An annotated bibliography on convergence of matrix products and the theory of joint/generalized spectral radius

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July 19, 2021

The proposed text is an attempt to create a (permanently updated) list of publications on the convergence of infinite matrix products and the rate of their growth/decrease as the number of factors tends to infinity.

Of course, the works mentioned below mainly reflect my personal interests. I may miss some works that other people consider "very important" or may not know about some of them, and I may also insert links to works that are of interest only to me.

Annotations to the mentioned works are taken from abstracts/summaries of the publications themselves, as well as from the MathSciNet and zbMATH databases. Sometimes annotations are slightly edited to combine all available sources.

The works mentioned in this text cover the following topics.

Joint/generalized spectral radius. General concept

- Joint/generalized spectral radius theory in itself: [3, 55, 67, 78, 79, 92, 117, 146, 154, 159, 168, 171, 191, 192, 205, 208, 216, 272, 274, 311, 315, 332, 340, 342, 348, 352, 356, 391, 404, 431, 443, 447];
- properties of and bounds for joint/generalized spectral radius: [58, 61, 66, 152, 174, 176, 180, 224, 230, 233, 236, 290, 366, 413, 429, 440, 443];
- Lagarias–Wang finiteness conjecture/property: [53, 130, 132, 140, 164, 165, 182, 185, 198, 215, 227, 235, 247, 255, 258, 303, 304, 326, 329, 370, 387, 449];
- Berger–Wang formula for the joint spectral radius: [288, 335, 397, 437, 443];
- normed finiteness property: [201];
- etc. [219, 284, 405, 419].

Applications

The number of works in which the theory of convergence of infinite matrix products is applied to other problems is growing very fast last years. So, I am forced to restrict the corresponding list of references to the publications mostly reflecting my personal interests.

- Computational mathematics, wavelets: [37, 44, 58, 59, 61, 107, 111, 112, 126, 158, 306, 368];
- asynchronous/parallel methods of computation: [5–8, 12, 23, 29, 109, 124, 414];

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- control theory, automata, consensus, asynchronous data exchange systems: [17–22, 25, 32, 34, 35, 39–41, 45, 80, 96, 137, 143, 147, 151, 154, 173, 181, 193, 270, 299, 306, 350, 353, 358, 380, 381, 395, 423];
- entropy games, matrix multiplication games: [361, 429];
- switched and iterated function systems: [31, 41, 93, 148, 188, 190, 209, 229, 250, 254, 260, 261, 265, 276, 278, 281, 283, 292, 295, 297, 302, 307, 312, 319–321, 327, 330, 341, 345, 346, 351, 357, 359, 360, 365, 367, 371, 373, 374, 378, 379, 384, 386, 388, 389, 393, 400, 412, 420, 421, 425, 427, 433, 439, 441, 442, 446, 450];
- linear inclusions: [26–28, 52, 57, 97, 119, 122, 149, 163, 244, 273, 275, 382];
- matrix and operator cocycles: [288, 305, 396];
- probability: [47, 87, 264, 331, 394, 409, 422];
- combinatorics: [99, 114, 127, 186, 217, 226];
- mathematical economics, arbitrage: [234, 280, 301];
- etc. [129, 200, 221, 362, 406, 436].

Alternative characterizations and definitions

- p-radius: [54, 76, 169, 206, 229, 262, 341, 342, 372, 373, 417];
- joint spectral radius in max-algebras: [56, 85, 104, 166, 177, 204, 220, 257, 267, 317, 383, 401];
- lower spectral radius or joint spectral subradius: [159, 176, 182, 217, 256, 286, 349];
- trace characterizations: [108, 241];
- minimax joint spectral radius: [415]

Convergence of infinite matrix products

This section contains publications in which the problem of convergence of infinite matrix products is considered as a special case of the general theory.

- [5, 9, 13–15, 32, 41–43, 48, 51, 63, 69, 73, 74, 79, 80, 84, 90, 91, 94, 95, 98, 100, 102, 110, 111, 116, 121, 125, 130, 133, 142, 189, 190, 207, 246, 285, 323, 334, 339, 376, 377, 398, 402, 407, 428, 432].

Semigroups of matrices and the theory of Banach algebras

Joint/generalized spectral radius from the point of view of general theory of matrix semigroups and theory of Banach algebras.

- Joint spectrum and key facts: [42, 62, 74, 150, 155, 178, 394, 444];
- extremal and Barabanov norms, antinorms, König chains, growth rate: [128, 134, 214, 248, 308, 355, 363, 386, 386, 430, 448];
- invariant spaces, simultaneous triangularization and reducibility of multiplicative semigroups: [16, 24, 36, 38, 67, 71, 72, 77, 88, 115, 123, 228];
- multiplicativity and submultiplicativity of the spectral radius: [60, 77, 392];
- joint/generalized spectral radius of operators in Banach spaces/algebras: [62, 76, 103, 117, 118, 135, 150, 268, 288, 290, 293, 296, 396, 397, 431]
- etc. [11, 310].

Complexity, decidability and primitivity

- Complexity and NP-hardness of problems related to the calculation of the joint/generalized spectral radius: [81–83, 104, 106, 171, 175, 212, 216, 226, 277, 300, 309, 322, 327, 350, 360, 409, 422];
- decidability of the boundedness/convergence problem for infinite matrix products: [33, 41, 105, 120, 138, 144, 199, 246, 313, 344, 352];
- mortality problem: [68, 70, 120, 131];
- primitivity of families of matrices, Černý conjecture: [286, 291, 347, 399, 424].

Algorithms

This section describes some of the algorithms used to compute the joint/generalized spectral radius and related problems.

- Theoretical approach: [89, 113, 139, 148, 156, 157, 167, 170, 187, 202, 203, 229, 242, 262, 318, 324, 328, 356, 373, 384, 426, 429];
- more or less practical realizations: [64, 162, 197, 210, 218, 231, 232, 245, 263, 264, 271, 279, 300, 308, 334, 343, 375, 416, 434, 438, 441];

Special classes of matrices or laws of composing matrix products

- Low-dimensional or rank-one matrices: [22, 41, 68, 131, 153, 196, 211, 225, 227, 237, 243, 247, 266, 282, 287, 314, 337];
- non-negative matrices and matrices sharing invariant cone: [160, 184, 199, 223, 224, 238–240, 286, 291, 318, 410, 426, 445];
- rational/integer/sign/binary matrices: [70, 82, 105, 131, 138, 198, 199, 227, 246, 363, 449];
- asynchronous systems/computations: [5, 22, 23, 25, 31, 32, 34, 35, 40, 41, 47, 94, 109, 124];
- Markov chains of matrices, constrained matrix products, constrained switching systems: [160, 186, 281, 331, 335, 336, 358, 374, 379, 412, 440, 441];
- mixtures of matrices, matrices with independent column/row uncertainty, product families of matrices: [17–20, 41, 143, 212, 318, 361, 369, 370, 389, 390, 415];
- matrix cocycles: [153, 225].

Measure theoretic/ergodic methods

- measure theoretic/ergodic methods in the theory of joint/generalized spectral radius: [100, 132, 194, 195, 236, 249–253, 281, 288, 304, 316, 330, 342, 354, 364, 385, 403, 408, 409, 411, 418, 422, 435];

Products of random stochastic matrices

- convergence of products of random stochastic matrices (this is a specific area of mathematics with its own methods, it will be mentioned below perfunctory): [4, 10, 30, 46, 50, 222, 251, 269, 298, 319, 325, 331, 447];

Mathematical tools and methods

Some math tools that have proven useful when exploring joint/generalized spectral radius.

- Fritz John's ellipsoid theorem: [2, 75, 101, 259, 333];
- extremal and Barabanov norms: [26–28, 136, 137, 161, 172, 206, 213, 235, 248, 275, 289, 294, 338, 448];
- invariant cones: [179];
- lifting approach, lifted polytope methods: [139, 203, 271, 334, 335, 440];
- max version of the Perron-Frobenius theorem: [85];
- polytope norms: [161, 183, 213];
- sum-of-squares (SOS) approach/programming: [187, 189, 190, 203, 242, 297, 327, 328, 371, 412, 433, 438, 441];
- transients, quasi–controllability, overshooting measure: [49, 65, 86, 168, 205];
- etc. [1, 141, 145].

References

1941

[1] I. Gelfand, *Normierte Ringe*, Rec. Math. [Mat. Sbornik] N. S. **9 (51)** (1941), 3–24. MR 0004726. Zbl 67.0406.02.

Ein normierter Ring R ist ein (bzgl. des Körpers der komplexen Zahlen) linearer, normierter vollständiger Raum im Sinne von Banach, in dem eine Multiplikation erklärt ist, die ihn zu einem kommutativen Ring macht, in dem ein Einheitselement e vom Betrag 1 existiert und stets $\|x \cdot y\| \leq \|x\| \cdot \|y\|$ gilt. Ist R ein Körper, so ist R gleich dem Körper der komplexen Zahlen. Dies wird mit funktionentheoretischen Sätzen bewiesen. Eine Funktion $x(\lambda)$, λ komplex, $x \in R$, heißt analytisch in einem Bereich, wenn dort $\lim\limits_{h\to 0} \dfrac{x(\lambda+h)-x(\lambda)}{h}$ im Sinn der Konvergenz nach der Norm existiert. Die Sätze der gewöhnlichen Funktionentheorie übertragen sich auf diese Funktionen leicht, da für jedes lineare Funktional $f(x)$ die Funktion $f(x(\lambda))$ analytisch im üblichen Sinn wird. Jedes Ideal von R ist in einem maximalen Ideal M enthalten, dessen Restklassenring R/M dem Körper der komplexen Zahlen isomorph ist. Jedem Element $x \in R$ wird so die Funktion $x(M)$ zugeordnet, wobei $x(M)$ die komplexe Zahl ist, die die Restklasse von x in R/M darstellt. Ist $x(M)$ nirgends Null, so besitzt x ein inverses Element. Die Funktion $x(M)$ ist dann und nur dann identisch Null, wenn x im Durchschnitt aller maximalen Ideale von R liegt. Notwendig und hinreichend dafür ist, daß x ein verallgemeinertes nilpotentes Element ist, für das $\sqrt[n]{\| x^n \|} \to 0$ geht.

Die Menge $\mathfrak{M}$ aller maximalen Ideale von R wird zu einem topologischen Raum, wenn man als eine Umgebung $U(M_0)$ alle $M \in \mathfrak{M}$ mit $|\, x_i(M) - x_i(M_0)\,| < \varepsilon$, $i = 1, \dots, n, x_1, \dots, x_n$ aus R, einführt. $\mathfrak{M}$ ist bikompakt und genügt den Hausdorffschen Axiomen, die $x(M)$ sind stetig auf $\mathfrak{M}$, also ist R homomorph in den Ring $C(\mathfrak{M})$ aller auf $\mathfrak{M}$ stetigen Funktionen abgebildet. Diese Abbildung ist dann und nur dann eine Isomorphie, wenn R kein Radikal, d. h. keine verallgemeinerten nilpotenten Elemente, enthält. Ist R separabel, so ist $\mathfrak{M}$ metrisierbar. Besitzt R eine endliche Anzahl von Erzeugenden, so ist $\mathfrak{M}$ eine abgeschlossene und beschränkte Teilmenge des n-dimensionalen Raumes. Zwei normierte Ringe ohne Radikal, die algebraisch isomorph sind, sind auch stetig isomorph. R ist dann und nur dann direkte Summe von Idealen, die Ringe mit Einselementen sind, wenn $\mathfrak{M}$ nicht zusammenhängend ist (R muß dabei aber mit x stets ein y enthalten, dessen Funktion $y(M)$ konjugiert komplex zu $x(M)$ ist). Untersuchungen über analytische Funktionen von Elementen des Ringes R.

1948

[2] F. John, *Extremum problems with inequalities as subsidiary conditions*, Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, Interscience Publishers, Inc., New York, N. Y., 1948, pp. 187–204. MR 0030135. Zbl 0034.10503.

This paper deals with an extension of Lagrange's multiplier rule to the case, where the subsidiary conditions are inequalities instead of equations. Only extrema of differentiable functions of a finite number of variables will be considered. There may however be an infinite number of inequalities prescribed. Lagrange's rule for the situation considered here differs from the ordinary one, in that the multipliers may always be assumed to be positive. This makes it possible to obtain sufficient conditions for the occurence or a minimum in terms of the first derivatives only. See also [333].

1960

[3] G.-C. Rota and G. Strang, *A note on the joint spectral radius*, Nederl. Akad. Wetensch. Proc. Ser. A 63 = Indag. Math. **22** (1960), 379–381. MR 0147922. Zbl 0095.09701.

Für eine beschränkte Untermenge B einer normierten Algebra $\mathfrak{A}$ mit Einheitselement wird der vereinigte (joint) Spelctralradius durch $r(B) = \lim_{n\to\infty} \sup_{t\in P_n} \|T\|^{1/n}$ definiert; hierin ist P_n die Menge aller Produkte mit n Faktoren aus B. Ist $\mathfrak{N}$ die Menge der (multiplikativen) Normen von $\mathfrak{A}$, die zu der gegebenen Norm von $\mathfrak{A}$ äquivalent sind (in dem Sinne, daß $k\|T\| \le N(T) \le k^{-1}\|T\|$ für eine Konstante $k \neq 0$ und alle T in $\mathfrak{A}$, so ist $r(B) = \inf_{N\in\mathfrak{N}} \sup_{t\in B} N(T)$. Dieser Satz wird mit Hilfe eines Lemmas bewiesen, aus dem die Autoren überdies noch eine Bemerkung über die Fortsetzbarkeit einer auf der Unteralgebra $\mathfrak{A}' \subset \mathfrak{A}$ definierton Norm gewinnen.

1963

[4] J. Wolfowitz, *Products of indecomposable, aperiodic, stochastic matrices*, Proc. Amer. Math. Soc. **14** (1963), 733–737, `doi:10.1090/S0002-9939-1963-0154756-3`. MR 0154756. Zbl 0116.35001.

A finite square matrix $P = \{p_{ij}\}$ is called stochastic if $p_{ij} \ge 0$ for all i, j, and $\sum_j p_{ij} = 1$ for all i. A stochastic matrix P is called indecomposable and aperiodic (SIA) if $Q = \lim_{n\to\infty} P^n$ exists and all the rows of Q are the same. SIA matrices are defined differently in books on probability theory.

We define $\delta(P)$ by $\delta(P) = \max_j \max_{i_1,i_2} |p_{i_1 j} - p_{i_2 j}|$. Thus $\delta(P)$ measures, in a certain sense, how different the rows of P are. If the rows of P are identical, $\delta(P) = 0$ and conversely.

Let $A_1, \dots, A_k$ be any square matrices of the same order. By a word (in the A's) of length t we mean the product of t A's (repetitions permitted). The object of this note is to prove the following assertion:

Theorem. *Let $A_1, \dots, A_k$ be square stochastic matrices of the same order such that any word in the A's is SIA. For any $\varepsilon > 0$ there exists an integer $\nu(e)$ such that any word B (in the A's) of length $n \ge \nu(e)$ satisfies $\delta(B) < \varepsilon$.*

In words, the result is that any sufficiently long word in the A's has all its rows approximately the same.

1969

[5] D. Chazan and W. Miranker, *Chaotic relaxation*, Linear Algebra Appl. **2** (1969), 199–222, `doi:10.1016/0024-3795(69)90028-7`. MR 0251888. Zbl 0225.65043.

In this paper we consider relaxation methods for solving linear systems of equations. These methods are suited for execution on a parallel system of processors. They have the feature of allowing a minimal amount of communication of computational status between the computers, so that the relaxation process, while taking on a chaotic appearance, reduces programming and processor time of a bookkeeping nature. We give a precise characterization of chaotic relaxation, some examples of divergence, and conditions guaranteeing convergence.

1970

[6] F. Robert, *Méthodes itératives "série parallèle"*, C. R. Acad. Sci. Paris Sér. A-B **271** (1970), A847–A850. MR 0269076. Zbl 0209.17602.

Il s'agit dans cette note, de présenter une classe d'algorithmes d'itérations linéaires, pour la résolution d'un système linéaire de $n = q \cdot k$ équations, adaptés à un calcul en parallèle mettant en oeuvre q ordinateurs. On appelle procédé série parallèle $\{k, q\}$ un tel procédé, qui est finalement le suivant: les q ordinateurs traitent simultanément les q premières composantes (chacun en prend une en charge) selon la méthode de Jacobi puis mettent en commun leurs résultats, puis traitent ensuite en parallèle les q composantes suivantes etc. Il est clair que le procédé $\{1, n\}$ est alors celui de Jacobi habituel, tandis que le procédé $\{n, 1\}$ est celui de Gauss-Seidel. L'étude de la contraction (qui implique la convergence) relativement à une norme vectorielle permet de donner une condition (utilisable numériquement) de convergence, et d'obtenir les résultats suivants: 1) Pour une M-matrice, tous les procédés série parallèle convergent. 2) A fortiori, pour une M-matrice, tous les procédés série-parallèle sont convergents. Si, de plus, k'est multiple de k, le procédé $\{k', q'\}$ (avec évidemment $k'q' = n = kq$) converge plus vite que le procédé $\{k, q\}$. On complète donc ainsi le résultat classique de comparaison de Jacobi et Gauss-Seidel, qui est cas particulier pour $k' = n$ et $k = 1$. Il en résulte que pour une M-matrice, les procédés ainsi définis convergent au moins aussi vite que celui de Jacobi, mais ne convergent pas plus vite que celui de Gauss-Seidel.

1974

[7] J.-C. Miellou, *Itérations chaotiques à retards*, C. R. Acad. Sci. Paris Sér. A **278** (1974), 957–960 (english). MR 362887. Zbl 0314.65028.

D. Chazan and *W. Miranker* [5] proved that the chaotic iteration for the linear equation $u = Au + r$ converges if the spectral radius $\rho(|A|) < 1$. Here this result is generalized to non-linear equations $u = F(u)$, where $F : D(F) \subset E \to E$ and a $E = \prod_{i=1}^{\alpha} E_i$ is the product of Banach spaces E_i, with norms $\| \ \|_i$ $(i = 1, \dots, \alpha)$. Taking on E the vectorial norm $\|(u_1, \dots, u_\alpha)\| = (\|u_1\|_1, \dots, \|u_\alpha\|_\alpha)$ it is assumed that there is a nonnegative $\alpha \times \alpha$-matrix T, $\rho(T) < 1$, such that for a fixed point u^* of F in $D(F)$ and all $v \in D(F)$ $\|F(u^*) - F(v)\| \le T(\|u^* - v\|)$ holds. Then it is shown that the chaotic iteration for $u = F(u)$, defined as in the paper of Chazan-Mlranker, converges to u^* for any starting vector near u^*. Bounds for the convergence rate are given.

1975

[8] J.-C. Miellou, *Itérations chaotiques à retards; études de la convergence dans le cas d'espaces partiellement ordonnés*, C. R. Acad. Sci. Paris Sér. A-B **280** (1975), A233–A236 (english). MR 391499. Zbl 0332.65060.

Here is given a general result of convergence in the case of partially ordered spaces for certain algorithms, which one applies to a problem of quasi-variational inequalities.

1976

[9] J. Hajnal, *On products of non-negative matrices*, Math. Proc. Camb. Philos. Soc. **79** (1976), 521–530, `doi:10.1017/S030500410005252X`. MR 396628. Zbl 0333.15013.

The main theme of this paper is that under wide conditions, the product of a large number of square non-negative matrices is close to a positive matrix of rank 1. (A non-negative matrix is one whose elements are non-negative real numbers). The results may be viewed as an extension of the Perron-Probenius theory for powers of non-negative matrices. Previous results in this field are reviewed in E. Seneta's book "Non-negative matrices" (1973). The present paper applies Birkhoff's theory of positive matrices as operators possessing a contraction property. The tendency of non-negative matrix products towards rank 1 is referred to as ergodicity (a term derived in this context from the theory of non-homogeneous Markov chains) and is here defined in such a manner as to apply to sequences of matrix products in which successive terms are formed by inserting an additional matrix in any place (and not merely at the extreme right or left). — The paper presents results for two sorts of sequences of products called “ergodic sequences” and “ergodic sets”. An ergodic sequence must contain a subsequence of matrices having no zero elements. The paper proves such results as the following. Suppose that (i) $A_1, A_2, \ldots$ are non-negative matrices each of which has at least one strictly positive element in each row and each column and (ii) $\sum_k \frac{m_k}{M_k} = \infty$ (where m_k and M_k are the smallest and largest element of A_k). Then if H_n is a product of the first n of the matrices A_k multiplied in any order, the sequence $\{H_n\}$ is a product of the first n of the matrices A_k multiplied in any order, the sequence $\{H_n\}$ is ergodic, i.e. will be arbitrarily dose to matrix of rank 1 for sufficiently large n. — The theorems concerning “ergodic sets” generalize results which have emerged in some fields of application such as the weak ergodicity theorem (Coale-Lopez) theorem) of demography or the Markov systems in Paz’e work on probabilistic automata.

1977

[10] J. M. Anthonisse and H. Tijms, *Exponential convergence of products of stochastic matrices*, J. Math. Anal. Appl. **59** (1977), no. 2, 360–364, `doi:10.1016/0022-247X(77)90114-7`. MR 0443092. Zbl 0381.15011.

This paper considers a finite set of stochastic matrices of finite order. Conditions are given under which any product of matrices from this set converges to a constant stochastic matrix. Also, it is shown that the convergence is exponentially fast.

1977/78

[11] A. Mandel and I. Simon, *On finite semigroups of matrices*, Theoret. Comput. Sci. **5** (1977/78), no. 2, 101–111, `doi:10.1016/0304-3975(77)90001-9`. MR 473070. Zbl 0368.20049.

Finite semigroups of n by n matrices over the naturals are characterized both by algebraic and combinatorial methods. Next we show that the cardinality of a finite semigroup S of n by n matrices over a field is bounded by a function depending only on n, the number of generators of S and the maximum cardinality of its subgroups. As a consequence, given n and k, there exist, up to isomorphism, only a finite number of finite semigroups of n by n matrices over the rationals, generated by at most k elements. Among other applications to Automaton Theory, we show that it is decidable whether the behavior of a given $N - \Sigma$ automaton is bounded.

1978

[12] G. M. Baudet, *Asynchronous iterative methods for multiprocessors*, J. Assoc. Comput. Mach. **25** (1978), no. 2, 226–244, `doi:10.1145/322063.322067`. MR 0494894. Zbl 0372.68015.

A class of asynchronous iterative methods is presented for solving a system of equations. Existing iterative methods are identified in terms of asynchronous iterations, and new schemes are introduced corresponding to a parallel implementation on a multiprocessor system with no synchronization between cooperating processes. A sufficient condition is given to guarantee the convergence of any asynchronous iterations, and results are extended to include iterative methods with memory. Asynchronous iterative methods are then evaluated from a computational point of view, and bounds are derived for the efficiency. The bounds are compared with actual measurements obtained by running various asynchronous iterations on a multiprocessor, and the experimental results show clearly the advantage of purely asynchronous iterative methods.

1979

[13] R. K. Brayton and C. H. Tong, *Stability of dynamical systems: a constructive approach*, IEEE Trans. Circuits and Systems **26** (1979), no. 4, 224–234, `doi:10.1109/TCS.1979.1084637`. MR 525235. Zbl 0413.93048.

A set A of $n \times n$ complex matrices is stable if for every neighborhood of the origin $U \subset C^n$, there exists another neighborhood of the origin V, such that for each $M \in A'$ (the set of finite products of matrices in A), $MV \subset U$. Matrix and Lyapunov stability are related.

Theorem. *A set of matrices A is stable if and only if there exists a norm, $|\cdot|$, such that for all $M \in A$, and all $z \in C^n$, $|Mz| < |z|$.*

The norm is a *Lyapunov function* for the set A. It need not be smooth; using smooth norms to prove stability can be inadequate. A novel central result is a constructive algorithm which can determine the stability of A based on the following assertion:

Theorem. *$A = \{M_0, M_1, \ldots, M_{m-1}\}$ is a set of m distinct complex matrices. Let W_0 be a bounded neighborhood of the origin. Define for $k > 0$,*

$$W_k = \text{convex hull } \left[\bigcup_{i=0}^{\infty} M_{k'}^i W_{k-1}\right]$$

where $k' = (k-1) \mod m$. Then A is stable if and only if $W^* \equiv \cup_{i=1}^{\infty} W_i$ is bounded.

The constructive algorithm represents a convex set by its extreme points and uses linear programming to construct the successive W_k. Sufficient conditions for the finiteness of constructing W_k from W_{k-1}, and for stopping the algorithm when either the set is proved stable or unstable are presented. A is generalized to be any convex set of matrices. A dynamical system of differential equations is stable if a corresponding set of matrices – associated with the logarithmic norms of the matrices of the linearized equations – is stable. The concept of the stability of a set of matrices is related to the existence of a matrix norm. Such a norm is an induced matrix norm if and only if the set of matrices is maximally stable (i.e., it cannot be enlarged and remain stable).

[14] J. E. Cohen, *Contractive inhomogeneous products of nonnegative matrices*, Math. Proc. Camb. Philos. Soc. **86** (1979), 351–364, `doi:10.1017/S0305004100056176`. MR 538757. Zbl 0415.60013.

J. Hajnal [9] showed that under wide conditions a sequence of products $H(1,q) = M_q \cdots M_1$, $q = 1, 2, \ldots$, of square non-negative matrices M_q approaches a sequence of positive matrices of rank 1. We call a product $H(1,q)$ inhomogeneous if its factors $M_1, \ldots, M_q$ are not necessarily all equal to one another. When the matrices M_q are members of an 'ergodic set', and x and y are positive vectors, the projective distance $d(H(1,q)x, H(1,q)y)$ decays at least exponentially fast as q increases. An important condition on an ergodic set is that any product of g members from the set be a matrix in which all elements are (strictly) positive, where g is some fixed positive integer.

1980

[15] R. K. Brayton and C. H. Tong, *Constructive stability and asymptotic stability of dynamical systems*, IEEE Trans. Circuits and Systems **27** (1980), no. 11, 1121–1130, `doi:10.1109/TCS.1980.1084749`. MR 594156. Zbl 0458.93047.

In an earlier paper, the authors presented an algorithm for constructing a Lyapunov function for a dynamical system. In this paper, we present theorems which allow the algorithm to be used in proving the asymptotic stability of dynamical systems, both difference and differential equations. The notion of an asymptotically stable set of matrices is introduced, and is shown to be a sufficient condition for the algorithm's termination in a finite number of steps. The instability stopping criterion is strengthened and the efficiency of the algorithm is improved in a number of ways. We investigate the tightness of our method by applying it to two-dimensional systems for which necessary and sufficient conditions for stability are known.

1982

[16] P. S. Guinand, *On quasinilpotent semigroups of operators*, Proc. Amer. Math. Soc. **86** (1982), no. 3, 485–486, `doi:10.2307/2044453`. MR 671220. Zbl 0503.47038.

A pair of operators A, B is constructed such that the semigroup generated by them consists of operators which are nilpotent of index 3. The sum $A + B$, however, is not quasinilpotent.

As noted in [135] this implies that the joint spectral radius of the set $\{A, B\}$ is strictly greater than zero, while the generalized spectral radius of the same set of operators is equal to zero. Thus, in this case the Berger-Wang theorem [42] is not valid.

1983

[17] A. F. Kleptsyn, V. S. Kozyakin, M. A. Krasnosel'skij, and N. A. Kuznetsov, *Effect of small synchronization errors on stability of complex systems. I*, Autom. Remote Control **44** (1983), no. 7, 861–867 (english). Zbl 0539.93067.

A necessary and sufficient condition is given for the asymptotic stability of the origin for n-dimensional autonomous discrete linear systems with small phase lags in switching. Furthermore, a sufficient condition for the origin to be absolutely stable with respect to phase errors is presented.

1984

[18] A. F. Kleptsyn, V. S. Kozyakin, M. A. Krasnosel'skij, and N. A. Kuznetsov, *Effect of small synchronization errors on stability of complex systems. II*, Autom. Remote Control **45** (1984), no. 3, 309–314 (english). Zbl 0618.93057.

[For part I see [17]]. Classes of linear systems with digital elements are considered that are insensitive to synchronization errors.

[19] A. F. Kleptsyn, V. S. Kozyakin, M. A. Krasnosel'skij, and N. A. Kuznetsov, *Effect of small synchronization errors on stability of complex systems. III*, Autom. Remote Control **45** (1984), no. 8, 1014–1018 (english). Zbl 0604.93047.

[For the previous parts see [17, 18]].

Sufficiency criteria are presented for the asymptotic stability of the origin for systems governed by n-dimensional autonomous discrete-time linear equations with desynchronized frequency of switching amongst the state components.

[20] A. F. Kleptsyn, V. S. Kozyakin, M. A. Krasnosel'skij, and N. A. Kuznetsov, *Stability of desynchronized systems*, Sov. Phys., Dokl. **29** (1984), 92–94 (english). MR 734942. Zbl 0593.93046.

The concept of phase and frequency desynchronized systems is introduced. These systems arise in cases when their state variables are switched independently (by using independent functional devices, desynchronized elements such as extrapolators, memory elements etc.). For such systems asymptotic stability theorems are produced.

[21] A. F. Kleptsyn, M. A. Krasnosel'skiĭ, N. A. Kuznetsov, and V. S. Kozjakin, *Desynchronization of linear systems*, Math. Comput. Simulation **26** (1984), no. 5, 423–431, `doi:10.1016/0378-4754(84)90106-X`. MR 762920. Zbl 0552.93051.

The studies of dynamics inherent in control systems which incorporate sampled data elements such as extrapolators, keys, memory elements, etc. are usually reduced to the analysis of difference equations. However, this approach tends to disregard the inevitable small desynchronization of times when sampled elements are connected. In some systems this desynchronization does not influence the stability; otheres may be destabilized by any infinitesimal desynchronization; finally, some unstable synchronized systems can become asymptotically stable following the introduction of any infinitesimal desynchronization. These facts account for many phenomena observed in the engineering practice that looked enigmatic when the mathematical models ignored small desynchronization. In designing the controllers the stability can be achieved by introduction of lags into the system.

The paper investigates linear discrete-time (sampled-data) systems with respect to stability. The systems are assumed to be desynchronized in time, i.e. the time instants for sampling the different state variables are incommensurable. Two types of desynchronization are distinguished: a phase one and a frequency one. Whereas phase-desynchronization describes sampling instants t_i to be $t_i = nh + \tau_i$, frequency-desynchronization describes them as $t_i = n(h + \delta_i)$. Various theorems for determining the stability of such systems are given without proofs, which will be published in another paper. It is appreciable that six examples illustrate the application of the results. Here it turns out, that a system may be stable if it is synchronized and may become unstable if small frequency shifts are introduced: If the frequency shifts become larger and larger, it is interesting to observe stability for the same system again.

1985

[22] A. F. Kleptsyn, *Stability of desynchronized complex systems of a special type*, Avtomat. i Telemekh. (1985), no. 4, 169–171. MR 807476.

1986

[23] B. Lubachevsky and D. Mitra, *A chaotic asynchronous algorithm for computing the fixed point of a nonnegative matrix of unit spectral radius*, J. Assoc. Comput. Mach. **33** (1986), no. 1, 130–150, `doi:10.1145/4904.4801`. MR 820102. Zbl 0641.65033.

One of the major problems in parallel computations is the communication between processors and their synchronization. The algorithm for the computation of a positive eigenvector to the unit eigenvalue of a positive, irreducible matrix of unit spectral radius described in this paper makes only mild assumptions with regard to the synchronization of the processors. The number and speed of the processors may vary within certain bounds. Geometric rate of convergence in a projective metric is demonstrated. Examples and counter examples which violate some of the assumptions and are not convergent are included, as well as numerical experiments which show the possible speed-up factor.

Given a nonnegative, irreducible matrix P of spectral radius unity, there exists a positive vector $\mathbf{r}$ such that $\mathbf{r} = \mathbf{r}P$. If P also happens to be stochastic, then $\mathbf{r}$ gives the stationary distribution of the Markov chain that has state-transition probabilities given by the elements of P. This paper gives an algorithm for computing $\mathbf{r}$ that is particularly well suited for parallel processing. The main attraction of our algorithm is that the timing and sequencing restrictions on individual processors are almost entirely eliminated and, consequently, the necessary coordination between processors is negligible and the enforced idle time is also negligible. Under certain mild and easily satisfied restrictions on P and on the implementation of the algorithm, $x(\cdot)$, the vectors of computed values are proved to converge to within a positive, finite constant of proportionality of $\mathbf{r}$. It is also proved that a natural measure of the projective distance of $x(\cdot)$ from $\mathbf{r}$ vanishes geometrically fast, and at a rate for which a lower bound is given. We have conducted extensive experiments on random matrices P, and the results show that the improvement over the parallel implementation of the synchronous version of the algorithm is substantial, sometimes exceeding the synchronization penalty to which the latter is always subject.

[24] G. Szép, *Simultaneous triangularization of projector matrices*, Acta Math. Hungar. **48** (1986), no. 3-4, 285–288, `doi:10.1007/BF01951353`. MR 861844. Zbl 0613.15009.

It is known that every matrix can be transformed into upper triangular form with unitary transformation. Now consider the problem of simultaneous triangularization of two matrices. Considering the results mentioned by H. Shapiro it seems to be useful to find simple necessary and sufficient conditions which can be easily verified for special matrices to be simultaneously triangularisable with unitary transformation.

Theorem. *Assume that C and D are projector matrices with complex elements. Then they can be simultaneously transformed into upper triangular form with unitary transformation if and only if $CD - DC$ is nilpotent.*

1988

[25] Ye. A. Asarin, V. S. Kozyakin, M. A. Krasnosel'skiĭ, N. A. Kuznetsov, and A. V. Pokrovskiĭ, *On some new types of mathematical models of complex systems*, Modelling and adaptive control (Sopron, 1986), Lecture Notes in Control and Inform. Sci., vol. 105, Springer, Berlin, 1988, pp. 10–26, `doi:10.1007/BFb0043174`. MR 958694. Zbl 0648.93025.

The paper is aimed at consideration of two new models whose study has just begun. Desynchronized linear models are introduced as discrete linear models with the state coordinates changing at different times. Desynchronization is suggested as the easiest way to attain stability for some systems. Moreover, the limit hysteresis nonlinearities are introduced and the averaging principle is studied.

[26] N. E. Barabanov, *Lyapunov indicator of discrete inclusions. I*, Autom. Remote Control **49** (1988), no. 2, 152–157. MR 940263. Zbl 0665.93043.

An equivalence is proved between the negativity of the, Lyapunov indicator corresponding to a discrete inclusion defined by $F(y) = \{Ay,\, A \in \mathcal{U}\}$, a polyhedral set:

$$\mathcal{U} = \{A + \sum_{i=1}^{m} b_i \nu_i c_i^T, \quad 0 \le \nu_i \le \mu_i, \quad b_i, c_i \in \mathbb{R}^n\},$$

and the absolute stability of the discrete control system:

$$x_{k+1} = Ax_k + \sum_{i=1}^{m} b_i \xi_k^i, \quad \sigma_k^i = c_i^T x_k, \quad \xi_k^i = \phi_i(\sigma_k^i, k)$$

in a class of nonlinearities that satisfy the conditions $0 \le \phi_i(\sigma, k)\sigma \le \mu_i \sigma^2$ for any $\sigma \in R$, $i = 1, \ldots, m$, $k = 0, 1, 2, \ldots$. It is also shown that the existence of an unbounded solution of a nonsingular inclusion is equivalent to a positive Lyapunov indicator.

[27] N. E. Barabanov, *The Lyapunov indicator of discrete inclusions. II*, Autom. Remote Control **49** (1988), no. 3, 283–287. MR 943889. Zbl 0665.93044.

The concept of conjugate inclusion is formulated and its properties are studied. The problem of determination of the Lyapunov indicator for an original inclusion is reduced to the analysis of a conjugate autonomous system of discrete equations.

[28] N. E. Barabanov, *The Lyapunov indicator of discrete inclusions. III*, Autom. Remote Control **49** (1988), no. 5, 558–565. MR 952665. Zbl 0665.93045.

Certain algebraic formulas characterizing the Lyapunov indicator ρ of a discrete inclusion are presented. For $\rho < 0$ it is shown that the inclusion state space can be embedded in a space of larger dimension in which the cubic norm decreases along any solution of the inclusion that corresponds to the original one. A similar necessary and sufficient condition is derived for the case $\rho \ge 0$ too. An algorithm which allows to determine the sign of ρ in a finite number of steps is also formulated.

1989

[29] D. P. Bertsekas and J. N. Tsitsiklis, *Parallel and distributed computation. Numerical methods*, Prentice Hall, Englewood Cliffs. NJ, 1989. MR 3587745. Zbl 0743.65107.

Parallel processing has confirmed its usefulness already by solving a number of important problems of numerical mathematics. From advent of parallel computers these have been used for a broad variety of large-sized and time demanding applications. Model numerical problems serve in benchmarks to test every new parallel computer system appearing on the market.

From a mathematical point of view, the monograph deals with analysis, design and implementation of parallel numerical algorithms. On a space of more than one hundred pages an adequate introductory knowledge about general aspects of parallel processing is given. The kernel of the book is structured into two main parts. The first part is devoted to synchronous algorithms while the second one is oriented towards methods which are based on the asynchronous execution principle.

There are five chapters which describe parallel synchronous algorithms. A detailed attention is paid to solving linear systems of equations with general and special matrices in the first chapter of them. The next chapter treats nonlinear problems. The synchronous part concludes chapters for the shortest path problem, dynamic programming and network flow analysis.

A class of totally asynchronous methods with some theoretical formulations is the subject of the first chapter in the asynchronous part of the book. The subsequent chapter is devoted to partially asynchronous algorithms, i.e. when some amount of synchronization is introduced in the algorithm. The final chapter of the monograph presents a design of an asynchronous network of processors for realization of a general type of parallel algorithms.

It is to recommend the book to those readers who deal seriously with solving numerical problems on advanced computers with parallel architecture. It is not just a “cooking book” containing algorithmic recipes but there can be found enough from the theory of parallel numerics itself.

[30] H. Cohn, *Products of stochastic matrices and applications*, Internat. J. Math. Math. Sci. **12** (1989), no. 2, 209–233, `doi:10.1155/S0161171289000268`. MR 994904. Zbl 0673.15010.

The paper deals with aspects of the limit behaviour of products of nonidentical finite or countable stochastic matrices (P_n). Applications are given to nonhomogeneous Markov models as positive chains, some classes of finite chains considered by Doeblin and weakly ergodic chains.

This paper is a streamlined survey of the literature of non-homogeneous Markov chains from the standpoint of tail σ-fields. The original papers were in the main the author's own [Math. Proc. Camb. Phil. Soc. 92, 527–534 (1982); Adv. Appl. Prob. 8, 502–516 (1976) and 9, 542–552 (1977) and 13, 388–401 (1981)].

1990

[31] E. A. Asarin, M. A. Krasnoselskii, V. S. Kozyakin, and N. A. Kuznetsov, *On modelling systems with non-synchronously operating impulse elements*, Math. Comput. Model. **14** (1990), no. C, 70–73, `doi:10.1016/0895-7177(90)90149-H`. Zbl 0744.93082.

Systems with impulse elements operating by some reason non-synchronously are often used in control science, technique, economics, ecology etc. To analyze such desynchronized systems one needs new constructions. Some results connected with new models and computer methods of stability analysis are set forth. The methods of stability analysis of synchronized systems are well developed. As a computer experiment has demonstrated all the conceivable effects of synchronized systems loosing, gaining or conserving stability are possible. So desynchronized systems require the development of special methods of stability analysis. In some simple cases the conditions of stability can be expressed by formulae. In other situations numerical methods of stability analysis of desynchronized systems are to be developed. Three situations of this kind are discussed in detail. 1. Phase and frequency desynchronized systems, i.e. linear systems with periodically switched components are considered. 2. It is discussed, what is to be done if a system is sensitive to desynchronization (looses stability under desynchronization). A method to do away with such a sensitivity is proposed. 3. Stochastically desynchronized systems are considered.

[32] V. S. Kozyakin, *Absolute stability of systems with asynchronous sampled-data elements*, Autom. Remote Control **51** (1990), no. 10, 1349–1355 (english). MR 1088520. Zbl 0737.93055.

The author considers the problem of asynchronous operation of elements in sampled-data systems. The concepts of absolute neutral and asymptotic stability with respect to different asynchronous operation classes are introduced and results characterizing such properties of asynchronous systems are presented.

[33] V. S. Kozyakin, *Algebraic unsolvability of problem of absolute stability of desynchronized systems*, Autom. Remote Control **51** (1990), no. 6, 754–759 (english). MR 1071607. Zbl 0737.93056.

In complex control systems containing sampled-data elements, it is possible that these elements operate asynchronously. In some cases asynchronous character of operation of sampled-data elements does not influence stability of system. In other cases any small desynchronization of the updating moments of sampled-data elements leads to dramatic changes of dynamics of a control system, and the system loses stability. Last years there is begun intensive studying of the effects connected with asynchronous operation of control systems; both necessary, and sufficient stability conditions for various classes of asynchronous systems were obtained. At the same time no one succeed in finding general, effectively verified criteria of stability of asynchronous systems, similar to known for synchronous systems. The problem on stability of linear asynchronous systems has appeared more difficult than the problem on stability of synchronous systems. In the paper, attempt of formal explanation of complexity of the stability analysis problem for linear asynchronous systems is undertaken. It is shown that there are no criteria of absolute stability of linear asynchronous systems consisting of a finite number of arithmetic operations.

[34] V. S. Kozyakin, *Stability analysis of asynchronous systems by methods of symbolic dynamics*, Sov. Phys., Dokl. **35** (1990), no. 3, 218–220. MR 1075682. Zbl 0713.93045.

The dynamics of autonomous control systems with sampled-data elements is described, by the equations

$$x_i(T^{ij}+0) = f_i[x_1(T^{ij}-0), \dots, x_k(T^{ij}-0)], \quad i = 1, \dots, k,$$

where each vector function $x_i(t)$ with values in R_{n_i} is constant on the intervals $T^{ij} < t < T^{ij+1}$. In the classical theory of sampled-data systems, the switching times T^{ij} are the same for all components x_i, i.e., $T^{1j} = T^{2j} = \cdots = T^{kj}$. A rich arsenal of methods for the analysis of the dynamics of the systems under consideration has been developed for this case.

An important class are phase-frequency asynchronous systems with switching times $T^{ij} = \phi_i + j\omega_i$. *A. F. Kleptsyn* [22] and *A. F. Kleptsyn*, *M. A. Krasnosel'skij*, *N. A. Kuznetsov* and *V. S. Kozyakin* [21] noted that the stability of these systems is closely linked with the properties of sequences of symbols called shift texts. This observation has led to important theorems on stability of linear two-component phase-frequency asynchronous control systems.

[35] V. S. Kozyakin, *Stability of phase-frequency desynchronized systems under a perturbation of the switching instants of the components*, Autom. Remote Control **51** (1990), no. 8, 1034–1040 (english). MR 1080599. Zbl 0735.93067.

The paper studies a control problem for systems with discrete elements which has found little attention in the past. Two distance subsystems are assumed to exchange information periodically. This is done non-synchronously, because such subsystem is assumed to have its individual periodicity. Asymptotic stability of the zero equilibrium position of the overall system is investigated in presence of small perturbations of the periods and the phases of switching of the subsystems.

[36] H. Radjavi, *On reducibility of semigroups of compact operators*, Indiana Univ. Math. J. **39** (1990), no. 2, 499–515, `doi:10.1512/iumj.1990.39.39028`. MR 1089051. Zbl 0691.47034.

A multiplicative semigroup $\mathcal{S}$ of compact operators on a Hilbert space is said to be reducible if its members have a common nontrivial invariant subspace. Sufficient conditions for reducibility are given. One sample corollary is that if 1 is the unique element of largest modulus in the spectrum of every member of $\mathcal{S}$, then $\mathcal{S}$ is reducible; further information is obtained about the lattice of invariant subspaces of $\mathcal{S}$. Another corollary concerns the case in which the operators in $\mathcal{S}$ have matrices with nonnegative entries: if every member of such an $\mathcal{S}$ has spectral radius 1, then $\mathcal{S}$ is reducible. Examples of simple irreducible semigroups are also presented.

1991

[37] I. Daubechies and J. C. Lagarias, *Two-scale difference equations. I. Existence and global regularity of solutions*, SIAM J. Math. Anal. **22** (1991), no. 5, 1388–1410, `doi:10.1137/0522089`. MR 1112515. Zbl 0763.42018.

A two-scale difference equation is a functional equation of the form $f(x) = \sum_{n=0}^{N} c_n f(\alpha x - \beta_n)$, where $\alpha > 1$ and $\beta_0 < \beta_1 < \cdots < \beta_n$, are real constants, and c_n are complex constants. Solutions of such equations arise in spline theory, in interpolation schemes for constructing curves, in constructing wavelets of compact support, in constructing fractals, and in probability theory. This paper studies the existence and uniqueness of L^1-solutions to such equations. In particular, it characterizes L^1-solutions having compact support. A time-domain method is introduced for studying the special case of such equations where $\{\alpha, \beta_0, \dots, \beta_n\}$ are integers, which are called lattice two-scale difference equations. It is shown that if a lattice two-scale difference equation has a compactly supported solution in $C^m(\mathbb{R})$, then $m < (\beta_n - \beta_0)/(\alpha - 1) - 1$.

Two examples are considered: the de Rham continuous nowhere-differentiable function [communicated by *Y. Meyer* (1987)] and the Lagrange interpolation functions of Deslauriers and Dubuc [*G. Deslauriers* and *S. Dubuc*, Constructive Approximation 5, No. 1, 49–68 (1989)].

[38] D. Hadwin, E. Nordgren, M. Radjabalipour, H. Radjavi, and P. Rosenthal, *On simultaneous triangularization of collections of operators*, Houston J. Math. **17** (1991), no. 4, 581–602, URL `https://www.math.uh.edu/~hjm/v017n4/0581HADWIN.pdf`. MR 1147275. Zbl 0784.47032.

Let G be a Banach algebra of operators and $\operatorname{Rad} G$ the Jacobson radical of G. The authors show that if $G/\operatorname{Rad} G$ is commutative, weakly closed and contains a maximal Abelian self-adjoint algebra, then it is triangularizable. They also show that a direct integral of weakly closed algebras is hypertriangularizable if and only if almost every of the algebras is hypertriangularizable. The analogous result for triangularizability is false. Theorems on the existence of invariant subspaces yield theorems on triangularizability. They show that the problem of existence of invariant subspaces for algebras that contain a bilateral and an injective backward bilateral weighted shift is related to the periodicity of the weights of the weighted shift.

[39] V. S. Kozyakin, *Perturbation of linear asynchronous systems*, Sov. Phys., Dokl. **36** (1991), no. 1, 16–17. MR 1102773. Zbl 0752.93064.

A system W is described by its state space $X = \{x = (x_1, x_2, \ldots, x_N)^T : x_i \in \mathbb{R}^{n_i}\}$ and dynamics $x(0) = 0$, $x(n+1) = A_{w(n)}x(n) + F(n)$, $F(n) \in X_{w(n)}$, $n = 0, 1, \ldots$, where $w = \{w(n)\}_{n=1}^{\infty}$ is a sequence of subsets $w(n) \subseteq \{1, \ldots, N\}$. Here $A = [a_{ij}]_{i,j=1}^{N}$ is a fixed block matrix, $A_{w(n)}$ is obtained from A by replacing its block rows corresponding to indices $i \notin w(n)$ with the corresponding rows of the identity matrix, and

$$X_{w(n)} = \{(x_1, \ldots, x_n)^T \in X : \quad x_i = 0 \quad \text{if} \quad i \notin w(n)\}.$$

The system W with the matrix A is called Perron absolutely stable if there is $\beta < \infty$ such that for any sequence w of subsets in $\{1, \ldots, N\}$ and any choice of $F(n) \in X_{w(n)}$ satisfying $\|F(n)\| \leq 1$, the state $x(n)$ satisfies the inequality $\|x(n)\| \leq \beta$ for all n. Perron absolutely stability is characterized in terms of existence of a norm in X with special properties.

[40] V. S. Kozyakin, *Stability of linear desynchronized systems with unsymmetric matrices*, Autom. Remote Control **52** (1991), no. 7, 928–933 (english). MR 1139401. Zbl 0744.93083.

A new criterion of asymptotic stability is established for linear desynchronized systems. We show that the problem of preserving absolute asymptotic stability in the presence of small matrix perturbation in linear desynchronized systems is equivalent to the problem of stability in the presence of persistent perturbations.

1992

[41] E. A. Asarin, V. S. Kozyakin, M. A. Krasnosel'skiĭ, and N. A. Kuznetsov, *Analiz ustoichivosti rassinkhronizovannykh diskretnykh sistem*, Nauka, Moscow, 1992, URL `http://eqworld.ipmnet.ru/ru/library/books/AsarinKozyakinKrasnoselskijKuznecov1992ru.pdf`, in Russian, English title: "Stability analysis of desynchronized (asynchronous) discrete-time systems". MR 1693324.

Control systems with asynchronous data exchange are considered. The stability theory of such systems (for various desynchronization classes) is presented. Some results may be considered as a new chapter of the theory of matrices. The correctness of the proposed models is also investigated. Many open problems are set forth. For engineers, mathematicians and control theorists. In Russian.

[42] M. A. Berger and Y. Wang, *Bounded semigroups of matrices*, Linear Algebra Appl. **166** (1992), 21–27, `doi:10.1016/0024-3795(92)90267-E`. MR 1152485. Zbl 0818.15006.

In this note two conjectures of *I. Daubechies* and *J. C. Lagarias* [43] are proved. The first asserts that if Σ is a bounded set of matrices such that all left infinite products converge, then Σ generates a bounded semigroup. The second asserts the equality of two differently defined joint spectral radii for a bounded set of matrices. One definition involves the conventional spectral radius, and one involves the operator norm.

[43] I. Daubechies and J. C. Lagarias, *Sets of matrices all infinite products of which converge*, Linear Algebra Appl. **161** (1992), 227–263, `doi:10.1016/0024-3795(92)90012-Y`. MR 1142737. Zbl 0746.15015.

An infinite product $\prod_{i=1}^{\infty} \mathsf{M}_i$ of matrices converges (on the right) if $\lim_{i\to\infty} \mathsf{M}_1 \cdots \mathsf{M}_i$ exists. A set $\Sigma = \{\mathsf{A}_i : i \geq 1\}$ of $n \times n$ matrices is called an *RCP set* (right-convergent product set) if all infinite products with each element drawn from Σ converge. Such sets of matrices arise in constructing self-similar objects like von Koch's snowflake curve, in various interpolation schemes, in constructing wavelets of compact support, and in studying nonhomogeneous Markov chains. This paper gives necessary conditions and also some sufficient conditions for a set Σ to be an RCP set. These are conditions on the eigenvalues and left eigenspaces of matrices in Σ and finite products of these matrices. Necessary and sufficient conditions are given for a finite set Σ to be an RCP set having a limit function $\mathsf{M}_\Sigma(d) = \prod_{i=1}^{\infty} \mathsf{A}_{d_i}$, where $\mathsf{d} = (d_1, \ldots, d_n, \ldots)$, which is a continuous function on the space of all sequences d with the sequence topology. Finite RCP sets of column-stochastic matrices are completely characterized. Some results are given on the problem of algorithmically deciding if a given set Σ is an RCP set.

[44] I. Daubechies and J. C. Lagarias, *Two-scale difference equations. II. Local regularity, infinite products of matrices and fractals*, SIAM J. Math. Anal. **23** (1992), no. 4, 1031–1079, `doi:10.1137/0523059`. MR 1166574. Zbl 0788.42013.

This paper studies solutions of the functional equation

$$f(x) = \sum_{n=0}^{N} c_n f(kx - n),$$

where $k \geq 2$ is an integer, and $\sum_{n=0}^{N} c_n = k$. Part I showed that equations of this type have at most one L^1-solution up to a multiplicative constant, which necessarily has compact support in $[0, N/k - 1]$. This paper gives a time-domain representation for such a function $f(x)$ (if it exists) in terms of infinite products of matrices (that vary as x varies). Sufficient conditions are given on $\{c_n\}$ for a continuous nonzero L^1-solution to exist. Additional conditions sufficient to guarantee $f \in C^r$ are also given. The infinite matrix product representations is used to bound from below the degree of regularity of such an L^1-solution and to estimate the Hölder exponent of continuity of the highest-order well-defined derivative of $f(x)$. Such solutions $f(x)$ are often smoother at some points than others. For certain $f(x)$ a hierarchy of fractal sets in $\mathbb{R}$ corresponding to different Hölder exponents of continuity for $f(x)$ is described.

[45] V. S. Kozyakin and A. V. Pokrovskij, *The role of controllability type properties in the study of asynchronous dynamic systems*, Sov. Phys., Dokl. **37** (1992), no. 5, 213–215. MR 1198568. Zbl 0790.93018.

Analysis of the stability of so-called synchronous systems has recently attracted considerable attention. In this paper we study the geometric properties of the family of modes for synchronous systems that have special properties of Kalman-controllability type. We show that relations between the properties of stability, asymptotic stability, and instability in the indicated family are analogous to the relations between the corresponding properties for the family of solutions of a linear differential equation or difference equation in the case of simplicity of the leading eigenvalues of the matrices that define these equations. The results stated below are useful for analysis of classical problems on absolute stability. Our approach is similar to the principle of the absence of bounded modes in the problem of absolute stability.

[46] A. Leizarowitz, *On infinite products of stochastic matrices*, Linear Algebra Appl. **168** (1992), 189–219, `doi:10.1016/0024-3795(92)90294-K`. MR 1154468. Zbl 0748.15023.

For a given sequence $\{Q_0, Q_1, \ldots\}$ of $n \times n$ stochastic matrices let $\{T_0, T_1, \ldots\}$ be a sequence of products taken in a certain order of multiplication. One says that weak ergodicity obtains for this order of multiplication if $(T_N)_{ik} - (T_N)_{jk} \to 0$ as $N \to \infty$ for every i, j and k. One says that strong ergodicity obtains for this order of multiplication if weak ergodicity obtains and $(T_N)_{ij}$ converges to a limit as $N \to \infty$ for every i and j.

Conditions are established for weak ergodicity of products taken in an arbitrary order, and strong ergodicity of the backward products $M_N = Q_N Q_{N-1} \cdots Q_1 Q_0$. Strong ergodicity is studied for products taken in an arbitrary order.

1993

[47] E. Asarin, P. Diamond, I. Fomenko, and A. Pokrovskii, *Chaotic phenomena in desynchronized systems and stability analysis*, Comput. Math. Appl. **25** (1993), no. 1, 81–87, `doi:10.1016/0898-1221(93)90214-G`. MR 1192675. Zbl 0774.93057.

Complex systems tend to be desynchronized, as part and parcel of their internal organization and internal connections. One way that this may arise is from quite small mismatching of operating times of system components. On the other hand, lack of synchronization can be built into the system. For example, “chaotic” iterations and asynchronous algorithms are exploited in parallel processing. Too, a system may be susceptible of change by numerous factors at different, asynchronous times. In all cases, it is important to understand the effect that desynchronization can have on the stability of the system and the convergence properties of processes.

We consider the symbolic dynamics of desynchronized switching times and extract a numerical quantity whose values determine the stability characteristics of the system. An important role in the proof is played by Hilbert’s projective metric.

1994

[48] R. Bru, L. Elsner, and M. Neumann, *Convergence of infinite products of matrices and inner-outer iteration schemes*, Electron. Trans. Numer. Anal. **2** (1994), no. Dec., 183–193 (electronic), URL `http://etna.mcs.kent.edu/volumes/1993-2000/vol2/abstract.php?vol=2&pages=183-193`. MR 1308895. Zbl 0852.65035.

We develop conditions under which a product $\prod_{i=0}^{\infty} T_i$ of matrices chosen from a possibly infinite set of matrices $\mathcal{S} = \{T_j \mid j \in J\}$ converges. We obtain the following conditions which are sufficient for the convergence of the product: There exists a vector norm such that all matrices in $\mathcal{S}$ are nonexpansive with respect to this norm and there exists a subsequence $\{i_k\}_{k=0}^{\infty}$ of the sequence of the nonnegative integers such that the corresponding sequence of operators $\{T_{i_k}\}_{k=0}^{\infty}$ converges to an operator which is paracontracting with respect to this norm. We deduce the continuity of the limit of the product of matrices as a function of the sequences $\{i_k\}_{k=0}^{\infty}$. But more importantly, we apply our results to the question of the convergence of inner-outer iteration schemes for solving singular consistent linear systems of equations, where the outer splitting is regular and the inner splitting is weakly regular.

[49] V. S. Kozyakin, N. A. Kuznetsov, and A. V. Pokrovskii, *Transients in quasi–controllable systems. Overshooting, stability and instability*, Control Engineering Practice **2** (1994), no. 6, 1080, `doi:10.1016/0967-0661(94)91809-0`, `arXiv:0909.4372`.

Families of regimes for control systems are studied possessing the so called quasi-controllability property that is similar to the Kalman controllability property. A new approach is proposed to estimate the degree of transients overshooting in quasi-controllable systems. This approach is conceptually related with the principle of bounded regimes absence in the absolute stability problem. Its essence is in obtaining of constructive a priori bounds for degree of overshooting in terms of the so called quasi-controllability measure. It is shown that relations between stability, asymptotic stability and instability for quasi-controllable systems are similar to those for systems described by linear differential or difference equations in the case when the leading eigenvalue of the corresponding matrix is simple. The results are applicable for analysis of transients, classical absolute stability problem, stability problem for desynchronized systems and so on.

[50] R. Lima and M. Rahibe, *Exact Lyapunov exponent for infinite products of random matrices*, J. Phys. A **27** (1994), no. 10, 3427–3437, `doi:10.1088/0305-4470/27/10/019`, `arXiv:chao-dyn/9407013`. MR 1282183. Zbl 0830.15018.

Despite significant work since the original paper by H. Furstenberg (1963), explicit formulae for Lyapunov exponents of infinite products of random matrices are available only in very few cases. In this work, we get a rigorous explicit formula for the Lyapunov exponent for some binary infinite products of random 2×2 real matrices. All these products are constructed using only two types of matrices, A and B, which are chosen according to a stochastic process. The matrix A is singular,

namely, its determinant is zero. This formula is derived by using a particular decomposition for the matrix B, which allows us to write the Lyapunov exponent as a sum of convergent series. The key point is the computation of all integer powers of B, which is achieved by a suitable change of frame. The computation then follows by looking at each of the special types of B (hyperbolic, parabolic and elliptic). Finally, we show with an example, that the Lyapunov exponent is a discontinuous function of the given parameter.

[51] D. P. Stanford and J. M. Urbano, *Some convergence properties of matrix sets*, SIAM J. Matrix Anal. Appl. **15** (1994), no. 4, 1132–1140, doi:10.1137/S0895479892228213. MR 1293908. Zbl 0807.15014.

A set $\mathcal{A} = \{A_j : j \in J\}$ of $n \times n$ matrices is pointwise convergent provided each n-vector x can be steered to zero by iterated multiplication by matrices in $\mathcal{A}$. The convergence is uniform if the sequence of multipliers may be chosen independently of x. This paper discusses conditions related to convergence for sets of diagonal, triangular, and general matrices, real and complex. It generalizes known conditions for convergence of a single matrix and characterizes convergence of a set of diagonal matrices in terms of semipositivity of a matrix derived from the set.

1995

[52] N. E. Barabanov, *Stability of inclusions of linear type*, American Control Conference, Proceedings of the 1995, vol. 5, June 1995, pp. 3366–3370, doi:10.1109/ACC.1995.532231.

