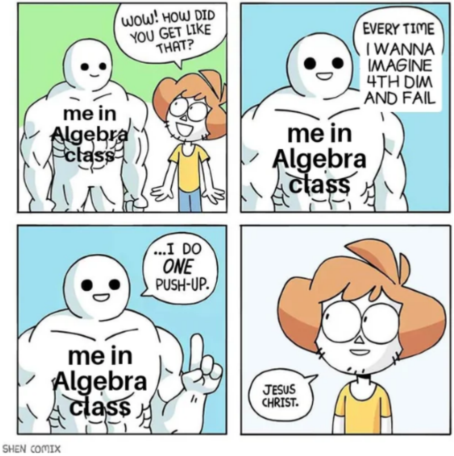

Introduction to Optimization

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Data Science

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Surprise online test tomorrow at 6:00 am

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- Syllabus – modules I and II
- Time to prepare ~ 20 hours

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Nobody:

Mom: *"Don't drink too much coffee...*

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How many hours for each module?

Another example

$$
\begin{aligned}
\text{maximize}\quad & x_1 + x_2 \\
\text{subject to}\quad & x_2 - x_1 \leq 1 \\
& x_1 + 6x_2 \leq 15 \\
& 4x_1 - x_2 \leq 10 \\
& x_1 \geq 0,\ x_2 \geq 0.
\end{aligned}
$$

(continue on board)

Another example

$$\begin{aligned} \text{maximize } & x_1 + x_2 \\ \text{subject to } & x_2 - x_1 \le 1 \\ & x_1 + 6x_2 \le 15 \\ & 4x_1 - x_2 \le 10 \\ & x_1 \ge 0, \ x_2 \ge 0. \end{aligned}$$

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To-do:

- show vector c and feasible region

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To-do:

- show vector c and feasible region *unique solution*

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(continue on board)

To-do:

- show vector c and feasible region *unique solution*
- replace $c = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$ with $\begin{bmatrix} \frac{1}{6} & 1 \end{bmatrix}^\top$

Another example

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- remove 2^{nd} and 3^{rd}

Another example

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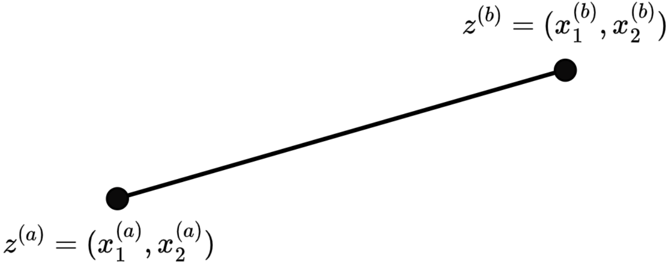
To-do:

- show vector c and feasible region *unique solution*
- replace $c = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$ with $\begin{bmatrix} \frac{1}{6} & 1 \end{bmatrix}^\top$ *infinitely many solutions*
- reverse 1^{st} and 3^{rd} *infeasible*
- remove 2^{nd} and 3^{rd} *unbounded*

Why it suffices to look at *corner points*?

$\max f(z)$ subject to some linear constraints where $f(z) = c^\top z$



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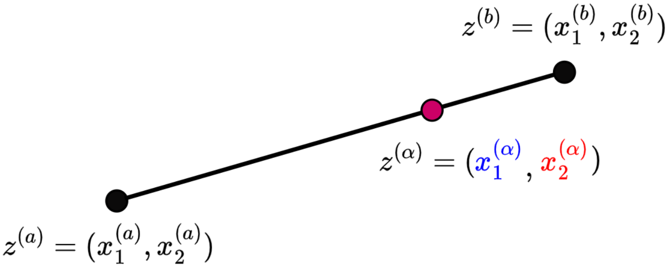

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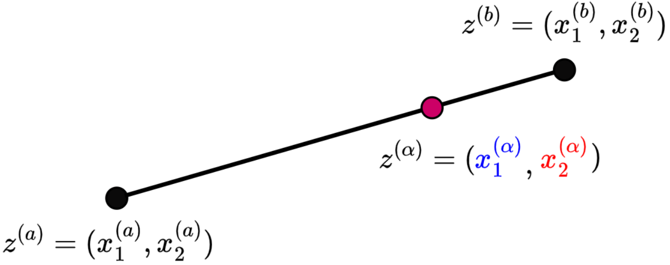

$$z^{(\alpha)} = \alpha z^{(a)} + (1-\alpha) z^{(b)} \quad ; \; 0 \le \alpha \le 1.$$

