e., why for all $x, y \in \psi(A^\omega)$ we have $x \leq y \implies g(x) \leq g(y)$? Indeed, $g(x) = \varphi(\gamma_x), g(y) = \varphi(\gamma_y)$ for some $\gamma_x \in \psi^{-1}(x)$ and $\gamma_y \in \psi^{-1}(y)$. Now, since $x \leq y$, we have $x = \psi(\gamma_x) \leq \psi(\gamma_y) = y$. By taking the contraposition of (6) we get that $g(x) = \varphi(\gamma_x) \leq \varphi(\gamma_y) = g(y)$, as required.

Finally, we show that any $g\colon \psi(A^\omega) \to \mathbb{R}$ with $\varphi = g \circ \psi$ must be continuous. For that we show that $|g(x) - g(y)| \leq |x - y|$ for all $x, y \in \psi(A^\omega)$. Take any $\alpha, \beta \in A^\omega$ with $x = \psi(\alpha)$ and $y = \psi(\beta)$. By prefix-monotonicity of φ we have that either $\varphi(w\alpha) \geq \varphi(w\beta)$ for all $w \in A^*$ or $\varphi(w\alpha) \leq \varphi(w\beta)$ for all $w \in A^*$. Up to swapping x and y we may assume that the first option holds. Then

$$\psi(\alpha) - \psi(\beta) = \sum_{w \in A^*} \left(\frac{1}{|A|+1}\right)^{|w|} [\varphi(w\alpha) - \varphi(w\beta)] \geq \varphi(\alpha) - \varphi(\beta) \geq 0.$$

On the left here we have $x - y$, and on the right we have $\varphi(\alpha) - \varphi(\beta) = g \circ \psi(\alpha) - g \circ \psi(\beta) = g(x) - g(y)$.

F Proof of Lemma 19

For brevity, we will omit φ in the notation $\mathrm{sh}[a, \varphi]$. We will also use a notation

$$\mathrm{sh}[a_1 a_2 \ldots a_n] = \mathrm{sh}[a_1] \circ \mathrm{sh}[a_2] \circ \ldots\ldots \circ \mathrm{sh}[a_n]$$

for $n \in \mathbb{N}, a_1 a_2 \ldots a_n \in A^n$. In particular, sh[empty string] will denote the identity function.

It is enough to show that

- ***(a)*** for any $s \in V$ and for any $\mathcal{P} \in \mathsf{Cons}(s, \sigma^*)$ we have

$$\varphi \circ \mathsf{lab}(\mathcal{P}) \geq \mathbf{x}_s^*.$$

- ***(b)*** for any $s \in V$ and for any $\mathcal{P} \in \mathsf{Cons}(s, \tau^*)$ we have

$$\varphi \circ \mathsf{lab}(\mathcal{P}) \leq \mathbf{x}_s^*.$$

Indeed, from these two inequalities we obtain $\mathit{Guarantee}[\sigma^*](s) \geq \mathbf{x}_s^* \geq \mathit{Guarantee}[\tau^*](s)$ for any $s \in V$, but on the other hand $\mathit{Guarantee}[\sigma^*](s) \leq \varphi \circ \mathsf{lab}(\mathcal{P}_s^{\sigma^*, \tau^*}) \leq \mathit{Guarantee}[\tau^*](s)$.

We only show ***(a)***, a proof of ***(b)*** is similar. Let e_n be the nth edge of $\mathcal{P}$ for $n \geq 1$. Define $s_n = \mathsf{target}(e_1 e_2 \dots e_n)$ and $V_n = \mathrm{sh}[\mathsf{lab}(e_1 \dots e_n)](\mathbf{x}_{s_n}^*)$. We also set $s_0 = s$ and $V_0 = \mathbf{x}_s^*$. Note that due to continuity of φ we have that $\lim_{n\to\infty} V_n = \varphi \circ \mathsf{lab}(\mathcal{P})$. Indeed, $\mathbf{x}_{s_n}^* \in \varphi(A^\omega)$, so there exists $\beta_n \in A^\omega$ with $\mathbf{x}_{s_n}^* = \varphi(\beta_n)$. Hence $\varphi(\mathsf{lab}(e_1 \dots e_n)\beta_n) = \mathrm{sh}[\mathsf{lab}(e_1 \dots e_n)](\varphi(\beta_n)) = \mathrm{sh}[\mathsf{lab}(e_1 \dots e_n)](\mathbf{x}_{s_n}^*) = V_n$. On the other hand, the n-length prefix of $\mathsf{lab}(e_1 \dots e_n)\beta_n$ coincides with the n-length prefix of $\mathsf{lab}(\mathcal{P})$. Hence $\varphi(\mathsf{lab}(e_1 \dots e_n)\beta_n) = V_n$ converges to $\varphi \circ \mathsf{lab}(\mathcal{P})$ as $n \to \infty$, as required.

So ***(a)*** is equivalent to $\lim_{n\to\infty} V_n \geq V_0$. To show this, we demonstrate that $V_{n+1} \geq V_n$ for every n. Indeed, assume first that $s_n \in V_{\mathrm{Max}}$. Then $e_{n+1} = \sigma^*(s_n)$ (so e_{n+1} is $\mathbf{x}$-tight) and $\mathsf{target}(e_{n+1}) = s_{n+1}$. This gives us $\mathrm{sh}[\mathsf{lab}(e_{n+1})](\mathbf{x}_{s_{n+1}}^*) = \mathbf{x}_{s_n}^*$. After applying the function $\mathrm{sh}[\mathsf{lab}(e_1 e_2 \dots e_n)]$ to this equality, we obtain $V_{n+1} = V_n$.

Now, if $s_n \in V_{\mathrm{Min}}$, then $\mathrm{sh}[\mathsf{lab}(e_{n+1})](\mathbf{x}_{s_{n+1}}^*) \geq \mathbf{x}_{s_n}^*$ by (3). The function $\mathrm{sh}[\mathsf{lab}(e_1 e_2 \dots e_n)]$ is composed of non-decreasing functions (because φ is prefix-monotone and because of Observation 16). Hence after applying this function to $\mathrm{sh}[\mathsf{lab}(e_{n+1})](\mathbf{x}_{s_{n+1}}^*) \geq \mathbf{x}_{s_n}^*$ we obtain our desired inequality $V_{n+1} \geq V_n$.

G Proof of Theorem 20

For the sake of readability we will use a notation similar to one that we used in Appendix F. First, we will denote f_i by $f[i]$. Moreover, we will use the following notation

$$f[a_1 a_2 \dots a_n] = f[a_1] \circ f[a_2] \circ \dots \circ f[a_n]$$

for $n \in \mathbb{N}, a_1 a_2 \dots a_n \in \{1, 2, \dots, m\}^n$ (in particular f[empty string] denotes the identity function).

▶ **Lemma 31.** *The condition **(a)** of Theorem 20 is equivalent to the following condition: for every $\varepsilon > 0$ for all but finitely many $w \in \{1, 2, \dots, m\}^*$ we have*

$$\mathsf{diam}(f[w](K)) \leq \varepsilon.$$

Proof. Assume that the condition **(a)** of Theorem 20 holds. Take any $\varepsilon > 0$. Call $w \in \{1, 2, \dots, m\}^*$ *bad* if $\mathsf{diam}(f[w](K)) > \varepsilon$. We have to show that the number of bad w if finite. Assume for contradiction that the number of bad w is infinite. Observe that any prefix of a bad w is also bad. Indeed, if $w = uv$ for some $u, v \in \{1, 2, \dots, m\}^*$, then $f[w](K) = f[u] \circ f[v](K) \subseteq f[u](K)$. Hence by Kőnig's Lemma there exists $\alpha = a_1 a_2 a_3 \dots \in \{1, 2, \dots, m\}^\omega$ such any finite prefix of α is bad. Observe that

$$\liminf_{n\to\infty} \mathsf{diam}(f[a_1 a_2 \dots a_n](K)) \geq \varepsilon,$$

contradiction with the condition ***(a)*** of Theorem 20.

The other direction of the lemma is obvious. ◀

Proof of *(a)* $\implies$ *(b)*. Define

$$d\colon K \times K \to [0, +\infty), \qquad d(x,y) = \sup_{w \in \{1,\ldots,m\}^*} \left(2 - 2^{-|w|}\right) \cdot \left| f[w](x) - f[w](y) \right|. \tag{7}$$

First, we obviously have $d(x,x) = 0$ and $d(x,y) = d(y,x)$. Notice also that $d(x,y) \geq |x-y|$, so $d(x,y) > 0$ for $x \neq y$. In turn, $d(x,y) \leq d(x,z) + d(z,y)$ because first we can write a similar inequality for every $w \in \{1,2,\ldots,m\}^*$ in (7), and then it remains to notice that the supremum of the sums is at most the sum of the supremums. These considerations show that d is a metric.

