

1. HEAT EQUATION

We deal here only with the simplest form of the ‘heat equation’, namely:

$$(1.1) \quad u_t = c^2 u_{xx}, \quad 0 < x < l, \quad 0 < t < \infty.$$

To briefly specify the simplest physical situation in which such an equation arises, consider a thin rod placed along the x -axis and let $u(x, t)$ represent the temperature in the rod at position x and time t . Under ideal conditions (like a perfect insulator, no external heat source, uniform rod material), one can show that temperature varies according to the heat equation (1.1) above. Here we shall deal with this equation together with the boundary conditions:

$$(1.2) \quad u(0, t) = u(l, t) = 0, \quad \forall t$$

along-with initial condition:

$$(1.3) \quad u(x, 0) = \phi(x).$$

First step: Separation of Variables. The first step consists of trying for ‘separated’ solutions to the general heat equation (1.1), of the form:

$$u(x, t) = F(x)G(t).$$

We shall impose the *homogeneous* boundary conditions a bit later. First, we get ahead with substituting into the heat equation, to get:

$$FG' = c^2 F''G.$$

Dividing both sides by $c^2 FG$, we get an equation wherein one side is a function of x -alone and the other, a function of t -alone, thereby forcing both sides to be constant; denoting this constant for some convenience by $-\lambda$, we have:

$$(5) \quad \frac{G'}{c^2 G} = \frac{F''}{F} = -\lambda.$$

Note that from the boundary conditions we have: $F(0) = F(l) = 0$.

Solving for $F(x)$. We know from the theory of eigenvalue problems, that the λ as above, cannot be negative; so, we need only consider the case when λ is positive and when it is zero. However, to keep it self-contained, we deal with all cases and split the analysis accordingly below.

Case 1: $\lambda = 0$. Then $F'' = 0 \Rightarrow F = ax + b$. Using $F(0) = F(l) = 0$ we get $a = b = 0$. So in this case we do not get any non-trivial solutions.

Case 2: $\lambda < 0$. Write $\lambda = -\mu^2$. Then

$$F'' - \mu^2 F = 0 \Rightarrow F = C_1 e^{\mu x} + C_2 e^{-\mu x}.$$

Applying $F(0) = 0$ gives $C_1 + C_2 = 0$. Applying $F(l) = 0$ leads to $C_1 e^{\mu l} + C_2 e^{-\mu l} = 0$ which forces $C_1 = C_2 = 0$. Thus, we do not get any non-trivial solutions in this case as well.

Case 3: $\lambda > 0$. Let $\lambda = \beta^2$. Then

$$(6) \quad F'' + \beta^2 F = 0.$$

The solution is

$$F(x) = A \cos \beta x + B \sin \beta x.$$

Using $F(0) = 0$ gives $A = 0$. Using $F(l) = 0$ gives $B \sin(\beta l) = 0$, hence

$$\beta = \frac{n\pi}{l}, \quad n = 1, 2, 3, \dots$$

Thus,

$$F_n(x) = \sin\left(\frac{n\pi x}{l}\right).$$

Solving for $G(t)$. From (5),

$$\frac{G'}{c^2 G} = \beta^2 \quad \Rightarrow \quad G' + c^2 \beta^2 G = 0.$$

Hence,

$$G_n(t) = B_n e^{-c^2 \beta^2 t} = B_n e^{-c^2 (n\pi/l)^2 t}.$$

Thus, separated solutions to the heat equation together with *homogeneous* boundary conditions are of the form:

$$(7) \quad u_n(x, t) = B_n \sin\left(\frac{n\pi x}{l}\right) e^{-c^2 (n\pi/l)^2 t}.$$

It must be noted that this needn't satisfy the initial condition and that, is address below next.

General Solution. First note that we may get more solutions to the heat equation plus the BCs, by taking the sum of the separated solutions:

$$(1.4) \quad u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) e^{-c^2 (n\pi/l)^2 t}.$$

We assume here that the sum converges and we can differentiate as needed to verify the heat equation term-by-term. More importantly next, we impose the initial condition $u(x, 0) = \phi(x)$ which leads to:

$$\Rightarrow \phi(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right).$$

Thus B_n 's are actually the Fourier sine coefficients:

$$(1.5) \quad B_n = \frac{2}{l} \int_0^l \phi(x) \sin\left(\frac{n\pi x}{l}\right) dx.$$

In other words, if we choose the numbers B_n in the general solution (1.4) as specified by (1.5), then we get a particular solution to the heat equation that not only satisfies the BCs but also the initial condition.

1.1. Derivation of the Heat Equation. Suppose we have this bar(or, rod) of length l around the x - axis. So $x = 0$ and $x = l$ are end of the bar. We are considering flow of heat in a uniform bar under the assumptions.

- (1) Sides of the bar are insulated so that no heat flows via them.
- (2) The loss of heat by conduction or radiation is negligible.
- (3) Homogenous material, strength and uniform cross-section.

Since our bar is thin, the temperature $u = u(x, t)$ can be considered as constant on any given cross-section and so depends on the horizontal position along x - axis. so u is a function of position x and time t .

Consider a subsection D of the bar say at x_0 . Now the total amount of heat $H = H(t)$ in D is

$$(1.6) \quad H(t) = \int_{x_0}^{x_1} c\rho u(x, t) dx$$

where, c is the specific heat of the material(or, the quantity of heat energy needed to raise the temperature of a unit quantity of the material by 1 degree of temperature) and ρ is the density of the material.

Now differentiate (1.6), we get

$$(1.7) \quad \frac{dH}{dt} = c\rho \int_{x_0}^x u_t(x, t) dx$$

Net change of heat H in D is just the state at which heat enters D the rate at which heat leaves D is

$$(1.8) \quad \frac{dH}{dt} = -ku_x(x_0, t) - (-ku_x(x_1, t))$$

where k is the thermal conductivity. Here, minus sign appears since there will be +ve flow of heat from left to right only if temperature is greater to the left of $x = x_0$. So the last equation is actually

$$(1.9) \quad \frac{dH}{dt} = ku_x(x_1, t) - ku_x(x_0, t) = k \int_{x_0}^{x_1} u_{xx}(x, t) dt$$

From (1.7), the last displayed equation becomes

$$c\rho \int_{x_0}^x u_t(x, t) dx = k \int_{x_0}^x u_{xx}(x, t) dt$$

Now differentiate w.r.t x , we obtain is

$$c\rho u_t = ku_{xx}$$

Then we get finally our heat equation

$$\boxed{u_t = \alpha^2 u_{xx}, \quad \text{where } \alpha = \sqrt{\frac{k}{c\rho}}}$$