## ASSIGNMENT - 3

(1) Writing $u = u(x,y)$ in polar coordinates as $v(r,\theta)$, derive the following expression for the Laplacian in polar coordinates:

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

(Hint: Recall the relationship between Cartesian and polar variables given by $x = r\cos(\theta), y = r\sin(\theta)$ and use chain rule).

*Solution*. Writing $u(x,y) = v(r,\theta)$ with

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

we compute the Laplacian using the chain rule.

First,

$$r = \sqrt{x^2+y^2}, \qquad \theta = \arctan\left(\frac{y}{x}\right),$$

so

$$r_x = \cos\theta, \quad r_y = \sin\theta, \qquad \theta_x = -\frac{\sin\theta}{r}, \quad \theta_y = \frac{\cos\theta}{r}.$$

Thus

$$u_x = v_r\cos\theta - \frac{1}{r}v_\theta\sin\theta, \qquad u_y = v_r\sin\theta + \frac{1}{r}v_\theta\cos\theta.$$

Differentiating again and simplifying,

$$u_{xx} + u_{yy} = v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta}.$$

Since $v(r,\theta) = u(x,y)$, this gives the Laplacian in polar coordinates:

$$\boxed{\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.}$$

□

(2) Verify that the function $u(x,t) = f(x-at)$ satisfies the following PDE:

$$u_t + au_x = 0.$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function.

Solution: Direct substitution and computation; will not be done here.

(3) (The linear transport equation) Find the solution to the initial value problem:

$$u_t + cu_x = 0,$$

subject to the initial condition $u(x,0) = \phi(x)$, where $c$ is a constant.

Solution: Done in class fully.

(4) Solve the PDE $u_x + xu_y = u$ subject to the condition $u(0, y_0) = y_0$.

Answer: Method discussed in class leads to

$$u(x,y) = e^x \left(y - \frac{x^2}{2}\right)$$

as the final answer.

(5) Radial solutions to Laplace equation are of great interest. These are solutions of the form $u(x,y) = v(r)$, where as usual $r = \sqrt{x^2+y^2}$, so $u$ is independent of $\theta$.

(a) Show that for a radial functions $u(x,y) = v(r)$, the Laplace equation reduces to the ODE:

$$v''(r) + \frac{1}{r}v'(r) = 0.$$

(b) Solve the above ODE and find all radial harmonic functions defined on $\mathbb{R}^2 \setminus \{0\}$. Recall that a function $u$ is said to be harmonic if it satisfies the Laplace equation $\Delta u = 0$.

(c) Which of these radial solutions extend to a harmonic function on all of $\mathbb{R}^2$ (including $r = 0$)? Explain your answer.

*Solution.* We look for radial solutions of the Laplace equation

$$\Delta u = 0 \qquad \text{in } \mathbb{R}^2.$$

**For (1):**Assume $u$ is radial, i.e.,

$$u(x,y) = v\Big(\sqrt{x^2+y^2}\Big).$$

Let $r = \sqrt{x^2+y^2}$. Then

$$\frac{\partial u}{\partial x} = v'(r)\frac{x}{r}, \qquad \frac{\partial u}{\partial y} = v'(r)\frac{y}{r}.$$

Differentiating again:

$$\frac{\partial^2 u}{\partial x^2} = v''(r)\frac{x^2}{r^2} + v'(r)\frac{r^2-x^2}{r^3}, \qquad \frac{\partial^2 u}{\partial y^2} = v''(r)\frac{y^2}{r^2} + v'(r)\frac{r^2-y^2}{r^3}.$$

Adding these gives

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = v''(r) + \frac{1}{r}v'(r), \quad r > 0.$$

So the Laplace equation becomes

$$v''(r) + \frac{1}{r}v'(r) = 0, \quad r > 0.$$

**For (2):** Let $w(r) = v'(r)$. Then

$$w'(r) + \frac{1}{r}w(r) = 0 \quad \Longrightarrow \quad \frac{w'(r)}{w(r)} = -\frac{1}{r}.$$

