How to approach a Probability Problem:-

Step 0:- Identify the Random Experiment.

Step 1:- Identify the Sample Space (Ω)

Step 2:- Identify the event of interest

Step 3:- Calculate the probability of each member of the sample space

Step 4:- Calculate the probability of the event of interest.

What is the probability the 2 people in a class of 105 have the same birthday?

Random Experiment:

Pick a Seq. of 105 numbers between 1 to 365 independently and uniformly at random

Sample Space:-

$$\Omega = \{1, 2, \ldots, 365\}^{105}$$

Event of interest:-

$$E = \{x_1, x_2, \ldots x_n \in \Omega : \exists\, i,j \; (i \neq j) \wedge (x_i = x_j)\}$$

$\bar{E}$ = set of all sequences where no 2 entries are the same.

Probability of each outcome:

$$P(\bar{x}) = \left(\frac{1}{365}\right)^{105} \text{ for every } \bar{x} \in \Omega$$

Probability of event:

$$P(E) = |E| \cdot P(\bar{x}) = \frac{|E|}{(365)^{105}} = 1 - P(\bar{E})$$

$$P(\bar{E}) = \frac{|\bar{E}|}{(365)^{105}}$$

$$= \frac{{}^{365}P_{105}}{(365)^{105}}$$

$$= \frac{\cancel{(365)}\,(364)!}{(260)!\,(365)^{\cancel{105}\ 104}}$$

$$= \frac{(364)\cdots\cdots(259)}{(365)\cdots\cdots(365)}$$

For $n = 365$, $k = 105$

$$P(\bar{E}) = \frac{{}^{n}P_{k}}{n^{k}}$$

$$= \frac{n}{n} \cdot \frac{n-1}{n} \cdot \cdots\cdots \cdot \frac{n-(k-1)}{n}$$

$$= 1 \cdot \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right)$$

$$\leq e^{-1/n} e^{-2/n} \cdots e^{-\frac{k-1}{n}}$$

$$P(\bar{E}) \leq e^{-\frac{1}{n}\left(\frac{k(k-1)}{2}\right)} \leq e^{-\frac{k^2}{2n}}$$

Putting values back in

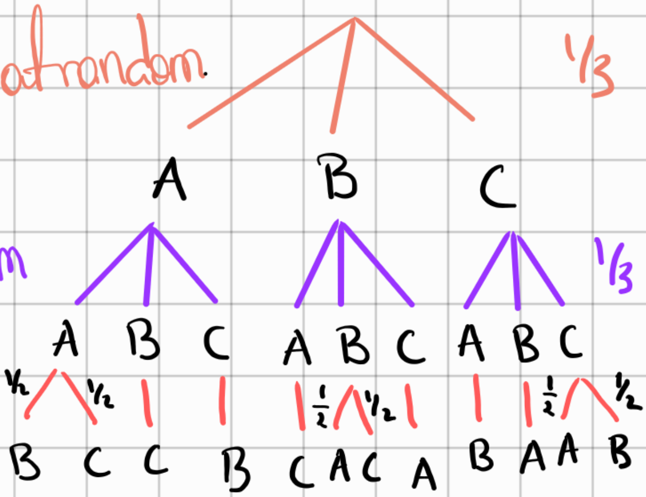
$$P(\bar{E}) \leq e^{-\frac{1}{365}\left(\frac{105 \cdot 104}{2}\right)}$$

$$1 - P(E) \leq e^{-\frac{1}{365}\left(\frac{105 \cdot 104}{2}\right)}$$

$$P(E) \geq 1 - e^{-\frac{1}{365}\left(\frac{105 \cdot 104}{2}\right)}$$

Monty Hall Problem:-

1. Host randomly picks a door at random.
2. Contestant picks a door at random
3. Host opens an empty door



$$\Omega = \{AAB, AAC, ABC, ACB, BAC, BBA, BBC, BCA, CAB, CBA, CCA, CCB\}$$

E = When staying wins.

