CS2020A Discrete Mathematics

Tutorial 2 Submission

Submitted By

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Prove the following or give a counterexample.

Theorem 1: For every positive integer n, the product $n(n+\frac{1}{2})(n+1)$ is a multiple of 3.

Proof: Given product is $n(n+\frac{1}{2})(n+1)$. This can written as

$$\begin{aligned} n(n+\frac{1}{2})(n+1) &= n(\frac{2n+1}{2})(n+1) \\ &= \frac{n(n+1)(2n+1)}{2} \\ &= \frac{n(n+1)(2n+1)}{6} \cdot 3 \\ &= 3\left[\frac{n(n+1)(2n+1)}{6}\right] \\ &= 3(\Sigma n^2) \\ &= 3t, \text{ where } t \in \mathbb{Z} \end{aligned}$$

The sum of squares of all natural numbers untill n will always be a natural number. Thus, the given product will always be a multiple of 3.

Theorem 2: The product of any k consecutive integers is a multiple of $k!$ (k-factorial)

Proof: Let k consecutive integers be $n+1$, $n+2$,..., $n+k$,where $n \in \mathbb{Z}$. We can write the product of these numbers as follows:

$$(n+1)(n+2)\dots(n+k) = \frac{(n+k)!}{n!}$$

On dividing this product with $k!$, we get the following.

$$\begin{aligned} \frac{(n+1)(n+2)\dots(n+k)}{k!} &= \frac{(n+k)!}{n! \cdot k!} \\ &= \frac{(n+k)!}{k! \cdot (n+k-k)!} \\ &= {}^{n+k}C_k \text{ or } \binom{n+k}{k} \end{aligned}$$

We know that ${}^{n+k}C_k$ will always be an integer. This means that the product of k consecutive integers taken is always perfectly divisible by $k!$, as no remainder is left during the process.

$\therefore$ The product of any k consecutive integers is a multiple of $k!$ (k-factorial).

Theorem 3: For every two positive integers n and k, $n^k - n$ is a multiple of k.

Proof: Let us take a Counter Example: n=2 and k=4.

$$n^k - n = 2^4 - 2 = 16 - 2 = 14$$

14 is not a multiple of 4
Therefore, $n^k - n$ is not multiple of k for any n and k.
$\therefore$ Theorem 3 is wrong.

Theorem 4: Every positive integer (greater than 1) can be expressed as a product of prime numbers.
Proof: Suppose Theorem 4 is false. Then there exists positive integers which cannot be expressed as a product of primes. Let n be the smallest such counterexample. Complete the above proof.

Solution:

Case 1: n is prime
In this case, we have nothing to prove because n itself is a prime factor of itself.

Case 2: n is not prime
This means that n can be expressed as

$$n = a \times b \text{ (where a,b} \in \mathbb{Z} \text{ and } 1 \leq a, b < n)$$

$\because a, b < n$
$\therefore$ a and b can be expressed as a product of primes
i.e, for prime numbers $p_1, p_2, ...p_k$ and $q_1, q_2, ...q_l$

$$a = p_1 p_2 .... p_k$$
$$b = q_1 q_1 .... q_l$$

$$\therefore n = p_1 p_2 ... p_k q_1 q_2 ... q_l$$

Thus, even in this case n can be expressed as a product of primes.

Theorem 5: There are infinitely many prime numbers.

Proof: Let there be a finite number of prime numbers, $p_1, p_2, p_3..., p_n$
Now we can construct a new number q as

$$q = (p_1.p_2.p_3...p_n) + 1$$

Now on dividing q with any of the prime numbers, it will always give remainder 1.

So, now we have 2 cases:

Case 1: q is a prime number
In this case, we can clearly see that q has to be larger than $p_1, p_2, ...p_n$. So this contradicts our earlier assumption that there are finite number of primes.

Case 2: q is a composite number
On dividing q with any of the prime numbers $p_1, p_2, ...p_n$, it always gives the remainder as 1. But since every number can be written as a product of prime numbers (Fundamental Theorem of Arithmetic), there must exist some primes other than $p_1, p_2, ...p_n$ which divide q. So this contradicts our earlier statement saying there are finite number of prime numbers.

$\therefore$ We can say that there are infinitely many prime numbers.

Theorem 6: Every prime number larger than 3 is just one away from 3!

Proof: We know that $3! = 6$ Now every single positive integer can be represented as

$$\begin{gathered} 6n \\ 6n+1 \\ 6n+2 \\ 6n+3 \\ 6n+4 \\ 6n+5 \end{gathered}$$

where $n \in \mathbb{N} \cup \{0\}$
Now from here we can clearly see that the following are composite numbers, as they have factors other than 1 and itself.

$$\begin{aligned} 6n &= 6 \cdot n \\ 6n+2 &= 2(3n+1) \\ 6n+3 &= 3(2n+1) \\ 6n+4 &= 2(3n+2) \end{aligned}$$

Now if we take $n = 0$, $6n+2$ gives 2 and $6n+3$ gives 3. So, other than these prime numbers, all other primes have to be expressed as one of these 2 options:

$$\begin{aligned} &6n+1 \\ &6n+5 = 6(n+1) - 1 \end{aligned}$$

So all prime numbers can be written as $(6n \pm 1)$ except 2 and 3.

$\therefore$ All prime numbers greater than 3 can be written as one away from 6, i.e., 3!.