Introduced several new concepts for the differential inclusions of linear type. The main result concerns the existence of an extremal norm for nonsingular inclusions. It allows the author to present the numerical algorithm for calculating the Lyapunov index up to arbitrary accuracy. For 3-ordered automatic control systems the author describes all extremal solutions and in such a way the author solves the problem of absolute stability in terms of coefficients of transfer functions

[53] M. A. Berger and Y. Wang, *The finiteness conjecture for the generalized spectral radius of a set of matrices*, Linear Algebra Appl. **214** (1995), 17–42, doi:10.1016/0024-3795(93)00052-2. MR 1311628. Zbl 0818.15007.

Let $\Sigma = \{A_1, A_2, \ldots\}$ be a set of $n \times n$ matrices, $\rho(A_j)$ the spectral radius of A_j, $\overline{\rho}_k(\Sigma) = \sup \rho(A_1 \cdots A_k)$, $\overline{\rho}(\Sigma) = \lim_{k\to\infty} \sup \overline{\rho}_k(\Sigma)^{1/k}$ the "generalized spectral radius" (cf. *I. Doubechies* and *J. C. Lagarias* [53]) $\widehat{\rho}_k(\Sigma) = \sup \|A_1 \cdots A_k\|$, $\widehat{\rho}(\Sigma) = \lim_{k\to\infty} \sup \widehat{\rho}_k(\Sigma)^{1/k}$ the "joint spectral radius" (cf. *G.-C. Rota* and *G. Strang* [3]). Then $\widehat{\rho} = \overline{\rho}$ for a finite set Σ (cf. *M. A. Berger* and *Y. Wang* [42]). Finiteness conjecture (FC): If Σ is finite, there is a k such that $\widehat{\rho}(\Sigma) = \overline{\rho}(\Sigma) = \overline{\rho}_k(\Sigma)^{1/k}$. Normed finiteness conjecture (NFC) for a given operator norm: If Σ is finite and $\|A_j\|_{\mathrm{op}} \leq 1$, then either $\widehat{\rho}(\Sigma) < 1$ or $\widehat{\rho}(\Sigma) = \overline{\rho}(\Sigma) = \overline{\rho}_k(\Sigma)^{1/k} = 1$ for some k.

Results in this paper: FC is true iff NFC is true for all operator norms. NFC is proved for a relatively large class of operator norms. For polytope norms and for the Euclidean norm, explicit upper bounds are given for the least k having $\overline{\rho}(\Sigma) = \overline{\rho}_k(\Sigma)^{1/k}$, implying upper bounds for generalized critical exponents for these norms.

[54] R. Bhatia and T. Bhattacharyya, *On the joint spectral radius of commuting matrices*, Studia Math. **114** (1995), no. 1, 29–38, URL http://matwbn.icm.edu.pl/ksiazki/sm/sm114/sm11413.pdf. MR 1330215. Zbl 0830.47002.

For a commuting n-tuple of matrices we introduce the notion of a joint spectral radius with respect to the p-norm and prove a spectral radius formula.

[55] L. Elsner, *The generalized spectral-radius theorem: an analytic-geometric proof*, Linear Algebra Appl. **220** (1995), 151–159, proceedings of the Workshop "Nonnegative Matrices, Applications and Generalizations" and the Eighth Haifa Matrix Theory Conference (Haifa, 1993), doi:10.1016/0024-3795(93)00320-Y. MR 1334574. Zbl 0828.15006.

Let Σ be a bounded set of complex matrices, $\Sigma^m = \{A_1 \cdots A_m : A_i \in \Sigma\}$. The generalized spectral-radius theorem states that $\rho(\Sigma) = \hat{\rho}(\Sigma)$, where $\rho(\Sigma)$ and $\hat{\rho}(\Sigma)$ are defined as follows:

$\rho(\Sigma) = \limsup \rho_m(\Sigma)^{1/m}$, where $\rho_m(\Sigma) = \sup\{\rho(A) : A \in \Sigma^m\}$ with $\rho(A)$ the spectral radius; $\hat{\rho}(\Sigma) = \limsup \rho_m(\Sigma)^{1/m}$, where $\hat{\rho}_m(\Sigma) = \sup\{\|A\| : A \in \Sigma^m\}$ with $\| \ \|$ any matrix norm.

The theorem in the title was proved for real matrices by *M. A. Berger* and *Y. Wang* [42], using results from the theory of rings. In the present paper the theorem is proved by tools from analysis and geometry (convexity) as well as by some elementary results from the Berger-Wang paper, which is in some ways simpler than the first proof by Berger and Wang.

[56] S. Gaubert, *Performance evaluation of* $(\max, +)$ *automata*, IEEE Trans. Automat. Control **40** (1995), no. 12, 2014–2025, `doi:10.1109/9.478227`. MR 1364950. Zbl 0855.93019.

Automata with multiplicities over the $(\max, +)$ semiring can be used to represent the behavior of timed discrete-event systems. This formalism, which extends both conventional automata and $(\max, +)$ linear representations, covers a class of systems with synchronization phenomena and variable schedules. Performance evaluation is considered in the worst, mean, and optimal cases. A simple algebraic reduction is provided for the worst case. The last two cases are solved for the subclass of deterministic series (recognized by deterministic automata). Deterministic series frequently arise due to the finiteness properties of $(\max, +)$ linear projective semigroups. The mean performance is given by the Kolmogorov equation of a Markov chain. The optimal performance is given by a Hamilton-Jacobi-Bellman equation.

[57] L. Gurvits, *Stability of discrete linear inclusion*, Linear Algebra Appl. **231** (1995), 47–85, `doi:10.1016/0024-3795(95)90006-3`. MR 1361100. Zbl 0845.68067.

Let $M = \{A_i\}$ be a set of linear operators on $\mathbb{R}^n$. The discrete linear inclusion DLI(M) is the set of possible trajectories $(x_i : i \geq 0)$ such that $x_n = A_{i_n} A_{i_{n-1}} \cdots A_{i_1} x_0$ where $x_0 \in \mathbb{R}^n$ and $A_{i_j} \in M$. We study several notions of stability for DLI(M), including absolute asymptotic stability (AAS), which is that all products $A_{i_n} \cdots A_{i_1} \to 0$ as $n \to \infty$. We mainly study the case that M is a finite set. We give criteria for the various forms of stability. Two new approaches are taken: one relates the question of AAS of DLI(M) to formal language theory and finite automata, while the second connects the AAS property to the structure of a Lie algebra associated to the elements of M. More generally, the discrete linear inclusion DLI(M) makes sense for M contained in a Banach algebra $\mathcal{B}$. We prove some results for AAS in this case, and give counterexamples showing that some results valid for finite sets of operators on $\mathbb{R}^n$ are not true for finite sets M in a general Banach algebras $\mathcal{B}$.

In particular, an example is given demonstrating that the Berger-Wang formula [42] need not be true for finite sets of linear operators on an infinite dimensional Hilbert space.

[58] C. Heil and G. Strang, *Continuity of the joint spectral radius: application to wavelets*, Linear algebra for signal processing (Minneapolis, MN, 1992), IMA Vol. Math. Appl., vol. 69, Springer, New York, 1995, pp. 51–61, `doi:10.1007/978-1-4612-4228-4_4`. MR 1351732. Zbl 0823.15009.

The joint spectral radius is the extension to two or more matrices of the (ordinary) spectral radius $\rho(A) = \max |\lambda_i(A)| = \lim \|A_m\|^{1/m}$. The extension allows matrix products $\prod_m$ taken in all orders, so that norms and eigenvalues are difficult to estimate. We show that the limiting process does yield a continuous function of the original matrices – this is their joint spectral radius. Then we describe the construction of wavelets from a dilation equation with coefficients c_k. We connect the continuity of those wavelets to the value of the joint spectral radius of two matrices whose entries are formed from the c_k.

[59] R.-Q. Jia, *Subdivision schemes in L_p spaces*, Adv. Comput. Math. **3** (1995), no. 4, 309–341, `doi:10.1007/BF03028366`. MR 1339166. Zbl 0833.65148.

Subdivision schemes play an important role in computer graphics and wavelet analysis. In this paper we are mainly concerned with convergence of subdivision schemes in L_p spaces ($1 \leq p \leq \infty$). We characterize the L_p-convergence of a subdivision scheme in terms of the p-norm joint spectral radius of two matrices associated with the corresponding mask. We also discuss various properties of the limit function of a subdivision scheme, such as stability, linear independence, and smoothness.

[60] W. E. Longstaff and H. Radjavi, *On permutability and submultiplicativity of spectral radius*, Canad. J. Math. **47** (1995), no. 5, 1007–1022, `doi:10.4153/CJM-1995-053-x`. MR 1350647. Zbl 0844.47003.

Let $r(T)$ denote the spectral radius of the operator T, acting on a complex, Hilbert space H. Let $\mathcal{S}$ be a multiplicative semigroup of operators on H. We say that r is *permutable* on $\mathcal{S}$ if $r(ABC) = r(BAC)$, for every $A, B, C \in \mathcal{S}$. We say that r is submultiplicative on $\mathcal{S}$ if $r(AE) < r(A)r(B)$, for every $A, B \in \mathcal{S}$. It is known that, if r is permutable on $\mathcal{S}$, then it is submultiplicative. We show that the converse holds in each of the following cases: (i) $\mathcal{S}$ consists of compact operators (ii) $\mathcal{S}$ consists of normal operators (iii) $\mathcal{S}$ is generated by orthogonal projections.

[61] M. Maesumi, *Optimum unit ball for joint spectral radius: an example from four-coefficient MRA*, Approximation theory VIII, Vol. 2 (College Station, TX, 1995), Ser. Approx. Decompos., vol. 6, World Sci. Publ., River Edge, NJ, 1995, pp. 267–274, URL `https://www.math.lamar.edu/faculty/maesumi/papers/Optimum.pdf`. MR 1471793. Zbl 0927.42025.

We give the exact value of joint spectral radius for certain matrices by explicitly constructing an optimum unit ball and an operator norm. Our example matrices are associated with 4-coefficient dilation equations which generate a multiresolution analysis. The method proposed here can be used in the search for orthogonal solution of the dilation equation with the highest Hölder exponent. It can also be used to decide if a given product of matrices is optimal, i.e., it satisfies the finiteness conjecture. The optimal ball is generated as the convex hull of action of semigroup of matrices, normalized by their joint spectral radius, on extreme points of a set of unit diameter which is maximal, convex, symmetric, compact and invariant under the optimal product.

[62] P. Rosenthal and A. Sołtysiak, *Formulas for the joint spectral radius of non-commuting Banach algebra elements*, Proc. Amer. Math. Soc. **123** (1995), no. 9, 2705–2708, `doi:10.2307/2160564`. MR 1257123. Zbl 0849.46034.

In 1960, *G.-C. Rota* and *G. Strang* [3] proposed a formula for the joint spectral radius of an n-tuple (or, more generally, any bounded set) of Banach algebra elements. A different formula was investigated by *M. A. Berger* and *Y. Wang* [42]. We consider relations between these formulas and the "geometric spectral radius".

1996

[63] W.-J. Beyn and L. Elsner, *Connecting paracontractivity and convergence of products*, Preprint // Sonderforschungsbereich 343, Diskrete Strukturen in der Mathematik 96-074, Universität Bielefeld, 1996, URL `https://www.math.uni-bielefeld.de/~beyn/AG_Numerik/html/en/preprints/sfb_01_09.html`.

In [43] the LCP-property of a finite set Σ of square complex matrices was introduced and studied by *I. Daubechies* and *J. C. Lagarias.* Σ is an LCP-set if all left infinite products formed from matrices in Σ are convergent. It had been shown earlier that a set Σ paracontracting with respect to a fixed norm is an LCP-set. Here we prove a converse statement If Σ is an LCP-set with a continuous limit function then there exists a norm such that all matrices in Σ are paracontracting with respect to this norm. In addition we introduce the stronger property of l-paracontractivity. It is shown that common l-paracontractivity of a set of matrices has a simple characterization. It turns out that in the above mentioned converse statement the norm can be chosen such that all matrices are l-paracontracting. It is shown that for Σ consisting of two projectors the LCP-property is equivalent to l-paracontractivity, even without requiring continuity.

[64] G. Gripenberg, *Computing the joint spectral radius*, Linear Algebra Appl. **234** (1996), 43–60, `doi:10.1016/0024-3795(94)00082-4`. MR 1368770. Zbl 0863.65017.

This paper presents algorithms for finding an arbitrarily small interval that contains the joint spectral radius of a finite set of matrices. It also presents a numerical criterion for verifying in certain cases that the joint spectral radius is the maximum of the spectral radii of the given matrices. Error bounds are derived for the case where calculations are done with finite precision and the matrices are not known exactly. The algorithms are implemented and applied to estimate Hölder exponents of the orthonormal wavelets ${}_N\phi$ constructed by Daubechies for $3 \le N \le 8$.

[65] V. S. Kozyakin and A. V. Pokrovskii, *Estimates of amplitudes of transient regimes in quasi-controllable discrete systems*, CADSEM Report 96–005, Deakin University, Geelong, Australia, 1996, `arXiv:0908.4138`.

Families of regimes for discrete control systems are studied possessing a special *quasi–controllability property* that is similar to the Kalman controllability property. A new approach is proposed to estimate the amplitudes of transient regimes in quasi–controllable systems. Its essence is in obtaining of constructive a priori bounds for degree of overshooting in terms of the quasi–controllability measure. The results are applicable for analysis of transients, classical absolute stability problem and, especially, for stability problem for desynchronized systems.

[66] M. Maesumi, *An efficient lower bound for the generalized spectral radius of a set of matrices*, Linear Algebra Appl. **240** (1996), 1–7, `doi:10.1016/0024-3795(94)00171-5`. MR 1387282. Zbl 0848.15017.

For a finite collection Σ of $q \times q$ matrices, its, generalized spectral radius $\rho(\Sigma)$ is defined as the lim sup of the maximal n^{th} root of the spectral radius of words of length n from Σ as $n \to \infty$. This concept has applications to wavelets. For fixed n, the naive computation of $\rho_n(\Sigma) = \max_{A\in\Sigma^n} |\rho(A)|^{\frac{1}{n}}$ takes $|\Sigma|^n$ spectral radii evaluations to find the maximum. This paper shortens the process to $|\Sigma|^n/n$ evaluations by observing that cyclically rearranged words from Σ^n have the same characteristic polynomial, hence spectral radius, by eliminating products that are a power of shorter products and by using the Dedekind-Liouville transforms from arithmetic number theory on the MacMahon formula for counting cyclically different words of length n.

[67] V. Yu. Protasov, *The joint spectral radius and invariant sets of linear operators*, Fundam. Prikl. Mat. **2** (1996), no. 1, 205–231, in Russian. MR 1789006. Zbl 0899.47002.

This paper concerns the properties of the joint spectral radius of several linear n-dimensional operators:

$$\widehat{\rho}(A_1,\dots,A_k) = \lim_{m\to\infty} \max_{\sigma} \|A_{\sigma(1)}\cdots A_{\sigma(m)}\|^{\frac{1}{m}}, \quad \sigma : \{1,\dots,m\} \to \{1,\dots,k\}.$$

The theorem of Dranishnikov-Konyagin on the existence of an invariant convex set M for several linear operators is proved. $\mathrm{Conv}(A_1M,\dots,A_kM) = \lambda M$, $\lambda = \widehat{\rho}(A_1,\dots,A_k)$.

The paper concludes with several boundary propositions on the construction of invariant sets some properties of the invariant sets and algorithm of finding the joint spectral radius with estimation of its difficulty.

[68] Y. Saouter, *The mortality of a pair of 2x2 matrices is decidable*, Rapport de recherche RR-2842, INRIA, 1996, URL `https://hal.inria.fr/inria-00073848`.

A pair of matrices is said to be mortal if there is a serie of these matrices for which the product is the null matrix. A recent result have established that the general problem of the mortality of a pair of integral matrices is undecidable. In this article, we prove by using only linear algebra that the mortality of a pair of 2×2 integral matrices is decidable.

1997

[69] W.-J. Beyn and L. Elsner, *Infinite products and paracontracting matrices*, Electron. J. Linear Algebra **2** (1997), 1–8 (electronic), `doi:10.13001/1081-3810.1006`. MR 1443067. Zbl 0893.65019.

Infinite products of matrices arise naturally when chaotic iteration procedures are applied to solve consistent linear systems. The convergence of such products, when each factor belongs to a finite set of matrices Σ, is ensured when they are paracontracting with respect to a fixed norm in the underlying vector space. The paper addresses the reciprocal statement and a general affirmative answer is given if a supplementary hypothesis is satisfied, namely that the set of fixed points for each matrix in Σ is the same for all of them. This hypothesis is shown to be equivalent to the continuity of the function that associates to each sequence of indices varying in the index set of Σ, the corresponding matrix product limit. For the particular case where Σ consists of two projectors, the equivalence is established without the supplementary hypothesis.

[70] V. D. Blondel and J. N. Tsitsiklis, *When is a pair of matrices mortal?*, Inform. Process. Lett. **63** (1997), no. 5, 283–286, `doi:10.1016/S0020-0190(97)00123-3`. MR 1475343. Zbl 1337.68123.

A set of matrices over the integers is said to be k-mortal (with k positive integer) if the zero matrix can be expressed as a product of length k of matrices in the set. The set is said to be mortal if it is k-mortal for some finite k. We show that the problem of deciding whether a pair of 48×48 integer matrices is mortal is undecidable, and that the problem of deciding, for a given k, whether a pair of matrices is k-mortal is NP-complete.

[71] R. Drnovšek, *An irreducible semigroup of idempotents*, Studia Math. **125** (1997), no. 1, 97–99, `doi:10.4064/sm-125-1-97-99`. MR 1455626. Zbl 0886.47005.

A semigroup of bounded idempotents with no nontrivial closed subspaces is constructed. This answers a question which was open for some time.

[72] R. Drnovšek, *On reducibility of semigroups of compact quasinilpotent operators*, Proc. Amer. Math. Soc. **125** (1997), no. 8, 2391–2394, `doi:10.1090/S0002-9939-97-04108-7`. MR 1422865. Zbl 0883.47001.

The following generalization of Lomonosov's invariant subspace theorem is proved:

Theorem. *Let S be a multiplicative semigroup of compact operators on a Banach space such that $\hat{r}(S_1, \ldots, S_n) = 0$ for every finite subset $\{S_1, \ldots, S_n\}$ of S, where $\hat{r}$ denotes the Rota-Strang spectral radius. Then S is reducible.*

This result implies that the following assertions are equivalent:

(A) *For each infinite-dimensional complex Hilbert space $\mathcal{H}$, every semigroup of compact quasinilpotent operators on $\mathcal{H}$ is reducible.*

(B) *For every complex Hilbert space $\mathcal{H}$, for every semigroup of compact quasinilpotent operators on $\mathcal{H}$, and for every finite subset $\{S_1, \ldots, S_n\}$ of S it holds that $\hat{r}(S_1, \ldots, S_n) = 0$.*

The question whether the assertion (A) is true was considered by Nordgren, Radjavi and Rosenthal in 1984, and it seems to be still open.

[73] L. Elsner and S. Friedland, *Norm conditions for convergence of infinite products*, Linear Algebra Appl. **250** (1997), 133–142, `doi:10.1016/0024-3795(95)00423-8`. MR 1420574. Zbl 0868.40001.

A set Σ of $n \times n$ complex-valued matrices is defined to be an LCP set if every infinite left product of members of Σ is convergent. Necessary and sufficient conditions for a set of matrices to be an LCP set have been provided by *I. Daubechies* and *J. C. Lagarias* [43], but their result involves the use of an infinite number of conditions. Straightforward conditions tantamount to the LCP property are given for a two-member set Σ as follows: The set $\Sigma = \{A_1, A_2\}$ is an LCP set if and only if a norm $|\cdot|$ exists on C^n such that, for $i = 1, 2$, (i) $|A_i| \leq 1$, (ii) the intersection of the spectrum of A_i with the unit circle is either empty or $\{1\}$, and (iii) $|A_1 A_2 x| = |x|$ when and only when $A_1 x = A_2 x = x$.

[74] L. Gurvits and L. Rodman, *Convergence of polynomially bounded semigroups of matrices*, SIAM J. Matrix Anal. Appl. **18** (1997), no. 2, 360–368, `doi:10.1137/S089547989528939X`. MR 1437336. Zbl 0876.15013.

A set of $n \times n$ matrices is called pointwise convergent if for each vector x there is an infinite product A_w of matrices from the set with $A_w x \to 0$ as the length of w goes to infinity. It is called uniformly convergent if the factors in A_w do not depend on x. It is shown that if the matrix set generates a polynomially bounded semigroup with $\sup_{|w|=k} |A_w|_2 \leq Ck^p$, then the two notions of convergence coincide. This result is also proved for certain sets of nonlinear maps on finite-dimensional real or complex vector spaces.

[75] R. Howard, *The John ellipsoid theorem*, 1997, URL `https://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.34.3825`.

This is a lecture given in the functional analysis seminar at the University of South Carolina. Contents: 1. Introduction. 2. Proof of the theorem. 3. The case of centrally symmetric convex bodies. 4. Proof of the uniquness of the John ellipsoid. 4.1. Examples of where the inequalities are sharp.

[76] V. Müller, *On the joint spectral radius*, Ann. Polon. Math. **66** (1997), 173–182, URL `https://users.math.cas.cz/~muller/papers.htm`, volume dedicated to the memory of Włodzimierz Mlak. MR 1438337. Zbl 0877.46037.

We prove the ℓ_p-spectral radius formula for n-tuples of commuting Banach algebra elements. This generalizes results of some earlier papers.

[77] M. Omladič and H. Radjavi, *Irreducible semigroups with multiplicative spectral radius*, Linear Algebra Appl. **251** (1997), 59–72, `doi:10.1016/0024-3795(95)00544-7`. MR 1421265. Zbl 0937.47043.

The authors study irreducible multiplicative semigroups of finite-dimensional linear operators with some additional properties. They show that each of the properties – submultiplicativity of the spectral radius, its multiplicativity, or its constancy, as well as a certain property called the Rota condition – implies that every member of such a semigroup is, except for a scalar coefficient, similar to the direct sum of an isometry and a nilpotent operator.

[78] V. Yu. Protasov, *A generalized joint spectral radius. A geometric approach*, Izv. Math. **61** (1997), no. 5, 995–1030, `doi:10.1070/im1997v061n05ABEH000161`. MR 1486700. Zbl 0893.15002.

The properties of the joint spectral radius with an arbitrary exponent $p \in [1, +\infty]$ are investigated for a set of finite-dimensional linear operators $A_1, \dots, A_k$,

$$\widehat{\rho}_p = \lim_{n\to\infty} \left(\frac{1}{k^n} \sum_{\sigma} \|A_{\sigma(1)} \cdots A_{\sigma(n)}\|^p \right)^{\frac{1}{pn}}, \quad p < \infty,$$

$$\widehat{\rho}_\infty = \lim_{n\to\infty} \max_{\sigma} \|A_{\sigma(1)} \cdots A_{\sigma(n)}\|^{\frac{1}{n}},$$

where the summation and maximum extend over all maps

$$\sigma \colon \{1, \dots, n\} \to \{1, \dots, k\}.$$

Using the operation of generalized addition of convex sets, we extend the Dranishnikov-Konyagin theorem on invariant convex bodies, which has hitherto been established only for the case $p = \infty$. The paper concludes with some assertions on the properties of invariant bodies and their relationship to the spectral radius $\widehat{\rho}_p$. The problem of calculating $\widehat{\rho}_p$ for even integers p is reduced to determining the usual spectral radius for an appropriate finite-dimensional operator. For other values of p, a geometric analogue of the method with a pre-assigned accuracy is constructed and its complexity is estimated.

[79] M.-H. Shih, J.-W. Wu, and C.-T. Pang, *Asymptotic stability and generalized Gelfand spectral radius formula*, Linear Algebra Appl. **252** (1997), 61–70, `doi:10.1016/0024-3795(95)00592-7`. MR 1428628. Zbl 0873.15012.

Let $\Sigma \subset \mathbb{C}^{n\times n}$ be a bounded set. Denote by Σ^m the set of all products of m matrices from Σ and by Σ' – the semigroup, generated by Σ. The set Σ is said to be asymptotically stable if there is a constant $\alpha \in (0, 1)$, such that there are bounded neighborhoods $U, V \subset \mathbb{C}^n$ of the origin for which $AV \subset \alpha^m U$ for all $A \in \Sigma^m$, $m = 1, 2, \dots$ (i.e. if there exists a norm $|\cdot|$ in $\mathbb{C}^n$ such that $|A| \leq \alpha < 1$ for all $A \in \Sigma$). It is shown that the asymptotic stability of Σ is equivalent to any of the following three conditions: (i) $\widehat{\rho}(\Sigma) = \limsup_{m\to\infty} (\sup_{A\in\Sigma^m} |A|)^{1/m} < 1$, (ii) $\rho(\Sigma) = \limsup_{m\to\infty} (\sup_{A\in\Sigma^m} \rho(A))^{1/m} < 1$, (iii) there exists $\alpha > 0$, such that $\rho(A) \leq \alpha < 1$ for all $A \in \Sigma'$, where $\rho(A)$ is the spectral radius of A. The generalized Gelfand spectral radius formula $\rho(\Sigma) = \widehat{\rho}(\Sigma)$ follows immediately from the above results.

[80] Y.-F. Su, A. Bhaya, E. Kaszkurewicz, and V. S. Kozyakin, *Further results on stability of asynchronous discrete-time linear systems*, Proc. 36th IEEE Conf. Decision and Control, vol. 1, 1997, pp. 915–920, `doi:10.1109/CDC.1997.650759`.

Focuses on the stability problem of discrete-time asynchronous linear systems, which can be viewed as linear systems with time-varying delays. Within this context, a stronger version of the necessity part of the classical Chazan-Miranker theorem [5] is proved and new results for special classes of system matrices are also presented.

[81] O. Toker, *On the complexity of the robust stability problem for linear parameter varying systems*, Automatica J. IFAC **33** (1997), no. 11, 2015–2017, `doi:10.1016/S0005-1098(97)00129-5`. MR 1486901. Zbl 0914.93046.

In this paper, it is shown that the problem of checking robust stability of linear parameter varying (LPV) systems is NP-hard, and therefore, it is rather unlikely to find polynomial-time solution procedures for this problem. In the frequency-domain structured uncertainty case, it is known that the robust stability problem is NP-hard (Toker and Özbay, 1995; Token 1995; Poljak and Rohn, 1993; Nemirovski, 1993; Braatz et al., 1994), but allowing the uncertain blocks to be time varying gives a computationally tractable problem (Shamma, 1994; Poolla and Tikku, 1995), which can be solved by convex optimization techniques. In the parametric uncertainty case, NP-hardness of the robust stability problem has been shown in (Poljak and Rohn, 1993; Nemirovski, 1993). The results of this paper show that, allowing the uncertain parameters to be time varying, does not give a computationally simpler problem, i.e. it remains NP-hard, and hence it is rather unlikely to find computationally tractable solution procedures for this problem. On the other hand, as far as the existence of an algorithm is considered, there is still no known (non-polynomial time) algorithm for the robust stability problem of linear parameter varying systems (Lagarias and Wang, 1995), and the well-known Tarski's theorem (Tarski, 1951) does not provide a solution procedure (Kozyakin, 1990). Recently, there has been some developments in the direction of constructing non-polynomialtime algorithms for a related problem, called the joint spectral radius (JSR) computation problem (Lagarias and Wang, 1995). We also comment on the use of these results for developing a non-polynomial-time algorithm for testing robust stability of linear parameter varying systems.

The author considers linear parameter varying systems

$$x(k+1) = \left(A_0 + \sum_{i=1}^{n} r_i(k) A_i\right) x(k),$$

where $\{r_i(k)\}$ is a sequence such that $\|r_i(k)\|_\infty \leq 1$ for each $i = 1, \ldots, n$. The system is said to be stable if for each choice of the sequences $\{r_i(k)\}$ and each initial condition the solution is bounded. The main result is that the problem of checking stability for such a system is NP-hard.

[82] J. N. Tsitsiklis and V. D. Blondel, *The Lyapunov exponent and joint spectral radius of pairs of matrices are hard — when not impossible — to compute and to approximate*, Math. Control Signals Systems **10** (1997), no. 1, 31–40, `doi:10.1007/BF01219774`. MR 1462278. Zbl 0888.65044.

The authors show that the joint spectral radius and the generalized spectral radius of two integer matrices are not approximable in polynomial time, and that two related quantities – the lower spectral radius and the largest Lyapunov exponent – are not algorithmically approximable. Some applications of these results are discussed.

[83] J. N. Tsitsiklis and V. D. Blondel, *Lyapunov exponents of pairs of matrices. A correction: "The Lyapunov exponent and joint spectral radius of pairs of matrices are hard — when not impossible — to compute and to approximate"*, Math. Control Signals Systems **10** (1997), no. 4, 381, `doi:10.1007/BF01211553`. MR 1486730.

1998

[84] T. Ando and M.-H. Shih, *Simultaneous contractibility*, SIAM J. Matrix Anal. Appl. **19** (1998), no. 2, 487–498 (electronic), `doi:10.1137/S0895479897318812`. MR 1614074. Zbl 0912.15033.

Let $\mathcal{C}$ be a set of $n \times n$ complex matrices. For $m = 1, 2, \ldots$, $\mathcal{C}^m$ is the set of all products of matrices in $\mathcal{C}$ of length m. Denote by $\hat{r}(\mathcal{C})$ the joint spectral radius of $\mathcal{C}$, that is,

$$\hat{r}(\mathcal{C}) \stackrel{\text{def}}{=} \limsup_{m\to\infty} [\sup_{A\in\mathcal{C}^m} \|A\|]^{\frac{1}{m}}.$$

We call $\mathcal{C}$ *simultaneously contractible* if there is an invertible matrix S such that

$$\sup\{\|S^{-1}AS\|;\ A \in \mathcal{C}\} < 1,$$

where $\|\cdot\|$ is the spectral norm. This paper is primarily devoted to determining the optimal joint spectral radius range for simultaneous contractibility of bounded sets of $n \times n$ complex matrices, that is, the maximum subset J of $[0, 1)$ such that if $\mathcal{C}$ is a bounded set of $n \times n$ complex matrices and $\hat{r}(\mathcal{C}) \in J$, then $\mathcal{C}$ is simultaneously contractible. The central result proved in this paper is that this maximum subset is $[0, \frac{1}{\sqrt{n}})$. Our method of proof is based on a matrix-theoretic version of complex John's ellipsoid theorem and the generalized Gelfand spectral radius formula.

[85] R. B. Bapat, *A max version of the Perron-Frobenius theorem*, Linear Algebra Appl. **275/276** (1998), 3–18, `doi:10.1016/S0024-3795(97)10057-X`. MR 1628380. Zbl 0941.15020.

In one variant of max algebra, the two binary operations are multiplication and maximization. There is a corresponding matrix theory, wherein the maximum takes the place of the sum in all matrix operations. In this context one can speak of maxeigenvalues and maxeigenvectors, and the purpose of this (largely expository) paper is to put some already-known maxeigentheory into a coherent framework, and to trace out more clearly its connections to the traditional eigentheory. Of particular concern is the max version of the Perron-Frobenius theorem:

Theorem. *For any nonnegative irreducible matrix $A = (a_{ij})$, there is a positive vector x such that $\max_j a_{ij}x_j = \mu(A)x_i$ for $i = 1, 2, \ldots, n$, where $\mu(A)$ is the maximum geometric mean of a circuit in the weighted directed graph corresponding to A.*

This theorem, which we refer to as the max version of the Perron-Frobenius Theorem, is well-known in the context of matrices over the max algebra and also in the context of matrix scalings. In the present work, which is partly expository, we bring out the intimate connection between this result and the Perron-Frobenius theory. We present several proofs of the result, some of which use the Perron-Frobenius Theorem. Structure of max eigenvalues and max eigenvectors is described. Possible ways to unify the Perron-Frobenius Theorem and its max version are indicated. Some inequalities for $\mu(A)$ are proved.

The paper offers five different proofs of this result, some of them new, reflecting the various methods of proof which have been applied to the classical Perron-Frobenius theorem. It also presents two different generalizations which unify the traditional and the max forms of this theorem. One of these, in particular, allows for an interpolation between the two by defining a matrix product with the traditional $\sum_{j=1}^n a_{ij}b_{jr}$ replaced by the sum of the k largest products $a_{ij}b_{jr}$, for k a number between 1 and n. If k is 1, this is the max product, while $k = n$ yields the traditional product.

[86] V. S. Kozyakin, N. A. Kuznetsov, and A. V. Pokrovskii, *Quasi-controllability and estimates of amplitudes of transient regimes in discrete systems*, Computational Engineering in Systems Applications, CESA'98, April 1–4, 1998 (Hammamet, Tunisia) (P. Borne, M. Ksouri, and A. El Kamel, eds.), vol. 1, IMACS-IEEE Multiconference, Ecole Centrale de Lille, France, CD-ROM, 1998, pp. 266–271, `arXiv:0909.4374`.

Families of regimes for discrete control systems are studied possessing a special quasi-controllability property that is similar to the Kalman controllability property. A new approach is proposed to estimate the amplitudes of transient regimes in quasi-controllable systems. Its essence is in obtaining of constructive a priori bounds for degree of overshooting in terms of the quasi-controllability measure. The results are applicable for analysis of transients, classical absolute stability problem and, especially, for stability problem for desynchronized systems.

[87] V. S. Kozyakin, N. A. Kuznetsov, and A. A. Vladimirov, *Matrix methods in Skorokhod problems*, Computational Engineering in Systems Applications, CESA'98, April 1–4, 1998 (Hammamet, Tunisia) (P. Borne, M. Ksouri, and A. El Kamel, eds.), vol. 2, IMACS-IEEE Multiconference, Ecole Centrale de Lille, France, CD-ROM, 1998, pp. 323–328, `doi:10.13140/2.1.4241.1203`.

The *Skorokhod problem* is a determined mathematical model used for the construction and analysis of constrained processes, both determined and stochastic, such as queing networks, processor sharing in communication networks, stochastic approximation schemes for problems with constraints, etc. The model we deal with consists of a convex polyhedral set Z in a finite-dimensional space $\mathbb{R}^n$ and a family of "reflection vectors" d_i associated with $(n-1)$-dimensional faces F_i of Z. According to certain rules of reflection, an output $x(t) \subseteq Z$ is generated for each continuous input $u(t), u(0) \in Z$. The corresponding input-output operator, if it exists, is called the *Skorokhod map.* The properties of the Skorokhod problem such as existence and uniqueness of an output for any admissible input, and different continuity properties of the associated Skorokhod map can be studied in terms of different types of stability of finite families of special $n \times n$-matrices, namely, the projection matrices onto the hyperplanes L_i parallel to faces F_i along the vectors d_i. We present both necessary and sufficient conditions of some of the above properties and also establish new relations between such notions as absolute stability, BV-stability, Perron stability, etc., of finite sets of projections and, more generally, of arbitrary $n \times n$-matrices.

[88] L. Livshits, G. MacDonald, B. Mathes, and H. Radjavi, *Reducible semigroups of idempotent operators*, J. Operator Theory **40** (1998), no. 1, 35–69, URL `https://www.theta.ro/jot/archive/1998-040-001/1998-040-001-002.html`. MR 1642522. Zbl 0995.47002.

We study the existence of common invariant subspaces for semigroups of idempotent operators. It is known that in finite dimensions every such semigroup is simultaneously triangularizable. The question of the existence of even one non-trivial invariant subspace is still open in infinite dimensions.

Working with semigroups of idempotent operators in Hilbert/Banach vector space settings, we exploit the connection between the purely algebraic structure and the operator structure to show that the answer is affirmative in a number of cases.

[89] M. Maesumi, *Calculating joint spectral radius of matrices and Hölder exponent of wavelets*, Approximation theory IX, Vol. 2 (Nashville, TN, 1998), Innov. Appl. Math., Vanderbilt Univ. Press, Nashville, TN, 1998, pp. 205–212. MR 1744409. Zbl 0916.42026.

The joint spectral radius (jsr) of a bounded collection of matrices $\mathcal{M}$ is the smallest positive number r such that $\mathcal{M}/r$ generates a norm-bounded semigroup. This quantity can be used to determine Hölder regularity of compactly supported wavelets. The finiteness conjecture of Lagarias and Daubechies states that if $\mathcal{M}$ is finite then a certain optimal product P of n elements of $\mathcal{M}$ attains the maximal growth rate, $\rho(P) = \mathrm{jsr}(\mathcal{M})^n$. We describe an algorithm which is conjectured to terminate iff P is an optimal product. Experimental results of the algorithm relating to the Hölder exponent of four-coefficient multiresolution analyses are presented.

[90] L. Mate, *On the infinite product of operators in Hilbert space*, Proc. Amer. Math. Soc. **126** (1998), no. 2, 535–543, `doi:10.1090/S0002-9939-98-04067-2`. MR 1415333. Zbl 0893.47001.

We give a necessary and sufficient condition for a certain set of infinite products of linear operators to be zero. We shall investigate also the case when this set of infinite products converges to a non-zero operator. The main device in these results is a weighted version of the König Lemma for infinite trees in graph theory.

[91] M. Neumann and H. Schneider, *Partial norms and the convergence of general products of matrices*, ArXiv.org e-Print archive, February 1998, `arXiv:math/9802108`.

Motivated by the theory of inhomogeneous Markov chains, we determine a sufficient condition for the convergence to 0 of a general product formed from a sequence of real or complex matrices. When the matrices have a common invariant subspace H, we give a sufficient condition for the convergence to 0 on H of a general product. Our result is applied to obtain a condition for the weak ergodicity of an inhomogeneous Markov chain. We compare various types of contractions which may be defined for a single matrix, such as paracontraction, l–contraction, and H–contraction, where H is an invariant subspace of the matrix.

[92] V. Yu. Protasov, *A generalization of the joint spectral radius: the geometrical approach*, Facta Univ. Ser. Math. Inform. (1998), no. 13, 19–23. MR 2015883. Zbl 1058.15011.

The author generalizes his earlier result on the joint spectral radius of the finite set of linear operators in the Euclidean space.

[93] I. A. Sheĭpak, *Nontrivial fractals in the plane, and linear operators with a joint spectral radius equal to* 1, Math. Notes **63** (1998), no. 5, 701–705, `doi:10.1007/BF02312855`. MR 1683656. Zbl 0938.37028.

Properties of invariant bodies of a family of linear operators $A_1, \dots, A_k : \mathbb{R}^n \to \mathbb{R}^n$ have been studied in the paper of *V. Yu. Protasov* [67], which also contains an algorithm that calculates the joint spectral radius of these operators. In the examples considered in the paper of Protasov (loc. cit), the dimensions (both topological and Hausdorff) of the invariant sets on the plane are equal to either 0 or 1. The main goal of this paper is to construct a class of operators with invariant sets having fractional Hausdorff dimension.

[94] Y. Su, A. Bhaya, E. Kaszkurewicz, and V. S. Kozyakin, *Further results on convergence of asynchronous linear iterations*, Linear Algebra Appl. **281** (1998), no. 1-3, 11–24, `doi:10.1016/S0024-3795(98)10030-7`. MR 1645319. Zbl 0935.65021.

This paper focuses on the convergence problem of asynchronous linear iterations. A stronger version of the necessity part of the theorem of *D. Chazan* and *W. Miranker* [5] is proved and new results for special classes of iteration matrices are also presented.

[95] D. B. Szyld, *The mistery of asynchronous iterations convergence when the spectral radius is one*, Report 98–102, College of Science and Technology, Temple University, 1998, URL `https://math.temple.edu/~szyld/reports/mystery.pdf`.

D. Chazan and *W. Miranker* [5], and other authors since, have shown that a necessary and sufficient condition for asynchronous iterations to converge is that the spectral radius of the absolute value of the iteration matrix is strictly less than one. Nevertheless, several authors, including *B. Lubachevsky* and *D. Mitra* [23], show convergence for asynchronous iterations for singular matrices and matrices representing Markov chains when the spectral radius of the nonnegative iteration matrix is exactly one. In this note, this apparent contradiction is resolved. It is shown that in fact, the spectral radius less than one is a sufficient condition. The necessity is a more subtle issue. Under certain conditions, with spectral radius equal to one, convergence can indeed be achieved.

[96] F. Wirth, *Dynamics of time-varying discrete-time linear systems: spectral theory and the projected system*, SIAM J. Control Optim. **36** (1998), no. 2, 447–487, `doi:10.1137/S0363012996299600`. MR 1616502. Zbl 0942.93023.

We study structural properties of linear time-varying discrete-time systems. At first an associated system on projective space is introduced as a basic tool to understand the linear dynamics. We study controllability properties of this system and characterize in particular the control sets and their cores. Sufficient conditions for an upper bound on the number of control sets with nonempty interior are given. Furthermore, exponential growth rates of the linear system are studied. Using finite-time controllability properties in the cores of control sets the Floquet spectrum of the linear system may be described. In particular, the closure of the Floquet spectrum is contained in the Lyapunov spectrum.

This article lays a foundation for the study of structural properties of linear time-varying discrete time control processes. The basic idea in this approach is to study the system on the projective space (Bogolyubov's projection introdued by Has'minskii). For deterministic, continuous-time systems

this program was carried out by F. Colonius and W. Kliemann. In this paper the author studies asymptotic properties of the system on the projective space, shows the existence of controls with universal properties and develops tools to analyze different dynamical notions of ‘spectra’. The author studies controllability properties of the system in terms of ‘control sets and their cores’. Sufficient conditions for an upper bound on the number of control sets with non-empty interior are given. Using finite time controllability properties in the cores of control sets, the Floquet spectrum of the linear system may be described. In particular, the author shows that the closure of the Floquet spectrum is contained in the Lyapunov spectrum.

[97] F. Wirth, *On the calculation of time-varying stability radii*, Internat. J. Robust Nonlinear Control **8** (1998), no. 12, 1043–1058, `doi:10.1002/(SICI)1099-1239(1998100)8:12<1043::AID-RNC364>3.3.CO;2-8`. MR 1647729. Zbl 0959.93044.

The paper addresses the problem of computing the maximal Lyapunov exponent of a discrete inclusion. This problem is formulated as an “average yield optimal control” problem. It is shown that the maximum value of this optimization problem can be approximated by the maximal value of a “discounted optimal control problem”. This result is used to obtain convergence rates of an algorithm for computing the time-varying stability radii.

1999

[98] L. Máté, *On infinite composition of affine mappings*, Fund. Math. **159** (1999), no. 1, 85–90, URL `https://eudml.org/doc/212321`. MR 1669710. Zbl 0939.47006.

Let $\{F_i : i = 1, \ldots, N\}$ be affine mappings of $\mathbb{R}^n$. It is well known that if there exists $j \leq 1$ such that for every $\sigma_1, \ldots, \sigma_j \in \{1, \ldots, N\}$ the composition $F_{\sigma_1} \circ \cdots \circ F_{\sigma_j}$ is a contraction, then for any infinite sequence $\sigma_1, \sigma_2, \ldots \in \{1, \ldots, N\}$ and any $z \in \mathbb{R}^n$, the sequence $F_{\sigma_1} \circ \cdots \circ F_{\sigma_n}(z)$ is convergent and the limit is independent of z. We prove the following converse result:

Theorem. *If $F_{\sigma_1} \circ \cdots \circ F_{\sigma_n}(z)$ is convergent for any $z \in \mathbb{R}^n$ and any $\sigma = \{\sigma_1, \sigma_2, \ldots\}$ belonging to some subshift Σ of N symbols (and the limit is independent of z), then there exists $j \geq 1$ such that for every $\sigma = \{\sigma_1, \sigma_2, \ldots\} \in \Sigma$ the composition $F_{\sigma_1} \circ \cdots \circ F_{\sigma_j}$ is a contraction.*

This result can be considered as a generalization of the main theorem of *I. Daubechies* and *J. C. Lagarias* [43, p. 239]. The proof involves some easy but non-trivial combinatorial considerations. The most important tool is a weighted version of the König Lemma for infinite trees in graph theory.

[99] B. E. Moision, A. Orlitsky, and P. H. Siegel, *Bounds on the rate of codes which forbid specified difference sequences*, Global Telecommunications Conference, 1999. GLOBECOM ’99 (Rio de Janeireo), vol. 1b, 1999, pp. 878–882, `doi:10.1109/GLOCOM.1999.830204`.

Certain magnetic recording applications call for a large number of sequences whose differences do not include certain disallowed patterns. We show that the number of such sequences increases exponentially with their length and that the exponent, or capacity, is the logarithm of the joint spectral radius of an appropriately defined set of matrices. We derive new algorithms for determining the joint spectral radius of sets of nonnegative matrices and combine them with existing algorithms to determine the capacity of several sets of disallowed differences that arise in practice.

[100] M. Neumann and H. Schneider, *The convergence of general products of matrices and the weak ergodicity of Markov chains*, Linear Algebra Appl. **287** (1999), no. 1-3, 307–314, special issue celebrating the 60th birthday of Ludwig Elsner, `doi:10.1016/S0024-3795(98)10196-9`. MR 1662874. Zbl 0943.15011.

The paper concerns the infinite general products of real or complex matrices. General means that the matrices to form the product are taken in arbitrary order from an infinite sequence of matrices $\{A_i\}$. In the case of stochastic matrices, such general products have been considered by *E. Seneta* [179].

In two main theorems, the authors give sufficient conditions for such a product to be bounded and to be convergent to zero, respectively. These conditions use the matrix submultiplicative norm μ

and are based on the convergence of $\sum_i[\max(\mu(A_i), 1) - 1]$ and divergence of $\sum_i[1 - \min(\mu(A_i), 1)]$, respectively.

The proofs are based on classical theorems related with the products of positive numbers. The results obtained are used to deduce a condition for weak ergodicity of inhomogeneous Markov chain. Also the ergodicity coefficient based on the matrix norm is compared with other ergodicity coefficients considered, for example, in a paper by *A. Rhodius* [Linear Algebra Appl. 194, 71–83 (1993)].

[101] M.-H. Shih, *Fritz John's convexity theorem and discrete dynamics*, Trends Math. **2** (1999), no. 1, 66–68, URL `https://www.yumpu.com/en/document/read/5014803/fritz-johns-convexity-theorem-and-discrete-dynamics`.

We consider the result of great impact by Fritz John in his work on Convex Geometry. John showed in 1948 that the boundary of any convex region centrally symmetric with respect to a point in $\mathbf{R}^n$ lies between two concentric homothetic ellipsoids of ratio $1/\sqrt{n}$. This result has become very important for Geometric Algorithms. The purpose of this talk is to discuss its matrix-theoretic version and its new application to Discrete Dynamics. The content is mainly taken from papers: *T. Ando* and *M.-H. Shih* [84], *M.-H. Shih* [102].

[102] M.-H. Shih, *Simultaneous Schur stability*, Linear Algebra Appl. **287** (1999), no. 1-3, 323–336, special issue celebrating the 60th birthday of Ludwig Elsner, `doi:10.1016/S0024-3795(98)10071-X`. MR 1662876. Zbl 0948.15016.

The notion of simultaneous Schur stability for a set of $n \times n$ complex matrices is introduced. For a set $\Sigma \subset \mathbb{C}^{n\times n}$, $\mathcal{S}(\Sigma)$ denotes the multiplicative semigroup generated by Σ. For an $n \times n$ complex matrix A, $\sigma(A)$ and $r(A)$ denote the spectrum of A and the spectral radius of A, respectively. A set $\Sigma \subset \mathbb{C}^{n\times n}$ is said to be simultaneously Schur stable if $r(A) \leq 1$, for every $A \in \mathcal{S}(\Sigma)$ and $1 \notin \sigma(A)$, for every $A \in \overline{\mathcal{S}(\Sigma)}$. A set $\Sigma \subset \mathbb{C}^{n\times n}$ is said to be asymptotically stable if there is a norm $\|\cdot\|$ on $\mathbb{C}^n$ and $\alpha > 0$ such that $\|A\| \leq \alpha < 1$, for every $A \in \Sigma$. The main theorem states:

Theorem. *For a bounded set Σ of $n \times n$ complex matrices, Σ is asymptotically stable iff Σ is simultaneously Schur stable.*

This result is used to give an analytic-combinatorial proof of the generalized Gelfand spectral radius formula.

Another interesting application of the main theorem is a simplified proof of a result concerning infinite products of matrices. It is emphasized that in a compact multiplicative semigroup of $n \times n$ complex matrices, the essential nature of simultaneous Schur stability is that zero is the only projection in the semigroup.

Finally, three conjectures are derived from the finiteness conjecture for the generalized spectral radius.

[103] Yu. V. Turovskii, *Volterra semigroups have invariant subspaces*, J. Funct. Anal. **162** (1999), no. 2, 313–322, `doi:10.1006/jfan.1998.3368`. MR 1682061. Zbl 0921.47008.

The article is devoted to the solution of the Volterra semigroup problem. This problem was raised by E. Nordgren, H. Radjavi and P. Rosenthal [Indiana Univ. Math. J. 33, 271–275 (1984)] and has the following form: does every (multiplicative) semigroup of compact quasinilpotent operators on an infinite dimensional Banach space have a nontrivial invariant subspace?

The problem is solved with no additional conditions related to nuclearity, positivity, special spectral behavior and so on. Some consequences are also considered.

2000

[104] V. D. Blondel, S. Gaubert, and J. N. Tsitsiklis, *Approximating the spectral radius of sets of matrices in the max-algebra is NP-hard*, IEEE Trans. Automat. Control **45** (2000), no. 9, 1762–1765, `doi:10.1109/9.880644`. MR 1792275. Zbl 0990.93073.

The lower and average spectral radii measure, respectively, the minimal and average growth rates of long products of matrices taken from a finite set. The logarithm of the average spectral radius is traditionally called Lyapunov exponent. When one performs these products in the max-algebra, we obtain quantities that measure the performance of discrete event systems. We show that approximating the lower and average max-algebraic spectral radii is NP-hard.

[105] V. D. Blondel and J. N. Tsitsiklis, *The boundedness of all products of a pair of matrices is undecidable*, Systems Control Lett. **41** (2000), no. 2, 135–140, `doi:10.1016/S0167-6911(00)00049-9`. MR 1831027. Zbl 0985.93042.

We show that the boundedness of the set of all products of a given pair Σ of rational matrices is undecidable. Furthermore, we show that the joint (or generalized) spectral radius $\rho(\Sigma)$ is not computable because testing whether $\rho(\Sigma) \leq 1$ is an undecidable problem. As a consequence, the robust stability of linear systems under time-varying perturbations is undecidable, and the same is true for the stability of a simple class of hybrid systems. We also discuss some connections with the so-called "finiteness conjecture". Our results are based on a simple reduction from the emptiness problem for probabilistic finite automata, which is known to be undecidable.

[106] V. D. Blondel and J. N. Tsitsiklis, *A survey of computational complexity results in systems and control*, Automatica J. IFAC **36** (2000), no. 9, 1249–1274, `doi:10.1016/S0005-1098(00)00050-9`. MR 1834719. Zbl 0989.93006.

The purpose of this paper is twofold: (a) to provide a tutorial introduction to some key concepts from the theory of computational complexity, highlighting their relevance to systems and control theory, and (b) to survey the relatively recent research activity lying at the interface between these fields. We begin with a brief introduction to models of computation, the concepts of undecidability, polynomial-time algorithms, NP-completeness, and the implications of intractability results. We then survey a number of problems that arise in systems and control theory, some of them classical, some of them related to current research. We discuss them from the point of view of computational complexity and also point out many open problems. In particular, we consider problems related to stability or stabilizability of linear systems with parametric uncertainty, robust control, time-varying linear systems, nonlinear and hybrid systems, and stochastic optimal control.

[107] M. Bröker and X. Zhou, *Characterization of continuous, four-coefficient scaling functions via matrix spectral radius*, SIAM J. Matrix Anal. Appl. **22** (2000), no. 1, 242–257, `doi:10.1137/S0895479897323750`. MR 1779727. Zbl 0964.42017.

We characterize the existence of continuous solutions of a four-coefficient dilation equation in terms of the usual spectral radius of a matrix. The criteria for the existence of such a solution can be very quickly examined. As a result we give an affirmative answer to a conjecture raised by Colella and Heil in 1992. Moreover, using our criteria we find the smoothest compactly supported four-coefficient orthogonal scaling function and thus the smoothest compactly supported orthonormal wavelet generated by this scaling function.

In particular, there is given the following formula for computation of the generalized spectral radius

Lemma 4. *Suppose C_0 and C_1 are two 2×2 matrices. If $\det(C_0) \leq 0$ or $\det(C_1) \leq 0$, then*

$$\rho(C_0, C_1) = \sup_{i+j\geq 1,\ i,j\geq 0} \left(\rho(C_0^i C_1^j)\right)^{\frac{1}{i+j}},$$

where $\rho(C_0, C_1)$ is the generalized spectral radius of the family of matrices $\{C_0, C_1\}$.

[108] Q. Chen and X. Zhou, *Characterization of joint spectral radius via trace*, Linear Algebra Appl. **315** (2000), no. 1-3, 175–188, `doi:10.1016/S0024-3795(00)00149-X`. MR 1774967. Zbl 0970.15020.

The joint spectral radius for a bounded collection of the, square matrices with complex entries and of the same size is characterized by the trace of matrices. This characterization allows us to give some estimates concerning the computation of the joint spectral radius.

For a bounded set Σ of square matrices the joint spectral radius $\rho(\Sigma)$ is defined as the limit superior of the supremum over the set $\mathcal{L}_n$ of all matrix products of length n from the collection Σ

of the n^{th} root of the product matrix norm, while the generalized spectral radius $\widetilde{\rho}(\Sigma)$ is the same expression involving the spectral radii of the matrix products instead.

Inequalities such as $\sup \rho(A)^{1/n} \leq \widetilde{\rho}(\Sigma) \leq \rho(\Sigma) \leq \sup \|A\|^{1/n}$ are known. For a finite set Σ of matrices this paper establishes that $\rho(\Sigma) = \limsup_{n\to\infty} \sup_{A\in\mathcal{L}_n} |\mathrm{trace}(A)|^{1/n}$ and gives estimates for the joint spectral radius for sets generated by two matrices in terms of the trace. These concepts apply to cascade algorithms for wavelets with compact support.

[109] A. Frommer and D. B. Szyld, *On asynchronous iterations*, J. Comput. Appl. Math. **123** (2000), no. 1-2, 201–216, numerical analysis 2000, Vol. III. Linear algebra, `doi:10.1016/S0377-0427(00)00409-X`. MR 1798526. Zbl 0967.65066.

Theory of asynchronous iterations, basically as extensions of synchronous convergence theorems to asynchronous iteration models for shared memory parallel computers. Application to linear systems, nonlinear systems (Newton) and waveform relaxation. However, for practical use "wasted operations" as well as synchronization and communication on distributed memory computers would be of interest.

[110] O. Holtz, *On convergence of infinite matrix products*, Electron. J. Linear Algebra **7** (2000), 178–181, `arXiv:math/0512590`. MR 1781470. Zbl 0964.15030.

The author shows that the convergence of the product

$$P = \prod_{n\in N} \begin{bmatrix} I & B_n \\ 0 & C_n \end{bmatrix},$$

where C_n are (uniform) contracting submatrices, is equivalent with the convergence of the subsequence $X_n = B_n(I - C_n)^{-1}$ and

$$P = \prod_{n\in N} \begin{bmatrix} I & \lim\limits_{n\to\infty} X_n \\ 0 & 0 \end{bmatrix}.$$

[111] E. Kaszkurewicz and A. Bhaya, *Matrix diagonal stability in systems and computation*, Birkhäuser Boston Inc., Boston, MA, 2000, `doi:10.1007/978-1-4612-1346-8`. MR 1733604. Zbl 0951.93058.

The book presents a collection of results, observations, and examples related to dynamical systems described by linear and nonlinear ordinary differential and difference equations. In particular, it considers dynamical systems that are susceptible to analysis by the Lyapunov approach. The book consists of 6 chapters. The first chapter is devoted to examples that originate from different applications and illustrate the way in which some special classes of dynamical systems dovetail with the concepts of matrix diagonal stability and the associated diagonal-type Lyapunov functions. The second chapter presents the matrix theory concepts of diagonal stability and also of D-stability and gives the properties of these classes of matrices. Chapter 3 introduces classes of dynamical systems that admit diagonal-type Lyapunov functions and gives the basic stability results. Chapter 4 shows how Jacobi-type iterative methods for the solution of linear and nonlinear equations and, more specifically, asynchronous versions of these methods can be analysed, based on diagonal-type Lyapunov functions. Chapter 5 shows and discusses the occurrence of the diagonal structure, introduced in the first chapter, in several classes of dynamical systems that include neural networks, digital filters, passive PLC circuits, and ecosystems. Finally, chapter 6 discusses various applications of dynamical systems in which diagonally stable structures can be used advantageously.

The book provides an essential reference for new methods and analysis related to dynamical systems described by linear and nonlinear ordinary differential equations and difference equations. As such, it is addressed to researchers, professionals, and graduates in applied mathematics, control engineering, stability of dynamical systems, convergence of algorithms, and scientific computation. Familiarity with linear algebra and matrix theory, as well as difference and differential equations, is the mathematical background expected from its readers. A prior knowledge of systems and control theory, including Lyapunov stability theory, is also expected, at least in sufficient measure to provide motivation for the problems studied.

[112] M. Maesumi, *Joint spectral radius and Hölder regularity of wavelets*, Comput. Math. Appl. **40** (2000), no. 1, 145–155, Approximation in mathematics (Memphis, TN, 1997), `doi:10.1016/S0898-1221(00)00148-6`. MR 1768971. Zbl 0948.42019.

We give preliminary results on the Hölder exponent of wavelets of compact support. In particular, we give a nearly complete map of this exponent for the family of four-coefficient multiresolution analyses and determine the smoothest one. This will resolve two conjectures by Colella and Heil. Wavelets of compact support can be generated via infinite products of certain matrices. The rate of growth of these products determines the regularity of the wavelet. This rate can be determined via joint, generalized, or common spectral radius of the given set of matrices. We outline a method for calculating this radius for a given set of matrices. The method relies on guessing the particular finite optimal product which satisfies the finiteness conjecture and exhibits the fastest growth. Then, we generate an optimal unit ball by taking the convex hull of the action of the semigroup of matrices, scaled by their joint radius, on the invariant ball of the scaled optimal product. If this process terminates in a finite number of steps and the convex hull does not grow, then the guessed optimal product is confirmed and the joint radius is determined.

[113] M. Maesumi, *Calculating the spectral radius of a set of matrices*, Wavelet analysis and multiresolution methods (Urbana-Champaign, IL, 1999), Lecture Notes in Pure and Appl. Math., vol. 212, Dekker, New York, 2000, pp. 255–272. MR 1777996. Zbl 0965.42021.

The paper presents an algorithm for calculating the joint or generalized spectral radius of a set of matrices. The algorithm is conjectured to verify, in a finite number of steps, if a certain product is the optimal product satisfying the finiteness conjecture of Daubechies and Lagarias. The algorithm has been applied to matrices arising from the regularity analysis of multiresolution analyses and compactly supported wavelets. In particular, four- and six-coefficient MRA's with the highest Hölder exponent are presented.

[114] V. Yu. Protasov, *Asymptotics of the partition function*, Mat. Sb. **191** (2000), no. 3, 65–98, `doi:10.1070/sm2000v191n03ABEH000464`. MR 1773255.

Some estimates for the joint and the lower spectral radii of the operators having a common invariant cone are obtained which then used in investigation of the following problem.