$$= \left\{ \begin{matrix} AAB, AAC, \\ BBA, BBC, \\ CCA, CCB \end{matrix} \right\} \rightarrow P(x) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{18}$$

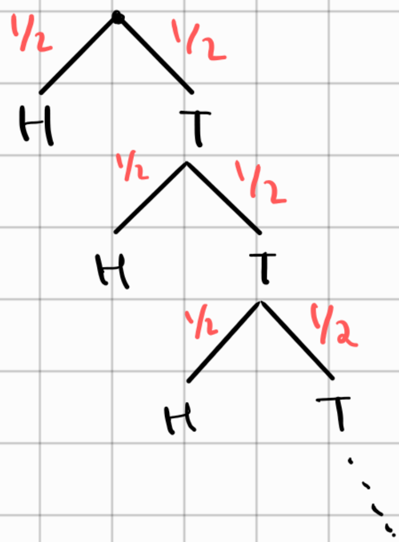
$$P(E) = \left(\frac{1}{18}\right) 6 = \frac{1}{3}$$

$$P(\bar{E}) = \frac{2}{3}$$

Coin tossing problem: (LLM)

E - First player wins

$$\Omega = \{H, TH, TTH, \dots\}$$



$$E = \{H, T^2H, T^4H, \dots\}$$

$$P(T^kH) = \frac{1}{2^{k+1}}$$

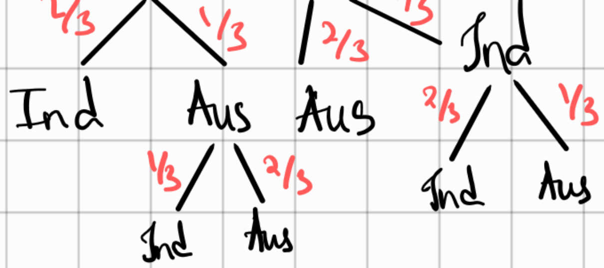
$$P(E) = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \dots = \frac{1/2}{3/4} = \frac{2}{3}$$

3-Match Series:

If a team wins a match, the probability that they win the next match increases to $\frac{2}{3}$

E - First team wins

$$P(E) = \left[\frac{1}{2}\cdot\frac{2}{3} + \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{3}\right] 2$$
$$= \frac{7}{9}.$$



Definitions:-

* A discrete probability space is a pair (Ω, p) where Ω is a countable set and p is a function from Ω to $[0,1]$ such that

$$\sum_{\omega \in \Omega} p(\omega) = 1$$

(a) Ω is called the sample space
(b) p is called the probability mass function.
(c) Any subset of Ω is called an event.
(d) We extend $p : \Omega \to [0,1]$ to a fn.

$$P : 2^{\Omega} \to [0,1] \text{ by}$$ [2^{Ω} is the powerset of Ω]

$$P(E) = \sum_{\omega \in E} p(\omega) \text{ for every } E \subseteq \Omega$$

P is called the probability function.

* Some properties of P.

(i) $P(\Omega) = 1$
(ii) $P(\emptyset) = 0$
(iii) $(A \subseteq B) \rightarrow (P(A) \leq P(B))$
(iv) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ [Special case of inclusion exclusion principle]
(v) $P(A \cup B) \leq P(A) + P(B)$

* Common pmf's:-

$p(0) = \alpha$

(i) Bernoulli-α : $\Omega = \{0,1\}$. $p(1) = 1-\alpha$

$0 \leq \alpha \leq 1$

(ii) Uniform : $\Omega = \{1,2,\dots,n\}$. $p(x) = \frac{1}{n} \ \forall x \in \Omega$

(iii) Binomial : $\Omega = \{0,1,\dots,n\}$ $p(k) = \frac{{}^nC_k}{2^n} \ \forall k \in \Omega$

(iv) Geometric : $\Omega = \{1,2,3,\dots\}$ $p(i) = (1-r)r^{i-1} \ \forall i \in \Omega$

