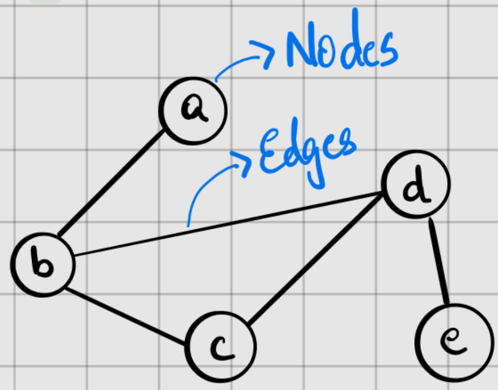
GRAPH THEORY

Graph:-



* Applications of Graphs:-

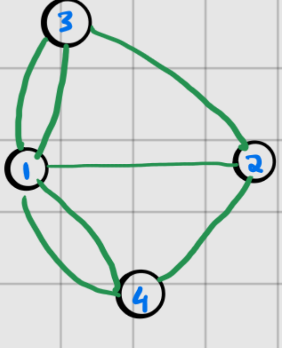
1. Computer Network (CS)
2. Transportation (CE)
3. Electrical Circuits (EE)
4. Hydraulic/Pneumatic Circuits (ME)
5. Social Network } (DS)
6. Neural Network } (DS)
7. Molecular Structure (CHE)
8. Ferro Magnetism (PHY)
9. Protein interaction } (BIO)
10. Disease Spreading } (BIO)
11. Railway Ticketing System
12. Maps (Routing).

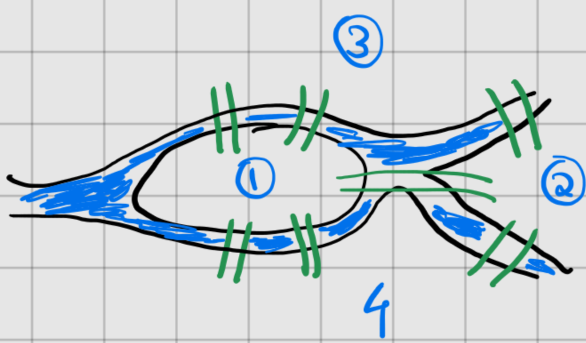
Reference:-

Hankins Ch-12

LLM - Ch-12, Ch-10.1, 10.2

* Kongsberg's Problem:





→ Every landmass which isn't the start or the end shouldn't have odd no of bridges

→ (Called an Eulerian Tour)

* You can travel through all the edges of a graph exactly once, only if ~~atmost~~ * 2 vertices have an odd degree.
* if and?
→ Only possible if graph is fully connected.
0 * (or) → There exists no graph with only 1 vertex with odd edges

* Connectivity :-

A graph G is connected if for every pair of vertices in G, there exists a path connecting them.

Thm 1 :- There are no graphs with exactly 1 odd degree vertex.

Thm 1' :- No. of odd degree vertices in a graph is always even. (Handshaking Lemma)

Proof :- Let G be a graph with vertices $v_1, v_2, \ldots, v_n$.

$$\sum_{i=1}^{n} \text{degree}(v_i) = 2(\text{No. of edges})$$

Thus there can't be odd no of. odd degree vertices.

(Also revis Bhabyam's Proof)

Thm 2 :- A connected graph has a Eulerian tour iff atmost 2 vertices have odd degree.

Thm 2' :- A connected graph has a Eulerian circuit iff all the vertices have even degree.

→ If you get stuck, it will only be at the starting point.

↳ Probably covered a sub graph. So, go back to a vertex with yet to be travelled edges, go through them

intuitive Proof.

and come back. Repeat until all paths are covered.

* Take, Eulerian ckt, with max no. of edges. If it covers all edges we are done, else, above pt shows it can be increased, contradicting our point.

Formal Definition:-

Simple Directed Graph:

→ A Simple Directed Graph G is a pair of sets (V, A) s.t $A \subseteq V \times V$.

* V is called the vertex set, A is called the arc set.
* $n(V \times V) = n^2$
* No. of possible graphs $= 2^{n^2}$

Simple Undirected Graph:

→ A Simple Directed Graph G is a pair of sets (V, E) s.t $E \subseteq \binom{V}{2}$

* V is called the vertex set, E is called the edge set

→ $\binom{V}{2}$ is the set of all 2-sized subsets of V.

