1. Wave Equation

Here we deal with the wave equation given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1.1}$$

where c is some (non-zero) constant. Such an equation arises in describing the vibrations of an elastic string. If the length of the string is L and the string is tied/fixed at $x = 0$ and $x = L$, we have the homogeneous boundary conditions [1]

$$u(0,t) = 0 = u(L,t) \tag{1.2}$$

for each fixed t. We may determine the shape of the (wave in the) string given by $u(x,t)$, once we know the initial deflection and initial velocity given by $\varphi(x)$ and $\psi(x)$ respectively. This means that at $t = 0$, we have the initial conditions [2]

$$\begin{cases} u(x,0) &= \varphi(x) \\ u_t(x,0) &= \psi(x). \end{cases} \tag{1.3}$$

Statement of the problem: The problem now is to determine $u = u(x,t)$ satisfying not only the (general) wave equation (1.1) but also the boundary conditions (1.2) and the initial conditions (1.3).

Essence of the method to be used: As a first step to determine such an u, we identify simple solutions to the general equation (1.1) of the form:

$$u(x,t) = F(x)G(t). \tag{1.4}$$

As u is a product of functions depending only on one variable, the method is called 'separation of variables'. We need to identify F, G such that (1.1) is satisfied. There will be infinitely many such F, G but it may well happen the corresponding u as in (1.4) may not *also* satisfy BCs and ICs. Towards that we first note that a linear combination of such product solutions is again a solution; in-fact we could even take 'infinite' linear combinations. This will lead to fairly general solutions out of which, we extract out the function that satisfies both the wave equation as well as the BCs and ICs.

Step-1: Substituting u as in (1.4) into the wave equation, we get:

$$F(x)G''(t) = c^2 F''(x)G(t).$$

If we rewrite this as:

$$\frac{1}{c^2}\left(\frac{G''(t)}{G(t)}\right) = \frac{F''(x)}{F(x)},$$

the LHS is a function of t-alone whereas the RHS is a function of x-alone. As x, t are independent variables, the only way this can happen is that both sides reduce to constant. If we denote this constant by $-\lambda$ for convenience, we have

$$\frac{1}{c^2}\left(\frac{G''(t)}{G(t)}\right) = \frac{F''(x)}{F(x)} = -\lambda.$$

[1] 'Boundary Conditions' will be abbreviated as BCs

[2] 'Initial Conditions' will be abbreviated as ICs

It follows that we have:

$$F'' + \lambda F \ = \ 0 \tag{1.5}$$

$$G'' + c^2 \lambda G \ = \ 0. \tag{1.6}$$

That is, the general wave equation PDE (1.1) has been split up into a pair of ODEs as above.

We now ask what do the BCs and ICs translate to? For that, just substitute $u = F(x)G(t)$ into those conditions, to get

$$\begin{aligned} u(0,t) = F(0)G(t) \ &= 0, \\ u(L,t) = F(L)G(t) \ &= 0. \end{aligned}$$

As $u \not\equiv 0$, G cannot be identically zero. Therefore, the above equations give

$$F(0) = 0 = F(L). \tag{1.7}$$

We may want to also rewrite the other pair of conditions namely the initial conditions in-terms of F, G. But we shall postpone that, to first: determine F satisfying (1.5) **along-with** the *homogeneous* boundary conditions (1.7). Noting that the this is one of the standard eigenvalue problems, which we quickly recap here. We split the analysis into 3 cases depending on the sign of λ, to determine which signs and values are possible.

Case-1: $\lambda = 0$.
If $\lambda = 0$, ODE (1.5) reduces to $F'' = 0$ whose solutions are only linear functions $F(x) = ax + b$. The BCs (1.7) then forces both $a, b = 0$, thereby $F = 0$ (and $u = 0$ as well) i.e., no *non-trivial* solutions.

Case-2: $\lambda < 0$.
Write $\lambda = -\mu^2$, so that it is visible that λ is negative. The (1.5) then reads: $F'' - \mu^2 F = 0$. Its characteristic polynomial $m^2 - \mu^2 = 0$ has roots $\pm\mu$. The BCs (1.7) then give the following simultaneous equations for the numbers A, B:

$$\begin{aligned} F(0) = A + B \ &= 0 \\ F(L) = Ae^{\mu L} + Be^{-\mu L} \ &= 0 \end{aligned}$$

Multiplying the first throughout by $e^{-\mu L}$ and subtracting from the second gives

$$A(e^{\mu L} - e^{-\mu L}) \ = \ A(e^{2\mu L} - 1) \ = 0.$$

As $\mu, L \neq 0$, we get $A = 0$. The equation $A + B = 0$ then forces $B = 0$ as well.

