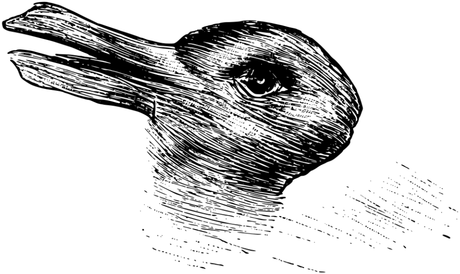
Welche Thiere gleichen einander am meisten?



Kaninchen und Ente.

Introduction to Optimization

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Data Science

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The Shire market

maximize profit

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Hobbits craft three magical goods using two scarce resources:

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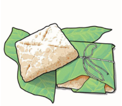

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a $\min$ and a $\max$

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Primal:

$$\begin{aligned} \min \quad & c^\top x \\ \text{subject to} \quad & Ax \geq b, \\ & x \geq 0. \end{aligned}$$

Dual:

$$\begin{aligned} \max \quad & y^\top b \\ \text{subject to} \quad & y^\top A \leq c^\top, \\ & y \geq 0. \end{aligned}$$

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Dual of the dual?

Asymmetric form of duality

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- Weak duality $\quad$ "$\max \leq \min$"

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- Strong duality — "$c^\top x^* = (y^*)^\top b$"
- Complementary slackness — *iff conditions for optimality*

Duality Theorems

1. Weak Duality: $\max \leq \min$

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■

Another perspective of duality

Back to the first example

$$\begin{aligned} \text{maximize } & 4x_1 + 5x_2 \\ \text{subject to } & x_1 + x_2 \leq 20 \\ & 3x_1 + 4x_2 \leq 72 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

Another perspective of duality

Back to the first example

$$\begin{aligned}\text{maximize } & 4x_1 + 5x_2 \\ \text{subject to } & x_1 + x_2 \leq 20 \\ & 3x_1 + 4x_2 \leq 72 \\ & x_1 \geq 0 \\ & x_2 \geq 0\end{aligned}$$

How did we conclude that 92 was **optimal**?

Another perspective of duality

Back to the first example

$$
\begin{aligned}
\text{maximize } & c_1x_1 + c_2x_2 \\
\text{subject to } & a_{11}x_1 + a_{12}x_2 \le b_1 \\
& a_{21}x_1 + a_{22}x_2 \le b_2 \\
& x_i \ge 0, i = 1, 2.
\end{aligned}
$$

Another perspective of duality

Back to the first example

maximize $c_1 x_1 + c_2 x_2$

subject to $a_{11} x_1 + a_{12} x_2 \leq b_1$

$a_{21} x_1 + a_{22} x_2 \leq b_2$

$x_i \geq 0, i = 1, 2.$

How do we find the **best upper bound** on $c^\top x$?

best = smallest

Another perspective of duality

Back to the first example

$$\begin{aligned} \text{maximize} \quad & c_1x_1 + c_2x_2 \\ \text{subject to} \quad & a_{11}x_1 + a_{12}x_2 \le b_1 \\ & a_{21}x_1 + a_{22}x_2 \le b_2 \\ & x_i \ge 0, i = 1, 2. \end{aligned}$$

How do we find the **best upper bound** on $c^\top x$?

best = smallest

(complete on board)

The 9 possibilities of the primal-dual pair

. . . fill using weak duality: $\max \leq \min$

(P): $\min\ c^\top x$ **subject to** $Ax \geq b, x \geq 0.$

(D): $\max\ y^\top b$ **subject to** $y^\top A \leq c^\top, y \geq 0.$

(P) \ (D)	unbounded	feasible & finite	infeasible
unbounded			
feasible & finite			
infeasible			

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(P) \ (D)	unbounded	feasible & finite	infeasible
unbounded			
feasible & finite			
infeasible			