Note that the supremum in (7) is always attained on some $w \in \{1,2,\ldots,m\}^*$. Indeed, if $d(x,y) = 0$, then it is attained already on the empty string. Assume now that $d(x,y) > 0$. By Lemma 31 all but finitely many $w \in \{1,2,\ldots,m\}^*$ satisfy $\mathsf{diam}\big(f[w](K)\big) \leq d(x,y)/3$. So only finitely many terms in (7) are bigger than $2d(x,y)/3$, and hence the supremum (which is $d(x,y) > 2d(x,y)/3$) must be attained on one of them.

This already implies that $f[i]$ is d-contracting for every $i \in \{1,2,\ldots,m\}$. Indeed, take any $x, y \in K$. Then for some $w \in \{1,2,\ldots,m\}^*$ we have:

$$d\big(f[i](x), f[i](y)\big) = (2 - 2^{-|w|}) \cdot \left| f[w]\big(f[i](x)\big) - f[w]\big(f[i](y)\big) \right|.$$

We have to show that if $x \neq y$, then $d\big(f[i](x), f[i](y)\big) < d(x,y)$. If $d\big(f[i](x), f[i](y)\big) = 0$, there is nothing to prove. Otherwise the quantity

$$\left| f[w]\big(f[i](x)\big) - f[w]\big(f[i](y)\big) \right|$$

is positive. Therefore we can write:

$$\begin{aligned} d\big(f[i](x), f[i](y)\big) &< (2 - 2^{-|w|-1}) \cdot \left| f[w]\big(f[i](x)\big) - f[w]\big(f[i](y)\big) \right| \\ &= (2 - 2^{-|wi|}) \cdot |f[wi](x) - f[wi](y)| \leq d(x,y). \end{aligned}$$

It remains to show that d is continuous. Consider any $(x_0, y_0) \in K \times K$. We have to show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $(x,y) \in K \times K$ with $|x - x_0| + |y - y_0| \leq \delta$ we have $|d(x,y) - d(x_0,y_0)| \leq \varepsilon$.

By Lemma 31 there exists $n \in \mathbb{N}$ such that for all $w \in \{1,2,\ldots,m\}^*$ with $|w| \geq n$ we have:

$$\mathsf{diam}\big(f[w](K)\big) \leq \varepsilon/6.$$

In particular, this means that in (7) all the terms corresponding to $w \in \{1,2,\ldots,m\}^*$ with $|w| \geq n$ are at most $\varepsilon/3$. Hence for every $(x,y) \in K \times K$ we have that $d(x,y)$ is $(\varepsilon/3)$-close to $d_n(x,y)$, where

$$d_n(x,y) = \max_{w \in A^*, |w| < n} \left(2 - 2^{-|w|}\right) \cdot \left| f[w](x) - f[w](y) \right|.$$

Now, notice that the function d_n is continuous (as a composition of finitely many continuous functions). Hence there exist $\delta > 0$ such that for all $(x,y) \in K \times K$ with $|x - x_0| + |y - y_0| \leq \delta$ we have $|d_n(x,y) - d_n(x_0,y_0)| \leq \varepsilon/3$. Obviously, for all such (x,y) we also have $|d(x,y) - d(x_0,y_0)| \leq \varepsilon$.

▶ Remark 32. Note that if $f_1, \ldots, f_m$ are non-decreasing, then in this construction we have $d(s,t) \leq d(x,y)$ for all $x, s, t, y \in K$ with $x \leq s \leq t \leq y$, and this proves the last paragraph of Theorem 20. Indeed, in this case $f[w]$ for $w \in \{1,2,\ldots,m\}^*$ are all composed of non-decreasing functions. Hence we have $f[w](x) \leq f[w](s) \leq f[w](t) \leq f[w](y)$ and $|f[w](s) - f[w](t)| \leq |f[w](x) - f[w](y)|$. By (7) this gives us $d(s,t) \leq d(x,y)$.

Proof of *(b)* $\implies$ *(a)*. We show that for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for all $w \in \{1, 2, \ldots, m\}^*$ with $|w| \geq n$ it holds that

$$\mathsf{diam}\big(f[w](K)\big) \leq \varepsilon.$$

Obviously, this implies ***(a)***.

Define $T = \{(x, y) \in K \times K \mid |x - y| \geq \varepsilon\}$. Note that T is a compact set. A function $d(x, y)/|x - y|$ is continuous on T (on T we never have 0 in the denominator). Hence there exists

$$z = \min_{(x,y)\in T} d(x, y)/|x - y|.$$

Observe that $z > 0$. Indeed, for some $(x, y) \in T$ we have $z = d(x, y)/|x - y|$. By definition of T we have $|x - y| \geq \varepsilon$. Hence $x \neq y$ and $d(x, y)$ is positive, as well as z.

Now, define $S = \{(x, y) \in K \times K \mid d(x, y) \geq z \cdot \varepsilon\}$. Again, S is a compact set. Consider a function:

$$h(x, y) = \max_{i\in\{1,\ldots,m\}} \frac{d\big(f[i](x), f[i](y)\big)}{d(x, y)}.$$

The function h is continuous on S (again, we never have 0 in its denominator on S). Hence there exists

$$\lambda = \max_{(x,y)\in S} h(x, y).$$

The function h is non-negative, so $\lambda \geq 0$. Let us show that $\lambda < 1$. Indeed, for some $(x, y) \in S$ we have $\lambda = h(x, y)$. By definition of h for some $i \in \{1, 2, \ldots, m\}$ we have:

$$\lambda = \frac{d\big(f[i](x), f[i](y)\big)}{d(x, y)}.$$

Since $(x, y) \in S$, we have $d(x, y) \geq z \cdot \varepsilon > 0$. Hence $x \neq y$. Now, $f[i]$ is d-contracting. Therefore $d\big(f[i](x), f[i](y)\big) < d(x, y)$ and $\lambda < 1$.

Define $D = \sup_{x,y\in K} d(x, y)$ and take any $n \in \mathbb{N}$ such that

$$\lambda^n < \frac{z\varepsilon}{D}$$

(if $D = 0$, then K consists of a single point and there is nothing to prove). We claim that for any $w \in \{1, 2 \ldots, m\}^*$ with $|w| \geq n$ we have $\mathsf{diam}\big(f[w](K)\big) \leq \varepsilon$. Since $f[w'](K)$ is a subset of $f[w](K)$ whenever w is a prefix of w', we only have to deal with w of length exactly n. Let us first establish that:

$$\sup_{x,y\in K} d\big(f[w](x), f[w](y)\big) \leq z\varepsilon. \tag{8}$$

Indeed, take any $x, y \in K$ and $w = a_1a_2 \ldots a_n \in \{1, 2, \ldots, m\}^n$. Define $w_{\geq i} = a_ia_{i+1} \ldots a_n$ for $i = 1, \ldots, n$, and let $w_{\geq n+1}$ be the empty string. Since $f[w_{\geq} i] = f[a_i] \circ f[w_{\geq i+1}]$ and since $f[a_i]$ is d-contracting, we have

$$d\big(f[w_{\geq i}](x), f[w_{\geq i}](y)\big) \leq d\big(f[w_{\geq i+1}](x), f[w_{\geq i+1}](y)\big).$$

In fact, if the right-hand side is at least $z\varepsilon$, then by definition of λ the left-hand side is at least $1/\lambda$ times smaller than the right-hand side. So if for contradiction $d\big(f[w](x), f[w](y)\big) > z\varepsilon$, then $d\big(f[w](x), f[w](y)\big)$ is at least $(1/\lambda)^n$ times smaller than $d(x, y) \leq D$. On the other hand by definition of n we have $\lambda^n D < z\varepsilon$, contradiction.

It remains to derive $\mathsf{diam}\big(f[w](K)\big) \leq \varepsilon$ from (8). We show the contraposition: if $|f[w](x) - f[w](y)| > \varepsilon$ for some $x, y \in K$, then $d\big(f[w](x), f[w](y)\big) > z\varepsilon$. Indeed, $|f[w](x) - f[w](y)| > \varepsilon$ means that $(f[w](x), f[w](y)) \in T$, so

$$\frac{d\big(f[w](x), f[w](y)\big)}{|f[w](x) - f[w](y)|} \geq \min_{(x,y)\in T} \frac{d(x, y)}{|x - y|} = z.$$

Therefore $d\big(f[w](x), f[w](y)\big) \geq z \cdot |f[w](x) - f[w](y)| > z\varepsilon$.

H Proof of Proposition 21

We use the same abbreviations regarding the notation $\mathrm{sh}[a, \varphi]$ as described in the beginning of Appendix F.