Conclusion so far: No non-trivial solutions in case λ is negative (and in case $\lambda = 0$). So, the only possibility is the next case.

Case-3: λ is positive, which we write as β^2. The equations (1.5) and (1.6) then have the following general solutions:

$$\begin{aligned} F(x) &= C\cos(\beta x) + D\sin(\beta x) \\ G(t) &= A\cos(\beta ct) + B\sin(\beta ct). \end{aligned}$$

The first of the BCs in (1.7) imply $C = 0$ while the second gives $D\sin(\beta L) = 0$ which gives: $\beta L = n\pi$. So $\beta = n\pi/L$. This means that the eigenvalues for the BVP for F are precisely:

$$\lambda_n \;:=\; \left(\frac{n\pi}{L}\right)^2. \tag{1.8}$$

The corresponding eigenfunctions are given by

$$F_n(x) := \sin\left(\frac{n\pi x}{L}\right)$$

for $n \in \mathbb{N}$. Therefore, there are an infinite number of separated solutions of (1.1) alongwith (1.2) for each n. They are

$$u_n(x,t) \;=\; \left(A_n\cos\frac{n\pi ct}{L} \;+\; B_n\sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L}, \tag{1.9}$$

where A_n, B_n are arbitrary constants. Call the function of t appearing within the braces as $G_n(t)$ and the remaining function of x as $F_n(x)$. While this is for non-negative integers n, the same holds for negative n as well, except for a sign change before the sine term. Now, notice that any finite sum

$$u(x,t) \;=\; \sum_{n=1}^{N}\left(A_n\cos\frac{n\pi ct}{L} \;+\; B_n\sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L}$$

is also a solution of (1.1) alongwith the *homogeneous* boundary conditions (1.2) but may **NOT** be a solution which satisfies the initial conditions (1.3) as well – for instance, it may well happen that the *given* functions φ, ψ are not 'trigonometric polynomials' i.e., a (finite) linear combination of sines and cosines as above [3]. However, theorems of Fourier series guarantee that in fairly great degree of generality i.e., under 'mild hypothesis' on the functions φ, ψ, they can be represented by such trigonometric expressions if we allow the sum to be infinite; Fourier series to be a bit more precise. In order to determine a solution which satisfies all the conditions (so, ICs in particular), we are therefore led to first consider the infinite series:

$$u(x,t) \;=\; \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi ct}{L} \;+\; B_n\sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L}. \tag{1.10}$$

Here we need to determine A_n, B_n such that the ICs (1.3) are in particular satisfied. Assuming term-by-term differentiation is valid, we have by differentiating the above with

[3] Before all this, we may ask: why must we seek solutions any more complex than $F(x)G(t)$ at all? Note that the first one of the initial conditions namely, $u(x,0) = \varphi(x)$ gives $F(x)G(0) = \varphi(x)$ while the second gives $F(x)G'(0) = \psi(x)$. So, as soon as $G(0), G'(0)$ are non-zero, this means that solution of the form $F(x)G(t)$ will necessarily *fail* to exist as soon as φ and ψ are not the same except for a (multiplicative) constant.

respect to t that:

$$u_t(x,t) \;=\; \sum_{n=1}^{\infty} \frac{n\pi c}{L}\left(B_n \cos\frac{n\pi ct}{L} \;-\; A_n \sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L}.$$

Note that with the above expressions for $u(x,t)$ and its t-partial derivative, the ICs lead (using $\sin(0)=0, \cos(0)=1$) to the following equalities:

$$\varphi(x) \;=\; \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{L}), \tag{1.11}$$

$$\psi(x) \;=\; \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin(\frac{n\pi x}{L}). \tag{1.12}$$

This means that the A_n's that we must choose in (1.10) to arrive at the solution to the wave equation satifying all BCs and ICs, are precisely the *Fourier coefficients* of φ i.e.,

$$A_n \;=\; \frac{1}{L/2}\int_0^L \varphi(x)\sin\frac{n\pi x}{L}dx. \tag{1.13}$$

Likewise, the B_n's are precisely given by

$$B_n \;=\; \frac{2}{n\pi c}\int_0^L \psi(x)\sin\frac{n\pi x}{L}dx. \tag{1.14}$$

Conclusion: The solution to the wave equation satisfying all the BCs and ICs is given by (1.10) wherein the numbers A_n, B_n are precisely given by (1.13) and (1.14) respectively.