Given a pair of positive integers m and d such that $2 \leqslant m \leqslant d$, for integer $n \geqslant 0$ the quantity $b_{m,d}(n)$, called the partition function is considered; this by definition is equal to the cardinality of the set

$$\Big\{(a_0, a_1, \dots) : n = \sum_k a_k m^k,\ a_k \in \{0, \dots, d-1\},\ k \geqslant 0\Big\}.$$

The properties of $b_{m,d}(n)$ and its asymptotic behaviour as $n \to \infty$ are studied. A geometric approach to this problem is put forward. It is shown that

$$C_1 n^{\lambda_1} \leqslant b_{m,d}(n) \leqslant C_2 n^{\lambda_2},$$

for sufficiently large n , where C_1 and C_2 are positive constants depending on m and d, and $\lambda_1 = \underline{\lim}_{n\to\infty} \dfrac{\log b(n)}{\log n}$ and $\lambda_2 = \overline{\lim}_{n\to\infty} \dfrac{\log b(n)}{\log n}$ are characteristics of the exponential growth of the partition function. For some pair (m, d) the exponents λ_1 and λ_2 are calculated as the logarithms of certain algebraic numbers; for other pairs the problem is reduced to finding the joint spectral radius of a suitable collection of finite-dimensional linear operators. Estimates of the growth exponents and the constants $C_1$and C_2 are obtained.

[115] H. Radjavi, P. Rosenthal, and V. Shulman, *Operator semigroups with quasinilpotent commutators*, Proc. Amer. Math. Soc. **128** (2000), no. 8, 2413–2420, `doi:10.1090/S0002-9939-00-05622-7`. MR 1706985. Zbl 1099.47503.

R. M. Guralnick has shown [Linear Multilinear Algebra 9, 133–148 (1980)] that a multiplicative semigroup of matrices $\mathcal{S}$ over any field is triangularizable over an extension field if $ST - TS$ is nilpotent for all $S, T \in \mathcal{S}$. Starting from this result (which is re-proved in the paper for the simpler case of the complex field), the authors investigate the possible analogues in the infinite-dimensional case. Using a recent result of *Yu. V. Turovskii* [103], the authors show that if $\mathcal{S}$ is a semigroup of compact operators in a Hilbert space such that $ST - TS$ is quasinilpotent for all $S, T \in \mathcal{S}$, then $\mathcal{S}$ is triangularizable. In the case of arbitrary Banach spaces, the authors show that any strongly compact group of operators having quasinilpotent.

[116] J. Shen, *Compactification of a set of matrices with convergent infinite products*, Linear Algebra Appl. **311** (2000), no. 1-3, 177–186, `doi:10.1016/S0024-3795(00)00080-X`. MR 1758212. Zbl 0958.15021.

The author generalizes and unifies some known results on a set Σ of square matrices (real or complex) with right-convergent-products (RCPs). Here Σ is said to be an RCP set if for any sequence $(A_1, A_2, \dots) \in \Sigma^\infty$, $\lim_{n\to\infty} A_1A_2\cdots A_n$ exists. A König chain of Σ with respect to a given norm $\|\cdot\|$ is a sequence $(A_1, A_2, \dots) \in \Sigma^\infty$ such that $\|A_1A_2\cdots A_m\|^{1/m} \geq \hat{\rho}$, $m = 1, 2, \dots$.

In Section 2 the author studies some important properties that are shared by both a bounded matrix set and its closure. In Section 3 he proves the existence of a König chain in a compact set of matrices, and then shows applications by reproducing some major known results.

[117] V. S. Shulman and Yu. V. Turovskiĭ, *Joint spectral radius, operator semigroups, and a problem of W. Wojtyński*, J. Funct. Anal. **177** (2000), no. 2, 383–441, `doi:10.1006/jfan.2000.3640`. MR 1795957. Zbl 0980.47008.

Recently, the second author proved a remarkable theorem: *Every semigroup of compact quasinilpotent (i.e. Volterra) Banach space operators has a nontrivial invariant subspace* [103]. The essence of the proof is that the algebra generated by the semigroup also consists of quasinilpotent operators. Algebras of compact quasinilpotent operators have invariant subspaces, according to V. I. Lomonosov's theory. The main tool used by Turovskii was the joint spectral radius technique developed by him.

The present paper continues the research on the connection between invariant subspace theory and the joint spectral radius technique.

W. Wojtyński [Stud. Math. 49, 263–273 (1977)] showed that locally finite Lie algebras of compact operators have invariant subspaces. The question remained open for general Lie algebras of Volterra operators. This question is answered in the affirmative here.

The authors use semigroups of principle operators (operators whose spectral radius coincides with their essential spectral radius). It is shown that if one such semigroup contains the group $\left\{e^{tT}, t \in \mathbb{R}\right\}$ for some nonzero Volterra operator T, then it has a hyperinvariant subspace. This gives the existence of hyperinvariant subspaces for Lie algebras of Volterra operators.

The paper contains many accompanying results which are of independent interest. In particular, it is proved that the joint spectral radius is a subharmonic function: let D be a domain on the complex plane $\mathbb{C}$ and $\lambda \in D \to M(\lambda)$ be a map into the set of bounded subsets of a Banach algebra A. The map $\lambda \to M(\lambda)$ is said *to be analytic* if there exists a family F of analytic function $f : D \to A$ such that $M(\lambda) = \{f(\lambda) : f \in F\}$ for all $\lambda \in D$. The family F is called *an analytic family for the map* $\lambda \to M(\lambda)$. We say that F is continuous at a point $\lambda_0 \in D$ if $\sup\{\|f(\lambda) - f(\lambda_0)\| : f \in F\} \to 0$ under $\lambda \to 0$, $\lambda \in D$. Also, F is *continuous* on D if it is continuous at every point of D. Sometimes it is written $F(\lambda)$ instead of $M(\lambda)$.

Theorem 3.5. *Let $\lambda \to M(\lambda)$, $\lambda \in D$, be an analytic map into the set of bounded subsets of A with analytic family F. If F is continuous on D then the functions $\lambda \to \log\rho(M(\lambda))$ and $\lambda \to \rho(M(\lambda))$ are subharmonic on D.*

[118] Yu. V. Turovskii and V. S. Shul'man, *Joint spectral radius and invariant subspaces*, Funct. Anal. Appl. **34** (2000), no. 2, 156–158, `doi:10.1007/BF02482436`. MR 1773855. Zbl 0978.47004.

The paper is a review of some recent results of the authors (no proofs are given) on existence of nontrivial invariant and hyperinvariant subspaces for some families of Banach space operators. For a bounded set M of elements of a normed algebra A put $\|M\| = \{\|a\| : a \in M\}$. Let M^n be the set of all products of n elements of M. $\rho(M) = \inf \|M^n\|^{1/n}$ is called the spectral radius of M. The essential spectral radius of M is the spectral radius of the image of M in the Calkin algebra. Most of the results of the paper contain conditions on $\rho(M)$ and/or $\rho_e(M)$. The authors gave answers to questions 4 and 5 posed by *W. Wojtynski* [Stud. Math. 59, 263–273 (1977)].

[119] A. Vladimirov, L. Elsner, and W.-J. Beyn, *Stability and paracontractivity of discrete linear inclusions*, Linear Algebra Appl. **312** (2000), no. 1-3, 125–134, `doi:10.1016/S0024-3795(00)00094-X`. MR 1759327. Zbl 0964.15032.

A square matrix A is called paracontracting (PC) with respect to a given norm if $Ax \neq x \Rightarrow \|Ax\| < \|x\|$. A set Σ of matrices is PC if all elements of Σ are PC with respect to the same norm. The main result of this paper is the proof of a conjecture (cf. *W.-J. Beyn* and *L. Elsner* [69]) that a finite set Σ of $n \times n$ matrices is PC iff each infinite trajectory $x_{j+1} = A_j x_j$ ($A_j \in \Sigma$) converges to a point $x^* \in \mathbb{R}^n$, i.e. all infinite products of elements of Σ are convergent. Moreover, it is shown that this is equivalent to $\lim_{j\to\infty} \|x_{j+1} - x_j\| = 0$ and also to the fact that Σ is product bounded, i.e. $\exists C > 0$ such that $\|A_1 \cdots A_m\| < C$ for all finite products of (not necessarily different) elements of Σ.

2001

[120] V. D. Blondel, O. Bournez, P. Koiran, C. H. Papadimitriou, and J. N. Tsitsiklis, *Deciding stability and mortality of piecewise affine dynamical systems*, Theoret. Comput. Sci. **255** (2001), no. 1-2, 687–696, `doi:10.1016/S0304-3975(00)00399-6`. MR 1819103. Zbl 0973.68067.

In this paper we study problems such as: given a discrete time dynamical system of the form $x(t+1) = f(x(t))$ where $f : R^n \to R^n$ is a piecewise affine function, decide whether all trajectories converge to 0. We show in our main theorem that this attractivity problem is undecidable as soon as $n \geq 2$. The same is true of two related problems: Stability (is the dynamical system globally asymptotically stable?) and mortality (do all trajectories go through 0?). We then show that attractivity and stability become decidable in dimension 1 for continuous functions.

[121] I. Daubechies and J. C. Lagarias, *Corrigendum/addendum to: "Sets of matrices all infinite products of which converge" [Linear Algebra Appl.* **161** *(1992), 227–263; MR1142737 (93f:15006)]*, Linear Algebra Appl. **327** (2001), no. 1-3, 69–83, `doi:10.1016/S0024-3795(00)00314-1`. MR 1823340. Zbl 0978.15024.

This corrigendum/addendum supplies corrected statements and proofs of some results in the paper [43]. These results concern special kinds of bounded semigroups of matrices. It also reports on progress on the topics of this paper made in the last eight years.

[122] P. Diamond and V. I. Opojtsev, *Stability of linear difference and differential inclusions*, Autom. Remote Control **62** (2001), no. 5, 695–703, `doi:10.1023/A:1010258420380`. MR 1852745. Zbl 1082.34509.

If S is an n-dimensional linear space and A the matrix of a linear transformation $\mathcal{A} : S \to S$, then the space S can always be imbedded into an N-dimensional linear space E, $N > n$, such that there exists an extension $\mathcal{B} : E \to E$ of $\mathcal{A}$ with a nonnegative matrix B in some base. This property and some of its applications are the subject of this paper. In particular, the authors study the extensions B of the matrix A for which the spectral radius $\rho(B)$ is close to $\rho(A)$. The inclusion $x^{k+1} \in \mathbb{A}x^k$ is also considered, where $\mathbb{A}$ is a compact set of matrices, and it is proved that this is asymptotically stable if and only if $\mathbb{A}$ can be extended to a set $\mathbb{B}$ of nonnegative matrices B with $\|B\|_1$ or $\|B\|_\infty < 1$. Similar results are derived for differential inclusions.

[123] R. Drnovšek, *Common invariant subspaces for collections of operators*, Integral Equations Operator Theory **39** (2001), no. 3, 253–266, `doi:10.1007/BF01332655`. MR 1818060. Zbl 0994.47008.

The paper concerns common invariant subspaces for sets of operators which are finitely quasinilpotent at some vector. A set $\mathcal{C}$ of bounded linear operators on a Banach space X is said to be finitely quasinilpotent at $x_0 \in X$ if for every finite subset $\mathcal{F}$ of $\mathcal{C}$ one has

$$\limsup_{n\to\infty} \Big(\sup_{T_1,\dots,T_n \in \mathcal{F}} |T_1 \cdots T_n x_0| \Big)^{1/n} = 0.$$

Theorem 3.3. *If a set $\mathcal{C}$ of operators on X is finitely quasinilpotent at some non-vector and $\mathcal{C}$ contains a non-zero compact operator, then there exists a nontrivial subspace of X which is invariant to each operator belonging either to $\mathcal{C}$ or to the commutant of $\mathcal{C}$.*

Theorem 3.5. *The same fact is valid in the case when $\mathcal{C}$ ($\neq \{0\}$) is finitely quasinilpotent at some non-zero vector and there exists a non-zero compact operator in the commutant of $\mathcal{C}$.*

The proof of Theorem 3.3 is based on Hilden's "ping-pong" method of proof of the celebrated Lomonosov's invariant subspace theorem, exposed by *A. J. Michaels* [Adv. Math. 25, 56–58 (1977)].

The last part of the paper includes versions of the above described results for sets of positive operators on Banach lattices. In particular, one extends a famous theorem of *B. de Pagter* [Math. Z. 192, 149–153 (1986)].

[124] A. Frommer and P. Spiteri, *On linear asynchronous iterations when the spectral radius of the modulus matrix is one*, Topics in numerical analysis, Comput. Suppl., vol. 15, Springer, Vienna, 2001, pp. 91–104, `doi:10.1007/978-3-7091-6217-0_8`. MR 1874506. Zbl 0994.65034.

A classical result on linear asynchronous iterations states that convergence occurs if and only if the spectral radius of the modulus matrix is less than 1. The present paper shows that if one introduces very mild restrictions on the admissible asynchronous processes, one gets convergence for a larger class of matrices for which the spectral radius of the modulus matrix is allowed to be equal to 1. The mild restrictions are virtually always fulfilled in practical implementations. In this manner, our result contributes to the better understanding of the different hypotheses underlying mathematical models for asynchronous iterations.

[125] N. Guglielmi and M. Zennaro, *On the asymptotic properties of a family of matrices*, Linear Algebra Appl. **322** (2001), no. 1-3, 169–192, `doi:10.1016/S0024-3795(00)00228-7`. MR 1804119. Zbl 0971.15016.

The authors consider bounded families $\mathcal{F}$ of complex $n \times n$-matrices. The concept of "asymptotic order" is introduced and how the norm of products of matrices behaves as the number of factors goes to infinity is investigated. In the case of "defective" families $\mathcal{F}$, using the asymptotic order one gets a deeper knowledge of the asymptotic behaviour than by just considering the so-called "generalized spectral radius". With reference to the well-known finiteness conjecture for finite families, the paper introduces the concepts of "spectrum-maximizing product" and "limit spectrum-maximizing product", showing that, for finite families of 2×2-matrices, defectivity is equivalent to the existence of defective such limit products.

[126] R.-Q. Jia, K.-S. Lau, and D.-X. Zhou, *L_p solutions of refinement equations*, J. Fourier Anal. Appl. **7** (2001), no. 2, 143–167, `doi:10.1007/BF02510421`. MR 1817673. Zbl 1030.42031.

In the recent characterizations of the L_p-solution of the refinement equation in terms of the 'p-norm joint spectral radius', there are problems in choosing the initial function for iteration, and, in addition, in requiring stability of the refinable function. In this paper, we overcome these difficulties and give a more complete characterization of this nature. The criterion is constructive and can be implemented. It can be used to describe the regularity of the solution without assuming stability. This has significant advantages over the previous work. The corresponding results for vector refinement equations are also discussed.

[127] B. E. Moision, A. Orlitsky, and P. H. Siegel, *On codes that avoid specified differences*, IEEE Trans. Inform. Theory **47** (2001), no. 1, 433–442, `doi:10.1109/18.904557`. MR 1820392. Zbl 0998.94563.

Certain magnetic recording applications call for a large number of sequences whose differences do not include certain disallowed binary patterns. We show that the number of such sequences increases exponentially with their length and that the growth rate, or capacity, is the logarithm of the joint spectral radius of an appropriately defined set of matrices. We derive a new algorithm for determining the joint spectral radius of sets of nonnegative matrices and combine it with existing algorithms to determine the capacity of several sets of disallowed differences that arise in practice.

[128] M.-H. Shih, *König chain for compact matrix sets*, Linear Algebra Appl. **330** (2001), no. 1-3, 205–208, doi:10.1016/S0024-3795(01)00265-8. MR 1826655. Zbl 0982.15035.

The author considers a compact set Σ of complex $(n \times n)$-matrices. It is shown that Σ admits a so-called König chain, i.e., a sequence $\{A_k\}$ in Σ such that $\|A_1 A_2 \cdots A_k\|^{1/m}$ is greater or equal to the joint spectral radius of Σ for all operator norms $\|\cdot\|$ (cf. *G.-C. Rota* and *G. Strang* [3]).

[129] A. Vladimirov and A. Rubinov, *Dynamics of positive multiconvex relations*, J. Convex Anal. **8** (2001), no. 2, 387–399, URL https://www.heldermann-verlag.de/jca/jca08/jca0207.pdf. MR 1915948. Zbl 1042.49027.

A notion of multiconvex relation as a union of a finite number of convex relations is introduced. For a particular case of multiconvex process, that is, a union of a finite set of convex processes, we define the notions of the joint and the generalized spectral radius in the same manner as for matrices. We prove the equivalence of these two values if all component processes are positive, bounded, and closed.

2002

[130] V. D. Blondel, J. Theys, and A. A. Vladimirov, *Switched systems that are periodically stable may be unstable*, Proc. of the Symposium MTNS (Notre-Dame, USA), 2002, URL https://www3.nd.edu/~mtns/papers/10181.pdf.

We prove the existence of two 2×2 real matrices such that all periodic products of these matrices converge to zero but there exists an infinite product that does not. We outline implications of this result for the stability of switched linear systems, and for the finiteness conjecture

[131] O. Bournez and M. Branicky, *The mortality problem for matrices of low dimensions*, Theory Comput. Syst. **35** (2002), no. 4, 433–448, doi:10.1007/s00224-002-1010-5. MR 1909045. Zbl 1016.68038.

In this paper we discuss the existence of an algorithm to decide if a given set of 2×2 matrices is mortal. A set $F = \{A_1, \ldots, A_m\}$ of square matrices is said to be mortal if there exist an integer $k \geq 1$ and some integers $i_1, i_2, \ldots, i_k \in \{1, \ldots, m\}$ with $A_{i_1} A_{i_2} \cdots A_{i_k} = 0$. We survey this problem and propose some new extensions. We prove the problem to be BSS-undecidable for real matrices and Turing-decidable for two rational matrices. We relate the problem for rational matrices to the entry-equivalence problem, to the zero-in-the-corner problem, and to the reachability problem for piecewise-affine functions. Finally, we state some NP-completeness results.

[132] T. Bousch and J. Mairesse, *Asymptotic height optimization for topical IFS, Tetris heaps, and the finiteness conjecture*, J. Amer. Math. Soc. **15** (2002), no. 1, 77–111 (electronic), doi:10.1090/S0894-0347-01-00378-2. MR 1862798. Zbl 1057.49007.

Given an Iterated Function System (IFS) of topical maps verifying some conditions, we prove that the asymptotic height optimization problems are equivalent to finding the extrema of a continuous functional, the average height, on some compact space of measures. We give general results to determine these extrema, and then apply them to two concrete problems. First, we give a new proof of the following assertion:

Theorem. *The densest heaps of two Tetris pieces are sturmian.*

Second, we construct an explicit counterexample to the Lagarias-Wang finiteness conjecture [53] for the joint spectral radius of a set of matrices.

[133] D. J. Hartfiel, *Nonhomogeneous matrix products*, World Scientific Publishing Co., Inc., River Edge, NJ, 2002. MR 1878339. Zbl 1011.15006.

The book is a research exposition the aim of which is to put together much of the basic work on nonhomogeneous matrix products, i.e., $A_k \cdots A_1$ with (at least two) different factors. The book consists of 13 chapters and an appendix in which a few classical results used in the book are given. Chapter 1 is an introduction containing some preliminary definitions and remarks. In Chapter 2 the projective metric in $\mathbb{R}^n$, Hausdorff metric and other functionals that can be used to study convergence of infinite products of matrices are explained. Chapter 3 is devoted to algebraic properties and convergence in semigroups of matrices. Chapter 4 deals with some nonnegative matrices (i.e., having nonnegative entries) and semigroups of such matrices. Ergodicity is studied in Chapter 5. Here ergodicity concerns sequences of products $A_1, A_2A_1, \ldots$, which appear more like rank 1 matrices as $k \to \infty$. In Chapter 6 some basic convergence results for infinite products of matrices are given, in Chapter 7 continuous convergence and some sequence spaces are studied. A matrix A is called paracontracting (PC) if $\|Ax\| < \|x\|$ whenever $Ax \neq x$. In Chapter 8 convergence of infinite products of PC matrices is studied. In Chapters 9 and 10 a matrix set Σ is considered and the convergence in Hausdorff metric of $\Sigma, \Sigma^2, \ldots$ is studied. In Chapter 12 slowly varying matrix products are considered. In Chapters 11 and 13 some applications are shown. These include, e.g., the construction of curves and fractals, solving demographic problems, applications in production systems and management structures. MATLAB codes for solving some of those problems are given in the ends of Chapters 11 and 13.

[134] Y.-C. Li and M.-H. Shih, *Contractibility of compact contractions in Hilbert space*, Linear Algebra Appl. **341** (2002), 369–378, special issue dedicated to Professor T. Ando, `doi:10.1016/S0024-3795(01)00489-X`. MR 1873634. Zbl 0999.15010.

For a *finite* set Σ of compact contractions in a complex Hilbert space $(H, \|\cdot\|)$, it is shown that $r(A) < 1$ for all A in the multiplicative semigroup generated by Σ if and only if there exists a positive integer N such that $\|A\| < 1$ for all A in the multiplicative semigroup generated by Σ with length greater than N. Here $r(A)$ denotes the spectral radius of A. As an application, an answer is given to an infinite-dimensional case of the finiteness conjecture for the generalized spectral radius attributed to *J. C. Lagarias* and *Y. Wang* [53].

[135] V. S. Shulman and Yu. V. Turovskiĭ, *Formulae for joint spectral radii of sets of operators*, Studia Math. **149** (2002), no. 1, 23–37, `doi:10.4064/sm149-1-2`. MR 1881714. Zbl 1016.47006.

The formula $\rho(M) = \max\{\rho_\chi(M), r(M)\}$ is proved for precompact sets M of weakly compact operators on a Banach space. Here $\rho(M)$ is the joint spectral radius (the Rota-Strang radius), $\rho_\chi(M)$ is the Hausdorff spectral radius (connected with the Hausdorff measure of noncompactness) and $r(M)$ is the Berger-Wang radius. The work is a further development of the ideas of earlier papers, namely *G.-C. Rota* and *G. Strang* [3] and *M. A. Berger* and *Y. Wang* [42].

[136] F. Wirth, *The generalized spectral radius and extremal norms*, Linear Algebra Appl. **342** (2002), 17–40, `doi:10.1016/S0024-3795(01)00446-3`. MR 1873424. Zbl 0996.15020.

Two properties of the generalized spectral radius are established, namely, its locally Lipschitz continuity on the space of compact irreducible sets of matrices and its strict monotonicity property. The author's approach is based on an important idea in the analysis of exponential stability of discrete inclusions that was introduced by *N. E. Barabanov* [26–28]. The author gives a new proof for Barabanov's result, which states that for irreducible sets of matrices an extremal norm always exists. Also, significant conditions for the existence of extremal norms are obtained.

[137] F. Wirth, *Parameter dependent extremal norms for linear parameter varying systems*, Fifteenth International Symposium on Mathematical Theory of Networks and Systems, August 12-16, 2002, University of Notre Dame, 2002, URL `https://www3.nd.edu/~mtns/papers/9024.pdf`.

We study families of time-varying linear systems with restrictions on the derivative of the parameter variation. This includes the systems usually considered in the area of linear parameter varying (LPV) systems. We show that it is possible to construct exact parameterized Lyapunov norms for a wide class of such systems. This may be used to derive (locally Lipschitz) continuous dependence of the exponential growth rate on the systems data. Furthermore, it is shown that the exponential growth rate may be approximated by exponential growth rates of periodic parameter variations.

2003

[138] V. D. Blondel and V. Canterini, *Undecidable problems for probabilistic automata of fixed dimension*, Theory Comput. Syst. **36** (2003), no. 3, 231–245, `doi:10.1007/s00224-003-1061-2`. MR 1962327. Zbl 1039.68061.

We prove that several problems associated to probabilistic finite automata are undecidable for automata whose number of input letters and number of states are fixed. As a corollary of one of our results we prove that the problem of determining if the set of all products of two 47×47 matrices with nonnegative rational entries is bounded is undecidable.

[139] V. D. Blondel and Yu. Nesterov, *Fast and precise approximations of the joint spectral radius*, EUEN/CORE – Center for operations research and econometrics, UCL, 2003, working paper, `doi:10.2139/ssrn.981383`.

In this paper, we introduce a procedure for approximating the joint spectral radius of a finite set of matrices with arbitrary precision. Our approximation procedure is based on semidefinite liftings and can be implemented in a recursive way. For two matrices even the first step of the procedure gives an approximation, whose relative quality is at least $1/\sqrt{2}$, that is, more than 70%. The subsequent steps improve the quality but also increase the dimension of the auxiliary problem from which this approximation can be found. In an improved version of our approximation procedure we show how a relative quality of $(1/\sqrt{2})^{1/k}$ can be obtained by evaluating the spectral radius of a single matrix of dimension $n^k(n^k+1)/2$ where n is the dimension of the initial matrices. This result is computationally optimal in the sense that it provides an approximation of relative quality $1-\varepsilon$ in time polynomial in $n^{1/\varepsilon}$ and it is known that, unless P=NP, no such algorithm is possible that runs in time polynomial in n and $1/\varepsilon$.

For the special case of matrices with non-negative entries we prove that

$$(1/2)^{1/k}\rho^{1/k}(A_1^{\otimes k} + A_2^{\otimes k}) \le \rho(A_1, A_2) \le \rho^{1/k}(A_1^{\otimes k} + A_2^{\otimes k})$$

where $A^{\otimes k}$ denotes the kth Kronecker power of A. An approximation of relative quality $(1/2)^{1/k}$ can therefore be obtained by computing the spectral radius of a single matrix of dimension n^k. From these inequalities it also follows that the spectral radius is given by the simple expression

$$\rho(A_1, A_2) = \lim_{k\to\infty} \|A_1^{\otimes k} + A_2^{\otimes k}\|^{1/k}$$

where it is somewhat surprising to notice that the right hand side does not directly involve any mixed products between the matrices A_1 and A_2.

[140] V. D. Blondel, J. Theys, and A. A. Vladimirov, *An elementary counterexample to the finiteness conjecture*, SIAM J. Matrix Anal. Appl. **24** (2003), no. 4, 963–970 (electronic), `doi:10.1137/S0895479801397846`. MR 2003315. Zbl 1043.15007.

The finiteness conjecture has recently been proved to be false (cf. *T. Bousch* and *J. Mairesse* [132]). The present paper provides an alternative proof of this fact. The authors prove that there exist (infinitely many) values of the real parameters a and b for which the matrices

$$a\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad b\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

have the following property: all infinite periodic products of the two matrices converge to zero, but there exists a nonperiodic product that doesn't. The proof is self-contained and fairly elementary; it uses only elementary facts from the theory of formal languages and from linear algebra. It is not constructive in that we do not exhibit any explicit values of a and b with the stated property; the problem of finding explicit matrices with this property remains open.

[141] J. Bochi, *Inequalities for numerical invariants of sets of matrices*, Linear Algebra Appl. **368** (2003), 71–81, `doi:10.1016/S0024-3795(02)00658-4`, `arXiv:math/0206128`. MR 1983195. Zbl 1031.15023.

This paper generalizes norm and spectral radius inequalities for single matrices to bounded sets of matrices. Namely, the following statement which follows from the Cayley-Hamilton theorem: if $A \in \mathbb{C}^{n,n}$ then $\|A^n\| \leq (2^n - 1)\rho(A)\|A\|^{n-1}$, can be extended to bounded subsets $\Sigma \subset \mathbb{C}^{n,n}$:

Theorem. $\|\Sigma^n\| \leq C\mathcal{R}(\Sigma)\|\Sigma\|^{n-1}$ *for a constant* $C = C(n)$*, every bounded subset* Σ *of* n *by* n *matrices, and every matrix norm on* $\mathbb{C}^{n,n}$.

Here Σ^n denotes the set of all products with n factors from Σ and $\mathcal{R}(\Sigma) = \lim_{n\to\infty} \|\Sigma^n\|^{1/n}$ in analogy to the spectral radius theorem $\rho(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$ for a single matrix A. A further result of the paper is as folows:

Theorem. *The inequality* $\mathcal{R}(\Sigma) \leq C_2 \max_{j\leq k} \rho(\Sigma^j)^{1/j}$ *is valid with universal constants* $C_2 = C_2(n)$ *and* $k = k(n)$ *depending only on the dimension* n*, but not on* $\Sigma \subset \mathbb{C}^{n,n}$.

The latter inequality implies the generalized spectral radius theorem of *M. A. Berger* and *Y. Wang* [42]. The proofs are topological and use geometric invariant theory, as well as the Cayley-Hamilton theorem.

[142] N. Guglielmi and M. Zennaro, *On the limit products of a family of matrices*, Linear Algebra Appl. **362** (2003), 11–27, doi:10.1016/S0024-3795(02)00460-3. MR 1955451. Zbl 1025.15040.

The authors [125] introduced the concept of the limit spectrum-maximizing product (l.s.m.p.) for a bounded family of complex $n \times n$ matrices and characterized the defectivity of finite families of 2×2 matrices by the existence of defective l.s.m. products. At the end of [loc. cit.] it was conjectured that this characterization should hold in general.

Here the authors prove that all nondefective bounded normalized families of matrices always have an l.s.m.p. and show that finite defective families of 3×3 matrices always have an spectrum-maximizing product (s.m.p.), but may not have a defective l.s.m.p. Thus, the conjecture occured to be false. Also, the authors construct a finite defective family of 4×4 matrices which does not have even any l.s.m.p., regardless its possible defectivity.

[143] V. S. Kozyakin, *Asynchronous systems: A short survey and problems*, Preprint 13/2003, Boole Centre for Research in Informatics, University College Cork — National University of Ireland, Cork, May 2003, URL https://www.researchgate.net/publication/228863199_Asynchronous_Systems_A_Short_Survey_and_Problems.

Looking at the dynamics of the system described by the equation $x(n+1) = f(x(n))$ one may say that coordinates of the vector $x = x_1, x_2, \ldots, x_N$ are updated synchronously. What happens with the system if coordinates of the vector x are updated asynchronously, i.e., if at a given moment n only coordinates with indices i from some set $w(n) \subseteq 1, 2, \ldots, N$ are changed in accordance with the law $x_i(n+1) = f_i(x(n))$ while others remain intact? This is the main topic which is discussed in the paper.

[144] V. S. Kozyakin, *Indefinability in o-minimal structures of finite sets of matrices whose infinite products converge and are bounded or unbounded*, Autom. Remote Control **64** (2003), no. 9, 1386–1400 (english), doi:10.1023/A:1026091717271. MR 2090805. Zbl 1078.93017.

This paper is concerned with the convergence and boundedness or unboundedness of the set of all possible matrix products with coefficients belonging to some finite set, i.e., the problem to which many problems of control theory and mathematics are reduced. The indefinability of this problem in o-minimal structures containing semialgebraic sets, which can be regarded as a characteristic for the complexity of the problem, is demonstrated. The result shows, in particular, that the solution of our problem cannot be found as a finite Boolean combination of conditions containing a finite number of ordinary arithmetical operations of addition, subtraction, and multiplication, as well as exponentiation and application of bounded analytic functions.

[145] G.-C. Rota, *Gian-Carlo Rota on analysis and probability*, Contemporary Mathematicians, Birkhäuser Boston Inc., Boston, MA, 2003, selected papers and commentaries, Edited by Jean Dhombres, Joseph P. S. Kung and Norton Starr. MR 1944526. Zbl 1159.01014.

This volume consists of selected papers by Gian-Carlo Rota, a rare mathematician who made major contributions to several areas of mathematics. Presented in Part I are his papers in analysis, written at the beginning of his career and having a continuing and pervasive influence today. Part II is devoted to his papers on convexity and probability theory, written towards the end of his career and containing many ideas that have yet to be fully developed. Comprehensive commentaries by experts in the field are included in every chapter. Accessible to both experts and beginners. Gian-Carlo Rota was one of those rare mathematicians who made major contributions to several areas of mathematics. Presented in the first part of this volume are reprints of his papers in analysis, which were written at the beginning of his career. These papers on differential equations, operator theory, ergodic theory, and other subjects have a continuing and pervasive influence. Reprints of his papers on convexity and probability theory are presented in the second part of the work. These were written towards the end of his career and contain many ideas that have yet to be fully developed. Comprehensive commentaries are included in every chapter. These survey articles detail work inspired by Rota's papers and also include discussions of many unsolved problems. As is customary with Rota's writings, the papers included in the volume – some published here for the first time – contain many fresh and unexpected ideas for further research. Thus, this volume will be of interest to both experts and beginners in the above-mentioned fields.

[146] G. Strang, *The joint spectral radius*, Gian-Carlo Rota on analysis and probability. Selected papers and commentaries (J. Dhombres, J. P. S. Kung, and N. Starr, eds.), Contemporary Mathematicians, Birkhäuser Boston Inc., Boston, MA, 2003, pp. 98–102.

The paper "A note on the joint spectral radius" [3] gives a natural definition of the joint spectral radius of two matrices A and B:

$$\rho(A, B) = \liminf_{N\to\infty} \text{ (largest norm of any product with } N \text{ factors)}^{1/N}.$$

The product can have A's and B's in any order (and the liminf is actually a limit). I was very proud to be a joint author with Gian-Carlo, but I am now a little ashamed that we never thought seriously about how to compute this number ρ. For a single matrix it equals the largest magnitude of the eigenvalues $\lambda(A)$. But products of A and B can produce norms and eigenvalues that are very hard to estimate (as N increases) from the two matrices. The Lyapunov exponent is a similar number, using averages over products of length N instead of maxima, and it suffers from the same difficulty (impossibility?) in actual computation. An equivalent definition of ρ is the infimum over all matrix norms of $\max(\|A\|, \|B\|)$. The definitions extend directly to sets of more than two matrices, and an l_p norm joint spectral radius has also proved useful [53, 59].

Every few years, Gian-Carlo would ask me whether anyone ever read our paper. After I had tenure, I could tell him the truth: "not often". Part of the reason may have been the relatively unfamiliar journal, and the nonexistence of the Internet, and even the atypically obscure language that he had chosen to express our (very simple) idea. In recent years I could change my answer! The joint spectral radius suddenly found application in wavelet theory, especially in the work of *Ingrid Daubechies* and *Jeff Lagarias* [43]. Of special interest is the question whether $\rho < 1$, so that products with more and more factors of A and B all go to zero.

[147] F. Wirth, *Further results on the Lyapunov spectrum of time-varying linear discrete-time systems*, Manuscript, 2003, URL `https://www.hamilton.ie/fabian/Archive/mtns-lyap.pdf`, Zentrum für Technomathematik, Universität Bremen.

We continue our analysis of the Lyapunov and Floquet spectrum of discrete-time systems begun in [96]. In particular we settle an open question by presenting an example of an invariant control set in which Lyapunov exponents may be realized that are not contained in the closure of the Floquet spectrum corresponding to the control set. Furthermore we show that under weak conditions the top Lyapunov exponents behaves strictly monotone under monotone growth of the families of time-varying systems. This implies a strong linearization result for stability radii of nonlinear system with respect to time-varying perturbations.

2004

[148] V. D. Blondel, Yu. Nesterov, and J. Theys, *Approximations of the rate of growth of switched linear systems*, Hybrid Systems: Computation and Control (Proceedings of the 7th International Workshop, HSCC 2004, Philadelphia, PA, USA, March 25-27, 2004), Lecture Notes in Computer Science, vol. 2993, Springer-Verlag, Berlin Heidelberg, 2004, pp. 173–186, doi:10.1007/978-3-540-24743-2_12. Zbl 1135.93333.

The joint spectral radius of a set of matrices is a measure of the maximal asymptotic growth rate that can be obtained by forming long products of matrices taken from the set. This quantity appears in a number of application contexts, in particular it characterizes the growth rate of switched linear systems. The joint spectral radius is notoriously difficult to compute and to approximate. We introduce in this paper the first polynomial time approximations of guaranteed precision. We provide an approximation $\hat{\rho}$ that is based on ellipsoid norms that can be computed by convex optimization and that is such that the joint spectral radius belongs to the interval $[\hat{\rho}/\sqrt{n}, \hat{\rho}]$ where n is the dimension of the matrices. We also provide a simple approximation for the special case where the entries of all the matrices are non-negative; in this case the approximation is proved to be within a factor at most m (m is the number of matrices) of the exact value.

[149] D. Cheban and C. Mammana, *Asymptotic stability of autonomous and non-autonomous discrete linear inclusions*, Bul. Acad. Ştiinţe Repub. Mold. Mat. (2004), no. 3, 41–52. MR 2148008. Zbl 1080.39006.

This paper deals with the absolute asymptotic stability of discrete linear inclusions of the form $x_{t+1} \in F(x_t)$, where $F(x) = \{A_1x, A_2x, \dots, A_mx\}$ and the A_i are linear bounded operators acting on a Banach space. The relations between absolute asymptotic stability, uniform asymptotic stability and uniform exponential stability are analyzed, and they are proved to be equivalent for compact inclusions.

[150] E. Yu. Emel'yanov and Z. Ercan, *A formula for the joint local spectral radius*, Proc. Amer. Math. Soc. **132** (2004), no. 5, 1449–1451, doi:10.1090/S0002-9939-03-07199-5. MR 2053352. Zbl 1062.47006.

The joint spectral radius for a bounded subset M of the algebra of all bounded linear operators on a Banach space X is defined by the formula $\rho(M) = \limsup_{n\to\infty} \|M^n\|^{1/n}$, where M^n denotes the set of all products $T_1T_2\cdots T_n$ ($T_j \in M$) and $\|M^n\| = \sup\{\|T\| : T \in M^n\}$. Its local counterpart is defined as follows: $\rho_x(M) = \limsup_{n\to\infty} \|M^nx\|^{1/n}$, where $x \in X$ and $\|M^nx\| = \sup\{\|Tx\| : T \in M^n\}$.

The main result of the present paper gives the following formula:

$$\rho_x(M) = \sup_{f\in X^*} \limsup_{n\to\infty} |f \circ M^n(x)|^{1/n},$$

where $|f \circ M^n(x)| = \sup\{|f(Tx)| : T \in M^n\}$. As a corollary to this result, the analogous formula for the joint spectral radius of a bounded set in a Banach algebra is obtained.

[151] V. Kozyakin, *A short introduction to asynchronous systems*, Proceedings of the Sixth International Conference on Difference Equations (Boca Raton, FL), CRC, 2004, pp. 153–165. MR 2092552. Zbl 1083.93032.

Looking at the dynamics of the system described by the equation

$$x(n+1) = f(x(n))$$

one may say that coordinates of the vector $x = \{x_1, x_2, \dots, x_N\}$ are updated *synchronously*. What happens with the system if coordinates of the vector x are updated *asynchronously*, i.e., if at a given moment n only coordinates with indices i from some set $\omega(n) \subseteq \{1, 2, \dots, N\}$ are changed in accordance with the law

$$x_i(n+1) = f_i(x(n))$$

while others remain intact? This is the main topic which is discussed below.

Examples of asynchronous systems are multiprocessor systems, distributed digital networks, discrete-time models of market economy, etc.

The main attention is be paid to discussion of the problem how asynchronism affects stability of the system. Examples show that all possible combinations of stability/instability for the pair 'synchronous/asynchronous system' may occur. And also, simple examples demonstrate that the problem of investigation of stability for asynchronous system is more complicate than for synchronous one, even in linear case. Nevertheless, in some situations asynchronous systems possess more robust properties than synchronous ones.

Formal explanation of this fact is be presented, and various methods of stability investigation for asynchronous system are discussed.

[152] F. Wirth, *A monotonicity property of the joint spectral radius*, 16th International Symposium on Mathematical theory of networks and systems (Heverlee), Katholieke Universiteit Leuven, 2004, URL `https://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.60.3589`.

We show that the joint spectral radius of a set of matrices is strictly increasing as a function of the data in the sense that if a set of matrices is contained in the relative interior of the convex hull of an irreducible set of matrices, then the joint spectral radius of the smaller set is strictly smaller than that of the larger set. This observation has some consequences in the theory of time-varying stability radii and their calculation. We show by example that, strict monotonicity notwithstanding, 0 may be a proximal normal of the joint spectral radius of some (finitely parameterized) matrix polytopes functions. This shows that the time-varying stability radius is not in general Lipschitz continuous when it is continuous.

[153] J.-C. Yoccoz, *Some questions and remarks about* $\mathrm{SL}(2,\mathbf{R})$ *cocycles*, Modern dynamical systems and applications, Cambridge Univ. Press, Cambridge, 2004, pp. 447–458, URL `https://www.college-de-france.fr/media/jean-christophe-yoccoz/UPL27219_SL_2_R__cocycles.pdf`. MR 2093316. Zbl 1148.37306.

There have been many deep results about cocycle maps in recent years, especially in the quasiperiodic case with the achievements of *H. Eliasson*, *J. Bourgain* and *R. Krikorian* amongst others. The questions that we address here are much more elementary: most of the time, we will be interested in locally constant cocycles with values in $SL(2,\mathbf{R})$ over a transitive subshift of finite type; we want to determine whether this cocycle map is uniformly hyperbolic and how it can bifurcate from uniform hyperbolicity. In our setting, parameter space is finite-dimensional, and we would like to describe it in the same way that one does for polynomials or rational maps, where hyperbolic components play a leading role in the picture. It appears that even in this very much simplified situation, several interesting questions appear.

2005

[154] N. Barabanov, *Lyapunov exponent and joint spectral radius: Some known and new results*, Proceedings of the 44th IEEE Conference on Decision and Control and European Control Conference 2005, Seville, Spain, December 12–15, 2005, pp. 2332–2337, `doi:10.1109/CDC.2005.1582510`.

The logarithm of joint spectral radius of a set of matrices coincides with Lyapunov exponent of corresponding linear inclusions. Main results about Lyapunov exponents of discrete time and continuous time linear inclusions are presented. They include the existence of extremal norm; relations between Lyapunov indices of dual inclusions; maximum principle for linear inclusions; algebraic criteria for stability of linear inclusions; algorithm to find out the sign of Lyapunov exponents. The main result is extended to linear inclusions with delays. The Aizerman problem for three-ordered timevarying continuous time systems with one nonlinearity is solved. The Perron-Frobenius theorem is extended for three-ordered continuous time linear inclusions.

[155] J. P. Bell, *A gap result for the norms of semigroups of matrices*, Linear Algebra Appl. **402** (2005), 101–110, `doi:10.1016/j.laa.2004.12.007`. MR 2141076. Zbl 1074.15033.

Let $\|\cdot\|$ be a matrix norm on $M_d(\mathbb{C})$ and, $\mathcal{A} = \{A_1, \ldots, A_e\}$ be a finite set of matrices in $M_d(\mathbb{C})$. Set

$$m_n(\mathcal{A}) = \max_{1\le i_1,\ldots,i_n\le e} \|A_{i_1} \cdots A_{i_n}\|.$$

The author shows that there is a gap in the possible growth of $m_n(\mathcal{A})$. The main result states:

Let $\mathcal{A}$ be given. Then either there is some constant $c > 1$ such that $m_n(\mathcal{A}) > c^n$ for all n sufficiently large, or $m_n(\mathcal{A}) = O(n^{d-1})$. Moreover, $m_n(\mathcal{A}) = O(n^{d-1})$ if and only if the eigenvalues of every matrix in the semigroup generated by $\mathcal{A}$ are all on or inside the unit circle.

As a consequence of this result, it follows, for example, that it is impossible to find a set of matrices $\mathcal{A}$ with $m_n(\mathcal{A}) \sim \exp(\sqrt{n})$.

[156] V. D. Blondel and Yu. Nesterov, *Computationally efficient approximations of the joint spectral radius*, SIAM J. Matrix Anal. Appl. **27** (2005), no. 1, 256–272 (electronic), `doi:10.1137/040607009`, `arXiv:math/0407485`. MR 2176820. Zbl 1089.65031.

The joint spectral radius of a set of matrices is a measure of the maximal asymptotic growth rate that can be obtained by forming long products of matrices taken from the set. This quantity appears in a number of application contexts but is notoriously difficult to compute and to approximate. We introduce in this paper a procedure for approximating the joint spectral radius of a finite set of matrices with arbitrary high accuracy. Our approximation procedure is polynomial in the size of the matrices once the number of matrices and the desired accuracy are fixed.

For the special case of matrices with non-negative entries we give elementary proofs of simple inequalities that we then use to obtain approximations of arbitrary high accuracy. From these inequalities it follows that the spectral radius of matrices with non-negative entries is given by the simple expression

$$\rho(A_1, \ldots, A_m) = \lim_{k\to\infty} \rho^{1/k}(A_1^{\otimes k} + \cdots + A_m^{\otimes k})$$

where it is somewhat surprising to notice that the right hand side does not directly involve any mixed product between the matrices ($A^{\otimes k}$ denotes the k-th Kronecker power of A).

For matrices with arbitrary entries (not necessarily non-negative) we introduce an approximation procedure based on semi-definite liftings that can be implemented in a recursive way. For two matrices, even the first step of the procedure gives an approximation whose relative accuracy is at least $1/\sqrt{2}$, that is, more than 70%. The subsequent steps improve the accuracy but also increase the dimension of the auxiliary problems from which the approximation can be found.

Our approximation procedures provide approximations of relative accuracy $1-\epsilon$ in time polynomial in $n^{(\ln m)/\epsilon}$, where m is the number of matrices and n is their size. These bounds are close from optimality since we show that, unless P=NP, no approximation algorithm is possible that provides a relative accuracy of $1-\epsilon$ and runs in time polynomial in n and $1/\epsilon$.

As a by-product of our results we prove that a widely used approximation of the joint spectral radius based on common quadratic Lyapunov functions (or on ellipsoid norms) has relative accuracy $1/\sqrt{m}$, where m is the number of matrices.

[157] V. D. Blondel, Yu. Nesterov, and J. Theys, *On the accuracy of the ellipsoid norm approximation of the joint spectral radius*, Linear Algebra Appl. **394** (2005), 91–107, `doi:10.1016/j.laa.2004.06.024`. MR 2100578. Zbl 1086.15020.

The authors consider two computable approximations of the joint spectral radius for a finite set $\mathcal{U}$ of matrices. The first approximation satisfies $\frac{1}{\sqrt{n}}\hat{\rho} \le \rho \le \hat{\rho}$ based on ellipsoid norms, where ρ is the joint spectral radius of $\mathcal{U}$, $\hat{\rho} = \inf_{P\succ 0} \max_{A_i\in\mathcal{U}} \|A_i\|_p$, $\|A\|_p$ is induced by the vector norm $\|x\|_p = \sqrt{x^TPx}$ and n is the dimension of the matrices. Moreover, for the special case of symmetric matrices, triangular matrices, or for sets of matrices that have a solvable Lie algebra, the equality $\rho = \hat{\rho}$ is satisfied. The other approximation is for the set of nonnegative matrices. In this case the approximation is proved to be within a factor at most m of the exact value, where m is the number of matrices in $\mathcal{U}$.

[158] M. Charina, C. Conti, and T. Sauer, *L_p-convergence of subdivision schemes: joint spectral radius versus restricted spectral radius*, Approximation theory XI: Gatlinburg 2004, Mod. Methods Math., Nashboro Press, Brentwood, TN, 2005, pp. 129–150. MR 2126678. Zbl 1074.65021.

We compare two approaches for investigating the L_p-convergence of multivariate scalar and vector subdivision schemes. The first approach is based on the so-called joint spectral radius, the second one on what we call the restricted spectral radius, a quantity that can be used to characterize the restricted contractivity of the corresponding difference subdivision schemes. We show that the two approaches are not only equivalent, but that in fact the joint spectral radius and the restricted spectral radius are equal for $1 \leq p \leq \infty$. One of the advantages of working with the restricted spectral radius is that the restricted p-norms can be computed by means of classical optimization methods, the restricted ∞-norm even by means of linear programming. This also allows for an estimation of the restricted spectral radii.

[159] A. Czornik, *On the generalized spectral subradius*, Linear Algebra Appl. **407** (2005), 242–248, `doi:10.1016/j.laa.2005.05.006`. MR 2161929. Zbl 1080.15008.

For $m \geq 1$, let Σ^m be the set of all products of length m of square complex matrices of some order k. Let $\rho(A)$ and $\|A\|$ be, respectively, the spectral radius and a matrix norm of a matrix A. In this note the author proves that the joint and the generalized spectral subradius are equal to infimum limits of $\inf_{A\in\Sigma^m} \|A\|^{1/m}$ and $\inf_{A\in\Sigma^m} \rho^{1/m}(A)$. This generalizes to the case of infinite set of matrices a formula due to *L. Gurvits* [57]. A new definition of stability of discrete linear inclusions is proposed.

[160] R. Gharavi and V. Anantharam, *An upper bound for the largest Lyapunov exponent of a Markovian product of nonnegative matrices*, Theoret. Comput. Sci. **332** (2005), no. 1-3, 543–557, `doi:10.1016/j.tcs.2004.12.025`. MR 2122519. Zbl 1066.60066.

We derive an upper bound for the largest Lyapunov exponent of a Markovian product of nonnegative matrices using Markovian type counting arguments. The bound is expressed as the maximum of a nonlinear concave function over a finite-dimensional convex polytope of probability distributions.

[161] N. Guglielmi, F. Wirth, and M. Zennaro, *Complex polytope extremality results for families of matrices*, SIAM J. Matrix Anal. Appl. **27** (2005), no. 3, 721–743 (electronic), `doi:10.1137/040606818`. MR 2208331. Zbl 1099.15023.

For finite families of complex $n \times n$ matrices, the focus is paid on those families that satisfy the so-called fitness conjecture, which was recently disproved in its more general formulation. The authors conjecture that the validity of the fitness conjecture for a finite family of nondetective type is equivalent to the existence of an extremal norm in the class of convex polytope norms. However, they are not able to prove the small complex polytope extremality theorem under some more restrictive hypotheses on the underlying family of matrices.

In addition, the obtained results assure a certain fitness property on the number of vertices of the unit ball of the extremal complex polytope norm, which could be very useful for the construction of suitable algorithms aimed at the actual computation of the spectral radius of the family.

Even though the obtained results are mostly theoretical in nature, they have potential impact in applications. This potential lies in the fact that, if there is a prior knowledge that a certain set $\mathcal{F}$ has an extremal polytope norm, then one could devise algorithms for the computation of the spectral radius $\rho(\mathcal{F})$ that rely on the computation of the extremal points of the unit ball of the norm.

[162] N. Guglielmi and M. Zennaro, *Polytope norms and related algorithms for the computation of the joint spectral radius*, Proceedings of the 44th IEEE Conference on Decision and Control and European Control Conference 2005, Seville, Spain, December 12–15, 2005, pp. 3007–3012, `doi:10.1109/CDC.2005.1582622`.

We address the problem of the computation of the spectral radius of a family of matrices. We briefly describe the extension of the concept of polytope to the complex space and outline the main geometric properties of such an object. Then we consider the norms determined by the complex polytopes and illustrate a possible algorithm for the approximation of the joint spectral radius of a family of matrices which is based on these complex polytope norms. As an example for our technique we consider the set of two matrices recently analyzed by Blondel, Nesterov and Theys to disprove the finiteness conjecture.

[163] L. Gurvits and A. Samorodnitsky, *A note on common quadratic Lyapunov functions for linear inclusions: Exact results and open problems*, Proceedings of the 44th IEEE Conference on Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC'05, 15-15 Dec. (Seville, Spain, Spain), IEEE, 2005, pp. 2350–2355, `doi:10.1109/CDC.2005.1582513`.

We prove several exact results on approximability of joint spectral radius by matrix norms induced by Euclidean norms. We point out, perhaps for the first time in this context, a difference between complex and real cases. New connections of joint spectral radius to convex geometry and combinatorics are established. Several open problems are posed.

[164] V. Kozyakin, *A dynamical systems construction of a counterexample to the finiteness conjecture*, Proceedings of the 44th IEEE Conference on Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC'05, 2005, pp. 2338–2343, `doi:10.1109/CDC.2005.1582511`.

In 1995 *J. C. Lagarias* and *Y. Wang* [53] conjectured that the generalized spectral radius of a finite set of matrices can be attained on a finite product of matrices. The first counterexample to this Finiteness Conjecture was given in 2002 by *T. Bousch* and *J. Mairesse* [132]. In 2003 *V. D. Blondel*, *J. Theys* and *A. A. Vladimirov* [140] proposed another proof of a counterexample to the Finiteness Conjecture which extensively exploited combinatorial properties of matrix products. In the paper, it is proposed one more proof of a counterexample of the Finiteness Conjecture fulfilled in a traditional manner of the theory of dynamical systems. It is presented description of the structure of trajectories with the maximal growing rate in terms of extremal norms and associated with them so-called extremal trajectories. The construction of the counterexample is based on a detailed analysis of properties of extremal norms of two-dimensional positive matrices in which the technique of the Gram symbols is essentially used. At last, notions and properties of the rotation number for discontinuous orientation preserving circle maps play significant role in the proof.

[165] V. S. Kozyakin, *Proof of a counterexample to the finiteness conjecture in the spirit of the theory of dynamical systems*, Preprint 1005, Weierstraß-Institut für Angewandte Analysis und Stochastik, Berlin, January 2005, URL `https://www.researchgate.net/publication/234109568_Proof_of_a_Counterexample_to_the_Finiteness_Conjecture_in_the_Spirit_of_the_Theory_of_Dynamical_Systems`.

In 1995 *J. C. Lagarias* and *Y. Wang* [53] conjectured that the generalized spectral radius of a finite set of square matrices can be attained on a finite product of matrices. The first counterexample to this Finiteness Conjecture was given in 2002 by *T. Bousch* and *J. Mairesse* [132] and their proof was based on measure-theoretical ideas. In 2003 *V. D. Blondel*, *J. Theys* and *A. A. Vladimirov* [140] proposed another proof of a counterexample to the Finiteness Conjecture which extensively exploited combinatorial properties of permutations of products of positive matrices. In the paper, it is proposed one more proof of a counterexample of the Finiteness Conjecture fulfilled in a rather traditional manner of the theory of dynamical systems. It is presented description of the structure of trajectories with the maximal growing rate in terms of extremal norms and associated with them so called extremal trajectories. The construction of the counterexample is based on a detailed analysis of properties of extremal norms of two-dimensional positive matrices in which the technique of the Gram symbols is essentially used. At last, notions and properties of the rotation number for discontinuous orientation preserving circle maps play significant role in the proof.

[166] Y.-Y. Lur, *On the asymptotic stability of nonnegative matrices in max algebra*, Linear Algebra Appl. **407** (2005), 149–161, `doi:10.1016/j.laa.2005.05.017`. MR 2161921. Zbl 1086.15016.

In the max algebra system, for $n \times n$ real matrices A and B, the product $A \otimes B$ has the (ij) entry defined by $\max_{1 \le k \le n} a_{ik} b_{kj}$, and for $x \in \mathbb{R}^n$, $A \otimes x$ has the ith component defined by $\max_{1 \le j \le n} a_{ij} x_j$. Fix a norm $\| \cdot \|$ on $\mathbb{R}^n$ and define $\eta(A) = \sup_{\|x\|=1, x \ge 0} \|A \otimes x\|$, $\hat{\eta}(A) = \lim_{k \to \infty} \sup \eta(big \otimes^k A)^{1/k}$. In this paper, the equivalence of the following conditions are proved: (i) $\eta(A) < 1$, (ii) $\hat{\eta}(A) < 1$, (iii) $\mu(A) < 1$; (iv) $\lim_{k \to \infty} \otimes^k A = 0$, where $\mu(A)$ is the maximum circuit geometric mean of the directed graph associated with A.

[167] M. Maesumi, *Construction of optimal norms for semi-groups of matrices*, Proceedings of the 44th IEEE Conference on Decision and Control and European Control Conference 2005, Seville, Spain, December 12–15, 2005, pp. 3013–3018, `doi:10.1109/CDC.2005.1582623`.

The notion of spectral radius of a set of matrices is a natural extension of spectral radius of a single matrix. The Finiteness Conjecture (FC) claims that among the infinite products made from the elements of a given finite set of matrices, there is a certain periodic product, made from the repetition of a finite product (the optimal product), whose rate of growth is maximal. FC has been disproved. In this paper it is conjectured that FC is almost always true, and an algorithm is presented to verify the optimality of a given product. The algorithm uses optimal norms, as a special subset of extremal norms. The algorithm has successfully calculated the spectral radius of the pair of matrices associated with compactly supported multi-resolution analyses and wavelets.

[168] E. Plischke, F. Wirth, and N. Barabanov, *Duality results for the joint spectral radius and transient behavior*, Proceedings of the 44th IEEE Conference on Decision and Control and European Control Conference 2005, Seville, Spain, December 12–15, 2005, pp. 2344–2349, `doi:10.1109/CDC.2005.1582512`.

For linear inclusions in discrete or continuous time several quantities characterizing the growth behavior of the corresponding semigroup are analyzed. These quantities are the joint spectral radius, the initial growth rate and (for bounded semigroups) the transient bound. It is recalled how these constants relate to one another and how they are characterized by various norms. A complete duality theory is developed in this framework, relating semigroups and dual semigroups and extremal or transient norms with their respective dual norms.

[169] V. Protasov, *Applications of the joint spectral radius to some problems of functional analysis, probability and combinatorics*, Proceedings of the 44th IEEE Conference on Decision and Control and European Control Conference 2005, Seville, Spain, December 12–15, 2005, pp. 3025–3030, `doi:10.1109/CDC.2005.1582625`.

In this paper we discuss applications of the joint spectral characteristics of finite dimensional linear operators such as joint spectral radius, lower spectral radius, p-radius, Lyapunov exponent etc. to some problems of functional analysis, fractal geometry, probability theory and combinatorial number theory.

[170] V. Protasov, *The geometric approach for computing the joint spectral radius*, Proceedings of the 44th IEEE Conference on Decision and Control and European Control Conference 2005, Seville, Spain, December 12–15, 2005, pp. 3001–3006, `doi:10.1109/CDC.2005.1582621`.

In this paper we describe the geometric approach for computing the joint spectral radius of a finite family of linear operators acting in finite-dimensional Eucledian space. The main idea is to use the invariant sets of of these operators. It is shown that any irreducible family of operators possesses a centrally-symmetric invariant compact set, not necessarily unique. The Minkowski norm generated by the convex hull of an invariant set (invariant body) possesses special extremal properties that can be put to good use in exploring the joint spectral radius. In particular, approximation of the invariant bodies by polytopes gives an algorithm for computing the joint spectral radius with a prescribed relative deviation ε. This algorithm is polynomial with respect to $1/\varepsilon$ if the dimension is fixed. Another direction of our research is the asymptotic behavior of the orbit of an arbitrary point under the action of all products of given operators. We observe some relations between the constants of the asymptotic estimations and the sizes of the invariant bodies. In the last section we give a short overview on the extension of geometric approach to the L_p-spectral radius.

[171] J. Theys, *Joint spectral radius: Theory and approximations*, Ph.D. thesis, Faculté des sciences appliquées, Département d'ingénierie mathématique, Center for Systems Engineering and Applied Mechanics, Université Catholique de Louvain, May 2005, URL `https://www.semanticscholar.org/paper/Joint-spectral-radius-%3A-theory-and-approximations%2F-Theys/7a041d4a8922f0a2d08303d3a1fd9c2fe4de05dd`.

The spectral radius of a matrix is a widely used concept in linear algebra. It expresses the asymptotic growth rate of successive powers of the matrix. This concept can be extended to sets of matrices, leading to the notion of "joint spectral radius". The joint spectral radius of a set of matrices was defined in the 1960's, but has only been used extensively since the 1990's. This concept is useful to study the behavior of multi-agent systems, to determine the continuity of wavelet basis functions or for expressing the capacity of binary codes. Although the joint spectral radius shares some properties with the usual spectral radius, it is much harder to compute, and the problem of approximating it is NP-hard. In this thesis, we first review theoretical results that lead to basic approximations of the joint spectral radius. Then, we list various specific cases where it is effectively computable, before presenting a specific type of sets of matrices, for which we solve the problem of computing it with a polynomial computational cost.

[172] F. Wirth, *On the structure of the set of extremal norms of a linear inclusion*, Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference 2005 Seville, Spain, December 12–15, 2005, pp. 3019–3024, `doi:10.1109/CDC.2005.1582624`.