Integrate:

$$\ln|w(r)| = -\ln r + C \quad \Longrightarrow \quad w(r) = \frac{C_1}{r}.$$

Integrate again:

$$v(r) = C_1 \ln r + C_2.$$

Hence in terms of $(x, y)$, the radial solution is

$$u(x,y) = C_1 \ln\sqrt{x^2+y^2} + C_2 = \frac{C_1}{2}\ln(x^2+y^2) + C_2.$$

**For (3):** The function $\ln|x|$ is not defined at $x = 0$ (it diverges to $-\infty$). Thus:

- If $C_1 \neq 0$, then $u(x) = C_1 \ln|x| + C_2$ is not defined at 0 and cannot be extended harmonically to all of $\mathbb{R}^2$.
- If $C_1 = 0$, then $u(x) = C_2$ is a constant function, which is harmonic everywhere.

All radial harmonic functions on $\mathbb{R}^2 \setminus \{0\}$ are

$$u(x) = C_1 \ln|x| + C_2.$$

The only radial harmonic functions on *all* of $\mathbb{R}^2$ (including 0) are the constants:

$$u(x) = C_2.$$

□

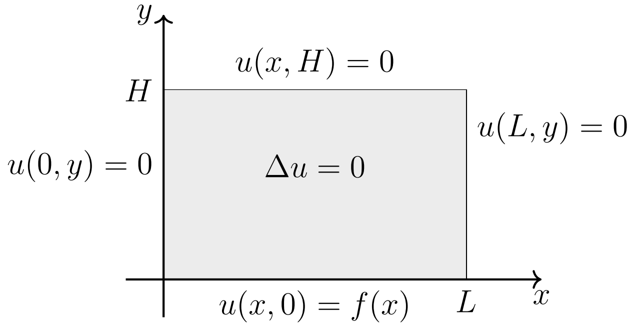
(6) Consider Laplace's equation in Cartesian coordinates, given by:

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L, \quad 0 < y < H.$$

Suppose the boundary conditions for this problem are:

$$u(0,y) = 0, \quad u(L,y) = 0, \quad u(x,0) = f(x), \quad u(x,H) = 0,$$

where $f(x)$ is a known function.




Solve Laplace's equation with these boundary conditions.

<u>*Solution*</u>. As with the heat and wave equations, we can solve this problem using the method of separation of variables. Let $u(x,y) = X(x)Y(y)$. Then, Laplace's equation becomes

$$X''Y + XY'' = 0$$

and we can separate the $x$ and $y$ dependent functions and introduce a separation constant, $\lambda$, which gives

$$\frac{X''}{X} = -\lambda \quad \text{and} \quad \frac{Y''}{Y} = -\lambda.$$

Thus, we are led to two differential equations:

$$X'' + \lambda X = 0 \quad \text{and} \quad Y'' - \lambda Y = 0.$$

From the boundary condition $u(0,y) = 0$, $u(L,y) = 0$, we have $X(0) = 0$, $X(L) = 0$. So, we have the usual eigenvalue problem for $X(x)$,

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(L) = 0.$$

The solutions to this problem are given by

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots$$

The general solution of the equation for $Y(y)$ is given by

$$Y(y) = c_1 e^{\sqrt{\lambda}y} + c_2 e^{-\sqrt{\lambda}y}.$$

The boundary condition $u(x,H) = 0$ implies $Y(H) = 0$. So, we have

$$c_1 e^{\sqrt{\lambda}H} + c_2 e^{-\sqrt{\lambda}H} = 0.$$

Thus,

$$c_2 = -c_1 e^{2\sqrt{\lambda}H}.$$

Inserting this result into the expression for $Y(y)$, we have

$$Y(y) = c_1 e^{\sqrt{\lambda}y} - c_1 e^{\sqrt{\lambda}(2H-y)}.$$

Simplifying, we get

$$Y(y) = c_1 e^{\sqrt{\lambda}H}\left(e^{\sqrt{\lambda}(y-H)} - e^{-\sqrt{\lambda}(y-H)}\right).$$

Thus,

$$Y(y) = c_1 e^{\sqrt{\lambda}H}\sinh\left(\sqrt{\lambda}(H-y)\right).$$

Having carried out this computation, we can now see that it would be better to guess this form in the future. So, for $Y(H) = 0$, one would guess a solution $Y(y) = \sinh\sqrt{\lambda}(H-y)$. Similarly, if $Y'(H) = 0$, one would guess a solution $Y(y) = \cosh\sqrt{\lambda}(H-y)$.

