Second Order Differential Equations :-

General form :-

$$\boxed{\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)} \quad ; \; x \in I$$

* The solution will be a 2-parameter family of curves

Initial Value Problem :-

The IVP of 2^{nd} Order LDE is to find the solution to ① satisfying the initial conditions. $y(x_0) = y_0, \; y'(x_0) = y_0' ; \; y_0, y_1 \in \mathbb{R}$

* Geometrically, Does there exist a solution to ① which passes through (x_0, y_0) having slope y_1.

Existance and Uniqueness of Solution :-

From ①, let p, q, r be continuous over a closed rect. $I = [a, b]$. If $x_0 \in I$ and $y_0, y_0' \in \mathbb{R}$, then IVP has a unique solution on I

Note :- The DE $y'' + p(x)y' + q(x)y = 0$ is called the reduced eqn corresponding to ①. —② [Non-homogeneous]

Solution :-

* General sol. of ② is a 2-parameter family of curves

* General sol of ② is a 2 parameter family of curves which represents the set of all sol. of ②. Denote as $y_g(x, c_1, c_2)$.

* Particular solution of ① is any fn satisfying ①. Denote as $y_p(x)$

(1) Then $y_g + y_p$ will be general solution of ①

(2) If y is any solution of ①, then there exist d_1, d_2 s.t

$$y = y_g(x, d_1, d_2) + y_p(x)$$

Proof:-

(1) $$\text{LHS} = (y_g + y_p)'' + p(x)(y_g + y_p)' + q(x)(y_g + y_p)$$

$$= (y_g'' + p(x)y_g' + q(x)y_g) + (y_p'' + p(x)y_p' + q(x)y_p)$$

$$= 0 + r(x)$$

$$= r(x)$$

$$= \text{RHS}$$

(2) Let y be any particular sol of ①.

Observe $y - y_p$ is a particular sol of ②

Also $y_g(x, c_1, c_2)$ is the general sol of ②. Then $\exists\ d_1, d_2$ s.t $y_g(x, d_1, d_2) = y - y_p$

$$\Rightarrow y = y_p(x) + y_g(x, d_1, d_2)$$

Gen. Sol of Reduced Eqⁿ :-

Thm:- If y_1 and y_2 are solutions to $y''+p(x)y'+q(x)y=0$, then $c_1y_1+c_2y_2$ is also a solution, for any $c_1, c_2 \in \mathbb{R}$.

* Let V be a function space containing all continous functions from I to $\mathbb{R}$. Let it be $C(I)$.
  $\Rightarrow$ In this space, Addition: For $f,g \in V$; $(f+g)$ as $(f+g)(x) = f(x)+g(x)$
  Scalar Mul.: For $\alpha \in \mathbb{R}, f \in V$; (αf) as $(\alpha f)(x) = \alpha \cdot f(x)$.

* Let S be the set of all sol. of ②. Then $S \subseteq C(I)$. S is a subspace of $C(I)$.

Aim: To generate a basis for $\{f,g\}$ of S

* $\{f,g\}$ are linearly independent if
  $cf+dg=0$ for some $c,d \in \mathbb{R} \Rightarrow c=d=0$.
* $\{f,g\}$ spans S if every element of S can be written as a linear combination of f and g.

$$\forall h \in S \; \exists \; c,d \in \mathbb{R} \left(cf(x)+dg(x)=h(x) \right)$$

Thm:- The eqⁿ $y''+p(x)y'+q(x)y=0$ has 2 L.I sol., which span the space of all solutions, i.e; dimension of space of all solutions is 2.

Proof:-

Choose a solution y_1 st $y_1(x_0)=1$, $y_1'(x_0)=0$

Choose a solution y_2 s.t $y_2(x_0)=0$, $y_2'(x_0)=1$

Claim:- y_1, y_2 are linearly independent

Let $c_1y_1(x)+c_2y_2(x)=0$; $c_1y_1'(x)+c_2y_2'(x)=0$

Evaluate at $x=x_0$. $\Rightarrow c_1(1)+c_2(0)=0 \Rightarrow c_1=0$

$\Rightarrow c_1(0)+c_2(1)=0 \Rightarrow c_2=0.$

Claim:- $y_1 y_2$ is a spanning set

Let $y \in S$, i.e; y is a solution of (2)

To show that: $\exists\, c, d \in \mathbb{R}$ s.t $y = cy_1 + dy_2$

$y' = cy_1' + dy_2'$

Let $c = y(x_0)$, $d = y'(x_0)$

$\rightarrow cy_1(x_0)+dy_2(x_0) = y(x_0)(1)+y'(x_0)(0) = y(x_0)$

$\rightarrow cy_1'(x_0)+dy_2'(x_0) = y(x_0)(0)+y'(x_0)(1) = y'(x_0)$

Both y and cy_1+dy_2 pass through $(x_0, y(x_0))$ and have slope $y'(x_0)$, for this choice of c,d.

Then by the uniqueness of solutions, $\boxed{y = cy_1+dy_2}$

$\rightarrow$ c,d can also be computed by considering the sys. of eq'ns

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} h(x_0) \\ h'(x_0) \end{bmatrix}$$

$\left[\begin{matrix} (y_1(x_0), y_1'(x_0)) \text{ and} \\ (y_2(x_0), y_2'(x_0)) \text{ are L.I} \end{matrix} \right]$

A - Wronskian Matrix

$$c = \frac{\begin{vmatrix} h(x_0) & y_2(x_0) \\ h'(x_0) & y_2'(x_0) \end{vmatrix}}{\det(A)} \qquad d = \frac{\begin{vmatrix} y_1(x_0) & h(x_0) \\ y_1'(x_0) & h'(x_0) \end{vmatrix}}{\det(A)}$$

Hence, $\{y_1, y_2\}$ is a basis of S.

$$\dim(S) = 2.$$

Thm: Let y_1, y_2 be any 2 L.I solutions to (2). Let $x_0 \in I$.
Then $(y_1(x_0), y_1'(x_0))$, $(y_2(x_0), y_2'(x_0))$ are L.I.

