1. **(Maximizing profit)** Miss Marple is preparing to sell handmade bookmarks and origami art pieces at the St. Mary Mead book fair. She wants to decide how many of each item to produce and bring in order to maximize her profit. She earns:

   - Rs. 10 profit for every bookmark sold,
   - Rs. 7 profit for every origami piece sold.

   However, Miss Marple faces the following constraints:

   - She has only enough paper to make up to 50 bookmarks.
   - The stall has limited display space: both bookmarks and origami pieces, along with some decorative props (like stands and lights), take up space. The total number of bookmarks, origami pieces, and display props cannot exceed 80 units.
   - Time is also a factor: it takes 2 minutes to make a bookmark and 5 minutes to make an origami piece. She has a maximum of 3 hours before the fair begins.

   Miss Marple must determine the optimal number of bookmarks (x_1), origami pieces (x_2), and display props (x_3) to bring to maximize her total profit, while staying within all constraints.

   i. Pose this as an LP. **(5 points)**

   Solution:

$$
\begin{aligned}
\text{maximize} \quad & 10x_1 + 7x_2 & \\
\text{subject to} \quad & x_1 \leq 50 & \text{(paper constraint)} \\
& x_1 + x_2 + x_3 \leq 80 & \text{(display space)} \\
& 2x_1 + 5x_2 \leq 180 & \text{(time limit)} \\
& x_1, x_2, x_3 \geq 0 & \text{(non-negativity)}
\end{aligned}
$$

   ii. Convert it to standard form. **(5 points)**

   Solution:

$$
\begin{aligned}
\text{maximize} \ & 10x_1 + 7x_2 \\
\text{subject to} \ & x_1 + 0x_2 + 0x_3 + s_1 = 50 \\
& x_1 + x_2 + x_3 + s_2 = 80 \\
& 2x_1 + 5x_2 + 0x_3 + s_3 = 180 \\
& x_1, x_2, x_3, s_1, s_2, s_3 \geq 0
\end{aligned}
$$

2. **(Weak duality)** Consider the following LPs.

(P) : $\min c^\top x$ subject to $Ax \geq b, x \geq 0$. **(D)** : $\max y^\top b$ subject to $y^\top A \leq c^\top, y \geq 0$.

i. Prove the weak duality lemma for the symmetric (P) and (D) above. **(5 points)**

Solution: Since y is feasible, we have

$$y^\top A \leq c^\top \quad \Longrightarrow \quad y^\top Ax \leq c^\top x \quad \text{for } x \geq 0.$$

Since x is feasible, we have

$$Ax \geq b \quad \Longrightarrow \quad y^\top Ax \geq y^\top b \quad \text{for } y \geq 0.$$

Putting the two inequalities together completes the proof. □

ii. Using weak duality lemma, prove that if x and y are feasible for (P) and (D), respectively, and

$$c^\top x = y^\top b,$$

then x and y are *optimal* for (P) and (D), respectively. **(5 points)**

Solution: Let $\widetilde{x}$ be an arbitrary feasible solution for (P). Since y is a feasible solution for (D), we have

$$y^\top b \leq c^\top \widetilde{x}. \qquad \text{(weak duality)}$$

Since $c^\top x = y^\top b$, we have

$$c^\top x \leq c^\top \widetilde{x}.$$

Since this is true for every feasible solution $\widetilde{x}$ of (P) and since x itself is feasible, we conclude that x is indeed a minimum. Using similar arguments, we conclude that y is optimal for (D).

3. **(Solving an LP)** Consider the LP

$$\textbf{(P)} : \max \; y^\top b \text{ subject to } y^\top A \leq c^\top, y \geq 0,$$

where

$$A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}; \qquad b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \qquad c = \begin{bmatrix} 2 & 2 & 5 \end{bmatrix}^\top.$$

i. Fill in the blanks to obtain the dual of the problem. **(6 points)**

Solution:

$$\begin{aligned} \textbf{(D)} : \quad \underline{\min} \quad & 2x_1 + \underline{2x_2} + \underline{5x_3} \\ \text{subject to } \; & \underline{-2x_1} + x_2 + \underline{x_3} \geq \underline{+1} \\ & \underline{x_1 - 2x_2 + x_3} \geq \underline{-1} \\ & \underline{x_1, x_2 \geq 0}. \end{aligned}$$

ii. Let $y^* = \begin{bmatrix} 1 & 4 \end{bmatrix}^\top$.