Since we already know the values of the eigenvalues $\lambda_n$ from the eigenvalue problem for $X(x)$, we have that the $y$-dependence is given by

$$Y_n(y) = \sinh\left(\frac{n\pi(H-y)}{L}\right), \quad n = 1, 2, \ldots$$

So, the product solutions are given by

$$u_n(x,y) = \sin\left(\frac{n\pi x}{L}\right)\frac{\sinh\frac{n\pi(H-y)}{L}}{\sinh\frac{n\pi H}{L}}, \quad n = 1, 2, \ldots$$

These solutions satisfy Laplace's equation and the three homogeneous boundary conditions in the problem.

The remaining boundary condition, $u(x,0) = f(x)$, still needs to be satisfied. Inserting $y = 0$ in the product solutions does not satisfy the boundary condition unless $f(x)$ is proportional to one of the eigen functions $X_n(x)$. So, we first write down the general solution as a linear combination of the product solutions,

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\frac{\sinh\frac{n\pi(H-y)}{L}}{\sinh\frac{n\pi H}{L}} \tag{6.3.3}$$

Now we apply the boundary condition, $u(x,0) = f(x)$, to find that

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \tag{6.3.4}$$

Defining $b_n = a_n \sinh\frac{n\pi H}{L}$, this becomes

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \tag{6.3.5}$$

We see that the determination of the unknown coefficients, $b_n$, is simply done by recognizing that this is a Fourier sine series. The Fourier coefficients are easily found as

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \tag{6.3.6}$$

Since $a_n = b_n/\sinh\frac{n\pi H}{L}$, we can finish solving the problem. The solution is

$$u(x,y) = \sum_{n=1}^{\infty} \frac{2}{L\sinh\frac{n\pi H}{L}}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \sin\left(\frac{n\pi x}{L}\right)\sinh\frac{n\pi(H-y)}{L} \tag{6.3.7}$$

where

$$a_n = \frac{2}{L\sinh\frac{n\pi H}{L}}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx. \tag{6.3.8}$$

□

(7) A rectangular plate has a width of 8 cm and is so long compared to its width that it may be considered infinite in length. The temperature along one short edge $y = 0$ is given by

$$u(x,0) = 100\sin\left(\frac{\pi x}{8}\right), \quad 0 < x < 8,$$

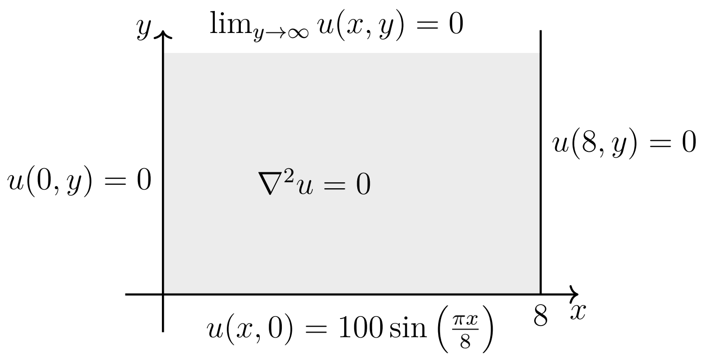
while the two long edges $x = 0$ and $x = 8$, as well as the other short edge ($y \to \infty$), are kept at 0°C. Find the steady-state temperature distribution $u(x, y)$ in the plate (recall that the steady state temperature satisfies Laplace equation).

