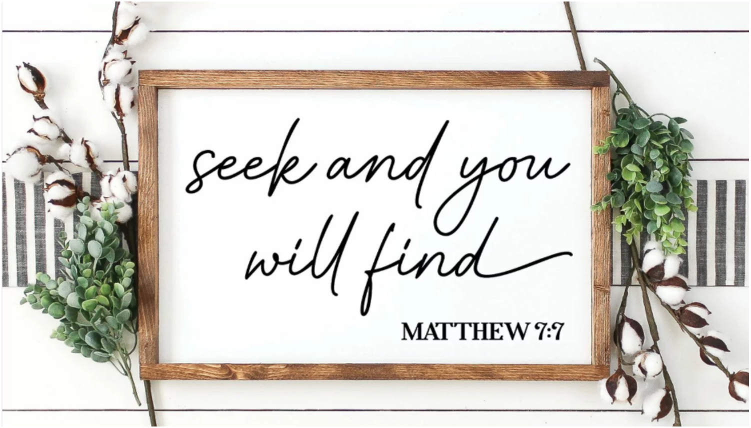

Introduction to Optimization

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- Iterative algorithm

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- Golden section (f)
- Fibonacci (f)
- Bisection (f')
- Secant (f')
- Newton's (f' and f'')

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  - ρ satisfies $1 - 2\rho = \rho(1 - \rho)$:

$$\boxed{\rho = \frac{3 - \sqrt{5}}{2} \approx 0.382}$$

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– Observe

$$\frac{\rho}{1 - \rho} = \frac{1 - \rho}{1}$$

– *Why the name?* If you divide a line into two parts ($a < b$) with a equal to ρ times the original length ($a + b$), then

$$\frac{a + b}{b} = \frac{b}{a}$$

Greek geometers called this the golden section

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The solution is given by

$$\begin{aligned}
\rho_1 &= 1 - \frac{F_N}{F_{N+1}} \\
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&\vdots \\
\rho_N &= 1 - \frac{F_1}{F_2}.
\end{aligned}$$

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3	3
4	5
5	8
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$\boxed{\text{Better reduction than golden section}}$ Why?

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uses only f'

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