Proof:

Consider $c_1, c_2 \in \mathbb{R}$.

$$c_1\big(y_1(x_0), y_1'(x_0)\big) + c_2\big(y_2(x_0), y_2'(x_0)\big) = 0$$

$$c_1 y_1(x_0) + c_2 y_2(x_0) = 0 \Rightarrow (c_1 y_1 + c_2 y_2)(x_0) = 0$$

$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = 0 \Rightarrow (c_1 y_1' + c_2 y_2')(x_0) = 0$$

Then by uniqueness, $c_1 y_1 + c_2 y_2 = 0 \Rightarrow \boxed{c_1 = c_2 = 0}$

Wronskian of two functions:

Let f, g be 2 differentiable functions on I. Then the Wronskian of f and g is defined as follows.

$$W(f,g)(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = (fg' - gf')(x)$$

* If f, g are L.D, the Wronskian will be zero.

Thm:- If y_1, y_2 are solutions (2), $W(y_1, y_2)(x)$ will either be zero or non-neg for all points

$$W(y_1, y_2)(x) = 0 \ \forall x$$

(or)

$$W(y_1, y_2)(x) \neq 0 \ \forall x$$

Proof:-

$$W = y_1 y_2' - y_2 y_1'$$

$$W' = y_1' y_2' + y_1 y_2'' - y_2' y_1' - y_2 y_1''$$

$$W' = y_1 y_2'' - y_2 y_1''$$

Also, $y_1'' + p(x) y_1' + q(x) y_1 = 0$ — (a)

$y_2'' + p(x) y_2' + q(x) y_2 = 0$ — (b)

Multiply (a) with y_2 and (b) with y_1

$$y_2 y_1'' + p(x) y_2 y_1' + q(x) y_2 y_1 = 0 \quad \text{— (c)}$$

$$y_1 y_2'' + p(x) y_1 y_2' + q(x) y_1 y_2 = 0 \quad \text{— (d)}$$

$\rightarrow$ (d) − (c)

$$W' + p(x)\left[y_1 y_2' - y_2 y_1'\right] = 0$$

$$W' + p(x) W = 0$$

This is a linear DE.

$$\rightarrow W = e^{-\int p\,dx}\left[c + \int 0\right]$$

$$\boxed{W = c\,e^{-\int p\,dx}}$$

Abel's formula.

This is zero at all points or never zero

Note:-

If $W(f,g)(x_1) = 0$, $W(f,g)(x_2) \neq 0$ for some $x_1, x_2 \in I$, then f, g are not solutions of the same DE. (2).

e.g:-

$$f(x) = x^2 \quad , \quad g(x) = \begin{cases} x^2 & \text{for } x \geq 0 \\ -x^2 & \text{for } x < 0 \end{cases} \quad \text{on } \mathbb{R}$$

i) $x_1 \geq 0$

$$W(f,g)(x_1) = x^2(2x) - x^2(2x) = 0$$

ii) $x_2 < 0$

$$W(f,g)(x_2) = x^2(-2x) - (-x^2)(2x) = 0$$

Thm:- Let y_1, y_2 be solutions of (2). Then

$$W(y_1, y_2)(x) = 0 \iff y_1, y_2 \text{ are L.D.}$$

Proof:-

WLOG, assume $y_1 \neq 0$.

$$\Rightarrow y_1(x) \neq 0 \ \forall x \in (c,d) \ ; \ (c,d) \subseteq I.$$

$\Rightarrow W(y_1, y_2) = 0$

$y_1 y_2' - y_2 y_1' = 0$

$$\frac{y_1 y_2' - y_2 y_1'}{y_1^2} = 0 \text{ on } (c,d).$$

$$\left(\frac{y_2}{y_1}\right)' = 0 \text{ on } (c,d).$$

$y_2 = k y_1$ on (c,d).

$y_2' = k y_1'$ on (c,d).

By Uniqueness, y_2 and cy_1 are sol^{n}s, then they coincide in all of I.

Thm:- Let y_1, y_2 be solutions of $y'' + py' + qy = 0$. The the following are equivalent.

* Let y_1, y_2 be L.I solutions of $y'' + py' + q = 0$. They can't vanish at the same point, as then the Wronskian would become zero at that point.

① y_1, y_2 are L.I.

② $W(y_1, y_2)(x) \neq 0 \ \forall x \in I$.

③ $(y_1(x), y_1'(x))$ and $(y_2(x), y_2'(x))$ are L.I $\forall x \in I$.

④ y_1, y_2 form a basis for all the solutions of $y'' + py' + qy = 0$

e.g:- i) $y_1 = x^2|x|$, $y_2 = x^3$

$\rightarrow y_1, y_2$ are L.I.

(a) $x \geq 0$

$$W = \begin{vmatrix} x^3 & x^3 \\ 3x^2 & 3x^2 \end{vmatrix} = 0$$

(b) $x < 0$

$$W = \begin{vmatrix} -x^3 & x^3 \\ -3x^2 & 3x^2 \end{vmatrix} = 0$$

$\Rightarrow W = 0$

So, y_1, y_2 cannot be sol^n of $y''+py'+qy=0$ for any cont. p and q.

ii) Show that there is no $y''+py'+q=0$ in $[0, \frac{\pi}{2}]$ s.t. $y_1 = \sin x$ and $y_2 = x - \pi$ are it's solutions

$$c_1 y_1 + c_2 y_2 = 0; \quad c_1 \sin x + c_2 (x - \pi) = 0 \quad \begin{cases} x = 2\pi \to c_2 = 0 \\ \to c_1 = 0 \end{cases}$$

L.I.

$$W(y_1, y_2) = \begin{vmatrix} \sin x & x - \pi \\ \cos x & 1 \end{vmatrix} = \sin x - (x - \pi)\cos x$$

$W(\pi) = 0$, which means y_1, y_2 can't be solutions.

iii) $y_1 = x$; $y_2 = \sin x$

$$W = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = x\cos x - \sin x$$

$W(0) = 0$ X

Methods to solve 2^nd order DE:

Reduction of order:

If the DE is of the form $F(x, y', y'') = 0$ or $F(y, y', y'') = 0$

I. $F(x, y', y'') = 0$ (Dep. var missing).

Assume $p = y'$

$$\Rightarrow \frac{dp}{dx} = y''$$

→ Then it will become $F(x, p, p') = 0$ → First order eq^n in p

→ After solving, the solution will then be a first order DE in y. Solve it to get solution.