(v) Poisson : $\Omega = \{0,1,2,\dots\}$ $p(i) = e^{-\lambda}\left[\frac{\lambda^i}{i!}\right] \forall i \in \Omega$

(vi) Generalized Binomial : $\Omega = \{0,1,2,\dots n\}$.

$$p(k) = \binom{n}{k}\alpha^k(1-\alpha)^{n-k} \quad \forall k \in \Omega$$

* Conditional Probability :-

If A and B are events on the same probability space (Ω, p) and $P(B) \neq 0$,

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow P(A \cup B/C) = P(A/C) + P(B/C) - P(A \cap B/C)$$

$$\boxed{\begin{array}{l} P_B(-) = P(-/B) \\ P_B : \Omega \to [0,1] \\ P_B(\omega) = \begin{cases} \frac{P(\omega)}{P(B)} & ; \omega \in B \\ 0 & ; \omega \notin B \end{cases} \end{array}}$$

* Law of Total Probability :-

$$P(A) = \sum_{i=1}^{n} P(E_i)P(E_i/A)$$

$\Rightarrow$ Medical test :-

Medical test:

A = A tested person has cancer
B = The person's test comes +ve

* 1 in 1000 people have cancer
* Test is 99% accurate.

$$P(A/B) = \frac{P(A\cap B)}{P(B)} = \frac{P(B/A)\,P(A)}{P(B/A)\cdot P(A) + P(B/\bar{A})\cdot P(\bar{A})}$$

[True positive]

$$= \frac{\frac{99}{100}\cdot\frac{1}{1000}}{\frac{99}{100}\cdot\frac{1}{1000} + \frac{1}{100}\cdot\frac{999}{1000}}$$

$$= \frac{99}{1098} = \frac{11}{122} \approx 9\%$$

⇒ Generally, medical tests talk with false positives and false negative for better clarity. This is because a test can be 99.9% accurate by just saying No all the time, given it is for a sufficiently rare disease.

* Baye's Rule:-

$$P(E_i/A) = \frac{P(A/E_i)\,P(E_i)}{\sum_{j=1}^{n} P(A/E_j)\,P(E_j)}$$

Law of total probability.

* Independence:-

⇒ We say 2 events from a probability space (Ω, P) are independent if

$P(A\cap B) = P(A)P(B)$ or $P(A/B) = P(A)$

e.g:- For a 6 sided die, $\Omega = \{1,2,3,4,5,6\}$, P - uniform

$$A = \{3,6\} \quad B = \{2,4,6\} \Rightarrow A \cap B = \{6\}$$

$$P(A) = \frac{1}{3} \quad P(B) = \frac{1}{2} \Rightarrow P(A \cap B) = \frac{1}{6}.$$