A systematic study of the set of extremal norms of an irreducible linear inclusion is undertaken. We recall basic methods for the construction of extremal norms, and consider the action of basic operations from convex analysis on these norms. It is shown that the set of extremal norms of an irreducible linear inclusion is a convex cone with a compact basis in an appropriate Banach space. Furthermore, the compact basis may be chosen to depend upper semi-continuously on the data. We explain that this is the reason for the local Lipschitz continuity of the joint spectral radius as a function of the data.

[173] F. Wirth, *The Gelfand formula for linear parameter-varying and linear switching systems*, Proceedings of the 16th IFAC World Congress (Czech Republic), vol. 16, part 1, IFAC, 2005, pp. 652–652, `doi:10.3182/20050703-6-CZ-1902.00653`.

It is shown that the Gelfand formula holds for a large class of families of linear time-varying systems, encompassing in particular standard formulations of linear parameter-varying and linear switching systems. By this result, the uniform exponential growth rate may be approximated to arbitrary precision by the growth rate of periodic systems within the family. This result extends classical results in the area of linear inclusions. The basic tool in the proof is a recent construction of parameter dependent Lyapunov functions for the family of linear time-varying systems that exactly characterize the exponential growth rate.

[174] F. Wirth, *The generalized spectral radius is strictly increasing*, Linear Algebra Appl. **395** (2005), 141–153, `doi:10.1016/j.laa.2004.07.013`. MR 2112880. Zbl 1070.15007.

Given a nonempty compact set of matrices $\mathcal{M} \subset \mathbb{K}^{n\times n}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Consider the discrete linear inclusion $x(t+1) \in \{Ax(t) : A \in \mathcal{M}\}$. A solution to this inclusion is a sequence $\{x(t)\}_{t\in\mathbb{N}}$, such that for every $t \in \mathbb{N}$ there is an $A(t) \in \mathcal{M}$ with $x(t+1) = A(t)x(t)$. The sets of products of length t are defined as: $\mathcal{S}_t = \{A(t-1)A(t-2)\cdots A(0) : A(s) \in \mathcal{M},\ s = 0, 1, \ldots, t-1\}$ and the semigroup given by $\mathcal{S} = \bigcup_{t=0}^{\infty} \mathcal{S}_t$.

Let $\rho(A)$ denote the spectral radius of A and let $\|\cdot\|$ be some operator norm on $\mathbb{K}^{n\times n}$. Define for $t \in \mathbb{N}$, $\bar{\rho}_t(\mathcal{M}) := \sup\{r(S_t)^{1/t} : S_t \in \mathcal{S}_t\}$ and $\hat{\rho}_t(\mathcal{M}) := \sup\{\|S_t\|^{1/t} : S_t \in \mathcal{S}_t\}$ Then the joint spectral radius and the generalized spectral radius are respectively defined as $\bar{\rho}(\mathcal{M}) := \lim_{t\to\infty} \sup \bar{\rho}_t(\mathcal{M})$ and $\hat{\rho}(\mathcal{M}) := \lim_{t\to\infty} \hat{\rho}_t(\mathcal{M})$. It is well known that $\bar{\rho}(\mathcal{M}) = \hat{\rho}(\mathcal{M})$. $\mathcal{M}$ is called irreducible, if only the trivial subspaces $\{0\}$ and $\mathbb{K}^n$ are invariant under all matrices $A \in \mathcal{M}$, otherwise $\mathcal{M}$ is called reducible. The semigroup $\mathcal{S}$ is irreducible if and only if the set $\mathcal{M}$ is irreducible.

The author proves that the generalized spectral radius of a compact set of matrices is a strictly increasing function of the set. The main tool in the proof is the observation that if $\mathcal{M}$ is convex, is not a singleton set, and the semigroup $\mathcal{S}$ generated by $\mathcal{M}$ satisfies $\sigma(S) \subset \{0\} \cup \{z \in \mathbb{C} : |z| = 1\}, \forall S \in \mathcal{S}(\mathcal{M})$, then $\mathcal{M}$ is reducible.

Some applications of this property in the area of time-varying stability radii are discussed. In particular, using the implicit function theorem sufficient conditions for Lipschitz continuity are derived.

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[175] V. D. Blondel, R. Jungers, and V. Protasov, *On the complexity of computing the capacity of codes that avoid forbidden difference patterns*, IEEE Trans. Inform. Theory **52** (2006), no. 11, 5122–5127, `doi:10.1109/TIT.2006.883615`, `arXiv:cs/0601036`. MR 2300380. Zbl 1320.94039.

Some questions related to the computation of the capacity of codes that avoid forbidden difference patterns are analysed. The maximal number of n-bit sequences whose pairwise differences do not contain some given forbidden difference patterns is known to increase exponentially with n; the coefficient of the exponent is the capacity of the forbidden patterns. In this paper, new inequalities for the capacity are given that allow for the approximation of the capacity with arbitrary high accuracy. The computational cost of the algorithm derived from these inequalities is fixed once the desired accuracy is given. Subsequently, a polynomial time algorithm is given for determining if the capacity of a set is positive while the same problem is shown to be NP-hard when the sets of forbidden patterns are defined over an extended set of symbols. Finally, the existence of extremal norms is proved for any set of matrices arising in the capacity computation. Based on this result, a second capacity approximating algorithm is proposed. The usefulness of this algorithm is illustrated by computing exactly the capacity of particular codes that were only known approximately

[176] A. Czornik and P. Jurgaś, *Some properties of the spectral radius of a set of matrices*, Int. J. Appl. Math. Comput. Sci. **16** (2006), no. 2, 183–188, URL `http://matwbn.icm.edu.pl/ksiazki/amc/amc16/amc1623.pdf`. MR 2238011. Zbl 1113.15009.

We show new formulas for the spectral radius and the spectral subradius of a set of matrices. The advantage of our results is that we express the spectral radius of any set of matrices by the spectral radius of a set of symmetric positive definite matrices. In particular, in one of our formulas the spectral radius is expressed by singular eigenvalues of matrices, whereas in the existing results it is expressed by eigenvalues.

[177] Y.-Y. Lur, *A max version of the generalized spectral radius theorem*, Linear Algebra Appl. **418** (2006), no. 1, 336–346, `doi:10.1016/j.laa.2006.02.014`. MR 2257600. Zbl 1108.15009.

The generalized spectral radius theorem states that the generalized spectral radius and the joint spectral radius of a bounded set of square complex matrices coincide. In this paper a max algebra version of the generalized spectral radius theorem is proposed.

Let Ψ be a bounded set of $n \times n$ nonnegative matrices in max algebra. In this paper we propose the notions of the max algebra version of the generalized spectral radius $\mu(\Psi)$ of Ψ, and the max algebra version of the joint spectral radius $\eta(\Psi)$ of Ψ. The max algebra version of the generalized spectral radius theorem $\mu(\Psi) = \eta(\Psi)$ is established. We propose the relationship between the generalized spectral radius $\rho(\Psi)$ of Ψ (in the sense of Daubechies and Lagarias) and its max algebra version $\mu(\Psi)$. Moreover, a generalization of Elsner and van den Driessche's lemma is presented as well.

[178] Y.-Y. Lur, *A note on a gap result for norms of semigroups of matrices*, Linear Algebra Appl. **419** (2006), no. 2-3, 368–372, `doi:10.1016/j.laa.2006.05.005`. MR 2277976. Zbl 1112.15016.

Let $\|\cdot\|$ be a matrix norm and Σ be a bounded set of $n \times n$ complex matrices. For $m \geq 1$, let Σ_m be the set of all products of length m of matrices in Σ. Let $S(\Sigma)$ denote the multiplicative semigroup generated by Σ. The generalized spectral radius of Σ, $\rho(\Sigma)$, is defined by

$\rho(\Sigma) = \limsup_{m\to\infty}[\rho_m(\Sigma)]^{\frac{1}{m}}$, where $\rho_m(\Sigma) = \sup_{A\in\Sigma_m}\rho(A)$, and $\rho(A)$ is the spectral radius of A. The joint spectral radius of Σ, $\hat\rho(\Sigma)$, is defined by $\hat\rho(\Sigma) = \limsup_{m\to\infty}[\hat\rho_m(\Sigma, \|\cdot\|)]^{\frac{1}{m}}$, where $\hat\rho_m(\Sigma, \|\cdot\|) = \sup_{A\in\Sigma_m}\|A\|$.

J. P. Bell [155] proved that there is a gap in the possible growth of $\hat\rho_m(\Sigma, \|\cdot\|)$ if Σ consists of finitely many $n\times n$ complex matrices. In this note, based on a lemma by Elsner, the author gives an elementary proof of Bell's theorem in bounded case. The author proves that either there is some constant $\alpha > 1$ such that $\hat\rho_m(\Sigma, \|\cdot\|) > \alpha^m$ for all m sufficiently large, or $\hat\rho_m(\Sigma, \|\cdot\|) = O(m^{n-1})$. Moreover, $\hat\rho_m(\Sigma, \|\cdot\|) = O(m^{n-1})$ if and only if the eigenvalues of every matrix in the semigroup generated by Σ are all on or inside the unit circle.

[179] E. Seneta, *Non-negative matrices and Markov chains*, Springer Series in Statistics, Springer, New York, 2006, revised reprint of the second (1981) edition [Springer-Verlag, New York; MR0719544]. MR 2209438. Zbl 1099.60004.

This book is a photographic reproduction of the book of the same title published in 1981, for which there has been continuing demand on account of its accessible technical level. Its appearance also helped generate considerable subsequent work on inhomogeneous products of matrices. This printing adds an additional bibliography on coefficients of ergodicity and a list of corrigenda.

[180] X. Zhou, *Estimates for the joint spectral radius*, Appl. Math. Comput. **172** (2006), no. 1, 332–348, `doi:10.1016/j.amc.2005.02.006`. MR 2197905. Zbl 1104.65033.

Estimates for the computation of the joint spectral radius defined by a bounded collection of square matrices and of same size will be given. In particular, the computational result shows that for matrices constructed by four-coefficient two-scale dilation equation our approach leads to the exact value of this joint spectral radius.

2007

[181] A. Czornik and A. Nawrat, *Generalized spectral radius and Lyapunov exponents of linear time varying systems*, IEEE International Conference on Control Applications, 2007. CCA 2007 (Singapore), IEEE, 2007, pp. 1303–1306, `doi:10.1109/CCA.2007.4389415`.

In this paper we consider discrete time varying linear systems with matrix coefficients in fixed set and we describe the set of all possible Lyapunov exponents for the system. We describe this set in terms of generalized spectral sub radius and the generalized spectral radius of the set of possible values of the coefficients.

[182] A. Czornik and P. Jurgaś, *Falseness of the finiteness property of the spectral subradius*, Int. J. Appl. Math. Comput. Sci. **17** (2007), no. 2, 173–178, `doi:10.2478/v10006-007-0016-1`. MR 2341291. Zbl 1126.93028.

By using only elementary facts from the theory of formal languages and from linear algebra, it is proved (but not in a constructive manner) that there exist infinitely many values of the real parameter α for which the exact value of the spectral subradius of the pair of matrices $\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$, $\alpha\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]$ cannot be calculated in a finite number of steps. This kind of falseness does not imply that no algorithm exists for an exact value computing of the spectral subradius of a finite set of real matrices in a finite number of steps. It simply states that it is impossible to set forth such an algorithm in the way that is suggested in the finiteness property of the spectral subradius. Thus, the problem of inventing such an algorithm is still open.

[183] N. Guglielmi and M. Zennaro, *Balanced complex polytopes and related vector and matrix norms*, J. Convex Anal. **14** (2007), no. 4, 729–766, URL `https://www.heldermann.de/JCA/JCA14/JCA144/jca14043.htm`. MR 2350813. Zbl 1128.52010.

In this paper we study the notion of *balanced complex polytope* as a generalization of a symmetric real polytope to the complex space $\mathbb{C}^n$. We pay particular attention to the geometric properties of such complex polytopes and of their counterparts in the *adjoint* form. In particular, we stress the differences occurring with respect to the well-known real case. We also introduce and discuss the related definitions of *complex polytope norm* and *adjoint complex polytope norm*.

[184] L. Gurvits, R. Shorten, and O. Mason, *On the stability of switched positive linear systems*, IEEE Trans. Automat. Control **52** (2007), no. 6, 1099–1103, doi:10.1109/TAC.2007.899057. MR 2329904. Zbl 1366.93436.

It was recently conjectured that the Hurwitz stability of, the, convex hull of a set of Metzler matrices is a necessary and sufficient condition for the asymptotic stability of the associated switched linear system under arbitrary switching. In this note, we show that (1) this conjecture is true for systems constructed from a pair of second-order Metzler matrices; (2) the conjecture is true for systems constructed from an arbitrary finite number of second-order Metzler matrices; and (3) the conjecture is in general false for higher order systems. The implications of our results, both for the design of switched positive linear systems, and for research directions that arise as a result of our work, are discussed toward the end of the note.

[185] V. S. Kozyakin, *Structure of extremal trajectories of discrete linear systems and the finiteness conjecture*, Autom. Remote Control **68** (2007), no. 1, 174–209, doi:10.1134/S0005117906040171. Zbl 1195.93082.

J. C. Lagarias and *Y. Wang* [53] conjectured that the generalized spectral radius of a finite set of square matrices can be attained on a finite product of matrices. The first counterexample to this Finiteness Conjecture was given by *T. Bousch* and *J. Mairesse* [132] and their proof was based on measure-theoretical ideas. *V. D. Blondel*, *J. Theys* and *A. A. Vladimirov* [140] proposed another proof of a counterexample to the Finiteness Conjecture which extensively exploited combinatorial properties of permutations of products of positive matrices. In the control theory, so as in the general theory of dynamical systems, the notion of generalized spectral radius is used basically to describe the rate of growth or decrease of the trajectories generated by matrix products. In this context, the above mentioned methods are not enough satisfactory (from the point of view of the author, of course) since they give no description of the structure of the trajectories with the maximal growing rate (or minimal decreasing rate). In connection with this, in 2005 *V. S. Kozyakin* [165] presented one more proof of the counterexample to the Finiteness Conjecture fulfilled in the spirit of the theory of dynamical systems. Unfortunately, the developed approach did not cover the class of matrices considered by Blondel, Theys and Vladimirov. The goal of the present paper is to compensate for this deficiency in the previous approach.

[186] B. Moision, A. Orlitsky, and P. H. Siegel, *On codes with local joint constraints*, Linear Algebra Appl. **422** (2007), no. 2-3, 442–454, doi:10.1016/j.laa.2006.11.002. MR 2305130. Zbl 1180.68207.

We study the largest number of sequences with the property that any two sequences do not contain specified pairs of patterns. We show that this number increases exponentially with the length of the sequences and that the exponent, or capacity, is the logarithm of the joint spectral radius of an appropriately defined set of matrices. We illustrate a new heuristic for computing the joint spectral radius and use it to compute the capacity for several simple collections. The problem of computing the achievable rate region of a collection of codes is introduced and it is shown that the region may be computed via a similar analysis.

[187] P. A. Parrilo and A. Jadbabaie, *Approximation of the joint spectral radius of a set of matrices using sum of squares*, Hybrid systems: computation and control, Lecture Notes in Comput. Sci., vol. 4416, Springer, Berlin, 2007, pp. 444–458, doi:10.1007/978-3-540-71493-4_35. MR 2363632. Zbl 1221.65098.

We provide an asymptotically tight, computationally efficient approximation of the joint spectral radius of a set of matrices using sum of squares (SOS) programming. The approach is based on a search for a SOS polynomial that proves simultaneous contractibility of a finite set of matrices. We provide a bound on the quality of the approximation that unifies several earlier results and is independent of the number of matrices. Additionally, we present a comparison between our approximation scheme and a recent technique due to Blondel and Nesterov, based on lifting of matrices. Theoretical results and numerical investigations show that our approach yields tighter approximations.

[188] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King, *Stability criteria for switched and hybrid systems*, SIAM Rev. **49** (2007), no. 4, 545–592, `doi:10.1137/05063516X`. MR 2375524. Zbl 1127.93005.

The study of the stability properties of switched and hybrid systems gives rise to a number of interesting and challenging mathematical problems. The objective of this paper is to outline some of these problems, to review progress made in solving them in a number of diverse communities, and to review some problems that remain open. An important contribution of our work is to bring together material from several areas of research and to present results in a unified manner. We begin our review by relating the stability problem for switched linear systems and a class of linear differential inclusions. Closely related to the concept of stability are the notions of exponential growth rates and converse Lyapunov theorems, both of which are discussed in detail. In particular, results on common quadratic Lyapunov functions and piecewise linear Lyapunov functions are presented, as they represent constructive methods for proving stability and also represent problems in which significant progress has been made. We also comment on the inherent difficulty in determining stability of switched systems in general, which is exemplified by NP-hardness and undecidability results. We then proceed by considering the stability of switched systems in which there are constraints on the switching rules, through both dwell-time requirements and state-dependent switching laws. Also in this case the theory of Lyapunov functions and the existence of converse theorems are reviewed. We briefly comment on the classical Lur'e problem and on the theory of stability radii, both of which contain many of the features of switched systems and are rich sources of practical results on the topic. Finally we present a list of questions and open problems which provide motivation for continued research in this area.

2008

[189] A. A. Ahmadi, *Non-monotonic Lyapunov functions for stability of nonlinear and switched systems: theory and computation*, Ph.D. thesis, Dept. of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, 2008, URL `https://dspace.mit.edu/handle/1721.1/44206`.

Lyapunov's direct method, which is based on the existence of a scalar function of the state that decreases monotonically along trajectories, still serves as the primary tool for establishing stability of nonlinear systems. Since the main challenge in stability analysis based on Lyapunov theory is always to find a suitable Lyapunov function, weakening the requirements of the Lyapunov function is of great interest. In this thesis, we relax the monotonicity requirement of Lyapunov's theorem to enlarge the class of functions that can provide certificates of stability. Both the discrete time case and the continuous time case are covered. Throughout the thesis, special attention is given to techniques from convex optimization that allow for computationally tractable ways of searching for Lyapunov functions. Our theoretical contributions are therefore amenable to convex programming formulations. In the discrete time case, we propose two new sufficient conditions for global asymptotic stability that allow the Lyapunov functions to increase locally, but guarantee an average decrease every few steps. Our first condition is nonconvex, but allows an intuitive interpretation. The second condition, which includes the first one as a special case, is convex and can be cast as a semidefinite program. We show that when non-monotonic Lyapunov functions exist, one can construct a more complicated function that decreases monotonically. We demonstrate the strength of our methodology over standard Lyapunov theory through examples from three different classes of dynamical systems. First, we consider polynomial dynamics where we utilize techniques from sum-of-squares programming. Second, analysis of piecewise and systems is performed. Here, connections to the method of piecewise quadratic Lyapunov functions are made.(cont.) Finally, we examine systems with arbitrary switching between a finite set of matrices. It will be shown that tighter bounds on the joint spectral radius can be obtained using our technique. In continuous time, we present conditions invoking higher derivatives of Lyapunov functions that allow the Lyapunov function to increase but bound the rate at which the increase can happen. Here, we build on previous work by Butz that provides a nonconvex sufficient condition for asymptotic stability using the first three derivatives of Lyapunov functions. We give a convex condition for asymptotic stability that includes the condition by Butz as a special case. Once again, we draw the connection to standard Lyapunov functions. An example of a polynomial vector field is given to show the potential advantages of using higher order derivatives over standard Lyapunov theory. We also discuss a theorem by Yorke that imposes minor conditions on the first and second derivatives to reject existence of periodic orbits, limit cycles, or chaotic attractors. We give some simple convex conditions that imply the requirement by Yorke and we compare them with those given in another earlier work. Before presenting our main contributions, we review some aspects of convex programming with more emphasis on semidefinite programming. We explain in detail how the method of sum of squares decomposition can be used to efficiently search for polynomial Lyapunov functions.

[190] A. A. Ahmadi and P. A. Parrilo, *Non-monotonic Lyapunov functions for stability of discrete time nonlinear and switched systems*, Proceedings of the 47th IEEE Conference on Decision and Control (CDC), 2008, pp. 614–621, `doi:10.1109/CDC.2008.4739402`.

We relax the monotonicity requirement of Lyapunov's theorem to enlarge the class of functions that can provide certificates of stability. To this end, we propose two new sufficient conditions for global asymptotic stability that allow the Lyapunov functions to increase locally, but guarantee an average decrease every few steps. Our first condition is non-convex, but allows an intuitive interpretation. The second condition, which includes the first one as a special case, is convex and can be cast as a semidefinite program. We show that when non-monotonic Lyapunov functions exist, one can construct a more complicated function that decreases monotonically. We demonstrate the strength of our methodology over standard Lyapunov theory through examples from three different classes of dynamical systems. First, we consider polynomial dynamics where we utilize techniques from sum-of-squares programming. Second, analysis of piecewise affine systems is performed. Here, connections to the method of piecewise quadratic Lyapunov functions are made. Finally, we examine systems with arbitrary switching between a finite set of matrices. It will be shown that tighter bounds on the joint spectral radius can be obtained using our technique.

[191] V. D. Blondel, *The birth of the joint spectral radius: an interview with Gilbert Strang*, Linear Algebra Appl. **428** (2008), no. 10, 2261–2264, `doi:10.1016/j.laa.2007.12.010`. MR 2405243. Zbl 1149.15001.

The definition of the joint spectral radius first appeared in 1960. It has resurfaced in a number of application areas, including wavelets, control theory, number theory, coding and sensor networks. In the interview the author discusses how this notion was actually born.

[192] V. D. Blondel, M. Karow, V. Protassov, and F. R. Wirth, *Special issue on the joint spectral radius: theory, methods and applications*, Linear Algebra Appl. **428** (2008), no. 10, 2259–2260, `doi:10.1016/j.laa.2008.01.016`. MR 2405242.

The joint spectral radius was first defined in 1960 in a paper by G.-C. Rota and G. Strang. It describes the maximal growth rate of arbitrary matrix products of matrices from a given set of matrices. As such it is a natural notion in matrix analysis. It turns out that what is easy to understand for a single matrix, in general poses a number of subtle problems for a set of matrices even if this set only contains two elements.

In the beginning the notion did not receive widespread attention, even though it is intimately related to the stability of time-varying systems and at the time there was considerable interest in this subject. The first edition of Cesari's monograph on stability of ordinary differential equations had just appeared in which this problem has a prominent place. Also Yoshizawa's monograph on the theory of Lyapunov functions only appeared 6 years later. Indeed the intimate link of the theory of the joint spectral radius with stability theory and in particular Lyapunov functions was not noted until in the late 80s Barabanov and also Pyatnitskii took up the study of this problem independently. However, at the time of the first definition of the joint spectral radius this connection was unnoticed.

In the early 90s this all changed. On one hand it was observed, that the question of convergence of products of matrices arises in a natural way in the theory of wavelets. On the other hand timevarying systems were receiving greater attention in the field of control and the study of linear inclusions was prominent. Driven by interest in these two fields, important progress has been made during the last decade. Further interesting applications of the joint spectral radius have been discovered. This quantity turned out to be useful in the analysis of the capacity of certain codes, in the stability analysis of numerical integration schemes and in the study of nonhomogeneous Markov chains. From a theoretical point of view the link between the joint spectral radius and extremal norms was explored in detail, regularity properties of the joint spectral radius as a map were discovered and

approximation schemes were shown. Probably the most famous results in this area are related to the finiteness property, which was conjectured to hold for all finite matrix sets. This conjecture was later disproved by showing that a counterexample must exist.1 More on this subject can be found in this issue. Finally, many different approaches to the calculation or maybe better approximation of the joint spectral radius were developed. These range from ideas using basic inequalities for the joint spectral radius, the numerical construction of extremal norms, the relation of the joint spectral radius to higher Kronecker powers, to discounted optimal control techniques based on the theory of Lyapunov exponents.

In this special issue we were fortunate to obtain submissions which present advances as well in new developments in the computation of the joint spectral radius as in theoretical issues.

We are grateful to all the contributors to this special issue and we hope that this collection will prove to be useful for researchers interested in the field. Clearly, there remain many open questions and new ones will arise as new applications are discovered.

[193] A. Czornik and P. Jurgaś, *Set of possible values of maximal Lyapunov exponents of discrete time-varying linear system*, Automatica J. IFAC **44** (2008), no. 2, 580–583, `doi:10.1016/j.automatica.2007.06.028`. MR 2530809. Zbl 1283.93170.

In this paper we consider discrete time varying linear systems with coefficients in fixed set of invertible matrices and we describe the set of all possible maximal Lyapunov exponents for the system. We show that the set includes an interval bounded by the generalized spectral subradius and the generalized spectral radius.

[194] X. Dai, Y. Huang, and M. Xiao, *Almost sure stability of discrete-time switched linear systems: a topological point of view*, SIAM J. Control Optim. **47** (2008), no. 4, 2137–2156, `doi:10.1137/070699676`. MR 2421343. Zbl 1165.93030.

We study the stability of discrete-time switched linear systems via symbolic topology formulation and the multiplicative ergodic theorem. A sufficient and necessary condition for μ_A-almost sure stability is derived, where μ_A is the Parry measure of the topological Markov chain with a prescribed transition $(0, 1)$-matrix A. The obtained μ_A-almost sure stability is invariant under small perturbations of the system. The topological description of stable processes of switched linear systems in terms of Hausdorff dimension is given, and it is shown that our approach captures the maximal set of stable processes for linear switched systems. The obtained results cover the stochastic Markov jump linear systems, where the measure is the natural Markov measure defined by the transition probability matrix. Two examples are provided to illustrate the theoretical outcomes of the paper.

[195] X. Dai, Y. Huang, and M. Xiao, *Topological formulation of discrete-time switched linear systems and almost sure stability*, 47th IEEE Conference on Decision and Control (CDC), 9-11 Dec. (Cuncun), IEEE, 2008, pp. 965–970, `doi:10.1109/CDC.2008.4738721`.

In this paper, we study the stability of discrete-time switched linear systems via symbolic topology formulation and the multiplicative ergodic theorem. A sufficient and necessary condition for μ_A-almost sure stability is derived, where μ_A is the Parry measure of the topological Markov chain with a prescribed transition $(0, 1)$-matrix A. The obtained μ_A-almost sure stability is invariant under small perturbations of the system. The topological description of stable processes of switched linear systems in terms of Hausdorff dimension is given, and it is shown that our approach captures the maximal set of stable processes for linear switched systems. The obtained results cover the stochastic Markov jump linear systems, where the measure is the natural Markov measure defined by the transition probability matrix.

[196] B. Fayad and R. Krikorian, *Exponential growth of product of matrices in SL$(2,\mathbb{R})$*, Nonlinearity **21** (2008), no. 2, 319–323, `doi:10.1088/0951-7715/21/2/007`. MR 2384551. Zbl 1142.37024.

In this paper the exponential growth of products of two matrices $A, B \in \mathrm{SL}(2,\mathbb{R})$ is investigated. It is proved, assuming A is a fixed hyperbolic matrix, that for Lebesgue almost every B, products of length n involving less than $n\alpha$, $0 \leq \alpha < 1/2$ matrices B are uniformly bounded from below by γn for some $\gamma > 1$.

[197] N. Guglielmi and M. Zennaro, *An algorithm for finding extremal polytope norms of matrix families*, Linear Algebra Appl. **428** (2008), no. 10, 2265–2282, `doi:10.1016/j.laa.2007.07.009`. MR 2405244. Zbl 1139.65027.

The problem of the computation of the joint spectral radius of a finite set of matrices is considered. We present an algorithm which, under some suitable assumptions, is able to check if a certain product in the multiplicative semigroup is spectrum maximizing. The algorithm proceeds by attempting to construct a suitable extremal norm for the family, namely a complex polytope norm.

As examples for testing our technique, we first consider the set of two 2-dimensional matrices recently analyzed by *V. D. Blondel*, *J. Theys* and *A. A. Vladimirov* [140] to disprove the finiteness conjecture, and then a set of 3-dimensional matrices arising in the zero-stability analysis of the 4-step backward differentiation formula for ordinary differential equations.

[198] R. M. Jungers and V. D. Blondel, *On the finiteness property for rational matrices*, Linear Algebra Appl. **428** (2008), no. 10, 2283–2295, `doi:10.1016/j.laa.2007.07.007`. MR 2405245. Zbl 1148.15004.

We analyze the periodicity of optimal long products of matrices. A set of matrices is said to have the finiteness property if the maximal rate of growth of long products of matrices taken from the set can be obtained by a periodic product. It was conjectured a decade ago that all finite sets of real matrices have the finiteness property. This "finiteness conjecture" is now known to be false but no explicit counterexample is available and in particular it is unclear if a counterexample is possible whose matrices have rational or binary entries. In this paper, we prove that all finite sets of nonnegative rational matrices have the finiteness property if and only if *pairs* of *binary matrices* do and we state a similar result when negative entries are allowed. We also show that all pairs of 2×2 binary matrices have the finiteness property. These results have direct implications for the stability problem for sets of matrices. Stability is algorithmically decidable for sets of matrices that have the finiteness property and so it follows from our results that if all pairs of binary matrices have the finiteness property then stability is decidable for nonnegative rational matrices. This would be in sharp contrast with the fact that the related problem of boundedness is known to be undecidable for sets of nonnegative rational matrices.

[199] R. M. Jungers, V. Protasov, and V. D. Blondel, *Efficient algorithms for deciding the type of growth of products of integer matrices*, Linear Algebra Appl. **428** (2008), no. 10, 2296–2311, `doi:10.1016/j.laa.2007.08.001`. MR 2405246. Zbl 1145.65030.

For a given set Σ of matrices, the joint spectral radius of Σ, denoted $\rho(\Sigma)$, is defined by the limit $\rho(\Sigma) = \lim_{t\to\infty} \max\{\|A_1 \cdots A_t\|^{1/t} : A_t \in \Sigma\}$ independently of any norm. For any finite set Σ of $n \times n$ matrices with nonnegative integer entries, the authors show that there is a polynomial algorithm that decides between the four cases: $\rho = 0$, $\rho = 1$ and bounded, $\rho = 1$ and polynomial growth, and $\rho > 1$. The polynomial solvability for $\rho > 1$ is somewhat surprising.

[200] R. M. Jungers, V. Yu. Protasov, and V. D. Blondel, *Computing the growth of the number of overlap-free words with spectra of matrices*, LATIN 2008: Theoretical Informatics (Proceedings of the 8th Latin American Symposium, Búzios, Brazil, April 7–11, 2008), Lecture Notes in Computer Science, vol. 4957, Springer Berlin Heidelberg, 2008, pp. 84–93, `doi:10.1007/978-3-540-78773-0_8`. MR 2472727. Zbl 1136.68471.

Overlap-free words are words over the alphabet $A = \{a, b\}$ that do not contain factors of the form $xvxvx$, where $x \in A$ and $v \in A^*$. We analyze the asymptotic growth of the number u_n of overlap-free words of length n. We obtain explicit formulas for the minimal and maximal rates of growth of u_n in terms of spectral characteristics (the lower spectral radius and the joint spectral radius) of one set of matrices of dimension 20. Using these descriptions we provide estimates of the rates of growth that are within 0.4% and 0.03% of their exact value. The best previously known bounds were within 11% and 3% respectively. We prove that u_n actually has the same growth for "almost all" n. This "average" growth is distinct from the maximal and minimal rates and can also be expressed in terms of a spectral quantity (the Lyapunov exponent). We use this expression to estimate it.

[201] Y.-C. Li and M.-H. Shih, *The normed finiteness property of compact contraction operators*, Linear Algebra Appl. **428** (2008), no. 10, 2319–2323, `doi:10.1016/j.laa.2007.08.030`. MR 2405248. Zbl 1162.47011.

We study the joint spectral radius given by a finite set of compact operators on a Hilbert space. It is shown that the normed finiteness property holds in this case, that is, if all the compact operators are contractions and the joint spectral radius is equal to 1 then there exists a finite product that has a spectral radius equal to 1. We prove an additional statement in that the requirement that the joint spectral radius be equal to 1 can be relaxed to the asking that the maximum norm of finite products of a length norm is equal to 1. The length of this product is related to the dimension of the subspace on which the set of operators is norm preserving.

Specifically, let Σ be a nonempty finite set of (linear) compact contraction operators on a Hilbert space $\mathcal{H}$, let Σ^m $(m = 1, 2, \ldots)$ be the set of all products of operators in Σ of length m, and let $\widetilde{\rho}(\Sigma)$ denote the joint spectral radius of Σ, that is, $\widetilde{\rho}(\Sigma) = \limsup_{m\to\infty}[\sup(\|A\| : A \in \Sigma^m)]^{1/m}$. The authors prove for such Σ that $\|Ax\| < \|x\|$ with some $A \in \Sigma^m$ and $(0 \neq)x \in \mathcal{H}$, that if $\widetilde{\rho}(\Sigma) = 1$ or there exists a finite product of the unit norm, then there exists a finite product with spectral radius equal to 1.

[202] M. Maesumi, *Optimal norms and the computation of joint spectral radius of matrices*, Linear Algebra Appl. **428** (2008), no. 10, 2324–2338, `doi:10.1016/j.laa.2007.09.036`. MR 2408030. Zbl 1138.65030.

The notion of spectral radius of a set of matrices is a natural extension of spectral radius of a single matrix. The finiteness conjecture (FC) claims that among the infinite products made from the elements of a given finite set of matrices, there is a certain periodic product, made from the repetition of the optimal product, whose rate of growth is maximal. FC has been disproved.

In this paper it is conjectured that FC is almost always true, and an algorithm is presented to verify the optimality of a given product. The algorithm uses optimal norms, as a special subset of extremal norms. Several conjectures related to optimal norms and non-decomposable sets of matrices are presented. The algorithm has successfully calculated the spectral radius of several parametric families of pairs of matrices associated with compactly supported multi-resolution analyses and wavelets. The results of related numerical experiments are presented.

[203] P. A. Parrilo and A. Jadbabaie, *Approximation of the joint spectral radius using sum of squares*, Linear Algebra Appl. **428** (2008), no. 10, 2385–2402, `doi:10.1016/j.laa.2007.12.027`. MR 2408034. Zbl 1151.65032.

We provide an asymptotically tight, computationally efficient approximation of the joint spectral radius of a set of matrices using sum of squares (SOS) programming. The approach is based on a search for an SOS polynomial that proves simultaneous contractibility of a finite set of matrices. We provide a bound on the quality of the approximation that unifies several earlier results and is independent of the number of matrices. Additionally, we present a comparison between our approximation scheme and earlier techniques, including the use of common quadratic Lyapunov functions and a method based on matrix liftings. Theoretical results and numerical investigations show that our approach yields tighter approximations.

[204] A. Peperko, *On the max version of the generalized spectral radius theorem*, Linear Algebra Appl. **428** (2008), no. 10, 2312–2318, `doi:10.1016/j.laa.2007.08.041`. MR 2405247. Zbl 1144.15006.

The generalized spectral radius theorem states that the generalized spectral radius and the joint spectral radius of a bounded set of square complex matrices coincide. A max algebra version of the generalized spectral radius theorem was proposed by *Y.-Y. Lur* [177]. This paper gives a short proof of it by generalizing a result about the relationship between the maximum circuit geometric mean and Hadamard powers of a matrix, which was proved by *S. Friedland* [Linear Algebra Appl. 74, 173–178 (1986)].

[205] E. Plischke and F. Wirth, *Duality results for the joint spectral radius and transient behavior*, Linear Algebra Appl. **428** (2008), no. 10, 2368–2384, `doi:10.1016/j.laa.2007.12.009`. MR 2408033. Zbl 05268361.

For linear inclusions in discrete or continuous time several quantities characterizing the growth behavior of the corresponding semigroup are analyzed. These quantities are the joint spectral radius, the initial growth rate and (for bounded semigroups) the transient bound. It is discussed how these constants relate to one another and how they are characterized by various norms. A complete duality theory is developed in this framework, relating semigroups and dual semigroups and extremal or transient norms with their respective dual norms.

[206] V. Yu. Protasov, *Extremal L_p-norms of linear operators and self-similar functions*, Linear Algebra Appl. **428** (2008), no. 10, 2339–2356, `doi:10.1016/j.laa.2007.09.023`. MR 2408031. Zbl 1147.15023.

We prove that for any $p \in [0, \infty]$ a finite irreducible family of linear operators possesses an extremal norm corresponding to the p-radius of these operators. As a corollary, we derive a criterion for the L_p-contractibility property of linear operators and estimate the asymptotic growth of orbits for any point. These results are applied to the study of functional difference equations with linear contractions of the argument (self-similarity equations). We obtain a sharp criterion for the existence and uniqueness of solutions in various functional spaces, compute the exponents of regularity, and estimate moduli of continuity. This, in particular, gives a geometric interpretation of the p-radius in terms of spectral radii of certain operators in the space $L_p[0, 1]$.

[207] M.-H. Shih and C.-T. Pang, *Simultaneous Schur stability of interval matrices*, Automatica J. IFAC **44** (2008), no. 10, 2621–2627, `doi:10.1016/j.automatica.2008.02.026`. MR 2531059. Zbl 1155.93416.

Interval matrix structures are ubiquitous in nature and engineering. Ordinarily, in an uncertain system there is associated with a set of coupled interval matrices, a basic issue of exploring its asymptotic stability. Here we introduce the notion of simultaneous Schur stability by linking the concepts of the majorant and the joint spectral radius, and prove the asymptotic stability of a set of interval matrices governed by simultaneous Schur stability. The present result may lead to the stability analysis of discrete dynamical interval systems.

[208] V. S. Shulman and Yu. V. Turovskii, *Application of topological radicals to calculation of joint spectral radii*, ArXiv.org e-Print archive, May 2008, `arXiv:0805.0209`.

It is shown that the joint spectral radius $\rho(M)$ of a precompact family M of operators on a Banach space X is equal to the maximum of two numbers: the joint spectral radius $\rho_e(M)$ of the image of M in the Calkin algebra and the Berger-Wang radius $r(M)$ defined by the formula

$$r(M) = \limsup_{n\to\infty} \left(\sup \{\rho(a) : a \in M^n\}^{1/n} \right).$$

Some more general Banach-algebraic results of this kind are also proved. The proofs are based on the study of special radicals on the class of Banach algebras.

[209] S. E. Tuna, *Growth rate of switched homogeneous systems*, Automatica J. IFAC **44** (2008), no. 11, 2857–2862, `doi:10.1016/j.automatica.2008.03.017`. MR 2527207. Zbl 1152.93492.

We consider discrete-time homogeneous systems under arbitrary switching and study their growth rate, the analogue of joint spectral radius for switched linear systems. We show that a system is asymptotically stable if and only if its growth rate is less than unity. We also provide an approximation algorithm to compute growth rate with arbitrary accuracy.

[210] C. Vagnoni, *Algorithms for the computation of the joint spectral radius*, Ph.D. thesis, Dipartimento di Matematica Pura ed Applicata, Scuola di Dottorato di Ricerca in Scienze Matematiche Indirizzo Matematica Computazionale, Università degli Studi di Padova, 2008, URL `http://paduaresearch.cab.unipd.it/408/`.

The asymptotic behaviour of the solutions of a discrete linear dynamical system is related to the spectral radius R of its associated family $\mathcal{F}$. In particular, a system is stable if $R \leq 1$ and there exists an extremal norm for the family $\mathcal{F}$. This kind of systems is important for a large number of applications. In particular, we mention the stability analysis of numerical methods for ordinary differential equations.

In the last decades some algorithms have been proposed in order to find real extremal norms of polytope type in the case of finite families. However, recently it has been observed that it is more useful to consider complex polytope norms, which are norms whose unit ball is a balanced complex polytope.

In this work, using the theory developed by *N. Guglielmi* and *M. Zennaro* in [183], we extend the algorithm for the construction (i.e. for the geometric representation) of the unit ball of real polytope norms to the complex space. In order to succeed in our purpose, we first needed to get a deeper theoretical knowledge of the balanced complex polytopes. However, due to the extreme increase in complexity of the geometry of such objects with the dimension n of the space, we have confined ourselves to face the two-dimensional case.

In particular, we have given original theoretical results on the geometry of two-dimensional balanced complex polytopes in order to present the first two efficient algorithms, one for the construction of a balanced complex polytope in the two-dimensional space and one for the computation of the complex polytope norm of a two-dimensional vector starting from the knowledge of the boundary of the corresponding unit ball.

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[211] A. Avila and T. Roblin, *Uniform exponential growth for some* $\mathrm{SL}(2,\mathbb{R})$ *matrix products*, J. Mod. Dyn. **3** (2009), no. 4, 549–554, `doi:10.3934/jmd.2009.3.549`. MR 2587085. Zbl 1189.37060.

Given a hyperbolic matrix $H \in SL(2,\mathbb{R})$, we prove that for almost every $R \in SL(2,\mathbb{R})$, any product of length n of H and R grows exponentially fast with n provided the matrix R occurs less than $o(\frac{n}{\log n \log\log n})$ times.

[212] V. D. Blondel and Yu. Nesterov, *Polynomial-time computation of the joint spectral radius for some sets of nonnegative matrices*, SIAM J. Matrix Anal. Appl. **31** (2009), no. 3, 865–876, `doi:10.1137/080723764`. MR 2538656. Zbl 1201.65051.

We propose two simple upper bounds for the joint spectral radius of sets of nonnegative matrices. These bounds, the joint column radius and the joint row radius, can be computed in polynomial time as solutions of convex optimization problems. We show that these bounds are within a factor $1/n$ of the exact value, where n is the size of the matrices. Moreover, for sets of matrices with independent column uncertainties or with independent row uncertainties, the corresponding bounds coincide with the joint spectral radius. In these cases, the joint spectral radius is also given by the largest spectral radius of the matrices in the set. As a by-product of these results, we propose a polynomial-time technique for solving Boolean optimization problems related to the spectral radius. We also describe economics and engineering applications of our results.

[213] N. Guglielmi and M. Zennaro, *Finding extremal complex polytope norms for families of real matrices*, SIAM J. Matrix Anal. Appl. **31** (2009), no. 2, 602–620, `doi:10.1137/080715718`. MR 2530266. Zbl 1197.93129.

The paper considers finite families $\mathcal{F}$ of real $n \times n$ matrices and, in particular, is focused on the computation of the joint spectral radius $\rho(\mathcal{F})$ through detection of an extremal norm in the class of complex polytope norms whose unit balls are balanced complex polytopes with a finite essential system of vertices. Such a finiteness property is very useful in view of the construction of efficient computational algorithms. More precisely, the paper improves the results obtained in a previous paper by the authors [161], which gave some conditions on the family $\mathcal{F}$ to guarantee the existence of an extremal complex polytope norm. Unfortunately, they exclude unnecessarily many interesting cases of real families. Therefore, here the conditions given in the previous paper in question are relaxed in order to provide a more satisfactory treatment of the real case. A modification of the algorithm [197] for finding extremal polytope norms of matrix families is put forward, its computational relevance is discussed and illustrative numerical examples are given.

[214] J. Jachymski, *König chains for submultiplicative functions and infinite products of operators*, Trans. Amer. Math. Soc. **361** (2009), no. 11, 5967–5981, doi:10.1090/S0002-9947-09-04909-5. MR 2529921. Zbl 1195.47009.

Let Σ be a set and let Σ^n (resp., Σ^∞) denote the Cartesian product of n (resp., ∞) copies of Σ. Let $P_{(1,2,\dots,n)}$ denote the projection of Σ^∞ onto Σ^n, that is,

$$P_{(1,2,\dots,n)}(\sigma_1,\sigma_2,\dots) = (\sigma_1,\sigma_2,\dots,\sigma_n), \qquad \sigma = (\sigma_1,\sigma_2,\dots) \in \Sigma^\infty.$$

The shift operator $s : \Sigma^\infty \to \Sigma^\infty$ is defined as $s(\sigma_1,\sigma_2,\sigma_3,\dots) = (\sigma_2,\sigma_3,\dots)$. A subset K of Σ^∞ is said to be shift invariant if $s(K) \subset K$. For a shift invariant set $K \subset \Sigma^\infty$, let $B_n(K) = P_{(1,2,\dots,n)}(K)$ for each positive integer n, and let $B(K) = \bigcup_n B_n(K)$. A function non-negative Φ on $B(K)$ is said to be submultiplicative if

$$\Phi(\sigma_1,\sigma_2,\dots,\sigma_n) \le \Phi(\sigma_1,\sigma_2,\dots,\sigma_j)\Phi(\sigma_{j+1},\dots,\sigma_n), \qquad 1 \le j \le n-1,$$

for each $n \ge 2$ and $(\sigma_1,\sigma_2,\dots,\sigma_n) \in B_n(K)$. A sequence $(\sigma_1,\sigma_2,\dots) \in \Sigma^\infty$ is said to be a König chain for $\{X_n : n = 1,2,\dots\}$ if $(\sigma_1,\dots,\sigma_n) \in X_n$ for $n = 1,2,\dots$, where $X_n \subset \Sigma^n$ for each positive integer n. The author gives some conditions under which there exists a König chain for a sequence of sets $\{X_n\}$, and also describes some situations for which a sequence $\{X_n\}$ does not have König chains. For a submultiplicative function Φ, let $\rho_n(\Phi) = \sup\{\Phi(w) : w \in B_n(K)\}$, where $K \subset \Sigma^\infty$ is shift invariant, then

$$\rho(\Phi) = \lim_{n\to\infty} \rho_n^{1/n}(\Phi)$$

is called the radius of Φ. An element $(\sigma_1,\sigma_2,\dots) \in K \subset \Sigma^\infty$ is called a König chain for Φ if $\Phi(\sigma_1,\sigma_2,\dots,\sigma_n) \ge (\rho(\Phi))^n$ for each positive integer n. The author proves that, if Σ is a compact topological space, K is a nonempty closed and shift invariant subset of Σ^∞, and if Φ is a positive real valued submultiplicative function on $B(K)$ such that $\Phi|_{B_n(K)}$ is upper semicontinuous, then there exists a König chain for Φ. This result leads to an improvement of a result of *L. Mate* [98]. The author also investigates *Hutchinson systems* of self-maps of a metric space, and he obtains an extension of a result due to *R. P. Kelisky* and *T. J. Rivlin* [Pac. J. Math. 21, 511–520 (1967)] concerning iterates of the Bernstein operators on the Banach space $C([0,1])$.

[215] R. M. Jungers and V. Yu. Protasov, *Counterexamples to the complex polytope extremality conjecture*, SIAM J. Matrix Anal. Appl. **31** (2009), no. 2, 404–409, doi:10.1137/080730652. MR 2530256. Zbl 1202.15027.

The authors disprove a recent conjecture due to *N. Guglielmi*, *F. Wirth*, and *M. Zennaro* [161] stating that any nondefective set of matrices having the finiteness property has an extremal complex polytope norm. The main result to disprove this conjecture is the following:

Theorem 4. *There exists an irreducible pair of* 3×3 *orthogonal matrices* A_0, A_1 *for which there is no complex polytope* $P \subset \mathbb{C}^3$ *such that* $A_iP \subset P$, $i = 0,1$.

Also, there exists a nondefective pair of 3×3 *matrices* A_0, A_1 *with nonnegative entries for which* $\rho(\{A_0,A_1\}) = \rho(A_0) = \rho(A_1) = 1$, *but there is no nondegenerate complex polytope* $P \subset \mathbb{C}^3$ *such that* $A_iP \subset P$, $i = 0,1$.

Here $\rho(\cdot)$ is the joint spectral radius.

The authors give two counterexamples to show that the conjecture is false even if the set of matrices is supposed to admit the positive orthant as an invariant cone, or even if the set of matrices is assumed to be irreducible.

[216] R. Jungers, *The joint spectral radius*, Lecture Notes in Control and Information Sciences, vol. 385, Springer-Verlag, Berlin, 2009, Theory and applications, `doi:10.1007/978-3-540-95980-9`. MR 2507938.

This monograph is based on the Ph.D. Thesis of the author. Its goal is twofold:

First, it presents most research work that has been done during his Ph.D., or at least the part of the work that is related with the joint spectral radius. This work was concerned with theoretical developments (part I) as well as the study of some applications (part II). As a second goal, it was the author's feeling that a survey on the state of the art on the joint spectral radius was really missing in the literature, so that the first two chapters of part I present such a survey. The other chapters mainly report personal research, except Chapter 5 which presents an important application of the joint spectral radius: the continuity of wavelet functions.

The first part of this monograph is dedicated to theoretical results. The first two chapters present the above mentioned survey on the joint spectral radius. Its minimum-growth counterpart, the *joint spectral subradius*, is also considered. The next two chapters point out two specific theoretical topics, that are important in practical applications: the particular case of nonnegative matrices, and the Finiteness Property.

The second part considers applications involving the joint spectral radius. We first present the continuity of wavelets. We then study the problem of the capacity of codes submitted to forbidden difference constraints. Then we go to the notion of overlap-free words, a problem that arises in combinatorics on words. We then end with the problem of trackability of sensor networks, and show how the theoretical results developed in the first part allow to solve this problem efficiently.

[217] R. M. Jungers, V. Yu. Protasov, and V. D. Blondel, *Overlap-free words and spectra of matrices*, Theoret. Comput. Sci. **410** (2009), no. 38-40, 3670–3684, `doi:10.1016/j.tcs.2009.04.022`, `arXiv:0709.1794`. MR 2553320. Zbl 1171.68035.

Overlap-free words are words over the binary alphabet $A = \{a, b\}$ that do not contain factors of the form $xvxvx$, where $x \in A$ and $v \in A^*$. We analyze the asymptotic growth of the number u_n of overlap-free words of length n as $n \to \infty$. We obtain explicit formulas for the minimal and maximal rates of growth of u_n in terms of spectral characteristics (the joint spectral subradius and the joint spectral radius) of certain sets of matrices of dimension 20×20. Using these descriptions we provide new estimates of the rates of growth that are within 0.4% and 0.03% of their exact values. The best previously known bounds were within 11% and 3%, respectively. We then prove that the value of un actually has the same rate of growth for "almost all" natural numbers n. This average growth is distinct from the maximal and minimal rates and can also be expressed in terms of a spectral quantity (the Lyapunov exponent). We use this expression to estimate it. In order to obtain our estimates, we introduce new algorithms to compute the spectral characteristics of sets of matrices. These algorithms can be used in other contexts and are of independent interest.

[218] V. S. Kozyakin, *On the computational aspects of the theory of joint spectral radius*, Dokl. Math. **80** (2009), no. 1, 487–491, `doi:10.1134/S1064562409040097`. MR 2573049. Zbl 1190.93035.

One of the most prominent tool to compute the the joint spectral radius of a matrix set is the so-called generalized Gelfand formula which represents the joint spectral radius as a limit of the weighted norms of matrix products with factors taken from the given matrix set. Unfortunately, the range of applicability of this formula is substantially restricted by a lack of estimates for the rate of convergence to the corresponding limit. In the paper this deficiency is made up to some extent. We establish explicit computable estimates of the joint spectral radius by the norms of matrix products with factors taken from the given matrix set, and propose two computational iterative relaxation procedures to approximate the joint spectral radius of matrix sets.

[219] V. Kozyakin, *On accuracy of approximation of the spectral radius by the Gelfand formula*, Linear Algebra Appl. **431** (2009), no. 11, 2134–2141, `doi:10.1016/j.laa.2009.07.008`, `arXiv:0810.2856`. MR 2567820. Zbl 1177.15012.

The famous Gelfand formula for the spectral radius of a matrix is of great importance in various mathematical constructions. Unfortunately, the range of applicability of this formula is substantially

restricted by a lack of estimates for the rate of convergence of the quantities $\|A^n\|^{1/n} \to \rho(A)$. In the paper this deficiency is made up to some extent. By using the Bochi inequalities we establish explicit computable estimates for the rate of convergence of the quantities $\|A^n\|^{1/n} \to \rho(A)$. The obtained estimates are then extended for evaluation of the joint spectral radius of matrix sets.

[220] Y.-Y. Lur and W.-W. Yang, *Continuity of the generalized spectral radius in max algebra*, Linear Algebra Appl. **430** (2009), no. 8-9, 2301–2311, `doi:10.1016/j.laa.2008.12.007`. MR 2508296. Zbl 1168.15005.

Let $\|\cdot\|$ be an induced matrix norm associated with a monotone norm on $\mathbb{R}^n$ and β be the collection of all nonempty closed and bounded subsets of $n \times n$ nonnegative matrices under this matrix norm. For $\Psi, \Phi \in \beta$, the Hausdorff metric H between Ψ and Φ is given by $H(\Psi,\Phi) = \max\{\sup_{A\in\Psi}\inf_{B\in\Phi}\|A-B\|,\ \sup_{B\in\Phi}\inf_{A\in\Psi}\|A-B\|\}$. The max algebra system consists of the set of nonnegative numbers with sum $a \otimes b = \max\{a,b\}$ and the standard product ab for $a, b \geq 0$. For $n \times n$ nonnegative matrices A, B their product is denoted by $A \otimes B$, where $[A \otimes B]_{ij} = \max_{1\leq k\leq n} a_{ik}b_{kj}$. For each $\Psi \in \beta$, the max algebra version of the generalized spectral radius of is $\mu(\Psi) = \limsup_{m\to\infty}[\sup_{A\in\Psi^m_\otimes}\mu(A)]^{1/m}$, where $\Psi^m_\otimes = \{A_1 \otimes A_2 \otimes \cdots \otimes A_m : A_i \in \Psi\}$. Here $\mu(A)$ is the maximum circuit geometric mean. In this paper we prove that the max algebra version of the generalized spectral radius is continuous on the Hausdorff metric space (β, H). The notion of the max algebra version of simultaneous nilpotence of matrices is also proposed. Necessary and sufficient conditions for the max algebra version of simultaneous nilpotence of matrices are presented as well.

[221] É. Olivier and A. Thomas, *Projective convergence of columns for inhomogeneous products of matrices with nonnegative entries*, ArXiv.org e-Print archive, August 2009, `arXiv:0908.4171`.

Let P_n be the n-step right product $A_1 \cdots A_n$, where $A_1, A_2, \ldots$ is a given infinite sequence of $d \times d$ matrices with nonnegative entries. In a wide range of situations, the normalized matrix product $P_n/\|P_n\|$ does not converge and we shall be rather interested in the asymptotic behavior of the normalized columns $P_nU_i/\|P_nU_i\|$, where $U_1, \ldots, U_d$ are the canonical $d \times 1$ vectors. Our main result in Theorem A gives a sufficient condition (**C**) over the sequence $A_1, A_2, \ldots$ ensuring the existence of *dominant columns* of P_n, having the same projective limit V: more precisely, for any rank n, there exists a partition of $\{1, \ldots, d\}$ made of two subsets $J_n \neq \emptyset$ and J_n^c such that each one of the sequences of normalized columns, say $P_nU_{j_n}/\|P_nU_{j_n}\|$ with $j_n \in J_n$ tends to V as n tends to $+\infty$ and are *dominant* in the sense that the ratio $\|P_nU_{j'_n}/\|P_nU_{j_n}\|$ tends to 0, as soon as $j'_n \in J_n^c$. The existence of sequences of such *dominant columns* implies that for any probability vector X with positive entries, the probability vector $P_nX/\|P_nX\|$, converges as n tends to $+\infty$. Our main application of Theorem A (and our initial motivation) is related to an *Erdős problem* concerned with a family of probability measures μ_β (for $1 < \beta < 2$ a real parameter) fully supported by a subinterval of the real line, known as *Bernoulli convolutions*. For some parameters β (actually the so-called PV-numbers) such measures are known to be *linearly representable*: the μ_β-measure of a suitable family of nested generating intervals may be computed by means of matrix products of the form P_nX, where A_n takes only finitely many values, say $A(\mathtt{0}), \ldots, A(\mathtt{a})$, and X is a probability vector with positive entries. Because, $A_n = A(\xi_n)$, where $\xi = \xi_1\xi_2\cdots$ is a sequence (one-sided infinite word) with $\xi_n \in \{\mathtt{0}, \ldots, \mathtt{a}\}$, we shall write $P_n = P_n(\xi)$ the dependence of the n-step product with ξ: when the convergence of $P_n(\xi)X/\|P_n(\xi)X\|$ is uniform w.r.t. ξ, a sharp analysis of the measure μ_β (Gibbs structures and multifractal decomposition) becomes possible. However, most of the matrices involved in the decomposition of these Bernoulli convolutions are large, sparse and it is usually not easy to prove the condition (**C**) of Theorem A. To illustrate the technics, we consider one parameter β for which the matrices are neither too small nor too large and we verify condition (**C**): this leads to the Gibbs properties of μ_β.

[222] A. Thomas, *Can an infinite left-product of nonnegative matrices be expressed in terms of infinite left-products of stochastic ones?*, ArXiv.org e-Print archive, August 2009, `arXiv:0908.3538`.

If a left-product $M_n \cdots M_1$ of square complex matrices converges to a nonnull limit when $n \to \infty$ and if the M_n belong to a finite set, it is clear that there exists an integer n_0 such that the

M_n, $n \geq n_0$, have a common right-eigenvector V for the eigenvalue 1. Now suppose that the M_n are nonnegative and that V has positive entries. Denoting by Δ the diagonal matrix whose diagonal entries are the entries of V, the stochastic matrices $S_n = \Delta^{-1} M_n \Delta$ satisfy $M_n \cdots M_{n_0} = \Delta S_n \cdots S_{n_0} \Delta^{-1}$, so the problem of the convergence of $M_n \cdots M_1$ reduces to the one of $S_n \cdots S_{n_0}$. In this paper we still suppose that the M_n are nonnegative but we do not suppose that V has positive entries. The first section details the case of the 2×2 matrices, and the last gives a first approach in the case of $d \times d$ matrices.

2009/10

[223] V. Yu. Protasov, R. M. Jungers, and V. D. Blondel, *Joint spectral characteristics of matrices: a conic programming approach*, SIAM J. Matrix Anal. Appl. **31** (2009/10), no. 4, 2146–2162, `doi:10.1137/090759896`. MR 2678961. Zbl 1203.65093.

The joint spectral radius $\hat{\rho}(\mathcal{M})$ of a set of $n \times n$ real matrices $\mathcal{M}$ is the exponent of the maximal asymptotic growth of products of matrices from this set when the length of the products grows. The joint spectral subradius $\check{\rho}(\mathcal{M})$ is the minimal growth counterpart, that is:

$$\hat{\rho}(\mathcal{M}) = \lim_{k \to \infty} \max\{\|A_{d_1} \cdots A_{d_k}\|^{1/k} : A_i \in \mathcal{M}\},$$

$$\check{\rho}(\mathcal{M}) = \lim_{k \to \infty} \min\{\|A_{d_1} \cdots A_{d_k}\|^{1/k} : A_i \in \mathcal{M}\}.$$

The authors propose a new method to compute the joint spectral radius and the joint spectral subradius of a set of matrices $\mathcal{M}$. The efficiency of the new algorithm is demonstrated by applying it to several problems in combinatorics, number theory and discrete mathematics.

The authors first restrict their attention to matrices that leave a cone invariant. The accuracy of their algorithm, depending on geometric properties of the invariant cone, is estimated. Then they extend their method to arbitrary sets of matrices by a lifting procedure, and demonstrate the efficiency of the new algorithm by applying it to several problems in combinatorics, number theory, and discrete mathematics.

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[224] Yu. A. Al'pin, *Bounds for the joint spectral radii of a set of nonnegative matrices*, Math. Notes **87** (2010), no. 1, 12–14, `doi:10.1134/S0001434610010025`. MR 2730378. Zbl 1202.15022.

For a finite set Σ of nonnegative matrices of order n the upper and lower joint spectral radius is defined. Bounds for these magnitudes are given in terms of the following matrix $S = (s_{ij})$, where $s_{ij} = \max\{\sum_k a_{ik} : A \in \Sigma, a_{ij} > 0\}$ and in terms of the similarly defined matrix S^-. It is shown that the upper joint spectral radius is bounded by the max spectral radius of S and that a similar result holds for the lower joint spectral radius. This generalizes results for the case of one matrix. In the proof methods of max algebra are used.