Let us first demonstrate that $\mathrm{sh}[a]$ is continuous for every $a \in A$. Consider any sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of $\varphi(A^\omega)$, converging to $x \in \varphi(A^\omega)$. We shall show that $\mathrm{sh}[a](x_n)$ converges to $\mathrm{sh}[a](x)$. Assume for contradiction that for some $\varepsilon > 0$ there are infinitely many n with $|\mathrm{sh}[a](x_n) - \mathrm{sh}[a](x)| > \varepsilon$. By taking the corresponding subsequence we may assume that all n satisfy this inequality. Let $\beta_n \in A^\omega$ be such that $x_n = \varphi(\beta_n)$. Due to compactness of A^ω there exists $\beta \in A^\omega$ such that for any open set $\mathcal{S} \subseteq A^\omega$ with $\beta \in \mathcal{S}$ we also have $\beta_n \in \mathcal{S}$ for infinitely many n. By applying this to sets of the form $\mathcal{S} = uA^\omega$, where u is a finite prefix of β, we get that for every $k \in \mathbb{N}$ there exists $n_k \geq k$ such that the first k elements of β_{n_k} and β coincide. Due to continuity of φ this means that $\lim_{k\to\infty} \varphi(\beta_{n_k}) = \lim_{k\to\infty} x_{n_k} = \varphi(\beta) = x$. By the same argument we also have $\lim_{k\to\infty} \varphi(a\beta_{n_k}) = \varphi(a\beta)$. But $\varphi(a\beta_{n_k}) = \mathrm{sh}[a](x_{n_k})$, $\varphi(a\beta) = \mathrm{sh}[a](x)$, and $|\mathrm{sh}[a](x_n) - \mathrm{sh}[a](x)| > \varepsilon$ for every n, contradiction.

Now, let us show that $\mathrm{sh}[a], a \in A$ satisfy item ***(a)*** of Theorem 20. By definition of $\mathrm{sh}[a]$ it is enough to establish that

$$\lim_{n\to\infty} \mathsf{diam}\big(\varphi(a_1 a_2 \ldots a_n A^\omega)\big) = 0$$

for any $a_1 a_2 a_3 \ldots \in A^\omega$. This is a simple consequence of the continuity of φ. Indeed, assume for contradiction that for some $a_1 a_2 a_3 \ldots \in A^\omega$ we have $\mathsf{diam}\big(\varphi(a_1 a_2 \ldots a_n A^\omega)\big) > \varepsilon$ for infinitely many n. Then for infinitely many n there exist $\beta_n, \gamma_n \in a_1 a_2 \ldots a_n A^\omega$ with $|\varphi(\beta_n) - \varphi(\gamma_n)| \geq \varepsilon$. At the same time, by continuity of φ, both $\varphi(\beta_n)$ and $\varphi(\gamma_n)$ must converge to $\varphi(a_1 a_2 a_3 \ldots)$, contradiction.

I Deriving Proposition 18 via a fixed point argument

We use the same abbreviations regarding the notation $\mathrm{sh}[a, \varphi]$ as described in the beginning of Appendix F.

Define a mapping $T \colon K^V \to K^V$, where $K = \varphi(A^\omega)$, as follows:

$$T(\mathbf{x})_u = \max_{e\in E, \mathsf{source}(e)=u} \mathrm{sh}[\mathsf{lab}(e)](\mathbf{x}_{\mathsf{target}(e)}), \qquad \text{for } u \in V_{\mathrm{Max}},$$

$$T(\mathbf{x})_u = \min_{e\in E, \mathsf{source}(e)=u} \mathrm{sh}[\mathsf{lab}(e)](\mathbf{x}_{\mathsf{target}(e)}), \qquad \text{for } u \in V_{\mathrm{Min}}.$$

Recall that K is a compact set (because A^ω is compact and φ is continuous). It is enough to show that T has a fixed point. The functions $\mathrm{sh}[a], a \in A$ are continuous by Proposition 21, so T is also continuous. Now, again by Proposition 21 and by ***(a)*** $\Longrightarrow$ ***(b)*** direction of

Theorem 20 there exists a continuous metric $d\colon K \times K \to [0, +\infty)$ such that for all $a \in A$ the function $\mathrm{sh}[a]$ is d-contracting. The payoff φ is prefix-monotone, so the functions $\mathrm{sh}[a]$ for $a \in A$ are non-decreasing (Observation 16), and hence by the last paragraph of Theorem 20 we may assume that for every $x, s, t, y \in K$ with $x \le s \le t \le y$ we have $d(s,t) \le d(x,y)$.

Define a metric $D\colon K^V \times K^V \to [0, +\infty)$ as follows:

$$D(\mathbf{x}, \mathbf{y}) = \max_{u \in V} d(\mathbf{x}_u, \mathbf{y}_u).$$

It is enough to show $D(T(\mathbf{x}), T(\mathbf{y})) < D(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in K^V, \mathbf{x} \neq \mathbf{y}$. Indeed, assume that we have it and consider a point $\mathbf{x}^* \in K^V$ minimizing $D(\mathbf{x}, T(\mathbf{x}))$ (such $\mathbf{x}^*$ exists because $D(\mathbf{x}, T(\mathbf{x}))$ is continuous and $K^V \times K^V$ is a compact set). If $\mathbf{x}^* \neq T(\mathbf{x}^*)$, then $D(T(\mathbf{x}^*), T \circ T(\mathbf{x}^*)) < D(\mathbf{x}^*, T(\mathbf{x}^*))$, contradiction.

Now, take any $\mathbf{x}, \mathbf{y} \in K^V, \mathbf{x} \neq \mathbf{y}$. Let $u \in V$ be such that $D(T(\mathbf{x}), T(\mathbf{y})) = d(T(\mathbf{x})_u, T(\mathbf{y})_u)$. Assume without loss of generality that $u \in V_{\mathrm{Max}}$. Also, up to swapping $\mathbf{x}$ and $\mathbf{y}$, we may assume that $T(\mathbf{x})_u \le T(\mathbf{y})_u$. Take $e \in E$ such that $\mathsf{source}(e) = u$ and

$$T(\mathbf{y})_u = \mathrm{sh}[\mathsf{lab}(e)](\mathbf{y}_{\mathsf{target}(e)}).$$

Denote $\mathsf{target}(e) = w$. Then we have:

$$\mathrm{sh}[\mathsf{lab}(e)](\mathbf{x}_w) \le T(\mathbf{x})_u \le T(\mathbf{y})_u = \mathrm{sh}[\mathsf{lab}(e)](\mathbf{y}_w).$$

Due to our additional requirement about d we obtain:

$$d(T(\mathbf{x})_u, T(\mathbf{y})_u) \le d\big(\mathrm{sh}[\mathsf{lab}(e)](\mathbf{x}_w), \mathrm{sh}[\mathsf{lab}(e)](\mathbf{y}_w)\big).$$

If $\mathbf{x}_w = \mathbf{y}_w$, then $0 = d(T(\mathbf{x})_u, T(\mathbf{y})_u) = D(T(\mathbf{x}), T(\mathbf{y})) < D(\mathbf{x}, \mathbf{y})$, because $\mathbf{x} \neq \mathbf{y}$. Now, if $\mathbf{x}_w \neq \mathbf{y}_w$, then due to the fact that the function $\mathrm{sh}[\mathsf{lab}(e)]$ is d-contracting, we have:

$$d\big(\mathrm{sh}[\mathsf{lab}(e)](\mathbf{x}_w), \mathrm{sh}[\mathsf{lab}(e)](\mathbf{y}_w)\big) < d(\mathbf{x}_w, \mathbf{y}_w) \le D(\mathbf{x}, \mathbf{y}),$$

as required.

J Proof of Theorem 22

No new ingredients are needed, we just connect the facts that we have already established with each other.

Assume first that $\varphi\colon \{1, 2, \ldots, m\}^\omega \to \mathbb{R}$ is continuous and positionally determined. Then φ is prefix-monotone by Theorem 8. By Proposition 15 we have $\varphi = g \circ \psi$ for some continuous prefix-monotone shift-deterministic payoff $\psi\colon \{1, 2, \ldots, m\}^\omega \to \mathbb{R}$ and some continuous non-decreasing $g\colon \psi(\{1, 2, \ldots, m\}^\omega) \to \mathbb{R}$. We set $K = \psi(\{1, 2, \ldots, m\}^\omega)$ (note that K is compact due to continuity of ψ) and $f_i = \mathrm{sh}[i, \psi]$. By Observation 16 the functions $f_1, f_2, \ldots, f_m$ are non-decreasing. By Proposition 21 the functions $f_1, f_2, \ldots, f_m$ are continuous and satisfy item ***(a)*** of Theorem 20. So by the ***(a)*** $\Longrightarrow$ ***(b)*** direction of Theorem 20 there exists a continuous metric $d\colon K \times K \to [0, +\infty)$ such that $f_1, f_2, \ldots, f_m$ are all d-contracting. Next, note that $\psi(a_1 a_2 a_3 \ldots)$ must be a unique point belonging to the intersection

$$\bigcap_{n=1}^{\infty} f_{a_1} \circ f_{a_2} \circ \ldots \circ f_{a_n}(K),$$

for every $a_1 a_2 a_3 \ldots \in \{1, 2, \ldots, m\}^\omega$. Indeed, on one hand for any $x \in K$ the quantity $f_{a_1} \circ f_{a_2} \circ \ldots \circ f_{a_n}(x)$ converges to the unique element of this intersection as $n \to \infty$. On the other hand, $f_{a_1} \circ f_{a_2} \circ \ldots \circ f_{a_n}(x) = \mathrm{sh}[a_1, \psi] \circ \mathrm{sh}[a_2, \psi] \circ \ldots \circ \mathrm{sh}[a_n, \psi](\psi(\beta)) = \psi(a_1 a_2 \ldots a_n \beta)$ converges to $\psi(a_1 a_2 a_3 \ldots)$ as $n \to \infty$, due to continuity of ψ (here $\beta \in \{1, 2, \ldots, m\}^\omega$ is such that $x = \psi(\beta)$, it exists because K is the image of ψ). These considerations show that choosing $K, f_1, \ldots, f_m, d, g$ on Steps 1–5 of Theorem 22 will give us φ.