$\Rightarrow$ We say events A, B, C are mutually independent if

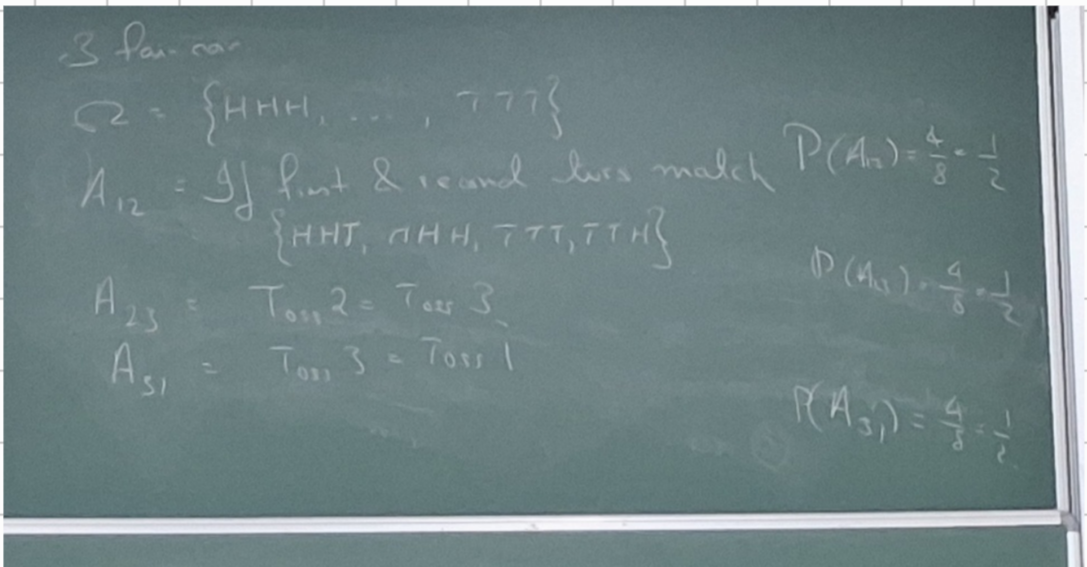
$$\begin{aligned} P(A \cap B) &= P(A) \cdot P(B) \\ P(B \cap C) &= P(B) \cdot P(C) \\ P(C \cap A) &= P(C) \cdot P(A) \\ P(A \cap B \cap C) &= P(A) \cdot P(B) \cdot P(C). \end{aligned}$$

* Mutual Independence:-

Events $A_1, \ldots, A_n$ over the probability space (Ω, P) are said to be mutually independent if for every subcollection of events, the probability of the intersection is equal to the product of the individual probabilities.

$\forall I \subseteq \{1, \ldots, n\}$

$$P\left(\bigcap_{i \in I} A_i\right) = \prod_{i \in I} P(A_i)$$



$A_{12} \cap A_{23} = \{TTT, HHH\}$ $\quad P(A_{12} \cap A_{23}) = \frac{2}{8} = \frac{1}{4}$

$= P(A_{12})P(A_{23})$

A_{12}, A_{23}, A_{31} are pair-wise independent

$$P(\underbrace{A_{12} \cap A_{23} \cap A_{31}}_{\{HHH, TTT\}}) = \frac{2}{8} = \frac{1}{4} \neq P(A_{12})P(A_{23})P(A_{31})$$

2 fair coin

$\Omega = \{HH, \dots, TT\}$

$A_1 = \{HT, HH\}$ (Toss 1 = head) $\quad \forall i,j \quad P(A_i) = \frac{1}{2}$

$A_2 = \{HH, TH\}$ (Toss 2 = head) $\quad P(A_i \cap A_j) = \frac{1}{4}$

$A_3 = \{HH, TT\}$ (Toss 1 = Toss 2) $\quad = P(A_i)P(A_j)$

$P(A_1 \cap A_2 \cap A_3) \overset{?}{=} 1/8$

$\neq$

$\Omega = \{6, 12, 15, 45, 10, 20\}$ $\quad P$: uniform

$A_2 =$ multiples of 2 in $\Omega = \{6, 12, 10, 20\}$

$A_3 =$ " 3 " $= \{6, 12, 15, 45\}$

$A_5 =$ " 5 " $= \{15, 45, 10, 20\}$

$\Omega = \{30, 7, 11, 2^1, \dots, 2^8, 3^1, \dots, 3^8, 5^1, \dots, 5^8\}$ p-uniform

$A_2 =$ multiples of 2 $= \{30, 2^1 \dots 2^8\}$ $\quad P(A_2) = \frac{1}{3}$

$A_3 =$ multiples of 3 $= \{30, 3^1, \dots, 3^8\}$ $\quad P(A_3) = \frac{1}{3}$

$A_5 =$ multiples of 5 $= \{30, 5^1, \dots, 5^8\}$ $\quad P(A_5) = \frac{1}{3}$

$$P(A_2 \cap A_3) = \frac{1}{27} \neq \frac{1}{3} \cdot \frac{1}{3}$$

$$P(A_2 \cap A_3 \cap A_5) = \frac{1}{27} = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}$$