- If (P) is unbounded, then $c^\top x^* = -\infty$

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unbounded			
feasible & finite			
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(P) \ (D)	unbounded	feasible & finite	infeasible
unbounded	✗		
feasible & finite			
infeasible			

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(P) \ (D)	unbounded	feasible & finite	infeasible
unbounded	✗	✗	
feasible & finite			
infeasible			

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(P) \ (D)	unbounded	feasible & finite	infeasible
unbounded	✗	✗	$\Uparrow$
feasible & finite			
infeasible			

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The 9 possibilities of the primal-dual pair

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(P) \ (D)	unbounded	feasible & finite	infeasible
unbounded	✗	✗	$\Uparrow$
feasible & finite			
infeasible			

- If (D) is unbounded, then $(y^*)^\top b = +\infty$

The 9 possibilities of the primal-dual pair

. . . fill using weak duality: $\max \leq \min$

(P): $\min\, c^\top x$ **subject to** $Ax \geq b,\, x \geq 0.$

(D): $\max\, y^\top b$ **subject to** $y^\top A \leq c^\top,\, y \geq 0.$

(P) \ (D)	unbounded	feasible & finite	infeasible
unbounded	✗	✗	$\Uparrow$
feasible & finite			
infeasible			

- If (D) is unbounded, then $(y^*)^\top b = +\infty \underset{\text{weak duality}}{\Longrightarrow}$ (P) is infeasible

The 9 possibilities of the primal-dual pair

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(P): $\min c^\top x$ **subject to** $Ax \ge b, x \ge 0.$ **(D)**: $\max y^\top b$ **subject to** $y^\top A \le c^\top, y \ge 0.$

(P) \ (D)	unbounded	feasible & finite	infeasible
unbounded	✗	✗	$\Uparrow$
feasible & finite	✗		
infeasible			

- If (D) is unbounded, then $(y^*)^\top b = +\infty \underset{\text{weak duality}}{\Longrightarrow}$ (P) is infeasible

The 9 possibilities of the primal-dual pair

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(P) \ (D)	unbounded	feasible & finite	infeasible
unbounded	✗	✗	$\Uparrow$
feasible & finite	✗		
infeasible	$\Leftarrow$		

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(P) \ (D)	unbounded	feasible & finite	infeasible
unbounded	✗	✗	$\Uparrow$
feasible & finite	✗		
infeasible	$\Leftarrow$		

- Both (P) and (D) infeasible –

The 9 possibilities of the primal-dual pair

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(P) \ (D)	unbounded	feasible & finite	infeasible
unbounded	✗	✗	$\Uparrow$
feasible & finite	✗		
infeasible	$\Leftarrow$		

- Both (P) and (D) infeasible – **possible!** (see E.g. 17.7 in textbook)

The 9 possibilities of the primal-dual pair

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(P) \ (D)	unbounded	feasible & finite	infeasible
unbounded	✗	✗	$\Uparrow$
feasible & finite	✗		
infeasible	$\Leftarrow$		*possible*

Corollary to weak duality

Weak Duality: $\max \leq \min$

Lemma 1 – Suppose x_0 and y_0 are feasible solutions for (P) and (D), respectively. If

$$c^\top x_0 = y_0^\top b,$$

then x_0 and y_0 are optimal for (P) and (D), respectively.

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Proof – Let x be an arbitrary feasible solution for (P).

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Proof – Let x be an arbitrary feasible solution for (P). Since y_0 is a feasible solution for (D),

$$y_0^\top b \leq c^\top x.$$

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Since $c^\top x_0 = y_0^\top b$, we have

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Corollary to weak duality

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(Exer: complete the proof) □

Corollary to weak duality

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Since $c^\top x_0 = y_0^\top b$, we have

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(Exer: complete the proof) □

$\boxed{\text{Converse?}}$

Duality Theorems (contd.)

Theorem – If (P) has an optimal solution, then so does (D)

Duality Theorems (contd.)