In turn, assume that φ were obtained in these 5 steps. By Theorem 8 we only have to show that φ is continuous and prefix-monotone. For continuity of φ one can first establish continuity of ψ and then refer to continuity of g. To show continuity of ψ we just have to note that the functions $f_1, f_2, \ldots, f_m$, defining ψ, satisfy by definition item ***(b)*** of Theorem 20. Hence they also satisfy item ***(a)***, which means that for any $a_1 a_2 a_3 \ldots \in \{1, 2, \ldots, m\}^\omega$ the diameter of $\psi\left(a_1 a_2 \ldots a_n \{1, 2, \ldots, m\}^\omega\right)$ tends to 0 as $n \to \infty$. Now the continuity of ψ is immediate. To show the prefix-monotonocity of φ we only have to do so for ψ, because non-decreasing functions preserve prefix-monotonicity. As for ψ, it is easy to derive from continuity of $f_1, \ldots, f_m$ the following:

$$\psi(i\alpha) = f_i(\psi(\alpha)) \qquad \text{for any } i \in \{1, \ldots, m\}, \alpha \in \{1, \ldots, m\}^\omega.$$

Since $f_1, f_2, \ldots, f_m$ are non-decreasing, this easily implies prefix-monotonicity.

K Proof of Proposition 25

It is sufficient to establish the following lemma.

▶ **Lemma 33.** *There exist a finite set A, a continuous positionally determined payoff $\varphi \colon A^\omega \to \mathbb{R}$ and three pairs $(\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3) \in A^\omega \times A^\omega$ of ultimately periodic sequences such that:*
- *for all $i \in \{1, 2, 3\}$ we have $\varphi(\alpha_i) > \varphi(\beta_i)$*
- *for every multi-discounted payoff $\psi \colon A^\omega \to \mathbb{R}$ there exists $i \in \{1, 2, 3\}$ such that $\psi(\alpha_i) \le \psi(\beta_i)$.*

Indeed, assume that this lemma is proved. Consider a game graph from Figure 2 consisting of three pairs of “lassos”. The only optimal strategy of Max there with respect to φ is to go to the left from v_1, v_2 and v_3. On the other hand, any multi-discounted payoff has an optimal strategy which for some $i \in \{1, 2, 3\}$ goes to the right from v_i.

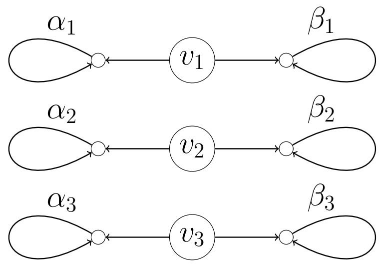

Figure 2 All nodes are owned by Max. Left lassos are colored by, respectively, α_1, α_2 and α_3; right lassos are colored by, respectively, β_1, β_2 and β_3 (we take into account the ultimate periodicity of α_i, β_i).

To show Lemma 33, we observe that following property of the multi-discounted payoffs.

▶ **Observation 34.** *Let A be a finite set and $\psi\colon A^\omega \to \mathbb{R}$ be a multi-discounted payoff. Then there are no $a, b \in A, \gamma \in A^\omega$ such that*

$$\begin{aligned}\psi(a\gamma) &> \psi(b\gamma),\\ \psi(aa\gamma) &< \psi(bb\gamma),\\ \psi(aaa\gamma) &> \psi(bbb\gamma).\end{aligned}$$

Proof. Assume for contradiction that such a, b, γ exist. Set $\lambda = \lambda(a)$, $\mu = \lambda(b)$, $u = w(a), v = w(b)$ and $x = \psi(\gamma)$. Then $\lambda, \mu \in [0, 1)$ and

$$\lambda x + u > \mu x + v, \tag{9}$$
$$\lambda^2 x + (1+\lambda)u < \mu^2 x + (1+\mu)v, \tag{10}$$
$$\lambda^3 x + (1+\lambda+\lambda^2)u > \mu^3 x + (1+\mu+\mu^2)v. \tag{11}$$

Multiply (9) by $\lambda + \mu + \lambda\mu$, multiply (10) by $-(1+\lambda+\mu)$, multiply (11) by 1 and take the sum. It can be checked that this will give us $0 > 0$. ◀

To finish a proof of Lemma 33 we construct a continuous positionally determined payoff $\varphi\colon \{1,2,3\}^\omega \to \mathbb{R}$ such that:

$$\begin{aligned}\varphi(13^\omega) &> \varphi(23^\omega),\\ \varphi(113^\omega) &< \varphi(223^\omega),\\ \varphi(1113^\omega) &> \varphi(23^\omega).\end{aligned}$$

For that we use a method described in Theorem 22. Namely, we set $K = [0, 1]$ and $d(x, y) = |x - y|$. Next, we let $f_1 = \frac{x}{2}$, $f_3 = \frac{x}{2} + \frac{1}{2}$. These two functions are clearly d-contracting. Finally, we let f_2 be a piece-wise linear function whose graph has the following break-points:

$$(0,0),\ (0.26, 0.11),\ (0.49, 0.26),\ (1, 0.49).$$

Observe that its slope is always from $[0, 1)$, so f_2 is also d-contracting. We set

$$\varphi(a_1a_2a_3\ldots) = \lim_{n\to\infty} f_{a_1} \circ f_{a_2} \circ \ldots \circ f_{a_n}(0).$$

By Theorem 22 we have that φ is a continuous positionally determined payoff. Now, it is easy to see that $\varphi(3^\omega) = 1$ and

$$\begin{aligned}\varphi(13^\omega) = 0.5 &> \varphi(23^\omega) = 0.49,\\ \varphi(113^\omega) = 0.25 &< \varphi(223^\omega) = 0.26,\\ \varphi(1113^\omega) = 0.125 &> \varphi(2223^\omega) = 0.11.\end{aligned}$$

L Proof of Theorem 26

First we establish two necessary conditions on payoffs that are positionally determined in MDPs.

▶ **Proposition 35.** *Let A be a finite set and $\varphi\colon A^\omega \to \mathbb{R}$ be a continuous payoff which is positionally determined in MDPs. Then there are no $a \in A$, $\beta, \gamma, \delta \in A^\omega$, $(p_1, p_2, p_3), (q_1, q_2, q_3) \in [0, +\infty)^3$ such that $p_1 + p_2 + p_3 = q_1 + q_2 + q_3 = 1$ and*

$$\begin{aligned}p_1\varphi(\beta) + p_2\varphi(\gamma) + p_3\varphi(\delta) &> q_1\varphi(\beta) + q_2\varphi(\gamma) + q_3\varphi(\delta),\\ p_1\varphi(a\beta) + p_2\varphi(a\gamma) + p_3\varphi(a\delta) &< q_1\varphi(a\beta) + q_2\varphi(a\gamma) + q_3\varphi(a\delta).\end{aligned}$$

▶ **Proposition 36.** *If a payoff is positionally determined in MDPs, then it is prefix-monotone.*

In fact, Proposition 36 is already proved. Indeed, in Appendix C we have shown that for any continuous payoff which is not prefix-monotone there exists a game graph *with all the nodes belonging to Max* where φ is not positional. This game graph is a deterministic MDP, so any continuous payoff which is not prefix monotone is not MDP-positional.

To finish a proof of Theorem 26 we show the following technical claim.

▶ **Proposition 37.** *Let A be a finite set and $\varphi\colon A^\omega \to \mathbb{R}$ be a continuous prefix-monotone payoff. Assume that there are no $a \in A$, $\beta, \gamma, \delta \in A^\omega$, $(p_1, p_2, p_3), (q_1, q_2, q_3) \in [0, +\infty)^3$ such that $p_1 + p_2 + p_3 = q_1 + q_2 + q_3 = 1$ and*

$$p_1\varphi(\beta) + p_2\varphi(\gamma) + p_3\varphi(\delta) > q_1\varphi(\beta) + q_2\varphi(\gamma) + q_3\varphi(\delta),$$
$$p_1\varphi(a\beta) + p_2\varphi(a\gamma) + p_3\varphi(a\delta) < q_1\varphi(a\beta) + q_2\varphi(a\gamma) + q_3\varphi(a\delta).$$

Then φ is a multi-discounted payoff.

