1. Laplace Equation

If a heat or wave process is stationary (independent of time), then $u_t = 0$ and $u_{tt} = 0$. Then, the known heat and wave equations (in one dimension) boil down to

$$u_{xx} = 0 \Longrightarrow u(x) = A + Bx, \text{ where } A, B \in \mathbb{R}.$$

The Laplace equation in two dimensions is

$$\Delta u = u_{xx} + u_{yy} = 0, \text{ where } u : \mathbb{R}^2 \to \mathbb{R}$$

A solution to the Laplace equation is called **a harmonic function**.

1.1. Laplace Equation in Cartesian coordinate.

Consider the problem in two variables with boundary conditions :

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < L, ; 0 < y < H, \\ u(0, y) = u(L, y) = 0, \\ u(x, 0) = f(x), \ u(x, H) = 0, \end{cases} \tag{1.1}$$

Let the solution be of the form $u(x, y) = X(x)Y(y)$. Then, from the separation of variables, we get

$$\begin{aligned} X''Y + Y''X &= 0 \\ \Longrightarrow -\frac{X''}{X} &= \frac{Y''}{Y} = \lambda \\ \Longrightarrow X'' + \lambda X = 0 &\text{ and } Y'' - \lambda Y = 0 \end{aligned}$$

From the boundary conditions $u(0, y) = u(L, y) = 0$, we have $X(0) = X(L) = 0$. Then, this is our usual eigenvalue problem. Solving this for $\lambda > 0, \lambda = \mu^2$, we get

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \Rightarrow \mu = \frac{n\pi}{L}$$

The general solution for Y is $Y(y) = C_1 e^{\sqrt{\lambda} y} + C_2 e^{-\sqrt{\lambda} y}$. Since $u(x, H) = 0$, it follows that $Y(H) = 0 \Rightarrow C_1 e^{\sqrt{\lambda} H} + C_2 e^{-\sqrt{\lambda} H} = 0 \Rightarrow C_2 = -C_1 e^{2\sqrt{\lambda} H}$. Hence, we get

$$\begin{aligned} Y(y) &= C_1 e^{\sqrt{\lambda} y} - -C_1 e^{2\sqrt{\lambda} H} e^{-\sqrt{\lambda} y} \\ &= C_1 e^{\sqrt{\lambda} H} \left(e^{-\sqrt{\lambda} H} e^{\sqrt{\lambda} y} - e^{\sqrt{\lambda} H} e^{-\sqrt{\lambda} y} \right) \\ &= C_1 e^{\sqrt{\lambda} H} \left(e^{-\sqrt{\lambda}(H-y)} - e^{\sqrt{\lambda}(H-y)} \right) \\ &= -2C_1 e^{\sqrt{\lambda} H} \sinh \sqrt{\lambda}(H - y) \end{aligned}$$

Thus, the main solution is

$$u_n(x, y) = \sin \frac{n\pi x}{L} \sinh \frac{n\pi(H - y)}{L}, \ n = 1, 2, \cdots$$

These u_n's satisfy the Laplace equation and the 3 homogeneous conditions as outlined in the problem. The remaining condition $u(x, 0) = f(x)$ still needs to be satisfied.

Write the general solution as a linear combination of the product solutions

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi(H-y)}{L}\right) \tag{1.2}$$

Now, using that initial condition, the last equation becomes,

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi H}{L}\right) = \sum_{n=1}^{\infty} \left(a_n \sinh\left(\frac{n\pi H}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right) \tag{1.3}$$

Recognize this as a Fourier sine series; the Fourier coefficients are

$$a_n \sinh\left(\frac{n\pi H}{L}\right) = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x$$

and hence

$$a_n = \frac{2}{L\sinh\left(\frac{n\pi H}{L}\right)} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x$$

1.2. **Laplace equation in Polar coordinate.** Take $x = r\cos(\theta), y = r\sin(\theta)$ and $u = u(x,y)$ where u satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1.4}$$

Now, differentiate u with respect to x consecutively one and two times.

$$\begin{aligned} u_x &= u_r r_x + u_\theta \theta_x, \\ u_{xx} &= (u_r r_x)_x + (u_\theta \theta_x)_x = (u_r)_x r_x + u_r r_{xx} + (u_\theta)_x \theta_x + u_\theta \theta_{xx} \\ &= (u_{rr} r_x + u_{r\theta}\theta_x) r_x + u_r r_{xx} + (u_{\theta r} r_x + u_{\theta\theta}\theta_x)\theta_x + u_\theta \theta_{xx}. \end{aligned} \tag{1.5}$$