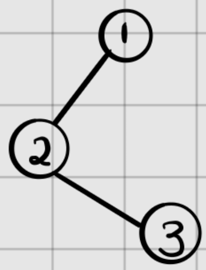
* $n\left[\binom{V}{2}\right] = {}^{n}C_{2}$

* No of possible graphs $= 2^{\binom{n}{2}}$.

⇒ Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are equal if $V_1 = V_2$ and $E_1 = E_2$

and $E_1 = E_2$.

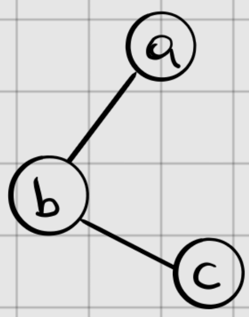
eg: $\underline{\underline{G_1}}$



$V_1 = \{1, 2, 3\}$

$E_1 = \{\{1,2\},\{2,3\}\}$

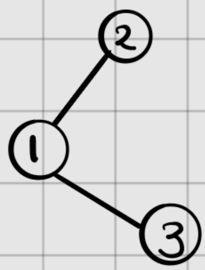
$\underline{\underline{G_2}}$



$V_1 = \{a, b, c\}$

$E_1 = \{\{a,b\},\{b,c\}\}$

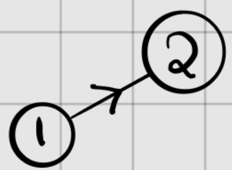
$\underline{\underline{G_3}}$



$V_1 = \{1, 2, 3\}$

$E_1 = \{\{1,2\},\{1,3\}\}$

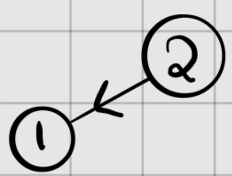
$\underline{\underline{D_1}}$



$V_1 = \{1, 2\}$

$E_1 = \{(1,2)\}$

$\underline{\underline{D_1}}$



$V_1 = \{1, 2\}$

$E_1 = \{(2,1)\}$

$\Rightarrow$ But, what if we want G_1, G_2, G_3 to be equivalent. For it, we can define it as follows.

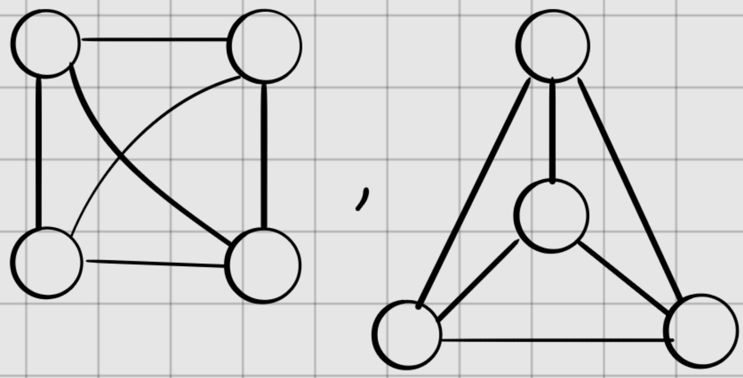
Isomorphic Graphs:

* If there exists a bijective function $f: V_1 \to V_2$, s.t.

$$\forall u,v \in V\left(\{u,v\} \in E_1 \iff \{f(u),f(v)\} \in E_2\right)$$

→ This effectively means f is a "relabelling" function

e.g.:



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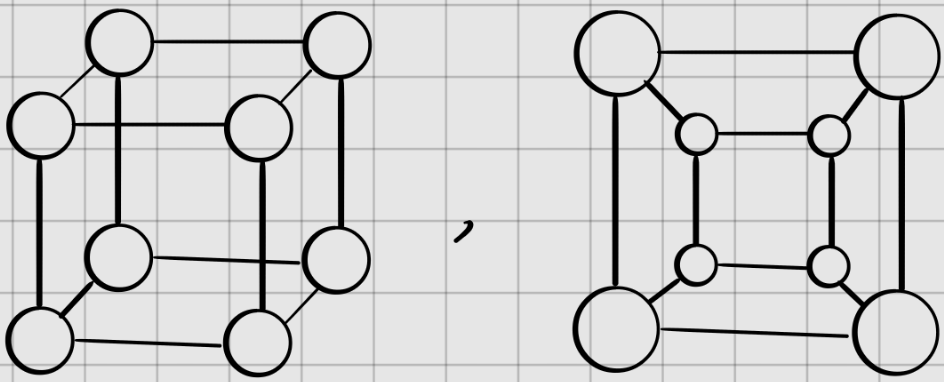

