Cantor - Dedekind - Schröder - Bernstein Theorem [1887-1898]

If there is an injective function from a set A to a set B and an injective fn from B to A, then there exists a bijective function from A to B.

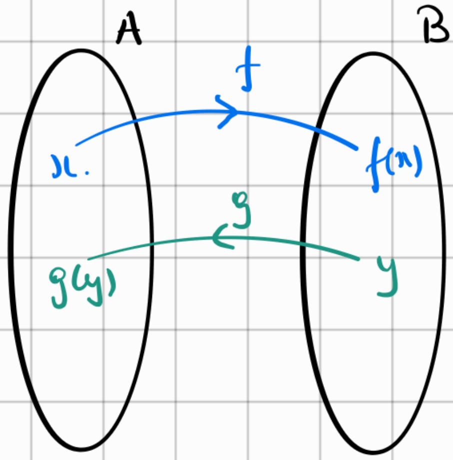
e.g:- $A = [0,1]$

$B = [0,1) = A \setminus \{1\}$

$f(x) = x/2$, $g(x) = x$.

The bijective function from A to B is $h(x) = \begin{cases} x/2 & \text{if } x = 1/2^k, k \in \mathbb{N} \\ x & \text{otherwise.} \end{cases}$

Proof:-



$V = A \cup B$

$E = E_f \cup E_g$

$E_f = \{(x, f(x)) : x \in A\}$ → A-perfect matching

$E_g = \{(y, g(y)) : y \in B\}$ → B-perfect matching

* A bijection from A to B are a perfect matching

* G has 4 types of components

T_1 : Even cycle

T_2 : Doubly ∞ directed path

T_3 : One sided ∞ directed path starting from A.

T_4: One sided ∞ directed path starting from B.

* Define $h: A \to B$ as

$$h(x) = \begin{cases} f(x), & x \text{ is in a } T_1 \text{ component} \\ f(x), & x \text{ is in a } T_2 \text{ component} \\ f(x), & x \text{ is in a } T_3 \text{ component} \\ g^{-1}(x), & x \text{ is in a } T_4 \text{ component} \end{cases}$$

↳ for all nodes in T_4, every ele of A has a preimage in B, hence $g^{-1}(x)$ exists.

$$h(x) = \begin{cases} f(x), & x \text{ is in a } T_1/T_2/T_3 \text{ component} \\ g^{-1}(x), & x \text{ is in a } T_4 \text{ component.} \end{cases}$$

* Claim:- $h(x)$ is injective

Suppose $h(x) = h(y)$

If $x, y \in T_1/T_2/T_3$

$f(x) = f(y) \Rightarrow x = y$

If $x, y \in T_4$

$g^{-1}(x) = g^{-1}(y) \Rightarrow x = y$ [Apply g on b.s]

If x and y are in diff. components,

h(x) and h(y) are in diff components, hence can never be eq.

Cardinality [Comparing sizes of Sets]:-

Definition:-

1. A set A is said to be smaller or equal in size to a set B if there exists an injective function from A to B. [Denoted as $A \preccurlyeq B$]

2. A set A is said to be equal in size to a set B if there exists a bijective function from A to B [Denoted as $A \equiv B$]

3. A set A is said to be strictly smaller in size to a set B if $|A| \leq |B|$ and $|A| \neq |B|$. [Denoted as $A \prec B$].

Bad notation

Theorem:- $(|A| \leq |B|)$ and $(|B| \leq |A|) \Rightarrow (|A| = |B|)$.

(Restatement of Cantor-Dedekind-(too many names) thm).

$\Rightarrow$ Prove that $|\mathbb{N}^+| < |\mathbb{N}|$

$$f: \mathbb{N}^+ \to \mathbb{N}$$
$$x \mapsto x$$

Now, show that $|\mathbb{N}^+| = |\mathbb{N}|$.

$$f: \mathbb{N}^+ \to \mathbb{N}$$
$$x \mapsto x-1$$

even no.s

$\Rightarrow$ Prove that $|\mathbb{N}| = |2\mathbb{N}|$

$f(x) = 2x$ is a bijection from $\mathbb{N}$ to $2\mathbb{N}$.

