JMC1 Group project: Maps, Sets and Fractals

Jeremy Bradley and Mark Wheelhouse

Summary

This project is to be attempted in groups of 3-4. It is deliberately open-ended, however this sheet will specify numerical investigations that should be attempted as part of the project. Up to 70% of the credit for this project can be gained from correctly attempting, documenting and presenting the basic investigations below (see the Assessment section for more details). You may use any language you like to perform these experiments, but you may like to take the opportunity to learn a new environment or language that is appropriate to the task.

The outputs of this project will be a project report of up to 15 pages (this is a strict limit) and a 15-20 minute presentation on your work, to be presented at the end of the Spring Term. The project report should be typeset in LaTeX [1] and should contain *no more than* 8 pages of diagrams. You can find a simple LaTeX template file (`template.tex`) in the repository provided for this exercise. You are free to choose the format of your presentation, but it should support and build on your project report.

Important: citation and referencing

Please do not copy material from Wikipedia or other resources uncited, we will be checking both your report and presentation and you will get into trouble[1]. By all means use these on-line resources to aid your understanding (some pages are better than others) and provide you with links into other references, but when you are forming mathematical or descriptive explanations always try to use your own words. If in doubt cite the source – this applies to pictures as well as text and equations. Books, papers and articles tend to carry more weight than crowd-sourced material as it will tend to have been through a more formal process of peer-review. You will get more credit for pictures that you have generated yourself (so make it clear that you have) than ones you have imported with citation (although sometimes this is necessary). If we have doubts we may ask you to show us your code generating a particular picture. Ensure that you have written your own core algorithm code, although you can use supporting libraries. You are advised to use BibTeX [2] for your referencing and you should take a look at a service such as `mendeley.com` to help you maintain your bibliography across a group of people.

The Investigation

This project requires an investigation into the non-linear dynamics of discrete maps, leading to the exploration of a fractal landscape. This is meant to be fun, but there is some very serious maths

[1]The first instance of plagiarism during an Imperial degree can result in a written warning; the second, no matter how small, can result in exclusion – so it really is not worth it.

underpinning the field that you are now in a good position to be able to start to appreciate.

The Verhulst process

The Verhulst process is a one dimensional discrete map that was designed to model bounded population growth. It has several forms but can be written as:

$$x_{k+1} = x_k + r x_k (1 - x_k) \qquad \text{for } k \geq 0 \qquad (1)$$

where $x_k \geq 0$. It is discrete as it evolves for discrete values of k; and 1 dimensional as there is only a single evolving quantity x_k. r is a real, positive, fixed parameter of the system and its value has a dramatic effect on the dynamics of the process.

Investigate what happens to x_k for suitably large k (with an initial value of $x_0 = 0.1$) for different values of r, $0 \leq r < 2.57$. You should find that for some range of r the process converges on a fixed point (a value of x_k such that $x_k = x_{k+1}$). Whereas, for other values of r, the process eventually oscillates between 2 or more fixed-points.

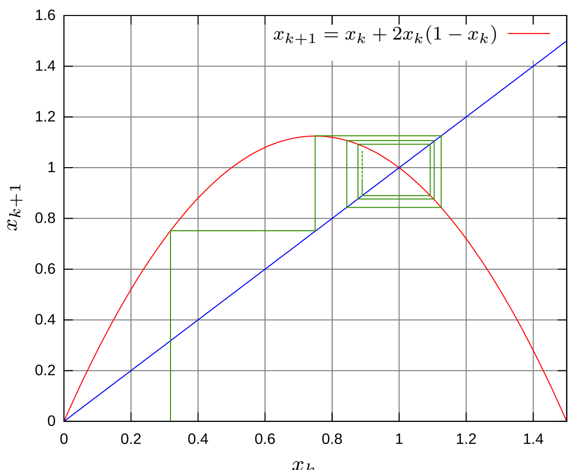

Figure 1: The Verhulst mapping function: $f(x) = x + rx(1 - x)$ for $r = 2$ and the line $y = x$. Successive iterations of the map reflect in the $y = x$ line to give the classic spider diagram (in green).