e.g:- Solve $xy'' - y' = 3x^2$

$$y' = p \,;\, y'' = p'$$

$$\Rightarrow x\frac{dp}{dx} - p = 3x^2$$

$$\frac{dp}{dx} + \left(\frac{-1}{x}\right)p = 3x$$

$$p = x\left(3x + c\right) = 3x^2 + cx.$$

$$\frac{dy}{dx} = 3x^2 + cx$$

$$\boxed{y = x^3 + k_1x^2 + k_2}$$

II. $F(y, y', y'') = 0$ (indep. var missing).

Assume $p = y' \Rightarrow y'' = \frac{dp}{dx} = p\frac{dp}{dy}$

$$\Rightarrow F\left(y, p, p\frac{dp}{dy}\right) = 0$$

eg:- Solve $y'' + k^2 y = 0$

Put $y' = p$

$$y'' = p\frac{dp}{dy}$$

$$\Rightarrow p\frac{dp}{dy} + k^2 y = 0$$

$$p\,dp + k^2 y\,dy = 0$$

$$p^2 = k^2 y^2 + c_1$$

$$\frac{dy}{dx} = \sqrt{k^2y^2 + c_1}$$

$$\frac{dy}{\sqrt{k^2y^2 + c_1}} = dx.$$

⋮

Finding L.I Solutions of $y'' + p(x)y' + q(x)y = 0$:-

If y_1 is a non-zero solution of ②, then find another solution y_2 of ② which is L.I with y_1.

* Then the general solution of ② is $c_1y_1 + c_2y_2$.

Method:-

→ Let y_1 be a non-zero sol.
→ y_2 is something that can be obtained by multiplying y_1 with a non-const. function.
→ So, assume $V(x)\cdot y_1(x)$ is a solution and put it in ② to obtain $V(x)$.

Aim: To find $V(x)$ s.t Vy_1 is a sol. to ②, given y_1 is also a sol.

⟹ Let $y_2 = Vy_1$; $y_2' = V'y_1 + Vy_1'$

$$y_2'' = V''y_1 + 2V'y_1' + Vy_1''$$

$$\Rightarrow V''y_1 + 2V'y_1' + Vy_1'' + p(x)(V'y_1 + Vy_1') + q(x)Vy_1 = 0$$

$$V''y_1 + V'(2y_1' + p(x)y_1) + V\underbrace{(y_1'' + p(x)y_1' + q(x)y_1)}_{0} = 0$$

$$V''y_1 + V'(2y_1' + py_1) = 0$$

$$V''y_1 = -V'(2y_1' + py_1)$$

$$\frac{V''}{V'} = -\left(\frac{2y_1'}{y_1} + p\right)$$

$$\int \frac{V''}{V'}\,dx = \int -\frac{2y_1'}{y_1} - p\,dx$$

$$\log(V') = -2\log(y_1) - \int p\,dx$$

$$\log V' + \log y_1^2 = -\int p\,dx$$

$$V'y_1^2 = e^{-\int p\,dx}$$

$$v' = \frac{1}{y_1^2} e^{-\int p\,dx}$$

$$\boxed{v = \int \frac{1}{y_1^2} e^{-\int p\,dx}\, dx}$$

$$\boxed{y_2 = v y_1 = \left(\int \frac{1}{y_1^2} e^{-\int p\,dx}\, dx \right) y_1}$$

$$\Rightarrow \omega(y_1 y_2) = y_1 y_2' - y_2 y_1'$$

$$= y_1 (v y_1' + v' y_1) - v y_1 y_1'$$

$$= v' y_1^2$$

$$= \frac{1}{\cancel{y_1^2}} e^{-\int p\,dx} \cdot \cancel{y_1^2}$$

$$= e^{-\int p\,dx} \neq 0$$

So, y_1 and y_2 are L.I. solutions to (2).

e.g:- i) Solve $x^2 y'' + x y' - y = 0$

$y_1 = x$ is a solution

$$y_2 = x \left(\int \frac{1}{x^2} e^{-\int \frac{1}{x} dx} dx \right)$$

$$= x \left(\int \frac{1}{x^2} \cdot \frac{1}{x}\, dx \right)$$

x^{-2}

$= x \cdot \frac{x}{-2}$

$$\boxed{y_2 = \frac{-1}{2x}}$$

General solution is $\boxed{y(x) = c_1 x + \frac{c_2}{x}}$.

ii) Given that $y_1 = 1/x$ is a sol. of $x^2y'' + 4xy' + 2y = 0$.
Find a L.I solution for $x \geq 1$

$$y_2 = \frac{v}{x}.$$

$$= \frac{1}{x}\int x^2 e^{-\int \frac{4}{x}dx}\, dx$$

$$= \frac{1}{x}\int x^2 \frac{1}{x^4}\, dx$$

$$= \frac{1}{x}\int \frac{1}{x^2}\, dx$$

$$y_2 = \frac{-1}{x^2}$$

Claim:- (2) cannot have a polynomial solution of degree ≥ 2.

Proof:- Let $y = a_n x^n + \dots + a_1 x + a_0$ be a sol to (2).
Substitute y in (2).

$$(n(n-1)a_n x^{n-2} + \dots) + p(na_n x^{n-1} + \dots) + q(a_n x^n + \dots) = 0$$

Coeff x^n

$qa_n = 0 \Rightarrow q = 0$

Coeff x^{n-1}

$$pna_n = 0 \Rightarrow p = 0$$

$$\Rightarrow y'' = 0 \Rightarrow \underline{\underline{y = ax+b}}$$

$\Rightarrow$ Now we check if the exponential function can be a solution

Consider :- $y = e^{mx}$

$$m^2 e^{mx} + p\, m e^{mx} + q\, e^{mx} = 0$$

$$e^{mx}(m^2 + pm + q) = 0 \qquad [p, q \text{ are constants}]$$

So, for $y = e^{mx}$ to be a solution, $m^2 + pm + q = 0$ Auxillary equation or Characteristic equation

Case-I :- $m^2 + pm + q = 0$ has distinct roots (Let m_1, m_2).