$\Rightarrow$ Prove that $|\mathbb{N}| = |\mathbb{Z}|$

$$f: \mathbb{N} \to \mathbb{Z}$$

$$f(x) = \begin{cases} -k, & x = 2k \\ k, & x = 2k+1 \end{cases}$$

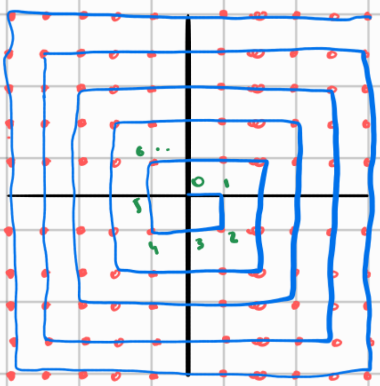
$g : \mathbb{Z} \to \mathbb{N}$

$$g(x) = \begin{cases} 2x, & x \geq 0 \\ -2x-1, & x < 0 \end{cases}$$

$\Rightarrow$ Prove that $|\mathbb{Z}| = |\mathbb{Z} \times \mathbb{Z}|$

$f(i) = (i, 0)$ is an injection from $\mathbb{Z}$ to $\mathbb{Z} \times \mathbb{Z}$

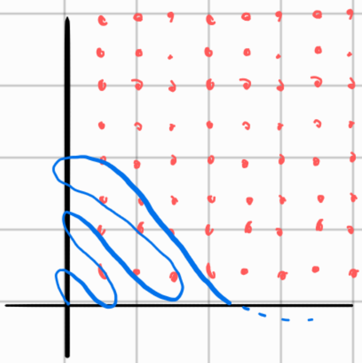
$g(i,j)$ = rank of (i,j) in the spiral scan of $\mathbb{Z} \times \mathbb{Z}$



$\Rightarrow$ Prove that $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$

$f(n) = (n, 0)$

$g(i,j) =$



Diagonal scan rank

$\Rightarrow f: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$, $g: \mathbb{Z} \to \mathbb{N}$ (injective), $h: \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ (injective)

$$f((i,j)) = h((g(i), g(j))) \Rightarrow \text{injective}$$

$\Rightarrow |\mathbb{N}| = |\mathbb{Q}|$

$f: \mathbb{N} \to \mathbb{Q}$

$x \mapsto x$

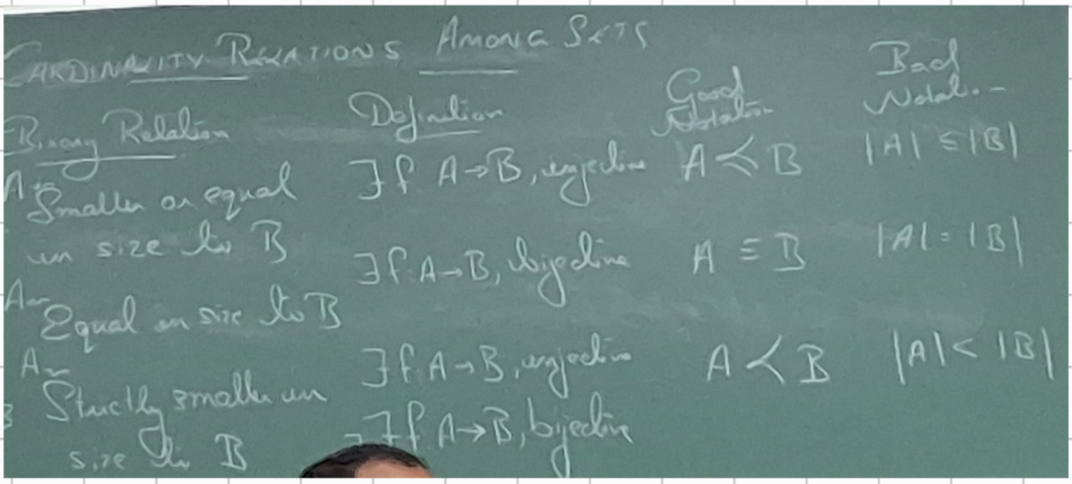
$g: \mathbb{Q} \to \mathbb{N}$

$\frac{p}{q} \mapsto 2^{|p|} 3^{|q|}$

(p,q)

$\mathbb{Q} \xrightarrow[g]{} \mathbb{Z}\times\mathbb{Z} \xrightarrow[R]{} \mathbb{N}$

$g(p/q) = (p,q)$



Definition:- A set is called

(a) finite if $A \prec \{1,2,\dots,n\}$ for some $n \in \mathbb{N}$

(b) countable if $A \preccurlyeq \mathbb{N}$

(c) uncountable if not countable.