1. Plot the fixed point(s) of the Verhulst process against r for $0 \leq r < 2.57$ and find the first few r-values that capture the moment of so-called *period doubling* [3] in the number of fixed points[2].
2. Is there a relationship between the values of r that see a period-doubling in x_k?
3. Plot the fixed point(s) and then 200 or so consecutive points of x_k for large k of the process for $1 \leq r < 3$. What happens when $r > 2.57$?
4. Are the above features the same if the initial value x_0 is varied?

Other non-linear maps

Attempt a similar investigation (parts 1 - 4 above) for:

1. The logistic map: $x_{k+1} = r x_k (1 - x_k)$, $k \geq 0$
2. The trigonometric map: $x_{k+1} = r \sin x_k$, $k \geq 0$

Investigate common features between these maps and the Verhulst map. If you find any common relationship, comment on why this is surprising. Finally, pick one of the above maps and show that it displays fractal properties.

[2] We refer to this sequence later as the period-doubling cascade.

Extra: As part of the additional work in this project, you could look at some more esoteric maps, such as the Gaussian map or the Tent map. You might also like to discuss chaotic behaviour (not necessarily formally) with respect to these one dimensional processes and show how it can be applied to one of the maps in outline for a particular range of parameters. You could try to devise your own chaotic map. You could construct your own investigation of two-dimensional chaotic maps, such as the Hénon map and the Gingerbreadman Map!

The Mandelbrot set

The Mandelbrot set, which many of you will have seen before, is a set of points in the complex plane. It is based around the complex, one dimensional map:

$$z_{k+1} = z_k^2 + c \qquad \text{for } k \geq 0 \tag{2}$$

where $z_k, c \in \mathbb{C}$ and $z_0 = 0$. The Mandelbrot set is defined to be the set of points $c \in \mathbb{C}$ for which Eq. (2) converges in the limit as $k \to \infty$. This definition can be refined into a computed test that states that a point c lies within the set if:

$$|z_k| \leq 2 \qquad \text{for all } k \geq 0 \tag{3}$$

In Fig. 2, the set of such c-values is shown by the central black area of the diagram.

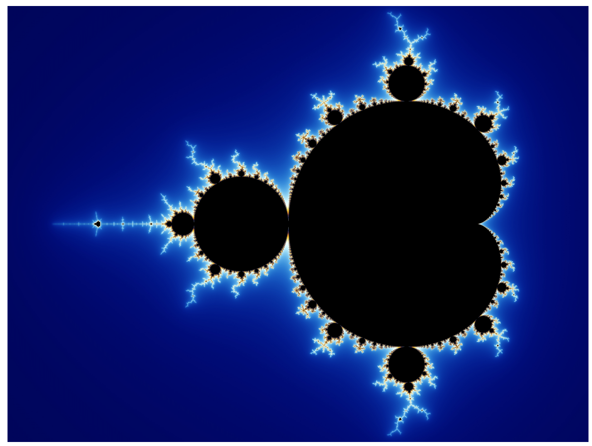

Figure 2: The famous beetle-like image of the Mandelbrot set from [4].

Attempt the following activities:

1. Reproduce Fig. 2 for $-1.2 \leq \operatorname{Im}(c) \leq 1.2$ and $-2.1 \leq \operatorname{Re}(c) \leq 0.7$ (by all means use your artistic input in selecting colours)[3]
2. Explain what the colours mean in your reproduction of the Mandelbrot set diagram
3. The Mandelbrot set is a compact set. Give a formal definition of a compact set.
4. Explain why, in Eq. (3), we restrict $|z_k| \leq 2$.
5. For each c-point, there is an associated Julia set of z-points. After researching Julia sets, pick a small number of values of c and plot the associated Julia sets.
6. Demonstrate the relationship between the sequence of real points $c_k = a_k + 0i$ in the Mandelbrot set and the period-doubling cascade a_k of the logistic map.
7. Verify numerically Boll's results (as is done in [5]) where, given that n_c is the least number of iterations of Eq. (2) required to have z_k exit the disc of radius 2, i.e. $n_c = \min(k : |z_k| > 2)$,

   (a) for $c = -0.75 + \epsilon i$ show that $\epsilon \times n_c \to \pi$ as $\epsilon \to 0$

   (b) for $c = 0.25 + \epsilon + 0i$ show that $\sqrt{\epsilon} \times n_c \to \pi$ as $\epsilon \to 0$

   Choose values of ϵ other than 10^{-n}.

[3] Note that $\operatorname{Re}(c)$ and $\operatorname{Im}(c)$ refer to the real and imaginary parts of c, respectively.

