PROBLEM SHEET 1

(1) Classify the following ODE's, by stating their order and whether they are linear or non-linear.

(i) $4\left(\frac{dy}{dx}\right)^2 + \frac{d^3y}{dx^3} = y,$

(ii) $y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^{-1} + y = 0,$

(iii) $y''(t) - y'(t) = \tan t$

(iv) $(1+y^2)\frac{d^2y}{dt^2} + t\frac{d^6y}{dx^6} + y = e^t.$

(Answers: (i) Order 3, Non-linear, (ii) Order 2, Non-linear, (iii) Order 2, Linear (iv) Order 6, Non-linear).

(2) Consider the one parameter family of circles with center at $(c, 0)$ and unit radius. The family is represented by the implicit relation

$$(x-c)^2 + y^2 = 1, \text{where } c \text{ is a real constant}$$

Determine a differential equation for which these family of curves represents an implicit solution (Ans: $(yy')^2 + y^2 = 1$)

(3) Solve $y'(t) + y(t) = 5\sin(2t)$. (Answer: $y(t) = Ce^{-t} + \sin(2t) - 2\cos(2t)$)

(4) Solve

$$\left(\frac{dy}{dx}\right)\log x \; - \; \frac{y}{x} = 0.$$

(Answer: $y(x) = c\log x$)

(5) For what values of the initial value $y(0) = y_0$ of the solution to the ODE:

$$y'(t) = y(t) + 1 + 3\sin(t),$$

does the solution remain bounded for all time? (Answer: $y_0 = -5/2$).

(6) (A first order ODE with discontinuous coefficients): There are times when the coefficient $p(x)$ in the Linear 1-st order ODE in standard/normal form:

$$\frac{dy}{dx} + p(x)y = x,$$

may not be continuous but may just have a jump discontinuity and may still have a 'reasonable' solution. This exercise is to do one example towards this. Consider the IVP where the ODE is as above with

$$p(x) = \begin{cases} 1, & \text{for } 0 \leq x \leq 2, \\ 3, & \text{for } x > 2. \end{cases}$$

and where the initial condition is $y(0) = 1$. Do the following.

(i) Find the general solution for $0 \leq x \leq 2$.

(ii) Write down the solution to the given IVP.

(iii) Write down the general solution for $x > 2$.

(iv) Now choose the constant in the general solution at part-(iii) above, so that the solutions to parts – (ii) and (iii), agree at $x = 2$. By 'patching the solutions together at $x = 2$', do we get a continuous function? a differentiable function? Does the patched-up solution 'badly' fail to solve the ODE? – sketch the graph of the solution over the interval $[0, 5]$.

(7) Solve the following separable ODE : $x'xe^t + 1 = 0$, for the function $x(t)$. What is the domain on which the solution is well-defined ?

(8) Consider the ODE: $\sqrt{t}dx + \sqrt{x}dt = 0$, for t, x varying in $\mathbb{R}^+$, the set of all strictly positive real numbers and for the unknown function $x(t)$. Show that every solution is defined only on an interval $I \subset \mathbb{R}^+$ of finite length. Next, suppose we allow non-negative valued $x(t)$ i.e., $x(t)$ is allowed to take the value 0 at some point $t_0 \in \mathbb{R}^+$. Show then that we get solutions which are defined for all $t \in \mathbb{R}^+$.

(9) Discuss the applicability of Picard's theorem to the I.V.P : $x'(t) = \sqrt{x(t)}$ with $x(1) = 0$.

(10) Find the family of all solutions to the ODE: $x'(t) = \frac{x^2+tx}{t^2}$.

(11) Solve $\dfrac{df}{dt} = f^2 + 3f - 4$.

(12) Show that $y(x) := e^{x^2} \int_0^x e^{-t^2} dt$ is a solution of the ODE $y' = 2xy + 1$.

(13) Solve the I.V.P

$$\frac{dy}{dx} = -\left(\frac{x^2 - xy + y^2}{x^2}\right)$$

with initial condition: $y(1) = 0$

(14) In the following equations, determine for which values of b, the equation is exact and solve for those values

(i) $(tx^2 + bt^2x)dt + (t + x)t^2dx = 0$

(ii) $(xe^{2xt} + t)dt + bte^{2xt}dx = 0$.

(15) Solve $y' + \frac{3}{x}y = \frac{2}{x^2}$ with initial value $y(1) = 2$.

(16) Consider the differential equation:

$$(x^3 + xy^2 - y)dx + (y^3 + yx^2 + x)dy = 0. \tag{0.1}$$

Let M, N denote the coefficients i.e., $M(x,y) = x^3 + xy^2 - y$ and $N = y^3 + yx^2 + x$. Show that:

$$\frac{\partial M}{\partial y} = 2xy - 1 \neq 2xy + 1 = \frac{\partial N}{\partial x}$$

which implies that (0.1) is not exact. However, if we rewrite equation (0.1) as

$$xdx + ydy + \frac{xdy - ydx}{x^2 + y^2} = 0$$

then show that:

$$\frac{\partial M}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial N}{\partial x}.$$

which seems to imply that (0.1) is exact. Is there a contradiction somewhere here? Explain.

(17) Show that every separable equation $y' = f(x)g(y)$ can be written as an exact equation (and verify that it is indeed exact) on any convex domain where g does not vanish. Using this rewrite $y' = xy$ as an exact equation and solve it, stating the conditions for the solution to be well-defined.

(18) A function u is said to be a harmonic function if

$$(\Delta u)(x,y) = \frac{\partial^2 u}{\partial x^2 u} + \frac{\partial^2 u}{\partial y^2} = 0$$

Show if u is harmonic then $-u_y\, dx + u_x\, dy = 0$ is an exact equation on any convex (more generally, any simply connected) domain.

(19) Show that the following first order ODE:

$$(5xy^2 - 2y)dx + (3x^2y - x)dy = 0$$

is not exact. Verify that x^3y is an integrating factor for it.

(20) Show that the following first order ODE:

$$(x - 2xy + e^y)dx + (y - x^2 + xe^y)dy = 0$$

is indeed exact and write down its solution (in implicit form as usual).

(21) Show that the first order ODE

$$(3x^2y + 2xy + y^3)dx + (x^2 + y^2)dy = 0$$

is not exact. Find an integrating factor for it and thereby solve the above ODE.

(<u>Answer</u>: The solution is implicitly given by $e^{3x}x^2y + e^{3x}y^3 = c$ where c as usual, is an arbitrary constant).

(22) Show that the first order ODE

$$x^2y^3 + x(1 + y^2)\frac{dy}{dx} = 0$$

is not exact. Find an integrating factor for it and thereby solve the above ODE.

(Hint: An integrating factor is given by $\mu(x,y) = 1/xy^3$ and the solution is implicitly given by $\frac{1}{2}x^2 - \frac{1}{2y^2} + \log|y| = c$ where c as usual, is an arbitrary constant).

(23) Find the equation of the curve through the point (1,0), whose slope at (x,y) is $\frac{y-1}{x^2+x}$. (Answer: $(y-1)(x+1)+2x=0$)

(24) Solve $x^3\frac{dy}{dx} = x^2y - 2y^3$.

(25) Find the orthogonal trajectories of the 1-parameter families (i) of parabolas $y = cx^2$, (ii) of curves given by $y = cx^5$ and (iii) of the curves given by $y^2 = cx^3$.