1. Weak Duality 2. Strong Duality: $c^\top x^* = (y^*)^\top b$

Theorem – If (P) has an optimal solution, then so does (D) and the optimal values are equal.

Duality Theorems (contd.)

1. Weak Duality
2. Strong Duality: $c^\top x^* = (y^*)^\top b$

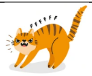
Theorem – If (P) has an optimal solution, then so does (D) and the optimal values are equal.

(P) \ (D)	**unbounded**	**feasible & finite**	**infeasible**
unbounded	✗	✗	$\Uparrow$
feasible & finite	✗		
infeasible	$\Leftarrow$		*possible*

Duality Theorems (contd.)

1. Weak Duality 2. Strong Duality: $c^\top x^* = (y^*)^\top b$

Theorem – If (P) has an optimal solution, then so does (D) and the optimal values are equal.

(P) \ (D)	unbounded	feasible & finite	infeasible
unbounded	✗	✗	$\Uparrow$
feasible & finite	✗		✗
infeasible	$\Leftarrow$		*possible*

Duality Theorems (contd.)

1. Weak Duality 2. Strong Duality: $c^\top x^* = (y^*)^\top b$

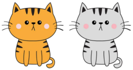
Theorem – If (P) has an optimal solution, then so does (D) and the optimal values are equal.

(P) \ (D)	**unbounded**	**feasible & finite**	**infeasible**
unbounded	✗	✗	$\Uparrow$
feasible & finite	✗		✗
infeasible	$\Leftarrow$	✗	*possible*

Duality Theorems (contd.)

1. Weak Duality 2. Strong Duality: $c^\top x^* = (y^*)^\top b$

Theorem – If (P) has an optimal solution, then so does (D) and the optimal values are **equal**.

(P) \ (D)	**unbounded**	**feasible & finite**	**infeasible**
unbounded	✗	✗	$\Uparrow$
feasible & finite	✗		✗
infeasible	$\Leftarrow$	✗	*possible*

Duality Theorems (contd.)

1. Weak Duality   2. Strong Duality   3. Complementary slackness

(P): $\min\ c^\top x$ s.t. $Ax \geq b, x \geq 0.$ **(D)**: $\max\ y^\top b$ s.t. $y^\top A \leq c^\top, y \geq 0.$

Duality Theorems (contd.)

1. Weak Duality   2. Strong Duality   3. Complementary slackness

(P): $\min\ c^\top x$ s.t. $Ax \geq b, x \geq 0.$ **(D)**: $\max\ y^\top b$ s.t. $y^\top A \leq c^\top, y \geq 0.$

Theorem – The feasible solutions x and y of (P) and (D), respectively, are optimal *if and only if*

i. $(c^\top - y^\top A)x = 0$

ii. $y^\top (Ax - b) = 0.$

Duality Theorems (contd.)

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ii. $y^\top (Ax - b) = 0.$

(symmetric case: proof on board)

Duality Theorems (contd.)

1. Weak Duality 2. Strong Duality 3. Complementary slackness

(P): $\min\ c^\top x$ s.t. $Ax = b, x \geq 0.$ **(D)**: $\max\ y^\top b$ s.t. $y^\top A \leq c^\top$.

Theorem – The feasible solutions x and y of (P) and (D), respectively, are optimal *if and only if*

i. $(c^\top - y^\top A)x = 0$

ii. $y^\top (Ax - b) = 0.$

(symmetric case: proof on board) (asymmetric case: left as exercise)

Back to Shire

Hobbits' $\max$ and Gandalf's $\min$

Resources: Timber (12 units), Thread (9 units)

Units of each product: y_1, y_2, y_3

Product	**Timber** (x_1)	**Thread** (x_2)	**Profit (coins)**
Lembas Bread (y_1)	5	1	11
Elven Cloak (y_2)	2	1	8
Mithril Ring (y_3)	1	2	7