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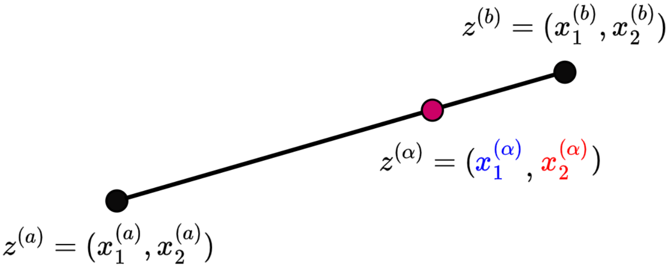

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That is,

$$x_1^{(\alpha)} = \alpha x_1^{(a)} + (1-\alpha) x_1^{(b)}$$
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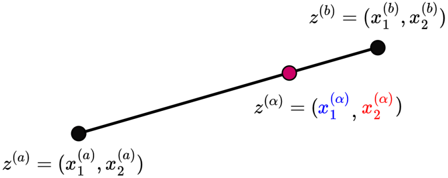

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Objective function value at $z^{(\alpha)}$:

$$f(z^{(\alpha)}) = c^\top z^{(\alpha)}$$

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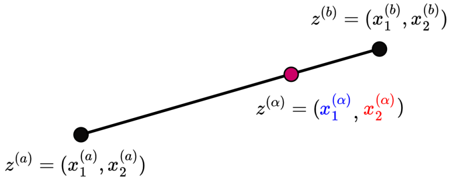

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$$\begin{aligned} f(z^{(\alpha)}) &= c^\top z^{(\alpha)} \\ &= c_1 x_1^{(\alpha)} + c_2 x_2^{(\alpha)} \end{aligned}$$

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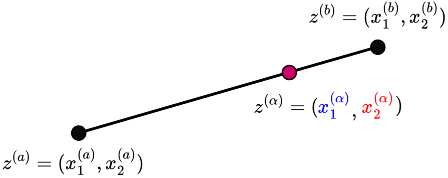

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Objective function value at $z^{(\alpha)}$:

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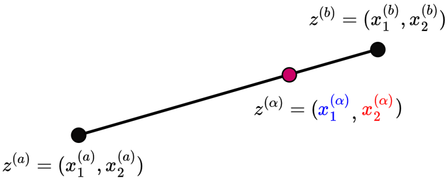

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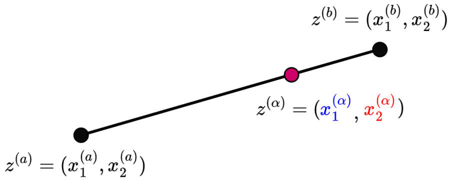

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\end{aligned}$$

Top 10 algorithms from the 20^{th} century

Computing in Science & Engineering, American Institute of Physics and the IEEE Computer Society, Jan 2000.