Then $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ are linearly independent solution of (2). $\left[\frac{e^{m_1 x}}{e^{m_2 x}} = e^{(m_1 - m_2)x} \neq \text{const.}\right]$

General sol is $\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$

Case-II :- $m^2 + pm + q = 0$ has complex roots (let $m_1 = a + ib$, $m_2 = a - ib$)

Then $e^{m_1 x}$ and $e^{m_2 x}$ are complex valued solution of (2).

$$y_1 = e^{m_1 x} = e^{ax}(\cos bx + i \sin bx)$$
$$y_2 = e^{m_2 x} = e^{ax}(\cos bx - i \sin bx)$$

To extract real valued solutions, we can take a lin combination of ... and ... (... value ...)

combination of y_1 and y_2 ($\because$ Vector space).

$$\Rightarrow \frac{y_1+y_2}{2} = e^{ax}\cos bx$$

$$\Rightarrow \frac{y_1-y_2}{2i} = e^{ax}\sin bx$$

} are the L.I solutions of (2).

General Solution is $\boxed{y = e^{ax}(c_1\cos bx + c_2\sin bx)}$.

Case-III :- $m^2+pm+q=0$ has only 1 root. $\left(m = \frac{-p}{2}\right)$

$$y_1 = e^{mx} = e^{-\frac{p}{2}x}$$

Use. $y_2 = \left(\int \frac{1}{y_1^2} e^{-\int p\,dx}\,dx\right) y_1$

$$\Rightarrow y_2 = e^{-px/2}\left(\int e^{px} e^{-\int p\,dx}\,dx\right)$$

$$y_2 = xe^{-px/2} = xe^{mx}$$

General Solution is $\boxed{y = e^{mx}(c_1 + c_2x)}$

eg:- (i) $y'' + y' - 2y = 0$

Aux. eq^n is $m^2+m-2=0$

$m^2+2m-m-2=0$

$(m+2)(m-1)=0$

$m=1,-2$

Gen. Sol is $y = c_1e^{x} + c_2e^{-2x}$

(ii) $y''+2y'+5y=0$

$$m^2+2m+5=0$$

$$m=\frac{-2\pm\sqrt{4-20}}{2}=-1\pm 2i$$

Compare with $a\pm ib \Rightarrow a=-1$
$b=2$

Gen. Sol is $y=e^{-x}\left[c_1\cos 2x+c_2\sin 2x\right]$

(iii) $y''-4y'+4y=0$

$$m^2-4m+4=0$$

$$\underline{\underline{m=2}}.$$

Gen Sol is $y=e^{2x}\left[c_1+c_2x\right]$

Cauchy-Euler Equations:-

$$x^2y''+axy'+by=0$$

Here, we consider $y=x^m$ [As x^m will not change at the end. It is being compensated].

Put $y=x^m$ in $x^2y''+axy'+by=0$

$$\Rightarrow x^2m(m-1)x^{m-2}+ax\,mx^{m-1}+bx^m=0$$

$$\Rightarrow x^m\left(m(m-1)+am+b\right)=0$$

$$\Rightarrow m^2+(a-1)m+b=0 \quad \text{[Auxillary Eq}^n\text{]}.$$

Case-I:- $m^2+(a-1)m+b=0$ has distinct roots m_1, m_2.

$\Rightarrow x^{m_1}, x^{m_2}$ are L.I solutions of $x^2y'' + axy' + by = 0$

General Solution is $\boxed{y = c_1 x^{m_1} + c_2 x^{m_2}}$

Case-II :- $m^2 + (a-1)m + b = 0$ has complex roots $\alpha + i\beta$, $\alpha - i\beta$.

Then $x^{\alpha + i\beta}$, $x^{\alpha - i\beta}$ are solutions

$$x^{m_1} = x^{\alpha + i\beta} = e^{\ln x(\alpha + i\beta)}$$
$$= x^{\alpha} \cdot e^{i\beta \ln x}$$
$$= x^{\alpha}\left[\cos(\beta \ln x) + i \sin(\beta \ln x)\right]$$

$$x^{m_2} = x^{\alpha - i\beta} = e^{\ln x(\alpha - i\beta)}$$
$$= x^{\alpha} \cdot e^{-i\beta \ln x}$$
$$= x^{\alpha}\left[\cos(\beta \ln x) - i \sin(\beta \ln x)\right]$$

$$y_1 = \frac{x^{m_1} + x^{m_2}}{2} = x^{\alpha}\cos(\beta \ln x)$$

$$y_2 = \frac{x^{m_1} + x^{m_2}}{2i} = x^{\alpha}\sin(\beta \ln x)$$

General Solution is $\boxed{y = x^{\alpha}\left[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)\right]}$

Case-III :- $m^2 + (a-1)m + b = 0$ has one root $\left[m = \frac{1-a}{2}\right]$

$$\Rightarrow y_1 = x^{\frac{1-a}{2}}$$

Use $y_2 = \left(\int \left(\frac{1}{\ldots} e^{-\int p\,dx} dx\right)\right) y_1$

$$\Rightarrow y_2 = x^{\frac{1-a}{2}} \cdot \int x^{a-1} e^{-\int \frac{a}{x} dx} dx$$

$$y_2 = x^{\frac{1-a}{2}} \int x^{a-1} \cdot \frac{1}{x^a} dx$$

$$y_2 = x^{\frac{1-a}{2}} \ln x .$$

General Solution is $\boxed{y = x^{\frac{1-a}{2}} \left[c_1 + c_2 \ln x\right]}$

e.g: (i) $x^2 y'' - 3xy' + 3y = 0$

$$m^2 - 4m + 3 = 0$$

$$m = 1, 3$$

Gen. Sol is $y = c_1 x + c_2 x^3$.

(ii) $x^2 y'' - 3x y' + 5y = 0$

$$m^2 - 4m + 5 = 0$$

$$m = \frac{4 \pm \sqrt{16 - 20}}{2} = 2 \pm i$$

$$\alpha = 2, \beta = 1$$

Gen Sol is $y = x^2 \left[c_1 \cos(\ln x) + c_2 \sin(\ln x)\right]$

Solution to Non-Homogenous DE:-

$$y'' + p(x)y' + q(x)y = r(x) \quad - (1)$$

$$y'' + p(x)y' + q(x)y = 0 \quad - (2)$$

Recall:- If y_1, y_2 are L.I solutions of (2) and y_p is a particular solution of (1), then

$$y = c_1y_1 + c_2y_2 + y_p \text{ is the general solution of (1).}$$

Method of Variation of parameters:-

→ We have $y = c_1y_1 + c_2y_2$ as a general solution of (2).