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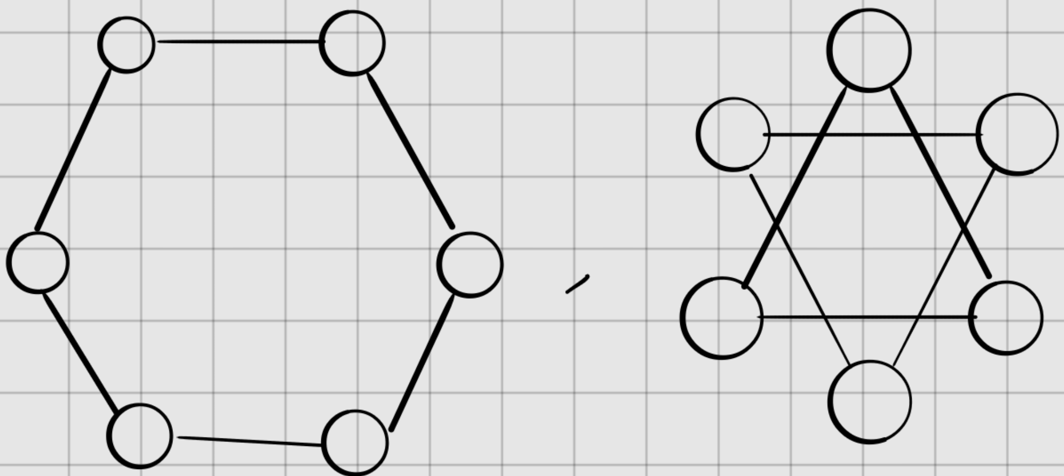

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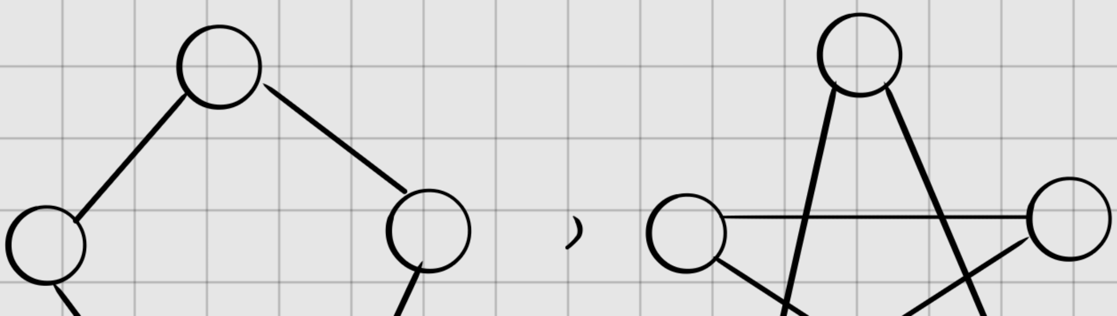

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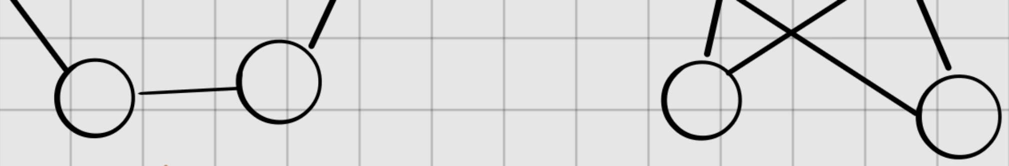


Graph Invariants:

* A property of graph that is preserved under isomorphism

SubGraphs:

