ASSIGNMENT - 3

(1) Writing $u = u(x, y)$ in polar coordinates as $v(r, \theta)$, derive the following expression for the Laplacian in polar coordinates:

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad = \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

(Hint: Recall the relationship between Cartesian and polar variables given by $x = r\cos(\theta), y = r\sin(\theta)$ and use chain rule).

(2) Verify that the function $u(x, t) = f(x - at)$ satisfies the following PDE:

$$u_t + au_x = 0.$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function.

(3) (The linear transport equation) Find the solution to the initial value problem:

$$u_t + cu_x = 0,$$

subject to the initial condition $u(x, 0) = \phi(x)$, where c is a constant.

(4) Solve the PDE $u_x + xu_y = u$ subject to the condition $u(0, y_0) = y_0$.

(5) Radial solutions to Laplace equation are of great interest. These are solutions of the form $u(x, y) = v(r)$, where as usual $r = \sqrt{x^2 + y^2}$, so u is independent of θ.

(a) Show that for a radial functions $u(x, y) = v(r)$, the Laplace equation reduces to the ODE:

$$v''(r) + \frac{1}{r}v'(r) = 0.$$

(b) Solve the above ODE and find all radial harmonic functions defined on $\mathbb{R}^2 \setminus \{0\}$. Recall that a function u is said to be harmonic if it satisfies the Laplace equation $\Delta u = 0$.

(c) Which of these radial solutions extend to a harmonic function on all of $\mathbb{R}^2$ (including $r = 0$)? Explain your answer.

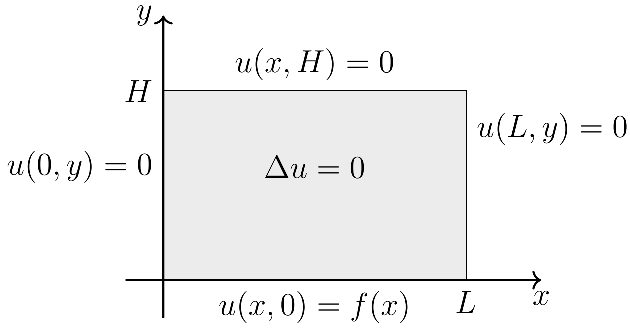
(6) Consider Laplace's equation in Cartesian coordinates, given by:

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L, \quad 0 < y < H.$$

The boundary conditions for this problem are:

$$u(0,y)=0,\quad u(L,y)=0,\quad u(x,0)=f(x),\quad u(x,H)=0,$$

where $f(x)$ is a known function.



Solve Laplace's equation with these boundary conditions.

(7) A rectangular plate has a width of 8 cm and is so long compared to its width that it may be considered infinite in length. The temperature along one short edge $y=0$ is given by

$$u(x,0)=100\sin\left(\frac{\pi x}{8}\right),\quad 0<x<8,$$

while the two long edges $x=0$ and $x=8$, as well as the other short edge ($y\to\infty$), are kept at 0°C. Find the steady-state temperature distribution $u(x,y)$ in the plate (recall that the steady state temperature satisfies Laplace equation).

(8) Solve the following Wave equation :

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(x,t) &= c^2\frac{\partial^2 u}{\partial x^2}(x,t), && 0<x<1,\ t>0,\\
u(0,t) &= u(1,t)=0, && t>0,\\
u(x,0) &= x(1-x), && 0<x<1,\\
u_t(x,0) &= 0, && 0<x<1.
\end{aligned}$$

(9) Consider the following Initial-Boundary value problem :

$$\begin{cases}
u_{tt}-u_{xx}=0, & 0<x<\pi,\ t>0,\\
u(x,0)=x, & 0<x<\pi,\\
u_t(x,0)=0, & 0<x<\pi,\\
u_x(0,t)=0,\quad u_x(\pi,t)=0, & t\ge 0.
\end{cases}$$

Find $u(x,t)$ in the form

$$u(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t)\cos nx + b_n(t)\sin nx.$$

(10) Solve the following BVP associated with the heat equation:

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = 20, \qquad u(0,t) = 0, \qquad u(L,t) = 0.$$

(11) Find a solution to the following partial differential equation:

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = f(x), \qquad u(-L,t) = u(L,t), \qquad \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t).$$