[225] A. Avila, J. Bochi, and J.-C. Yoccoz, *Uniformly hyperbolic finite-valued* $\mathrm{SL}(2,\mathbb{R})$*-cocycles*, Comment. Math. Helv. **85** (2010), no. 4, 813–884, `doi:10.4171/CMH/212`, `arXiv:0808.0133`. MR 2718140. Zbl 1201.37032.

We consider finite families of $SL(2,\mathbb{R})$ matrices whose products display uniform exponential growth. These form open subsets of $(SL(2,\mathbb{R}))^N$, and we study their components, boundary, and complement. We also consider the more general situation where the allowed products of matrices satisfy a Markovian rule.

More precisely, N will be an integer bigger than 1, and the base $X = \Sigma \subset N^{\mathbb{Z}}$ will be a transitive subshift of finite type (also called topological Markov chain), equipped with the shift map $\sigma : \Sigma \to \Sigma$. We will only consider cocycles defined by a map $A : \Sigma \to SL(2,\mathbb{R})$ depending only on the letter in position zero. The parameter space will be therefore the product $(SL(2,\mathbb{R}))^N$. The parameters $(A_1, \ldots, A_N)$ which correspond to a uniformly hyperbolic cocycle form an open set $\mathcal{H}$ which is the object of our study: we would like to describe its boundary, its connected components, and its complement. Roughly speaking, we will see that this goal is attained for the full shift on

two symbols, and that new phenomena appear with at least 3 symbols which make such a complete description much more difficult and complicated.

[226] V. D. Blondel and R. M. Jungers, *Long products of matrices*, Combinatorics, Automata and Number Theory, Encyclopedia of Mathematics and its Applications, no. 135, Cambridge University Press, Cambridge, 2010, pp. 530–562, `doi:10.1017/CBO9780511777653.012`.

The *joint spectral radius* of a set of matrices is the maximal growth rate that can be obtained by forming long products of matrices taken in the set. This quantity and its minimal growth counterpart, the *joint spectral subradius*, have proved useful for studying several problems from combinatorics and number theory. For instance, they characterize the growth of certain classes of languages, the capacity of forbidden difference constraints on languages, and the trackability of sensor networks. In Section 11.2 we describe some of these applications.

While the joint spectral radius and related notions have applications in combinatorics and number theory, these disciplines have in turn been helpful to improve our understanding of problems related to the joint spectral radius. As an example, we present in Section 11.3 a central result that has been proved with the help of techniques from combinatorics on words: the falseness of the finiteness conjecture.

In practice, computing a joint spectral radius is not an easy task. As we will see, this quantity is NP-hard to approximate in general, and the simple question of knowing, given a set of matrices, if its joint spectral radius is larger than one is even algorithmically undecidable. However, in recent years, approximation algorithms have been proposed that perform well in practice. Some of these algorithms run in exponential time while others provide no accuracy guarantee. In practice, by combining the advantages of the different algorithms, it is often possible to obtain satisfactory estimates.

[227] A. Cicone, N. Guglielmi, S. Serra-Capizzano, and M. Zennaro, *Finiteness property of pairs of* 2×2 *sign-matrices via real extremal polytope norms*, Linear Algebra Appl. **432** (2010), no. 2-3, 796–816, `doi:10.1016/j.laa.2009.09.022`. MR 2577718. Zbl 1186.15006.

This paper deals with the joint spectral radius of a finite set of matrices. We say that a set of matrices has the finiteness property if the maximal rate of growth, in the multiplicative semigroup it generates, is given by the powers of a finite product.

Here we address the problem of establishing the finiteness property of pairs of $n \times n$ sign-matrices. Such problem is related to the conjecture that pairs of sign-matrices fulfil the finiteness property for any dimension. This would imply, by a recent result by Blondel and Jungers, that finite sets of rational matrices fulfil the finiteness property, which would be very important in terms of the computation of the joint spectral radius. The technique used in this paper could suggest an extension of the analysis to $n \times n$ sign-matrices, which still remains an open problem.

As a main tool of our proof we make use of a procedure to find a so-called real extremal polytope norm for the set. In particular, we present an algorithm which, under some suitable assumptions, is able to check if a certain product in the multiplicative semigroup is spectrum maximizing.

For pairs of sign-matrices we develop the computations exactly and hence are able to prove analytically the finiteness property. On the other hand, the algorithm can be used in a floating point arithmetic and provide a general tool for approximating the joint spectral radius of a set of matrices.

[228] H. Gessesse, A. I. Popov, H. Radjavi, E. Spinu, A. Tcaciuc, and V. G. Troitsky, *Bounded indecomposable semigroups of non-negative matrices*, Positivity **14** (2010), no. 3, 383–394, `doi:10.1007/s11117-009-0024-5`. MR 2680502. Zbl 1204.15041.

Given a semigroup $\mathfrak{S}$ of non-negative, $n \times n$, indecomposable matrices (i.e. for every pair $i, j \leq n$ there exists $S \in \mathfrak{S}$ such that $S_{ij} \neq 0$), in this paper it is mainly shown that if there exists a pair k, l such that $\{(S)_{kl} : S \in \mathfrak{S}\}$ is bounded, then, after a simultaneous diagonal similarity, all the entries are in $[0, 1]$. Further, quantitative versions of this result as well as extensions to infinite-dimensional cases are provided, and finally, the continuous case (convolution semigroup) is examined.

[229] R. M. Jungers and V. Yu. Protasov, *Weak stability of switching dynamical systems and fast computation of the p-radius of matrices*, Proceedings of 49th IEEE Conference on Decision and Control (CDC) (Atlanta, GA), 2010, pp. 7328–7333, `doi:10.1109/CDC.2010.5717653`.

The stability of a switching linear dynamical system is ruled by the so-called joint spectral radius of the set of matrices characterizing the dynamical system. In some situations, the system is not stable in the classical sense, but might still be stable in a weaker meaning. We introduce the new notion of weak stability or L_p-stability of a switched dynamical system based on the so-called p-radius of the set of matrices. The p-radius characterizes the average rate of growth of norms of matrices in a multiplicative semigroup. This quantity has found several applications in the recent years. We analyze the computability of this quantity, and we describe a series of approximations that converge to the p-radius with a priori computable accuracy. For nonnegative matrices, this gives efficient approximation schemes for the p-radius computation. We finally show the efficiency of our methods on several practical examples.

[230] V. Kozyakin, *An explicit Lipschitz constant for the joint spectral radius*, Linear Algebra Appl. **433** (2010), no. 1, 12–18, `doi:10.1016/j.laa.2010.01.028`, `arXiv:0909.3170`. MR 2645060. Zbl 1198.15006.

F. Wirth [136] has proved that the joint spectral radius of irreducible compact sets of matrices is locally Lipschitz continuous as a function of the matrix set. In the paper, an explicit formula for the related Lipschitz constant is obtained.

[231] V. Kozyakin, *Iterative building of Barabanov norms and computation of the joint spectral radius for matrix sets*, Discrete Contin. Dyn. Syst. Ser. B **14** (2010), no. 1, 143–158, `doi:10.3934/dcdsb.2010.14.143`, `arXiv:0810.2154`. MR 2644257. Zbl 1201.65067.

An extremal norm satisfying the following theorem is called a Barabanov norm [26–28].

Theorem. *Let the, matrix set $\mathcal{A} = \{A_1, \ldots, A_r\}$ be irreducible. Then the quantity ρ is the joint (generalized) spectral radius of $\mathcal{A}$ if and only if there exist a norm $\|\cdot\|$ in $\mathbb{R}^m$ such that $\rho\|x\| = \max\{\|A_1x\|\}, \{\|A_2x\|, \ldots, \|A_rx\|\}$.*

The author used this norm in earlier papers (e.g., *V. Kozyakin* [185]), to disprove the Lagarias-Wang finiteness conjecture [53]. The geometrical properties of the unit balls of specific Barabanov norms play a decisive role in the corresponding constructions and in other applications.

Because the Barabanov norms are only defined by an implicit procedure it is very difficult to visualize the shapes of the related unit balls. Thus, in the present work an iteration procedure is created that allows to build Barabanov norms for irreducible matrix sets. Simultaneously the joint spectral radius of these sets is computed. The convergence of the iteration is formulated in a main theorem and proved with the help of some lemmas. Computational errors of the joint spectral radius can be a posteriori determined. The algorithm is demonstrated by two examples. The corresponding computed unit spheres are shown in figures. Some open problems about the accuracy of the Barabanov norm, about the convergence rate of the joint spectral radius and other are formulated at the end of the paper.

[232] V. Kozyakin, *Max-Relaxation iteration procedure for building of Barabanov norms: Convergence and examples*, ArXiv.org e-Print archive, February 2010, `arXiv:1002.3251`.

The problem of construction of Barabanov norms for analysis of properties of the joint (generalized) spectral radius of matrix sets has been discussed in a number of publications. In earlier author's works the method of Barabanov norms was the key instrument in disproving the Lagarias-Wang Finiteness Conjecture [53]. The related constructions were essentially based on the study of the geometrical properties of the unit balls of some specific Barabanov norms. In this context the situation when one fails to find among current publications any detailed analysis of the geometrical properties of the unit balls of Barabanov norms looks a bit paradoxical. Partially this is explained by the fact that Barabanov norms are defined nonconstructively, by an implicit procedure. So, even in simplest cases it is very difficult to visualize the shape of their unit balls. The present work may be treated as the first step to make up this deficiency. In the paper an iteration procedure is considered that allows to build numerically Barabanov norms for the irreducible matrix sets and simultaneously to compute the joint spectral radius of these sets.

[233] V. Kozyakin, *On explicit a priori estimates of the joint spectral radius by the generalized Gelfand formula*, Differ. Equ. Dyn. Syst. **18** (2010), no. 1-2, 91–103, `doi:10.1007/s12591-010-0010-1`, `arXiv:0810.2157`. MR 2670076. Zbl 1219.15019.

In various problems of control theory, non-autonomous and multivalued dynamical systems, wavelet theory and other fields of mathematics information about the rate of growth of matrix products with factors taken from some matrix set plays a key role. One of the most prominent quantities characterizing the exponential rate of growth of matrix products is the so-called joint or generalized spectral radius. In the work some explicit a priori estimates for the joint spectral radius with the help of the generalized Gelfand formula are obtained. These estimates are based on the notion of the measure of irreducibility (quasi-controllability) of matrix sets previously introduced and investigated by *V. S. Kozyakin* and *A. V. Pokrovskii* [45, 65].

[234] V. Kozyakin, B. O'Callaghan, and A. Pokrovskii, *Sequences of arbitrages*, ArXiv.org e-Print archive, April 2010, `arXiv:1004.0561`.

The goal of this article is to understand some interesting features of sequences of arbitrage operations, which look relevant to various processes in Economics and Finances. In the second part of the paper, analysis of sequences of arbitrages is reformulated in the linear algebra terms. This admits an elegant geometric interpretation of the problems under consideration linked to the asynchronous systems theory. We feel that this interpretation will be useful in understanding more complicated, and more realistic, mathematical models in economics.

[235] I. D. Morris, *Criteria for the stability of the finiteness property and for the uniqueness of Barabanov norms*, Linear Algebra Appl. **433** (2010), no. 7, 1301–1311, `doi:10.1016/j.laa.2010.05.007`. MR 2680257. Zbl 1202.15028.

A set of matrices is said to have the *finiteness property* if the maximal rate of exponential growth of long products of matrices drawn from that set is realized by a periodic product. The extent to which the finiteness property is prevalent among finite sets of matrices is the subject of ongoing research. In this article, we give a condition on a finite irreducible set of matrices which guarantees that the finiteness property holds not only for that set, but also for all sufficiently nearby sets of equal cardinality. We also prove a theorem giving conditions under which the Barabanov norm associated to a finite irreducible set of matrices is unique up to multiplication by a scalar, and show that in certain cases these conditions are also persistent under small perturbations.

[236] I. D. Morris, *A rapidly-converging lower bound for the joint spectral radius via multiplicative ergodic theory*, Adv. Math. **225** (2010), no. 6, 3425–3445, `doi:10.1016/j.aim.2010.06.008`, `arXiv:0906.0260`. MR 2729011. Zbl 1205.15032.

We use ergodic theory to prove a quantitative version of a theorem of *M. A. Berger* and *Y. Wang* [42], which relates the joint spectral radius of a set of matrices to the spectral radii of finite products of those matrices. The proof rests on a theorem asserting the existence of a continuous invariant splitting for certain matrix cocycles defined over a minimal homeomorphism and having the property that all forward products are uniformly bounded.

Given $\mathbf{A}$ a bounded nonempty set of $d \times d$ complex matrices. The joint spectral radius of $\mathbf{A}$ introduced by Rota and Strang is

$$\varrho(\mathbf{A}) := \lim_{n\to\infty} \sup\{\|A_n, \dots A_1\|^{1/n} : A_i \in \mathbf{A}\}$$

where $\|\cdot\|$ denotes any norm on $\mathbb{C}^d$. Berger-Wang formula asserts that

$$\varrho(\mathbf{A}) = \lim_{n\to\infty} \sup\{\rho(A_n, \dots A_1)^{1/n} : A_i \in \mathbf{A}\}$$

where $\rho(A)$ denotes the spectral radius of a matrix A. The rate of convergence of Berger-Wang formula is studied. A main result asserts that for any positive real number r

$$\varrho(\mathbf{A}) - \max_{1\le k\le n} \varrho_k^-(\mathbf{A}) = O\left(\frac{1}{n^r}\right)$$

where

$$\varrho_n^-(\mathbf{A}) := \sup\{\rho(A_n \cdots A_1)^{1/n} : A_i \in \mathbf{A}\}.$$

A more general result is obtained when $\mathbf{A}$ is nonempty compact. The proof rests on a structure theorem for continuous matrix cocycles over minimal homomorphisms having the property that all forward products are uniformly bounded. Possible extensions are discussed. A comprehensive list of references is given.

[237] B. Mößner, *On the joint spectral radius of matrices of order 2 with equal spectral radius*, Adv. Comput. Math. **33** (2010), no. 2, 243–254, `doi:10.1007/s10444-009-9130-y`. MR 2659589. Zbl 1197.15007.

Many applications like stability analysis of stochastic dynamical systems, or in approximation theory use the joint spectral radius to verify the convergence or smoothness of subdivision algorithms. The joint spectral radius of two matrices $A_1, A_2 \in \mathbb{R}^{d\times d}$ is

$$\operatorname{jsr}(A_1, A_2) := \lim_{n\to\infty} \max\{\|T_1T_2\cdots T_n\|^{1/n} \mid T_i = A_1 \text{ or } A_2\}$$

and it measures the growth (or decrease) rate of products of matrices taken from a set.

In the Introduction of this paper, an accurate presentation of the problem of computing spectral radius and of the literature around this subject is given. The author shows the difficulty of determining the joint spectral radius and the few results that were produced in this direction starting from 1990 by *V. Kozyakin* [33] that proved that $\left\{(A,B) \in \left(\mathbb{R}^{2\times 2}\right)^2 \mid \operatorname{jsr}(A,B) < 1\right\}$ is not a semi-algebraic set. Some of the known results are about the polynomial-time approximation algorithms for the joint spectral radius (*V. D. Blondel* and *J. N. Tsitsiklis* showed in 1997 that until P=NP there is no such an approximation algorithm [82]. The joint spectral radius seems to be known in the case of pairs of matrices of order 2 both when one has negative determinant (*M. Bröker* and *X. Zhou* [107]) and when of the matrix is singular (*R. Lima* and *M. Rahibe* [50]). Other examples were given by *T. N. T. Goodman*, *C. A. Micchelli* and *J. D. Ward* [in: Recent advances in wavelet analysis. Boston, MA: Academic Press, Inc. Wavelet Anal. Appl. 3, 335–360 (1994)] and by *L. Villemoes* [SIAM J. Math. Anal. 25, No. 5, 1433–1460 (1994)].

The main result of the paper is the computation of two palindromic matrices L, P of order 2. It is proved that $\operatorname{jsr}(L,R) = \max\left(\rho(L), \sqrt{\rho(L,R)}\right)$ where $\rho(A)$ is the spectral radius of the matrix A: $\rho(A) := \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$.

Moreover, the technique of the proof of this result is used by the author to generalize the computation of the spectral radius the similar matrices with certain relation between the entries.

At the end of the paper, some examples and an application of the joint spectral radius formula on a subdivision scheme is given.

[238] V. Yu. Protasov, *Semigroups of nonnegative matrices*, Russ. Math. Surv. **65** (2010), no. 6, 1186–1188, `doi:10.1070/RM2010v065n06ABEH004722`. MR 2779367. Zbl 1217.20037.

Let $\mathcal{S}$ be an arbitrary multiplicative semigroup of $n \times n$ matrices, all of which are non-negative, that is, have non-negative entries. In particular, $\mathcal{S}$ can be defined as the set of all products of a given family of non-negative matrices. Under what conditions does $\mathcal{S}$ contain a strictly positive matrix? In the special case when $\mathcal{S}$ is generated by one matrix, that is, $\mathcal{S} = \{A^k, k \in \mathbb{N}\}$, several combinatorial and analytical criteria are known. In addition, for this case there are many results on the minimal positive power of the matrix and on the general structure of matrices that have no positive powers. In this paper we derive a criterion for the existence of a positive element in a given matrix semigroup (Proposition 1). As a corollary we prove the polynomial solvability of this problem for semigroups generated by finite families of matrices (Theorem 1).

[239] L. Rodman, H. Seyalioglu, and I. M. Spitkovsky, *On common invariant cones for families of matrices*, Linear Algebra Appl. **432** (2010), no. 4, 911–926, `doi:10.1016/j.laa.2009.10.004`, `arXiv:0903.0444`. MR 2577636. Zbl 1188.15028.

The authors consider the existence and construction of common invariant cones for families of real matrices.

Let $\mathcal{A} = A_1, A_2, \ldots, A_n$ be a family of 2×2 real matrices. The authors obtain necessary and necessary and sufficient conditions for the existence of an $\mathcal{A}$-invariant proper cone. They analyze different cases:

(i) when all the matrices A_j share a dominant eigenvector u;

(ii) when all the elements of $\mathcal{A}_1$ are diagonalizable matrices, being

$$\mathcal{A}_1 = \mathcal{A} \cup \{A_iA_j : A_i, A_j \in \mathcal{A}, \quad \det A_i < 0, \det A_j < 0 \text{ and } A_iA_j \neq cI\};$$

(iii) when there is none, one or two dominant eigenlines (one dimensional eigenspace) corresponding to non-diagonalizable matrices in $\mathcal{A}_1$;. . . .

For matrices of arbitrary size $m \times m$, the authors give an existence criterion for (and actually a construction of) a common invariant cone of a family $\mathcal{A} = A_1, A_2, \ldots, A_n$ under the assumption that all the elements A i can be put in a diagonal form by the same similarity transformation. They also provide some sufficient conditions for such a cone to exist when the matrices share the dominant eigenvector.

Finally, the authors present several examples illustrating the theoretical results and their limitations.

[240] A. Thomas, *Infinite products of nonnegative* 2×2 *matrices by nonnegative vectors*, ArXiv.org e-Print archive, June 2010, `arXiv:1006.4050`.

Given a finite set $\{M_0, \ldots, M_{d-1}\}$ of nonnegative 2×2 matrices and a nonnegative column-vector V, we associate to each $(\omega_n) \in \{0, \ldots, d-1\}^{\mathbb{N}}$ the sequence of the column-vectors $\frac{M_{\omega_1}\cdots M_{\omega_n}V}{\|M_{\omega_1}\cdots M_{\omega_n}V\|}$. We give the necessary and sufficient condition on the matrices M_k and the vector V for this sequence to converge for all $(\omega_n) \in \{0, \ldots, d-1\}^{\mathbb{N}}$ such that $\forall n,\ M_{\omega_1} \ldots M_{\omega_n} V \neq \binom{0}{0}$.

[241] J. Xu, *On the trace characterization of the joint spectral radius*, Electron. J. Linear Algebra **20** (2010), 367–375, `doi:10.13001/1081-3810.1381`. MR 2679735. Zbl 1217.15016.

A characterization of the joint spectral radius, due to *Q. Chen* and *X. Zhou* [108], relies on the limit superior of the k-th root of the dominant trace over products of matrices of length k. In this note, a sufficient condition is given such that the limit superior takes the form of a limit. This result is useful while estimating the joint spectral radius. Although it applies to a restricted class of matrices, it appears to be relevant to many realistic situations.

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[242] A. A. Ahmadi, R. Jungers, P. A. Parrilo, and M. Roozbehani, *Analysis of the joint spectral radius via Lyapunov functions on path-complete graphs*, Proceedings of the 14th international conference on Hybrid systems: computation and control (HSCC'11) (New York, NY, USA), ACM, 2011, pp. 13–22, `doi:10.1145/1967701.1967706`. MR 3207287. Zbl 1364.93376.

We study the problem of approximating the joint spectral radius (JSR) of a finite set of matrices. Our approach is based on the analysis of the underlying switched linear system via inequalities imposed between multiple Lyapunov functions associated to a labeled directed graph. Inspired by concepts in automata theory and symbolic dynamics, we define a class of graphs called path-complete graphs, and show that any such graph gives rise to a method for proving stability of the switched system. This enables us to derive several asymptotically tight hierarchies of semidefinite programming relaxations that unify and generalize many existing techniques such as common quadratic, common sum of squares, maximum/minimum-of-quadratics Lyapunov functions. We characterize all path-complete graphs consisting of two nodes on an alphabet of two matrices and compare their performance. For the general case of any set of $n \times n$ matrices we propose semidefinite programs of modest size that approximate the JSR within a multiplicative factor of $1/\sqrt[4]{n}$ of the true value. We establish a notion of duality among path-complete graphs and a constructive converse Lyapunov theorem for maximum/minimum-of-quadratics Lyapunov functions.

[243] J. Allen, B. Seeger, and D. Unger, *On the size of the resonant set for the products of* 2×2 *matrices*, Involve **4** (2011), no. 2, 157–166, `doi:10.2140/involve.2011.4.157`, `arXiv:1106.2903`. MR 2876196. Zbl 1233.37029.

For $\theta \in [0, 2\pi)$, consider the rotation matrix R_θ and

$$h = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}, \quad \lambda > 1.$$

Let $W_n(\theta)$ denote the product of m R_θ's and n h's with the condition $m \leq [\epsilon n]$ $(0 < \epsilon < 1)$. We analyze the measure of the set of θ for which $\|W_n(\theta)\| \geq \lambda^{\delta n}$ $(0 < \delta < 1)$. This can be regarded as a model problem for the so-called Bochi-Fayad conjecture.

[244] M. Barnsley and A. Vince, *The eigenvalue problem for linear and affine iterated function systems*, Linear Algebra Appl. **435** (2011), no. 12, 3124–3138, doi:10.1016/j.laa.2011.05.011, arXiv:1004.5040. MR 2831601. Zbl 1229.15008.

The eigenvalue problem for a linear function L centers on solving the eigen-equation $Lx = \lambda x$. This paper generalizes the eigenvalue problem from a single linear function to an iterated function system F consisting of possibly an infinite number of linear or affine functions. The eigen-equation becomes $F(X) = \lambda X$, where $\lambda > 0$ is real, X is a compact set, and $F(X) = \bigcup_{f \in F} f(X)$. The main result is that an irreducible, linear iterated function system F has a unique eigenvalue λ equal to the joint spectral radius of the functions in F and a corresponding eigenset S that is centrally symmetric, star-shaped, and full dimensional. Results of Barabanov and of Dranishnikov-Konyagin-Protasov on the joint spectral radius follow as corollaries.

[245] C.-T. Chang and V. Blondel, *Approximating the joint spectral radius using a genetic algorithm framework*, Proceedings of the 18th IFAC World Congress, vol. 18, part 1, IFAC, 2011, pp. 8681–8686, doi:10.3182/20110828-6-IT-1002.01412.

The joint spectral radius of a set of matrices is a measure of the maximal asymptotic growth rate of products of matrices in the set. This quantity appears in many applications but is known to be difficult to approximate. Several approaches to approximate the joint spectral radius involve the construction of long products of matrices, or the construction of an appropriate extremal matrix norm. In this article we present a brief overview of several recent approximation algorithms and introduce a genetic algorithm approximation method. This new method does not give any accuracy guarantees but is quite fast in comparison to other techniques. The performances of the different methods are compared and are illustrated on some benchmark examples. Our results show that, for large sets of matrices or matrices of large dimension, our genetic algorithm may provide better estimates or estimates for situations where these are simply too expensive to compute with other methods. As an illustration of this we compute in less than a minute a bound on the capacity of a code avoiding a given forbidden pattern that improves the bound currently reported in the literature.

[246] A. Cicone, *Spectral properties of families of matrices*, Ph.D. thesis, Università degli Studi dell'Aquila, Facoltà di Scienze, Dipartimento di Matematica Pura ed Applicata, L'Aquila, 2011, URL https://users.math.msu.edu/users/cicone/papers/antoniociconethesis.pdf.

During the past few decades there has been, an increasing interest in studying the behavior of long products generated using matrices of a given generic family and in particular in analysing the maximal growth rate of these products. This study can be done considering the generalization of the spectral radius of a matrix to the case of a family of matrices, which is called *joint spectral radius* or simply *spectral radius* and it was first introduced by Rota and Strang in the three pages paper "A note on the joint spectral radius" [3]. To be precise this generalization can be formulated in many different ways, but for the families of matrices which are common in applications, i.e. bounded and finite, all the possible generalizations coincide with each other in a unique value that is called joint spectral radius, as explained in Chapter 4.

The joint spectral radius analysis proves to be useful in many different contexts like, for example, in the construction of wavelets of compact support, in analysing the asymptotic behavior of solutions of linear difference equations with variable coefficients, in the coordination of autonomous agents and many others. The same quantity, however, can prove to be hard to compute and can lead even to undecidable problems. In this thesis we present all the known generalizations of spectral radius, the properties, theoretical results and challenges associated with them and an algorithm for the exact evaluation of the joint spectral radius. We make use of this algorithm to prove a finiteness conjecture about 2×2 sign-matrices proposed recently by Blondel, Jungers and Protasov.

[247] X. Dai, *A criterion of simultaneously symmetrization and spectral finiteness for a finite set of real 2-by-2 matrices*, ArXiv.org e-Print archive, November 2011, `arXiv:1111.2108`.

In this paper, we consider the simultaneously symmetrization and spectral finiteness for a finite set of real 2-by-2 matrices.

[248] X. Dai, *Extremal and Barabanov semi-norms of a semigroup generated by a bounded family of matrices*, J. Math. Anal. Appl. **379** (2011), no. 2, 827–833, `doi:10.1016/j.jmaa.2010.12.059`. MR 2784362. Zbl 1215.15025.

Let $S = \{S_i\}_{i \in \mathcal{I}}$ be, an, arbitrary family of complex n-by-n matrices, where $1 \le n < \infty$. Let $\hat{\rho}(S)$ denote the joint spectral radius of S, defined as

$$\hat{\rho}(S) = \limsup \left\{ \sup_{(i_1,\dots,i_l)\in\mathcal{I}^l} \|S_{i_1} \cdots S_{i_l}\|^{1/l} \right\},$$

which is independent of the norm $\|\cdot\|$ used here. A semi-norm $\|\cdot\|_*$ on $\mathbb{C}^n$ is called "extremal" of S, if it satisfies

$$\|x\|_* \not\equiv 0 \qquad \text{and} \qquad \|x \cdot S_i\|_* \le \hat{\rho}(S)\|x\|_* \quad \forall x = (x_1,\dots,x_n) \in \mathbb{C}^n \text{ and } i \in \mathcal{I}.$$

In this paper, using an elementary analytic approach the author shows that if S is bounded in $\mathbb{C}^{n\times n}$, then there always exists, for S, an extremal semi-norm $\|\cdot\|_*$ on $\mathbb{C}^n$; if additionally S is compact in $(\mathbb{C}^{n\times n}, \|\cdot\|)$, this extremal semi-norm has the "Barabanov-type property", i.e., to any $x \in \mathbb{C}^n$, one can find an infinite sequence $i_\cdot : \mathbb{N} \to \mathcal{I}$ with $\|x \cdot S_{i_1} \cdots S_{i_k}\|_* = \hat{\rho}(S)^k \|x\|_*$ for each $k \ge 1$. This implies and generalizes the Barabanov's Norm Theorem, Berger-Wang's Formula and Elsner's Reduction Theorem.

[249] X. Dai, *Optimal state points of the subadditive ergodic theorem*, Nonlinearity **24** (2011), no. 5, 1565–1573, `doi:10.1088/0951-7715/24/5/009`. MR 2785982. Zbl 1254.37007.

Let $\boldsymbol{S} = \{S_1,\dots,S_K\}$ be a finite set of $d \times d$ complex matrices with the joint spectral radius 1. In this paper, as a corollary of an ergodic theorem proved there, it is proved the fillowing:

Corollary 4. *If $\boldsymbol{S}$ is periodically switched stable and $\sigma = (i_1, i_2, \dots)$ in $\{1,\dots,K\}^{\mathbb{N}}$ is a switching law which generates a Markovian measure under the standard shift transformation, then $S_{i_1} \cdots S_{i_n} \to 0$ as $n \to \infty$.*

[250] X. Dai, *Weakly Birkhoff recurrent switching signals, almost sure and partial stability of linear switched dynamical systems*, J. Differential Equations **250** (2011), no. 9, 3584–3629, `doi:10.1016/j.jde.2011.01.029`. MR 2773179. Zbl 1208.93045.

Let $\mathcal{I}$ be a separable metric space, not necessarily compact, and $\boldsymbol{S} \colon \mathcal{I} \to \mathbb{C}^{d\times d}$ be a continuous bounded function with the joint spectral radius 1. We denote by $\Sigma_{\mathcal{I}}^{+} = \{\sigma_\cdot \colon \mathbb{N} \to \mathcal{I}\}$ the product space $\mathcal{I}^{\mathbb{N}}$ and let $\theta_+ \colon \sigma_\cdot \mapsto \sigma_{\cdot+1}$ be the standard shift transformation. One of the main theorems is the following:

Theorem C'. *If μ is an ergodic measure of θ_+ and the support of μ contains a stable switching law $\sigma_\cdot$ for $\boldsymbol{S}$, then $\boldsymbol{S}$ is μ-almost surely exponentially stable.*

[251] X. Dai, Y. Huang, and M. Xiao, *Periodically switched stability induces exponential stability of discrete-time linear switched systems in the sense of Markovian probabilities*, Automatica J. IFAC **47** (2011), no. 7, 1512–1519, `doi:10.1016/j.automatica.2011.02.034`. MR 2889251. Zbl 1219.93142.

The conjecture that periodically switched stability implies absolute asymptotic stability of random infinite products of a finite set of square matrices, has recently been disproved under the guise of the finiteness conjecture. In this paper, we show that this conjecture holds in terms of Markovian probabilities. More specifically, let $S_k \in \mathbb{C}^{n\times n}$, $1 \leq k \leq K$, be arbitrarily given K matrices and $\Sigma_K^+ = \{(k_j)_{j=1}^{\infty} \mid 1 \leq k_j \leq K \text{ for each } j \geq 1\}$, where $n, K \geq 2$. Then we study the exponential stability of the following discrete-time switched dynamics S:

$$x_j = S_{k_j} \cdots S_{k_1} x_0, \quad j \geq 1 \text{ and } x_0 \in \mathbb{C}^n$$

where $(k_j)_{j=1}^{\infty} \in \Sigma_K^+$ can be an arbitrary switching sequence. For a probability row-vector $\mathbf{p} = (p_1, \dots, p_K) \in \mathbb{R}^K$ and an irreducible Markov transition matrix $\mathsf{P} \in \mathbb{R}^{K\times K}$ with $\mathbf{p}\mathsf{P} = \mathbf{p}$, we denote by $\mu_{\mathbf{p},\mathsf{P}}$ the Markovian probability on Σ_K^+ corresponding to $(\mathbf{p}, \mathsf{P})$. By using symbolic dynamics and ergodic-theoretic approaches, we show that, if S possesses the periodically switched stability then, (i) it is exponentially stable $\mu_{\mathbf{p},\mathsf{P}}$-almost surely; (ii) the set of stable switching sequences $(k_j)_{j=1}^{\infty} \in \Sigma_K^+$ has the same Hausdorff dimension as Σ_K^+. Thus, the periodically switched stability of a discrete-time linear switched dynamics implies that the system is exponentially stable for "almost" all switching sequences.

[252] X. Dai, Y. Huang, and M. Xiao, *Pointwise stabilization of discrete-time stationary matrix-valued Markovian chains*, ArXiv.org e-Print archive, July 2011, `arXiv:1107.0132`.

We study the pointwise stabilizability of a discretetime, time-homogeneous, and stationary Markovian jump linear system. By using measure theory, ergodic theory and a splitting theorem of state space we show in a relatively simple way that if the system is essentially product-bounded, then it is pointwise convergent if and only if it is pointwise exponentially convergent, which provides an important characteristic of pointwise convergence under the framework of symbolic dynamics.

[253] X. Dai, Y. Huang, and M. Xiao, *Realization of joint spectral radius via ergodic theory*, Electron. Res. Announc. Math. Sci. **18** (2011), 22–30, `doi:10.3934/era.2011.18.22`. MR 2817401. Zbl 1218.15004.

Based on the classic multiplicative ergodic theorem and the semi-uniform subadditive ergodic theorem, we show that *there always exists at least one ergodic Borel probability measure such that the joint spectral radius of a finite set of square matrices of the same size can be realized almost everywhere with respect to this Borel probability measure.* The existence of at least one ergodic Borel probability measure, in the context of the joint spectral radius problem, is obtained in a general setting.

[254] X. Dai, Y. Huang, and M. Xiao, *Stability criteria via common non-strict Lyapunov matrix for discrete-time linear switched systems*, ArXiv.org e-Print archive, August 2011, `arXiv:1108.0239`.

Let $\boldsymbol{S} = \{S_1, S_2\} \subset \mathbb{R}^{d\times d}$ have a common, but not necessarily strict, Lyapunov matrix (i.e. there exists a symmetric positive-definite matrix P such that $P - S_k^T P S_k \geq 0$ for $k = 1, 2$). Based on a splitting theorem of the state space $\mathbb{R}^d$ [252], we establish several stability criteria for the discrete-time linear switched dynamics

$$x_n = S_{\sigma_n} \cdots S_{\sigma_1}(x_0), \quad x_0 \in \mathbb{R}^d \text{ and } n \geq 1$$

governed by the switching signal $\sigma\colon \mathbb{N} \to \{1, 2\}$. More specifically, let $\rho(A)$ stand for the spectral radius of a matrix $A \in \mathbb{R}^{d\times d}$, then the outline of results obtained in this paper are:

(1) *For the case $d = 2$, $\boldsymbol{S}$ is absolutely stable (i.e., $\|S_{\sigma_n} \cdots S_{\sigma_1}\| \to 0$ driven by all switching signals σ) if and only if $\rho(S_1), \rho(S_2)$ and $\rho(S_1S_2)$ all are less than 1;*

(2) *For the case $d = 3$, $\boldsymbol{S}$ is absolutely stable if and only if $\rho(A) < 1$ $\forall A \in \{S_1, S_2\}^{\ell}$ for $\ell = 1, 2, 3, 4, 5, 6,$ and 8.*

This further implies that for any $\boldsymbol{S} = \{S_1, S_2\} \subset \mathbb{R}^{d\times d}$ with the generalized spectral radius $\rho(\boldsymbol{S}) = 1$ where $d = 2$ or 3, if $\boldsymbol{S}$ has a common, but not strict in general, Lyapunov matrix, then $\boldsymbol{S}$ possesses the spectral finiteness property.

[255] X. Dai and V. Kozyakin, *Finiteness property of a bounded set of matrices with uniformly sub-peripheral spectrum*, J. Commun. Technol. Electron. **56** (2011), no. 12, 1564–1569, `doi:10.1134/S1064226911120096`, `arXiv:1106.2298`.

In the paper, a simple condition guaranteing the finiteness, property, for a bounded set $\boldsymbol{S} = \{S_k\}_{k\in K}$ of real or complex $d \times d$ matrices, is presented. It is shown that existence of a sequence of matrix products $\boldsymbol{S}_{\sigma(n_\ell)}$ of length n_ℓ for $\boldsymbol{S}$ with $n_\ell \to \infty$ such that the spectrum of each matrix $\boldsymbol{S}_{\sigma(n_\ell)}$ is uniformly sub-peripheral and

$$\rho(\boldsymbol{S}) := \sup_{n\geq 1} \sup_{i_1,\dots,i_n\in K} \sqrt[n]{\rho(S_{i_1}\cdots S_{i_n})} = \lim_{\ell\to+\infty} \sqrt[n_\ell]{\rho(\boldsymbol{S}_{\sigma(n_\ell)})},$$

guarantees the spectral finiteness property for $\boldsymbol{S}$.

[256] F. M. Dekking and B. Kuijvenhoven, *Differences of random Cantor sets and lower spectral radii*, J. Eur. Math. Soc. (JEMS) **13** (2011), no. 3, 733–760, `doi:10.4171/JEMS/266`, `arXiv:0811.0525`. MR 2781931. Zbl 05873843.

We investigate the question under which conditions the algebraic difference between two independent random Cantor sets C_1 and C_2 almost surely contains an interval, and when not. The natural condition is whether the sum $d_1 + d_2$ of the Hausdorff dimensions of the sets is smaller (no interval) or larger (an interval) than 1. Palis conjectured that generically it should be true that $d_1 + d_2 > 1$ should imply that $C_1 - C_2$ contains an interval. We prove that for 2-adic random Cantor sets generated by a vector of probabilities (p_0, p_1) the interior of the region where the Palis conjecture does not hold is given by those p_0, p_1 which satisfy $p_0 + p_1 > \sqrt{2}$ and $p_0p_1(1 + p_0^2 + p_1^2) < 1$. We furthermore prove a general result which characterizes the interval/no interval property in terms of the lower spectral radius of a set of 2×2 matrices.

[257] B. B. Gursoy and O. Mason, *$P^1_{\max}$ and $S_{\max}$ properties and asymptotic stability in the max algebra*, Linear Algebra Appl. **435** (2011), no. 5, 1008–1018, `doi:10.1016/j.laa.2011.02.054`. MR 2807214. Zbl 1223.15034.

A matrix $A \in \mathbb{R}^{n\times n}_+$ is said to be a $P^1_{\max}$-matrix if $\operatorname{per}_{\max}(B) \leq 1$ for any principal submatrix B of A. Here

$$\operatorname{per}_{\max}(B) = \max_{\sigma\in S_n} \otimes_{i=1}^n a_{i,\sigma(i)}$$

is the max-algebraic permanent function. The authors define the class of $P^1_{\max}$-matrices and obtain its characterization. Then using the approach of *Y. Song, M. S. Gowda, G. Ravindran* (On some properties of P-matrix sets. Linear Algebra Appl., **290** (1999), pp. 237–246) in the new max-algebraic context these results are extended for the sets of matrices. To do this the authors introduce row-$P^1_{\max}$-property and $S_{\max}$-property by the analogy with the corresponding definitions in the conventional arithmetics. It is proved that $S_{\max}$ property for a set of matrices is related to the stability of its max-convex hull. The above results are then applied to the study of stability questions for discrete-time systems and certain systems of difference equations over max-algebras.

[258] K. G. Hare, I. D. Morris, N. Sidorov, and J. Theys, *An explicit counterexample to the Lagarias-Wang finiteness conjecture*, Adv. Math. **226** (2011), no. 6, 4667–4701, `doi:10.1016/j.aim.2010.12.012`, `arXiv:1006.2117`. MR 2775881. Zbl 1218.15005.

The joint spectral radius of a finite set of real $d \times d$ matrices is defined to be the maximum possible exponential rate of growth of long products of matrices drawn from that set. A set of matrices is said to have the *finiteness property* if there exists a periodic product which achieves this maximal rate of growth. *J. C. Lagarias* and *Y. Wang* [53] conjectured in 1995 that every finite set of real $d \times d$ matrices satisfies the finiteness property. However, *T. Bousch* and *J. Mairesse* [132] proved in 2002 that counterexamples to the finiteness conjecture exist, showing in particular that there exists a family of pairs of 2×2 matrices which contains a counterexample. Similar results were subsequently given by V. D. Blondel, J. Theys and A. A. Vladimirov and by V. S. Kozyakin, but no explicit counterexample to the finiteness conjecture has so far been given. The purpose of this

paper is to resolve this issue by giving the first completely explicit description of a counterexample to the Lagarias-Wang finiteness conjecture. Namely, for the set

$$\mathsf{A}_{\alpha_*} := \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \alpha_* \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$$

we give an explicit value of

$$\alpha_* \simeq 0.749326546330367557943961948091344672091327370236064317358024\ldots$$

such that A_{α_*} does not satisfy the finiteness property.

[259] S.-Y. Hsu and M.-H. Shih, *A proof of Fritz John ellipsoid theorem*, J. Nonlinear Convex Anal. **12** (2011), no. 1, 1–4. MR 2816404. Zbl 1215.15026.

We present a simple proof of *Fritz John*'s ellipsoid theorem [cf. Fritz John. Collected papers. Volumes 1, 2. Contemporary Mathematicians, 543–560 (1985)] using a projection theorem proved by the Hahn-Banach theorem.

[260] J. Hu, J. Shen, and W. Zhang, *Generating functions of switched linear systems: Analysis, computation, and stability applications*, IEEE Trans. Automat. Control **56** (2011), no. 5, 1059–1074, `doi:10.1109/TAC.2010.2067590`. MR 2815911. Zbl 1368.93281.

In this paper, a unified framework is proposed to study the exponential stability of discrete-time switched linear systems and, more generally, the exponential growth rates of their trajectories under three types of switching rules: arbitrary switching, optimal switching, and random switching. It is shown that the maximum exponential growth rates of system trajectories over all initial states under these three switching rules are completely characterized by the radii of convergence of three suitably defined families of functions called the strong, the weak, and the mean generating functions, respectively. In particular, necessary and sufficient conditions for the exponential stability of the switched linear systems are derived based on these radii of convergence. Various properties of the generating functions are established, and their relations are discussed. Algorithms for computing the generating functions and their radii of convergence are also developed and illustrated through examples.

[261] Y. Huang, J. Luo, T. Huang, and M. Xiao, *The set of stable switching sequences for discrete-time linear switched systems*, J. Math. Anal. Appl. **377** (2011), no. 2, 732–743, `doi:10.1016/j.jmaa.2010.11.053`. MR 2769170. Zbl 1214.93089.

We study the characterization of the asymptotical stability for discrete-time switched linear systems. We first translate the system dynamics into a symbolic setting under the framework of symbolic topology. Then by using the ergodic measure theory, a lower bound estimate of Hausdorff dimension of the set of asymptotically stable sequences is obtained. We show that the Hausdorff dimension of the set of asymptotically stable switching sequences is positive if and only if the corresponding switched linear system has at least one asymptotically stable switching sequence. The obtained result reveals an underlying fundamental principle: a switched linear system either possesses uncountable numbers of asymptotically stable switching sequences or has none of them, provided that the switching is arbitrary. We also develop frequency and density indexes to identify those asymptotically stable switching sequences of the system.

[262] R. M. Jungers and V. Yu. Protasov, *Fast methods for computing the p-radius of matrices*, SIAM J. Sci. Comput. **33** (2011), no. 3, 1246–1266, `doi:10.1137/090777906`. MR 2813238. Zbl 1236.65036.

The p-radius characterizes the average rate of growth of norms of matrices in a multiplicative semigroup. This quantity has found several applications in recent years. The paper is concerned with the numerical approximation of the p-radius of a set of matrices. The authors prove that the complexity of its approximation increases exponentially with p and propose a series of approximations that converge to the p-radius with a priori computable accuracy.

[263] V. Kozyakin, *A relaxation scheme for computation of the joint spectral radius of matrix sets*, J. Difference Equ. Appl. **17** (2011), no. 2, 185–201, `doi:10.1080/10236198.2010.549008`, `arXiv:0810.4230`. MR 2783343. Zbl 1214.65015.

The problem of computation of the joint (generalized) spectral radius of matrix sets has been discussed in a number of publications. In this paper, an iteration procedure is proposed that allows to build numerically Barabanov norms for the irreducible matrix sets and simultaneously to compute the joint spectral radius of these sets.

[264] K. Li, *Theories and ultra efficient computation of joint spectral radius for estimating first passage time distribution of Markov set-chain*, IEEE Transactions on Automatic Control **56** (2011), no. 12, 2951–2956, `doi:10.1109/TAC.2011.2161791`. MR 2906684. Zbl 1368.60076.

This technical note is concerned with the tail distribution of the first passage time of Markov set chains (MSC). An original two-part idea—a more progressive relation and a sortedness test—is conceived to characterize such chains. The theoretical construction based on this idea further results in an algorithm that can compute the tightest exponent bound of the tail distribution for high-dimensional problem instances with surprising ease. To understand the computational implication of this algorithm, note that the problem is equivalent to computing the joint spectral radius (JSR) of a special independent column polytope (one that defines Markov set chains) of nonnegative matrices. In this context, the reported algorithm can compute the exact JSR value for cases of 100×100 matrices in less than a second in Matlab. Problems of this size is far beyond the scope of known JSR techniques. It is worth noting that the fields of MSC and JSR have not had significant overlap as one may expect, despite their conceptual akiness. Meanwhile, the present technical note is a contribution that belongs to both fields.

[265] T. Monovich and M. Margaliot, *A second-order maximum principle for discrete-time bilinear control systems with applications to discrete-time linear switched systems*, Automatica J. IFAC **47** (2011), no. 7, 1489–1495, `doi:10.1016/j.automatica.2011.02.025`. MR 2889248. Zbl 1220.49013.

A powerful approach for analyzing the stability of continuous-time switched systems is based on using optimal control theory to characterize the “most unstable” switching law. This reduces the problem of determining stability under arbitrary switching to analyzing stability for the specific “most unstable” switching law. For discrete-time switched systems, the variational approach received considerably less attention. This approach is based on using a first-order necessary optimality condition in the form of a Maximum Principle (MP), and typically this is not enough to completely characterize the “most unstable” switching law. In this paper, we provide a simple and self-contained derivation of a second-order necessary optimality condition for discrete-time bilinear control systems. This provides new information that cannot be derived using the first-order MP. We demonstrate several applications of this second-order MP to the stability analysis of discrete-time linear switched systems.

[266] I. D. Morris, *Rank one matrices do not contribute to the failure of the finiteness property*, ArXiv.org e-Print archive, September 2011, `arXiv:1109.4648`.

The joint spectral radius of a bounded set of $d \times d$ real or complex matrices is defined to be the maximum exponential rate of growth of products of matrices drawn from that set. A set of matrices is said to satisfy the finiteness property if this maximum rate of growth occurs along a periodic infinite sequence. In this note we give some sufficient conditions for a finite set of matrices to satisfy the finiteness property in terms of its rank one elements. We show in particular that if a finite set of matrices does not satisfy the finiteness property, then the subset consisting of all matrices of rank at least two is nonempty, does not satisfy the finiteness property, and has the same joint spectral radius as the original set. We also obtain an exact formula for the joint spectral radii of sets of matrices which contain at most one element not of rank one, generalising a recent result of X. Dai.

[267] A. Peperko, *On the continuity of the generalized spectral radius in max algebra*, Linear Algebra Appl. **435** (2011), no. 4, 902–907, `doi:10.1016/j.laa.2011.02.015`. MR 2807242. Zbl 1221.15020.

Given a bounded set Ψ of $n \times n$ non-negative matrices, let $\rho(\Psi)$ and $\mu(\Psi)$ denote the generalized spectral radius of Ψ and its max version, respectively. We show that

$$\mu(\Psi) = \sup_{t \in (0,\infty)} \left(n^{-1} \rho(\Psi^{(t)}) \right)^{1/t},$$

where $\Psi^{(t)}$ denotes the Hadamard power of Ψ. We apply this result to give a new short proof of a known fact that $\mu(\Psi)$ is continuous on the Hausdorff metric space (β, H) of all nonempty compact collections of $n \times n$ non-negative matrices.

[268] A. Peperko, *On the functional inequality for the spectral radius of compact operators*, Linear Multilinear Algebra **59** (2011), no. 4, 357–364, `doi:10.1080/03081080903489985`. MR 2802518. Zbl 1223.47007.

Let $A_1, A_2, \ldots, A_n$ be matrices of the same order and let r denote the spectral radius. Functions $F : \mathbb{R}_+^n \to \mathbb{R}$ satisfying

$$r(F(A_1, A_2, \ldots, A_n)) \leq F(r(A_1), r(A_2), \ldots, r(A_n))$$

for all non-negative matrices $A_1, A_2, \ldots, A_n$ were characterized by *L. Elsner, D. Hershkowitz* and *A. Pincus* [Linear Algebra Appl. 129, 103–130 (1990)]. The author generalizes this result to the setting of infinite non-negative matrices that define compact operators on a Banach sequence space.

[269] V. Yu. Protasov, *Invariant functions for the Lyapunov exponents of random matrices*, Sb. Math. **202** (2011), no. 1, 101–126, `doi:10.1070/SM2011v202n01ABEH004139`. MR 2796828. Zbl 1239.60004.

A new approach to the study of Lyapunov exponents of random matrices is presented. We prove that any family of nonnegative $(d \times d)$-matrices has a continuous concave invariant functional on $\mathbb{R}_+^d$. Under some standard assumptions on the matrices, this functional is strictly positive, and the coefficient corresponding to it is equal to the largest Lyapunov exponent. As a corollary we obtain asymptotics for the expected value of the logarithm of norms of matrix products and of their spectral radii. Another corollary gives new upper and lower bounds for the Lyapunov exponent, and an algorithm for computing it for families of nonnegative matrices. We consider possible extensions of our results to general nonnegative matrix families and present several applications and examples.

[270] M. Przedwojski, K. Galkowski, P. Bauer, and E. Rogers, *On the stability and control of discrete linear systems with clock synchronisation errors*, Internat. J. Control **84** (2011), no. 9, 1491–1499, `doi:10.1080/00207179.2011.604164`. MR 2842730. Zbl 1230.93052.

This article considers discrete linear time-invariant systems that can be decomposed into subsystems whose states are synchronised by a common clock with a signal that reaches them with delays. In particular, stability for the case where all subsystems have the same sampling frequency, but different switching times, is investigated. In contrast to previous work, the approach taken here models the set of system matrices that arise using a polytopic uncertainty description, which has seen extensive application in robust control theory for linear systems. Stabilisation is then achieved by state feedback and a method to handle the combinatorial explosion of the number of polytope vertices is developed and illustrated using an example from swarm system navigation.

[271] G. Vankeerberghen, J. Hendrickx, R. Jungers, C. T. Chang, and V. Blondel, *The JSR Toolbox*, MATLAB® Central, 2011, URL `https://www.mathworks.com/matlabcentral/fileexchange/33202-the-jsr-toolbox`.

The Joint Spectral Radius of a set of matrices characterizes the maximal asymptotic rate of growth of a product of matrices taken in this set, when the length of the product increases. It is known to be very hard to compute. In recent years, many different methods have been proposed to approximate it. These methods have different advantages, depending on the application considered, the type of matrices considered, the desired accuracy or running time, etc. The goal of this toolbox is to provide the practioner with the best available methods, and propose an easy tool for the researcher to compare the different methods.

The JSR Toolbox is a MATLAB toolbox that provides a large set of methods for the approximation of the joint spectral radius, but also includes several helper functions, for example for comparison or analysis purposes, and several demonstration functions. One important feature is that the toolbox offers a "default" algorithm that computes bounds on the joint spectral radius by combining several approaches presented in this article. This may be useful if one does not know what algorithm is more suitable. The behavior of this algorithm may also be parameterized as needed, for instance by setting a maximal computation time or by fine-tuning a particular step in an algorithm. The main steps of the default algorithm are the following:

- Try to transform the problem into a set of smaller independent problems. This is possible when the matrices in the set Σ are simultaneously blocktriangularizable.
- If the matrices are nonnegative, start with the pruning algorithm in order to get some bounds $[\beta^-, \beta^+]$ on the joint spectral radius, then compute the joint conic radius, using the positive orthant as cone, and the ellipsoidal norm approximation using $[\beta^-, \beta^+]$ as initial bounds.
- If some matrices have negative entries, start with a variant of Gripenberg's algorithm in order to get some initial bounds and a candidate product. This variant may rescale the matrices during the computation in order to avoid overflows. After this first step, compute the ellipsoidal norm approximation and, if needed, try to certify optimality or to find a better candidate product using a balanced complex polytope method or a conitope method, which is a lifted polytope method.

[272] J. Xu and M. Xiao, *A characterization of the generalized spectral radius with Kronecker powers*, Automatica J. IFAC **47** (2011), no. 7, 1530–1533, `doi:10.1016/j.automatica.2011.04.007`. MR 2889254. Zbl 1227.15013.

Based on Turán's power sum theory, in this note a result presented by *V. D. Blondel* and *Y. Nesterov* [156] is extended in the following form:
Theorem. *The generalized spectral radius of* $\Sigma = \{A_1, \ldots, A_m\} \subset \mathbb{C}^{n\times n}$ *is characterized by* $\rho(\Sigma) = \limsup_{k\to\infty} \rho^{1/k}(A_1^{\otimes k} + \cdots + A_m^{\otimes k})$*, where* $A_i^{\otimes k}$ *is the kth Kronecker power of* A_i*.*

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[273] A. A. Ahmadi, R. M. Jungers, P. A. Parrilo, and M. Roozbehani, *When is a set of LMIs a sufficient condition for stability?*, IFAC Proceedings Volumes **45** (2012), no. 13, 313–318, 7th IFAC Symposium on Robust Control Design, `doi:10.3182/20120620-3-DK-2025.00098`, `arXiv:1201.3227`.

We study stability criteria for discrete time switching systems. We investigate the structure of sets of LMIs that are a sufficient condition for stability (i.e., such that any switching system which satisfies these LMIs is stable). We provide an exact characterization of these sets. As a corollary, we show that it is PSPACE-complete to recognize whether a particular set of LMIs implies the stability of a switching system.

[274] A. A. Ahmadi and P. A. Parrilo, *Joint spectral radius of rank one matrices and the maximum cycle mean problem*, Proceedings of the 51st Annual Conference on Decision and Control (CDC), 10-13 Dec., IEEE, 2012, pp. 731–733, `doi:10.1109/CDC.2012.6425992`.

We prove several exact results on approximability of joint spectral radius by matrix norms induced by Euclidean norms. We point out, perhaps for the first time in this context, a difference between complex and real cases. New connections of joint spectral radius to convex geometry and combinatorics are established. Several open problems are posed.

[275] M. Ait Rami, V. S. Bokharaie, O. Mason, and F. R. Wirth, *Extremal norms for positive linear inclusions*, 20th International Symposium on Mathematical Theory of Networks and Systems, MTNS2012, 9–13 July (Melbourne), The University of Melburne, 2012, URL `https://fwn06.housing.rug.nl/mtns2012/Full%20Paper/MTNS2012_0177_paper.pdf`.

We consider the joint spectral radius of sets of matrices for discrete or continuous positive linear inclusions and study associated extremal norms. We show that under a matrix-theoretic notion of

irreducibility there exist absolute extremal norms. This property is used to extend regularity results for the joint spectral radius. In particular, we see that in the case of positive systems irreducibility in the sense of nonnegative matrices, which is weaker than the usual representation theoretic concept, is sufficient for local Lipschitz properties of the joint spectral radius.

[276] M. F. Barnsley and A. Vince, *Real projective iterated function systems*, J. Geom. Anal. **22** (2012), no. 4, 1137–1172, `doi:10.1007/s12220-011-9232-x`, `arXiv:1003.3473`. MR 2965365. Zbl 1256.28002.

This paper contains four main results associated with an attractor of a projective iterated function system (IFS). The first theorem characterizes when a projective IFS has an attractor which avoids a hyperplane. The second theorem establishes that a projective IFS has at most one attractor. In the third theorem the classical duality between points and hyperplanes in projective space leads to connections between attractors that avoid hyperplanes and repellers that avoid points, as well as hyperplane attractors that avoid points and repellers that avoid hyperplanes. Finally, an index is defined for attractors which avoid a hyperplane. This index is shown to be a nontrivial projective invariant.

[277] C.-T. Chang, *Heuristic optimization methods for three matrix problem*, Ph.D. thesis, Université catholique de Louvain, Louvain School of Engineering, Louvain-la-Neuve, Belgium, 2012, URL `https://perso.uclouvain.be/chia-tche.chang/research.php`.

Optimization is a major field in applied mathematics. Many applications involve the search of the best solution to a problem according to some criterion. Depending on the considered optimization problem, finding the best solution is not always possible in a reasonable amount of time. Heuristic algorithms are often used when the problem is too difficult to solve exactly. These methods are used to speed up the search for a good solution but they do not guarantee that an optimal solution will be found. In this thesis, we explore such heuristic approaches for three different matrix problems. First, we study the minimum-volume bounding box problem, which consists in finding the smallest rectangular parallelepiped enclosing a given set of points in the three-dimensional space. This problem appears for example in collision detection, which is a very important issue in computational geometry and computer graphics. In these applications, a solution has to be determined in a very short amount of time. We propose a new hybrid algorithm able to approximate optimal bounding boxes at a low computational cost. In particular, it is several orders of magnitude faster than the only currently known exact algorithm. Second, we investigate the subset selection problem. Given a large set of features, we want to choose a small subset containing the most relevant features while removing the redundant ones. This problem has applications in data mining since this can be seen as a dimensionality reduction problem. We develop several windowed algorithms that tackle the subset selection problem for the maximum-volume criterion, which is NP-hard. Finally, we address the topic of the approximation of the joint spectral radius. This quantity characterizes the growth rate of product of matrices and is NP-hard to approximate. The joint spectral radius appears in many fields, including system theory, graph theory, combinatorics, language theory... We present an experimental study of existing approaches and propose a new genetic-based algorithm that is able to find bounds on the joint spectral radius in a short amount of time.

[278] Y. Chitour, P. Mason, and M. Sigalotti, *On the marginal instability of linear switched systems*, Systems Control Lett. **61** (2012), no. 6, 747–757, `doi:10.1016/j.sysconle.2012.04.005`. MR 2929512. Zbl 1250.93112.

Stability properties for continuous-time linear switched systems are at first determined by the (largest) Lyapunov exponent associated with the system, which is the analogue of the joint spectral radius for the discrete-time case. The purpose of this paper is to provide a characterization of marginally unstable systems, i.e., systems for which the Lyapunov exponent is equal to zero and there exists an unbounded trajectory, and to analyze the asymptotic behavior of their trajectories. Our main contribution consists in pointing out a resonance phenomenon associated with marginal instability. In the course of our study, we derive an upper bound of the state at time t, which is polynomial in t and whose degree is computed from the resonance structure of the system. We also derive analogous results for discrete-time linear switched systems.

[279] A. Cicone and V. Protasov, *Joint spectral radius computation*, MATLAB® Central, 2012, URL `https://www.mathworks.com/matlabcentral/fileexchange/36460-joint-spectral-radius-computation`.

The algorithm allows to evaluate lower and upper bounds for the JSR of a set of matrices and estimates its candidate spectrum maximizing products (s.m.p). It is based on ellipsoidal norms and a branch and bound technique.

[280] R. Cross, V. Kozyakin, B. O'Callaghan, A. Pokrovskii, and A. Pokrovskiy, *Periodic sequences of arbitrage: A tale of four currencies*, Metroeconomica **63** (2012), no. 2, 250–294, `doi:10.1111/j.1467-999X.2011.04140.x`, `arXiv:1112.5850`. Zbl 1242.91152.

This paper investigates arbitrage chains involving four currencies and four foreign exchange trader-arbitrageurs. In contrast with the three-currency case, we find that arbitrage operations when four currencies are present may appear periodic in nature, and not involve smooth convergence to a 'balanced' ensemble of exchange rates in which the law of one price holds. The goal of this article is to understand some interesting features of sequences of arbitrage operations, features of which might well be relevant in other contexts in finance and economics.