Back to Shire

Hobbits' $\max$ and Gandalf's $\min$

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Mithril Ring (y_3)	1	2	7

$$
\begin{aligned}
\textbf{(D):}\quad \text{maximize}\quad & 11y_1 + 8y_2 + 7y_3 \\
\text{subject to}\quad & 5y_1 + 2y_2 + 1y_3 \leq 12 \\
& 1y_1 + 1y_2 + 2y_3 \leq 9 \\
& y_1, y_2, y_3 \geq 0.
\end{aligned}
$$

Back to Shire

Hobbits' max and Gandalf's min

Resources: Timber (12 units), Thread (9 units)

Units of each product: y_1, y_2, y_3

Prices (coins per unit): x_1, x_2

Product	**Timber** (x_1)	**Thread** (x_2)	**Profit (coins)**
Lembas Bread (y_1)	5	1	11
Elven Cloak (y_2)	2	1	8
Mithril Ring (y_3)	1	2	7

(P) :

$$\begin{aligned} \text{minimize } & 12x_1 + 9x_2 \\ \text{subject to } & 5x_1 + 1x_2 \geq 11 \\ & 2x_1 + 1x_2 \geq 8 \\ & 1x_1 + 2x_2 \geq 7 \\ & x_1, x_2 \geq 0. \end{aligned}$$

(D):

$$\begin{aligned} \text{maximize } & 11y_1 + 8y_2 + 7y_3 \\ \text{subject to } & 5y_1 + 2y_2 + 1y_3 \leq 12 \\ & 1y_1 + 1y_2 + 2y_3 \leq 9 \\ & y_1, y_2, y_3 \geq 0. \end{aligned}$$

Back to Shire

Hobbits' max and Gandalf's min

Resources: Timber (12 units), Thread (9 units)

Units of each product: y_1, y_2, y_3 **Prices (coins per unit):** x_1, x_2

Product	**Timber** (x_1)	**Thread** (x_2)	**Profit (coins)**
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- Solve (P) graphically

Back to Shire

Hobbits' $\max$ and Gandalf's $\min$

Resources: Timber (12 units), Thread (9 units)

Units of each product: y_1, y_2, y_3

Prices (coins per unit): x_1, x_2

Product	**Timber** (x_1)	**Thread** (x_2)	**Profit (coins)**
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$$\begin{aligned} \text{maximize } & 11y_1 + 8y_2 + 7y_3 \\ \text{subject to } & 5y_1 + 2y_2 + 1y_3 \leq 12 \\ & 1y_1 + 1y_2 + 2y_3 \leq 9 \\ & y_1, y_2, y_3 \geq 0. \end{aligned}$$

- Solve (P) graphically
- Let's have $y = (0, 5, 2)$?

Back to Shire

Hobbits' $\max$ and Gandalf's $\min$

Resources: Timber (12 units), Thread (9 units)

Units of each product: y_1, y_2, y_3

Prices (coins per unit): x_1, x_2

Product	**Timber** (x_1)	**Thread** (x_2)	**Profit (coins)**
Lembas Bread (y_1)	5	1	11
Elven Cloak (y_2)	2	1	8
Mithril Ring (y_3)	1	2	7

(P) :

$$\begin{aligned} \text{minimize } & 12x_1 + 9x_2 \\ \text{subject to } & 5x_1 + 1x_2 \geq 11 \\ & 2x_1 + 1x_2 \geq 8 \\ & 1x_1 + 2x_2 \geq 7 \\ & x_1, x_2 \geq 0. \end{aligned}$$

(D):

$$\begin{aligned} \text{maximize } & 11y_1 + 8y_2 + 7y_3 \\ \text{subject to } & 5y_1 + 2y_2 + 1y_3 \leq 12 \\ & 1y_1 + 1y_2 + 2y_3 \leq 9 \\ & y_1, y_2, y_3 \geq 0. \end{aligned}$$

· Solve (P) graphically

· Solve (D) graphically (using x^*)