Generalize. After proving a statement, seek to prove a more general statement. Weaken the hypothesis or strengthen the conclusion. Apply the idea of the argument in another similar-enough circumstance. Unify our understanding of diverse situations. Seek the essence of a phenomenon.

Exercises

1.1 Prove that the square of any odd number is odd. (Assume that a positive integer is odd if and only if it has the form $2k + 1$.)

1.2 Master several alternative proofs of the irrationality of $\sqrt{2}$, such as those available at www.cut-the-knot.org/proofs/sq_root.shtml, and present one of the proofs to the class.

1.3 Prove that $\sqrt[4]{2}$ is irrational. Give a direct argument, but kindly also deduce it as a corollary of theorem 1.

1.4 Prove that $\sqrt[m]{2}$ is irrational for every integer $m \geq 2$.

1.5 Prove that $\sqrt{5}$ and $\sqrt{7}$ are irrational. Prove that $\sqrt{p}$ is irrational, whenever p is prime.

1.6 Prove that $\sqrt{20}$ is irrational as a corollary of the fact that $\sqrt{5}$ is irrational.

1.7 Prove that $\sqrt{2m}$ is irrational, whenever m is odd.

1.8 In the geometric proof that $\sqrt{2}$ is irrational, what are the side lengths of the smaller squares that arise in the proof? Using those expressions and some elementary algebra, construct a new algebraic proof that $\sqrt{2}$ is irrational. [Hint: Assume that $p^2 = q^2 + q^2$ and that this is the smallest instance in the positive integers. Now consider $2q - p$ and $p - q$.]

1.9 Criticize this "proof." Claim. $\sqrt{n}$ is irrational for every natural number n. Proof. Suppose toward contradiction that $\sqrt{n} = p/q$ in lowest terms. Square both sides to conclude that $nq^2 = p^2$. So p^2 is a multiple of n, and therefore p is a multiple of n. So $p = nk$ for some k. So $nq^2 = (nk)^2 = n^2k^2$, and therefore $q^2 = nk^2$. So q^2 is a multiple of n, and therefore q is a multiple of n, contrary to the assumption that p/q is in lowest terms. □

1.10 Criticize this "proof." Claim. $\sqrt{14}$ is irrational. Proof. We know that $\sqrt{14} = \sqrt{2} \cdot \sqrt{7}$, and we also know that $\sqrt{2}$ and $\sqrt{7}$ are each irrational, since 2 and 7 are prime. Thus, $\sqrt{14}$ is the product of two irrational numbers and therefore irrational. □

1.11 For which natural numbers n is $\sqrt{n}$ irrational? Prove your answer. [Hint: Consider the prime factorization of n, and consider especially the exponents of the primes in that prime factorization.]

1.12 Prove the unifying theorem mentioned at the end the chapter, namely, that $\sqrt[k]{n}$ is irrational unless n is itself an integer kth power.

1.13 Prove that the irrational real numbers are exactly those real numbers that are a different distance from every rational number. Is it also true if you swap "rational" and "irrational"?