1. Prove that for any matrix $A \in \mathbb{R}^{m\times n}$,

$$\text{rank } A + \text{nullity } A = n,$$

   where $\text{nullity}(A) := \dim \mathcal{N}(A)$.

2. Prove that every vector in the null space of A is orthogonal to every vector in the row space of A.

3. Show that the dimension of the column space of A is equal to the dimension of the row space of A.

4. Prove that $\mathcal{N}(A^T A) = \mathcal{N}(A)$ for any matrix $A \in \mathbb{R}^{m\times n}$.

5. Prove that the intersection of the null space of A and the row space of A contains only the zero vector, i.e.,

$$\mathcal{N}(A) \cap \text{row}(A) = \{\mathbf{0}\}.$$

6. Let $A \in \mathbb{R}^{m\times n}$ and consider the system $A\mathbf{x} = \mathbf{b}$.
   (a) Prove that if $\mathbf{b}$ lies in the column space of A, then the system has at least one solution.
   (b) Further, prove that if the columns of A are linearly independent, then this solution is unique.

7. Given

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 7 \\ 9 \end{bmatrix},$$

   determine if $\mathbf{b}$ lies in the column space of A. If yes, find the vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$.

8. For

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix},$$

   (a) By inspection, determine the row rank.
   (b) Find a basis for the column space of A.
   (c) Find a basis for the null space of A.