1. Let $A \in \mathbb{R}^{m \times n}$, $m \geq n$. Show that
$$\mathcal{N}(A) = \mathcal{N}(A^\top A).$$

2. Given $A \in \mathbb{R}^{m \times n}$, $m \geq n$, show that the vector $h \in \mathrm{Col}(A)$ that is the closest to a vector $b \in \mathbb{R}^m$ satisfies
$$a^\top (h - b) = 0$$
for all $a \in \mathrm{Col}(A)$ and vice versa.

3. Consider the matrix
$$A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
and a vector $b \in \mathbb{R}^2$.

   (a) Let
   $$b = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$
   (i) Plot the vector b and the column space of A.
   (ii) Compute and plot the least squares solution $\hat{x}$ that minimizes $\|Ax - b\|$.
   (iii) Is b in the column space of A? What do you observe about the residual vector $r = b - A\hat{x}$?

   (b) Let
   $$b = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$
   (i) Repeat the steps above: plot b, the column space of A, and the least squares solution $\hat{x}$.
   (ii) Plot the residual vector $r = b - A\hat{x}$ and verify that it is orthogonal to the column space of A.
   (iii) Compare this case to part (a) and comment on the difference in geometry.

4. A spring is stretched to lengths $L = 3, 4,$ and 5 cm under applied forces $F = 1, 2,$ and 4 N, respectively. Assuming Hooke's law
$$L = a + bF,$$
estimate the length to which the spring would be stretched when the force applied is 3 N.

5. We are given two mixtures, A and B. Mixture A contains 30% gold, 40% silver, and 30% platinum, whereas mixture B contains 10% gold, 20% silver, and 70% platinum (all percentages of weight). We wish to determine the ratio of the weight of mixture A to the weight of mixture B such that we have as close as possible to a total of 5 ounces of gold, 3 ounces of silver, and 4 ounces of platinum. Formulate and solve the problem using the linear least-squares method.

6. We are performing an experiment to calculate the gravitational constant g as follows. We drop a ball from a certain height and measure its distance from the original point at certain time instants. The results of the experiment are shown in the following table.

Time (seconds)	1.00	2.00	3.00
Distance (metres)	5.00	19.50	44.00

The equation relating the distance s and the time t at which s is measured is given by: $s = \frac{1}{2}gt^2$.

(a) Find the least-squares estimate of g (in metre per second squared) using the experimental results from the table above.

(b) Now, suppose that we take an additional measurement at time 4.00 seconds, and obtain a distance of 78.5 metres. Use the recursive least-squares algorithm to calculate an updated least-squares estimate of g.

7. You are modeling a linear relationship between input t and output y using the model:

$$y = \alpha t + \beta$$

Using the following four data points of the form (t, y): $(1, 3),\ (2, 5),\ (3, 7),\ (4, 9)$, the parameter estimate was computed using least squares to yield : $x^{(0)} = (\alpha, \beta) = (2, 1)$.

Now, three new data points arrive:

$$(t_1, y_1) = (1, 5), \quad (t_2, y_2) = (2, 4), \quad (t_3, y_3) = (3, 6)$$

(a) **Batch Update:** Use all three new data points together to compute $x^{(1)}$.

(b) **Step-by-step Update:** Starting from $x^{(0)}$ and G_0, update the parameter estimate using RLS with one new data point at a time. Verify that the final estimate matches the $x^{(1)}$ obtained in the previous part.

8. Verify the Sherman-Morrison-Woodbury formula.