Simplex algorithm – bird's eye view

Algorithms, Dasgupta-Papadimitriou-Vazirani [online]

```
Let v be any vertex of the feasible region.
while ∃ a neighbor v' of v with better objective value:
        set v = v'
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Both tasks are easy if the vertex is at the origin!

But what if our current vertex u is elsewhere?

Transform u *into the origin, by shifting the coordinate system from the usual* $(x_1, x_2, \cdots, x_n)$ *to the "local view" from* u*.*

The two tasks at the origin

"Both tasks are easy if the vertex is at the origin!"

On each iteration, the algorithm has **two** tasks:

1. Check whether the current vertex is optimal (and if so, halt).
2. Determine where to move next.

Consider the LP:

$$\text{maximize } c^\top x \text{ subject to } Ax \geq b, x \geq 0 \quad ; \; x \in \mathbb{R}^n.$$

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Origin is feasible $\implies$ Origin is a vertex

since it is the unique point at which the n inequalities

$x_1 \geq 0, \; x_2 \geq 0, \; \cdots, \; x_n \geq 0$ are tight.

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The origin is optimal if and only if all $c_i \leq 0$.

Increase some x_i for which $c_i > 0$.

Increase by how much? Until we hit some other constraint.

Solve using simplex algorithm

"*always operate at the origin*"

$$\max\ 2x_1 + 5x_2 \text{ subject to}$$

$$2x_1 - x_2 \le 4 \quad (1)$$

$$x_1 + 2x_2 \le 9 \quad (2)$$

$$-x_1 + x_2 \le 3 \quad (3)$$

$$x_1 \ge 0 \quad (4)$$

$$x_2 \ge 0 \quad (5)$$

Solve using simplex algorithm

"*always operate at the origin*"

Current vertex: (4),(5) **Objective value**:

$$\max\ 2x_1 + 5x_2 \text{ subject to}$$

$$2x_1 - x_2 \leq 4 \qquad (1)$$

$$x_1 + 2x_2 \leq 9 \qquad (2)$$

$$-x_1 + x_2 \leq 3 \qquad (3)$$

$$x_1 \geq 0 \qquad (4)$$

$$x_2 \geq 0 \qquad (5)$$

Solve using simplex algorithm

"*always operate at the origin*"

Current vertex: (4),(5) **Objective value**: 0

$$\max\ 2x_1 + 5x_2 \text{ subject to}$$

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$$-x_1 + x_2 \leq 3 \qquad (3)$$

$$x_1 \geq 0 \qquad (4)$$

$$x_2 \geq 0 \qquad (5)$$

Solve using simplex algorithm

"*always operate at the origin*"

Current vertex: (4),(5)

Objective value: 0

Move:

$$\max\ 2x_1 + 5x_2 \text{ subject to}$$

$$2x_1 - x_2 \leq 4 \qquad (1)$$

$$x_1 + 2x_2 \leq 9 \qquad (2)$$

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Solve using simplex algorithm

*"always operate at the **origin**"*

Current vertex: (4),(5) **Objective value**: 0

Move: increase x_2

$$\max\ 2x_1 + 5x_2 \text{ subject to}$$

$$\begin{aligned}
2x_1 - x_2 &\leq 4 && (1)\\
x_1 + 2x_2 &\leq 9 && (2)\\
-x_1 + x_2 &\leq 3 && (3)\\
x_1 &\geq 0 && (4)\\
x_2 &\geq 0 && (5)
\end{aligned}$$

Solve using simplex algorithm

"*always operate at the origin*"

Current vertex: (4),(5) **Objective value**: 0

Move: increase x_2

(5) is released, (3) is tightened

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Solve using simplex algorithm

"*always operate at the origin*"

Current vertex: (4),(5)

