CS2020A Discrete Mathematics

Tutorial 8 Submission

Submitted By

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Defn: An undirected graph G is called connected if there is a path between any two vertices in G.

Problem 1. What is the minimum number of edges in a connected undirected graph on n vertices? Justify.

Solution. The minimum number of edges in a connected undirected graph of n vertices is $n-1$.

Proof: Using Structural Induction

Base Case: For $n = 2$ vertices, the minimum number of edges is 1 which satisfies the statement, as $2 - 1 = 1$

Induction Hypothesis: Let the minimum number of edges in a connected undirected graph of k vertices be $k - 1$.

Now let there be two graphs with k vertices which are connected using minimum number of edges. So each graph has $k - 1$ edges.

Now to connect these two graphs, we just need one edge, from any one vertex of the first graph to any of the vertex of the second graph.

This would make a connected graph containing $2k$ vertices with minimum number of edges.
So the number of edges used is $(k-1)+(k-1)+1 = 2k-1$

Therefore, from structural induction we can conclude that the minimum number of vertices required connect n vertices is $n-1$.

Defn: A simple undirected graph $G = (V, E)$ is called bipartite if the vertex set V can be partitioned into two parts V_1 and V_2 such that all the edges in E have one endpoint in V_1 and the other in V_2.

Problem 2. What is the maximum number of edges possible in a (simple undirected) bipartite graph on n vertices? Justify.

Solution. Let $G = (V, E)$ be a bipartite graph with n vertices. The vertex set V is partitioned into two disjoint sets V_1 and V_2. Let $|V_1| = k$, so $|V_2| = n - k$.

To maximize the number of edges, every vertex in V_1 must be connected to every vertex in V_2. The number of edges is then $n(E) = k(n - k)$. The product of two numbers with a fixed sum $(k + (n - k) = n)$ is maximized when the numbers are as close to each other as possible. This means we must partition the n vertices as evenly as possible.

Case 1: n is even.
The most even partition is into two sets of size $n/2$.

$$n(E)_{max} = \frac{n}{2} \times \frac{n}{2} = \frac{n^2}{4}$$

Case 2: n is odd.
The most even partition is into sets of size $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$.

$$n(E)_{max} = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil = \left(\frac{n-1}{2} \right) \left(\frac{n+1}{2} \right) = \frac{n^2 - 1}{4}$$

Both results can be combined into a single expression using the floor function. The maximum number of edges is:

$$n(E)_{max} = \left\lfloor \frac{n^2}{4} \right\rfloor$$

Defn: A simple undirected graph $G = (V, E)$ is called k-colourable if one can colour all the vertices in V with at most k different colours so that the colors of any two vertices u and v, such that $\{u, v\} \in E$ are different.

Problem 3. Consider the infinite graph $G = (\mathbb{Z}, E)$ where $\{i, j\} \in E$ iff $(i - j)$ is not a multiple of 5. What is the smallest k for which G is k-colorable? Justify.

Solution. We know that any integer can be expressed as
$5n$
$5n + 1$
$5n + 2$
$5n + 3$
or, $5n + 4$
where $n \in \mathbb{Z}$
Let us divide the integers into the above mentioned groups.
In the graph the elements of the group $5n$ are connected to all the elements except for the elements in $5n$ itself. The same case can applied for the elements in all the other groups.
Thus, the minimum number of colors, k for which the graph G is k-colorable is equal to the number of such groups i.e, **5** in this case.

Defn: A cycle in a graph $G = (V, E)$ is a sequence $v_1, v_2, \ldots, v_k$ of vertices such that $v_k = v_1$ and for each i from 1 to $k - 1$, $\{v_i, v_{i+1}\} \in E$. The length of a cycle is the length of the above sequence.

Problem 4. Prove that a graph in which every vertex has degree at least two contains a cycle.

Solution. Let the graph $G = (V, E)$ containing $v_1, v_2, \ldots, v_k$ as vertices be the longest connected graph where each vertex has a degree of at least 2 but does **not** contain a cycle.
Since v_k also has a degree ≥ 2, there can be 2 cases:

Case 1: v_k is connected to a vertex other than the vertices in G. But this leads to a contradiction to the assumption that G is the longest such connected graph.

Case 2: v_k is connected to one of the vertices of G other than v_k itself. But this again contradicts the assumption that G doesn't contain a cycle.

Therefore, the assumption is false. A graph in which every vertex has degree at least 2 contains a cycle.

Defn: An n-cycle is a graph consisting of a single cycle of length n and nothing else.

Problem 5. For what values of n is the n-cycle 2-colorable?

Solution. For an n-cycle to be 2-colourable, the colours should alternate around the cycle, with v_k and v_{k+1} having different colours and v_1 and v_n also having different colours. So, if we give colour 1 to all the odd vertices, $v_1, v_3, v_5 \ldots$, and colour 2 to all the even vertices, $v_2, v_4, v_6 \ldots$, then the vertex n has to have colour 2.
So, **n should be even** for the n-cycle to be 2-colourable.

Theorem: The following statements are equivalent for a simple undirected graph $G = (V, E)$.

a) G is bipartite.

b) G is 2-colorable.

c) G does not contain a cycle of odd length.

Problem 6. Prove or give a counterexample.

Solution. We can show statements a, b, c are equivalent if (a $\leftrightarrow$ b) and (b $\leftrightarrow$ c). Because, then (a $\leftrightarrow$ c) aswell.

(i) **a $\leftrightarrow$ b**

- a $\rightarrow$ b
  If $G = (V, E)$ is bipartite, with V partitioned into V_1 and V_2, we can color all the vertices in V_1 one color and the vertices in V_2 another color, so that adjacent vertices receive different colors. So, G is 2-colorable.
- a $\leftarrow$ b
  If G is 2-colorable, let V_1 be set of vertices with color 1 and V_2 be the set of vertices with color 2. By definition of a 2-colorable graph, no edge will have 2 endpoints in only V_1 or only V_2. So V_1 and V_2 is a bi-partition of V and thus G is bipartite.

(ii) **b $\leftrightarrow$ c**

- b $\rightarrow$ c
  For contradiction, assume G contains an odd cycle of vertices $v_1 v_2 \ldots v_k v_1$. Now, for any 2-colorable graph, the colors alternate around the cycle, leaving v_k with the same color as v_1 when k is odd. This shouldn't be possible as v_1 and v_k are adjacent vertices. Hence, a 2-colorable graph will never have an odd cycle.
- b $\leftarrow$ c
  If G does not have an odd cycle, then it contains either an even cycle or no cycle at all. If it has no cycle, then it is always 2-colorable. As we can color a vertex with color 1, its adjacent vertices color 2 and now treat each of the branches from these new vertices as their own graphs, meaning they can each be 2-colorable as well. If the graph contains an even cycle, then the subgraph with even cycle can be colored with alternating colors, and then multiple even cycle graphs and non-cyclic graphs can be joined to achieve the original graph, which is still 2-colorable.