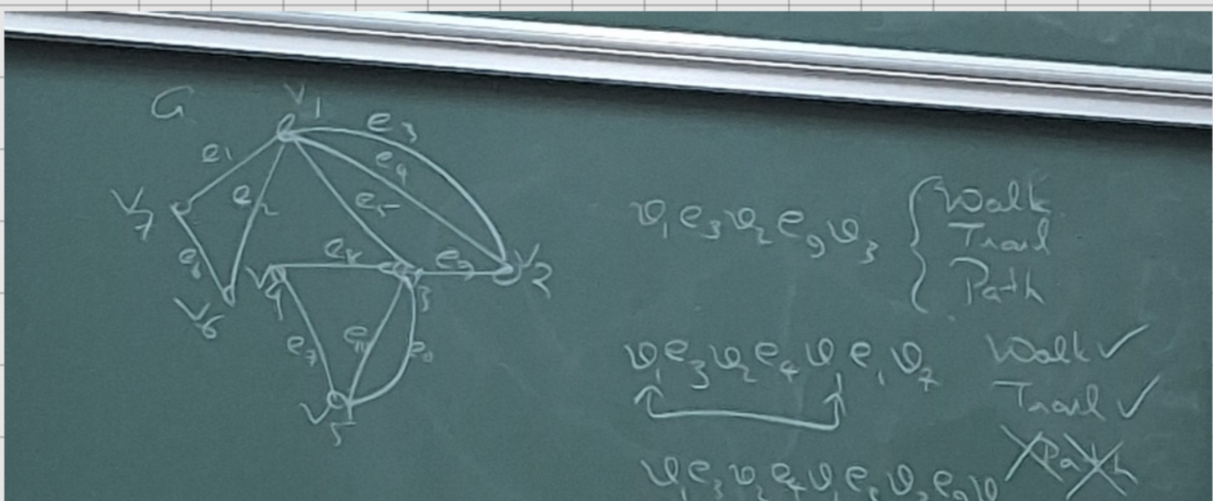
* $G_1 = (V_1, E_1)$ is a subgraph of $G_2 = (V_2, E_2)$ if $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$.

$\Rightarrow$ We informally say that G_2 contains G_1 if G_1 is a subgraph of G_2

* The no. of non-isomorphic graphs for n-vertices will lie b/w

$$\frac{2^{\binom{n}{2}}}{n!} \quad \text{and} \quad 2^{\binom{n}{2}}$$

GRAPH CONNECTIVITY:



(Undir.)

Walk:- It is an alternating seq. of vertices & edges starting and ending with a vertex s.t every edge in this sequence is preceded by one of it's end-points and succeeded by the other endpoint.

Trail:- It is a walk with no repeated edges

Path:- It is a trail, with no repeated vertices
or walk

- ⇒ Length of a walk is the no. of edges in it.
- ⇒ Distance b/w 2 vertices in a graph is the length of a shortest path b/w those 2 vertices.
- ⇒ If there is no path b/w 2 vertices, their distance is defined to be ∞.
- ⇒ Distance b/w u, v in a graph G is denoted as $d_G(u,v)$.

Theorem 1:- A shortest walk b/w two vertices is a path.

Proof:- If a walk has a repetition of a vertex, then there will exist a shorter walk that will not have this repetition. So, the shortest walk will have no vertex repeated, meaning it's a path. Hence, proved using WOP.

Theorem 2:- For any 3 vertices u, v, w in a graph G;

$$d_G(u,w) \leq d_G(u,v) + d_G(v,w)$$

Proof:- We can go from u to w by going from u to v and the v to w. Concatenating the shortest u,v path and the shortest v,w path will give a u,w walk of length RHS. As it is a walk, the shortest u,w path will have length atmost RHS.

⇒ An undirected graph is connected if there exists a path b/w every pair of vertices.

⇒ A connected component in an undir. graph G is a maximal connected subgraph of G.

⇒ In a directed graph,

→ It is said to be strongly connected if there is a path b/w (u,v) and a path b/w (v,u).

→ It is said to be weakly connected if the underlying undir. graph is connected.

Closed Walk:- A walk which starts and ends at the same vertex.

Tour:- A closed walk.

Circuit:- A closed trail

Cycle:- A closed walk of positive length in which no vertex repeats except the start being the same as end.

Theorem:- If a graph has an odd length tour, then it has an odd cycle.

Proof:-

Theorem:- If a graph contains an odd tour, then it contains

Theorem:- If a graph contains an odd tour, then it contains an odd cycle.

Proof:- Let T be the odd tour of shortest length in G. If no intermediate vertex is repeated in T, the T itself is a cycle, otherwise we can split T into smaller tours, one of which is odd. This contradicts the choice of T.