*Solution.* The steady-state temperature satisfies Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 8,\ y > 0,$$

with boundary conditions:

$$u(0, y) = 0, \quad u(8, y) = 0, \quad u(x, 0) = 100 \sin\left(\frac{\pi x}{8}\right), \quad \lim_{y\to\infty} u(x, y) = 0.$$




Assume a solution of the form

$$u(x, y) = X(x)Y(y).$$

Substituting into Laplace's equation gives

$$X''(x)Y(y) + X(x)Y''(y) = 0 \quad \Longrightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$$

This leads to the ordinary differential equations:

$$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0.$$

The boundary conditions along $x$ are:

$$X(0) = 0, \quad X(8) = 0.$$

The eigen functions and eigenvalues are

$$X_n(x) = \sin\left(\frac{n\pi x}{8}\right), \quad \lambda_n = \left(\frac{n\pi}{8}\right)^2, \quad n = 1, 2, 3, \ldots$$

The $y$-equation is

$$Y_n'' - \lambda_n Y_n = 0 \quad \Longrightarrow \quad Y_n(y) = A_n e^{\sqrt{\lambda_n} y} + B_n e^{-\sqrt{\lambda_n} y}.$$

Since the temperature must remain finite as $y \to \infty$, we discard the growing exponential term: that is, $A_n = 0$ and hence

$$Y_n(y) = B_n e^{-\sqrt{\lambda_n} y} = B_n e^{-\frac{n\pi}{8} y}.$$

So, the general solution $u$ is given by

$$u(x,y)=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi x}{8}\right)e^{-\frac{n\pi}{8}y}.$$

At $y=0$:

$$u(x,0)=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi x}{8}\right)=100\sin\left(\frac{\pi x}{8}\right).$$

Comparing Fourier sine series, we get

$$B_1=100,\quad B_n=0,\ n>1.$$

$$\boxed{u(x,y)=100\ \sin\left(\frac{\pi x}{8}\right)\ e^{-\frac{\pi}{8}y}\quad {}^\circ\mathrm{C},\quad 0<x<8,\ y>0}$$

This solution satisfies all boundary conditions and represents the steady-state temperature distribution of the plate. □

(8) Solve the following Wave equation :

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(x,t)&=c^2\frac{\partial^2 u}{\partial x^2}(x,t), && 0<x<1,\ t>0,\\
u(0,t)&=u(1,t)=0, && t>0,\\
u(x,0)&=x(1-x), && 0<x<1,\\
u_t(x,0)&=0, && 0<x<1.
\end{aligned}$$

*Solution*. We know that if we have

$$\left\{\begin{array}{l}
\dfrac{\partial^2 u}{\partial t^2}=c^2\dfrac{\partial^2 u}{\partial x^2},\ 0<x<L,\ t>0\\
u(0,t)=u(L,t)=0\ \text{(BCs)}\\
u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x)\ \text{(ICs)}
\end{array}\right. \tag{0.1}$$

Comparing the given system with the general form (0.1), we get

$$L=1,\quad \varphi(x)=x(1-x),\quad \psi(x)=0.$$

Then, the general solution is

$$\begin{aligned}
u(x,t)&=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[\left(A_n\cos\left(\frac{n\pi ct}{L}\right)+B_n\sin\left(\frac{n\pi ct}{L}\right)\right],\\
&=\sum_{n=1}^{\infty}\sin(n\pi x)\left[\left(A_n\cos(n\pi ct)+B_n\sin(n\pi ct)\right],
\end{aligned}\tag{0.2}$$

with the coefficients

$$A_n=\frac{2}{L}\int_0^L\varphi(x)\sin(\frac{n\pi x}{L})\,\mathrm{d}x=2\int_0^1 x(1-x)\sin(n\pi x)\,dx\quad\text{and}\tag{0.3}$$

$$B_n=\frac{2}{n\pi c}\int_0^L\psi(x)\sin(\frac{n\pi x}{L})\mathrm{d}x=\frac{2}{n\pi c}\int_0^1 0\cdot\sin(n\pi x)\,dx=0.$$