Since $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$. Now, we have $r_x = \dfrac{x}{\sqrt{x^2+y^2}} = x/r$ and

$$\theta_x = \frac{1}{1+(\frac{y}{x})^2} \times (-\frac{y}{x^2}) = -\frac{y}{r^2}.$$

Then,

$$r_{xx} = \frac{r - x r_x}{r^2} = \frac{1}{r} - \frac{x^2}{r^3} = \frac{r^2 - x^2}{r^3} = \frac{y^2}{r^3} \text{ and } \theta_{xx} = -y(-\frac{2}{r^3}) r_x = \frac{2xy}{r^4}.$$

Hence, putting all these values in the equation (1.5), we arrive at

$$\begin{aligned} u_{xx} &= u_{rr}\frac{x^2}{r^2} + u_{r\theta}\left(-\frac{y}{r^2}\right) \times \frac{x}{r} + u_r\left(\frac{y^2}{r^3}\right) + u_{\theta r}\left(\frac{x}{r}\right) \times \left(-\frac{y}{r^2}\right) + u_{\theta\theta}\left(\frac{y^2}{r^4}\right) + u_\theta\left(\frac{2xy}{r^4}\right) \\ &= u_{rr}\left(\frac{x^2}{r^2}\right) + u_{r\theta}\left(-\frac{2xy}{r^3}\right) + u_r\left(\frac{y^2}{r^3}\right) + u_{\theta\theta}\left(\frac{y^2}{r^4}\right) + u_\theta\left(\frac{2xy}{r^4}\right) \quad \text{(since } u_{r\theta} = u_{\theta r}\text{)} \end{aligned} \tag{1.6}$$

Now, we have $r_y = \dfrac{y}{\sqrt{x^2+y^2}} = \dfrac{y}{r}$ and $\theta_y = \dfrac{1}{1+(\frac{y}{x})^2} \times \frac{1}{x} = \dfrac{x}{r^2}$.

Then,

$$r_{yy} = \frac{r - ur_y}{r^2} = \frac{1}{r} - \frac{y^2}{r^3} = \frac{r^2 - y^2}{r^3} = \frac{x^2}{r^3} \quad \text{and} \quad \theta_{yy} = -\frac{2x}{r^3} \times r_y = -\frac{2xy}{r^4}.$$

So, just like in the previous case, putting all these values in (1.5) with respect to y, we get

$$\begin{aligned} u_{yy} &= u_{rr}\left(\frac{y^2}{r^2}\right) + u_{r\theta}\left(\frac{x}{r^2}\right)\left(\frac{y}{r}\right) + u_r\left(\frac{x^2}{r^3}\right) + u_{\theta r}\left(\frac{y}{r}\right)\left(\frac{x}{r^2}\right) + u_{\theta\theta}\left(\frac{x^2}{r^4}\right) + u_\theta\left(-\frac{2xy}{r^4}\right) \\ &= u_{rr}\left(\frac{y^2}{r^2}\right) + u_{r\theta}\left(\frac{2xy}{r^3}\right) + u_r\left(\frac{x^2}{r^3}\right) + u_{\theta\theta}\left(\frac{x^2}{r^4}\right) + u_\theta\left(-\frac{2xy}{r^4}\right). \end{aligned} \tag{1.7}$$

Combining (1.6) and (1.7), and then substituting in (1.4), we obtain

$$0 = u_{xx} + u_{yy} = u_{rr}\left(\frac{x^2+y^2}{r^2}\right) + u_r\left(\frac{x^2+y^2}{r^3}\right) + u_{\theta\theta}\left(\frac{x^2+y^2}{r^4}\right).$$

Hence, we obtain the expression for the Laplace equation in polar coordinates as

$$\boxed{\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.}$$

1.3. **Solving Laplace equation in a Disk.** Consider the following Dirichlet problem on a circle as

$$u_{xx} + u_{yy} = 0 \text{ for } x^2+y^2 < a^2, \tag{1.8}$$

$$u = h(\theta) \text{ for } x^2+y^2 = a^2 \tag{1.9}$$

with radius a and boundary data.