→ We try to find 2 functions v_1 and v_2 s.t $v_1y_1 + v_2y_2$ is a particular solution of (1).

⇒ Consider $y = v_1y_1 + v_2y_2$ is a solution of (1).

$$y' = (v_1y_1' + v_2y_2') + (v_1'y_1 + v_2'y_2)$$

→ Now, we take $v_1'y_1 + v_2'y_2 = 0$, so as to simplify the second derivative

$$y'' = (v_1y_1'' + v_2y_2'') + (v_1'y_1' + v_2'y_2')$$

Substituting in (1);

$$\Rightarrow (v_1y_1'' + v_2y_2'' + v_1'y_1' + v_2'y_2') + p(v_1y_1' + v_2y_2' + v_1'y_1 + v_2'y_2) + q(v_1y_1 + v_2y_2) = r.$$

$$v_1\underbrace{(y_1'' + py_1' + qy_1)}_{0} + v_2\underbrace{(y_2'' + py_2' + qy_2)}_{0} + v_1'y_1' + v_2'y_2' + p\underbrace{(v_1'y_1 + v_2'y_2)}_{0} = r$$

$$v_1'y_1' + v_2'y_2' = r$$

So, now our goal is to solve for

$$v_1' y_1 + v_2' y_2 = 0$$

$$v_1' y_1' + v_2' y_2' = r(x)$$

This is a system of lin. eqⁿˢ.

$$\underbrace{\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}}_{W(y_1, y_2)} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ r(x) \end{bmatrix}$$

Using Cramer's Rule,

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ r(x) & y_2' \end{vmatrix}}{W(y_1, y_2)} = \frac{-y_2 r}{W(y_1, y_2)}, \quad v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r(x) \end{vmatrix}}{W(y_1, y_2)} = \frac{y_1 r}{W(y_1, y_2)}$$

$$\boxed{v_1 = \int \frac{-y_2 r}{W(y_1, y_2)} \, dx}$$

$$\boxed{v_2 = \int \frac{y_1 r}{W(y_1, y_2)} \, dx.}$$

e.g:- i) Solve $y'' + y = \operatorname{Cosec} x$

First solve $y'' + y = 0$ (Aux. eqⁿ $m^2 + 1 = 0$)

$$y_1 = \sin x, \quad y_2 = \cos x, \quad W = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$V_1 = \int \frac{-\cos x \operatorname{cosec} x}{-1}\,dx \qquad\qquad = -1$$

$$= \int \cot x \, dx$$

$$= \log|\sin x|$$

$$V_2 = \int \frac{\sin x \operatorname{cosec} x}{-1}\,dx$$

$$= -x$$

So, a particular sol is $y = \sin x \log|\sin x| - x\cos x$.

Gen. solution is

$$y = c_1 \sin x + c_2 \cos x + \log|\sin x| \cdot \sin x - x\cos x.$$

Method of Undetermined coeffs:- (p, q Constants, r(x) is Trig/Poly/Exp)

① $y'' + py' + qy = ke^{ax}$; $k, a \in \mathbb{R}$

We can say, $y = Ae^{ax}$ is a solution.

$$\Rightarrow A\, a \cdot a\, \cancel{e^{ax}} + pAa\cancel{e^{ax}} + qA\cancel{e^{ax}} = k\cancel{e^{ax}}$$

$$A(a^2 + pa + q) = k$$

Case-I :- a is not a root of $m^2 + pm + q = 0$

Then $A = \dfrac{k}{a^2 + pa + q}$

$$\boxed{ke^{ax}}$$

$$\boxed{y_p = \frac{\quad}{a^2+pa+q}}$$ is a particular sol of (1).

Case-II :- a is one of the roots of $m^2+pm+q=0$. [a is not a double root]

As we cannot find an A, we try something else.

Try $y = Axe^{ax}$

$y' = Aaxe^{ax} + Ae^{ax}$

$y'' = Aa^2xe^{ax} + 2Aae^{ax}$

$$\Rightarrow Aa^2x\cancel{e^{ax}} + 2Aa\cancel{e^{ax}} + p(Aax\cancel{e^{ax}} + A\cancel{e^{ax}}) + q(Ax\cancel{e^{ax}}) = k\cancel{e^{ax}}$$

$$A\left(\cancel{(a^2+pa+q)}^{0}x + (2a+p)\right) = k$$

$$A = \frac{k}{2a+p}$$

$$\boxed{y_p = \frac{kxe^{ax}}{2a+p}}$$

Case-III :- a is the only root of $m^2+pm+q=0$ [a is a double root]

Now take $y = Ax^2e^{ax}$

$y' = Aax^2e^{ax} + 2Axe^{ax}$

$y'' = Aa^2x^2e^{ax} + 4Aaxe^{ax} + 2Ae^{ax}$

$$\Rightarrow A\cancel{(a^2+pa+q)}^{0}x^2e^{ax} + 2A\cancel{(2a+p)}^{0}xe^{ax} + 2Ae^{ax} = ke^{ax}$$

$$A = \frac{k}{2}$$

$$\boxed{y_p = \frac{kx^2e^{ax}}{2}}$$

e.g: (i) $y'' - 3y' - 4y = 3e^{2x}$

Aux:- $m^2 - 3m - 4 = 0$

$m = 4, -1$

Gen sol of ②: $c_1e^{4x} + c_2e^{-x}$

$$y_p = \frac{3e^{2x}}{(2)^2 + (-3)(2) - 4} = \frac{-e^{2x}}{2}.$$

Gen sol is $c_1e^{4x} + c_2e^{-x} - \dfrac{e^{2x}}{2}$

(ii) $y'' + 5y' + 6y = e^{-3x}$

Aux:- $m^2 + 5m + 6 = 0$

$m = -3, -2.$

Gen sol is $c_1e^{-3x} + c_2e^{-2x} - xe^{-3x}$

② $y'' + py' + qy = f(x)e^{ax}$, where $f(x)$ is a polynomial of degree n.
$a \in \mathbb{R}^n$.

Case-I:- a is not a root of $m^2 + pm + q = 0$.

Take $y_p(x) = g(x)e^{ax}$, $g(x)$ is a polynomial of degree n.

$$= (a_n x^n + \dots + a_0) e^{ax}$$

Sub this in the eq^n, compare coeffs to find $a_n, \dots, a_0$.