L.1 Proof of Proposition 35

Assume for contradiction that such $a, \beta, \gamma, \delta, (p_1, p_2, p_3), (q_1, q_2, q_3)$ exist. Similarly to a proof of the "only if" part of Theorem 8, due to continuity of φ we may assume that β, γ and δ are ultimately periodic. We construct an A-labeled MDP $\mathcal{M}$ where φ has no optimal positional strategy. To define $\mathcal{M}$ first consider an A-labeled game graph from Figure 3.

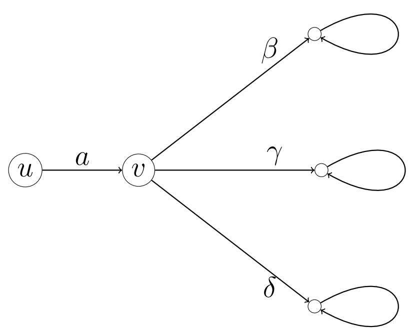

Figure 3 A graph for an MDP where φ has no optimal positional strategy.

In this graph there are exactly 3 infinite paths ("lassos") $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ that start at v. We label edges in such a way that $\mathsf{lab}(\mathcal{P}_1) = \beta, \mathsf{lab}(\mathcal{P}_2) = \gamma, \mathsf{lab}(\mathcal{P}_3) = \delta$. Clearly, this is possible due to the ultimate periodicity of β, γ and δ.

Next, we turn this graph into an MDP (formally, nodes of the graph will be states of the MDP). There will be two actions available at the node v. Both will be distributed on the three successors of v, one with probabilities p_1, p_2, p_3, and the other with probabilities q_1, q_2, q_3 (in the descending order if one looks at Figure 3). For each node different from v there will be only one action with the source in this node, leading with probability 1 to its unique successor.

Note that each possible transition is along some edge of the graph from Figure 3, and the label of this edge will also serve as the label of the transition. This concludes a description of $\mathcal{M}$.

To show that φ is not MDP-positional in $\mathcal{M}$, note that in $\mathcal{M}$ there are exactly 2 positional strategies, σ_p and σ_q, corresponding to two actions available at v. We show that none of these two strategies is optimal.

It is easy to see that:

$$\mathbb{E}\varphi \circ \mathsf{lab}\big(\mathcal{P}_u^{\sigma_p}\big) = p_1 \cdot \varphi(a\beta) + p_2 \cdot \varphi(a\gamma) + p_3 \cdot \varphi(a\delta),$$
$$\mathbb{E}\varphi \circ \mathsf{lab}\big(\mathcal{P}_v^{\sigma_p}\big) = p_1 \cdot \varphi(\beta) + p_2 \cdot \varphi(\gamma) + p_3 \cdot \varphi(\delta),$$
$$\mathbb{E}\varphi \circ \mathsf{lab}\big(\mathcal{P}_u^{\sigma_q}\big) = q_1 \cdot \varphi(a\beta) + q_2 \cdot \varphi(a\gamma) + q_3 \cdot \varphi(a\delta),$$
$$\mathbb{E}\varphi \circ \mathsf{lab}\big(\mathcal{P}_v^{\sigma_q}\big) = q_1 \cdot \varphi(\beta) + q_2 \cdot \varphi(\gamma) + q_3 \cdot \varphi(\delta).$$

Due to our assumptions about $(p_1, p_2, p_3), (q_1, q_2, q_3)$ we obtain:

$$\mathbb{E}\varphi \circ \mathsf{lab}\big(\mathcal{P}_u^{\sigma_p}\big) < \mathbb{E}\varphi \circ \mathsf{lab}\big(\mathcal{P}_u^{\sigma_q}\big), \qquad \mathbb{E}\varphi \circ \mathsf{lab}\big(\mathcal{P}_v^{\sigma_p}\big) > \mathbb{E}\varphi \circ \mathsf{lab}\big(\mathcal{P}_v^{\sigma_q}\big).$$

Therefore, neither σ_p nor σ_q is optimal.

L.2 Proof of Proposition 37

If $\varphi(\gamma) = \varphi(\delta)$ for all $\beta, \gamma \in A^\omega$, then clearly φ is multi-discounted (one can define $\lambda(a) = 0, w(a) = \varphi(\gamma)$ for all $a \in A$ and for an arbitrary $\gamma \in A^\omega$). So in what follows we fix any $\gamma, \delta \in A^\omega$ with $\varphi(\gamma) \neq \varphi(\delta)$. First we derive from the conditions of Proposition 37 the following:

▶ **Lemma 38.** *For any $a \in A$ there exist $\lambda(a), w(a) \in \mathbb{R}$ such that for any $\beta \in A^\omega$ we have:*

$$\varphi(a\beta) = \lambda(a)\varphi(\beta) + w(a).$$

Proof. The following system in (λ, w) has a unique solution:

$$\begin{pmatrix} \varphi(a\gamma) \\ \varphi(a\delta) \end{pmatrix} = \begin{pmatrix} \varphi(\gamma) & 1 \\ \varphi(\delta) & 1 \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ w \end{pmatrix}, \tag{12}$$

(because $\varphi(\gamma) \neq \varphi(\delta)$). Let its solution be $(\lambda(a), w(a))$. We show that $\varphi(a\beta) = \lambda(a)\varphi(\beta) + w(a)$ for all $\beta \in A^\omega$. Let us first show that

$$\det \begin{pmatrix} 1 & 1 & 1 \\ \varphi(\beta) & \varphi(\gamma) & \varphi(\delta) \\ \varphi(a\beta) & \varphi(a\gamma) & \varphi(a\delta) \end{pmatrix} = 0. \tag{13}$$

Indeed, otherwise there exists a vector $(x, y, z) \in \mathbb{R}^3$ such that

$$\begin{pmatrix} 1 & 1 & 1 \\ \varphi(\beta) & \varphi(\gamma) & \varphi(\delta) \\ \varphi(a\beta) & \varphi(a\gamma) & \varphi(a\delta) \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \tag{14}$$

Let $P_1, P_2, P_3, Q_1, Q_2, Q_3$ be any positive real numbers such that $x = P_1 - Q_1, y = P_2 - Q_2, z = P_3 - Q_3$. From the first equality in (14) it follows that there exists $S > 0$ such that

$$S = P_1 + P_2 + P_3 = Q_1 + Q_2 + Q_3.$$

Define $p_i = P_i/S, q_i = Q_i/S$ for $i \in \{1, 2, 3\}$. Observe that $a, \beta, \gamma, \rho, (p_1, p_2, p_3), (q_1, q_2, q_3)$ violate the conditions of Proposition 37 (this can be seen from the second and the third equality in (14)), contradiction. Therefore (13) is proved.

The first two rows of the matrix from (13) are linearly independent because $\varphi(\gamma) \neq \varphi(\delta)$. Hence the third one must be a linear combination of the first two. I.e., there must exist $\lambda, w \in \mathbb{R}$ such that

$$(\varphi(a\beta), \varphi(a\gamma), \varphi(a\delta)) = \lambda(\varphi(\beta), \varphi(\gamma), \varphi(\delta)) + w(1, 1, 1).$$

From the second and the third coordinate we conclude that (λ, w) must be a solution to (12), so $\lambda = \lambda(a), w = w(a)$. Now, by looking at the first coordinate we obtain that $\varphi(a\beta) = \lambda(a)\varphi(\beta) + w(a)$, as required. ◀

From now on let $\lambda(a), w(a)$ for $a \in A$ be as in Lemma 38. Let us show that $\lambda(a) \in [0, 1)$ for all $a \in A$.

Assume first that for some $a \in A$ we have $\lambda(a) < 0$. Without loss of generality we may also assume that $\varphi(\gamma) < \varphi(\delta)$. Then $\varphi(a\gamma) = \lambda(a)\varphi(\gamma) + w(a) > \lambda(a)\varphi(\delta) + w(a) = \varphi(a\delta)$. Therefore φ is not prefix-monotone, contradiction.

Next, assume that $\lambda(a) \geq 1$ for some $a \in A$. Consider the following two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ of real numbers:

$$x_n = \varphi(\underbrace{aa\ldots a}_{n}\gamma), \qquad y_n = \varphi(\underbrace{aa\ldots a}_{n}\delta).$$

By definition, we set $x_0 = \varphi(\gamma)$ and $y_0 = \varphi(\delta)$. Note that by our choice of γ and δ we have $x_0 \neq y_0$. Next, since φ is continuous, we have:

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \varphi(aaa\ldots). \tag{15}$$

On the other hand, we can directly compute x_n and y_n by repeatedly applying Lemma 38:

$$x_n = \lambda(a)^n x_0 + w(a)(1 + \lambda(a) + \ldots + \lambda(a)^{n-1}), \tag{16}$$
$$y_n = \lambda(a)^n y_0 + w(a)(1 + \lambda(a) + \ldots + \lambda(a)^{n-1}). \tag{17}$$

We will show that $\lambda(a) \geq 1$ contradicts (15–17).