Degree:-

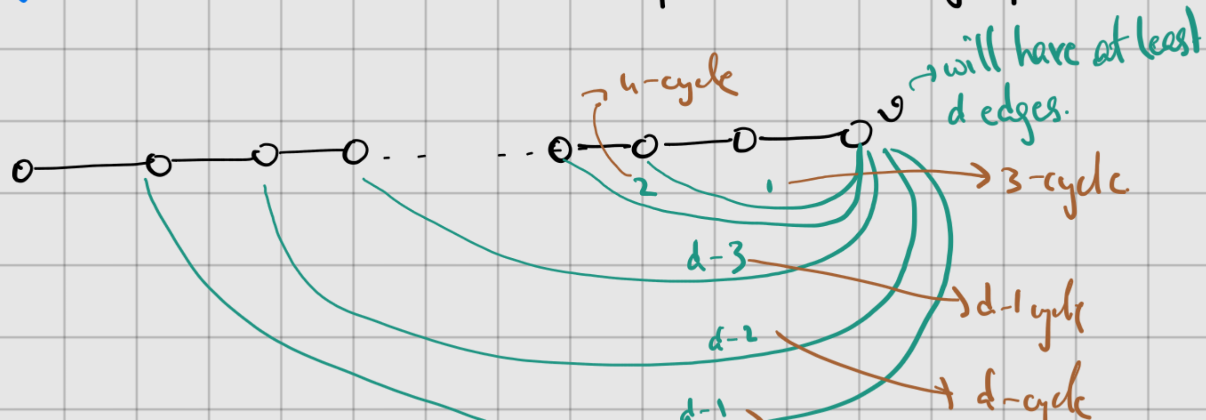
Minimum degree (δG)
Maximum " (ΔG)
Average " ($\bar{d} G$) $= \frac{2m}{n}$

Theorem:- If a graph has a min. degree of atleast 2, then it has a cycle

Proof:- Trail the graph starting from an arbitrary vertex and not retracing an edge immediately until you get a vertex repeated.

Theorem:- If a graph has a min. degree of atleast d, $d \geq 2$, then it has a cycle of atleast d+1.

Proof:- Start with the maximal path on the graph.



Hamiltonian Cycle:-

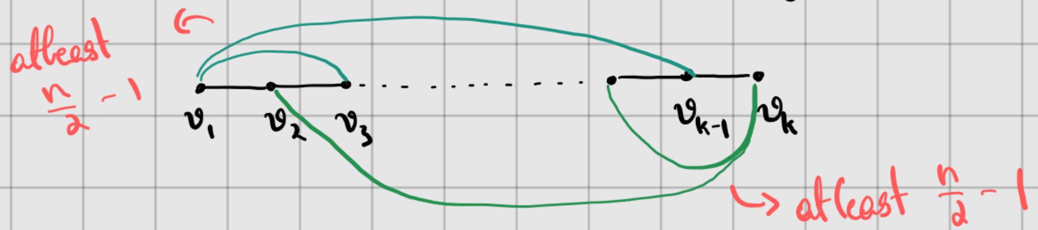
A Hamiltonian cycle in a graph is a cycle that contains every vertex of the graph.

(Drac's Thm).

Theorem:- If a finite simple graph of n vertices ($n \geq 3$) has a min degree of $n/2$ then it has a Hamiltonian Cycle.

Proof:

Let $P = v_1, v_2, \dots, v_k$ be a longest path in G.



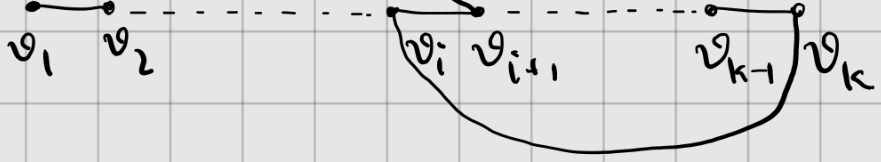
Consider an edge $\{v_i, v_{i+1}\}$

→ It is left happy if $\{v_i, v_k\} \in E$

→ " " right " " $\{v_1, v_{i+1}\} \in E$

→ " " doubly " " it is left and right happy

* Observe there are atleast $n/2$ left happy edges and $n/2$ right happy edges. As we have only $n-1$ edges, the will always be atleast one doubly happy edge.



* We can see,

$$C = v_1, v_2, \ldots, v_i, v_k, v_{k-1}, \ldots, v_{i+1}, v_1$$

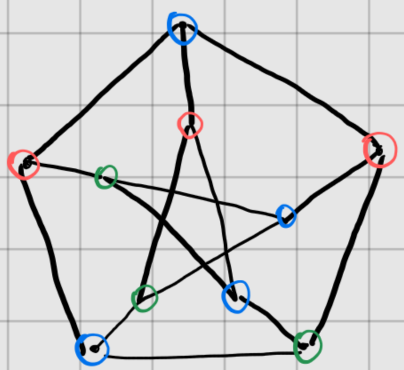
is a cycle.

