1. Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

2. Given

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 3 & 1 & 0 & 2 \\ -1 & 4 & 1 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix},$$

determine if b lies in the column space of A. If yes, find a vector x such that $Ax = b$.

3. Prove that for any matrix $A \in \mathbb{R}^{m\times n}$,

$$\operatorname{rank} A + \operatorname{nullity} A = n,$$

where $\operatorname{nullity}(A) := \dim \mathcal{N}(A)$.

4. We are performing an experiment to estimate the acceleration due to gravity on planet Niflheim. A die is dropped from rest, and its falling distance is measured at different times. The results are shown below:

Time (seconds)	0.50	1.50	2.50
Distance (metres)	1.20	10.80	31.25

The relationship between distance s and time t is given by:

$$s = \frac{1}{2}at^2,$$

where a is the acceleration due to gravity on Niflheim (in metres per second squared).

(a) Find the least-squares estimate of a using the measurements from the table above.

(b) Suppose an additional measurement is taken at $t = 3.50$ seconds, and the distance recorded is 61.00 metres. Use the recursive least-squares algorithm to update the estimate of a.

5. Solve the following optimization problem:

$$\begin{array}{ll} \text{minimize} & \|x - x_0\| \\ \text{subject to} & \begin{bmatrix} 2 & -1 & 0 \end{bmatrix} x = 4 \end{array}$$

where

$$x_0 = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}^\top.$$

6. Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, with $m < n$ and $\operatorname{rank} A = m$. Show that

$$x^* = A^\top (AA^\top)^{-1} b$$

is the only vector in $\operatorname{row}(A)$ satisfying $Ax^* = b$.

7. Question 12.24 in textbook[1] :
   "derive a recursive least-squares algorithm where we remove (instead of add) a data point."

8. Given the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 4 \end{bmatrix},$$

   compute the generalized inverse $A^{\dagger}$ of A.

[1]Chong, E. K. P., & Zak, S. H. (2013). An Introduction to Optimization (4th ed.). Wiley.