[281] X. Dai, *A Gel'fand-type spectral-radius formula and stability of linear constrained switching systems*, Linear Algebra Appl. **436** (2012), no. 5, 1099–1113, `doi:10.1016/j.laa.2011.07.029`, `arXiv:1107.0124`. MR 2890907. Zbl 1237.15010.

Using ergodic theory, in this paper we present a Gel'fand-type spectral radius formula which states that the joint spectral radius is equal to the generalized spectral radius for a matrix multiplicative semigroup $\boldsymbol{S}^+$ restricted to a subset that need not carry the algebraic structure of $\boldsymbol{S}^+$. This generalizes the Berger-Wang formula. Using it as a tool, we study the absolute exponential stability of a linear switched system driven by a compact subshift of the one-sided Markov shift associated to $\boldsymbol{S}$ (cf. *I. D. Morris* [288]).

[282] X. Dai, Y. Huang, J. Liu, and M. Xiao, *The finite-step realizability of the joint spectral radius of a pair of $d \times d$ matrices one of which being rank-one*, Linear Algebra Appl. **437** (2012), no. 7, 1548–1561, `doi:10.1016/j.laa.2012.04.053`, `arXiv:1106.0870`. MR 2946341. Zbl 1250.15011.

We study the finite-step realizability of the, joint/generalized spectral radius of a pair of real $d \times d$ matrices $\{\mathrm{S}_1, \mathrm{S}_2\}$, one of which has rank 1, where $2 \leq d < +\infty$. Let $\rho(A)$ denote the spectral radius of a square matrix A. Then we prove that there always exists a finite-length word $(i_1^*, \dots, i_\ell^*) \in \{1,2\}^\ell$, for some finite $\ell \geq 1$, such that

$$\sqrt[\ell]{\rho(\mathrm{S}_{i_1^*} \cdots \mathrm{S}_{i_\ell^*})} = \sup_{n \geq 1} \left\{ \max_{(i_1,\dots,i_n) \in \{1,2\}^n} \sqrt[n]{\rho(\mathrm{S}_{i_1} \cdots \mathrm{S}_{i_n})} \right\};$$

that is to say, there holds the spectral finiteness property for $\{\mathrm{S}_1, \mathrm{S}_2\}$. This implies that stability is algorithmically decidable for $\{\mathrm{S}_1, \mathrm{S}_2\}$.

[283] E. Fornasini and M. E. Valcher, *Stability and stabilizability criteria for discrete-time positive switched systems*, IEEE Trans. Automat. Control **57** (2012), no. 5, 1208–1221, `doi:10.1109/TAC.2011.2173416`. MR 2923880. Zbl 1369.93526.

In this paper we consider the class of discrete-time switched systems switching between autonomous positive subsystems. First, sufficient conditions for testing stability, based on the existence of special classes of common Lyapunov functions, are investigated, and these conditions are mutually related, thus proving that if a linear copositive common Lyapunov function can be found, then a quadratic positive definite common function can be found, too, and this latter, in turn, ensures the existence of a quadratic copositive common function. Secondly, stabilizability is introduced and characterized. It is shown that if these systems are stabilizable, they can be stabilized by means of a periodic switching sequence, which asymptotically drives to zero every positive initial state.

Conditions for the existence of state-dependent stabilizing switching laws, based on the values of a copositive (linear/ quadratic) Lyapunov function, are investigated and mutually related, too.

Finally, some properties of the patterns of the stabilizing switching sequences are investigated, and the relationship between a sufficient condition for stabilizability (the existence of a Schur convex combination of the subsystem matrices) and an equivalent condition for stabilizability (the existence of a Schur matrix product of the subsystem matrices) is explored.

[284] N. Guglielmi and M. Zennaro, *On the asymptotic regularity of a family of matrices*, Linear Algebra Appl. **436** (2012), no. 7, 2093–2104, `doi:10.1016/j.laa.2011.11.017`. MR 2889978. Zbl 1247.15006.

In the paper bounded families $\mathcal{F}$ of complex $n \times n$ matrices are considered. Sufficient conditions are given under which the sequence $\{\bar{\rho}_k(\mathcal{F})^{1/k}\}_{k\geq 1}$, where $\bar{\rho}_k(\mathcal{F})$ is the supremum of the spectral radii of all possible products of k matrices chosen in $\mathcal{F}$, is convergent to its supremum $\rho(\mathcal{F})$, the so-called (generalized) spectral radius of $\mathcal{F}$. It is also illustrated a possible practical application.

Let $\mathcal{F}$ be a bounded family of $n \times n$ complex matrices and $\Sigma_k(\mathcal{F}) = \{A_1 \cdots A_k : A_i \in \mathcal{F}\}$. For each $k \geq 1$, define the number $\bar{\rho}_k(\mathcal{F}) = \sup_{Q\in\Sigma_k(\mathcal{F})} \rho(Q)$ where $\rho(\cdot)$ denotes the spectral radius of a matrix. The generalized spectral radius of $\mathcal{F}$ is $\rho(\mathcal{F}) = \limsup_{k\to\infty} \bar{\rho}_k(\mathcal{F})^{1/k}$. $\mathcal{F}$ is said to be asymptotically regular if

$$\rho(\mathcal{F}) = \lim_{k\to\infty} \bar{\rho}_k(\mathcal{F})^{1/k}.$$

$\mathcal{F}$ is said to have finiteness property if there exists $k^* \geq 1$ and a $P \in \Sigma_{k^*}(\mathcal{F})$ such that

$$\rho(\mathcal{F}) = \bar{\rho}_{k^*}(\mathcal{F})^{1/k^*} = \rho(P)^{1/k^*}$$

and the special matrix P is called a spectrum-maximizing product for $\mathcal{F}$. Theorem 3.1 states a sufficient condition on a matrix P and $k^* > 1$ such that $\lim_{k\to\infty} \bar{\rho}_k(\mathcal{F})^{1/k} \geq \rho(P)^{1/k^*}$. As a consequence, Corollary 3.1 states a sufficient condition on a $\mathcal{F}$ with finiteness property to be asymptotically regular. At the end of the article, there is an application Theorem 5.2 which states that if $\mathcal{F}$ is a bounded family of nonnegative matrices with finiteness property and there exists a primitive spectrum-maximizing product, then $\mathcal{F}$ is asymptotically regular.

[285] Y. S. Hanna and S. F. Ragheb, *On the infinite products of matrices*, Advances in Pure Mathematics **5** (2012), no. 2, 349–353, `doi:10.4236/apm.2012.25050`.

In different fields in space researches, scientists are in need to deal with the product of matrices. In this paper, we develop conditions under which a product $\prod_{i=0}^{\infty}$ of matrices chosen from a possibly infinite set of matrices $M = \{P_j,\ j \in J\}$ converges. There exists a vector norm such that all matrices in M are no expansive with respect to this norm and also a subsequence $\{i_k\}_{k=0}^{\infty}$ of the sequence of nonnegative integers such that the corresponding sequence of operators $\{P_{ik}\}_{k=0}^{\infty}$ converges to an operator which is paracontracting with respect to this norm. The continuity of the limit of the product of matrices as a function of the sequences $\{i_k\}_{k=0}^{\infty}$ is deduced. The results are applied to the convergence of inner-outer iteration schemes for solving singular consistent linear systems of equations, where the outer splitting is regular and the inner splitting is weak regular.

[286] R. M. Jungers, *On asymptotic properties of matrix semigroups with an invariant cone*, Linear Algebra Appl. **437** (2012), no. 5, 1205–1214, `doi:10.1016/j.laa.2012.04.006`. MR 2942343. Zbl 1258.15015.

Two joint spectral characteristics are studied for finitely generated matrix semigroups, when the matrices share one or two invariant cones. For a bounded set $\Sigma \in \mathbb{R}^{n\times n}$, these characteristics are the joint spectral radius $\rho(\Sigma)$ and the joint spectral subradius $\check{\rho}(\Sigma)$ respectively,

$$\rho(\Sigma) = \lim_{t\to\infty} \sup\{\|A_1 \cdots A_t\|^{1/t} : A_i \in \Sigma\},\ \check{\rho}(\Sigma) = \lim_{t\to\infty} \inf\{\|A_1 \cdots A_t\|^{1/t} : A_i \in \Sigma\}.$$

It is well-known that $\rho(\Sigma)$ is continuous w.r.t. the Hausdorff distance, but $\check{\rho}(\Sigma)$ is not continuous. In this paper the continuity of the joint spectral subradius is proved in the neighborhood of sets of matrices that leave an embedded pair of cones invariant.

A convex closed cone K' is embedded in a cone $K \subset \mathbb{R}^{n\times n}$ if $(K' \setminus \{0\}) \subset \mathrm{int}K$ and in this case one says that $\{K, K'\}$ is an embedded pair. The main theorem is the following:

Theorem. *If Σ is a compact set in $\mathbb{R}^{n\times n}$ which leaves an embedded pair of cones invariant and (Σ_k) is a sequence of sets in $\mathbb{R}^{n\times n}$ that converges to Σ in the Hausdorff metric, then $\check{\rho}(\Sigma_k) \to \check{\rho}(\Sigma)$ as $k \to \infty$.*

The author denotes by Σ^t the set of products of length t of matrices from the bounded set Σ of matrices which leave a cone K invariant. He proves that if there exists $A \in \Sigma$ which is K-primitive (i.e. $\exists m \in \mathbb{N}$ such that $A^m(K \setminus \{0\}) \subset \mathrm{int}K$), then both the maximal trace $\max\limits_{A\in\Sigma^t}\{\mathrm{tr}^{1/t}(A)\}$ and the averaged maximal spectral radius $\max\limits_{A\in\Sigma^t}\{\rho^{1/t}(A)\}$ converge to the joint spectral radius as $t \to \infty$.

[287] J. Liu and M. Xiao, *Computation of joint spectral radius for network model associated with rank-one matrix set*, Neural Information Processing. Proceedings of the 19th International Conference, ICONIP 2012, Doha, Qatar, November 12-15, 2012, Part III, Lecture Notes in Computer Science, vol. 7665, Springer Berlin Heidelberg, 2012, pp. 356–363, doi:10.1007/978-3-642-34487-9_44.

In the paper, we prove that any finite set of rank-one matrices has the finiteness property by making use of (invariant) extremal norm. An explicit formula for the computation of joint/generalized spectral radius of such type of matrix sets is derived. Several numerical examples from current literature are provided to illustrate our theoretical conclusion.

[288] I. D. Morris, *The generalised Berger-Wang formula and the spectral radius of linear cocycles*, J. Funct. Anal. **262** (2012), no. 3, 811–824, doi:10.1016/j.jfa.2011.09.021, arXiv:0906.2915. MR 2863849. Zbl 1254.47006.

Using multiplicative ergodic theory we prove two formulae describing the relationships between different joint spectral radii for sets of bounded linear operators acting on a Banach space. In particular we recover a formula previously proved by V. S. Shulman and Yu. V. Turovskiĭ using operator-theoretic ideas. As a byproduct of our method we answer a question of J. E. Cohen on the limiting behaviour of the spectral radius of a measurable matrix cocycle.

Let $\mathcal{X}$ be a Banach space and let $B_{\mathcal{X}}$ be its unit ball. The Hausdorff measure of noncompactness for an operator $A \in \mathcal{B}(\mathcal{X})$ is defined to be $\|A\|_{\chi} = \inf\{\varepsilon > 0 : A(B_{\mathcal{X}})$ is covered by a finite set of ε-balls$\}$. The finite rank approximation seminorm for A is defined to be $\|A\|_f = \inf\{\|A - F\| : \mathrm{rank}\, F < \infty\}$. For a bounded set of operators $\mathbf{A}$ on $\mathcal{X}$ let $\mathbf{A}^n = \{A_{i_1} \cdots A_{i_n} : A_i \in \mathbf{A}\}$, $n \in \mathbb{N}$. The authors consider the following joint spectral radii: the Rota-Strang joint spectral radius $\hat{\varrho}(\mathbf{A}) = \lim_{n\to\infty} \sup\{\|A\|^{1/n} : A \in \mathbf{A}^n\}$, $\varrho_{\chi}(\mathbf{A}) = \lim_{n\to\infty} \sup\{\|A\|_{\chi}^{1/n} : A \in \mathbf{A}^n\}$, $\varrho_f(\mathbf{A}) = \lim_{n\to\infty} \sup\{\|A\|_f^{1/n} : A \in \mathbf{A}^n\}$, and $\varrho_r(\mathbf{A}) = \limsup_{n\to\infty} \sup\{\rho(A)^{1/n} : A \in \mathbf{A}^n\}$, where $\rho(A)$ is the spectral radius of the operator A.

Let T be a continuous transformation of a compact metric space X and $\mathcal{X}$ be a Banach space. A linear cocycle over T is a function $\mathcal{A} : X \times \mathbb{N} \to \mathcal{B}(\mathcal{X})$ such $\mathcal{A}(x, n+m) = \mathcal{A}(T^m x, n)\mathcal{A}(x, m)$ for all $x \in X$ and $n, m \in \mathbb{N}$. The main result of the paper is the following:

Theorem. *For a continuous cocycle $\mathcal{A}$ over T, the following formulas hold true:*

$$\lim_{n\to\infty} \sup_{x\in X} \|\mathcal{A}(x,n)\|^{1/n} = \max\left\{\limsup_{n\to\infty} \sup_{x\in X} \rho(\mathcal{A}(x,n))^{1/n},\ \lim_{n\to\infty} \sup_{x\in X} \|\mathcal{A}(x,n)\|_{\chi}^{1/n}\right\},$$

$$\lim_{n\to\infty} \sup_{x\in X} \|\mathcal{A}(x,n)\|_{\chi}^{1/n} = \lim_{n\to\infty} \sup_{x\in X} \|\mathcal{A}(x,n)\|_{f}^{1/n}.$$

The following formulas for the spectral radii of a precompact and nonempty set $\mathbf{A}$ of operators on a Banach space $\mathcal{X}$ are obtained as a consequence of this theorem:

$$\hat{\varrho}(\mathbf{A}) = \max\{\varrho_{\chi}(\mathbf{A}),\ \varrho_r(\mathbf{A})\}, \quad \varrho_{\chi}(\mathbf{A}) = \varrho(\mathbf{A}).$$

The first of these formulas was proved earlier by different methods by *V. S. Shulman* and *Y. V. Turovskii* [117, 135].

[289] I. D. Morris, *A new sufficient condition for the uniqueness of Barabanov norms*, SIAM J. Matrix Anal. Appl. **33** (2012), no. 2, 317–324, `doi:10.1137/110837826`, `arXiv:1109.4649`. MR 2970208. Zbl 1256.15009.

The joint spectral radius $\rho(\mathcal{A})$ of a bounded, set, $\mathcal{A}$ of d-by-d matrices over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$) is defined by *G.-C. Rota* and *G. Strang* [3], to be $\limsup_{n\to\infty}\{\|A_{i_1}\cdots A_{i_n}\|^{1/n} : A_{i_j} \in \mathcal{A}\}$. $\mathcal{A}$ is said to be irreducible if there is no proper subspace of $\mathbb{K}^d$ which is left invariant under all elements of $\mathcal{A}$. *N. E. Barabanov* showed [26–28] that, for any compact irreducible set $\mathcal{A}$ of $M_d(\mathbb{K})$, there is associated a norm $|||.|||$ on $\mathbb{K}^d$ such that $\rho(\mathcal{A})|||v||| = \sup\{|||Av||| : A \in \mathcal{A}\}$ holds for all vectors v in $\mathbb{K}^d$. The present paper is concerned with a sufficient condition for the uniqueness of such a Barabanov norm for $\mathcal{A}$. The main result is as follows:

Theorem 2.1. *If $\mathcal{A}$ is a nonempty bounded irreducible subset of $M_d(\mathbb{K})$ such that the limit semigroup $\&(\mathcal{A}) \equiv \bigcap_{m=1}^{\infty} \left(\overline{\bigcup_{n=m}^{\infty} \rho(A)^{-n}\mathcal{A}^n}\right)$, where $\mathcal{A}^n = \{A_{i_1}\cdots A_{i_n} : A_{i_j} \in \mathcal{A}\}$, is such that for every pair of nonzero vectors v_1 and v_2 in $\mathbb{K}^d$, there exist B_1 and B_2 in $\&(\mathcal{A})$ and λ in $\mathbb{K}$ such that $B_1v_1 = \lambda v_2$ and $B_2v_2 = \lambda^{-1}v_1$, then the Barabanov norm for $\mathcal{A}$ is unique up to a positive scalar constant.*

Examples of $\mathcal{A}$ consisting of 2-by-2 real matrices are given to illustrate this condition and to contrast with the ones previously obtained by the author. A theoretical application of the main theorem is also given:

Theorem 4.1. *There exists a nonempty open set $\mathcal{V} \subset \mathcal{O}_2(\mathbb{R}^2)$ such that the sets*

$$\mathcal{V}_1 := \{(A_1, A_2) \in \mathcal{V} \colon \{A_1, A_2\} \text{ has a unique Barabanov norm}\},$$

$$\mathcal{V}_2 := \{(A_1, A_2) \in \mathcal{V} \colon \{A_1, A_2\} \text{ does not have a unique Barabanov norm}\}$$

are both dense in $\mathcal{V}$.

This is in contrast to a previous result of the author that there is a nonempty open set u of pairs of 2-by-2 real matrices for which every pair in u has a unique Barabanov norm.

[290] A. Peperko, *Bounds on the generalized and the joint spectral radius of Hadamard products of bounded sets of positive operators on sequence spaces*, Linear Algebra Appl. **437** (2012), no. 1, 189–201, `doi:10.1016/j.laa.2012.02.022`. MR 2917439. Zbl 1243.15011.

Suppose that $A, B, A_1, \ldots, A_m$ are nonnegative $n\times n$ matrices, and $\circ$ and ρ denote the Hadamard product and the spectral radius, respectively. It is a known fact that the spectral radius of the Hadamard product of two non-negative matrices is submultiplicative. *K. M. R. Audenaert* [Linear Algebra Appl. 432, No. 1, 366–368 (2010)] proved a conjecture of *X. Zhan* [Advanced Workshop on Trends and Developments in Linear Algebra, ICTP, Trieste (2009)] by establishing that

$$\rho(A \circ B) \le \rho^{1/2}((A \circ A)(B \circ B)) \le \rho(AB).$$

R. A. Horn and *F. Zhang* [Electron. J. Linear Algebra 20, 90–94 (2010)] showed that

$$\rho(A \circ B) \le \rho^{1/2}(AB \circ BA) \le \rho(AB).$$

Z. Huang [Linear Algebra Appl. 434, No. 2, 457–462 (2011)] proved that $\rho(A_1 \circ \cdots \circ A_m) \le \rho(A_1 \cdots A_m)$.

In the paper the author extends the results above to the setting of the generalized and the joint spectral radius of bounded sets of non-negative matrices that define bounded operators on Banach sequence spaces. He also proves the inequalities

$$\rho(A \circ B) \le \rho^{1/2}((A \circ A)(B \circ B)) \le \rho(AB \circ AB)^{1/4}\rho(BA \circ BA)^{1/4} \le \rho(AB)$$

and

$$\rho(A \circ B) \le \rho^{1/2}(AB \circ BA) \le \rho(AB \circ AB)^{1/4}\rho(BA \circ BA)^{1/4} \le \rho(AB)$$

in the case of the usual spectral radius of non-negative matrices.

[291] V. Yu. Protasov and A. S. Voynov, *Sets of nonnegative matrices without positive products*, Linear Algebra Appl. **437** (2012), no. 3, 749–765, `doi:10.1016/j.laa.2012.02.029`. MR 2921734. Zbl 1245.15033.

For an arbitrary irreducible set of nonnegative $d \times d$-matrices, we consider the following problem: does there exist a strictly positive product (with repetitions permitted) of those matrices? Under some general assumptions, we prove that if it does not exist, then there is a partition of the set of basis vectors of $\mathbb{R}^d$, on which all given matrices act as permutations. Moreover, there always exists a unique maximal partition (with the maximal number of parts) possessing this property, and the number of parts is expressed by eigenvalues of matrices. This generalizes well-known results of Perron-Frobenius theory on primitivity of one matrix to families of matrices. We present a polynomial algorithm to decide the existence of a positive product for a given finite set of matrices and to build the maximal partition. Similar results are obtained for scrambling products. Applications to the study of Lyapunov exponents, inhomogeneous Markov chains, etc. are discussed.

[292] J. Shen and J. Hu, *Stability of discrete-time switched homogeneous systems on cones and conewise homogeneous inclusions*, SIAM J. Control Optim. **50** (2012), no. 4, 2216–2253, `doi:10.1137/110845215`. MR 2974737. Zbl 1252.93068.

This paper presents a stability analysis of switched homogeneous systems on cones under arbitrary and optimal switching rules with extensions to conewise homogeneous or linear inclusions. Several interrelated approaches, such as the joint spectral radius approach and the generating function approach, are exploited to derive necessary and sufficient stability conditions and to develop suitable algorithms for stability tests. Specifically, the generalized joint spectral radius and the generalized joint lower spectral radius are introduced to characterize the radii of domains of strong and weak attraction. Furthermore, strong and weak generating functions and their radii of convergence are employed to derive stability conditions; their analytic properties, numerical approximations, and convergence analysis are established. Extensions to conewise homogeneous or linear inclusions are made to address state-dependent switching dynamics. Relations between different stability notions in the strong or weak sense are studied; Lyapunov techniques are used for stability analysis of the conewise linear inclusions.

[293] V. S. Shulman and Yu. V. Turovskii, *Topological radicals, III. Joint spectral radius*, ArXiv.org e-Print archive, August 2012, in Russian, `arXiv:1208.4592`.

We develop the theory of topological radicals, and in particular the theory of the hypocompact radical. The results are applied for obtaining convenient formulas of calculation of the joint spectral radius of a precompact family of elements in a Banach algebra.

[294] R. Teichner and M. Margaliot, *Explicit construction of a Barabanov norm for a class of positive planar discrete-time linear switched systems*, Automatica J. IFAC **48** (2012), no. 1, 95–101, `doi:10.1016/j.automatica.2011.09.028`. MR 2879415. Zbl 1244.93092.

We consider the stability under arbitrary switching of a discrete-time linear switched system. A powerful approach for addressing this problem is based on studying the "Most Unstable" Switching Law (MUSL). If the solution of the switched system corresponding to the MUSL converges to the origin, then the switched system is stable for any switching law. The MUSL can be characterized using optimal control techniques. This variational approach leads to a Hamilton–Jacobi–Bellman equation describing the behavior of the switched system under the MUSL. The solution of this equation is sometimes referred to as a Barabanov norm of the switched system. Although the Barabanov norm was studied extensively, it seems that there are few examples where it was actually computed in closed-form. In this paper, we consider a special class of positive planar discrete-time linear switched systems and provide a closed-form expression for a corresponding Barabanov norm and a MUSL. The unit circle in this norm is a parallelogram.

[295] S. Trenn and F. Wirth, *Linear switched DAEs: Lyapunov exponents, a converse Lyapunov theorem, and Barabanov norms*, 51st IEEE Conference on Decision and Control (CDC), 10-13 Dec., Maui, HI, USA, 2012, pp. 2666–2671, `doi:10.1109/CDC.2012.6426245`.

For linear switched differential algebraic equations (DAEs) we consider the problem of characterizing the maximal exponential growth rate of solutions. It is shown that a finite exponential growth rate exists if and only if the set of consistency projectors associated to the family of DAEs is product bounded. This result may be used to derive a converse Lyapunov theorem for switched DAEs. Under the assumption of irreducibility we show that a construction reminiscent of the construction of Barabanov norms is feasible as well.

[296] Yu. V. Turovskii and V. S. Shulman, *Topological radicals and joint spectral radius*, Funct. Anal. Appl. **46** (2012), no. 4, 287–304, `doi:10.1007/s10688-012-0036-y`, `arXiv:0805.0209`. MR 3075096. Zbl 1319.46038.

It is shown that the joint spectral radius $\rho(M)$ of a precompact family M of operators on a Banach space X is equal to the maximum of two numbers: the joint spectral radius $\rho_e(M)$ of the image of M in the Calkin algebra and the Berger-Wang radius $r(M)$ defined by the formula

$$r(M) = \limsup_{n\to\infty} \left(\sup \{\rho(a) : a \in M^n\}^{1/n} \right).$$

Some more general Banach-algebraic results of this kind are also proved. The proofs are based on the study of special radicals on the class of Banach algebras.

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[297] A. A. Ahmadi and R. M. Jungers, *Switched stability of nonlinear systems via SOS-convex Lyapunov functions and semidefinite programming*, Proceedings of the 52nd IEEE Annual Conference on Decision and Control (CDC), 2013, pp. 727–732, `doi:10.1109/CDC.2013.6759968`.

We introduce the concept of sos-convex Lyapunov functions for stability analysis of discrete time switched systems. These are polynomial Lyapunov functions that have an algebraic certificate of convexity, and can be efficiently found by semidefinite programming. We show that convex polynomial Lyapunov functions are universal (i.e., necessary and sufficient) for stability analysis of switched linear systems. On the other hand, we show via an explicit example that the minimum degree of an sos-convex Lyapunov function can be arbitrarily higher than a (non-convex) polynomial Lyapunov function. (The proof is omitted.) In the second part, we show that if the switched system is defined as the convex hull of a finite number of nonlinear functions, then existence of a non-convex common Lyapunov function is not a sufficient condition for switched stability, but existence of a convex common Lyapunov function is. This shows the usefulness of the computational machinery of sos-convex Lyapunov functions which can be applied either directly to the switched nonlinear system, or to its linearization, to provide proof of local switched stability for the nonlinear system. An example is given where no polynomial of degree less than 14 can provide an estimate to the region of attraction under arbitrary switching.

[298] D. Bajovic, J. Xavier, J. M. F. Moura, and B. Sinopoli, *Consensus and products of random stochastic matrices: Exact rate for convergence in probability*, IEEE Trans. Signal Process. **61** (2013), no. 10, 2557–2571, `doi:10.1109/TSP.2013.2248003`, `arXiv:1202.6389`. MR 3053826. Zbl 1393.90025.

Distributed consensus and other linear systems with system stochastic matrices W_k emerge in various settings, like opinion formation in social networks, rendezvous of robots, and distributed inference in sensor networks. The matrices W_k are often random, due to, e.g., random packet dropouts in wireless sensor networks. Key in analyzing the performance of such systems is studying convergence of matrix products $W_k W_{k-1} \cdots W_1$. In this paper, we find the exact exponential rate I for the convergence in probability of the product of such matrices when time k grows large, under the assumption that the W_k's are symmetric and independent identically distributed in time. Further, for commonly used random models like with gossip and link failure, we show that the rate I is found by solving a min-cut problem and, hence, easily computable. Finally, we apply our results to optimally allocate the sensors' transmission power in consensus+innovations distributed detection.

Our analysis reveals that the exponential rate of convergence in probability depends only on the statistics of the support graphs of the random matrices. Further, we show how to compute this rate

for commonly used random models: gossip and link failure. With these models, the rate is found by solving a min-cut problem, and hence it is easily computable. Finally, as an illustration, we apply our results to solving power allocation among networked sensors in a consensus+innovations distributed detection problem.

[299] V. Bokharaie and G. Parsaee, *An application of joint spectral radius in power control problem for wireless communications*, ArXiv.org e-Print archive, July 2013, `arXiv:1307.0555`.

Resource management, including power control, is one of the most essential functionalities of any wireless telecommunication system. Various transmitter power-control methods have been developed to deliver a desired quality of service in wireless networks. We consider two of these methods: Distributed Power Control and Distributed Balancing Algorithm schemes. We use the concept of joint spectral radius to come up with conditions for convergence of the transmitted power in these two schemes when the gains on all the communications links are assumed to vary at each time-step.

[300] C.-T. Chang and V. D. Blondel, *An experimental study of approximation algorithms for the joint spectral radius*, Numer. Algorithms **64** (2013), no. 1, 181–202, `doi:10.1007/s11075-012-9661-z`. MR 3090842. Zbl 1276.65021.

We describe several approximation algorithms for the joint spectral radius and compare their performance on a large number of test cases. The joint spectral radius of a set Σ of $n \times n$ matrices is the maximal asymptotic growth rate that can be obtained by forming products of matrices from Σ. This quantity is NP-hard to compute and appears in many areas, including in system theory, combinatorics and information theory. A dozen algorithms have been proposed this last decade for approximating the joint spectral radius but little is known about their practical efficiency. We overview these approximation algorithms and classify them in three categories: approximation obtained by examining long products, by building a specific matrix norm, and by using optimization-based techniques. All these algorithms are now implemented in a (freely available) MATLAB toolbox that was released in 2011. This toolbox allows us to present a comparison of the approximations obtained on a large number of test cases as well as on sets of matrices taken from the literature. Finally, in our comparison we include a method, available in the toolbox, that combines different existing algorithms and that is the toolbox's default method. This default method was able to find optimal products for all test cases of dimension less than four.

[301] R. Cross and V. S. Kozyakin, *Double exponential instability of triangular arbitrage systems*, Discrete Contin. Dyn. Syst. Ser. B **18** (2013), no. 2, 349–376, `doi:10.3934/dcdsb.2013.18.349`, `arXiv:1204.3422`. MR 2999081. Zbl 1260.91261.

If financial markets displayed the informational efficiency postulated in the efficient markets hypothesis (EMH), arbitrage operations would be self-extinguishing. The present paper considers arbitrage sequences in foreign exchange (FX) markets, in which trading platforms and information are fragmented. In [280] it was shown that sequences of triangular arbitrage operations in FX markets containing 4 currencies and trader-arbitrageurs tend to display periodicity or grow exponentially rather than being self-extinguishing. This paper extends the analysis to 5 or higher-order currency worlds. The key findings are that in a 5-currency world arbitrage sequences may also follow an exponential law as well as display periodicity, but that in higher-order currency worlds a double exponential law may additionally apply. There is an "inheritance of instability" in the higher-order currency worlds. Profitable arbitrage operations are thus endemic rather that displaying the self-extinguishing properties implied by the EMH.

[302] A. Czornik and M. Niezabitowski, *Controllability and stability of switched systems*, 18th International Conference on Methods and Models in Automation and Robotics (MMAR) (26-29 Aug. 2013, Miedzyzdroje, Poland), IEEE, 2013, pp. 16–21, `doi:10.1109/MMAR.2013.6669874`.

The study of properties of switched and hybrid systems gives rise to a number of interesting and challenging mathematical problems. This paper aim to briefly survey recent results on stability and controllability of switched linear systems. First, the stability analysis for switched systems is reviewed. We focus on the stability analysis for switched linear systems under arbitrary switching, and we highlight necessary and sufficient conditions for asymptotic stability.

[303] X. Dai, *Some criteria for spectral finiteness of a finite subset of the real matrix space* $\mathbb{R}^{d\times d}$, Linear Algebra Appl. **438** (2013), no. 6, 2717–2727, `doi:10.1016/j.laa.2012.09.026`, `arXiv:1206.2110`. MR 3008528. Zbl 1270.15006.

In this paper, we present some checkable criteria for the spectral finiteness of a finite subset of the real $d \times d$ matrix space $\mathbb{R}^{d\times d}$, where $2 \leq d < \infty$.

[304] X. Dai, Y. Huang, and M. Xiao, *Extremal ergodic measures and the finiteness property of matrix semigroups*, Proc. Amer. Math. Soc. **141** (2013), no. 2, 393–401, `doi:10.1090/S0002-9939-2012-11330-9`, `arXiv:1107.0123`. MR 2996944. Zbl 06141459.

Let $\mathbf{S} = \{S_1, \dots, S_K\}$ be a finite set of, complex, $d \times d$ matrices and, Σ_K^+ the compact space of all one-sided infinite sequences $i_.\colon \mathbb{N} \to \{1, \dots, K\}$. An ergodic probability μ_* of the Markov shift $\theta\colon \Sigma_K^+ \to \Sigma_K^+;\ i_. \mapsto i_{.+1}$, is called "extremal" for $\mathbf{S}$, if $\rho(\mathbf{S}) = \lim_{n\to\infty} \sqrt[n]{\|S_{i_1} \cdots S_{i_n}\|}$ holds for μ_*-a.e. $i_. \in \Sigma_K^+$, where $\rho(\mathbf{S})$ denotes the generalized/joint spectral radius of $\mathbf{S}$. Using extremal norm and Kingman subadditive ergodic theorem, it is shown:

Theorem. $\mathbf{S}$ *has the spectral finiteness property (i.e.* $\rho(\mathbf{S}) = \sqrt[n]{\rho(S_{i_1} \cdots S_{i_n})}$ *for some finite-length word* $(i_1, \dots, i_n)$*) if and only if for some extremal measure* μ_* *of* $\mathbf{S}$*, it has at least one periodic density point* $i_. \in \Sigma_K^+$.

[305] X. Dai, Y. Huang, and M. Xiao, *Spectral finiteness property and splitting of state space for matrix-valued cocycles*, ArXiv.org e-Print archive, August 2013, `arXiv:1308.6111`.

By proving and using a splitting of state space without the uniform product boundedness condition, it is studied the effectiveness of the computation of the joint spectral radius for a linear cocycle driven by a discrete-time continuous semiflow. Our result may be applied to Markovian jump linear systems.

[306] P. Dumas, *Joint spectral radius, dilation equations, and asymptotic behavior of radix-rational sequences*, Linear Algebra Appl. **438** (2013), no. 5, 2107–2126, `doi:10.1016/j.laa.2012.10.013`. MR 3005279. Zbl 1368.11005.

Radix-rational sequences are solutions of systems of recurrence equations based on the radix representation of the index. For each radix-rational sequence with complex values we provide an asymptotic expansion, essentially in the scale $N^\alpha \log^\ell N$. The precision of the asymptotic expansion depends on the joint spectral radius of the linear representation of the sequence of first-order differences. The coefficients are Hölderian functions obtained through some dilation equations, which are usual in the domains of wavelets and refinement schemes. The proofs are ultimately based on elementary linear algebra.

[307] M. Gaye, Y. Chitour, and P. Mason, *Properties of Barabanov norms and extremal trajectories associated with continuous-time linear switched systems*, 2013 IEEE 52nd Annual Conference on Decision and Control (CDC), IEEE, 2013, pp. 716–721, `doi:10.1109/CDC.2013.6759966`.

Consider continuous-time linear switched systems on $\mathbb{R}^n$ associated with compact convex sets of matrices. When the system is irreducible and the largest Lyapunov exponent is equal to zero, a Barabanov norm always exists. This paper deals with two sets of issues: (a) properties of Barabanov norms such as uniqueness up to homogeneity and strict convexity; (b) asymptotic behaviour of the extremal solutions of the system. Regarding Issue (a), we provide partial answers and propose two open problems motivated by appropriate examples. As for Issue (b), we establish, when $n = 3$, a Poincaré-Bendixson theorem under a regularity assumption on the set of matrices defining the system.

[308] N. Guglielmi and V. Protasov, *Exact computation of joint spectral characteristics of linear operators*, Found. Comput. Math. **13** (2013), no. 1, 37–97, `doi:10.1007/s10208-012-9121-0`, `arXiv:1106.3755`. MR 3009529. Zbl 06153962.

We address the problem of the exact computation of two joint spectral characteristics of a family of linear operators, the joint spectral radius (JSR) and the lower spectral radius (LSR), which are well-known different generalizations to a set of operators of the usual spectral radius of a linear operator. In this paper we develop a method which – under suitable assumptions – allows us to compute the JSR and the LSR of a finite family of matrices exactly. We remark that so far no algorithm has been available in the literature to compute the LSR exactly. The paper presents necessary theoretical results on extremal norms (and on extremal antinorms) of linear operators, which constitute the basic tools of our procedures, and a detailed description of the corresponding algorithms for the computation of the JSR and LSR (the last one restricted to families sharing an invariant cone). The algorithms are easily implemented, and their descriptions are short. If the algorithms terminate in finite time, then they construct an extremal norm (in the JSR case) or antinorm (in the LSR case) and find their exact values; otherwise, they provide upper and lower bounds that both converge to the exact values. A theoretical criterion for termination in finite time is also derived. According to numerical experiments, the algorithm for the JSR finds the exact value for the vast majority of matrix families in dimensions $\leq$20. For nonnegative matrices it works faster and finds the JSR in dimensions of order 100 within a few iterations; the same is observed for the algorithm computing the LSR. To illustrate the efficiency of the new method, we apply it to give answers to several conjectures which have been recently stated in combinatorics, number theory, and formal language theory.

[309] K. G. Hare, I. D. Morris, and N. Sidorov, *Extremal sequences of polynomial complexity*, Math. Proc. Cambridge Philos. Soc. **155** (2013), no. 2, 191–205, `doi:10.1017/S0305004113000157`, `arXiv:1201.6236`. MR 3091514. Zbl 1326.15031.

The joint spectral radius of a bounded set of $d \times d$ real matrices is defined to be the maximum possible exponential growth rate of products of matrices drawn from that set. For a fixed set of matrices, a sequence of matrices drawn from that set is called *extremal* if the associated sequence of partial products achieves this maximal rate of growth. An influential conjecture of *J. Lagarias* and *Y. Wang* [53] asked whether every finite set of matrices admits an extremal sequence which is periodic. This is equivalent to the assertion that every finite set of matrices admits an extremal sequence with bounded subword complexity. Counterexamples were subsequently constructed which have the property that every extremal sequence has at least linear subword complexity. In this paper we extend this result to show that for each integer $p \geq 1$, there exists a pair of square matrices of dimension $2^p(2^{p+1}-1)$ for which every extremal sequence has subword complexity at least $2^{-p^2}n^p$.

[310] M. Javaheri, *Maximally transitive semigroups of $n \times n$ matrices*, J. Math. Anal. Appl. **401** (2013), no. 2, 743–753, `doi:10.1016/j.jmaa.2012.12.047`. MR 3018024. Zbl 06156283.

We prove that, for every $n \geq 1$, there exists a pair of $n \times n$ matrices that generates a topologically n-transitive semigroup action on $\mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Equivalently, we construct dense 2-generator subsemigroups of $GL(n, \mathbb{K})$ for all $n \geq 1$.

[311] R. M. Jungers, *Joint spectral characteristics: a tale of three disciplines*, Developments in language theory. Proceedings of the 17th international conference, DLT 2013, Marne-la-Vallée, France, June 18–21, 2013, Lecture Notes in Computer Science, vol. 7907, Springer Berlin Heidelberg, 2013, pp. 27–28, `doi:10.1007/978-3-642-38771-5_3`.

Joint spectral characteristics describe the stationary behavior of a discrete time linear switching system. Well, that's what an electrical engineer would say. A mathematician would say that they characterize the asymptotic behavior of a semigroup of matrices, and a computer scientist would perhaps see them as describing languages generated by automata.

[312] J. Klamka, A. Czornik, and M. Niezabitowski, *Stability and controllability of switched linear dynamical systems*, Bull. Pol. Acad. Sci. Tech. Sci. **61** (2013), no. 3, 547–555, `doi:10.2478/bpasts-2013-0055`.

The study of properties of switched and hybrid systems gives rise to a number of interesting and challenging mathematical problems. This paper aims to briefly survey recent results on stability and controllability of switched linear systems. First, the stability analysis for switched systems is reviewed. We focus on the stability analysis for switched linear systems under arbitrary switching, and we highlight necessary and sufficient conditions for asymptotic stability. After that, we review the controllability results.

[313] V. S. Kozyakin, *Algebraic unsolvability of problem of absolute stability of desynchronized systems revisited*, ArXiv.org e-Print archive, January 2013, `arXiv:1301.5409`.

In the author's article [33], it was shown that in general for linear desynchronized systems there are no algebraic criteria of absolute stability. In this paper, a few misprints occurred in the original version of the article are corrected, and two figures are added.

[314] J. Liu and M. Xiao, *Rank-one characterization of joint spectral radius of finite matrix family*, Linear Algebra Appl. **438** (2013), no. 8, 3258–3277, `doi:10.1016/j.laa.2012.12.032`, `arXiv:1109.1356`. MR 3023275. Zbl 1267.15009.

Let $\mathcal{F}$ be a finite set of $n \times n$ complex matrices and for each $k \geq 1$ let $\mathcal{F}_k$ denote the set of all products $A_1 \cdots A_k$ of length k with each $A_i \in \mathcal{F}$. The joint spectral radius of $\mathcal{F}$ is defined as $\rho(\mathcal{F}) := \inf_{\|\ \|} \max_{A\in\mathcal{F}} \|A\|$ where the infimum is taken over all sub-multiplicative matrix norms. The authors are interested in methods of computing $\rho(\mathcal{F})$.

It can be shown that

$$\max_{A\in\mathcal{F}_k} \rho(A)^{1/k} \leq \rho(\mathcal{F}) \leq \max_{A\in\mathcal{F}_k} \|A\|^{1/k} \text{ for all } k \geq 1$$

where $\rho(A)$ denotes the spectral radius of A and $\|\ \|$ is any sub-multiplicative norm. Furthermore, the right hand side converges to $\rho(\mathcal{F})$ as $k \to \infty$ (see *G.-C. Rota* and *G. Strang* [3]) and the lim sup of the left hand side is also equal to $\rho(\mathcal{F})$. The set $\mathcal{F}$ is said to have the finiteness property if $\rho(\mathcal{F})$ is equal to $\rho(A)^{1/k}$ for some k and some $A \in \mathcal{F}_k$. Although not every finite set $\mathcal{F}$ has the finiteness property, there are some important classes of sets of matrices which do have this property, and when this holds there may be more efficient ways to compute the value of $\rho(\mathcal{F})$.

The first part of the present paper proves that if $\mathcal{F}$ contains at most one matrix of rank > 1, then $\mathcal{F}$ has the finiteness property. Moreover, if all matrices in $\mathcal{F}$ have rank 1 and k is the least value for which there there exists $A \in \mathcal{F}_k$ such that $\rho(\mathcal{F}) = \rho(A)^{1/k}$, then A is a product of k distinct matrices from $\mathcal{F}$. In this special case, for small sets $\mathcal{F}$, this gives an efficient way to compute $\rho(\mathcal{F})$ (see also *A. A. Ahmadi* and *P. A. Parrilo* [274]). To extend beyond this limited case, the authors consider the set $P(\mathcal{F}_k)$ of rank one approximations to the elements in $\mathcal{F}_k$ (which may be obtained using singular value decompositions) and prove that $\rho(\mathcal{F}) = \limsup_{k\to\infty} \rho(P(\mathcal{F}_k))^{1/k}$. In the special case where each matrix in $\mathcal{F}$ has nonnegative real entries and some product of matrices in $\mathcal{F}$ has all entries > 0, there is a simpler formula: $\rho(\mathcal{F})$ is equal to the limit of $\max_{A\in\mathcal{F}_k}$ $(\mathrm{tr}\ A)^{1/k}$ as $k \to \infty$. Numerical examples are given to show how these formulae may be used in practice.

[315] I. Morris and N. Sidorov, *On a Devil's staircase associated to the joint spectral radii of a family of pairs of matrices*, J. Eur. Math. Soc. (JEMS) **15** (2013), no. 5, 1747–1782, `doi:10.4171/JEMS/402`, `arXiv:1107.3506`. MR 3082242. Zbl 06203606.

The joint spectral radius of a finite set of real $d\times d$ matrices is defined to be the maximum possible exponential rate of growth of products of matrices drawn from that set. In previous work with K. G. Hare and J. Theys we showed that for a certain one-parameter family of pairs of matrices, this maximum possible rate of growth is attained along Sturmian sequences with a certain characteristic ratio which depends continuously upon the parameter. In this paper we answer some open questions from that paper by showing that the dependence of the ratio function upon the parameter takes the form of a Devil's staircase. We show in particular that this Devil's staircase attains every rational value strictly between 0 and 1 on some interval, and attains irrational values only in a set of Hausdorff dimension zero. This result generalises to include certain one-parameter families considered by other authors. We also give explicit formulas for the preimages of both rational and irrational numbers under the ratio function, thereby establishing a large family of pairs of matrices for which the joint spectral radius may be calculated exactly.

[316] I. D. Morris, *Mather sets for sequences of matrices and applications to the study of joint spectral radii*, Proc. Lond. Math. Soc. (3) **107** (2013), no. 1, 121–150, `doi:10.1112/plms/pds080`. MR 3083190. Zbl 06194569.

The joint spectral radius of a compact set of $d \times d$, matrices is defined to be the maximum possible exponential growth rate of products of matrices drawn from that set. In this article we investigate the ergodic-theoretic structure of those sequences of matrices drawn from a given set whose products grow at the maximum possible rate. This leads to a notion of Mather set for matrix sequences which is analogous to the Mather set in Lagrangian dynamics. We prove a structure theorem establishing the general properties of these Mather sets and describing the extent to which they characterise matrix sequences of maximum growth. We give applications of this theorem to the study of joint spectral radii and to the stability theory of discrete linear inclusions.

These results rest on some general theorems on the structure of orbits of maximum growth for subadditive observations of dynamical systems, including an extension of the semi-uniform subadditive ergodic theorem of Schreiber, Sturman and Stark, and an extension of a noted lemma of Y. Peres. These theorems are presented in the appendix.

[317] V. Müller and A. Peperko, *Generalized spectral radius and its max algebra version*, Linear Algebra Appl. **439** (2013), no. 4, 1006–1016, `doi:10.1016/j.laa.2012.09.024`. MR 3061751. Zbl 1281.15009.

Let $\Sigma \subset \mathbb{C}^{n\times n}$ and $\Psi \subset \mathbb{R}_+^{n\times n}$ be bounded subsets and let $\rho(\Sigma)$ and $\mu(\Psi)$ denote the generalized spectral radius of Σ and the max algebra version of the generalized spectral radius of Ψ, respectively. We apply a single matrix description of $\mu(\Psi)$ to give a new elementary and straightforward proof of the Berger-Wang formula in max algebra and consequently a new short proof of the original Berger-Wang formula in the case of bounded subsets of $n \times n$ non-negative matrices. We also obtain a new description of $\mu(\Psi)$ in terms of the Schur-Hadamard product and prove new trace and max-trace descriptions of $\mu(\Psi)$ and $\rho(\Sigma)$. In particular, we show that

$$\mu(\Psi) = \limsup_{m\to\infty} \, [\sup_{A\in\Psi_\otimes^m} \operatorname{tr}_\otimes(A)]^{1/m} = \limsup_{m\to\infty} \, [\sup_{A\in\Psi_\otimes^m} \operatorname{tr}(A)]^{1/m}$$

and

$$\rho(\Sigma) = \limsup_{m\to\infty} \, [\sup_{B\in\Sigma^m} \operatorname{tr}(|B|)]^{1/m} = \limsup_{m\to\infty} \, [\sup_{B\in\Sigma^m} \operatorname{tr}_\otimes(|B|)]^{1/m}$$

where $\operatorname{tr}_\otimes(A) = \max_{i=1,\dots,n} a_{ii}$ and $|B| = [|b_{ij}|]$.

[318] Yu. Nesterov and V. Yu. Protasov, *Optimizing the spectral radius*, SIAM J. Matrix Anal. Appl. **34** (2013), no. 3, 999–1013, `doi:10.1137/110850967`. MR 3073651. Zbl 1282.15029.

We suggest a new approach to finding the maximal and the minimal spectral radii of linear operators from a given compact family of operators, which share a common invariant cone (e.g., family of nonnegative matrices). In the case of families with the so-called product structure, this leads to efficient algorithms for optimizing the spectral radius and for finding the joint and lower spectral radii of the family. Applications to the theory of difference equations and to problems of optimizing the spectral radius of graphs are considered.

[319] M. Ogura and C. F. Martin, *Generalized joint spectral radius and stability of switching systems*, Linear Algebra Appl. **439** (2013), no. 8, 2222–2239, `doi:10.1016/j.laa.2013.06.028`. MR 3091300. Zbl 1280.93090.

This paper extends the notion of generalized joint spectral radius with exponents, originally defined for a finite set of matrices, to probability distributions. We show that, under a certain invariance condition, the radius is calculated as the spectral radius of a matrix that can be easily computed, extending the classical counterpart. Using this result we investigate the mean stability of switching systems. In particular we establish the equivalence of mean square stability, simultaneous contractibility in square mean, and the existence of a quadratic Lyapunov function. Also the stabilization of positive switching systems is studied. Numerical examples are given to illustrate the results.

[320] M. Ogura and C. F. Martin, *On the mean stability of a class of switched linear systems*, Proceedings of the 52th IEEE Conference on Decision and Control, CDC 2013, 2013, pp. 97–102, `arXiv:1409.6032`.

This paper investigates the mean stability of a class of discrete-time stochastic switched linear systems using the L^p-norm joint spectral radius of the probability distributions governing the switched systems. First we prove a converse Lyapunov theorem that shows the equivalence between the mean stability and the existence of a homogeneous Lyapunov function. Then we show that, when p goes to ∞, the stability of the pth mean becomes equivalent to the absolute asymptotic stability of an associated deterministic switched system. Finally we study the mean stability of Markovian switched systems. Numerical examples are presented to illustrate the results.

[321] M. Ogura and C. F. Martin, *Stability of switching systems and generalized joint spectral radius*, Proceedings of the European Control Conference, ECC 2013, 17–19 July (Zurich, Switzerland), IEEE, 2013, pp. 3185–3190, `doi:10.23919/ECC.2013.6669115`.

This paper studies the mean stability of stochastic switching linear systems. We first show that the mean stability is characterized by an extended version of so called generalized joint spectral radius. Then it is shown that, under an invariance condition, the quantity can be computed as the spectral radius of a certain matrix associated with the given switching system. Also we show that the mean square stability is equivalent to the existence of a Lyapunov function. Our results are illustrated by numerical examples.

[322] V. Yu. Protasov and R. M. Jungers, *Is switching systems stability harder for continuous time systems?*, Proceedings of the 52nd IEEE Conference on Decision and Control (CDC), 10–13 Dec. (Florence, Italy), IEEE, 2013, pp. 704–709, `doi:10.1109/CDC.2013.6759964`.

We analyse the problem of stability of a continuous time linear switching system (LSS) versus the stability of its Euler discretization. It is well-known that the existence of a positive τ for which the corresponding discrete time system with stepsize τ is stable implies the stability of the LSS. Our main goal is to obtain a converse statement, that is to estimate the discretization stepsize $\tau > 0$ up to a given accuracy $\varepsilon > 0$. This would lead to a method for deciding the stability of a continuous time LSS with a guaranteed accuracy. As a first step towards the solution of this problem, we show that for systems of matrices with real spectrum the parameter τ can be effectively estimated. We prove that in this special case, the discretized system is stable if and only if the Lyapunov exponent of the LSS is smaller than $-C\tau$, where C is an effective constant depending on the system. The proofs are based on applying Markov-Bernstein type inequalities for systems of exponents.

[323] V. Yu. Protasov and R. M. Jungers, *Lower and upper bounds for the largest Lyapunov exponent of matrices*, Linear Algebra Appl. **438** (2013), no. 11, 4448–4468, `doi:10.1016/j.laa.2013.01.027`. MR 3034543. Zbl 1281.65154.

We introduce a new approach to evaluate the largest Lyapunov exponent of a family of nonnegative matrices. The method is based on using special positive homogeneous functionals on $\mathbb{R}^d_+$, which gives iterative lower and upper bounds for the Lyapunov exponent. They improve previously known bounds and converge to the real value. The rate of convergence is estimated and the efficiency of the algorithm is demonstrated on several problems from applications (in functional analysis, combinatorics, and language theory) and on numerical examples with randomly generated matrices. The method computes the Lyapunov exponent with a prescribed accuracy in relatively high dimensions (up to 60). We generalize this approach to all matrices, not necessarily nonnegative, derive a new universal upper bound for the Lyapunov exponent, and show that a potential similar lower bound does not exist in general.

[324] V. Yu. Protasov and R. M. Jungers, *Convex optimization methods for computing the Lyapunov exponent of matrices*, 2013 European Control Conference (ECC), July 2013, pp. 3191–3196, `arXiv:1201.3218`.

We introduce a new approach to evaluate the largest Lyapunov exponent of a family of nonnegative matrices. The method is based on using special positive homogeneous functionals on $\mathbb{R}^d_+$, which gives iterative lower and upper bounds for the Lyapunov exponent. They improve previously known bounds and converge to the real value. The rate of converges is estimated and the efficiency of the algorithm is demonstrated on several problems from applications (in functional analysis, combinatorics, and language theory) and on numerical examples with randomly generated matrices. The method computes the Lyapunov exponent with a prescribed accuracy in relatively high dimensions (up to 60). We generalize this approach to all matrices, not necessarily nonnegative, derive a new universal upper bound for the Lyapunov exponent, and show that such a lower bound, in general, does not exist.

[325] A. Thomas, *Almost sure convergence of products of* 2×2 *nonnegative matrices*, ArXiv.org e-Print archive, February 2013, `arXiv:1302.4715`.

We study the almost sure convergence of the normalized columns in an infinite product of nonnegative matrices, and the almost sure rank one property of its limit points. Given a probability on the set of 2×2 nonnegative matrices, with finite support $\mathcal{A} = \{A(0), \dots, A(s-1)\}$, and assuming that at least one of the $A(k)$ is not diagonal, the normalized columns of the product matrix $P_n = A(\omega_1)\dots A(\omega_n)$ converge almost surely (for the product probability) with an exponential rate of convergence if and only if the Lyapunov exponents are almost surely distinct. If this condition is satisfied, given a nonnegative column vector V the column vector $\frac{P_nV}{\|P_nV\|}$ also converges almost surely with an exponential rate of convergence. On the other hand if we assume only that at least one of the $A(k)$ do not have the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, $ad \neq 0$, nor the form $\begin{pmatrix} 0 & b \\ d & 0 \end{pmatrix}$, $bc \neq 0$, the limit-points of the normalized product matrix $\frac{P_n}{\|P_n\|}$ have almost surely rank 1 – although the limits of the normalized columns can be distinct – and $\frac{P_nV}{\|P_nV\|}$ converges almost surely with a rate of convergence that can be exponential or not exponential.

[326] S. Wang and J. Wen, *The finiteness conjecture for the joint spectral radius of a pair of matrices*, Proceedings of the 9th International Conference on Computational Intelligence and Security (CIS), 2013, December 14–15 (Leshan, China), IEEE, 2013, pp. 798–802, `doi:10.1109/CIS.2013.174`.

A set of matrices is said to have the finiteness property if the maximal rate of growth of long products of matrices taken from the set can be obtained by a periodic product. We study the finite-step realizability of the joint/generalized spectral radius of a pair of $n \times n$ square matrices. Let $\Sigma = \{A, B\}$ where A, B are $n \times n$ matrices and B is a rank-one matrix. Then we have $\rho(\Sigma) = \max_{t,s} \rho(A^tB^s)^{\frac{1}{s+t}}$. That is to say, Σ have the finiteness property where the maximum is attained at (t, s) with the optimal sequence A^tB^s.

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[327] A. A. Ahmadi and R. M. Jungers, *On complexity of Lyapunov functions for switched linear systems*, IFAC Proceedings Volumes **47** (2014), no. 3, 5992–5997, 19th IFAC World Congress, `doi:10.3182/20140824-6-ZA-1003.02484`.

We show that for any positive integer d, there are families of switched linear systems — in fixed dimension and defined by two matrices only — that are stable under arbitrary switching but do not admit (i) a polynomial Lyapunov function of degree $\leq d$, or (ii) a polytopic Lyapunov function with $\leq d$ facets, or (iii) a piecewise quadratic Lyapunov function with $\leq d$ pieces. This implies that there cannot be an upper bound on the size of the linear and semidefinite programs that search for such stability certificates. Several constructive and non-constructive arguments are presented which connect our problem to known (and rather classical) results in the literature regarding the finiteness conjecture, undecidability, and non-algebraicity of the joint spectral radius. In particular, we show that existence of a sum of squares Lyapunov function implies the finiteness property of the optimal product.

[328] A. A. Ahmadi, R. M. Jungers, P. A. Parrilo, and M. Roozbehani, *Joint spectral radius and path-complete graph Lyapunov functions*, SIAM J. Control Optim. **52** (2014), no. 1, 687–717, `doi:10.1137/110855272`, `arXiv:1111.3427`. MR 3168608. Zbl 1292.93093.

We introduce the framework of path-complete graph Lyapunov functions for approximation of the joint spectral radius. The approach is based on the analysis of the underlying switched system via inequalities imposed among multiple Lyapunov functions associated to a labeled directed graph. Inspired by concepts in automata theory and symbolic dynamics, we define a class of graphs called path-complete graphs, and show that any such graph gives rise to a method for proving stability of the switched system. This enables us to derive several asymptotically tight hierarchies of semidefinite programming relaxations that unify and generalize many existing techniques such as common quadratic, common sum of squares, and maximum/minimum-of-quadratics Lyapunov functions. We compare the quality of approximation obtained by certain classes of path-complete graphs including a family of dual graphs and all path-complete graphs with two nodes on an alphabet of two matrices. We provide approximation guarantees for several families of path-complete graphs, such as the De Bruijn graphs, establishing as a byproduct a constructive converse Lyapunov theorem for maximum/minimum-of-quadratics Lyapunov functions.

[329] M. Charina, *Finiteness conjecture and subdivision*, Appl. Comput. Harmon. Anal. **36** (2014), no. 3, 522–526, `doi:10.1016/j.acha.2013.09.002`. MR 3175093. Zbl 1311.15023.

The finiteness conjecture by J. C. Lagarias and Y. Wang [53] states that the joint spectral radius of a finite set of square matrices is attained on some finite product of such matrices. This conjecture is known to be false in general. Nevertheless, we show that this conjecture is true for a big class of finite sets of square matrices used for the smoothness analysis of scalar univariate subdivision schemes with finite masks.

[330] X. Dai, *Robust periodic stability implies uniform exponential stability of Markovian jump linear systems and random linear ordinary differential equations*, J. Franklin Inst. **351** (2014), no. 5, 2910–2937, `doi:10.1016/j.jfranklin.2014.01.010`, `arXiv:1307.4209`. MR 3191925. Zbl 1372.93209.

In this paper, there are shown the following two statements.

(1) *A discrete-time Markovian jump linear system is uniformly exponentially stable if and only if it is robustly periodically stable, by using a Gel'fand-Berger-Wang formula proved here.*

(2) *A random linear ODE driven by a semiflow with closing by periodic orbits property is uniformly exponentially stable if and only if it is robustly periodically stable, by using Shantao Liao's perturbation technique and the semi-uniform ergodic theorems.*

The proofs involve ergodic theory in both of the above two cases. In addition, counterexamples are constructed to the robustness condition and to spectral finiteness of linear cocycle.

[331] X. Dai, T. Huang, and Y. Huang, *Exponential stability of nonhomogeneous matrix-valued Markovian chains*, ArXiv.org e-Print archive, January 2014, `arXiv:1401.6014`.

Let $\boldsymbol{\xi} = \{\xi_n\}_{n\geq 0}$ be a nonhomogeneous/nonstationary Markovian chain on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ valued in the state space $\boldsymbol{S}$ that consists of a finite number of real d-by-d matrices such that $\mathbb{P}(\{\xi_0 = S\}) > 0$ for each $S \in \boldsymbol{S}$. As usual, $\boldsymbol{\xi}$ is called *uniformly exponentially stable* if there exist two constants $C > 0$ and $0 < \lambda < 1$ so that for all $n \geq 1$, $\|\xi_0(\omega)\xi_1(\omega)\cdots\xi_{n-1}(\omega)\| \leq C\lambda^n$ for $\mathbb{P}$-a.e.$\omega \in \Omega$. In this note, we show that if the Markovian transition probability matrices of $\boldsymbol{\xi}$ have the same transition sign matrix for all times $n \geq 0$, then $\boldsymbol{\xi}$ is uniformly exponentially stable if and only if there are $\gamma < 1$ and $N > 0$ such that for each $n > N$, the spectral radii $\rho(S_{i_0}\cdots S_{i_{n-1}})$ are less than or equal to γ for all n-length closed sample paths $(S_{i_0}, \cdots, S_{i_{n-1}}) \in \boldsymbol{S}^n$ with $\mathbb{P}(\{\xi_0 = S_{i_0}, \ldots, \xi_{n-1} = S_{i_{n-1}}, \xi_n = S_{i_0}\}) > 0$.