First consider the case $\lambda(a) = 1$. Then x_n and y_n look as follows:

$$x_n = x_0 + nw(a), \qquad y_n = y_0 + nw(a).$$

If $w(a) \neq 0$, then the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are not convergent, contradiction with (15). If $w(a) = 0$, then one sequence converges to x_0, and the other to y_0. But $x_0 \neq y_0$, so this again gives a contradiction with (15).

Now consider the case $\lambda(a) > 1$. Then we can rewrite (16–17) as follows:

$$x_n = \lambda(a)^n \left(x_0 + \frac{w(a)}{\lambda(a) - 1}\right) - \frac{w(a)}{\lambda(a) - 1}, \qquad y_n = \lambda(a)^n \left(y_0 + \frac{w(a)}{\lambda(a) - 1}\right) - \frac{w(a)}{\lambda(a) - 1}.$$

Since $x_0 \neq y_0$, for at least one of these two expressions the coefficient before $\lambda(a)^n$ is non-zero. Hence either $\{x_n\}_{n\in\mathbb{N}}$ or $\{y_n\}_{n\in\mathbb{N}}$ diverges. This contradicts (15).

We have established that $\lambda(a) \in [0, 1)$ for every $a \in A$. All that remains to do is to show that φ satisfies (1). For that we again employ continuity of φ. Take any $\beta \in A^\omega$. Note that by Lemma 38 we have:

$$\varphi(a_1 a_2 \ldots a_n \beta) = \lambda(a_1) \cdot \ldots \cdot \lambda(a_n)\varphi(\beta) + \sum_{i=1}^{n} \lambda(a_1) \cdot \ldots \lambda(a_{i-1}) \cdot w(a_i). \tag{18}$$

We know that $\lambda(a_i)$ are all from $[0,1)$. Since the set A is finite, all $\lambda(a_i)$ are uniformly bounded away from 1. Hence the first term in the right-hand side of (18) converges to 0 as $n \to \infty$. On the other hand, the second term in the right-hand side of (18) converges to the series from the right-hand side of (1). Finally, due to continuity of φ, the left-hand side of (18) converges to $\varphi(a_1 a_2 a_3 \ldots)$. Thus, φ is multi-discounted.

M Proof of Lemma 27

By definition $Guarantee[\sigma](u)$ is the infimum of the image of $\varphi \circ \mathsf{lab}$ on the set $\mathsf{Cons}(u,\sigma)$. Now, the set $\mathsf{Cons}(u,\sigma)$ is exactly the set of infinite paths that start at u and consist only of edges from E^σ. So we can write:

$$\mathsf{Cons}(u,\sigma) = \bigcup_{\substack{e \in E^\sigma \\ \mathsf{source}(e)=u}} e\mathsf{Cons}(\mathsf{target}(e),\sigma).$$

The infimum of a union of finitely many sets is the minimum of the infimums of these sets. So we get:

$$Guarantee[\sigma](u) = \min_{\substack{e \in E^\sigma \\ \mathsf{source}(e)=u}} \inf \varphi \circ \mathsf{lab}\left(e\mathsf{Cons}(\mathsf{target}(e),\sigma)\right).$$

For any $a \in A, \mathcal{S} \subseteq A^\omega$, by definition of $\mathrm{sh}[a,\varphi]$, we can write:

$$\varphi\big(a\mathcal{S}\big) = \mathrm{sh}[a,\varphi]\big(\varphi(\mathcal{S})\big).$$

After applying this to $a = \mathsf{lab}(e), \mathcal{S} = \mathsf{lab}\big(\mathsf{Cons}(\mathsf{target}(e),\sigma)\big)$ we obtain:

$$\varphi \circ \mathsf{lab}\left(e\mathsf{Cons}(\mathsf{target}(e),\sigma)\right) = \mathrm{sh}[\mathsf{lab}(e),\varphi]\big(\varphi \circ \mathsf{lab}\left(\mathsf{Cons}(\mathsf{target}(e),\sigma)\right)\big).$$

Now, since $\mathrm{sh}[\mathsf{lab}(e),\varphi]$ is non-decreasing (by Observation 16) and continuous (by Proposition 21), we can interchange inf and $\mathrm{sh}[\mathsf{lab}(e),\varphi]$. This gives us:

$$\begin{aligned}\inf \mathrm{sh}[\mathsf{lab}(e),\varphi]&\big(\varphi \circ \mathsf{lab}\left(\mathsf{Cons}(\mathsf{target}(e),\sigma)\right)\big)\\ &= \mathrm{sh}[\mathsf{lab}(e),\varphi]\big(\inf \varphi \circ \mathsf{lab}\left(\mathsf{Cons}(\mathsf{target}(e),\sigma)\right)\big)\\ &= \mathrm{sh}[\mathsf{lab}(e),\varphi]\big(Guarantee[\sigma](\mathsf{target}(e))\big).\end{aligned}$$

By putting all this together we obtain:

$$Guarantee[\sigma](u) = \min_{\substack{e \in E^\sigma \\ \mathsf{source}(e)=u}} \mathrm{sh}[\mathsf{lab}(e),\varphi]\big(Guarantee[\sigma](\mathsf{target}(e))\big).$$

This is exactly what we need to prove. Indeed, for $u \in V_{\mathrm{Max}}$ the minimum is over a single edge, $e = \sigma(u)$. In turn, for $u \in V_{\mathrm{Min}}$ the minimum is over all edges that start at u.

N Proof of Lemma 28

We use the same abbreviations regarding the notation $\mathrm{sh}[a,\varphi]$ as described in the beginning of Appendix F.

For $\mathbf{x} \in \varphi(A^\omega)^V$, let the *modified cost* of an edge $e \in E$ with respect to $\mathbf{x}$ be the following quantity:

$$R^{\mathbf{x}}(e) = \mathrm{sh}[\mathsf{lab}(e)](\mathbf{x}_{\mathsf{target}(e)}) - \mathbf{x}_{\mathsf{source}(e)}.$$

We need the following "potential transformation lemma" (its analog for the discounted payoffs is well-known, see, e.g., [15, Lemma 3.6]).

▶ **Lemma 39.** *Take any* $\mathbf{x} \in \varphi(A^\omega)^V$. *Let* $\mathcal{P} = e_1e_2e_2\ldots$ *be an infinite path in* G. *Then there exists an infinite sequence of non-negative real numbers* $\lambda_1, \lambda_2, \lambda_3, \ldots$ *such that* $\lambda_1 = 1$ *and*

$$\varphi \circ \mathsf{lab}\big(\mathcal{P}\big) - \mathbf{x}_{\mathsf{source}(\mathcal{P})} = \sum_{n=1}^{\infty} \lambda_n \cdot R^{\mathbf{x}}(e_i).$$

Proof. For $u \in V$ let $\beta_u \in A^\omega$ be such that $\mathbf{x}_u = \varphi(\beta_u)$. Define $s_n = \mathsf{target}(e_1e_2\ldots e_n)$ for $n \geq 1$ and $s_0 = \mathsf{source}(\mathcal{P})$. By continuity of φ we have

$$\varphi \circ \mathsf{lab}\big(\mathcal{P}\big) = \lim_{n\to\infty} \varphi\big(\mathsf{lab}(e_1e_2\ldots e_n)\beta_{s_n}\big) = \lim_{n\to\infty} \mathrm{sh}[\mathsf{lab}(e_1e_2\ldots e_n)](\mathbf{x}_{s_n}).$$

Hence we obtain

$$\begin{aligned}
\varphi \circ \mathsf{lab}\big(\mathcal{P}\big) - \mathbf{x}_{s_0} &= \lim_{n\to\infty} \big(\mathrm{sh}[\mathsf{lab}(e_1e_2\ldots e_n)](\mathbf{x}_{s_n}) - \mathbf{x}_{s_0}\big) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \big(\mathrm{sh}[\mathsf{lab}(e_1e_2\ldots e_k)](\mathbf{x}_{s_k}) - \mathrm{sh}[\mathsf{lab}(e_1e_2\ldots e_{k-1})](\mathbf{x}_{s_{k-1}})\big) \\
&= \sum_{n=1}^{\infty} \big(\mathrm{sh}[\mathsf{lab}(e_1e_2\ldots e_n)](\mathbf{x}_{s_n}) - \mathrm{sh}[\mathsf{lab}(e_1e_2\ldots e_{n-1})](\mathbf{x}_{s_{n-1}})\big).
\end{aligned}$$

We can write each term in this series as:

$$\begin{aligned}
&\mathrm{sh}[\mathsf{lab}(e_1e_2\ldots e_n)](\mathbf{x}_{s_n}) - \mathrm{sh}[\mathsf{lab}(e_1e_2\ldots e_{n-1})](\mathbf{x}_{s_{n-1}}) \\
&\quad = \mathrm{sh}[\mathsf{lab}(e_1\ldots e_{n-1})]\big(\mathrm{sh}[\mathsf{lab}(e_n)](\mathbf{x}_{s_n})\big) - \mathrm{sh}[\mathsf{lab}(e_1e_2\ldots e_{n-1})](\mathbf{x}_{s_{n-1}}) \\
&\quad = \lambda_n \cdot \big(\mathrm{sh}[\mathsf{lab}(e_n)](\mathbf{x}_{s_n}) - \mathbf{x}_{s_{n-1}}\big),
\end{aligned}$$

for some $\lambda_n \in [0, +\infty)$, because $\mathrm{sh}[\mathsf{lab}(e_1\ldots e_{n-1})]$ is non-decreasing (for $n = 1$ we get $\lambda_1 = 1$ because sh[empty strong] is the identity function). It remains to notice that by definition:

$$\mathrm{sh}[\mathsf{lab}(e_n)](\mathbf{x}_{s_n}) - \mathbf{x}_{s_{n-1}} = R^{\mathbf{x}}(e_n)$$

(because $\mathsf{source}(e_n) = s_{n-1}, \mathsf{target}(e_n) = s_n$). ◀

We apply this lemma to the vector $\mathbf{g} = \{Guarantee[\sigma](u)\}_{u\in V}$. Note that by Lemma 27 we have $R^{\mathbf{g}}(e) \geq 0$ for every $e \in E^\sigma$. In turn, since e' is σ-violating, we have $R^{\mathbf{g}}(e') > 0$.