Case-II :- a is a root of $m^2 + pm + q = 0$. (not a double root)

Take $y_p(x) = x g(x) e^{ax}$, $g(x)$ is a polynomial of degree n.

$$= x (a_n x^n + \dots + a_0) e^{ax}$$

Sub this in the eq^n, compare coeffs to find $a_n, \dots, a_0$.

Case-III :- a is a root of $m^2 + pm + q = 0$. (a is a double root)

Take $y_p(x) = x^2 g(x) e^{ax}$, $g(x)$ is a polynomial of degree n.

$$= x^2 (a_n x^n + \dots + a_0) e^{ax}$$

Sub this in the eq^n, compare coeffs to find $a_n, \dots, a_0$.

e.g :- $y'' - 3y' = 3x^2 + 2$

Sol :- $m^2 - 3m = 0 \Rightarrow m = 0, 3.$

$$y_p = x g(x)$$

$$y_p = x(ax^2 + bx + c) = ax^3 + bx^2 + cx$$

$$y_p' = 3ax^2 + 2bx + c$$

$$y_p'' = 6ax + 2b$$

$$6ax + 2b - 3(3ax^2 + 2bx + c) = 3x^2 + 2.$$

$$-9ax^2 + (6a - 6b)x + 2b - 3c = 3x^2 + 2$$

$$a = b$$

$$a = \frac{-1}{3}$$

$$b = \frac{-1}{3}$$

③ $y'' + py' + q = e^{\alpha x}(k_1 \cos\beta x + k_2 \sin\beta x)$.

Case-I:- $\alpha + i\beta$ is not a root of $m^2 + pm + q = 0$

Take $y_p(x) = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

Sub this in the eq^n, compare coeffs to find A, B.

Case-II:- $\alpha + i\beta$ is a root of $m^2 + pm + q = 0$

Take $y_p(x) = xe^{\alpha x}(A\cos\beta x + B\sin\beta x)$

Sub this in the eq^n, compare coeffs to find A, B.

e.g:- (i) $y'' + 2y' + 2y = 4e^{x}\sin x$

$m^2 + 2m + 2 = 0 \Rightarrow m = -1 \pm i$

$\alpha + i\beta = 1 + i$ is not a root

$y_p = e^{x}(A\cos x + B\sin x)$

$y_p' = e^{x}((A+B)\cos x + (B-A)\sin x)$

$y_p'' = e^{x}(2B\cos x - 2A\sin x)$

$$2B\cos x - 2A\sin x + 2(A+B)\cos x + 2(B-A)\sin x + 2A\cos x + 2B\sin x$$
$$= 4\sin x$$

$2B + 2A + 2B + 2A = 0 \Rightarrow 4A + 4B = 0 \Rightarrow A = -B$

$-2A + 2B - 2A + 2B = 4 \Rightarrow 4B - 4A = 4 \Rightarrow B - A = 1$

$A = \frac{-1}{2}, \; B = \frac{1}{2}.$

$$y_p = \frac{e^x}{2}(\sin x - \cos x)$$

iii) $y'' + y = \sin x$

$m^2 + 1 = 0 \Rightarrow m = \pm i$

$\alpha + i\beta = i$ is a root

$$y_p = x(A\cos x + B\sin x)$$

$$y_p' = (A\cos x + B\sin x) + x(B\cos x - A\sin x)$$

$$y_p'' = (B\cos x - A\sin x) + (B\cos x - A\sin x) + x(-A\cos x - B\sin x)$$

$$\cos x(2B - \cancel{Ax} + \cancel{Ax}) + \sin x(-2A - \cancel{Bx} + \cancel{Bx}) = \sin x$$

$B = 0$

$A = \frac{-1}{2}$

$$y_p = \frac{-x}{2}\cos x$$

④ $y''+py'+q = f(x)e^{\alpha x}(k_1\cos\beta x + k_2\sin\beta x)$.

Case-I :- $\alpha+i\beta$ is not a root of $m^2+pm+q=0$.

$$y_p(x) = e^{\alpha x}\left([a_nx^n+\dots+a_0]\cos\beta x + [b_nx^n+\dots+b_0]\sin\beta x\right)$$

Case-II :- $\alpha+i\beta$ is a root of $m^2+pm+q=0$.

$$y_p(x) = x\,e^{\alpha x}\left([a_nx^n+\dots+a_0]\cos\beta x + [b_nx^n+\dots+b_0]\sin\beta x\right)$$

$\Rightarrow$ $y''+py'+qy = r_1(x)+\dots+r_n(x)$

$y_p = y_1(x)+\dots+y_n(x)$, where y_i is the particular sol of $y''+py'+qy = r_i(x)$

e.g :- (i) $y''-3y'-4y = \sin 4x + 2e^{4x} + e^{5x} - x$

$m^2-3m-4=0 \quad m=-1,4$

$$y_p = (A\cos 4x + B\sin 4x) + Cxe^{4x} + De^{5x} + Ex + F$$

Compare and Solve.

(ii) $y''+16y = \sin 4x + \cos x - 4\cos 4x + 4$

$m^2+16=0 \Rightarrow m = \pm 4i$

$$y_p = x(A\cos 4x + B\sin 4x) + (C\cos x + D\sin x) + E$$

Compare and Solve.

(iii) $y'' - 3y' - 4y = xe^{-x}$

$$m^2 - 3m - 4 = 0 \Rightarrow m = -1, 4.$$

$$y_p = e^{-x}\left[Ax^2 + Bx\right]$$

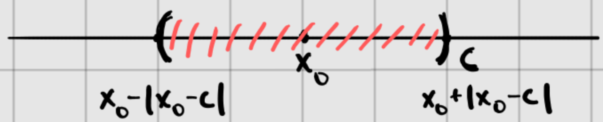
Power Series Method:-

Power Series:-

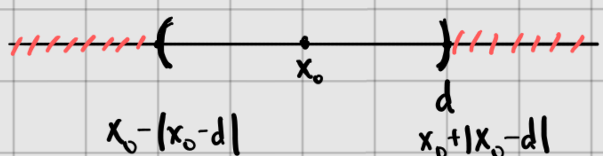
$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots \qquad -(1)$$

Recall:-

(1) If the power series converges at a point c, then the series converges for all x with $|x-x_0| < |c-x_0|$