[332] N. Guglielmi and M. Zennaro, *Stability of linear problems: Joint spectral radius of sets of matrices*, Current Challenges in Stability Issues for Numerical Differential Equations, Lecture Notes in Mathematics, Springer, 2014, pp. 265–313, `doi:10.1007/978-3-319-01300-8_5`. MR 3204994. Zbl 1318.65081.

It is well known that the stability analysis of step-by-step numerical methods for differential equations often reduces to the analysis of linear difference equations with variable coefficients. This

class of difference equations leads to a family $\mathcal{F}$ of matrices depending on some parameters and the behaviour of the solutions depends on the convergence properties of the products of the matrices of $\mathcal{F}$. To date, the techniques mainly used in the literature are confined to the search for a suitable norm and for conditions on the parameters such that the matrices of $\mathcal{F}$ are contractive in that norm. In general, the resulting conditions are more restrictive than necessary. An alternative and more effective approach is based on the concept of joint spectral radius of the family $\mathcal{F}$, $\rho(\mathcal{F})$. It is known that all the products of matrices of $\mathcal{F}$ asymptotically vanish if and only if $\rho(\mathcal{F}) < 1$. The aim of this chapter is that to discuss the main theoretical and computational aspects involved in the analysis of the joint spectral radius and in applying this tool to the stability analysis of the discretizations of differential equations as well as to other stability problems. In particular, in the last section, we present some recent heuristic techniques for the search of optimal products in finite families, which constitutes a fundamental step in the algorithms which we discuss. The material we present in the final section is part of an original research which is in progress and is still unpublished.

[333] F. John, *Extremum problems with inequalities as subsidiary conditions*, Traces and Emergence of Nonlinear Programming (G. Giorgi and T. H. Kjeldsen, eds.), Birkhäuser/Springer Basel AG, Basel, 2014, pp. 197–215, reprint of [2], `doi:10.1007/978-3-0348-0439-4_9`. MR 3204131.

This paper deals with an extension of Lagrange's multiplier rule to the case, where the subsidiary conditions are inequalities instead of equations. Only extrema of differentiable functions of a finite number of variables will be considered. There may however be an infinite number of inequalities prescribed. Lagrange's rule for the situation considered here differs from the ordinary one, in that the multipliers may always be assumed to be positive. This makes it possible to obtain sufficient conditions for the occurence or a minimum in terms of the first derivatives only.

[334] R. M. Jungers, A. Cicone, and N. Guglielmi, *Lifted polytope methods for computing the joint spectral radius*, SIAM J. Matrix Anal. Appl. **35** (2014), no. 2, 391–410, `doi:10.1137/130907811`, `arXiv:1207.5123`. MR 3188391. Zbl 1296.93067.

We describe new methods for deciding the stability of switching systems. The methods build on two ideas previously appeared in the literature: the polytope norm iterative construction, and the lifting procedure. Moreover, the combination of these two ideas allows us to introduce a pruning algorithm which can importantly reduce the computational burden. We prove several appealing theoretical properties of our methods like a finiteness computational result which extends a known result for unlifted sets of matrices, and provide numerical examples of their good behaviour.

[335] V. Kozyakin, *The Berger-Wang formula for the Markovian joint spectral radius*, Linear Algebra Appl. **448** (2014), 315–328, `doi:10.1016/j.laa.2014.01.022`, `arXiv:1401.2711`. MR 3182989. Zbl 06278067.

The Berger-Wang formula establishes equality between the joint and generalized spectral radii of a set of matrices. For matrix products whose multipliers are applied not arbitrarily but in accordance to some Markovian law, there are also known analogs of the joint and generalized spectral radii. However, the known proofs of the Berger-Wang formula hardly can be directly applied in the case of Markovian products of matrices since they essentially rely on the arbitrariness of appearance of different matrices in the related matrix products. Nevertheless, as has been shown by *X. Dai* [330] the Berger-Wang formula is valid for the case of Markovian analogs of the joint and the generalized spectral radii too, although the proof in this case heavily exploits the more involved techniques of multiplicative ergodic theory. In the paper we propose a matrix theory construction allowing deduce the Markovian analog of the Berger-Wang formula from the classical Berger-Wang formula.

[336] V. Kozyakin, *Matrix products with constraints on the sliding block relative frequencies of different factors*, Linear Algebra Appl. **457** (2014), 244–260, `doi:10.1016/j.laa.2014.05.016`, `arXiv:1403.5050`. MR 3230443. Zbl 1291.15027.

One of fundamental results of the theory of joint/generalized spectral radius, the Berger-Wang theorem, establishes equality between the joint and generalized spectral radii of a set of matrices.

Generalization of this theorem on products of matrices whose factors are applied not arbitrarily but are subjected to some constraints is connected with essential difficulties since known proofs of the Berger-Wang theorem rely on the arbitrariness of appearance of different matrices in the related matrix products. Recently, *X. Dai* [330] proved an analog of the Berger-Wang theorem for the case when factors in matrix products are formed by some Markov law.

We introduce the concepts of the joint and generalized spectral radii for products of matrices subjected to constraints on the sliding block relative frequencies of occurrences of different matrices, and prove an analog of the Berger-Wang theorem for this case.

[337] R. Luís and H. M. Oliveira, *Products of* 2×2 *matrices related to non autonomous Fibonacci difference equations*, Appl. Math. Comput. **226** (2014), 101–116, `doi:10.1016/j.amc.2013.09.075`, `arXiv:1308.1137`. MR 3144294. Zbl 1354.11012.

A technique to compute arbitrary products of a class of Fibonacci 2×2 square matrices is proved in this work. General explicit solutions for non autonomous Fibonacci difference equations are obtained from these products.

In the periodic non autonomous Fibonacci difference equations the monodromy matrix, the Floquet multipliers and the Binet's formulas are obtained. In the periodic case explicit solutions are obtained and the solutions are analyzed.

[338] O. Mason and F. Wirth, *Extremal norms for positive linear inclusions*, Linear Algebra Appl. **444** (2014), 100–113, `doi:10.1016/j.laa.2013.11.020`, `arXiv:1306.3814`. MR 3145832. Zbl 1285.15012.

For finite-dimensional linear semigroups which leave a proper cone invariant it is shown that irreducibility with respect to the cone implies the existence of an extremal norm. In case the cone is simplicial a similar statement applies to absolute norms. The semigroups under consideration may be generated by discrete-time systems, continuous-time systems or continuous-time systems with jumps. The existence of extremal norms is used to extend results on the Lipschitz continuity of the joint spectral radius beyond the known case of semigroups that are irreducible in the representation theory interpretation of the word.

[339] B. Mojškerc, *On the structure of finite-dimensional paracontractions*, Linear Algebra Appl. **446** (2014), 148–162, `doi:10.1016/j.laa.2014.01.006`. MR 3163134. Zbl 06277974.

A paracontraction with respect to a vector norm $\|\cdot\|$ is a matrix $A \in \mathcal{M}_n$ with the following property: either $Ax = x$ or $\|Ax\| < \|x\|$ for any $x \in \mathbb{C}^n$. Paracontractions arise naturally when observing various stability properties of products of matrices.

We give a characterization of paracontractions with respect to a strictly convex norm on $\mathbb{C}^n$. As an application we give a characterization of linear mappings on $\mathcal{M}_n$ which preserve paracontractions with respect to p-norms for $1 \leq p \leq \infty$ in both directions.

[340] C. Möller and U. Reif, *A tree-based approach to joint spectral radius determination*, Linear Algebra Appl. **463** (2014), 154–170, `doi:10.1016/j.laa.2014.08.009`. MR 3262394. Zbl 1300.15004.

We suggest a novel method to determine the joint spectral radius of finite sets of matrices by validating the finiteness property. It is based on finding a certain finite tree with nodes representing sets of matrix products. Our approach accounts for cases where one or several matrix products satisfy the finiteness property. Moreover, is potentially functional even for reducible sets of matrices.

[341] M. Ogura and R. M. Jungers, *Efficiently computable lower bounds for the p-radius of switching linear systems*, 53rd IEEE Conference on Decision and Control (CDC), 15-17 Dec. (Los Angeles, California, USA), IEEE, 2014, pp. 5463–5468, `doi:10.1109/CDC.2014.7040243`.

This paper proposes novel lower bounds on a quantity called L^p-norm joint spectral radius, or in short, p-radius, of a finite set of matrices. Despite its wide range of applications, (for example, to the stability of switching linear systems and the uniqueness of the equilibrium solutions of switching linear economical models), algorithms for computing the p-radius are only available in a very limited number of particular cases. We propose lower bounds that do not require any special structure on matrices and are formulated as the maximal spectral radius of a matrix family generated by weighting matrices via Kronecker products. We show on numerical examples that the proposed lower bounds can largely improve the existing ones.

[342] M. Ogura and C. F. Martin, *A limit formula for joint spectral radius with p-radius of probability distributions*, Linear Algebra Appl. **458** (2014), 605–625, `doi:10.1016/j.laa.2014.06.034`, `arXiv:1401.3026`. MR 3231838. Zbl 1294.15015.

In this paper we show a characterization of the joint spectral radius of a set of matrices as the limit of the p-radius of an associated probability distribution when p tends to ∞. Allowing the set to have infinitely many matrices, the obtained formula extends the results in the literature. Based on the formula, we then present a novel characterization of the stability of switched linear systems for an arbitrary switching signal via the existence of stochastic Lyapunov functions of any higher degrees. Numerical examples are presented to illustrate the results.

[343] G. Vankeerberghen, J. Hendrickx, and R. M. Jungers, *JSR: A toolbox to compute the joint spectral radius*, Proceedings of the 17th International Conference on Hybrid Systems: Computation and Control (New York, NY, USA), HSCC'14, ACM, 2014, pp. 151–156, `doi:10.1145/2562059.2562124`. Zbl 1364.65099.

We present a toolbox for computing the Joint Spectral Radius of a set of matrices, i.e., the maximal asymptotic growth rate of products of matrices taken in that set. The Joint Spectral Radius has a wide range of applications, including switched and hybrid systems, combinatorial words theory, or the study of wavelets. However, it is notoriously difficult to compute or approximate; it is actually uncomputable, and its approximation is NP-hard. The toolbox compiles several recent computation and approximation methods, and also contains an automatic blackbox method for inexperienced users, selecting the most appropriate methods based on an automatic study of the matrix set provided. The tool is implemented in Matlab and is freely downloadable (with documentation and demos) from Matlab Central [271].

[344] N. Vlassis and R. Jungers, *Polytopic uncertainty for linear systems: new and old complexity results*, Systems Control Lett. **67** (2014), 9–13, `doi:10.1016/j.sysconle.2014.02.001`, `arXiv:1310.1930`. MR 3183374. Zbl 1288.93056.

We survey the problem of deciding the stability or stabilizability of uncertain linear systems whose region of uncertainty is a polytope. This natural setting has applications in many fields of applied science, from Control Theory to Systems Engineering to Biology.

We focus on the algorithmic decidability of this property when one is given a particular polytope. This setting gives rise to several different algorithmic questions, depending on the nature of time (discrete/continuous), the property asked (stability/stabilizability), or the type of uncertainty (fixed/switching). Several of these questions have been answered in the literature in the last thirty years. We point out the ones that have remained open, and we answer all of them, except one which we raise as an open question. In all the cases, the results are negative in the sense that the questions are NP-hard.

As a byproduct, we obtain complexity results for several other matrix problems in Systems and Control.

[345] Y. Wang, *Stability of linear autonomous systems under regular switching sequences*, Master's thesis, Mechanical Sci & Engineering, University of Illinois at Urbana-Champaign, 2014, URL `https://www.ideals.illinois.edu/handle/2142/50614`.

In this work, we discuss the stability of a discrete-time linear autonomous system under regular switching sequences, whose switching sequences are generated by a Muller automaton. This system arises in various engineering problems such as distributed communication and automotive engine control. The asymptotic stability of this system, referred to as regular asymptotic stability, generalizes two well-known definitions of stability of autonomous discrete-time linear switched systems, namely absolute asymptotic stability (AAS) and shuffle asymptotic stability (SAS). We also extend these stability definitions to robust versions. We prove that absolute asymptotic stability, robust absolute asymptotic stability and robust shuffle asymptotic stability are equivalent to exponential stability. In addition, by using the Kronecker product, we prove that a robust regular asymptotic stability problem is equivalent to the conjunction of several robust absolute asymptotic stability problems.

[346] Y. Wang, N. Roohi, G. E. Dullerud, and M. Viswanathan, *Stability of linear autonomous systems under regular switching sequences*, Proceedings of the 53d IEEE Conference on Decision and Control (CDC), 15-17 Dec. (Los Angeles, California, USA), IEEE, 2014, pp. 5445–5450, doi:10.1109/CDC.2014.7040240.

In this work, we discuss the stability of a discrete-time linear autonomous system under regular switching sequences, whose switching sequences are generated by a Muller automaton. This system arises in various engineering problems such as distributed communication and automotive engine control. The asymptotic stability of this system, referred to as regular asymptotic stability, generalizes two well-known definitions of stability of autonomous discrete-time linear switched systems, namely absolute asymptotic stability (AAS) and shuffle asymptotic stability (SAS). We also extend these stability definitions to robust versions. We prove that absolute asymptotic stability, robust absolute asymptotic stability and robust shuffle asymptotic stability are equivalent to exponential stability. In addition, by using the Kronecker product, we prove that a robust regular asymptotic stability problem is equivalent to the conjunction of several robust absolute asymptotic stability problems.

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[347] V. D. Blondel, R. M. Jungers, and A. Olshevsky, *On primitivity of sets of matrices*, Automatica J. IFAC **61** (2015), 80–88, doi:10.1016/j.automatica.2015.07.026, arXiv:1306.0729. MR 3401692. Zbl 06523630.

A nonnegative matrix A is called primitive if A^k is positive for some integer $k > 0$. A generalization of this concept to sets of matrices is the following: a set of matrices $\mathcal{M} = \{A_1, A_2, \ldots, A_m\}$ is primitive if $A_{i_1} A_{i_2} \cdots A_{i_k}$ is positive for some indices $i_1, i_2, \ldots, i_k$. The concept of primitive sets of matrices is of importance in several applications, including the problem of computing the Lyapunov exponents of switching systems. In this paper, we analyze the computational complexity of deciding if a given set of matrices is primitive and we derive bounds on the length of the shortest positive product.

We show that while primitivity is algorithmically decidable, unless P=NP it is not possible to decide positivity of a matrix set in polynomial time. Moreover, we show that the length of the shortest positive sequence can be exponential in the dimension of the matrices. On the other hand, we give a simple combinatorial proof of the fact that when the matrices have no zero rows or zero columns, primitivity can be decided in polynomial time. This latter observation is related to the well-known 1964 conjecture of Černý on synchronizing automata. Moreover, we show that for such matrices the length of the shortest positive sequence is at most polynomial in the dimension.

[348] J. Bochi, *Another proof of the spectral radius formula*, 2015, URL http://www.mat.uc.cl/~jairo.bochi/docs/SRF.pdf, manuscript.

This note gives an elementary proof of the spectral radius formula in finite dimension, inspired by ideas from ergodic theory.

[349] J. Bochi and I. D. Morris, *Continuity properties of the lower spectral radius*, Proc. Lond. Math. Soc. (3) **110** (2015), no. 2, 477–509, doi:10.1112/plms/pdu058, arXiv:1309.0319. MR 3335285. Zbl 1311.15022.

The lower spectral radius, or joint spectral subradius, of a set of real $d \times d$ matrices is defined to be the smallest possible exponential growth rate of long products of matrices drawn from that set. The lower spectral radius arises naturally in connection with a number of topics including combinatorics on words, the stability of linear inclusions in control theory, and the study of random

Cantor sets. In this article we apply some ideas originating in the study of dominated splittings of linear cocycles over a dynamical system to characterize the points of continuity of the lower spectral radius on the set of all compact sets of invertible $d \times d$ matrices. As an application we exhibit open sets of pairs of 2×2 matrices within which the analogue of the Lagarias–Wang finiteness property for the lower spectral radius fails on a residual set, and discuss some implications of this result for the computation of the lower spectral radius.

[350] P.-Y. Chevalier, J. M. Hendrickx, and R. M. Jungers, *Efficient algorithms for the consensus decision problem*, SIAM J. Control Optim. **53** (2015), no. 5, 3104–3119, `doi:10.1137/140988024`, `arXiv:1409.6505`. MR 3403131. Zbl 1320.93073.

We address the problem of determining if a discrete time switched consensus system converges for any switching sequence and that of determining if it converges for at least one switching sequence. For these two problems, we provide necessary and sufficient conditions that can be checked in singly exponential time. As a side result, we prove the existence of a polynomial time algorithm for the first problem when the system switches between only two subsystems whose corresponding graphs are undirected, a problem that had been suggested to be NP-hard by Blondel and Olshevsky.

[351] Y. Chitour, M. Gaye, and P. Mason, *Geometric and asymptotic properties associated with linear switched systems*, J. Differential Equations **259** (2015), no. 11, 5582–5616, `doi:10.1016/j.jde.2015.07.001`, `arXiv:1409.4524`. MR 3397301. Zbl 1334.34035.

Consider continuous-time linear switched systems on $\mathbb{R}^n$ associated with compact convex sets of matrices. When the system is irreducible and the largest Lyapunov exponent is equal to zero, there always exists a Barabanov norm (i.e. a norm which is non increasing along trajectories of the linear switched system together with extremal trajectories starting at every point, that is trajectories of the linear switched system with constant norm). This paper deals with two sets of issues: (a) properties of Barabanov norms such as uniqueness up to homogeneity and strict convexity; (b) asymptotic behaviour of the extremal solutions of the linear switched system. Regarding Issue (a), we provide partial answers and propose four open problems motivated by appropriate examples. As for Issue (b), we establish, when $n = 3$, a Poincaré-Bendixson theorem under a regularity assumption on the set of matrices defining the system. Moreover, we revisit the noteworthy result of *N.E. Barabanov* [Dokl. Akad. Nauk 334(2), 154–155 (1994)] dealing with the linear switched system on $\mathbb{R}^3$ associated with a pair of Hurwitz matrices $\{A, A+bc^T\}$. We first point out a fatal gap in Barabanov's argument in connection with geometric features associated with a Barabanov norm. We then provide partial answers relative to the asymptotic behavior of this linear switched system.

[352] A. Cicone, *A note on the joint spectral radius*, ArXiv.org e-Print archive, February 2015, `arXiv:1502.01506`.

Last two decades have been characterized by an increasing interest in the analysis of the maximal growth rate of long products generated by matrices belonging to a specific set/family. The maximal growth rate can be evaluated considering a generalization of the spectral radius of a single matrix to the case of a set of matrices.

This generalization can be formulated in many different ways, nevertheless in the commonly studied cases of bounded or finite families all the possible generalizations coincide in a unique value that is usually called *joint spectral radius* or simply *spectral radius*. The joint spectral radius, however, can prove to be hard to compute and can lead even to undecidable problems. We present in this paper all the possible generalizations of the spectral radius, their properties and the associated theoretical challenges.

From an historical point of view the first two generalizations of spectral radius, the so–called joint and common spectral radius, were introduced by Rota and Strang in the three pages paper "*A note on the joint spectral radius*" published in 1960 [3]. After that more than thirty years had to pass before a second paper was issued on this topic: in 1992 *I. Daubechies* and *J. C. Lagarias* [43] published "*Sets of matrices all infinite products of which converge*" introducing the generalized spectral radius, conjecturing it was equal to the joint spectral radius (this was proven immediately after by *M. A. Berger* and *Y. Wang* [42]) and presenting examples of applications. From then on there has been a rapidly increasing interest on this subject and the more years pass the more the

number of mathematical branches and applications directly involved in the study of these quantities increases [191].

The study of infinite products convergence properties proves to be of primary interest in a variety of contexts: nonhomogeneous Markov chains, deterministic construction of functions and curves with self-similarities under changes in scale like the von Koch snowflake and the de Rham curves, two-scale refinement equations that arise in the construction of wavelets of compact support and in the dyadic interpolation schemes of Deslauriers and Dubuc, the asymptotic behavior of the solutions of linear difference equations with variable coefficients, coordination of autonomous agents, hybrid systems with applications that range from intelligent traffic systems to industrial process control, the stability analysis of dynamical systems of autonomous differential equations, computer–aided geometric design in constructing parameterized curves and surfaces by subdivision or refinement algorithms, the stability of asynchronous processes in control theory, the stability of desynchronised systems, the analysis of magnetic recording systems and in particular the study of the capacity of codes submitted to forbidden differences constraints, probabilistic automata, the distribution of random power series and the asymptotic behavior of the Euler partition function, the logarithm of the joint spectral radius appears also in the context of discrete linear inclusions as the Lyapunov indicator. For a more extensive and detailed list of applications we refer the reader to the Gilbert Strang's paper "*The Joint Spectral Radius*" [146] and to the doctoral theses by *R. Jungers* [216] and *J. Theys* [171].

The paper develops as following: in Section 1 we give notation and terminology used throughout this paper; Section 2 presents first a case of study associated with the asymptotic behavior analysis of the solutions of linear difference equations with variable coefficients, further, it contains the definitions and properties of all the possible generalizations of spectral radius for a set of matrices, in particular the irreducibility, nondefectivity and finiteness properties are discussed.

[353] A. Czornik and M. Niezabitowski, *Alternative formulae for lower general exponent of discrete linear time-varying systems*, J. Franklin Inst. **352** (2015), no. 1, 399–419, `doi:10.1016/j.jfranklin.2014.11.003`. MR 3292336. Zbl 1307.93237.

The Bohl exponents, similarly as Lyapunov exponents, are one of the most important numerical characteristics of dynamical systems used in control theory. Properties of the Lyapunov characteristics are well described in the literature. Properties of the Bohl exponents are much less investigated. In this paper we consider the so-called junior lower general exponent of discrete linear time-varying system and present some alternative formulae for it. We also discuss relations between lower Bohl exponents of the perturbed system and junior lower general exponent of the unperturbed system.

[354] X. Dai, Y. Huang, and M. Xiao, *Pointwise stability of discrete-time homogeneous matrix-valued Markovian processes*, IEEE Trans. Automat. Control **60** (2015), no. 7, 1898–1903, `doi:10.1109/TAC.2014.2361594`. MR 3365076. Zbl 1360.93736.

In this note we study the pointwise stability of a discrete-time, matrix-valued, and stationary Markovian jump linear system. When the system is restricted to a linear subspace, we show that it is pointwise convergent if and only if it is pointwise exponentially convergent.

[355] N. Guglielmi and M. Zennaro, *Canonical construction of polytope Barabanov norms and antinorms for sets of matrices*, SIAM J. Matrix Anal. Appl. **36** (2015), no. 2, 634–655, `doi:10.1137/140962814`. MR 3352607. Zbl 06444038.

Barabanov norms have been introduced by *N. E. Barabanov* [26] and constitute an important instrument in analyzing the joint spectral radius of a family of matrices and related issues. However, although they have been studied extensively, even in very simple cases it is very difficult to construct them explicitly (see, e.g., *V. S. Kozyakin* [231]). In this paper we give a canonical procedure to construct them exactly, which associates a polytope extremal norm — constructed by using the methodologies described by *N. Guglielmi*, *F. Wirth*, and *M. Zennaro* [161] and *N. Guglielmi* and *V. Protasov* [308] — to a polytope Barabanov norm. Hence, the existence of a polytope Barabanov norm has the same genericity of an extremal polytope norm. Moreover, we extend the result to polytope antinorms, which have been recently introduced to compute the lower spectral radius of a finite family of matrices having an invariant cone.

[356] C. Möller, *A new strategy for exact determination of the joint spectral radius*, Ph.D. thesis, Fachbereich Mathematik, Technische Universität Darmstadt, Germany, 2015, URL `https://tuprints.ulb.tu-darmstadt.de/4603/`.

Computing the joint spectral radius of a finite matrix family is, though interesting for many applications, a difficult problem. This work proposes a method for determining the exact value which is based on graph-theoretical ideas. In contrast to some other algorithms in the literature, the purpose of the approach is not to find an extremal norm for the matrix family. To validate that the finiteness property (FP) is satisfied for a certain matrix product, a tree is to be analyzed whose nodes code sets of matrix products. A sufficient, and in certain situations also necessary, criterion is given by existence of a finite tree with special properties, and an algorithm for searching such a tree is proposed. The suggested method applies in case of several FP-products as well and is not limited to asymptotically simple matrix families. In the smoothness analysis of subdivision schemes, joint spectral radius determination is crucial to detect Hölder regularity. The palindromic symmetry of matrices, which results from symmetric binary subdivision, is considered in the context of set-valued trees. Several illustrating examples explore the capabilities of the approach, consolidated by examples from subdivision.

[357] M. Philippe and R. M. Jungers, *Converse Lyapunov theorems for discrete-time linear switching systems with regular switching sequences*, Proceedings of the European Control Conference (ECC), 15-17 July 2015 (Linz, Austria), IEEE, 2015, `doi:10.1109/ECC.2015.7330816`, `arXiv:1410.7197`.

We present a stability analysis framework for the general class of discrete-time linear switching systems for which the switching sequences belong to a regular language. They admit arbitrary switching systems as special cases.

Using recent results of *X. Dai* [281] on the asymptotic growth rate of such systems, we introduce the concept of *multinorm* as an algebraic tool for stability analysis.

We conjugate this tool with two families of multiple quadratic Lyapunov functions, parameterized by an integer $T \geq 1$, and obtain converse Lyapunov Theorems for each.

Lyapunov functions of the first family associate one quadratic form per state of the automaton defining the switching sequences. They are made to decrease after every T successive time steps. The second family is made of the *path-dependent* Lyapunov functions of Lee and Dullerud. They are parameterized by an amount of memory $(T-1) \geq 0$.

Our converse Lyapunov theorems are finite. More precisely, we give sufficient conditions on the asymptotic growth rate of a stable system under which one can compute an integer parameter $T \geq 1$ for which both types of Lyapunov functions exist. As a corollary of our results, we formulate an arbitrary accurate approximation scheme for estimating the asymptotic growth rate of switching systems with constrained switching sequences.

[358] M. Philippe and R. M. Jungers, *A sufficient condition for the boundedness of matrix products accepted by an automaton*, Proceedings of the 18th International Conference on Hybrid Systems: Computation and Control (New York, NY, USA), HSCC'15, ACM, 2015, pp. 51–57, `doi:10.1145/2728606.2728610`. MR 3440811. Zbl 1366.68151.

We study the boundedness of products of matrices associated with words in a regular language. This question naturally arises in the stability analysis of switching systems with constrained switching sequences.

Our main contribution is a sufficient condition for the boundedness of all the possible products of matrices that may occur in a marginally unstable system. We show that this condition can be expressed in terms of products of finite lengths, and is therefore algorithmically checkable.

We then compare our condition with a second one, inspired by a lifting procedure introduced by Kozyakin, and prove that our condition is at least as powerful as this second one.

[359] V. Yu. Protasov and R. M. Jungers, *Resonance and marginal instability of switching systems*, Nonlinear Anal. Hybrid Syst. **17** (2015), 81–93, `doi:10.1016/j.nahs.2015.02.003`, `arXiv:1411.0497`. MR 3351031. Zbl 06514744.

We analyse the so-called Marginal Instability of linear switching systems, both in continuous and discrete time. This is a phenomenon of unboundedness of trajectories when the Lyapunov exponent is zero. We disprove two recent conjectures of *Chitour, Mason,* and *M. Sigalotti* [278] stating that for generic systems, the resonance is sufficient for marginal instability and for polynomial growth of the trajectories. The concept of resonance originated with the same authors is modified. A characterization of marginal instability under some mild assumptions on the system is provided. These assumptions can be verified algorithmically and are believed to be generic. Finally, we analyze possible types of fastest asymptotic growth of trajectories. An example of a marginally unstable pair of matrices with non-polynomial growth is given.

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[360] A. A. Ahmadi and R. M. Jungers, *Lower bounds on complexity of Lyapunov functions for switched linear systems*, Nonlinear Anal. Hybrid Syst. **21** (2016), 118–129, `doi:10.1016/j.nahs.2016.01.003`, `arXiv:1504.03761`. MR 3500076. Zbl 1382.93028.

We show that for any positive integer d, there are families of switched linear systems — in fixed dimension and defined by two matrices only — that are stable under arbitrary switching but do not admit (i) a polynomial Lyapunov function of degree $\leq d$, or (ii) a polytopic Lyapunov function with $\leq d$ facets, or (iii) a piecewise quadratic Lyapunov function with $\leq d$ pieces. This implies that there cannot be an upper bound on the size of the linear and semidefinite programs that search for such stability certificates. Several constructive and non-constructive arguments are presented which connect our problem to known (and rather classical) results in the literature regarding the finiteness conjecture, undecidability, and non-algebraicity of the joint spectral radius. In particular, we show that existence of an extremal piecewise algebraic Lyapunov function implies the finiteness property of the optimal product, generalizing a result of Lagarias and Wang. As a corollary, we prove that the finiteness property holds for sets of matrices with an extremal Lyapunov function belonging to some of the most popular function classes in controls.

[361] E. Asarin, J. Cervelle, A. Degorre, C. Dima, F. Horn, and V. Kozyakin, *Entropy games and matrix multiplication games*, 33rd Symposium on Theoretical Aspects of Computer Science, (STACS 2016) (Dagstuhl, Germany) (N. Ollinger and H. Vollmer, eds.), LIPIcs. Leibniz Int. Proc. Inform., vol. 47, Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2016, pp. 11:1–11:14 (english), `doi:10.4230/LIPIcs.STACS.2016.11`. MR 3539108. Zbl 1390.91016. game theory, entropy, joint spectral, radius

Two intimately related new classes of games are introduced and studied: entropy games (EGs) and matrix multiplication games (MMGs). An EG is played on a finite arena by two-and-a-half players: Despot, Tribune and the non-deterministic People. Despot wants to make the set of possible People's behaviors as small as possible, while Tribune wants to make it as large as possible. An MMG is played by two players that alternately write matrices from some predefined finite sets. One wants to maximize the growth rate of the product, and the other to minimize it. We show that in general MMGs are undecidable in quite a strong sense. On the positive side, EGs correspond to a subclass of MMGs, and we prove that such MMGs and EGs are determined, and that the optimal strategies are simple. The complexity of solving such games is in NP cap coNP.

[362] A. Babiarz, A. Czornik, and J. Klamka, *A generalization of Yamamoto's theorem relating eigenvalue moduli and singular values of a matrix*, AIP Conference Proceedings **1738**, 130006 (Rhodes, Greece), AIP Publishing, 2016, pp. 130006–1–130006–4, `doi:10.1063/1.4951922`.

In this paper we present a generalization of Yamamoto's theorem relating eigenvalue moduli and singular values of a matrix. In our generalization a single matrix is replaced by a bounded set of matrices. The main result of the paper includes also, as a special case, the equality between generalized spectral radius and joint spectral radius.

[363] J. P. Bell, M. Coons, and K. G. Hare, *Growth degree classification for finitely generated semigroups of integer matrices*, Semigroup Forum **92** (2016), no. 1, 23–44, `doi:10.1007/s00233-015-9725-1`, `arXiv:1410.5519`. MR 3448399. Zbl 1356.20034.

Let $\mathcal{A}$ be a finite set of $d \times d$ matrices with integer entries and let $m_n(\mathcal{A})$ be the maximum norm of a product of n elements of $\mathcal{A}$. In this paper, we classify gaps in the growth of $m_n(\mathcal{A})$; specifically, we prove that $\lim_{n\to\infty} \log m_n(\mathcal{A})/\log n \in \mathcal{B}Z_{\geqslant 0} \cup \{\infty\}$. This has applications to the growth of regular sequences as defined by Allouche and Shallit.

[364] J. Bochi and M. Rams, *The entropy of Lyapunov-optimizing measures of some matrix cocycles*, J. Mod. Dyn. **10** (2016), 255–286, `doi:10.3934/jmd.2016.10.255`, `arXiv:1312.6718`. MR 3538864. Zbl 1346.15030.

We consider one-step cocycles of 2×2 matrices, and we are interested in their Lyapunov-optimizing measures, i.e., invariant probability measures that maximize or minimize a Lyapunov exponent. If the cocycle is dominated, that is, the two Lyapunov exponents are uniformly separated along all orbits, then Lyapunov-optimizing measures always exist, and are characterized by their support. Under an additional hypothesis of nonoverlapping between the cones that characterize domination, we prove that the Lyapunov-optimizing measures have zero entropy. This conclusion certainly fails without the domination assumption, even for typical one-step $\mathrm{SL}(2,\mathbb{R})$-cocycles; indeed we show that in the latter case there are measures of positive entropy with zero Lyapunov exponent.

[365] S. Cong, *Stability analysis for planar discrete-time linear switching systems via bounding joint spectral radius*, Systems Control Lett. **96** (2016), 7–10, `doi:10.1016/j.sysconle.2016.06.012`. MR 3547649. Zbl 1347.93230.

A pair of 2-by-2 matrices can be transformed into the joint forms, which are bound together via an invariant that measures the deformation of their structures with respect to each other. With this, we obtain an approximation of the joint spectral radius of such a pair of matrices from above and then apply it to assessing stability for planar discrete-time linear switching systems.

[366] A. Czornik, P. Jurgaś, and M. Niezabitowski, *Estimation of the joint spectral radius*, Man-Machine Interactions 4 (A. Gruca, A. Brachman, S. Kozielski, and T. Czachórski, eds.), Advances in Intelligent Systems and Computing, vol. 391, Springer International Publishing, 2016, pp. 401–410 (English), `doi:10.1007/978-3-319-23437-3_34`.

The joint spectral radius of a set of matrices is a generalization of the concept of spectral radius of a matrix. Such notation has many applications in the computer science, and more generally in applied mathematics. It has been used, for example in graph theory, control theory, capacity of codes, continuity of wavelets, overlap-free words, trackable graphs. It is impossible to provide analytic formulae for this quantity and therefore any estimation are highly desired. The main result of this paper is to provide an estimation of the joint spectral radius in the terms of matrices norms and spectral radii.

[367] N. Guglielmi and V. Protasov, *Computing Lyapunov exponents of switching systems*, AIP Conference Proceedings **1738**, 020006 (Rhodes, Greece), AIP Publishing, 2016, pp. 020006–1–020006–4, `doi:10.1063/1.4951750`.

We discuss a new approach for constructing polytope Lyapunov functions for continuous-time linear switching systems. The method we propose allows to decide the uniform stability of a switching system and to compute the Lyapunov exponent with an arbitrary precision. The method relies on the discretization of the system and provides — for any given discretization stepsize — a lower and an upper bound for the Lyapunov exponent. The efficiency of the new method is illustrated by numerical examples. For a more extensive discussion we remand the reader to [161].

[368] N. Guglielmi and V. Yu. Protasov, *Invariant polytopes of sets of matrices with application to regularity of wavelets and subdivisions*, SIAM J. Matrix Anal. Appl. **37** (2016), no. 1, 18–52, `doi:10.1137/15M1006945`, `arXiv:1502.01192`. MR 3447127. Zbl 06548831.

We generalize the recent invariant polytope algorithm for computing the joint spectral radius and extend it to a wider class of matrix sets. This, in particular, makes the algorithm applicable to

sets of matrices that have finitely many spectrum maximizing products. A criterion of convergence of the algorithm is proved.

As an application we solve two challenging computational open problems. First we find the regularity of the Butterfly subdivision scheme for various parameters ω. In the "most regular" case $\omega = \frac{1}{16}$, we prove that the limit function has Hölder exponent 2 and its derivative is "almost Lipschitz" with logarithmic factor 2. Second we compute the Hölder exponent of Daubechies wavelets of high order.

[369] V. Kozyakin, *Hourglass alternative and constructivity of spectral characteristics of matrix products*, Workshop on switching dynamics & verification, January 28-29, Institut pour le Contrôle et la Décision de l'Idex Paris-Saclay, Paris, France, 2016 (english), `doi:10.13140/RG.2.1.4880.1042`.

Recently Blondel, Nesterov and Protasov proved that the finiteness conjecture holds for the generalized and the lower spectral radii of the sets of non-negative matrices with independent row/column uncertainty. We show that this result can be obtained as a simple consequence of the so-called hourglass alternative which can be used also to establish the minimax relations between the spectral radii of matrix products. Axiomatization of the statements that constitute the hourglass alternative makes it possible to define a new class of sets of positive matrices having the finiteness property, which includes the sets of non-negative matrices with independent row uncertainty. This class of matrices, supplemented by the zero and identity matrices, forms a semiring with the Minkowski operations of addition and multiplication of matrix sets, which gives means to construct new sets of non-negative matrices possessing the finiteness property for the generalized and the lower spectral radii.

The Hourglass alternative helps us to describe a new class of positive linear discrete-time switching systems for which the problems of stability or stabilizability can be resolved constructively. This class generalizes the class of systems with independently switching state vector components. The distinctive feature of this class is that, for any system from this class, its components or blocks can be arbitrarily connected in parallel or in series without loss of the 'constructive resolvability' property. It is shown also that, for such systems, it is possible to build constructively the individual positive trajectories with the greatest or the lowest rate of convergence to the zero.

[370] V. Kozyakin, *Hourglass alternative and the finiteness conjecture for the spectral characteristics of sets of non-negative matrices*, Linear Algebra Appl. **489** (2016), 167–185, `doi:10.1016/j.laa.2015.10.017`, `arXiv:1507.00492`. MR 3421844. Zbl 1326.15053.

Recently Blondel, Nesterov and Protasov proved [212, 318] that the finiteness conjecture holds for the generalized and the lower spectral radii of the sets of non-negative matrices with independent row/column uncertainty. We show that this result can be obtained as a simple consequence of the so-called hourglass alternative earlier used by the author and his companions to analyze the minimax relations between the spectral radii of matrix products. Axiomatization of the statements that constitute the hourglass alternative makes it possible to define a new class of sets of positive matrices having the finiteness property, which includes the sets of non-negative matrices with independent row uncertainty. This class of matrices, supplemented by the zero and identity matrices, forms a semiring with the Minkowski operations of addition and multiplication of matrix sets, which gives means to construct new sets of non-negative matrices possessing the finiteness property for the generalized and the lower spectral radii.

[371] B. Legat, R. M. Jungers, and P. A. Parrilo, *Generating unstable trajectories for switched systems via dual sum-of-squares techniques*, Proceedings of the 19th International Conference on Hybrid Systems: Computation and Control (New York, NY, USA), HSCC '16, ACM, 2016, pp. 51–60, `doi:10.1145/2883817.2883821`. Zbl 1364.93315.

The joint spectral radius (JSR) of a set of matrices characterizes the maximal asymptotic growth rate of an infinite product of matrices of the set. This quantity appears in a number of applications including the stability of switched and hybrid systems. Many algorithms exist for estimating the JSR but not much is known about how to generate an infinite sequence of matrices with an optimal asymptotic growth rate. To the best of our knowledge, the currently known algorithms select a small sequence with large spectral radius using brute force (or branch-and-bound variants) and repeats this sequence infinitely.

In this paper we introduce a new approach to this question, using the dual solution of a sum of squares optimization program for JSR approximation. Our algorithm produces an infinite sequence of matrices with an asymptotic growth rate arbitrarily close to the JSR. The algorithm naturally extends to the case where the allowable switching sequences are determined by a graph or finite automaton. Unlike the brute force approach, we provide a guarantee on the closeness of the asymptotic growth rate to the JSR. This, in turn, provides new bounds on the quality of the JSR approximation. We provide numerical examples illustrating the good performance of the algorithm.

[372] I. D. Morris, *An inequality for the matrix pressure function and applications*, Adv. Math. **302** (2016), 280–308, `doi:10.1016/j.aim.2016.07.025`, `arXiv:1507.00642`. MR 3545931. Zbl 1350.15005.

We prove an a priori lower bound for the pressure, or p-norm joint spectral radius, of a measure on the set of $d \times d$ real matrices which parallels a result of *J. Bochi* [141] for the joint spectral radius. We apply this lower bound to give new proofs of the continuity of the affinity dimension of a self-affine set and of the continuity of the singular-value pressure for invertible matrices, both of which had been previously established by D.-J. Feng and P. Shmerkin using multiplicative ergodic theory and the subadditive variational principle. Unlike the previous proof, our lower bound yields algorithms to rigorously compute the pressure, singular value pressure and affinity dimension of a finite set of matrices to within an a priori prescribed accuracy in finitely many computational steps. We additionally deduce a related inequality for the singular value pressure for measures on the set of 2×2 real matrices, give a precise characterisation of the discontinuities of the singular value pressure function for two-dimensional matrices, and prove a general theorem relating the zero-temperature limit of the matrix pressure to the joint spectral radius.

[373] M. Ogura, V. M. Preciado, and R. M. Jungers, *Efficient method for computing lower bounds on the p-radius of switched linear systems*, Systems Control Lett. **94** (2016), 159–164, `doi:10.1016/j.sysconle.2016.06.008`, `arXiv:1503.03034`. MR 3530610. Zbl 1344.93050.

This paper proposes lower bounds on a quantity called L^p-norm joint spectral radius, or in short, p-radius, of a finite set of matrices. Despite its wide range of applications to, for example, stability analysis of switched linear systems and the equilibrium analysis of switched linear economical models, algorithms for computing the p-radius are only available in a very limited number of particular cases. The proposed lower bounds are given as the spectral radius of an average of the given matrices weighted via Kronecker products and do not place any requirements on the set of matrices. We show that the proposed lower bounds theoretically extend and also can practically improve the existing lower bounds. A Markovian extension of the proposed lower bounds is also presented.

[374] M. Philippe, R. Essick, G. E. Dullerud, and R. M. Jungers, *Stability of discrete-time switching systems with constrained switching sequences*, Automatica **72** (2016), 242–250, `doi:10.1016/j.automatica.2016.05.015`, `arXiv:1503.06984`. MR 3542938. Zbl 1344.93088.

We introduce a novel framework for the stability analysis of discrete-time linear switching systems with switching sequences constrained by an automaton. The key element of the framework is the algebraic concept of multinorm, which associates a different norm per node of the automaton, and allows to exactly characterize stability. Building upon this tool, we develop the first arbitrarily accurate approximation schemes for estimating the constrained joint spectral radius $\hat{\rho}$, that is the exponential growth rate of a switching system with constrained switching sequences. More precisely, given a relative accuracy $r > 0$, the algorithms compute an estimate of $\hat{\rho}$ within the range $[\hat{\rho}, (1 + r)\hat{\rho}]$. These algorithms amount to solve a well defined convex optimization program with known time-complexity, and whose size depends on the desired relative accuracy $r > 0$.

[375] V. Yu. Protasov, *Spectral simplex method*, Math. Program. **156** (2016), no. 1-2, Ser. A, 485–511, `doi:10.1007/s10107-015-0905-2`. MR 3459208. Zbl 06562698.

We develop an iterative optimization method for finding the maximal and minimal spectral radius of a matrix over a compact set of nonnegative matrices. We consider matrix sets with product structure, i.e., all rows are chosen independently from given compact sets (row uncertainty sets). If all the uncertainty sets are finite or polyhedral, the algorithm finds the matrix with maximal/minimal spectral radius within a few iterations. It is proved that the algorithm avoids cycling and terminates within finite time. The proofs are based on spectral properties of rank-one corrections of nonnegative matrices. The practical efficiency is demonstrated in numerical examples and statistics in dimensions up to 500. Some generalizations to non-polyhedral uncertainty sets, including Euclidean balls, are derived. Finally, we consider applications to spectral graph theory, mathematical economics, dynamical systems, and difference equations.

[376] A. Vladimirov, *Continuous products of matrices*, ArXiv.org e-Print archive, March 2016, `arXiv:1603.00854`.

We answer the question if the continuous product of square matrices $M(t)$ over $t \in [0, 1]$ can be correctly defined. The case where all $M(t)$ are taken from a finite set Σ is studied. We find necessary and sufficient conditions on Σ that ensure the convergence of products $M(t_0 = 0)M(t_1) \cdots M(t_N = 1)$ as the partition $0 < t_1 < \cdots < 1$ refines. These conditions are properties LCP (left convergent product) and RCP (right convergent product) of the set Σ. That is, it suffices to require the convergence of all finite products $M_1M_2 \cdots M_K$ and $M_K \cdots M_2M_1$ as $K \to \infty$, where $M_i \in \Sigma$. The theory of joint spectral radius is heavily used.

[377] X. Wang and Z. Cheng, *Infinite products of uniformly paracontracting matrices*, Linear Multilinear Algebra **64** (2016), no. 5, 856–862, `doi:10.1080/03081087.2015.1063577`. MR 3479385. Zbl 1346.65018.

We define uniform paracontraction for an arbitrary set of $n \times n$ matrices and show that an infinite product of matrices drawn from a uniformly paracontracting set is convergent. Moreover, if the uniformly paracontracting set is finite and the matrices are drawn in a regulated way, the infinite product is exponentially convergent.

[378] Y. Wang, N. Roohi, G. E. Dullerud, and M. Viswanathan, *Stability analysis of switched linear systems defined by regular languages*, IEEE Trans. Autom. Control **PP** (2016), no. 99, 1–1, `doi:10.1109/TAC.2016.2599930`. MR 3641474. Zbl 1366.93558.

In this work, we study the stability of an autonomous discrete-time linear switched system whose switching sequences are generated by a Muller automaton. This system arises in various engineering problems such as distributed communication and automotive engine control. The asymptotic stability of this system, referred to as regular asymptotic stability (RAS), generalizes two well-known definitions of stability of autonomous discrete-time linear switched systems, namely absolute asymptotic stability (AAS) and shuffle asymptotic stability (SAS). We also extend these stability definitions to robust versions. We show that absolute asymptotic stability, robust absolute asymptotic stability and robust shuffle asymptotic stability are equivalent to exponential stability. In addition, by using the Kronecker product, we prove that a robust regular asymptotic stability problem is equivalent to the conjunction of several robust absolute asymptotic stability problems.

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[379] N. Athanasopoulos, K. Smpoukis, and R. M. Jungers, *Invariant sets analysis for constrained switching systems*, IEEE Control Systems Letters **1** (2017), no. 2, 256–261, `doi:10.1109/LCSYS.2017.2714840`. MR 4208542.

We study discrete time linear constrained switching systems with additive disturbances, in the general setting where the switching acts on the system matrices, the disturbance sets, and the state constraint sets. Our primary goal is to extend the existing invariant set constructions when the switching signal is constrained by a given automation. We achieve it by working with a relaxation of invariance, namely the multi-set invariance. By exploiting recent results on computing the stability metrics for these systems, we establish explicit bounds on the number of iterations required for each construction. Last, as an application, we develop new maximal invariant set constructions for the case of linear systems in far fewer iterations compared to the state-of-the-art.

[380] B. Balle, P. Gourdeau, and P. Panangaden, *Bisimulation metrics for weighted automata*, 44th International Colloquium on Automata, Languages, and Programming (ICALP 2017) (I. Chatzigiannakis, P. Indyk, F. Kuhn, and A. Muscholl, eds.), Leibniz International Proceedings in Informatics (LIPIcs), vol. 80, Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany, 2017, pp. 103:1–103:14, `doi:10.4230/LIPIcs.ICALP.2017.103`, `arXiv:1702.08017`. MR 3685843. Zbl 1442.68080.

We develop a new bisimulation (pseudo)metric for weighted finite automata (WFA) that generalizes Boreale's linear bisimulation relation. Our metrics are induced by seminorms on the state space of WFA. Our development is based on spectral properties of sets of linear operators. In particular, the joint spectral radius of the transition matrices of WFA plays a central role. We also study continuity properties of the bisimulation pseudometric, establish an undecidability result for computing the metric, and give a preliminary account of applications to spectral learning of weighted automata. See also [423].

[381] P.-Y. Chevalier, J. M. Hendrickx, and R. M. Jungers, *Tight bound for deciding convergence of consensus systems*, Systems Control Lett. **105** (2017), 78–83, `doi:10.1016/j.sysconle.2017.05.001`, `arXiv:1601.04975`. MR 3668032. Zbl 1372.93009.

We analyze the asymptotic convergence of all infinite products of matrices taken in a given finite set by looking only at *finite* or *periodic* products. It is known that when the matrices of the set have a common nonincreasing polyhedral norm, all infinite products converge to zero if and only if all infinite *periodic* products with periods smaller than a certain value converge to zero. Moreover, bounds on that value are available [53].

We provide a stronger bound that holds for both polyhedral norms and polyhedral seminorms. In the latter case, the matrix products do not necessarily converge to 0, but all trajectories of the associated system converge to a common invariant subspace. We prove that our bound is tight for all seminorms.

Our work is motivated by problems in *consensus systems*, where the matrices are *stochastic* (nonnegative with rows summing to one), and hence always share a same common nonincreasing polyhedral seminorm. In that case, we also improve existing results.

[382] X. Dai, T. Huang, Y. Huang, Y. Luo, G. Wang, and M. Xiao, *Chaotic behavior of discrete-time linear inclusion dynamical systems*, J. Franklin Inst. **354** (2017), no. 10, 4126–4155, `doi:10.1016/j.jfranklin.2017.03.010`, `arXiv:1308.4274`. MR 3651331. Zbl 1367.93270.

Given K real d-by-d nonsingular matrices $S_1, \ldots, S_K$, by extending the well-known Li-Yorke chaotic description of a deterministic nonlinear dynamical system to a discrete-time linear inclusion dynamical system: $x_n \in \{S_k x_{n-1}\}_{1\leq k\leq K}$ with $x_0 \in \mathbb{R}^d$ and $n \geq 1$, we study the chaotic characteristic of the state trajectory $(x_n(x_0, \sigma))_{n\geq1}$, governed by a switching law $\sigma\colon \mathbb{N} \to \{1, \ldots, K\}$, for any initial states $x_0 \in \mathbb{R}^d$. Two sufficient conditions are given so that for a "large" subset of the space of all possible switching laws σ, we have the sharp infinite oscillation as follows:

$$\liminf_{n\to+\infty} \|x_n(x_0, \sigma)\| = 0 \quad \text{and} \quad \limsup_{n\to+\infty} \|x_n(x_0, \sigma)\| = +\infty \quad \forall x_0 \in \mathbb{R}^d \setminus \{0\}.$$

This implies that there coexists at least one positive, one zero and one negative Lyapunov exponents and that the trajectories $(x_n(x_0, \sigma))_{n\geq1}$ are extremely sensitive to the initial states $x_0 \in \mathbb{R}^d$. We also show that a periodically stable linear inclusion system, which may be unbounded, does not have any such chaotic behavior.

[383] L. Daviaud, P. Guillon, and G. Merlet, *Comparison of max-plus automata and joint spectral radius of tropical matrices*, 42nd International Symposium on Mathematical Foundations of Computer Science (MFCS 2017) (Dagstuhl, Germany) (K. G. Larsen, H. L. Bodlaender, and J.-F. Raskin, eds.), Leibniz International Proceedings in Informatics (LIPIcs), vol. 83, Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2017, pp. 19:1–19:14, `doi:10.4230/LIPIcs.MFCS.2017.19`, `arXiv:1612.02647`. MR 3755312. Zbl 1441.68119.

Weighted automata over the tropical semiring $\mathbb{Z}_{\max} = (\mathbb{Z} \cup \{-\infty\}, \max, +)$ are closely related to finitely generated semigroups of matrices over $\mathbb{Z}_{\max}$. In this paper, we use results in automata theory to study two quantities associated with sets of matrices: the joint spectral radius and the ultimate rank. We prove that these two quantities are not computable over the tropical semiring, i.e. there is no algorithm that takes as input a finite set of matrices Γ and provides as output the joint spectral radius (resp. the ultimate rank) of Γ. On the other hand, we prove that the joint spectral radius is nevertheless approximable and we exhibit restricted cases in which the joint spectral radius and the ultimate rank are computable. To reach this aim, we study the problem of comparing functions computed by weighted automata over the tropical semiring. This problem is known to be undecidable and we prove that it remains undecidable in some specific subclasses of automata.

[384] S. Gaubert and N. Stott, *Tropical Kraus maps for optimal control of switched systems*, Proceedings of the 56th Annual Conference on Decision and Control (CDC), 12-15 Dec. (Melbourne, Victoria, Australia), IEEE, 2017, pp. 1330–1337, `doi:10.1109/CDC.2017.8263839`, `arXiv:1706.04471`.

Kraus maps (completely positive trace preserving maps) arise classically in quantum information, as they describe the evolution of noncommutative probability measures. We introduce tropical analogues of Kraus maps, obtained by replacing the addition of positive semidefinite matrices by a multivalued supremum with respect to the Löwner order. We show that non-linear eigenvectors of tropical Kraus maps determine piecewise quadratic approximations of the value functions of switched optimal control problems. This leads to a new approximation method, which we illustrate by two applications: 1) approximating the joint spectral radius, 2) computing approximate solutions of Hamilton-Jacobi PDE arising from a class of switched linear quadratic problems studied previously by McEneaney. We report numerical experiments, indicating a major improvement in terms of scalability by comparison with earlier numerical schemes, owing to the "LMI-free" nature of our method.

[385] J. T. A. Gomes, *Conjuntos maximizantes para sequências de funções e o Raio espectral conjunto*, Ph.D. thesis, Universidade Estadual de Campinas, Instituto de Matemática, Estatística e Computação Científica, Campinas, 2017, URL `https://bdtd.ibict.br/vufind/Record/CAMP_01c3b1af1f91b867f24a22ac0650e6a3`.

Em uma generalização da teoria de otimização ergódica para sequências de funções, desenvolvemos novas perspectivas no que concerne à caracterização das probabilidades maximizantes. Para isto obter, determinam-se condições suficientes que permitem destacar o suporte de tais medidas por meio de versões generalizadas do local de maximização e do conjunto de Aubry. Além de aprimorar resultados conhecidos na literatura da teoria de otimização ergódica sobre tais conjuntos maximizantes, tamb'em sugerimos extens oes do teorema de Atkinson e da noção de subação para o contexto das sequências de funções. Ao final, estudamos condições necessárias e suficientes que fornecem norma extremal de politopo e uma nova formulação para a propriedade da finitude do raio espectral conjunto.

In a generalized version of ergodic optimization theory for sequences of functions, we develop new perspectives concerning the characterization of maximizing probabilities. For this purpose, we provide sufficient conditions that allow us to detach the support of these measures by means of generalizations of the maximizing locus and the Aubry set. Besides the fact that we improve known results in the ergodic optimization theory literature on these maximizing sets, we also suggest extensions to Atkinson's theorem and to the notion of sub-action for the context of sequences of functions. At the end, we study necessary and sufficient conditions which provide an extremal polytope norm and a new formulation for the finiteness property of the joint spectral radius.

[386] N. Guglielmi, L. Laglia, and V. Protasov, *Polytope Lyapunov functions for stable and for stabilizable LSS*, Found. Comput. Math. **17** (2017), no. 2, 567–623, `doi:10.1007/s10208-015-9301-9`, `arXiv:1406.5927`. MR 3627457. Zbl 1361.93050.

We present a new approach for constructing polytope Lyapunov functions for continuous-time linear switching systems (LSS). This allows us to decide the stability of LSS and to compute the Lyapunov exponent with a good precision in relatively high dimensions. The same technique is also extended for stabilizability of positive systems by evaluating a polytope concave Lyapunov function ("antinorm") in the cone. The method is based on a suitable discretization of the underlying continuous system and provides both a lower and an upper bound for the Lyapunov exponent. The absolute error in the Lyapunov exponent computation (the distance between lower and upper bound) is estimated from above and proved to be linear in the dwell time. The practical efficiency of the new method is demonstrated in several examples and in the list of numerical experiments with randomly generated matrices of dimensions up to 10 (for general linear systems) and up to 100 (for positive systems). The development of the method is based on several theoretical results proved in the paper: the existence of monotone invariant norms and antinorms for positively irreducible systems, the equivalence of all contractive norms for stable systems and the linear convergence theorem.

[387] O. Jenkinson and M. Pollicott, *Joint spectral radius, Sturmian measures, and the finiteness conjecture*, Ergodic Theory Dynam. Systems (2017), 1–39, `doi:10.1017/etds.2017.18`, `arXiv:1501.03419`. MR 3868023. Zbl 1405.15028.

The joint spectral radius of a pair of 2×2 real matrices $(A_0, A_1) \in M_2(\mathbb{R})^2$ is defined to be $r(A_0, A_1) = \limsup_{n\to\infty} \max\{\|A_{i_1} \cdots A_{i_n}\|^{1/n} : i_j \in \{0, 1\}\}$, the optimal growth rate of the norm of products of these matrices. The Lagarias-Wang finiteness conjecture [53], asserting that $r(A_0, A_1)$ is always the nth root of the spectral radius of some length-n product $A_{i_1} \cdots A_{i_n}$, has been refuted by *T. Bousch* & *J. Mairesse* [132], with subsequent counterexamples presented by *V. Blondel*, *J. Theys* and *A. Vladimirov* [140], *V. Kozyakin* [164], *K. Hare*, *I. Morris*, *N. Sidorov* and *J. Theys* [258].

In this article we present a large class of finiteness counterexamples, proving that there exists an open subset of $M_2(\mathbb{R})^2$ with the property that each member (A_0, A_1) of the subset generates uncountably many counterexamples of the form (A_0, tA_1). In particular, it follows that the set of finiteness counterexamples in $M_2(\mathbb{R})^2$ is of Hausdorff dimension at least 7. Our methods employ ergodic theory, in particular the analysis of Sturmian invariant measures; this approach allows a short proof that the relation between the parameter t and the Sturmian parameter $\mathcal{P}(t)$ is a Devil's staircase.

[388] R. M. Jungers and P. Mason, *On feedback stabilization of linear switched systems via switching signal control*, SIAM J. Control Optim. **55** (2017), no. 2, 1179–1198, `doi:10.1137/15M1027802`, `arXiv:1601.08141`. MR 3633780. Zbl 1361.93051.

Motivated by recent applications in control theory, we study the feedback stabilizability of switched systems, where one is allowed to chose the switching signal as a function of $x(t)$ in order to stabilize the system. We propose new algorithms and analyze several mathematical features of the problem which were unnoticed up to now, to our knowledge. We prove complexity results, (in)equivalence between various notions of stabilizability, existence of Lyapunov functions, and we provide a case study for a paradigmatic example introduced by *D. P. Stanford* and *J. M. Urbano* [51].

[389] V. S. Kozyakin, *Constructive stability and stabilizability of positive linear discrete-time switching systems*, J. Commun. Technol. Electron. **62** (2017), no. 6, 686–693, `doi:10.1134/S1064226917060110`, `arXiv:1511.05665`.

We describe a new class of positive linear discrete-time switching systems for which the problems of stability or stabilizability can be resolved constructively. The systems constituting this class can be treated as a natural generalization of systems with the so-called independently switching state vector components. Distinctive feature of such systems is that their components can be arbitrarily "re-connected" in parallel or in series without loss of the "constructive resolvability" property for the problems of stability or stabilizability of a system. It is shown also that, for such systems, the individual positive trajectories with the greatest or the lowest rate of convergence to the zero can be built constructively.

[390] V. Kozyakin, *Minimax theorem for the spectral radius of the product of non-negative matrices*, Linear Multilinear Algebra **65** (2017), no. 11, 2356–2365 (english), `doi:10.1080/03081087.2016.1273877`, `arXiv:1603.05375`. MR 3740702. Zbl 1387.15017.

We prove the minimax equality for the spectral radius $\rho(AB)$ of the product of matrices $A \in \mathcal{A}$ and $B \in \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are compact sets of non-negative matrices of dimensions $N \times M$ and $M \times N$, respectively, satisfying the so-called hourglass alternative.