Objective value: 0

Move: increase x_2

STOP at $x_2 = ?$

(5) is released, (3) is tightened

$$\max\ 2x_1 + 5x_2 \text{ subject to}$$

$$2x_1 - x_2 \leq 4 \qquad (1)$$

$$x_1 + 2x_2 \leq 9 \qquad (2)$$

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Solve using simplex algorithm

"*always operate at the origin*"

Current vertex: (4),(5) **Objective value**: 0

Move: increase x_2

STOP at $x_2 = 3$

$$\max\ 2x_1 + 5x_2 \text{ subject to}$$

$$2x_1 - x_2 \le 4 \qquad (1)$$

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Current vertex: (4),(5)

Objective value: 0

Move: increase x_2

STOP at $x_2 = 3$

$\boxed{\textbf{New vertex: } (4),(3)}$

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Solve using simplex algorithm

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Current vertex: (4),(5)

Objective value: 0

Move: increase x_2

STOP at $x_2 = 3$

$\boxed{\textbf{New vertex: } (4),(3)}$

$$y_1 = x_1$$
$$y_2 = 3 + x_1 - x_2$$

$\max\ 2x_1 + 5x_2$ subject to

$2x_1 - x_2 \leq 4$	(1)
$x_1 + 2x_2 \leq 9$	(2)
$-x_1 + x_2 \leq 3$	(3)
$x_1 \geq 0$	(4)
$x_2 \geq 0$	(5)

Solve using simplex algorithm

"*always operate at the **origin***"

Current vertex: (4),(3) **Objective value**:

max $15 + 7y_1 - 5y_2$ subject to

$$y_1 + y_2 \leq 7 \quad (1)$$

$$3y_1 - 2y_2 \leq 3 \quad (2)$$

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$$y_1 \geq 0 \quad (4)$$

$$-y_1 + y_2 \leq 3 \quad (5)$$

Solve using simplex algorithm

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Current vertex: (4),(3) **Objective value**: 15

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Solve using simplex algorithm

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Current vertex: (4),(3)

Objective value: 15

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$$y_1 \geq 0 \qquad (4)$$

$$-y_1 + y_2 \leq 3 \qquad (5)$$

Solve using simplex algorithm

"*always operate at the **origin***"

Current vertex: (4),(3) **Objective value**: 15

Move: increase y_1

$$\max\ 15 + 7y_1 - 5y_2 \text{ subject to}$$

$$y_1 + y_2 \le 7 \qquad (1)$$

$$3y_1 - 2y_2 \le 3 \qquad (2)$$

$$y_2 \ge 0 \qquad (3)$$

$$y_1 \ge 0 \qquad (4)$$

$$-y_1 + y_2 \le 3 \qquad (5)$$

Solve using simplex algorithm

"*always operate at the origin*"

Current vertex: (4),(3)

Objective value: 15

Move: increase y_1

$\boxed{\text{(4) is released, (2) is tightened}}$

$$\max\ 15 + 7y_1 - 5y_2 \text{ subject to}$$

$$y_1 + y_2 \leq 7 \qquad (1)$$

$$3y_1 - 2y_2 \leq 3 \qquad (2)$$

$$y_2 \geq 0 \qquad (3)$$

$$y_1 \geq 0 \qquad (4)$$

$$-y_1 + y_2 \leq 3 \qquad (5)$$

Solve using simplex algorithm

"*always operate at the origin*"

Current vertex: (4),(3)

Objective value: 15

Move: increase y_1

STOP at $y_1 = ?$

$\boxed{\text{(4) is released, (2) is tightened}}$

$$\max\ 15 + 7y_1 - 5y_2 \text{ subject to}$$

$$y_1 + y_2 \leq 7 \qquad (1)$$

$$3y_1 - 2y_2 \leq 3 \qquad (2)$$

$$y_2 \geq 0 \qquad (3)$$

$$y_1 \geq 0 \qquad (4)$$

$$-y_1 + y_2 \leq 3 \qquad (5)$$

Solve using simplex algorithm

"*always operate at the origin*"

Current vertex: (4),(3) **Objective value**: 15