Let us at first show that

$$Guarantee[\sigma'](u) \geq Guarantee[\sigma](u) = \mathbf{g}_u$$

for every $u \in V$. For that we show that $\varphi \circ \mathsf{lab}(\mathcal{P}) \geq \mathbf{g}_u$ for any infinite path $\mathcal{P} = e_1e_2e_3\ldots \in \mathsf{Cons}(u, \sigma')$. Indeed, by Lemma 39 we can write:

$$\varphi \circ \mathsf{lab}(\mathcal{P}) - \mathbf{g}_u = \sum_{n=1}^{\infty} \lambda_n R^{\mathbf{g}}(e_n) \tag{19}$$

for some $\lambda_n \in [0, +\infty), \lambda_1 = 1$. All the edges of $\mathcal{P}$ are from $E^\sigma \cup \{e'\}$. Hence all the terms in this series, as we discussed, are non-negative, and so is the left-hand side.

To show $\sum_{u\in V} Guarantee[\sigma'](u) > \sum_{u\in V} Guarantee[\sigma](u)$ it is now enough to show that $Guarantee[\sigma'](u) > Guarantee[\sigma](u) = \mathbf{g}_u$ for some $u \in V$. For that we take $u = \mathsf{source}(e')$. Then the first edge of any $\mathcal{P} \in \mathsf{Cons}(u, \sigma')$ is e'. So the first term in (19) equals $R^{\mathbf{g}}(e')$. All the other terms, as we have discussed, are non-negative. Hence $Guarantee[\sigma'](u) \geq R^{\mathbf{g}}(e') + Guarantee[\sigma](u)$, and it remains to recall that $R^{\mathbf{g}}(e')$ is strictly positive.

O Proof of Proposition 29

By Lemma 27 a Max's positional strategy σ uses only edges that are $Guarantee[\sigma]$-tight. Hence for a non-optimal σ the vector $Guarantee[\sigma]$ cannot be a solution to Bellman's equations by Lemma 19. It remains to take σ' as in Lemma 28. An argument for positional strategies of Min is similar.

P Proof of Theorem 30

First in Subsection P.1 it is demonstrated that w.l.o.g. we may assume that φ is shift-deterministic (so that we can use Proposition 29) and that we are given an oracle which simply compares values of φ on ultimately periodic sequences. Then in Subsection P.2 we expose a framework of *recursively local-global functions* due to Björklund and Vorobyov. Finally, in Subsection P.3 we use this framework to show Theorem 30 in the assumptions of Subsection P.1.

P.1 Reducing to shift-deterministic payoffs

It is sufficient to establish Theorem 30 with the following assumptions.

▶ **Assumption 1.** *Payoff φ is continuous, positionally determined and **shift-deterministic**.*

▶ **Assumption 2.** *We are given an oracle which for $u, v, a, b \in A^*$ tells, whether $\varphi(u(v)^\omega) > \varphi(a(b)^\omega)$.*

To justify this, it is enough to show the following lemma.

▶ **Lemma 40.** *Let A be a finite set and let $\varphi\colon A^\omega \to \mathbb{R}$ be a continuous positionally determined payoff. Then there exist a continuous positionally determined shift-deterministic payoff $\psi\colon A^\omega \to \mathbb{R}$ and a non-decreasing function $g\colon \psi(A^\omega) \to \mathbb{R}$ such that $\varphi = g \circ \psi$ and*

$$\big[\exists w \in A^* \ \varphi(w\alpha) > \varphi(w\beta)\big] \iff \psi(\alpha) > \psi(\beta) \qquad \textit{for all } \alpha, \beta \in A^\omega.$$

Indeed, let φ be an arbitrary continuous positionally determined payoff and ψ be as in Lemma 40. Observe that an equilibrium for ψ is also an equilibrium for $\varphi = g \circ \psi$ (see a proof of Proposition 1). So to solve the strategy synthesis for φ it is enough to do so for ψ. Clearly, ψ satisfies Assumption 1. Finally, note that the oracle of Assumption 2 for ψ simply coincides on every input with the oracle we are given for φ in Theorem 30.

Proof of Lemma 40. We take ψ as in the proof of Proposition 15. It is established there that

- ψ is continuous, prefix-monotone (hence positionally determined) and shift-deterministic;
- $\varphi = g \circ \psi$ for some non-decreasing $g\colon \psi(A^\omega) \to \mathbb{R}$.

This information is sufficient to derive

$$\big[\exists w \in A^* \ \varphi(w\alpha) > \varphi(w\beta)\big] \implies \psi(\alpha) > \psi(\beta).$$

Indeed, if $g \circ \psi(w\alpha) = \varphi(w\alpha) > \varphi(w\beta) = g \circ \psi(w\beta)$ for some $w \in A^*$, then we also have $\psi(w\alpha) > \psi(w\beta)$, because g is non-decreasing. Due to prefix-monotonicity of ψ we also have $\psi(\alpha) \geq \psi(\beta)$. It remains to demonstrate that $\psi(\alpha) \neq \psi(\beta)$. Indeed, $\psi(\alpha) = \psi(\beta) \implies \psi(w\alpha) = \psi(w\beta)$ because ψ is shift-deterministic.

To demonstrate that

$$\psi(\alpha) > \psi(\beta) \implies \left[\exists w \in A^* \ \varphi(w\alpha) > \varphi(w\beta)\right]$$

we have to recall a construction of ψ. We can write:

$$\psi(\alpha) - \psi(\beta) = \sum_{w \in A^*} \left(\frac{1}{|A|+1}\right)^{|w|} \left[\varphi(w\alpha) - \varphi(w\beta)\right]$$

If $\varphi(w\alpha) \leq \varphi(w\beta)$ for all $w \in A^*$, then clearly $\psi(\alpha) \leq \psi(\beta)$. This is exactly the contraposition of an implication we have to prove. ◀

P.2 Recursively local-global functions

Fix $d \in \mathbb{N}$. A d-dimensional *structure* is a collection $\mathcal{S} = \{S_i\}_{i=1}^d$ of d non-empty finite sets $S_1, S_2, \ldots, S_d$. *Vertices* of $\mathcal{S}$ are elements of the Cartesian product $\prod_{i=1}^d S_i$. Two vertices $\sigma = (\sigma_1, \ldots, \sigma_d), \sigma' = (\sigma'_1, \ldots, \sigma'_d) \in \prod_{i=1}^d S_i$ of a structure $\mathcal{S}$ are called *neighbors* if there is exactly one $i \in \{1, 2, \ldots, d\}$ such that $\sigma_i \neq \sigma_i^*$.

A structure $\mathcal{S}' = \{S'_i\}_{i=1}^d$ is a *substructure* of a structure $\mathcal{S} = \{S_i\}_{i=1}^d$ if $S'_i \subseteq S_i$ for every $i \in \{1, 2, \ldots, d\}$. A substructure $\mathcal{S}' = \{S'_i\}_{i=1}^d$ of a structure $\mathcal{S} = \{S_i\}_{i=1}^d$ is called *a facet* of $\mathcal{S}$ if there exists $i \in \{1, 2, \ldots, d\}$ such that $|S'_i| = 1, |S_i| > 1$ and $\forall j \in \{1, 2, \ldots, d\} \setminus \{i\} \ S_j = S'_j$.

For two structures $\mathcal{S} = \{S_i\}_{i=1}^d$ and $\mathcal{S}' = \{S'_i\}_{i=1}^d$ such that $\forall i \in \{1, 2, \ldots, d\} \ S_i \not\subseteq S'_i$ we denote by $\mathcal{S} \setminus \mathcal{S}'$ the following structure: $\mathcal{S} \setminus \mathcal{S}' = \{S_i \setminus S'_i\}_{i=1}^d$.