Extra: By considering the location of Boll's results relative to the Mandelbrot set, see if you can find other occurrences of π at other likely points around the Mandelbrot set.

For the related complex map:

$$z_{k+1} = z_k^d + c \qquad \text{for } k \geq 0 \tag{4}$$

Explore/plot the different types of set obtained as d becomes fractional $0 < d < 1$; as d goes negative $-2 < d < 0$; as d goes to higher powers $d \in \{3, 4, \ldots, 8\}$. Are there period-doubling cascades and Boll-style results to be found in these various more exotic sets?

Getting Started

Your first task is to set up a group of *three or four* people for the project and choose one person to be the group leader. You need to enter this information into CATe as usual, but note that you do not need to actually submit any files.

The groups created on CATe will be used to assign shared group repositories on the department's `GitLab` server for this project. Once created, you will be able to clone a copy of your group's repository into your local workspace with the following command:

```
git clone https://gitlab.doc.ic.ac.uk:80/lab1415_spring/computeralgebra_<gnum>.git
```

replacing `<gnum>` with your group number, which can be found on the `GitLab` webpages (`https://gitlab.doc.ic.ac.uk`). You should work on the files in your local workspace, making regular commits back to this repository. Your final submission will be taken from this `GitLab` repository, so make sure that you push your work to it correctly.

Submission

As you work, you should *add*, *commit* and *push* your changes to your Git repository. Your `GitLab` repository should contain all of the source code for your project.

Please include a `README` file in the top level folder indicating which report figures were generated by which code projects – and what configuration was used for each figure, if appropriate. Separate parts of the project should be kept in separate folders.

To be sure that we review the correct version of your code you will need to submit a text file to CATe containing the *revision id* that you consider to be your final solution. You should check the state of your `GitLab` repository using the `LabTS` webpages. If you click through to the `computeralgebra` exercise you'll see a list of the different versions of your work that you have pushed. Next to each commit there is a buttin that will let your download the revision ID for that commit as a `cate_token` file.

You should download the `cate_token` for the commit that you want to submit as your "final" version. As a group you should submit your project report (`report.pdf`) and your chosen CATe submission token (`cate_token.txt`) to CATe by midnight on Friday 20th March 2015.

Assessment

As discussed above, 70% of the credit for this project comes from the investigation described above. This credit is divided as follows:

- 20% will be awarded for the correctness and completeness of your numerical exploration of the Verhulst process and other non-linear maps.
- 20% will be awarded for the correctness and completeness of your numerical exploration of the Mandelbrot set.
- 20% will be awarded for delivering an effective presentation of your project – in terms of clarity of description and explanation.
- Finally, 10% will be available for a suitably aesthetic selection of pictures in the report/presentation.

The remaining 30% of the credit for this project will be awarded for extra detail that you and your group document/present. This extra material can take the form of one or more of the following:

- historical background and development of the mathematical concepts and structures
- further mathematical detail surrounding the concepts below (there is plenty)
- description of algorithmic and programming optimisations that your group has researched and implemented (you should include details of any memory or time savings as a result of using these optimisations and comparisons with some base case)

Some suggested investigations are marked by an **Extra** paragraph. but are only suggestions. You are at liberty to attempt your own variations as above.

References

[1] E. Krishnan, A. M. Shan, T. Rishi, L. A. Ajith, and C. V. Radhakrishnan, "Online tutorials on LaTeX," 2002. `http://www.tug.org/tutorials/tugindia/`.

[2] A. Feder, "How to use BibTeX," 2006. `http://www.bibtex.org/Using/`.

[3] S. Lynch, *Dynamical Systems with Applications Using Mathematica*, ch. Nonlinear Discrete Dynamical Systems. No. 12, Birkhäuser, 2007. `http://www.phy.duke.edu/~hx3/physics/map.pdf`.

[4] W. Beyer, "The Mandelbrot set," 2006. `http://en.wikipedia.org/wiki/File:Mandel_zoom_00_mandelbrot_set.jpg`.

[5] A. Klebanoff, "π in the Mandelbrot set," *Fractals*, vol. 9, no. 4, pp. 393–402, 2001. `doi://10.1142/S0218348X01000828` or `http://www.doc.ic.ac.uk/~jb/teaching/jmc/pi-in-mandelbrot.pdf`.