Move: increase y_1

STOP at $y_1 = 1$

$$\max\ 15 + 7y_1 - 5y_2 \text{ subject to}$$

$$y_1 + y_2 \leq 7 \quad (1)$$

$$3y_1 - 2y_2 \leq 3 \quad (2)$$

$$y_2 \geq 0 \quad (3)$$

$$y_1 \geq 0 \quad (4)$$

$$-y_1 + y_2 \leq 3 \quad (5)$$

Solve using simplex algorithm

"*always operate at the origin*"

Current vertex: (4),(3)

Objective value: 15

Move: increase y_1

STOP at $y_1 = 1$

$\boxed{\textbf{New vertex}\text{: (2),(3)}}$

$$\max\ 15 + 7y_1 - 5y_2 \text{ subject to}$$

$$y_1 + y_2 \leq 7 \qquad (1)$$

$$3y_1 - 2y_2 \leq 3 \qquad (2)$$

$$y_2 \geq 0 \qquad (3)$$

$$y_1 \geq 0 \qquad (4)$$

$$-y_1 + y_2 \leq 3 \qquad (5)$$

Solve using simplex algorithm

"*always operate at the origin*"

Current vertex: (4),(3)

Objective value: 15

Move: increase y_1

STOP at $y_1 = 1$

New vertex: (2),(3)

$$z_1 = 3 - 3y_1 + 2y_2$$
$$z_2 = y_2$$

$$\max\ 15 + 7y_1 - 5y_2 \text{ subject to}$$

$$y_1 + y_2 \leq 7 \qquad (1)$$
$$3y_1 - 2y_2 \leq 3 \qquad (2)$$
$$y_2 \geq 0 \qquad (3)$$
$$y_1 \geq 0 \qquad (4)$$
$$-y_1 + y_2 \leq 3 \qquad (5)$$

Solve using simplex algorithm

"*always operate at the origin*"

Current vertex: (2),(3)

Objective value:

$$\max\ 22 - \frac{7}{3}z_1 - \frac{1}{3}z_2 \text{ subject to}$$

$$-\frac{1}{3}z_1 + \frac{5}{3}z_2 \le 6 \qquad (1)$$

$$z_1 \ge 0 \qquad (2)$$

$$z_2 \ge 0 \qquad (3)$$

$$\frac{1}{3}z_1 - \frac{2}{3}z_2 \le 1 \qquad (4)$$

$$\frac{1}{3}z_1 + \frac{1}{3}z_2 \le 4 \qquad (5)$$

Solve using simplex algorithm

"*always operate at the* ***origin***"

Current vertex: (2),(3) **Objective value:** 22

$$\max\ 22 - \frac{7}{3}z_1 - \frac{1}{3}z_2 \text{ subject to}$$

$$-\frac{1}{3}z_1 + \frac{5}{3}z_2 \leq 6 \qquad (1)$$

$$z_1 \geq 0 \qquad (2)$$

$$z_2 \geq 0 \qquad (3)$$

$$\frac{1}{3}z_1 - \frac{2}{3}z_2 \leq 1 \qquad (4)$$

$$\frac{1}{3}z_1 + \frac{1}{3}z_2 \leq 4 \qquad (5)$$

Solve using simplex algorithm

"*always operate at the origin*"

Current vertex: (2),(3) **Objective value**: 22

Move:

$$\max\ 22 - \frac{7}{3}z_1 - \frac{1}{3}z_2 \text{ subject to}$$

$$-\frac{1}{3}z_1 + \frac{5}{3}z_2 \leq 6 \qquad (1)$$

$$z_1 \geq 0 \qquad (2)$$

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$$\frac{1}{3}z_1 - \frac{2}{3}z_2 \leq 1 \qquad (4)$$

$$\frac{1}{3}z_1 + \frac{1}{3}z_2 \leq 4 \qquad (5)$$

Solve using simplex algorithm

"*always operate at the origin*"

Current vertex:

Objective value: 22

Move:

$$\max\ 22 - \frac{7}{3}z_1 - \frac{1}{3}z_2 \text{ subject to}$$

$$-\frac{1}{3}z_1 + \frac{5}{3}z_2 \leq 6 \qquad (1)$$

$$z_1 \geq 0 \qquad (2)$$

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$$\frac{1}{3}z_1 + \frac{1}{3}z_2 \leq 4 \qquad (5)$$

Solve using simplex algorithm

"*always operate at the* ***origin***"

Current vertex: (2),(3) **Objective value**: 22

Move: origin is optimal!
(all $c_i < 0$)

$$\max\ 22 - \frac{7}{3}z_1 - \frac{1}{3}z_2 \text{ subject to}$$

$$-\frac{1}{3}z_1 + \frac{5}{3}z_2 \leq 6 \qquad (1)$$

$$z_1 \geq 0 \qquad (2)$$

$$z_2 \geq 0 \qquad (3)$$

$$\frac{1}{3}z_1 - \frac{2}{3}z_2 \leq 1 \qquad (4)$$

$$\frac{1}{3}z_1 + \frac{1}{3}z_2 \leq 4 \qquad (5)$$