(d) countably infinite if A is countable but not finite.

Theorem:- $\mathbb{R}$ is uncountable [Cantor, 1890]

Proof:- To prove:- There is no injection from $\mathbb{R} \to \mathbb{N}$

Enough to show that there is no surjection from $\mathbb{N}$ to $\mathbb{R}$.

Suppose f is a surjection from $\mathbb{N} \to \mathbb{R}$

$$\left.\begin{aligned} f(0) &= z_0 \cdot d_{00} d_{01} d_{02} \ldots \\ f(1) &= z_1 \cdot d_{10} d_{11} d_{12} \ldots \\ f(2) &= z_2 \cdot d_{20} d_{21} d_{22} \ldots \\ &\vdots \end{aligned}\right\}$$ terminating dec. rep of $f(x)$ padded with 0's if there is non-uniqueness.

Now look at the real number

$0.e_0 e_1 e_2 \ldots$; where $e_i \in \{5, 6\}$; $e_i \neq d_{ii}$.

This number will not be in the Range of f as there will be atleast one decimal digit that will be different.

Hence, by contradiction; there exists no surjection.

$\therefore$ $\mathbb{R}$ is uncountable. [$\because \mathbb{N} < \mathbb{R}$]

Theorem: $P(\mathbb{N})$ is uncountable.

Proof: To prove: There is no surjection from $\mathbb{N}$ to $P(\mathbb{N})$.

Characteristic fn of a set $S \subseteq N$ is

$$\phi(S) = b_0 b_1 \ldots\ldots$$

where, $b_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}$

Suppose f is a surjection from $\mathbb{N}$ to $P(\mathbb{N})$.

$$\phi(f(0)) = b_{00} b_{01} \ldots\ldots$$
$$\phi(f(1)) = b_{10} b_{11} \ldots\ldots$$
$$\vdots$$

Chose a binary representation $e_0 e_1 e_2 \ldots\ldots$ s.t $e_i = \neg b_{ii}$.

This doesn't lie in the Range of $\phi \circ f$.

Hence, by contradiction, f is not a surjection.

(or)

$S = \{k : k \notin f(k)\}$ is a set in $P(\mathbb{N})$ that is not in the image of f. Hence, f is not a surjection.

$\therefore P(\mathbb{N})$ is uncountable.

Theorem:- There is no surjection from a set to it's power set. i.e, for any set A, $P(A)$ is uncountable

Proof:- Let A be any set and let $f : A \to P(A)$ be a surjective function.

Define the set,

$$S = \{k \in A : k \notin f(k)\}$$

As f is sujective, $\exists s \in A$ s.t $f(s) = S$

If $s \in S$, the by definition of S, $s \notin f(s) \Rightarrow s \notin S$.

else, if $s \notin S \Rightarrow s \notin f(s)$, then by def of S, $s \in S$.

Thus, by contradiction, there exists no sujective f.

Look at the no. of Membership problems possible for any subset of $\mathbb{N}$ → Uncountable

The no. of programs we can write = List of finite bin. strings → Countable.

So, there exists problems that cannot be solved using a comp. program.

Halting Problem:-

Theorem:- There does not exist a program which can decide if a program will halt on an input.

Proof:- Suppose Halt_tester exists.


```
def Invert (string x):
    res = Halt_tester(x, x)
    if (res == Halt):
        while (0<1):
            print("Hi")
    else:
```

```
while(Halt_tester(x,x) = Halt):
    print("Hi")
return.
```

else:

return.

Invert(Invert) — Halt ≡ Halt_tester(Invert, Invert) is Hang

Invert(Invert) — Hang ≡ Halt_tester(Invert, Invert) is Halt.

Russel's Paradox:-

* A set is a well defined collection of objects
* So, a set can contain itself.
* A set is called Normal if it is not a member of itself.

Let N be a set of all Normal sets.

$$\left.\begin{array}{l} \text{If } N \in N \text{, by def of } N,\ N \notin N \\ \text{If } N \notin N \text{, by def of } N,\ N \in N \end{array}\right\} \text{Contradiction}$$

LLM - Ch8 [8.1, 8.2]

Hamkins - Ch13