1. An engineer is designing the curved support surface of a new hanging bridge. The shape of this surface is modelled by the equation:
$$f(x, y) = x^2 + y^2 - 4x - 6y + 13.$$
Here, $f(x, y)$ denotes the height above the river at the point (x, y). According to the construction plan, the lowest point of the surface must be at least 2 meters above the river level to allow clearance for boats. Determine whether this design meets the clearance requirement. If it does, by how much does it exceed the required height? If not, by how much does it fall short? In either case, adjust the design equation (e.g., shift it upward/downward) so that it exactly meets the 2-meter clearance.

2. Consider the space $\mathbb{R}^3$. The intersection of the planes defined by
$$2x_1 + x_2 + 2x_3 = 9$$
and
$$5x_1 + 5x_2 + 7x_3 = 29$$
is a line. To identify the point on this line that is closest to the origin in Euclidean distance, we use two different methods.

   (i) All the points on the line satisfy the two equations – write the system of two equations in the form $Ax = b$ and find the solution x^* with the smallest norm.

   (ii) Use the method of Lagrange multipliers to find the minimum value of the square of the Euclidean norm of (x_1, x_2, x_3), namely,
$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2,$$
subject to the constraints given by the equations of the two planes. Verify that (x_1^*, x_2^*, x_3^*) you obtain is indeed the minimizer.

3. A mountain climber at the summit of a mountain wants to descend to a lower altitude as fast as possible. The altitude of the mountain is given approximately by
$$h(x, y) = 3000 - \frac{1}{10000}(5x^2 + 4xy + 2y^2) \text{ meters}$$
where x, y are horizontal coordinates on the earth (in meters), with the mountain summit located above the origin. In thirty minutes, the climber can reach any point (x, y) on a circle of radius 1000 m. What point should she travel to in order to get as far down as possible in 30 minutes?

4. A company manufactures x units of one item and y units of another. The total cost C of producing these two items is approximated by the function
$$C = 5x^2 + 2xy + 3y^2 + 800.$$
If the production quota for the total number of items (both types combined) is 39, find the minimum production cost.

5. Identify the points at the intersection of the curves
$$z^2 = x^2 + y^2$$
and
$$x - 2z = 3$$
that are, respectively, the closest to and the farthest from the origin in Euclidean distance.

6. Using the Lagrange (SOSC) conditions, show that the solution to the problem:

$$\min \frac{1}{2}\|x\|^2 \text{ subject to } Ax = b \ ; \qquad A \in \mathbb{R}^{m\times n}, m \leq n, \text{rank } A = m,$$

is given by

$$x^* = A^\top (AA^\top)^{-1} b.$$

7. Solve using the Lagrange multiplier method:

   i. $\min 2x_1^2 + x_2^2$ subject to $x_1 + x_2 = 1$.

   ii. $\min x_1^2 + x_2$ subject to $x_1^2 - x_2^2 = 1$.

   iii. $\min\ 2x_1^2 + x_1x_2 + x_2^2 + 500$ subject to $x_1 + x_2 = 200$.

8. **(KKT conditions)**

   i. Write the KKT conditions for $\max c^\top x$ subject to $Ax \leq b, x \geq 0$; $\qquad A \in \mathbb{R}^{m\times n}$.

   ii. Solve $\min x_1^2 + x_2^2 + 60x_1$ subject to $x_1 \geq 80, x_1 + x_2 \geq 120$.

   iii. Given $p_1, \ldots, p_n \geq 0$ such that $\sum_{i=1}^n p_i = 1$, solve

$$\min \sum_{i=1}^n p_i \ell_i \text{ subject to } \sum_{i=1}^n 2^{-\ell_i} \leq 1, \ell_i \geq 0.$$