Solution. We separate variables in polar coordinates as $u(r,\theta) = F(r)G(\theta)$, i.e.

$$\begin{aligned} 0 = u_{xx} + u_{yy} &= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \\ &= F''G + \frac{1}{r}F'G + \frac{1}{r^2}FG'' \end{aligned}$$

Dividing by FG and multiplying by r^2, we obtain

$$\begin{aligned} r^2\frac{F''}{F} + r\frac{F'}{F} + \frac{G''}{G} &= 0 \\ \implies r^2\frac{F''}{F} + r\frac{F'}{F} = -\frac{G''}{G} &= \lambda \text{ (say)} \end{aligned}$$

Thus,

$$G'' + \lambda G = 0 \tag{1.10}$$

$$r^2F'' + rF' - \lambda F = 0 \tag{1.11}$$

For G, we naturally consider periodic boundary conditions. So, let us consider

$$\begin{cases} G'' + \lambda G = 0 \\ G(-\pi) = G(\pi) \\ G'(-\pi) = G'(\pi) \end{cases} \tag{1.12}$$

Now, let us find the eigenvalues and eigenfunctions of this system.

Case-1: If $\lambda > 0, \lambda = \beta^2$, then $G'' + \beta^2 G = 0$. The general solution is

$$G = C\cos(\beta\theta) + D\sin(\beta\theta).$$

Since, $G(-\pi) = G(\pi)$,

$$C\cos(-\beta\pi) + D\sin(-\beta\pi) = C\cos(\beta\pi) + D\sin(\beta\pi) \Rightarrow 2D\sin(\beta\pi) = 0.$$

So, either $D = 0$ or $\sin(\beta\pi) = 0 \Rightarrow \beta = n(\in \mathbb{Z})$. Similarly, $G'(-\pi) = G'(\pi)$, which implies

$$\begin{aligned} -C\beta\sin(-\beta\pi) + D\beta\cos(\beta(-\pi)) &= -C\beta\sin(\beta\pi) + D\beta\cos(\beta\pi) \\ \Rightarrow 2C\beta\sin(\beta\pi) &= 0 \\ \Rightarrow C = 0 \text{ or } \beta &= n \end{aligned} \tag{1.13}$$

Then, the eigenvalues are $\lambda_n = n^2$, and the eigenfunctions are

$$G_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta), \quad n = 1, 2, \cdots.$$

Case 2: If, $\lambda = 0$, then $G'' = 0 \Rightarrow G(\theta) = C + D\theta$. Now, as $G(-\pi) = G(\pi)$, $C - D\pi = C + D\pi \Rightarrow D = 0$. Therefore, in this case, 0 is an eigenvalue with eigenfunction $G(\theta) = C_0($ constant$)$.

Case 3: If $\lambda < 0$ means $\lambda = -\beta^2$ for some $\beta > 0$., So we have $G'' - \beta^2 G = 0$. And the general solution is $G(\theta) = C_1 e^{\beta\theta} + C_2 e^{-\beta\theta}$. Using the boundary conditions, we obtain

$$\begin{aligned} C_1 e^{-\beta\pi} + C_2 e^{\beta\pi} &= C_1 e^{\beta\pi} + C_2 e^{-\beta\pi} \\ \Longrightarrow C_1 &= C_2 \end{aligned} \tag{1.14}$$

Again from $G'(-\pi) = G'(\pi)$, we get

$$\begin{aligned} C_1\beta e^{-\beta\pi} - C_2\beta e^{\beta\pi} &= C_1\beta e^{\beta\pi} - C_2\beta e^{-\beta\pi} \\ \Longrightarrow C_1 &= -C_2 \end{aligned} \tag{1.15}$$

From (1.14) and (1.15), $C_1 = C_2 = 0$ Thus, no eigenvalues and the final solutions of (1.12) are given by

$$G_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta), \; n = 0, 1, 2, \cdots$$

Now, $r^2F'' + rF' - \lambda_n F = 0$ (it's a Cauchy Euler equation) boils down to

$$\begin{aligned} m(m-1) + m - \lambda_n &= 0 \\ \Longrightarrow m^2 - n^2 &= 0 \\ \Longrightarrow m &= \pm n \end{aligned}$$

