Assignment-2

(1) Solve the following 2nd order ODEs by the "reduction of order"
(a) $y'' = \sqrt{1-(y')^2}$
(b) $\sqrt{x}y'' = (y')^2$
(c) $yy'' + y'^2 = 2x + 1$.

(2) State whether the following subsets of $C^2(\mathbb{R})$ are linearly independent with justification
(a) $\{\sin^2(x), \sqrt{2}(1-\cos 2x)\}$
(b) $\{2x, 3x^2, 5x - 8x^2\}$
(c) $\{\sin x, \cos x, \tan x\}$.

(3) Suppose that y_1 and y_2 are linearly independent solutions of $xy'' + 2y' + x(e^x)y = 0$ and that their Wronskian $W(y_1, y_2)$ has value 2 at $x = 1$. What is the value at $x = 5$?

(4) Determine the unique solution satisfying $z'' + 4z' + 4z = \cos^2 t$ and $z(0) = 2, z'(0) = 0$.

(5) Solve the following constant coefficient linear ODE
(a) $y'' - 6y' + 5y = 0$.
(b) $y'' - 6y' + 9y = 0$.
(c) $y'' + 4y' + 5y = 0$.
(d) $y'' + 25y = 0$.

(6) Solve the following I.V.Ps
(a) $y'' - 4y = 0$, with initial conditions $y(0) = 1$ and $y'(0) = 1$.
(b) $y'' - 4y' + 4y = 0$ with $y(0) = 0$ and $y'(0) = 3$.

(7) Can the functions $y_1(x) = \sin x$ and $y_2(x) = x - \pi$, be solutions to the same second order homogeneous LINEAR ODE?

(8) Solve the non-homogeneous equation:
(a) $y'' + 16y = 2\cos^2(x)$.
(b) $y'' - 5y' + 4y = e^{4x}$.

(9) Solve the following Cauchy-Euler equations;
(i) $2x^2y'' + 3xy' - y = 0$
(ii) $x^2y'' + 5xy' + 4y = 0$
(iii) $x^2y'' + xy' + y = 0$.

(10) Prove that the general Cauchy-Euler equation

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$$

can be transformed to an equation with constant coefficients and hence write down the form of the solutions to Cauchy – Euler equations.

(11) The equation

$$x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0 \tag{0.1}$$

is a special case of a LINEAR, homogeneous ODE:

$$x^2y'' + xy' + (x^2 - p^2)y = 0$$

known as Bessel's equation of importance in applications of differential equations. In this exercise, we only consider the case $p = 1/2$. First, verify that

$$y(x) = \frac{\sin x}{\sqrt{x}},$$

is a solution to this ODE on any subinterval I of the set of all the positive reals and find the general solution.

(12) The following 2^{nd}-order LINEAR homogeneous ODE:

$$(1 - x^2)y'' - 2xy' + p(p + 1)y = 0,$$

is known as LEGENDRE's equation. In this exercise, we only consider the case $p = 1$, then $y_1(x) = x$ is an obvious solution. Find the general solution. (we shall study this more in the course later).

(13) Find the general solution of $x'' - \frac{2}{t}x' + \frac{2}{t^2}x = t\sin t$.

(14) Find the general solution of $y'' - y' - 2y = 4x^2$.

(15) (Method of undetermined coefficients) Find the general solution to the non-homogeneous ODE: $y'' + 4y = 4t^2 + 10e^{-t}$.

(16) Find the general solution of $y''' + y' = \sec(x)$ using method of variation of parameters.

(17) Find the general solution of

$$y'' + 4y' + 4y = 3e^{-2x}.$$

(18) Find the general solution of

$$y'' + 4y' + 4y = 3xe^{-2x}.$$

(19) Determine the linear ODE of the least possible order whose set of all solutions is the linear span of the functions: $1, x^2, e^x$.

(20) Find the general solution of the non-homogeneous ODE $(x-1)y'' - xy' + y = (x-1)^2$ for $x > 1$, it being given that $y_1(x) = e^x$ is a solution of the associated homogeneous equation.