<u>Claim</u>:- C is a Hamiltonian Cycle.

<u>Proof</u>:- If C has an edge to a vertex that is not in C, we get a new longest path, which is a contradiction. So, P includes all the vertices from G. Hence, C is a Hamiltonian cycle.

GRAPH COLORING:

I am parjanya



Any graph which contains an odd cycle, cannot be colored with 2-colors.

Vertex Coloring:-

* A vertex coloring of a graph $G = (V, E)$ is a function from V to $\mathbb{N}$.
* A k-coloring of G is a fn from V to $\{0, \ldots, k-1\}$.
* A proper coloring of G is a vertex coloring $f: V \to \mathbb{N}$ s.t. $f(u) \neq f(v)\ \forall (u, v$ s.t $\{u, v\} \in E)$.

Chromatic Number:

* The smallest k for which G has a proper k-coloring
* Denoted by $\chi(G)$. [chi of G]

Theorem: For any simple undirected graph G,

$$\chi(G) \leq \Delta(G) + 1$$

max. degree of G.

Proof: Color Greeding

In any step while greeding, atmost $\Delta(G)$ colors are forbidden. So, the max. required will be $\Delta(G)+1$.

Cut Vertex: A cut vertex in a graph is a vertex whose removal increases the no. of components in the graph.

Bridge: An edge whose removal increases the no. of components in the graph.

⇒ A graph is called

(a) 2-vertex connected if G is connected and has no cut vertices

(b) 2-edge connected if G is connected and has no bridges

Theorem: Every 2-vertex connected graph on atleast 3 vertices is 2-edge connected.

k-vertex Connected: A connected graph which cannot be split by taking out any set of k-1 vertices (similar for k-edge con..)

TREES:-

Theorem:- The following are equivalent for an undirected graph G.

(i) G is connected and acyclic
(ii) G is maximally acyclic
(iii) G is minimally connected
(iv) G is connected and every edge of G is a bridge.
(v) There exists a unique path b/w every pair of vertices in G.

Proof:-

(i) → (ii)

If we add a new edge to G, then as there already exists a path b/w any two vertices, that path along with the new edge creates a cycle.
Hence, it is maximally acyclic.

maximal - adding an edge makes the property false.
minimal - removing an edge makes the property false.

(ii) → (iii)

If G is disconnected, adding an edge b/w a vertex of one component with the vertex of another componet, it will not become cyclic. So, it is not maximally acyclic. Hence G is connected.

Now, if we remove an edge in G, it increases the no. of components in G. Hence it is minimally connected.

(iii) → (iv).

(by definition of bridge)

(iv) → (v).

As G is connected, there exists a path b/w any 2 vertices

G is connected, there exists a path b/w any 2 vertices.

Suppose this path is not unique. Then there will be an edge that is only in one path and it is not a bridge.

Proof by contradiction

(v) → (i)

As path exists, it is connected.
If not acyclic, there will be multiple paths. Hence, G is acyclic.

Definition:- A tree is a connected acyclic graph.

Definition:- A leaf in a graph is a vertex with degree atmost 1.

Theorem:- Any tree on two or more vertices has atleast two leaves

Proof:- Consider a maximal path of the tree. Then the starting and ending of this path would be leaves.

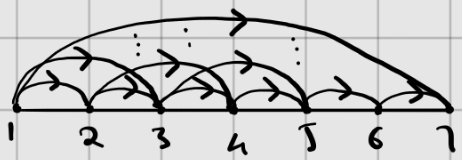
Theorem:- Any tree on two or more vertices has atleast $\Delta(G)$ leaves

Proof:- The tree will have $\Delta(G)$ subtrees, which will all then have atleast 1 leaf. (Refine this later)

Theorem:- Every tree on n vertices has exactly n-1 edges

Directed Acyclic Graphs (DAG):-

* The max no. of edges in a simple n-vertex DAG is $\binom{n}{2}$



All edges point in the same direction, so no cycle can exists.

Definitions:-

(1) A minimum vertex in a DAG is one from which we can reach every vertex. have a path

(2) A minimal vertex in a DAG is one which cannot be reached from any other vertex.