(2) If the power series diverges at a point d, then the power series diverges for all x with $|x-x_0| > |d-x_0|$



Radius of Convergence:-

Radius of convergence of $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ is defined as

$$R = \sup \left\{ |x - x_0| : \sum_{n=0}^{\infty} a_n (x-x_0)^n \text{ converges} \right\}$$

* The interval of convergence is $(x_0 - R, x_0 + R)$
* The power series converges at all points x with $|x - x_0| < R$ and it diverges at all points x with $|x - x_0| > R$.
* No conclusion can be drawn about convergence at endpoints $x_0 - R, x_0 + R$.

$$R = \frac{1}{\limsup |a_n|^{1/n}}, \quad R = \frac{1}{\lim \left| \frac{a_{n+1}}{a_n} \right|}$$

* We can define a real-valued fn over the interval of convergence

$$f : (x_0 - R, x_0 + R) \to \mathbb{R}.$$

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

* Here, f will be infinitely many differentiable.

$$f'(x) = a_1 + 2a_2(x - x_0) + \ldots\ldots + na_n(x-x_0)^{n-1} + \ldots\ldots$$

$$f'(x) = \sum_{n=1}^{\infty} na_n (x-x_0)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}$$

$$\vdots \qquad\qquad \vdots$$

$$f^{(k)}(x) = \sum_{n=k}^{\infty} \underbrace{\frac{n!}{(n-k)!}}_{{}^nP_k} (x - x_0)^{n-k}$$

* We can obtain the values of the coeffs $a_0, \cdots, a_n$ from f as follows

$$f(x_0) = a_0 \Rightarrow a_0 = f(x_0)$$

$$f'(x_0) = a_1 \Rightarrow a_1 = f'(x_0)$$

$$f''(x_0) = 2a_2 \Rightarrow a_2 = \frac{1}{2} f''(x_0)$$

$$\vdots \qquad\qquad \vdots$$

$$f^{(k)}(x_0) = k!\,a_k \Rightarrow \boxed{a_k = \frac{1}{k!} f^{(k)}(x_0)}$$

* Combining these results,

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(x_0)}{n!} (x - x_0)}$$

* Note:- Power series can be differentiated term by term and the Radius of convergence of the new power series is also the same.

* Note:- Power series representation around x_0 is unique, i.e;

$$\text{if } \sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} b_n (x-x_0)^n, \text{ then } a_n = b_n \ \forall n$$

* $$\sum_{n=0}^{\infty} a_n (x-x_0)^n + \sum_{n=0}^{\infty} b_n (x-x_0)^n = \sum_{n=0}^{\infty} (a_n + b_n)(x-x_0)^n$$

$\Rightarrow R \geq \min\{R_1, R_2\}$, where $R =$ RoC of sum

$R_1 =$ " " 1^{st} series

$R_2 =$ " " 2^{nd} series

* $\sum_{n\geq 0} a_n (x-x_0)^n \times \sum_{n\geq 0} b_n (x-x_0)^n = \sum_{n\geq 0} \left(\sum_{k=0} a_k b_{n-k}\right)(x-x_0)^n$

* Let $f: I \to \mathbb{R}$ be a fn and $x \in I$. Then f is said to be analytic around x_0 if there exists a $\delta > 0$ s.t

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad \text{for } x \in (x_0-\delta, x_0+\delta) \subseteq I.$$

eg: (1) polynomials

(2) e^x is analytic at 0 $\quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

(3) $\sin x$ is analytic at 0 $\quad \sin x = \sum_{n=0} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

(4) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$.

(5) Rational $\left[\frac{p(x)}{q(x)}\right]$ is analytic at x s.t $q(x) \neq 0$

To solve: $y'' + f(x)y' + g(x)y = r(x)$ — (1)

We are looking at a power series solution of (1) at x_0, i.e; does there exists coeffs $a_0, a_1, \ldots$ s.t $y = \sum_{n=0} a_n (x-x_0)^n$ is a solution of (1)

Theorem:-

Let f, g, r have power series representation at x_0 having RoC r_1, r_2, r_3 resp. Then (1) has a power series solution $y = \sum a_n (x-x_0)^n$ whose RoC is $\min\{r_1, r_2, r_3\}$.

i.e; if f, g, r are analytical at x_0, then (1) has a solution that is analytic at x_0.

* The point $x_0 \in I$ is said to be an ordinary point if the coefficient fns f, g, r are analytic at x_0.
* If x_0 is not an ordinary point, then it is a singular point.

Restate: If x_0 is an ordinary point of (1), then (1) admits a power series sol. at x_0.

METHOD:

When f, g, r are analytic at x_0

$$f(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^n, \quad g(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^n, \quad r(x) = \sum_{n=0}^{\infty} d_n (x-x_0)^n$$

Then ① has solution of the form,

$$y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

Now substitute the expressions in ①, obtain the value of a_i's by comparing coeffs of $(x-x_0)^j$.

e.g:- (1) $y'' + y = 0$

$$f(x) = 0, \quad g(x) = 1, \quad r(x) = 0$$

$x_0 = 0$ is an ordinary point.

Consider $\displaystyle y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-x_0)^{n-2}$$

$$= \sum_{m=0}^{\infty} (m+1)(m+2) a_{m+2} (x-x_0)^m$$

$\Rightarrow \quad y'' + y = 0$

$$(n+1)(n+2) a_{n+2} + a_n = 0$$

$$\frac{a_{n+2}}{a_n} = \frac{-1}{(n+1)(n+2)}$$

$$a_2 = \frac{-a_0}{2} = \frac{-a_0}{2!} \qquad a_3 = \frac{-a_1}{6}$$

$$a_4 = \frac{a_0}{24} = \frac{a_0}{4!} \qquad a_5 = \frac{a_1}{120}$$

$$\boxed{a_{2n} = \frac{(-1)^n a_0}{(2n)!}} \qquad \boxed{a_{2n+1} = \frac{(-1)^n a_1}{(2n+1)!}}$$

$$y = a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)$$

$$\boxed{y = a_0 \cos x + a_1 \sin x}$$

LEGENDRE'S EQUATION:-

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0, \quad -1 < x < 1, \quad p \in \mathbb{R}.$$

$$y'' - \Big(\underbrace{\frac{2x}{1-x^2}}\Big)y' + \underbrace{\frac{p(p+1)}{1-x^2}}y = 0$$

(Analytical at all x except 1, −1)

$f(x) = \dfrac{-2x}{1-x^2}$, $g(x) = \dfrac{p(p+1)}{1-x^2}$, $r(x) = 0$. are all analytic at $x_0 = 0$.