Solve using simplex algorithm

"*always operate at the origin*"

Current vertex: (2),(3) **Objective value**: $\boxed{22}$

Move: origin is optimal!
(all $c_i < 0$)

$$\max\ 22 - \frac{7}{3}z_1 - \frac{1}{3}z_2 \text{ subject to}$$

$$-\frac{1}{3}z_1 + \frac{5}{3}z_2 \leq 6 \quad (1)$$

$$z_1 \geq 0 \quad (2)$$

$$z_2 \geq 0 \quad (3)$$

$$\frac{1}{3}z_1 - \frac{2}{3}z_2 \leq 1 \quad (4)$$

$$\frac{1}{3}z_1 + \frac{1}{3}z_2 \leq 4 \quad (5)$$

Solve using simplex algorithm

"*always operate at the origin*"

Current vertex: (2),(3)

Objective value: $\boxed{22}$

Move: origin is optimal!
(all $c_i < 0$)

Solve (2),(3) in the original LP:

$$\max\ 22 - \frac{7}{3}z_1 - \frac{1}{3}z_2 \text{ subject to}$$

$$-\frac{1}{3}z_1 + \frac{5}{3}z_2 \leq 6 \qquad (1)$$

$$z_1 \geq 0 \qquad (2)$$

$$z_2 \geq 0 \qquad (3)$$

$$\frac{1}{3}z_1 - \frac{2}{3}z_2 \leq 1 \qquad (4)$$

$$\frac{1}{3}z_1 + \frac{1}{3}z_2 \leq 4 \qquad (5)$$

Solve using simplex algorithm

"*always operate at the origin*"

Current vertex: (2),(3)

Objective value: $\boxed{22}$

Move: origin is optimal!
(all $c_i < 0$)

Solve (2),(3) in the original LP:

$$\boxed{(x_1, x_2) = (1, 4)}$$

$\max 22 - \frac{7}{3} z_1 - \frac{1}{3} z_2$ subject to

$$-\frac{1}{3} z_1 + \frac{5}{3} z_2 \leq 6 \qquad (1)$$

$$z_1 \geq 0 \qquad (2)$$

$$z_2 \geq 0 \qquad (3)$$

$$\frac{1}{3} z_1 - \frac{2}{3} z_2 \leq 1 \qquad (4)$$

$$\frac{1}{3} z_1 + \frac{1}{3} z_2 \leq 4 \qquad (5)$$

Back to the first example

solve using simplex

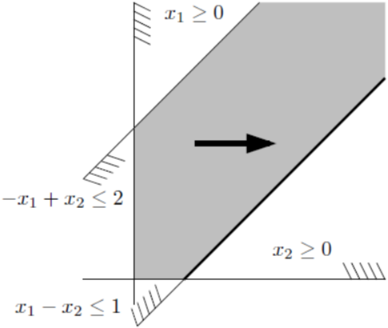
$$\begin{aligned} \text{maximize } & 4x_1 + 5x_2 \\ \text{subject to } & x_1 + x_2 \leq 20 \\ & 3x_1 + 4x_2 \leq 72 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

(complete on board)

Another interesting example

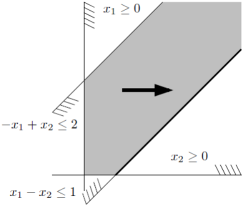
Unbounded LP

$$\begin{aligned} &\text{maximize } x_1 \\ &\text{subject to } x_1 - x_2 \le 1 && (1) \\ &\quad -x_1 + x_2 \le 2 && (2) \\ &\quad x_1 \ge 0 && (3) \\ &\quad x_2 \ge 0 && (4) \end{aligned}$$



Another interesting example

Unbounded LP



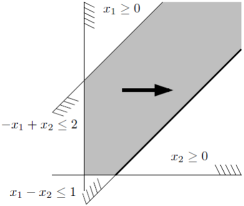
$$
\begin{array}{lll}
\text{maximize } x_1 & & \\
\text{subject to } x_1 - x_2 \leq 1 & & (1) \\
-x_1 + x_2 \leq 2 & & (2) \\
x_1 \geq 0 & & (3) \\
x_2 \geq 0 & & (4)
\end{array}
$$

Current vertex: (3),(4)

Objective value: 0

Another interesting example

Unbounded LP



$$\begin{aligned} \text{maximize } & x_1 & \\ \text{subject to } & x_1 - x_2 \leq 1 & (1) \\ & -x_1 + x_2 \leq 2 & (2) \\ & x_1 \geq 0 & (3) \\ & x_2 \geq 0 & (4) \end{aligned}$$

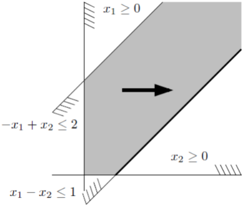
Current vertex: (3),(4)

Objective value: 0

Move: increase x_1

Another interesting example

Unbounded LP



$$
\begin{aligned}
\text{maximize } & x_1 & \\
\text{subject to } & x_1 - x_2 \leq 1 & (1)\\
& -x_1 + x_2 \leq 2 & (2)\\
& x_1 \geq 0 & (3)\\
& x_2 \geq 0 & (4)
\end{aligned}
$$

Current vertex: (3),(4)

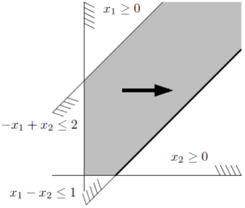
Objective value: 0

Move: increase x_1

STOP at $x_1 = 1$

Another interesting example

Unbounded LP



maximize x_1

subject to $x_1 - x_2 \leq 1$ (1)