Then, actually (0.2) becomes

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)\cos(n\pi ct) \tag{0.4}$$

Now,

$$\begin{aligned}\int_0^1 x\sin(n\pi x)\,dx &= \int_0^1 -\frac{1}{n\pi}x\,\frac{d}{dx}\cos(n\pi x)\,dx \\ &= -\frac{1}{n\pi}\Big[x\cos(n\pi x)\Big]_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)\,dx \\ &= -\frac{1}{n\pi}\cos(n\pi),\end{aligned}$$

and

$$\begin{aligned}-\int_0^1 x^2\sin(n\pi x)\,dx &= \frac{1}{(n\pi)^2}\int_0^1 x^2\frac{d^2}{dx^2}\sin(n\pi x)\,dx \\ &= \frac{1}{(n\pi)^2}\Big[x^2\frac{d}{dx}\sin(n\pi x)\Big]_0^1 - \frac{2}{(n\pi)^2}\int_0^1 x\frac{d}{dx}\sin(n\pi x)\,dx \\ &= \frac{1}{(n\pi)^2}\Big[n\pi x^2\cos(n\pi x)\Big]_0^1 - \frac{2}{(n\pi)^2}\Big[x\sin(n\pi x)\Big]_0^1 + \frac{2}{(n\pi)^2}\int_0^1 \sin(n\pi x)\,dx \\ &= \frac{\cos(n\pi)}{n\pi} - \frac{2}{(n\pi)^3}(\cos(n\pi)-1) \\ &= \frac{\cos(n\pi)(n^2\pi^2-2)+2}{n^3\pi^3}.\end{aligned}$$

Hence, combining these two into (0.3) reveal

$$\begin{aligned}A_n &= 2\int_0^1 x(1-x)\sin(n\pi x)\,dx \\ &= 2\left(\int_0^1 x\sin(n\pi x)\,dx - \int_0^1 x^2\sin(n\pi x)\,dx\right) \\ &= 2\left[-\frac{\cos(n\pi)}{n\pi} + \frac{\cos(n\pi)(n^2\pi^2-2)+2}{n^3\pi^3}\right] \\ &= \frac{4}{n^3\pi^3}\Big[1-\cos(n\pi)\Big] \\ &= \begin{cases}\dfrac{8}{n^3\pi^3}, & n \text{ odd}, \\ 0, & n \text{ even}.\end{cases}\end{aligned}$$

Therefore, the general solution (0.4) becomes

$$u(x,t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{8}{n^3\pi^3}\sin(n\pi x)\cos(n\pi ct).$$

□

(9) Consider the following Initial-Boundary value problem :

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < \pi, \ t > 0, \\ u(x,0) = x, & 0 < x < \pi, \\ u_t(x,0) = 0, & 0 < x < \pi, \\ u_x(0,t) = 0, \quad u_x(\pi,t) = 0, & t \geq 0. \end{cases}$$

Find $u(x,t)$ in the form

$$u(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t) \cos nx + b_n(t) \sin nx.$$

*Solution.* Functions $a_n(t)$ and $b_n(t)$ are determined by the boundary conditions:

$$u_x(x,t) = \sum_{n=1}^{\infty} \Big[ -a_n(t) n \sin nx + b_n(t) n \cos nx \Big],$$

$$0 = u_x(0,t) = \sum_{n=1}^{\infty} b_n(t) n \quad \Rightarrow \quad b_n(t) = 0.$$

Thus,

(2)
$$u(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t) \cos nx.$$

Substitute (2) into $u_{tt} - u_{xx} = 0$:

$$\frac{a_0''(t)}{2} + \sum_{n=1}^{\infty} a_n''(t) \cos nx + \sum_{n=1}^{\infty} a_n(t) n^2 \cos nx = 0,$$

$$a_0''(t) = 0, \qquad a_n''(t) + n^2 a_n(t) = 0,$$

whose general solutions are

(3)
$$a_0(t) = c_0 t + d_0, \qquad a_n(t) = c_n \sin nt + d_n \cos nt.$$

Also,

$$a_0'(t) = c_0, \qquad a_n'(t) = c_n n \cos nt - d_n n \sin nt.$$

Constants $c_n$ and $d_n$ are determined by the initial conditions:

$$x = u(x,0) = \frac{a_0(0)}{2} + \sum_{n=1}^{\infty} a_n(0) \cos nx = \frac{d_0}{2} + \sum_{n=1}^{\infty} d_n \cos nx,$$

$$0 = u_t(x,0) = \frac{a_0'(0)}{2} + \sum_{n=1}^{\infty} a_n'(0) \cos nx = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n n \cos nx.$$

