1. Find necessary and sufficient conditions on the reals a and b under which the linear program

$$\begin{aligned}\text{minimize } & ax_1 + bx_2 + cx_3 \\ \text{subject to } & \\ & x_1 + 2x_2 + 3x_3 \geq 4 \\ & x_i \geq 0,\ i = 1, 2, 3.\end{aligned}$$

   (a) infeasible (b) unbounded (c) has a unique optimal solution

2. Write the dual of the problem in the question above. For what values of a and b is the dual

   (a) infeasible (b) unbounded (c) has a unique optimal solution

3. Solve using the simplex tableau:

$$\begin{aligned}\max\ & x_1 + 4x_2 \\ \text{subject to } & 2x_1 + x_2 \leq 3 \\ & 3x_1 + 5x_2 \leq 9 \\ & x_1 + 3x_2 \leq 5 \\ & x \geq 0.\end{aligned}$$

4. Question 15.11 in textbook.

5. State and prove the *if and only if* cases of the complementary slackness condition.

6. Derive the dual of the problem:

$$\min c^\top x \text{ subject to } Ax \leq b.$$

   You may use the standard definition of dual.

7. Let $P \in \mathbb{R}^{n\times n}$ be a matrix with the property that each element is in the real interval $[0, 1]$, and the sum of the elements of each row is equal to 1; such a matrix is called a *stochastic matrix*. Now consider a vector $x > 0$ such that $x^\top e = 1$, where $e = [1\ \cdots\ 1]^\top$; call such a vector x a *probability vector*.

   i. Consider the primal LP:

$$\max x^\top e \text{ subject to } x^\top P = x^\top, x \geq 0.$$

   Write down the dual of this problem.

   ii. Show that the dual is infeasible.

   iii. Is the primal feasible/unbounded?

   iv. Use iii. to deduce the result: $\exists$ a vector $x > 0$ such that $x^\top P = x^\top$ and $x^\top e = 1$.

8. Consider the LP:

$$\begin{aligned}\min\ & c^\top x \\ \text{subject to } & Ax \geq b, \\ & x \geq 0.\end{aligned}$$

   Recall that we convert this LP into standard form by introducing slack variables. Are the two minimums the same? Prove.