So, we get $F_n(r) = Cr^n + Dr^{-n}, n = 1, 2, \cdots$ and hence the overall solution is

$$u_n(r,\theta) = F_n(r)G_n(\theta) = (C_n r^n + D_n r^{-n})(A_n\cos(n\theta) + B_n\sin(n\theta)), n = 1, 2, \cdots \tag{1.16}$$

But when $n = 0$, then we already have $G_0(\theta) = \text{constant}$, but what is $F_0(r)$. Then actually we get a reduced equation which is

$$\begin{aligned} r^2 F'' + rF' &= 0 \\ \implies m(m-1) + m &= 0 \\ \implies m &= 0 \end{aligned} \tag{1.17}$$

Hence, $F_0(r) = C + D\log(r)$, when $n = 0$. And the full solution then is

$$u_0(r,\theta) = F_0(r)G_0(\theta) = (C + D\log(r))C_0 \tag{1.18}$$

All these solutions (1.16), (1.18) are harmonic functions in the disk, except that half of them are not bounded at the origin ($r = 0$). Note we have only used boundary conditions on the variable θ but not in the variable r. So we find that the finiteness (boundedness) of the solution at $r = 0$ serves as our boundary, so we reject $\log(r)$ and r^{-n}. Now, summing the remaining solutions we get

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos(n\theta) + B_n \sin(n\theta)) \tag{1.19}$$

Now we use the boundary condition $r = 0$ that is $u(a,\theta) = b(\theta)$

$$\implies b(\theta) = A_0 + \sum_{n=1}^{\infty} a^n (A_n \cos(n\theta) + B_n \sin(n\theta)) \tag{1.20}$$

This is precisely the full Fourier series for $h(\theta)$. Therefore, the Fourier coefficients are

$$A_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi)\cos(n\phi)\,\mathrm{d}\phi \tag{1.21}$$

$$B_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi)\sin(n\phi)\,\mathrm{d}\phi \tag{1.22}$$

Therefore, (1.20) and (1.21) constitute a full solution to our problem.

□

1.4. **Laplacian in Spherical coordinate.** Let $x = r\cos(\theta)\sin(\phi), y = r\sin(\theta)\sin(\phi), z = r\cos(\phi)$. Now, by the chain rule,

$$\begin{aligned} 0 = \nabla^2 u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\ &= \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi} + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} \\ &= \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial u}{\partial \phi}\right)\right] \end{aligned}$$

Hence, the Dirichlet problem in spherical coordinates is

$$\begin{cases} \nabla^2 u = \dfrac{1}{r^2}\left[\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{\sin\phi}\dfrac{\partial}{\partial \phi}\left(\sin\phi\dfrac{\partial u}{\partial \phi}\right)\right] \\ u(r,\phi) = f(\phi) \\ \lim_{r\to\infty} u(r,\phi) = 0 \end{cases} \tag{1.23}$$

This PDE is taken by assuming that the solution is independent of θ, because the Dirichlet problem is independent of θ. This may be an electrostatic potential (or, $f(\phi)$) at which the sphere $S : r = R$ is kept.

Method : $u(r, \phi) = G(r)H(\phi)$. By the usual argument, we obtain

$$r^2 \frac{d^2G}{dr^2} + 2r\frac{dG}{dr} - kG = 0 \tag{1.24}$$

$$\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dH}{d\phi}\right) + kH = 0 \tag{1.25}$$

for $k = nG + 1$. Now, solutions for the (1.24) are $G_n(r) = r^n$. For the second equation, if we take $\cos\phi = \omega$, then $\sin^2\phi = 1 - \omega^2$. Then,

$$\frac{dH}{d\phi} = \frac{dH}{d\omega}\frac{d\omega}{d\phi} = -\sin\phi\frac{dH}{d\omega}$$

So, the equation (1.20) becomes

$$\frac{d}{d\omega}\left[(1-\omega^2)\frac{dH}{d\omega}\right] + \eta(n) = 0$$

which is nothing but **Legendre's equation** with $H = \mathcal{P}_n(\omega) = \mathcal{P}_n(\cos\phi)$. And the general solutions are the following :

(a) $u_n(r, \phi) = A_n r^n \mathcal{P}_n(\cos\phi)$

(b) $u_n^\alpha(r, \phi) = \dfrac{B_n}{r^{n+1}}\mathcal{P}_n(\cos\phi)$