By orthogonality, multiply both equations by $\cos mx$, including $m = 0$, and integrate:

$$\int_0^{\pi} x \, dx = d_0 \frac{\pi}{2}, \qquad \int_0^{\pi} x \cos mx \, dx = d_m \frac{\pi}{2},$$

$$\int_0^{\pi} 0 \, dx = c_0 \frac{\pi}{2}, \qquad \int_0^{\pi} 0 \cos mx \, dx = c_m m \frac{\pi}{2}.$$

Thus,

(4) $$d_0 = \pi, \qquad d_n = \frac{2}{n^2\pi^2}\Big(\cos n\pi - 1\Big), \qquad c_n = 0.$$

Then from (3) we get

$$a_0(t) = d_0 = \pi, \qquad a_n(t) = \frac{2}{n^2\pi^2}(\cos n\pi - 1)\cos nt.$$

Therefore, we finally have the solution (2) as

$$u(x,t) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(\cos n\pi - 1)\cos nt\cos nx}{n^2}$$

□

(10) Solve the following BVP associated with the heat equation:

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = 20, \qquad u(0,t) = 0, \qquad u(L,t) = 0.$$

*Solution*. First, the solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}.$$

The coefficients are given by

$$B_n = \frac{2}{L}\int_0^L 20\sin\left(\frac{n\pi x}{L}\right)\,dx = \frac{2}{L}\left(\frac{20L(1-\cos(n\pi))}{n\pi}\right) = \frac{40(1-(-1)^n)}{n\pi}.$$

If we plug these in we get the solution,

$$u(x,t) = \sum_{n=1}^{\infty}\frac{40(1-(-1)^n)}{n\pi}\sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}.$$

□

(11) Find a solution to the following partial differential equation:

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = f(x), \qquad u(-L,t) = u(L,t), \qquad \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t).$$

*Solution*. Assume the solution is of the form

$$u(x,t) = \varphi(x)G(t),$$

we get the following two ordinary differential equations that we need to solve:

$$\frac{dG}{dt} = -k\lambda G, \qquad \frac{d^2\varphi}{dx^2} + \lambda\varphi = 0,$$

$$\varphi(-L) = \varphi(L), \qquad \frac{d\varphi}{dx}(-L) = \frac{d\varphi}{dx}(L).$$

**Case 1 :** If $\lambda > 0$, then the general solution to the differential equation is:

$$\varphi(x) = c_1\cos(\sqrt{\lambda}\,x) + c_2\sin(\sqrt{\lambda}\,x).$$

Applying the first boundary condition and recalling that cosine is an even function and sine is an odd function gives us,

$$c_1 \cos(-L\sqrt{\lambda}) + c_2 \sin(-L\sqrt{\lambda}) = c_1 \cos(L\sqrt{\lambda}) + c_2 \sin(L\sqrt{\lambda})$$
$$-c_2 \sin(L\sqrt{\lambda}) = c_2 \sin(L\sqrt{\lambda})$$
$$0 = 2c_2 \sin(L\sqrt{\lambda})$$

At this stage we can't really say anything as either $c_2$ or sine could be zero. So, let's apply the second boundary condition and see what we get.

$$-\sqrt{\lambda}\, c_1 \sin(-L\sqrt{\lambda}) + \sqrt{\lambda}\, c_2 \cos(-L\sqrt{\lambda}) = -\sqrt{\lambda}\, c_1 \sin(L\sqrt{\lambda}) + \sqrt{\lambda}\, c_2 \cos(L\sqrt{\lambda})$$
$$\sqrt{\lambda}\, c_1 \sin(L\sqrt{\lambda}) = -\sqrt{\lambda}\, c_1 \sin(L\sqrt{\lambda})$$
$$2\sqrt{\lambda}\, c_1 \sin(L\sqrt{\lambda}) = 0$$

We get something similar. However, notice that if $\sin(L\sqrt{\lambda}) \neq 0$ then we would be forced to have $c_1 = c_2 = 0$ and this would give us the trivial solution which we don't want.