(a) Is y^* feasible for (P)? **Answer:** Yes/~~No~~ **(1 point)**

(b) Is y^* optimal for (P)? Check using complementary slackness. **(8 points)**

Hint – Which constraint of (P) does y^* satisfy with a slack?
Since $y_1^* > 0$ and $y_2^* > 0$, what can you say about the corresponding constraints in (D)?

Solution: The given y^* satisfies the **second** constraint with a slack. Therefore, if y^* is optimal for (P), then the optimal solution for (D), x^*, must satisfy

$$x_2^* = 0.$$

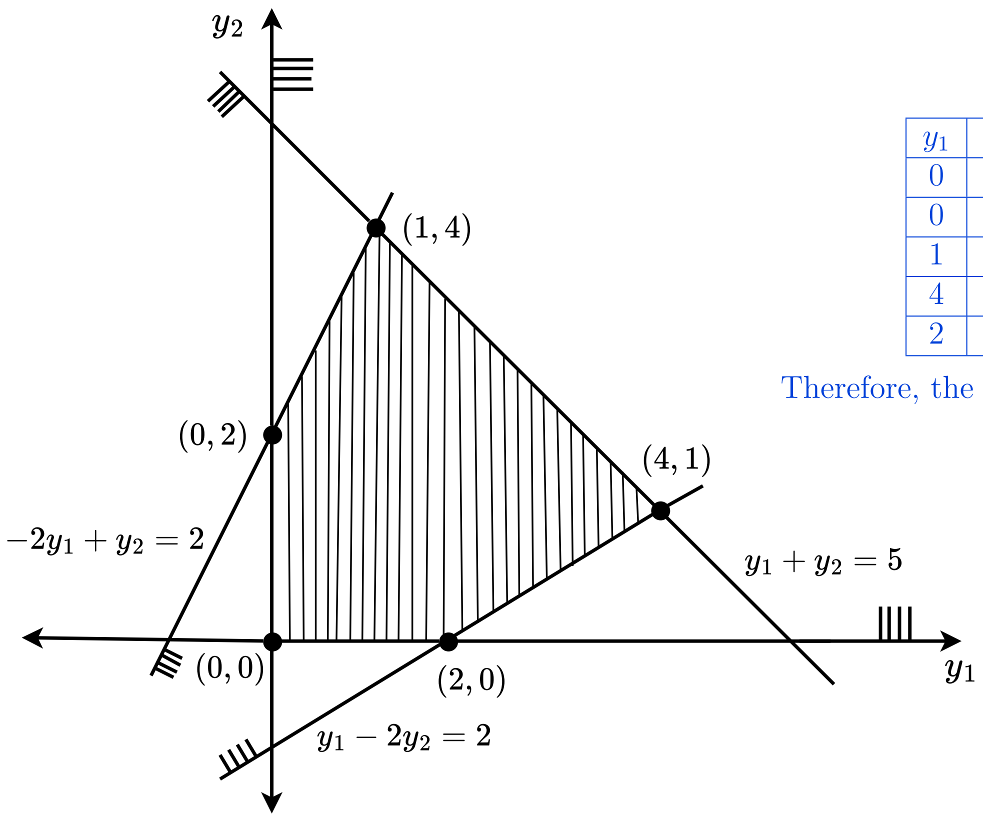
Also, since $y_1^* > 0$ and $y_2^* > 0$, we must have $Ax^* = b$. Putting these together, we have

$$\begin{aligned} -2x_1^* + x_3^* &= 1 \\ x_1^* + x_3^* &= -1 \end{aligned}$$

whereby $x^* = \begin{bmatrix} -\frac{2}{3} & 0 & -\frac{1}{3} \end{bmatrix}^\top$. However, observe that x^* is not feasible for (D). Hence, y^* cannot be optimal for (P).

iii. Solve (P) graphically. **(9 points)**

Caution: Label the lines, shade the feasible region, and label the corner points.
Construct the table containing the objective function values for all the corner points.



y_1	y_2	z
0	0	0
0	2	-2
1	4	-3
4	1	3
2	0	2

Therefore, the optimum is $(4,1)$.

iv. Using the (correct) solution obtained in iii. and the result in question 2.ii., check if $x^* = \begin{bmatrix} 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}^\top$ is optimal for (D). **(6 points)**

Solution: First, it is easy to verify that x^* is feasible for (D). From iii. we obtain $y^* = \begin{bmatrix} 4 & 1 \end{bmatrix}^\top$ and

$$(y^*)^\top b = 3.$$

For the given x^*, we have

$$c^\top x^* = 3.$$

Hence, by the result in Q2, we conclude that x^* is optimal for (D).