$-x_1 + x_2 \leq 2$ (2)

$x_1 \geq 0$ (3)

$x_2 \geq 0$ (4)

Current vertex: (3),(4)

Objective value: 0

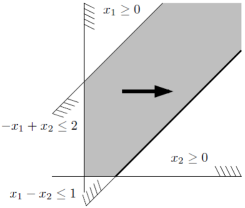
Move: increase x_1

STOP at $x_1 = 1$

New vertex: (1),(4)

Another interesting example

Unbounded LP



$$
\begin{aligned}
\text{maximize } & x_1 \\
\text{subject to } & x_1 - x_2 \leq 1 && (1) \\
& -x_1 + x_2 \leq 2 && (2) \\
& x_1 \geq 0 && (3) \\
& x_2 \geq 0 && (4)
\end{aligned}
$$

Current vertex: (3),(4)

Objective value: 0

Move: increase x_1

STOP at $x_1 = 1$

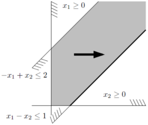
New vertex: (1),(4)

$$y_2 = x_2$$

$$y_1 = 1 - x_1 + x_2$$

Another interesting example

Unbounded LP



maximize x_1

subject to $x_1 - x_2 \leq 1$ (1)

$-x_1 + x_2 \leq 2$ (2)

$x_1 \geq 0$ (3)

$x_2 \geq 0$ (4)

Current vertex: (3),(4)

Objective value: 0

Move: increase x_1

STOP at $x_1 = 1$

New vertex: (1),(4)

$y_2 = x_2$

$y_1 = 1 - x_1 + x_2$

---

maximize $1 - y_1 + y_2$

subject to $y_1 \geq 0$ (1)

$y_1 \leq 3$ (2)

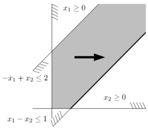
$y_1 - y_2 \leq 1$ (3)

$y_2 \geq 0$ (4)

Current vertex: (1),(4)

Another interesting example

Unbounded LP



maximize x_1

subject to $x_1 - x_2 \leq 1$ (1)

$-x_1 + x_2 \leq 2$ (2)

$x_1 \geq 0$ (3)

$x_2 \geq 0$ (4)

Current vertex: (3),(4)

Objective value: 0

Move: increase x_1

STOP at $x_1 = 1$

New vertex: (1),(4)

$y_2 = x_2$

$y_1 = 1 - x_1 + x_2$

---

maximize $1 - y_1 + y_2$

subject to $y_1 \geq 0$ (1)

$y_1 \leq 3$ (2)

$y_1 - y_2 \leq 1$ (3)

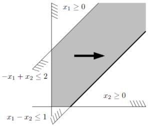
$y_2 \geq 0$ (4)

Current vertex: (1),(4)

Objective value: 1

Another interesting example

Unbounded LP



maximize x_1

subject to

$$x_1 - x_2 \leq 1 \quad (1)$$
$$-x_1 + x_2 \leq 2 \quad (2)$$
$$x_1 \geq 0 \quad (3)$$
$$x_2 \geq 0 \quad (4)$$

Current vertex: (3),(4)

Objective value: 0

Move: increase x_1

STOP at $x_1 = 1$

New vertex: (1),(4)

$$y_2 = x_2$$
$$y_1 = 1 - x_1 + x_2$$

---

maximize $1 - y_1 + y_2$

subject to

$$y_1 \geq 0 \quad (1)$$
$$y_1 \leq 3 \quad (2)$$
$$y_1 - y_2 \leq 1 \quad (3)$$
$$y_2 \geq 0 \quad (4)$$

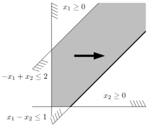
Current vertex: (1),(4)

Objective value: 1

Move: increase y_2

Another interesting example

Unbounded LP



$$
\begin{aligned}
\text{maximize } & x_1 & \\
\text{subject to } & x_1 - x_2 \leq 1 & (1)\\
& -x_1 + x_2 \leq 2 & (2)\\
& x_1 \geq 0 & (3)\\
& x_2 \geq 0 & (4)
\end{aligned}
$$

Current vertex: (3),(4)

Objective value: 0

Move: increase x_1

STOP at $x_1 = 1$

New vertex: (1),(4)

$y_2 = x_2$

$y_1 = 1 - x_1 + x_2$

$$
\begin{aligned}
\text{maximize } & 1 - y_1 + y_2 & \\
\text{subject to } & y_1 \geq 0 & (1)\\
& y_1 \leq 3 & (2)\\
& y_1 - y_2 \leq 1 & (3)\\
& y_2 \geq 0 & (4)
\end{aligned}
$$

Current vertex: (1),(4)

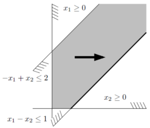
Objective value: 1

Move: increase y_2

STOP at $y_2 = ?$

Another interesting example

Unbounded LP



maximize x_1

subject to

$x_1 - x_2 \leq 1$ (1)

$-x_1 + x_2 \leq 2$ (2)

$x_1 \geq 0$ (3)

$x_2 \geq 0$ (4)

Current vertex: (3),(4)

Objective value: 0

Move: increase x_1

STOP at $x_1 = 1$