Let $\mathcal{S}$ be a structure and f be a function from the set of vertices of $\mathcal{S}$ to $\mathbb{R}$. A vertex σ of $\mathcal{S}$ is called a *local* maximum of f if $f(\sigma) \geq f(\sigma')$ for every neighbor σ' of σ in $\mathcal{S}$. A vertex σ is called a *global* maximum of f if $f(\sigma) \geq f(\sigma')$ for every vertex σ' of $\mathcal{S}$. The function f is called *local-global* if all its local maxima are global. The function f is called *recursively local-global* if all its restrictions to substructures of $\mathcal{S}$ are local-global.

Given a structure $\mathcal{S}$ and a function f from the set of vertices of $\mathcal{S}$ to $\mathbb{R}$, we are interested in finding a global maximum of f. In [1] Björklund and Vorobyov studied the following algorithm for this problem.

Algorithm 2 MSW($\mathcal{S}', \sigma$).

    Input: $\mathcal{S}' = \{S'_i\}_{i=1}^d$ is a substructure of $\mathcal{S}$ and σ is a vertex of $\mathcal{S}'$.
if $\forall i \in \{1, 2, \ldots, d\} \ |S'_i| = 1$ **then**
    output σ;
else
    choose a random facet F of $\mathcal{S}'$ such that σ is not a vertex of F;
    $\sigma^* \leftarrow$ MSW($\mathcal{S}' \setminus F, \sigma$);
    if $\exists$ *a neighbor* σ' *of* σ^* *in* $\mathcal{S}'$ *such that* $f(\sigma') > f(\sigma^*)$ *and* σ' *is a vertex of* F
    then
        output MSW(F, σ');
    else
        output σ^*;
    end
end

▶ **Theorem 41** (Theorem 5.1 in [1]). *Let $\mathcal{S} = \{S_i\}_{i=1}^d$ be a d-dimensional structure and $f\colon \prod_{i=1}^d S_i \to \mathbb{R}$ be a recursively local-global function. Then for any initial vertex σ of $\mathcal{S}$ the algorithm* MSW($\mathcal{S}, \sigma$) *outputs a global maximum of f in expected*

$$e^{O\left(\log m + \sqrt{d \log m}\right)} \ \textit{time},$$

where $m = \sum_{i=1}^{d} |S_i|$.

P.3 Deriving Theorem 30 with Assumptions 1 and 2

Let $G = (V = V_{\text{Max}} \sqcup V_{\text{Min}}, E), \mathsf{lab}\colon E \to A$ be an A-labeled game graph in which we want to solve the strategy synthesis. We will only show how to find an optimal positional strategy of Max, an argument for Min is similar.

Let $d = |V_{\text{Max}}|$ and $V_{\text{Max}} = \{u_1, u_2, \ldots, u_d\}$. Define $S_i = \{e \in E \mid \mathsf{source}(e) = u_i\}$. Consider a structure $\mathcal{S} = \{S_i\}_{i=1}^{d}$. Obviously, we may identify vertices of $\mathcal{S}$ with positional strategies of Max. Define

$$f\colon \prod_{i=1}^{d} S_i \to \mathbb{R}, \qquad f(\sigma) = \sum_{u \in V} \mathit{Guarantee}[\sigma](u).$$

▶ **Lemma 42.** *Any global maximum of f is an optimal positional strategy of Max.*

Proof. Let σ be a global maximum of f and σ^* be any optimal positional strategy of Max. By optimality of σ^* we have $\mathit{Guarantee}[\sigma^*](u) \geq \mathit{Guarantee}[\sigma](u)$ for every $u \in V$. On the other hand, σ maximizes the sum of the guarantees (over all positional strategies of Max), so we must have $\mathit{Guarantee}[\sigma^*](u) = \mathit{Guarantee}[\sigma](u)$ for every $u \in V$. This means that σ is also optimal. ◀

▶ **Lemma 43.** *The function f is recursively local-global.*

Proof. A fact that f is local-global is a simple consequence of Proposition 29 (note that by Assumption 1 our payoff satisfies the requirements of this proposition). Indeed, a strategy σ which is not a global maximum of f cannot be optimal. Then σ' as in Proposition 29 is a neighbor of σ with $f(\sigma') > f(\sigma)$, so σ cannot be a local maximum as well.

To show that in fact f is recursively local-global it is sufficient to note that substructures of $\mathcal{S}$ correspond to subgraphs of G, and for these subgraphs Proposition 29 still holds. ◀

Due to these two lemmas, Algorithm 2 for $\mathcal{S}$ and f as above will give us an optimal strategy of Max in expected

$$e^{O\left(\log m + \sqrt{d \log m}\right)} \text{ time,}$$

where $m = \sum_{i=1}^{d} |S_i|$. Note that d does not exceed the number of nodes of G and m does not exceed the number of edges, so Theorem 30 follows.

Still, Algorithm 2 involves comparisons of $f(\sigma')$ and $f(\sigma^*)$ for σ' and σ^* that are neighbors of each other. We should explain how to perform these comparisons using only the oracle from Assumption 2.

In other words, our task is to compare two sums

$$\sum_{u \in V} \mathit{Guarantee}[\sigma_1](u), \qquad \sum_{u \in V} \mathit{Guarantee}[\sigma_2](u)$$

for two positional strategies σ_1, σ_2 of Max that differ from each other in exactly one node. Assume that $v \in V_{\text{Max}}$ is such that $\{v\} = \{u \in V_{\text{Max}} \mid \sigma_1(u) \neq \sigma_2(u)\}$.

Let G^{σ_1} (respectively, G^{σ_2}) be a graph obtain from G by deleting all edges that are not consistent with σ_1 (respectively, σ_2). Next, let G^{σ_1,σ_2} be a graph of all edges that appear either in G^{σ_1} or in G^{σ_2}. Let us note for rigor that edges of $G^{\sigma_1}, G^{\sigma_2}$ and G^{σ_1,σ_2} are labeled in a same way as the corresponding edges in G, and a partition $V = V_{\text{Max}} \sqcup V_{\text{Min}}$ in these graphs is exactly the same as in G.

Observe that in G^{σ_1,σ_2} strategies σ_1, σ_2 are the only two positional strategies of Max (indeed, all nodes of Max except v have in G^{σ_1,σ_2} exactly one out-going edge, and v has exactly 2). One of them must be optimal in G^{σ_1,σ_2}. So either $\mathit{Guarantee}[\sigma_1](u) \geq \mathit{Guarantee}[\sigma_2](u)$ for all $u \in V$ or $\mathit{Guarantee}[\sigma_1](u) \leq \mathit{Guarantee}[\sigma_2](u)$ for all $u \in V$. This means that

$$\sum_{u\in V} \mathit{Guarantee}[\sigma_1](u) > \sum_{u\in V} \mathit{Guarantee}[\sigma_2](u)$$
$$\iff \exists u \in V \; \mathit{Guarantee}[\sigma_1](u) > \mathit{Guarantee}[\sigma_2](u).$$

So our task reduces to a task of comparing $\mathit{Guarantee}[\sigma_1](u)$ and $\mathit{Guarantee}[\sigma_2](u)$ for $u \in V$.

Let us first perform this assuming that we also have an oracle which can solve strategy synthesis for φ in *one-player game graphs* (i.e., in graphs where one of the players owns all the nodes). Then we can find an optimal Min's positional strategy τ_1 in G^{σ_1} and an optimal Min's positional strategy τ_2 in G^{σ_2} (indeed, in these two graphs all nodes of Max have exactly one out-going edge, so equivalently we can view these nodes as nodes of Min). Observe that τ_1 is an optimal response to σ_1 and τ_2 is an optimal response to σ_2, so we have:

$$\mathit{Guarantee}[\sigma_1](u) = \varphi \circ \mathsf{lab}\big(\mathcal{P}_u^{\sigma_1,\tau_1}\big), \qquad \mathit{Guarantee}[\sigma_2](u) = \varphi \circ \mathsf{lab}\big(\mathcal{P}_u^{\sigma_2,\tau_2}\big).$$

It remains to compare the value of φ on $\mathsf{lab}\big(\mathcal{P}_u^{\sigma_1,\tau_1}\big)$ and $\mathsf{lab}\big(\mathcal{P}_u^{\sigma_2,\tau_2}\big)$. We can do this via the oracle from Assumption 2 as these two sequences are ultimately periodic.

Of course, we are not done yet since we used an additional oracle. However, note that we do not need this oracle in the case when G is already one-player. Indeed, in this case there is exactly one infinite path consistent with σ, and hence $\mathit{Guarantee}[\sigma_1](u)$ and $\mathit{Guarantee}[\sigma_2](u)$ are simply values of φ on some ultimately periodic sequences. Hence, Theorem 30 is already proved for one-player game graphs without any clauses. Now, whenever in the construction above we query the "illegal" oracle we can instead use this one-player version of Theorem 30. Note that we always run it on one-player game graphs that have at most as many nodes and edges as G. So this only increases the expected running time by an additional factor of $e^{O\left(\log m + \sqrt{n \log m}\right)}$.