[391] I. D. Morris, *Generic properties of the lower spectral radius for some low-rank pairs of matrices*, Linear Algebra Appl. **524** (2017), 35–60, `doi:10.1016/j.laa.2017.02.023`, `arXiv:1510.00209`. MR 3630178. Zbl 1360.15025.

The lower spectral radius of a set of $d \times d$ matrices is defined to be the minimum possible exponential growth rate of long products of matrices drawn from that set. When considered as a function of a finite set of matrices of fixed cardinality it is known that the lower spectral radius can vary discontinuously as a function of the matrix entries. In a previous article the author and J. Bochi conjectured that when considered as a function on the set of all pairs of 2×2 real matrices, the lower spectral radius is discontinuous on a set of positive (eight-dimensional) Lebesgue measure, and related this result to an earlier conjecture of Bochi and Fayad. In this article we investigate the continuity of the lower spectral radius in a simplified context in which one of the two matrices is assumed to be of rank one. We show in particular that the set of discontinuities of the lower spectral radius on the set of pairs of 2×2 real matrices has positive seven-dimensional Lebesgue measure, and that among the pairs of matrices studied, the finiteness property for the lower spectral radius is true on a set of full Lebesgue measure but false on a residual set.

[392] V. Yu. Protasov and A. S. Voynov, *Matrix semigroups with constant spectral radius*, Linear Algebra Appl. **513** (2017), 376–408, `doi:10.1016/j.laa.2016.10.013`, `arXiv:1407.6568`. MR 3573807. Zbl 1359.15010.

Multiplicative matrix semigroups with constant spectral radius (c.s.r.) are studied and applied to several problems of algebra, combinatorics, functional equations, and dynamical systems. We show that all such semigroups are characterized by means of irreducible ones. Each irreducible c.s.r. semigroup defines walks on Euclidean sphere, all its nonsingular elements are similar (in the same basis) to orthogonal. We classify all nonnegative c.s.r. semigroups and arbitrary low-dimensional semigroups. For higher dimensions, we describe five classes and leave an open problem on completeness of that list. The problem of algorithmic recognition of c.s.r. property is proved to be polynomially solvable for irreducible semigroups and undecidable for reducible ones.

[393] S. Safavi and U. A. Khan, *Asymptotic stability of LTV systems with applications to distributed dynamic fusion*, IEEE Transactions on Automatic Control **62** (2017), no. 11, 5888–5893, `doi:10.1109/TAC.2017.2648501`, `arXiv:1412.8018`. MR 3730968. Zbl 1390.93506.

We investigate the behavior of Linear Time-Varying (LTV) systems with randomly appearing, sub-stochastic system matrices. Motivated by dynamic fusion over mobile networks, we develop conditions on the system matrices that lead to asymptotic stability of the underlying LTV system. By partitioning the sequence of system matrices into slices, we obtain the stability conditions in terms of slice lengths and introduce the notion of unbounded connectivity, i.e., the time-intervals, over which the multi-agent network is connected, do not have to be bounded as long as they do not grow faster than a certain exponential rate. We further apply the above analysis to derive the asymptotic behavior of a dynamic leader-follower algorithm.

[394] C. Sert, *Joint spectrum and large deviation principle for random matrix products*, C. R. Math. Acad. Sci. Paris **355** (2017), no. 6, 718–722, `doi:10.1016/j.crma.2017.04.015`. MR 3661555. Zbl 1373.60017.

The aim of this note is to announce some results about the probabilistic and deterministic asymptotic properties of linear groups. The first one is the analogue, for norms of random matrix products, of the classical theorem of Cramér on large deviation principles (LDP) for sums of iid real random variables. In the second result, we introduce a limit set describing the asymptotic shape of the powers $S^n = \{g_1. \dots .g_n \mid g_i \in S\}$ of a subset S of a semisimple linear Lie group G (e.g. $\mathrm{SL}(d, \mathbb{R})$). This limit set has applications, among others, in the study of large deviations.

[395] Y. Shang, *Subspace confinement for switched linear systems*, Forum Math. **29** (2017), no. 3, 693–699, `doi:10.1515/forum-2015-0188`. MR 3641673. Zbl 1360.15003.

In this note, we introduce the asymptotic subspace confinement problem, generalizing the usual concept of convergence in discrete-time linear systems. Instead of precise convergence, subspace confinement only requires the convergence of states to a certain subspace of the state space, offering useful flexibility and applicability. We establish a criterion for deciding the asymptotic subspace confinement, drawing upon a general finiteness result for the infinite product of matrices. Our results indicate that the asymptotic subspace confinement problem is algorithmically decidable when an invariant subspace for the set of matrices and some polytope norms are given.

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[396] L. Backes and D. Dragičević, *On the spectral radius of compact operator cocycles*, ArXiv.org e-Print archive, October 2018, `doi:10.1142/S021949372150026X`, `arXiv:1810.01762`.

We extend the notions of joint and generalized spectral radii to cocycles acting on Banach spaces and obtain a version of Berger-Wang's formula when restricted to the space of cocycles taking values in the space of compact operators. Moreover, we observe that the previous quantities depends continuously on the underlying cocycle.

[397] E. Breuillard and K. Fujiwara, *On the joint spectral radius for isometries of non-positively curved spaces and uniform growth*, ArXiv.org e-Print archive, April 2018, `arXiv:1804.00748`.

We recast the notion of joint spectral radius in the setting of groups acting by isometries on non-positively curved spaces and give geometric versions of results of Berger-Wang and Bochi valid for δ-hyperbolic spaces and for symmetric spaces of non-compact type. This method produces nice hyperbolic elements in many classical geometric settings. Applications to uniform growth are given, in particular a new proof and a generalization of a theorem of Besson-Courtois-Gallot.

[398] M. Brundu and M. Zennaro, *Invariant multicones for families of matrices*, Annali di Matematica (2018), 1–44, `doi:10.1007/s10231-018-0790-4`. MR 3927171. Zbl 1408.15015.

In this paper, we investigate sufficient conditions on the structure of the eigenspaces of a given finite family of matrices to assure the existence of an embedded pair of invariant multicones, which are the smallest and the biggest in a suitable and natural sense. Multicones, very similar structures to those known in the literature as 1-multicones, are quite natural generalizations of the classical cones. The conditions we find also suggest us a practical computational procedure for the actual construction of such invariant embedded pair.

[399] C. Catalano and R. M. Jungers, *The synchronizing probability function for primitive sets of matrices*, Developments in language theory, Lecture Notes in Comput. Sci., vol. 11088, Springer, Cham, 2018, pp. 194–205, `doi:10.1007/978-3-319-98654-8_16`. MR 3855942. Zbl 06983378.

Motivated by recent results relating synchronizing automata and primitive sets, we tackle the synchronization process and the related longstanding Černý conjecture by studying the primitivity phenomenon for sets of nonnegative matrices having neither zero-rows nor zero-columns. We formulate the primitivity process in the setting of a two-player probabilistic game and we make use of convex optimization techniques to describe its behavior. We report numerical results and supported by them we state a conjecture that, if true, would imply an upper bound of $n(n-1)$ on the reset threshold of a certain class of automata.

[400] A. Cicone, N. Guglielmi, and V. Yu. Protasov, *Linear switched dynamical systems on graphs*, Nonlinear Anal. Hybrid Syst. **29** (2018), 165–186, `doi:10.1016/j.nahs.2018.01.006`, `arXiv:1607.00415`. MR 3795595. Zbl 1391.37035.

We consider linear dynamical systems with a structure of a multigraph. The vertices are associated to linear spaces and the edges correspond to linear maps between those spaces. We analyse the asymptotic growth of trajectories (associated to paths along the multigraph), the stability and the stabilizability problems. This generalizes the classical linear switching systems and their recent extensions to Markovian systems, to systems generated by regular languages, etc. We show that an arbitrary system can be factorized into several irreducible systems on strongly connected multigraphs. For the latter systems, we prove the existence of invariant (Barabanov) multinorm and derive a method of its construction. The method works for a vast majority of systems and finds the joint spectral radius (Lyapunov exponent). Numerical examples are presented and applications to the study of fractals, attractors, and multistep methods for ODEs are discussed.

[401] N. Guglielmi, O. Mason, and F. Wirth, *Barabanov norms, Lipschitz continuity and monotonicity for the max algebraic joint spectral radius*, Linear Algebra Appl. **550** (2018), 37–58, `doi:10.1016/j.laa.2018.01.042`, `arXiv:1705.02008`. MR 3786246. Zbl 1390.15112.

We present several results describing the interplay between the max algebraic joint spectral radius (JSR) for compact sets of matrices and suitably defined matrix norms. In particular, we extend a classical result for the conventional algebra, showing that the JSR can be described in terms of induced norms of the matrices in the set. We also show that for a set generating an irreducible semigroup (in a cone-theoretic sense), a monotone Barabanov norm always exists. This fact is then used to show that the max algebraic JSR is locally Lipschitz continuous on the space of compact irreducible sets of matrices with respect to the Hausdorff distance. We then prove that the JSR is Hoelder continuous on the space of compact sets of nonnegative matrices. Finally, we prove a strict monotonicity property for the max algebraic JSR that echoes a fact for the classical JSR.

[402] V. Kozyakin, *On convergence of infinite matrix products with alternating factors from two sets of matrices*, Discrete Dyn. Nat. Soc. **2018** (2018), no. 9216760, 5, `doi:10.1155/2018/9216760`, `arXiv:1712.06356`. MR 3798756. Zbl 1417.93254.

We consider the problem of convergence to zero of matrix products $A_nB_n\cdots A_1B_1$ with factors from two sets of matrices, $A_i \in \mathcal{A}$ and $B_i \in \mathcal{B}$, due to a suitable choice of matrices $\{B_i\}$. It is assumed that for any sequence of matrices $\{A_i\}$ there is a sequence of matrices $\{B_i\}$ such that the corresponding matrix product $A_nB_n\cdots A_1B_1$ converges to zero. We show that in this case the convergence of the matrix products under consideration is uniformly exponential, that is, $\|A_nB_n\cdots A_1B_1\| \leq C\lambda^n$, where the constants $C > 0$ and $\lambda \in (0,1)$ do not depend on the sequence $\{A_i\}$ and the corresponding sequence $\{B_i\}$.

Other problems of this kind are discussed and open questions are formulated.

[403] I. D. Morris, *Ergodic properties of matrix equilibrium states*, Ergodic Theory Dynam. Systems **38** (2018), no. 6, 2295–2320, `doi:10.1017/etds.2016.117`, `arXiv:1603.01744`. MR 3833350. Zbl 1397.37033.

Given a finite irreducible set of real $d\times d$ matrices $A_1,\dots,A_M$ and a real parameter $s > 0$, there exists a unique shift-invariant equilibrium state on $\{1,\dots,M\}^{\mathbb{N}}$ associated to $(A_1,\dots,A_M,s)$. In this article we characterise the ergodic properties of such equilibrium states in terms of the algebraic properties of the semigroup generated by the associated matrices. We completely characterise when the equilibrium state has zero entropy, when it is fully supported, when it gives distinct Lyapunov exponents to the natural cocycle generated by $A_1,\dots,A_M$, and when it is a Bernoulli measure. We also give a general sufficient condition for the equilibrium state to be mixing, and give an example where the equilibrium state is ergodic but not totally ergodic.

[404] V. Müller and A. Peperko, *Lower spectral radius and spectral mapping theorem for suprema preserving mappings*, Discrete Contin. Dyn. Syst. **38** (2018), no. 8, 4117–4132, `doi:10.3934/dcds.2018179`, `arXiv:1712.00340`. MR 3814367. Zbl 06919411.

We study Lipschitz, positively homogeneous and finite suprema preserving mappings defined on a max-cone of positive elements in a normed vector lattice. We prove that the lower spectral radius of such a mapping is always a minimum value of its approximate point spectrum. We apply this result to show that the spectral mapping theorem holds for the approximate point spectrum of such a mapping. By applying this spectral mapping theorem we obtain new inequalites for the Bonsall cone spectral radius of max type kernel operators.

[405] E. Oregón-Reyes, *Negative curvature, matrix products, and ergodic theory*, Master's thesis, Faculty of Mathematics, Pontificia Universidad Católica de Chile, 2018, URL `http://www.mat.uc.cl/~jairo.bochi/docs/Oregon-Reyes_master_thesis.pdf`.

In the theses, some properties of sets of isometries of Gromov hyperbolic spaces are investigated and a new inequality about matrix products and a Berger-Wang formula is obtained.

[406] E. Oregón-Reyes, *Properties of sets of isometries of Gromov hyperbolic spaces*, Groups Geom. Dyn. **12** (2018), no. 3, 889–910, `doi:10.4171/GGD/468`, `arXiv:1606.01575`. MR 3844998. Zbl 1398.53049.

We prove an inequality concerning isometries of a Gromov hyperbolic metric space, which does not require the space to be proper or geodesic. It involves the joint stable length, a hyperbolic version of the joint spectral radius, and shows that sets of isometries behave like sets of 2×2 real matrices. Among the consequences of the inequality, we obtain the continuity of the joint stable length and an analogue of Berger-Wang theorem.

[407] A. Thomas, *Normalized image of a vector by an infinite product of nonnegative matrices*, ArXiv.org e-Print archive, August 2018, `arXiv:1808.09803`.

Let $P_n = A_1 \cdots A_n$, where $\mathcal{A} = (A_n)_{n\in\mathbb{N}}$ is a sequence of $d \times d$ matrices, and let V be a d-dimensional column-vector. We call "normalized image of V by the infinite product of the matrices A_n", the vector $\lim_{n\to\infty} P_nV/\|P_nV\|$ if exists. In general the sequence of matrices $n \mapsto P_n/\|P_n\|$ does not converge but, under some sufficient conditions specified in Theorem A the sequence of vectors $n \mapsto P_nV/\|P_nV\|$ converges. We use Theorem A to prove that certain sofic (i.e. linearly representable) measures satisfy the multifractal formalism.

[408] R. Zou, Y. Cao, and G. Liao, *Continuity of spectral radius over hyperbolic systems*, Discrete Contin. Dyn. Syst. **38** (2018), no. 8, 3977–3991, `doi:10.3934/dcds.2018173`. MR 3814361. Zbl 1396.37061.

The continuity of joint and generalized spectral radius is proved for Hölder continuous cocycles over hyperbolic systems. We also prove the periodic approximation of Lyapunov exponents for non-invertible non-uniformly hyperbolic systems, and establish the Berger-Wang formula for general dynamical systems.

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[409] J. M. Altschuler and P. A. Parrilo, *Lyapunov exponent of rank-one matrices: ergodic formula and inapproximability of the optimal distribution*, 2019 IEEE 58th Conference on Decision and Control (CDC), 11-13 Dec. (Nice, France, France), IEEE, 2019, pp. 4439–4445, `doi:10.1109/CDC40024.2019.9029462`, `arXiv:1905.07531`.

The *Lyapunov exponent* corresponding to a set of square matrices $\mathcal{A} = \{A_1, \dots, A_n\}$ and a probability distribution p over $\{1, \dots, n\}$ is $\lambda(\mathcal{A}, p) := \lim_{k\to\infty} \frac{1}{k} \mathbb{E} \log\|A_{\sigma_k} \cdots A_{\sigma_2} A_{\sigma_1}\|$, where σ_i are i.i.d. according to p. This quantity is of fundamental importance to control theory since it determines the asymptotic convergence rate $e^{\lambda(\mathcal{A},p)}$ of the stochastic linear dynamical system $x_{k+1} = A_{\sigma_k} x_k$. This paper investigates the following "design problem": given $\mathcal{A}$, compute the distribution p minimizing $\lambda(\mathcal{A}, p)$. Our main result is that it is NP-hard to decide whether there exists a distribution p for which $\lambda(\mathcal{A}, p) < 0$, i.e. it is NP-hard to decide whether this dynamical system can be stabilized.

This hardness result holds even in the "simple" case where $\mathcal{A}$ contains only rank-one matrices. Somewhat surprisingly, this is in stark contrast to the Joint Spectral Radius – the deterministic kindred of the Lyapunov exponent – for which the analogous optimization problem for rank-one matrices is known to be exactly computable in polynomial time.

To prove this hardness result, we first observe via Birkhoff's Ergodic Theorem that the Lyapunov exponent of rank-one matrices admits a simple formula and in fact is a quadratic form in p. Hardness of the design problem is shown through a reduction from the Independent Set problem. Along the way, simple examples are given illustrating that $p \mapsto \lambda(\mathcal{A}, p)$ is neither convex nor concave in general, and a connection is made to the fact that the Martin distance on the $(1, n)$ Grassmannian is not a metric. See. [422].

[410] U. Azfar, C. Catalano, L. Charlier, and R. M. Jungers, *A linear bound on the k-rendezvous time for primitive sets of NZ matrices*, Developments in language theory (P. Hofman and M. Skrzypczak, eds.), Lecture Notes in Comput. Sci., vol. 11647, Springer, Cham, 2019, pp. 59–73, doi:10.1007/978-3-030-24886-4_4. MR 3991875. Zbl 07117536.

A set of nonnegative matrices is called primitive if there exists a product of these matrices that is entrywise positive. Motivated by recent results relating synchronizing automata and primitive sets, we study the length of the shortest product of a primitive set having a column or a row with k positive entries (the k-RT). We prove that this value is at most linear w.r.t. the matrix size n for small k, while the problem is still open for synchronizing automata. We then report numerical results comparing our upper bound on the k-RT with heuristic approximation methods.

[411] J. Bochi and E. Garibaldi, *Extremal norms for fiber-bunched cocycles*, Journal de l'École polytechnique — Mathématiques **6** (2019), 947–1004, doi:10.5802/jep.109, arXiv:1808.02804. MR 4031530. Zbl 1441.37059.

In traditional Ergodic Optimization, one seeks to maximize Birkhoff averages. The most useful tool in this area is the celebrated Mañé Lemma, in its various forms. In this paper, we prove a non-commutative Mañé Lemma, suited to the problem of maximization of Lyapunov exponents of linear cocycles or, more generally, vector bundle automorphisms. More precisely, we provide conditions that ensure the existence of an extremal norm, that is, a Finsler norm with respect to which no vector can be expanded in a single iterate by a factor bigger than the maximal asymptotic expansion rate. These conditions are essentially irreducibility and sufficiently strong fiber bunching. Therefore we extend the classic concept of Barabanov norm, which is used in the study of the joint spectral radius. We obtain several consequences, including sufficient conditions for the existence of Lyapunov maximizing sets.

[412] F. Della Rossa and F. Dercole, *Tree-based algorithms for the stability of discrete-time switched linear systems under arbitrary and constrained switching*, IEEE Trans. Autom. Control **64** (2019), no. 9, 3823–3830, doi:10.1109/TAC.2018.2887142. MR 4003174. Zbl 07158456.

We present a direct approach to study the stability of discrete-time switched linear systems that can be applied to arbitrary switching, as well as when switching is constrained by a switching automaton. We explore the tree of possible matrix products, by pruning the subtrees rooted at contractions and looking for unstable repeatable products. Generically, this simple strategy either terminates with all contracting leafs—showing the system's asymptotic stability—or finds the shortest unstable and repeatable matrix product. Although it behaves in the worst-case as the exhaustive search, we show that its performance is greatly enhanced by measuring contractiveness w.r.t. sum-of-squares polynomial norms, optimized to minimize the largest expansion among the system's modes.

[413] M. Gil', *A bound for the joint spectral radius of operators in a Hilbert space*, Univers. J. Math. Appl. **2** (2019), 94–99, doi:10.32323/ujma.543952.

We suggest a bound for the joint spectral radius of a finite set of operators in a Hilbert space. In appropriate situations that bound enables us to avoid complicated calculations and gives a new explicit stability test for the discrete time switched systems. The illustrative example is given. Our results are new even in the finite dimensional case.

[414] H. Heaton and Y. Censor, *Asynchronous sequential inertial iterations for common fixed points problems with an application to linear systems*, J. Global Optim. **74** (2019), no. 1, 95–119, doi:10.1007/s10898-019-00747-4, arXiv:1808.04723. MR 3943617. Zbl 07069296.

The common fixed points problem requires finding a point in the intersection of fixed points sets of a finite collection of operators. Quickly solving problems of this sort is of great practical importance for engineering and scientific tasks (e.g., for computed tomography). Iterative methods for solving these problems often employ a Krasnosel'skiĭ-Mann type iteration. We present an Asynchronous Sequential Inertial (ASI) algorithmic framework in a Hilbert space to solve common fixed points problems with a collection of nonexpansive operators. Our scheme allows use of out-of-date iterates when generating updates, thereby enabling processing nodes to work simultaneously and without synchronization. This method also includes inertial type extrapolation terms to increase the speed of convergence. In particular, we extend the application of the recent "ARock algorithm" [Peng, Z. et al, SIAM J. on Scientific Computing **38**, A2851-2879, (2016)] in the context of convex feasibility problems. Convergence of the ASI algorithm is proven with no assumption on the distribution of delays, except that they be uniformly bounded. Discussion is provided along with a computational example showing the performance of the ASI algorithm applied in conjunction with a diagonally relaxed orthogonal projections (DROP) algorithm for estimating solutions to large linear systems.

[415] V. Kozyakin, *Minimax joint spectral radius and stabilizability of discrete-time linear switching control systems*, Discrete Contin. Dyn. Syst. Ser. B **24** (2019), no. 8, 3537–3556, `doi:10.3934/dcdsb.2018277`, `arXiv:1712.06805`. MR 3986244. Zbl 1427.93180.

To estimate the growth rate of matrix products $A_n \cdots A_1$ with factors from some set of matrices $\mathcal{A}$, such numeric quantities as the joint spectral radius $\rho(\mathcal{A})$ and the lower spectral radius $\check{\rho}(\mathcal{A})$ are traditionally used. The first of these quantities characterizes the maximum growth rate of the norms of the corresponding products, while the second one characterizes the minimal growth rate. In the theory of discrete-time linear switching systems, the inequality $\rho(\mathcal{A}) < 1$ serves as a criterion for the stability of a system, and the inequality $\check{\rho}(\mathcal{A}) < 1$ as a criterion for stabilizability.

For matrix products $A_nB_n \cdots A_1B_1$ with factors $A_i \in \mathcal{A}$ and $B_i \in \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are some sets of matrices, we introduce the quantities $\mu(\mathcal{A}, \mathcal{B})$ and $\eta(\mathcal{A}, \mathcal{B})$, called the lower and upper minimax joint spectral radius of the pair $\{\mathcal{A}, \mathcal{B}\}$, respectively, which characterize the maximum growth rate of the matrix products $A_nB_n \cdots A_1B_1$ over all sets of matrices $A_i \in \mathcal{A}$ and the minimal growth rate over all sets of matrices $B_i \in \mathcal{B}$. In this sense, the minimax joint spectral radii can be considered as generalizations of both the joint and lower spectral radii. As an application of the minimax joint spectral radii, it is shown how these quantities can be used to analyze the stabilizability of discrete-time linear switching control systems in the presence of uncontrolled external disturbances of the plant.

[416] T. Mejstrik, *Joint spectral radius and subdivision schemes*, Ph.D. thesis, Fakultät für Mathematik, Universität Wien, Besprechungszimmer 09. Stock, Oskar-Morgenstern-Platz 1, 2019, URL `http://www.tommsch.com/misc/diss.pdf`.

This thesis extends the matrix based approach to the setting of multiple subdivision schemes studied in [Sauer, 2012]. Multiple subdivision schemes, in contrast to stationary and nonstationary schemes, allow for level dependent subdivision weights and for level dependent choice of the dilation matrices. The latter property of multiple subdivision makes the standard definition of the transition matrices, crucial ingredient of the matrix approach in the stationary and non-stationary settings, inapplicable. We show how to avoid this obstacle and characterize the convergence of multiple subdivision schemes in terms of the joint spectral radius of certain square matrices derived from subdivision weights.

Albeit the characterization of the convergence of multiple subdivision schemes in terms of the joint spectral radius is elegant, the numerical computation of the joint spectral radius still poses big problems. In several papers of 2013 – 2016, Guglielmi and Protasov made a breakthrough in the problem of the joint spectral radius computation, developing the invariant polytope algorithm, which for most matrix families finds the exact value of the joint spectral radius. This algorithm found many applications in problems of functional analysis, approximation theory, combinatorics, etc. In this thesis we propose a modification of the invariant polytope algorithm making it roughly three times faster and suitable for higher dimensions. The modified version works for most matrix families of dimensions up to 25, for non-negative matrices the dimension is up to 3000.

Besides, we introduce a new, fast algorithm for computing good lower bounds for the joint spectral radius, which finds in most cases the exact value of the joint spectral radius in less than 5 seconds. Corresponding examples and statistics of numerical results are provided.

[417] I. D. Morris, *Fast approximation of the p-radius, matrix pressure or generalised Lyapunov exponent for positive and dominated matrices*, ArXiv.org e-Print archive, May 2019, `arXiv:1905.00749`.

If $A_1, \dots, A_N$ are real $d \times d$ matrices then the *p-radius*, *generalised Lyapunov exponent* or *matrix pressure* is defined to be the asymptotic exponential growth rate of the sum $\sum_{i_1,\dots,i_n=1}^{N} \|A_{i_n} \cdots A_{i_1}\|^p$, where p is a real parameter. Under its various names this quantity has been investigated for its applications to topics including wavelet regularity and refinement equations, fractal geometry and the large deviations theory of random matrix products. In this article we present a new algorithm for computing the p-radius under the hypothesis that the matrices are all positive, or more generally under the hypothesis that they satisfy a weaker condition called *domination*. This algorithm is based on interpreting the p-radius as the leading eigenvalue of a trace-class operator on a Hilbert space and estimating that eigenvalue via approximations to the Fredholm determinant of the operator. In this respect our method is closely related to the work of Z.-Q. Bai and M. Pollicott on computing the top Lyapunov exponent of a random matrix product. For pairs of positive matrices of low dimension our method yields substantial improvements over existing methods.

[418] I. D. Morris, *Lyapunov-maximising measures for pairs of weighted shift operators*, Ergodic Theory Dynam. Systems **39** (2019), no. 1, 225–247, `doi:10.1017/etds.2017.22`, `arXiv:1510.00162`. MR 3881131. Zbl 1402.37039.

Motivated by recent investigations of ergodic optimisation for matrix cocycles, we study the measures of maximum top Lyapunov exponent for pairs of bounded weighted shift operators on a separable Hilbert space. We prove that for generic pairs of weighted shift operators the Lyapunov-maximising measure is unique, and show that there exist pairs of operators whose unique Lyapunov-maximising measure takes any prescribed value less than $\log 2$ for its metric entropy. We also show that in contrast to the matrix case, the Lyapunov-maximising measures of pairs of bounded operators are in general not characterised by their supports: we construct explicitly a pair of operators, and a pair of ergodic measures on the 2-shift with identical supports, such that one of the two measures is Lyapunov-maximising for the pair of operators and the other measure is not. Our proofs make use of the Ornstein $\bar{d}$-metric to estimate differences in the top Lyapunov exponent of a pair of weighted shift operators as the underlying measure is varied.

[419] J. E. Pascoe, *The outer spectral radius and dynamics of completely positive maps*, ArXiv.org e-Print archive, May 2019, `arXiv:1905.09895`.

We examine a special case of an approximation of the joint spectral radius given by Blondel and Nesterov, which we call the outer spectral radius. The outer spectral radius is given by the square root of the ordinary spectral radius of the n^2 by n^2 matrix $\sum \overline{X_i} \otimes X_i$. We give an analogue of the spectral radius formula for the outer spectral radius which can be used to quickly obtain the error bounds in methods based on the work of Blondel and Nesterov. The outer spectral radius is used to analyze the iterates of a completely positive map, including the special case of quantum channels. The average of the iterates of a completely positive map approach to a completely positive map where the Kraus operators span an ideal in the algebra generated by the Kraus operators of the original completely positive map. We also give an elementary treatment of Popescu's theorems on similarity to row contractions in the matrix case, describe connections to the Parrilo-Jadbabaie relaxation, and give a detailed analysis of the maximal spectrum of a completely positive map.

[420] V. Yu. Protasov, *Comprehensive Lyapunov functions for linear switching systems*, Automatica J. IFAC **109** (2019), 108526, 7, `doi:10.1016/j.automatica.2019.108526`. MR 3992376. Zbl 1429.93317.

For a linear dynamical switching system, we analyse maximal asymptotic growth of trajectories depending on the initial point. Both discrete and continuous time systems in $\mathbb{R}^d$ are considered. We prove the existence of a Lyapunov norm in $\mathbb{R}^d$ with the following property: for every invariant linear subspace $L \subset \mathbb{R}^d$ of the system, the restriction of the norm on L provides a tight upper bound for the growth of trajectories on L. For this, we introduce the concept of the spectral normal form of a family of matrices. Properties of the comprehensive Lyapunov norms are analysed and methods of their construction are discussed.

[421] D. P. Reber, *Exponential stability of intrinsically stable dynamical networks and switched networks with time-varying time delays*, Master's thesis, Department of Mathematics, Brigham Young University, 2019, URL `https://scholarsarchive.byu.edu/etd/7136`.

Dynamic processes on real-world networks are time-delayed due to finite processing speeds and the need to transmit data over nonzero distances. These time-delays often destabilize the network's dynamics, but are difficult to analyze because they increase the dimension of the network.We present results outlining an alternative means of analyzing these networks, by focusing analysis on the Lipschitz matrix of the relatively low-dimensional undelayed network. The key criteria, intrinsic stability, is computationally efficient to verify by use of the power method. We demonstrate applications from control theory and neural networks.

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[422] J. M. Altschuler and P. A. Parrilo, *Lyapunov exponent of rank-one matrices: ergodic formula and inapproximability of the optimal distribution*, SIAM J. Control Optim. **58** (2020), no. 1, 510–528, `doi:10.1137/19M1264072`. MR 4068319. Zbl 1451.93408.

The *Lyapunov exponent* corresponding to a set of square matrices $\mathcal{A} = \{A_1, \ldots, A_n\}$ and a probability distribution p over $\{1, \ldots, n\}$ is $\lambda(\mathcal{A}, p) := \lim_{k\to\infty} \frac{1}{k} \mathbb{E} \log\|A_{\sigma_k} \cdots A_{\sigma_2} A_{\sigma_1}\|$, where σ_i are i.i.d. according to p. This quantity is of fundamental importance to control theory since it determines the asymptotic convergence rate $e^{\lambda(\mathcal{A},p)}$ of the stochastic linear dynamical system $x_{k+1} = A_{\sigma_k} x_k$. This paper investigates the following "design problem": given $\mathcal{A}$, compute the distribution p minimizing $\lambda(\mathcal{A}, p)$. Our main result is that it is NP-hard to decide whether there exists a distribution p for which $\lambda(\mathcal{A}, p) < 0$, i.e. it is NP-hard to decide whether this dynamical system can be stabilized.

This hardness result holds even in the "simple" case where $\mathcal{A}$ contains only rank-one matrices. Somewhat surprisingly, this is in stark contrast to the Joint Spectral Radius – the deterministic kindred of the Lyapunov exponent – for which the analogous optimization problem for rank-one matrices is known to be exactly computable in polynomial time.

To prove this hardness result, we first observe via Birkhoff's Ergodic Theorem that the Lyapunov exponent of rank-one matrices admits a simple formula and in fact is a quadratic form in p. Hardness of the design problem is shown through a reduction from the Independent Set problem. Along the way, simple examples are given illustrating that $p \mapsto \lambda(\mathcal{A}, p)$ is neither convex nor concave in general. We conclude with extensions to continuous distributions, exchangeable processes, Markov processes, and stationary ergodic processes. See. [409].

[423] B. Balle, P. Gourdeau, and P. Panangaden, *Bisimulation metrics and norms for real-weighted automata*, Information and Computation (2020), 104649, in Press, Corrected Proof, `doi:10.1016/j.ic.2020.104649`, `arXiv:1702.08017`.

We develop a new bisimulation (pseudo)metric for weighted finite automata (WFA) that generalizes Boreale's linear bisimulation relation. Our metrics are induced by seminorms on the state space of WFA. Our development is based on spectral properties of sets of linear operators. In particular, the joint spectral radius of the transition matrices of WFA plays a central role. We also study continuity properties of the bisimulation pseudometric, establish an undecidability result for computing the metric, and give a preliminary account of applications to spectral learning of weighted automata. See also [380].

[424] C. Catalano and R. M. Jungers, *The Synchronizing Probability Function for Primitive Sets of Matrices*, Internat. J. Found. Comput. Sci. **31** (2020), no. 6, 777–803, `doi:10.1142/S0129054120410051`. MR 4174088.

Motivated by recent results relating synchronizing DFAs and primitive sets, we tackle the synchronization process and the related longstanding Černý conjecture by studying the primitivity phenomenon for sets of nonnegative matrices having neither zero-rows nor zero-columns. We formulate the primitivity process in the setting of a two-player probabilistic game and we make use of convex optimization techniques to describe its behavior. We develop a tool for approximating and upper bounding the exponent of any primitive set and supported by numerical results we state a conjecture that, if true, would imply a quadratic upper bound on the reset threshold of a new class of automata.

[425] C. Chenavier, R. Ushirobira, and L. Hetel, *Normal forms of matrix words for stability analysis of discrete-time switched linear systems*, 2020 European Control Conference (ECC), 12-15 May (Saint Petersburg, Russia), IEEE, May 2020, pp. 1842–1846, `doi:10.23919/ECC51009.2020.9143862`.

In this paper, we propose a new method for investigating the stability of discrete-time switched linear systems by means of linear algebra techniques. Exponential stability of such systems is equivalent to the existence of solutions of linear matrix inequalities indexed by matrix words, i.e., products of matrices of the sub-systems). Our method consists in using the link between this characterization of exponential stability and linear dependency among matrix words. In particular, we introduce a criterion to reduce drastically the number of linear matrix inequalities, by removing redundant ones. This is achieved by eliminating matrix words that depend on the others. From this criterion, we also relate exponential stability to quadratic stability of another switched system. An example is given to illustrate our methods.

[426] A. Cvetković and V. Yu. Protasov, *The greedy strategy in optimizing the Perron eigenvalue*, Mathematical Programming (2020), `doi:10.1007/s10107-020-01585-z`, `arXiv:1807.05099`.

We address the problems of minimizing and of maximizing the spectral radius over a convex family of non-negative matrices. Those problems being hard in general can be efficiently solved for some special families. We consider the so-called product families, where each matrix is composed of rows chosen independently from given sets. A recently introduced greedy method works surprisingly fast. However, it is applicable mostly for strictly positive matrices. For sparse matrices, it often diverges and gives a wrong answer. We present the "selective greedy method" that works equally well for all non-negative product families, including sparse ones. For this method, we prove a quadratic rate of convergence and demonstrate its exceptional efficiency in numerical examples. In dimensions up to 2000, the matrices with minimal/maximal spectral radii in product families are found within a few iterations. Applications to dynamical systems and to the graph theory are considered.

[427] C. P. Dettmann, R. M. Jungers, and P. Mason, *Lower bounds and dense discontinuity phenomena for the stabilizability radius of linear switched systems*, Systems Control Lett. **142** (2020), 104737, 6, `doi:10.1016/j.sysconle.2020.104737`, `arXiv:2002.10369`. MR 4120003. Zbl 1451.93269.

We investigate the stabilizability of discrete-time linear, switched systems, when the sole control action of the controller is the switching signal, and when the controller has access to the state of the system in real time. Despite their importance in many control settings, no algorithm is known that allows to decide the stabilizability of such systems, and very simple examples have been known for long, for which the stabilizability question is open.

We provide new results allowing us to bound the so-called stabilizability radius, which characterizes the stabilizability property of discrete-time linear switched systems. These results allous to improve significantly the computation of the stabilizability radius for the above-mentioned examples. As a by-product, we exhibit a discontinuity property for this problem, which brings theoretical understanding of its complexity.

[428] D.-J. Feng, C.-H. Lo, and S. Shen, *Uniformity of Lyapunov exponents for non-invertible matrices*, Ergodic Theory and Dynamical Systems **40** (2020), no. 9, 2399–2433, `doi:10.1017/etds.2019.4`, `arXiv:1702.07251`. MR 4130809. Zbl 07228221.

Let $\mathbf{M} = (M_1, \ldots, M_k)$ be a tuple of real $d \times d$ matrices. Under certain irreducibility assumptions, we give checkable criteria for deciding whether $\mathbf{M}$ possesses the following property: there exist two constants $\lambda \in \mathbb{R}$ and $C > 0$ such that for any $n \in \mathbb{N}$ and any $i_1, \ldots, i_n \in \{1, \ldots, k\}$, either $M_{i_1} \cdots M_{i_n} = \mathbf{0}$ or $C^{-1}e^{\lambda n} \leq \|M_{i_1} \cdots M_{i_n}\| \leq Ce^{\lambda n}$, where $\|\cdot\|$ is a matrix norm. The proof is based on symbolic dynamics and the thermodynamic formalism for matrix products. As applications, we are able to check the absolute continuity of a class of overlapping self-similar measures on $\mathbb{R}$, the absolute continuity of certain self-affine measures in $\mathbb{R}^d$ and the dimensional regularity of a class of sofic affine-invariant sets in the plane.

Besides, we clarify the relations between different properties on Lyapunov exponent of a family of non-negative matrices.

[429] S. Gaubert and N. Stott, *A convergent hierarchy of non-linear eigenproblems to compute the joint spectral radius of nonnegative matrices*, Mathematical Control & Related Fields **10** (2020), no. 3, 573–590, `doi:10.3934/mcrf.2020011`, `arXiv:1805.03284`. MR 4128840. Zbl 07293645.

We show that the joint spectral radius of a finite, collection of nonnegative matrices can be bounded by the eigenvalue of a non-linear operator. This eigenvalue coincides with the ergodic constant of a risk-sensitive control problem, or of an entropy game, in which the state space consists of all switching sequences of a given length. We show that, by increasing this length, we arrive at a convergent approximation scheme to compute the joint spectral radius. The complexity of this method is exponential in the length of the switching sequences, but it is quite insensitive to the size of the matrices, allowing us to solve very large scale instances (several matrices in dimensions of order 1000 within a minute). An idea of this method is to replace a hierarchy of optimization problems, introduced by Ahmadi, Jungers, Parrilo and Roozbehani, by a hierarchy of nonlinear eigenproblems. To solve the latter eigenproblems, we introduce a projective version of Krasnoselskii–Mann iteration. This method is of independent interest as it applies more generally to the nonlinear eigenproblem for a monotone positively homogeneous map. Here, this method allows for scalability by avoiding the recourse to linear or semidefinite programming techniques.

[430] N. Guglielmi and M. Zennaro, *An antinorm theory for sets of matrices: Bounds and approximations to the lower spectral radius*, Linear Algebra Appl. **607** (2020), 89–117, `doi:10.1016/j.laa.2020.07.037`. MR 4137719. Zbl 07309723.

For the computation of the lower spectral radius of a finite family of matrices that shares an invariant cone, two recent papers by Guglielmi and Protasov [308] and Guglielmi and Zennaro [355] make use of so-called antinorms. Antinorms are continuous, nonnegative, positively homogeneous and superadditive functions defined on the cone and turn out to be related to the lower spectral radius of the family in a similar way as norms are related to the joint spectral radius. In this paper, we revisit the theory of antinorms in a systematic way, filling in some theoretical holes, correcting a common mistake present in the literature and adding some new properties and results. In particular, we prove that, under suitable assumptions, the lower spectral radius is characterized by a Gelfand type limit computed on an antinorm.

[431] E. Kissin, V. S. Shulman, and Yu. V. Turovskii, *From Lomonosov Lemma to radical approach in joint spectral radius theory*, The Mathematical Legacy of Victor Lomonosov, De Gruyter, Berlin, Boston, 2020, pp. 205–230, `doi:10.1515/9783110656756-015`, `arXiv:2005.02743`. Zbl 07279804.

In this paper we discuss the infinite-dimensional generalizations of the famous theorem of Berger-Wang (generalized Berger-Wang formulas) and give an operator-theoretic proof of I. Morris's theorem about coincidence of three essential joint spectral radius, related to these formulas. Further we develop Banach-algebraic approach based on the theory of topological radicals, and obtain some new results about these radicals.

[432] V. Kozyakin, *On boundedness of infinite matrix products with alternating factors from two sets of matrices*, ArXiv.org e-Print archive, October 2020, `arXiv:2010.03890`.

We consider the question of the boundedness of matrix products $A_nB_n\cdots A_1B_1$ with factors from two sets of matrices, $A_i \in \mathcal{A}$ and $B_i \in \mathcal{B}$, due to an appropriate choice of matrices $\{B_i\}$. It is assumed that for any sequence of matrices $\{A_i\}$ there is a sequence of matrices $\{B_i\}$ for which the sequence of matrix products $\{A_nB_n\cdots A_1B_1\}_{n=1}^{\infty}$ is norm bounded. Some situations are described in which in this case the norms of matrix products $A_nB_n\cdots A_1B_1$ are uniformly bounded, that is, $\|A_nB_n\cdots A_1B_1\| \le C$ for all natural numbers n, where $C > 0$ is some constant independent of the sequence $\{A_i\}$ and the corresponding sequence $\{B_i\}$. In the general case, the question of the validity of the corresponding statement remains open.

[433] B. Legat, P. Parrilo, and R. Jungers, *Certifying unstability of switched systems using sum of squares programming*, SIAM J. Control Optim. **58** (2020), no. 4, 2616–2638, `doi:10.1137/18M1173460`. MR 4142040. Zbl 07268459.

The joint spectral radius (JSR) of a set of matrices characterizes the maximal asymptotic growth rate of an infinite product of matrices of the set. This quantity appears in a number of applications including the stability of switched and hybrid systems. A popular method used for the stability analysis of these systems searches for a Lyapunov function with convex optimization tools.

We investigate dual formulations for this approach and leverage these dual programs for developing new analysis tools for the JSR. We show that the dual of this convex problem searches for the occupations measures of trajectories with high asymptotic growth rate. We both show how to generate a sequence of guaranteed high asymptotic growth rate and how to detect cases where we can provide lower bounds to the JSR. We deduce from it a new guarantee for the upper bound provided by the sum of squares lyapunov program. We end this paper with a method to reduce the computation of the JSR of low rank matrices to the computation of the constrained JSR of matrices of small dimension.

All results of this paper are presented for the general case of constrained switched systems, that is, systems for which the switching signal is constrained by an automaton.

[434] T. Mejstrik, *Algorithm 1011: Improved invariant polytope algorithm and applications*, ACM Trans. Math. Software **46** (2020), no. 3, Art. 29, 26, `doi:10.1145/3408891`, `arXiv:1812.03080`. MR 4161245.

In several papers of 2013 – 2016, Guglielmi and Protasov made a breakthrough in the problem of the joint spectral radius computation, developing the invariant polytope algorithm which for most matrix families finds the exact value of the joint spectral radius. This algorithm found many applications in problems of functional analysis, approximation theory, combinatorics, etc.. In this paper we propose a modification of the invariant polytope algorithm making it roughly 3 times faster and suitable for higher dimensions. The modified version works for most matrix families of dimensions up to 25, for non-negative matrices the dimension is up to three thousand. Besides we introduce a new, fast algorithm for computing good lower bounds for the joint spectral radius. The corresponding examples and statistics of numerical results are provided. Several applications of our algorithms are presented. In particular, we find the exact values of the regularity exponents of Daubechies wavelets of high orders and the capacities of codes that avoid certain difference patterns.

[435] R. Mohammadpour, *Lyapunov spectrum properties and continuity of the lower joint spectral radius*, ArXiv.org e-Print archive, January 2020, `arXiv:2001.03958`.

In this paper we study ergodic optimization and multifractal behavior of Lyapunov exponents for matrix cocycles. We show that the restricted variational principle holds for generic cocycles (in the sense of *C. Bonatti, X. Gómez-Mont, and M. Viana.* Généricité déxposants de Lyapunov nonnuls pour des produits déterministes de matrices. Annales de l'Institut Henri Poincare (C) Non Linear Analysis, vol. 20, Elsevier, 2003, pp. 579–624.) over mixing subshifts of finite type. We also show that the Lyapunov spectrum is equal to the closure of the set where the entropy spectrum is positive for such cocycles. Moreover, we show the continuity of the entropy spectrum at boundary of Lyapunov spectrum in the sense that $h_{top}(E(\alpha_t)) \to h_{top}(E(\beta(\mathcal{A}))$, where $E(\alpha) = \{x \in X : \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{A}^n(x)\| = \alpha\}$, for such cocycles.

We prove the continuity of the lower joint spectral radius for linear cocycles under the assumption that linear cocycles satisfy a cone condition.

[436] Yu. Nesterov and V. Yu. Protasov, *Computing closest stable nonnegative matrix*, SIAM J. Matrix Anal. Appl. **41** (2020), no. 1, 1–28, `doi:10.1137/17M1144568`. MR 4048011. Zbl 1448.65027.

Problem of finding the closest stable matrix for a dynamical system has many applications. It is well studied both for continuous and discrete-time systems, and the corresponding optimization problems are formulated for various matrix norms. As a rule, non-convexity of these formulations does not allow finding their global solutions. In this paper, we analyze positive discrete-time systems. They also suffer from non-convexity of the stability region, and the problem in the Frobenius norm or

in Euclidean norm remains hard for them. However, it turns out that for certain polyhedral norms, the situation is much better. We show, that for the distances measured in the max-norm, we can find exact solution of the corresponding nonconvex projection problems in polynomial time. For the distance measured in the operator ℓ_∞-norm or ℓ_1-norm, the exact solution is also efficiently found. To this end, we develop a modification of the recently introduced spectral simplex method. On the other hand, for all these three norms, we obtain exact descriptions of the region of stability around a given stable matrix. In the case of max-norm, this can be seen as an analogue of Kharitonov's theorem for non-negative matrices.

[437] E. Oregón-Reyes, *A new inequality about matrix products and a Berger-Wang formula*, J. École polytech. Math. **7** (2020), 185–200, `doi:10.5802/jep.114`, `arXiv:1710.00639`. MR 4054333. Zbl 07152734.

We prove an inequality relating the norm of a product of matrices $A_n \cdots A_1$ with the spectral radii of subproducts $A_j \cdots A_i$ with $1 \leq i \leq j \leq n$. Among the consequences of this inequality, we obtain the classical Berger-Wang formula as an immediate corollary, and give an easier proof of a characterization of the upper Lyapunov exponent due to I. Morris. As main ingredient for the proof of this result, we prove that for a large enough n, the product $A_n \cdots A_1$ is zero under the hypothesis that $A_j \cdots A_i$ are nilpotent for all $1 \leq i \leq j \leq n$. The main result is the following:

Theorem. *Given $d \in \mathbb{N}$, there exists an integer $N \geq 1$, and a constant $0 < \delta < 1$ such that, for every local field k and norm $\|\cdot\|$ in $M_d(k)$, there is a constant $C = C(k, d, \|\cdot\|) > 1$ satisfying the following inequality for all $A_1, \ldots, A_N \in M_d(k)$:*

$$\|A_N \cdots A_1\| \leq C \left(\prod_{1 \leq i \leq N} \|A_i\| \right) \max_{1 \leq \alpha \leq \beta \leq N} \left(\frac{\rho(A_\beta \cdots A_\alpha)}{\prod_{\alpha \leq i \leq \beta} \|A_i\|} \right)^\delta,$$

where the right hand side is treated as zero if one of the A_i is the zero matrix.

[438] J. Wang, M. Maggio, and V. Magron, *SparseJSR: A fast algorithm to compute joint spectral radius via sparse SOS decompositions*, ArXiv.org e-Print archive, August 2020, `arXiv:2008.11441`.

This paper focuses on the computation of joint spectral radii (JSR), when the involved matrices are sparse. We provide a sparse variant of the procedure proposed by Parrilo and Jadbabaie, to compute upper bounds of the JSR by means of sum-of-squares (SOS) relaxations. Our resulting iterative algorithm, called `SparseJSR`, is based on the *term sparsity* SOS (TSSOS) framework, developed by Wang, Magron and Lasserre, yielding SOS decompositions of polynomials with arbitrary sparse support. `SparseJSR` exploits the sparsity of the input matrices to significantly reduce the computational burden associated with the JSR computation. Our algorithmic framework is then successfully applied to compute upper bounds for JSR, on randomly generated benchmarks as well as on problems arising from stability proofs of controllers, in relation with possible hardware and software faults.

[439] Z. Wu and Q. He, *Optimal switching sequence for switched linear systems*, SIAM J. Control Optim. **58** (2020), no. 2, 1183–1206, `doi:10.1137/18M1197928`. MR 4091167. Zbl 07203554.

We study the following optimization problem over a dynamical system that consists of several linear subsystems: Given a finite set of $n \times n$ matrices and an n-dimensional vector, find a sequence of K matrices, each chosen from the given set of matrices, to maximize a convex function over the product of the K matrices and the given vector. This simple problem has many applications in operations research and control, yet a moderate-sized instance is challenging to solve to optimality for state-of-the-art optimization software. We propose a simple exact algorithm for this problem. Our algorithm runs in polynomial time when the given set of matrices has the *oligo-vertex* property, a concept we introduce in this paper for a set of matrices. We derive several sufficient conditions for a set of matrices to have the oligo-vertex property. Numerical results demonstrate the clear advantage of our algorithm in solving large-sized instances of the problem over one state-of-the-art global solver. We also pose several open questions on the oligo-vertex property and discuss its potential connection with the finiteness property of a set of matrices, which may be of independent interest.

[440] X. Xu and B. Acikmese, *Approximation of the constrained joint spectral radius via algebraic lifting*, IEEE Transactions on Automatic Control (2020), 1–1, (Early Access), `doi:10.1109/TAC.2020.3020580`, `arXiv:1807.03965`.

A linear constrained switching system is a discrete-time linear switched system whose switching sequences are constrained by a deterministic finite automaton. As a characterization of the asymptotic stability of a constrained switching system, the constrained joint spectral radius is difficult to compute or approximate. Using the semi-tensor product of matrices, we express dynamics of a deterministic finite automaton, an arbitrary switching system and a constrained switching system into their matrix forms, respectively, where the matrix expression of a constrained switching system can be seen as the matrix expression of a lifted arbitrary switching system. Inspired by this, we propose a lifting method for the constrained switching system, and prove that the constrained joint/generalized spectral radius of the constrained switching system is equivalent to the joint/generalized spectral radius of the lifted arbitrary switching system. Examples are provided to show the advantages of the proposed lifting method.

[441] Y. Zhang and X. Xu, *Finding trajectories with high asymptotic growth rate for linear constrained switching systems via a lift approach*, ArXiv.org e-Print archive, September 2020, `arXiv:2009.12948`.

This paper investigates how to generate a sequence of matrices with an asymptotic growth rate close to the constrained joint spectral radius (CJSR) of the constrained switching system whose switching sequences are constrained by a deterministic finite automaton. Based on a matrix-form expression, the dynamics of a constrained switching system are proved to be equivalent to the dynamics of a lifted arbitrary switching system. By using the dual solution of a sum-of-squares optimization program, an algorithm is designed to produce a sequence of matrices with an asymptotic growth rate that can be made arbitrarily close to the joint spectral radius (JSR) of the lifted arbitrary switching system, or equivalently the CJSR of the original constrained switching system. Several numerical examples are provided to illustrate the better performance of the proposed algorithm compared with existing ones.

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[442] G. Aazan, A. Girard, L. Greco, and P. Mason, *Stability of shuffled switched linear systems: A joint spectral radius approach*, HAL archives-ouvertes.fr. HAL Id: hal-03257026v1, June 2021, URL `https://hal.archives-ouvertes.fr/hal-03257026`.

A switching signal for a switched system is said to be shuffled if all modes of the system are activated infinitely often. In this paper, we develop tools to analyze stability properties of discrete-time switched linear systems driven by shuffled switching signals. We introduce the new notion of shuffled joint spectral radius (SJSR), which intuitively measures how much the state of the system contracts each time the signal shuffles (i.e. each time all modes have been activated). We show how this notion relates to stability properties of the associated switched systems. In particular, we show that some switched systems that are unstable for arbitrary switching signals can be stabilized by using switching signals that shuffle sufficiently fast and that the SJSR allows us to derive an expression of the minimal shuffling rate required to stabilize the system. We then present several approaches to compute lower and upper bounds of the SJSR using tools such as the classical joint spectral radius, Lyapunov functions and finite state automata. Several tightness results of the bounds are established. Finally, numerical experiments are presented to illustrate the main results of the paper.

[443] E. Breuillard, *On the joint spectral radius*, ArXiv.org e-Print archive, March 2021, to be submitted to a Springer volume in memory of Jean Bourgain, `arXiv:2103.09089`.

For a bounded subset S of $d \times d$ complex matrices, the Berger-Wang theorem and Bochi's inequality allow to approximate the joint spectral radius of S from below by the spectral radius of a short product of elements from S. Our goal is two-fold: we review these results, providing self-contained proofs, and we derive an improved version with explicit bounds that are polynomial in d. We also discuss other complete valued fields.

[444] E. Breuillard and C. Sert, *The joint spectrum*, J. Lond. Math. Soc. (2) **103** (2021), no. 3, 943–990, `doi:10.1112/jlms.12397`, `arXiv:1809.02404`. MR 4245827. Zbl 07360999.

We introduce the notion of *joint spectrum* of a compact set of matrices $S \subset \mathrm{GL}_d(\mathbb{C})$, which is a multi-dimensional generalization of the joint spectral radius. We begin with a thorough study of its properties (under various assumptions: irreducibility, Zariski-density, and domination). Several classical properties of the joint spectral radius are shown to hold in this generalized setting and an analogue of the Lagarias-Wang finiteness conjecture is discussed. Then we relate the joint spectrum to matrix valued random processes and study what points of it can be realized as Lyapunov vectors. We also show how the joint spectrum encodes all word metrics on reductive groups. Several examples are worked out in detail.

This paper relies extensively on colour figures. Some references to colour may not be meaningful in the printed version, and we refer the reader to the online version which includes the colour figures.

[445] C. Catalano, U. Azfar, L. Charlier, and R. M. Jungers, *A linear bound on the k-rendezvous time for primitive sets of NZ matrices*, Fundamenta Informaticae **180** (2021), no. 4, 289–314, `doi:10.3233/FI-2021-2043`.

A set of nonnegative matrices is called primitive if there exists a product of these matrices that is entrywise positive. Motivated by recent results relating synchronizing automata and primitive sets, we study the length of the shortest product of a primitive set having a column or a row with k positive entries, called its k-rendezvous time (k-RT), in the case of sets of matrices having no zero rows and no zero columns. We prove that the k-RT is at most linear w.r.t. the matrix size n for small k, while the problem is still open for synchronizing automata. We provide two upper bounds on the k-RT: the second is an improvement of the first one, although the latter can be written in closed form. We then report numerical results comparing our upper bounds on the k-RT with heuristic approximation methods.

[446] Y. Chitour, N. Guglielmi, V. Yu. Protasov, and M. Sigalotti, *Switching systems with dwell time: Computing the maximal Lyapunov exponent*, Nonlinear Anal. Hybrid Syst. **40** (2021), 101021, `doi:10.1016/j.nahs.2021.101021`, `arXiv:1912.10214`. MR 4219337.

We study asymptotic stability of continuous-time systems with mode-dependent guaranteed dwell time. These systems are reformulated as special cases of a general class of mixed (discrete-continuous) linear switching systems on graphs, in which some modes correspond to discrete actions and some others correspond to continuous-time evolutions. Each discrete action has its own positive weight which accounts for its time-duration. We develop a theory of stability for the mixed systems; in particular, we prove the existence of an invariant Lyapunov norm for mixed systems on graphs and study its structure in various cases, including discrete-time systems for which discrete actions have inhomogeneous time durations. This allows us to adapt recent methods for the joint spectral radius computation (Gripenberg's algorithm and the Invariant Polytope Algorithm) to compute the Lyapunov exponent of mixed systems on graphs.

[447] Y. Chitour, G. Mazanti, and M. Sigalotti, *On the gap between deterministic and probabilistic joint spectral radii for discrete-time linear systems*, Linear Algebra Appl. **613** (2021), 24–45, `doi:10.1016/j.laa.2020.12.013`, `arXiv:1812.08399`. MR 4192243. Zbl 07312092.

Given a discrete-time linear switched system $\Sigma(\mathcal{A})$ associated with a finite set $\mathcal{A}$ of matrices, we consider the measures of its asymptotic behavior given by, on the one hand, its deterministic joint spectral radius $\rho_{\mathrm{d}}(\mathcal{A})$ and, on the other hand, its probabilistic joint spectral radii $\rho_{\mathrm{p}}(\nu, P, \mathcal{A})$ for Markov random switching signals with transition matrix P and a corresponding invariant probability ν. Note that $\rho_{\mathrm{d}}(\mathcal{A})$ is larger than or equal to $\rho_{\mathrm{p}}(\nu, P, \mathcal{A})$ for every pair (ν, P). In this paper, we investigate the cases of equality of $\rho_{\mathrm{d}}(\mathcal{A})$ with either a single $\rho_{\mathrm{p}}(\nu, P, \mathcal{A})$ or with the supremum of $\rho_{\mathrm{p}}(\nu, P, \mathcal{A})$ over (ν, P) and we aim at characterizing the sets $\mathcal{A}$ for which such equalities may occur.

[448] T. Mejstrik and V. Yu. Protasov, *Elliptic polytopes and invariant norms of linear operators*, ArXiv.org e-Print archive, July 2021, `arXiv:2107.02610`.

We address the problem of constructing elliptic polytopes in $\mathbb{R}^d$, which are convex hulls of finitely many two-dimensional ellipses with a common center. Such sets arise in the study of spectral properties of matrices, asymptotics of long matrix products, in the Lyapunov stability, etc. The main issue in the construction is to decide whether a given ellipse is in the convex hull of others. The computational complexity of this problem is analysed by considering an equivalent optimisation problem. We show that the number of local extrema of that problem may grow exponentially in d. For $d = 2, 3$, it admits an explicit solution for an arbitrary number of ellipses; for higher dimensions, several geometric methods for approximate solutions are derived. Those methods are analysed numerically and their efficiency is demonstrated in applications.

[449] G. Panti and D. Sclosa, *The finiteness conjecture holds in* $(\mathrm{SL}_2\mathbb{Z}_{\geq 0})^2$, Nonlinearity **34** (2021), no. 8, 5234–5260, `doi:10.1088/1361-6544/ac0484`, `arXiv:2006.16899`.

Let A, B be matrices in $\mathrm{SL}_2\mathbb{R}$ having trace greater than or equal to 2. Assume the pair A, B is coherently oriented, that is, can be conjugated to a pair having nonnegative entries. Assume also that either A, B^{-1} is coherently oriented as well, or A, B have integer entries. Then the Lagarias-Wang finiteness conjecture holds for the set $\{A, B\}$, with optimal product in $\{A, B, AB, A^2B, AB^2\}$. In particular, it holds for every matrix pair in $\mathrm{SL}_2\mathbb{Z}_{\geq 0}$.

[450] Z. Wang and R. M. Jungers, *A data-driven method for computing polyhedral invariant sets of black-box switched linear systems*, IEEE Control Syst. Lett. **5** (2021), no. 5, 1843–1848, `doi:10.1109/LCSYS.2020.3044838`, `arXiv:2009.10984`. MR 4213953.

In this paper, we consider the problem of invariant set computation for black-box switched linear systems using merely a finite set of observations of system simulations. In particular, this paper focuses on polyhedral invariant sets. We propose a data-driven method based on the one step forward reachable set. For formal verification of the proposed method, we introduce the concept of almost-invariant sets for switched linear systems. The convexity-preserving property of switched linear systems allows us to conduct contraction analysis on almost-invariant sets and derive an a priori probabilistic guarantee. In the spirit of non-convex scenario optimization, we also establish a posteriori the level of violation on the computed set. The performance of our method is then illustrated by a switched system under arbitrary switching between two modes.

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