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CS2020A Discrete Mathematics

Tutorial 9 Submission

Submitted By

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1. Define walk, tour and cycle in a directed graph.

Solution. **Walk:** It is an alternating sequence of vertices and edges starting and ending with a vertex such that every edge in this sequence is preceded by one of it's end points and succeeded by the other end point.

Tour: It is a closed walk, meaning the staring and the end vertices are the same, and it should not have any repeated edges.

Cycle: It is a closed walk (starting and end vertex are the same), with no repeated vertices, except the start and end one.
We can also say that it is a tour with no repeated intermediate vertices.

2. An Euler Circuit in a directed graph G is a circuit (closed trail) which includes every edge of G. Equivalently, it is a tour (closed walk) which includes every edge exactly once.

Theorem 1. (Directed Euler) A directed graph G has an Euler Circuit if and only if G is __________ and __________.

Complete the theorem statement and prove it.

Please mention "strongly connected"

Solution. **connected** and **all vertices have in-degree equal to out-degree**

According to the definition of a circuit, the graph should start and end at the same vertex and no edge should be repeated.
Now, if the given directed graph has an Euler circuit, then it starts and end at the same vertex and each edge is traversed only once. This means that the graph has to be connected.

Now consider an arbitrary graph containing an Euler circuit. Consider a directed edge pointing to a vertex v_i. Now since the start and end vertex are the same, it has to get of this v_i vertex too. But it cannot repeat the edge it already traversed, so we need a new edge. So we always need one edge to enter a vertex and another edge to exit it. So therefore, the in degree should be equal to the out degree for any vertex.

3. Theorem 2. Any simple undirected graph which contains a tour of positive length contains a cycle.

Prove or give a counterexample.

Solution. Consider a simple undirected graph $G = (V, E)$ where $V = \{u, v\}$ and $E = \{\{u, v\}\}$.

The walk $u \to v \to u$ is a **tour** of positive length (it starts and ends at the same vertex and uses existing edges). However, G has no **cycle**, since a cycle in a simple undirected graph must contain at least three distinct vertices or two distinct edges.

Therefore, this graph is a **counterexample** to the theorem, since it contains a tour of positive length but no cycle.

4. **Theorem 3.** Any simple undirected graph with minimum degree at least 3 contains an even cycle.

Prove or give a counterexample.

Solution. Consider a simple undirected graph $G = (V, E)$ where V is the set containing all the vertices and E is the set containing all the edges.

Let us assume the longest connected path in G namely $P = v_1v_2...v_i...v_j...v_k$ where v_i and v_j are the other neighbours of v_1 (i.e. v_1 has edges with atleast v_2, v_i, v_j)

Case 1: $i \to odd, j \to odd$
When i and j are odd, the even cycle can be $v_1v_i...v_jv_1$
Proof: Since i and j are odd, $(j - i)$ or $(i - j)$ is even.
Also adding the 2 edges $v_1 - v_i$ and v_1v_j, we now have $(j - i + 2)$ edges.
Since $j - i$ is even, and $(Even + Even = Even)$; the cycle $v_1v_i...v_jv_1$ is a cycle of even length.

Case 2: $i \to even$, $j \to odd$
When i is even while j is odd, the even cycle can be $v_1v_2...v_iv_1$
Proof: Since i is even, $v_1 \to v_i$ is of odd length
Adding the edge $v_1 - v_i$, we now have $(i - 1 + 1 = i)$ edges
Since i was taken to be even, we are sure to have an even cycle $v_1v_2...v_iv_1$

From Case 2, we can imply that having atleast one even-indexed vertex connected to v_1 is enough to apply Case 2.

So cases like ($i \to odd$ and $j \to even$) and ($i \to even$ and $j \to even$ can be dealt with in a similar fashion as Case 2.

So we can now conclude that, **Any simple undirected graph with minimum degree at least 3 contains an even cycle**

5. Theorem 4. Let G be a simple undirected graph on at least three vertices. If $d(u) + d(v) \geq n$ for every pair of non-adjacent vertices u and v, then G has a Hamiltonian cycle.

Definition 1. $d(v)$ is the degree of vertex v and n is the number of vertices in G. Two vertices are non-adjacent if they do not form an edge.

Prove or give a counterexample.

Solution.
Let G be a simple undirected graph on $n \geq 3$ vertices such that $d(u) + d(v) \geq n$ for every pair of non-adjacent vertices u and v. Then G has a Hamiltonian cycle.

Let P be a **longest possible path** in G. Since P is maximal, no vertex outside P can be attached to its ends; otherwise we could extend P. Hence, all neighbors of v_1 and v_k lie inside P.

$$v_1 \to v_2 \to \cdots \to v_i \to \cdots \to v_j \to \cdots \to v_{k-1} \to v_k$$

We use the **left-happy / right-happy** argument.

- An edge $\{v_i, v_{i+1}\}$ is called **left-happy** if $\{v_i, v_k\} \in G$
- An edge $\{v_i, v_{i+1}\}$ is called **right-happy** if $\{v_1, v_{i+1}\} \in G$
- An edge $\{v_i, v_{i+1}\}$ is called **doubly-happy** if it is both left-happy and right-happy.

If v_1 and v_k are adjacent, then we already have a Hamiltonian Cycle as P is a longest possible path in G.
If v_1 and v_k are not adjacent, we can apply the hypothesis:

$$d(v_1) + d(v_k) \geq n.$$

Because P contains $k - 1 < n$ edges (because P contains $k \leq n$ vertices), there exists at-least one edge that is **doubly-happy**. Formally, there must exist some index i such that

$$v_1 \text{ is adjacent to } v_{i+1} \quad \text{and} \quad v_k \text{ is adjacent to } v_i.$$

Now, starting from v_1, we can go forward along P to v_i, jump to v_k, travel backward along P to v_{i+1}, and finally return to v_1. Let us call this path C.

$$v_1 \to v_2 \to \cdots \to v_i \to v_k \to v_{k-1} \to \cdots \to v_{i+1} \to v_1$$

Claim: C is a Hamiltonian Cycle in G.

Proof: If C had an edge to a vertex outside P, then we have a path longer than P, which is a contradiction.
Thus, C is the maximal connected component. As G is connected, C covers all the vertices of G, making it a Hamiltonian Cycle in G.