* Every minimum vertex in a DAG is minimal. Because if it is not minimal, then it be reached by some vertex, which in turn can be reached by the minimum vertex, forming a cycle.
* A DAG has atmost one minimum vertex.
* Every finite DAG has atleast one minimal vertex. (Starting vertex of a maximal path)

Topological sort:-

A linear extension / topological sort of a DAG G is an ordering of $V(G)$ s.t every edge is directed from an earlier edge to a later edge.

Theorem:- Every finite DAG has a topological sort.

Proof:- Take a minimal vertex and use induction.

$$topsort(G) = v\ \boxed{topsort(G)}\quad ;\ G' = G/\{v\}$$

v is a minimal vertex.

Matching:

Matching:-

A matching in a graph G is a set of edges which share no endpoints.

* A matching in a graph is called.

(1) Maximal if no proper superset of M is a matching
(2) Maximum if there is no matching larger that it in G.
(3) Perfect if it's size is $n/2$

* A matching in a bipartite graph G with parts L, R is called L-perfect (or L-saturating) if every vertex in L is contained in some edge of M.

⇒ L-perfect ⇒ Maximum matching (proof ?)

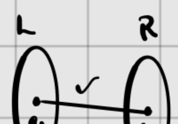
⇒ G does not have an L-perfect matching if one of the following occurs
(1) $|R| < |L|$
(2) L contains a degree 0 vertex [(1) is a special case of (3)]
(3) $\exists S \subseteq L$, s.t. $|N(S)| < |S|$ → set of neighbours of vertices in S.

⇒ A bipartite graph G with parts L and R is said to satisfy Hall's condition if

$$\boxed{\forall S \subseteq L \; |N(S)| \geq |S|}$$ Necessary condition.

Theorem:- [Hall's Marriage Thm, 1930]
A bipartite graph with parts L and R has an L-perfect matching iff it satisfies Hall's condition

Proof:-
(1) [if-part]
Induction on $|L|$.
Base Case:- $|L| = 1$



$M = \{\{a,b\}\}$

Induction hypothesis : true for $|L| \leq k$

Let G be a graph with $|L| = k+1$ and which satisfies Hall's cond^n

Case-(i):- Oversatisfies Hall's condition.

$$\forall S \subsetneq L \quad |N(S)| > |S|.$$

Consider $G' = G \setminus \{u,v\}$, $\{u,v\} \in E(G)$

$$|L'| = |L| - 1 = k$$

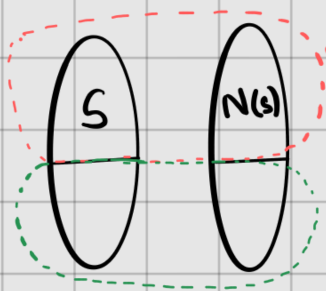
$$\forall S \subseteq L' \quad N_{G'}(S) = N_G(S) \setminus \{v\}$$

$$|N_{G'}(S)| = \underbrace{|N_G(S)|}_{\geq |S|+1} - 1 \qquad \text{as } |N_G(S)| > |S|$$

$$|N_{G'}(S)| \geq |S|$$

Case-(ii):- Exactly satisfies Hall's condition

$$\exists\, S \subsetneq L \quad |N(S)| = |S|$$



$\to G' = G$ isolated to S and $N(S)$

$\to G'' = L \setminus S$ and $R \setminus N(S)$

Now we try to prove that G' satisfies Hall's condition.

If $T \subseteq S$, $N_G(T) = N_{G'}(T)$

$$\Rightarrow |N_G(T)| = |N_{G'}(T)|$$
$$\geq |T|$$

Hence G' satisfies Hall's equation. Then by induction hypothesis, G' satisfies Hall's condition. $[M']$

Now we try to show that G'' satisfies Hall's condition.

Let $T \subseteq L \setminus S$

Consider $T \cup S$ in G

$$|N_G(T \cup S)| \geq |T \cup S|$$

$$|N_G(T) \cup N_G(S)| \geq |T \cup S|$$

$$|N_{G''}(T)| + \cancel{|N_G(S)|} \geq |T| + \cancel{|S|}$$

$$|N_{G''}(T)| \geq |T|$$

Hence G'' satisfies Hall's condition. G'' has an S-perfect matching. $[M'']$

$M = M' \cup M''$ is an L-perfect matching.

* For any 2 matching M_1, M_2, $M_1 \cup M_2$ will form a graph with paths and even cycles
  → If $|M_2| > |M_1|$, $M_1 \cup M_2$ will always contain a path like so