New vertex: (1),(4)

$y_2 = x_2$

$y_1 = 1 - x_1 + x_2$

---

maximize $1 - y_1 + y_2$

subject to

$y_1 \geq 0$ (1)

$y_1 \leq 3$ (2)

$y_1 - y_2 \leq 1$ (3)

$y_2 \geq 0$ (4)

Current vertex: (1),(4)

Objective value: 1

Move: increase y_2

STOP at $y_2 = ?$

$\boxed{\text{LP is unbounded}}$

One last example

Degeneracy

$$
\begin{aligned}
\text{maximize } & x_2 & \\
\text{subject to } & -x_1 + x_2 \leq 0 & (1) \\
& x_1 \leq 2 & (2) \\
& x_1 \geq 0 & (3) \\
& x_2 \geq 0 & (4)
\end{aligned}
$$

Current vertex: (3),(4)

One last example

Degeneracy

$$\begin{aligned} \text{maximize } & x_2 \\ \text{subject to } & -x_1 + x_2 \leq 0 && (1) \\ & x_1 \leq 2 && (2) \\ & x_1 \geq 0 && (3) \\ & x_2 \geq 0 && (4) \end{aligned}$$

Current vertex: (3),(4)

Objective value: 0

One last example

Degeneracy

$$
\begin{aligned}
\text{maximize } & x_2 \\
\text{subject to } & -x_1 + x_2 \leq 0 && (1) \\
& x_1 \leq 2 && (2) \\
& x_1 \geq 0 && (3) \\
& x_2 \geq 0 && (4)
\end{aligned}
$$

Current vertex: (3),(4)

Objective value: 0

Move: increase x_2

One last example

Degeneracy

maximize x_2

subject to $-x_1 + x_2 \leq 0$ (1)

$x_1 \leq 2$ (2)

$x_1 \geq 0$ (3)

$x_2 \geq 0$ (4)

Current vertex: (3),(4)

Objective value: 0

Move: increase x_2

One last example

Degeneracy

$$
\begin{aligned}
\text{maximize } & x_2 & \\
\text{subject to } & -x_1 + x_2 \leq 0 & (1)\\
& x_1 \leq 2 & (2)\\
& x_1 \geq 0 & (3)\\
& x_2 \geq 0 & (4)
\end{aligned}
$$

Current vertex: (3),(4)

Objective value: 0

Move: increase x_2

CAN'T!

One last example

Degeneracy

$$\begin{aligned} \text{maximize } & x_2 \\ \text{subject to } & -x_1 + x_2 \leq 0 && (1) \\ & x_1 \leq 2 && (2) \\ & x_1 \geq 0 && (3) \\ & x_2 \geq 0 && (4) \end{aligned}$$

Current vertex: (3),(4)

Objective value: 0

Move: increase x_2

CAN'T!

One last example

Degeneracy

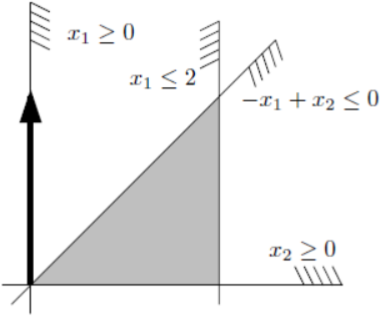
maximize	x_2	
subject to	$-x_1 + x_2 \leq 0$	(1)
	$x_1 \leq 2$	(2)
	$x_1 \geq 0$	(3)
	$x_2 \geq 0$	(4)

Current vertex: (3),(4)

Objective value: 0

Move: increase x_2

CAN'T!



One last example

Degeneracy

maximize x_2

subject to $-x_1 + x_2 \leq 0$ (1)

$x_1 \leq 2$ (2)

$x_1 \geq 0$ (3)

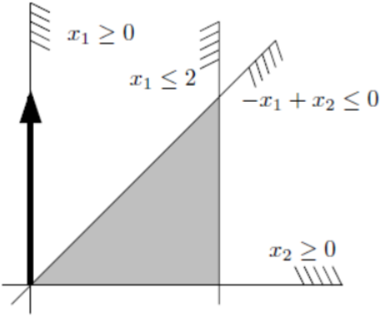
$x_2 \geq 0$ (4)

Current vertex: (3),(4)

Objective value: 0

Move: increase x_2

CAN'T!



Move: increase x_1 $\rightarrow$

LP in standard form

simplex tableau demands . . .

$$\begin{aligned} \text{minimize } & c^\top x \\ \text{subject to } & Ax = b, \\ & x \geq 0. \end{aligned}$$

LP in standard form

simplex tableau demands . . .

$$\begin{aligned} \text{minimize } & c^\top x \\ \text{subject to } & Ax = b, \\ & x \geq 0. \end{aligned}$$

Suppose we are given the inequality constraint

$$x_1 \leq 7.$$

LP in standard form

simplex tableau demands . . .

$$\begin{aligned} \text{minimize } & c^\top x \\ \text{subject to } & Ax = b, \\ & x \geq 0. \end{aligned}$$

Suppose we are given the inequality constraint

$$x_1 \leq 7.$$

Convert to an **equality constraint** – introduce a slack variable:

$$\begin{aligned} x_1 + s_1 &= 7 \\ s_1 &\geq 0. \end{aligned}$$