This means therefore that we must have

$$\sin(L\sqrt{\lambda}) = 0$$

which in turn means (from work in our previous examples) that the positive eigenvalues for this problem are,

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad n = 1, 2, 3, \ldots$$

Now, there is no reason to believe that $c_1 = 0$ or $c_2 = 0$. All we know is that they both can't be zero and so that means we in fact have two sets of eigenfunctions for this problem corresponding to positive eigenvalues. They are,

$$\varphi_n(x) = \cos\left(\frac{n\pi x}{L}\right), \qquad \varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right), \qquad n = 1, 2, 3, \ldots$$

**Case 2 :** If $\lambda = 0$, then the general solution in this case is

$$\varphi(x) = c_1 + c_2 x.$$

Applying the first boundary condition gives,

$$c_1 + c_2(-L) = c_1 + c_2(L)$$

$$2Lc_2 = 0 \quad \Rightarrow \quad c_2 = 0$$

The general solution is then,

$$\varphi(x) = c_1$$

and this will trivially satisfy the second boundary condition. Therefore $\lambda = 0$ is an eigenvalue for this BVP and the eigenfunctions corresponding to this eigenvalue is,

$$\varphi(x) = 1.$$

**Case 2 :** If $\lambda < 0$, then the general solution here is,

$$\varphi(x) = c_1 \cosh(\sqrt{-\lambda}\, x) + c_2 \sinh(\sqrt{-\lambda}\, x)$$

Applying the first boundary condition and using the fact that hyperbolic cosine is even and hyperbolic sine is odd gives,

$$\begin{aligned} c_1 \cosh(-L\sqrt{-\lambda}) + c_2 \sinh(-L\sqrt{-\lambda}) &= c_1 \cosh(L\sqrt{-\lambda}) + c_2 \sinh(L\sqrt{-\lambda}) \\ -c_2 \sinh(L\sqrt{-\lambda}) &= c_2 \sinh(L\sqrt{-\lambda}) \\ 2c_2 \sinh(L\sqrt{-\lambda}) &= 0 \end{aligned}$$

Now, in this case we are assuming that $\lambda < 0$ and so $L\sqrt{-\lambda} \neq 0$. This in turn tells us that $\sinh(L\sqrt{-\lambda}) \neq 0$. We therefore must have $c_2 = 0$. Let's now apply the second boundary condition to get,

$$\begin{aligned} \sqrt{-\lambda}\, c_1 \sinh(-L\sqrt{-\lambda}) &= \sqrt{-\lambda}\, c_1 \sinh(L\sqrt{-\lambda}) \\ 2\sqrt{-\lambda}\, c_1 \sinh(L\sqrt{-\lambda}) = 0 \quad &\Rightarrow \quad c_1 = 0 \end{aligned}$$

By our assumption on $\lambda$ we again have no choice here but to have $c_1 = 0$ and so for this boundary value problem there are no negative eigenvalues. Summarizing up then we have the following sets of eigenvalues and eigenfunctions and note that we've merged the $\lambda = 0$ case into the cosine case since it can be here to simplify things up a little.

$$\begin{aligned} \lambda_n &= \left(\frac{n\pi}{L}\right)^2, \qquad \varphi_n(x) = \cos\left(\frac{n\pi x}{L}\right), \qquad n = 0, 1, 2, 3, \ldots \\ \lambda_n &= \left(\frac{n\pi}{L}\right)^2, \qquad \varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right), \qquad n = 1, 2, 3, \ldots \end{aligned}$$

The time problem is we have,

$$G(t) = c\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}.$$

In this case we actually have two different possible product solutions that will satisfy the partial differential equation and the boundary conditions. They are,

$$\begin{aligned} u_n(x,t) &= A_n \cos\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}, \qquad n = 0, 1, 2, 3, \ldots \\ u_n(x,t) &= B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}, \qquad n = 1, 2, 3, \ldots \end{aligned}$$

The Principle of Superposition is still valid however and so a sum of any of these will also be a solution and so the solution to this partial differential equation is,

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}.$$

If we apply the initial condition to this we get,

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \; + \; \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right).$$

So the coefficients are given by,

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx, \qquad n = 1,2,3,\ldots$$

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx, \qquad n = 1,2,3,\ldots$$

□