1. Verify the following: For $x, b \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m\times n}$,
$$\begin{aligned} D(b^\top x) &= b^\top \\ D(b^\top Ax) &= b^T A \\ D(x^T Ax) &= x^\top (A + A^\top) \quad \text{for } m = n, \end{aligned}$$
and calculate the gradient of $f : \mathbb{R}^n \to \mathbb{R}$ defined by
$$f(x) = \frac{1}{2} x^\top Qx - b^\top x$$
where $Q \in \mathbb{R}^{n\times n}$ is symmetric, positive definite.

2. Perform two iterations leading to the minimization of
$$f(x_1, x_2) = x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_1^2 + x_2^2 + 3$$
using the steepest descent method with the starting point $x^{(0)} = [0 \;\; 0]^\top$. Also, determine an optimal solution analytically.

3. Question 8.7 in textbook.

4. Find the minimizer of
$$f(x_1, x_2) = \frac{1}{2}x^\top \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} x - x^\top \begin{bmatrix} -1 \\ 1 \end{bmatrix}; \; x \in \mathbb{R}^2$$
using the conjugate direction method with
$$x^{(0)} = [0 \;\; 0]^\top \; ; \qquad d^{(0)} = [1 \;\; 0]^\top \; ; \qquad d^{(1)} = \begin{bmatrix} -\frac{3}{8} & \frac{3}{4} \end{bmatrix}^\top .$$

5. In the conjugate direction algorithm, show that
$$g^{(k+1)^\top} d^{(i)} = 0$$
for all $k, 0 \le k \le n-1$, and $0 \le i \le k$.

6. Question 10.10 in textbook.

7. The rank one (Quasi-Newton) correction is given by
$$H_{k+1} = H_k + \frac{(\Delta x^{(k)} - H_k g^{(k)})(\Delta x^{(k)} - H_k g^{(k)})^\top}{\Delta g^{(k)\top}(\Delta x^{(k)} - H_k g^{(k)})}.$$
where $\Delta g^{(k)} = \nabla f(x^{(k+1)}) - \nabla f(x^{(k)})$ and $\Delta x^{(k)} = x^{(k+1)} - x^{(k)}$.
   (a) Show that if $\Delta g^{(k)\top}(\Delta x^{(k)} - H_k g^{(k)}) > 0$, then $H_{k+1} > 0$.
   (b) Let
$$f(x_1, x_2) = x_1^2 + \frac{1}{2}x_2^2 + 3.$$
   Apply the rank one algorithm to minimize f. Use $x^{(0)} = [1 \;\; 2]^\top$ and $H_0 = I_2$.

8. Show that for $0 < p, q < 1$,
$$p \log \frac{p}{q} + (1-p) \log \frac{1-p}{1-q} \ge 2(p-q)^2. \qquad (\log \text{ to the base } e)$$
Hint: Show that the partial derivative of the function
$$f(p, q) = p \log \frac{p}{q} + (1-p) \log \frac{1-p}{1-q} - 2(p-q)^2$$
with respect to q is negative for $q \le p$. The result follows since $f(p, p) = 0$.