So, the eqn has a power series solution $y = \sum_{n=0}^{\infty} a_n x^n$

$$\Rightarrow \quad y'' - x^2 y'' - 2xy' + p(p+1)y = 0$$

(Write all derivatives power series)

$n = 0 \Rightarrow 2\cdot 1 a_2 + p(p+1)a_0 = 0$

$n = 1 \Rightarrow 3\cdot 2 a_3 - 2a_1 + p(p+1)a_1 = 0$

$n \geq 2 \Rightarrow a_{n+2} = -(p-n)(p+n+1)$

$$\frac{\qquad}{(n+1)(n+2)} a_n$$

$$a_2 = \frac{-p(p+1)}{2!} a_0$$

$$a_3 = -\frac{(p-1)(p+2)}{3!} a_1$$

$$a_4 = \frac{-p(p-2)(p+1)(p+3)}{4!} a_0$$

$$a_5 = -\frac{(p-1)(p-3)(p+2)(p+4)}{5!} a_1$$

$$\vdots$$

$$y = \overbrace{a_0\left[1 - \frac{p(p+1)}{2!}x^2 - \frac{p(p-2)(p+1)(p+3)}{4!}x^4 + \cdots\right]}^{y_1}$$
$$\underbrace{+ a_1\left[x - \frac{(p-1)(p+2)}{3!}x^3 - \frac{(p-1)(p-3)(p+2)(p+5)}{5!}x^5 + \cdots\right]}_{y_2}$$

$= a_0 y_1 + a_1 y_2$; y_1, y_2 are L.I. and $\{y_1, y_2\}$ is the basis

* If p is an even integer, y_1 terminates and becomes a polynomial
* If p is an odd integer, y_2 terminates and becomes a polynomial
* RoC of both y_1, y_2 is 1.

$$\left|\frac{a_{n+2}}{a_n}\right| = \left|\frac{-(p-n)(p+n+1)}{(n+1)(n+2)}\right| = 1$$

* If $p = k$, $k \in \mathbb{Z}^+$, then coeff of x^{k+2} is 0

$$a_{k+2} = a_{k+4} = \cdots = 0.$$

(i) k is an even integer

$y_1 \rightarrow$ polynomial of degree k, with only even powers

$y_2 \rightarrow$ non-terminating series containg only odd powers

(ii) k is an odd integer

$y_1 \rightarrow$ non-terminating series containg only even powers

$y_2 \rightarrow$ polynomial of degree k, with only odd powers

* A polynomial solⁿ $P_k(x)$ is called a Legendre's polynomial if $P_k(1) = 1$.

* Consider $P_k(x)$.

We know,

$$a_{n+2} = \frac{-(k-n)(k+n+1)}{(n+2)(n+1)} a_n$$

So, $a_{k+2} = 0$, $a_k \neq 0$

Consider $n = k-2$.

$$a_k = \frac{-(k-(k-2))(k+k-2+1)}{(k-2+2)(k-2+1)} a_{k-2}$$

$$a_k = \frac{-2(2k-1)}{k(k-1)} a_{k-2}.$$

$$a_{k-2} = \frac{-k(k-1)}{2(2k-1)} a_k.$$

All a_{k-2l} is a multiple of a_k. So, the polynomial will be a multiple of a_k. So we chose a_k so that the polyn.

maps 1 to 1.

Choice: $\boxed{a_k = \frac{(2k)!}{2^k (k!)^2}}$

$$a_{k-2} = \frac{-\cancel{k}\cancel{(k-1)}}{\cancel{2}\cancel{(2k-1)}} \cdot \frac{\cancel{2k}\cancel{(2k-1)}(2k-2)!}{2^k\, \cancel{k}\cancel{(k-1)}(k-2)!\,\cancel{k}(k-1)!}$$

$$= \frac{-(2k-2)!}{2^k (k-1)!\,(k-2)!}$$

$$a_{k-4} = \frac{-\cancel{(k-2)}\cancel{(k-3)}}{\cancel{2}\cancel{(2k-3)}} \cdot \frac{-\cancel{2(k-1)}\cancel{(2k-3)}(2k-4)!}{2^k \underbrace{\cancel{(k-1)}\cancel{(k-2)}\cancel{(k-3)}(k-4)!}_{(k-2)!}}$$

$$= \frac{(2k-4)!}{2^k (k-2)!\,(k-4)!}$$

$$a_{k-2l} = \frac{(-1)^l (2k-2l)!}{2^k (k-l)!\,(k-2l)!}$$

$$P_k(x) = \sum_{l=0}^{[\frac{k}{2}]} \frac{(-1)^l (2k-2l)!}{2^k (k-l)!\,(k-2l)!} x^{k-2l}$$

* $P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x).$$

$\vdots$

Corollary:- Any polynomial solution of the Lengendre's Equation is a scalar multiple of P_n for some n.

Case I:

n is even

$P_n = a_0 y_1 + a_1 y_2$

$\underbrace{P_n - a_0 y_1}_{\text{even degree}} = \underbrace{a_1 y_2}_{\text{odd degree}} \Rightarrow a_1 = 0.$

$P_n = a_0 y_1$

Let y be a polynomial sol

$y = c_1 y_1 + c_2 y_2$

As n is even, $c_2 = 0$

$y = \frac{c_1}{a_0} P_n.$

Case I:

n is odd

$P_n = a_0 y_1 + a_1 y_2$

$\underbrace{P_n - a_1 y_2}_{\text{odd degree}} = \underbrace{a_0 y_1}_{\text{odd degree}} \Rightarrow a_0 = 0.$

$P_n = a_1 y_2$

Let y be a polynomial sol

$y = c_1 y_1 + c_2 y_2$

As n is odd, $c_1 = 0$

$y = \frac{c_2}{a_1} P_n.$

Rodrigue's formula:-

$$P_n(x) = \frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n}(x^2 - 1)^n$$

Theorem:- If f is a cont. fn. on $[-1,1]$, then

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x).$$

Regular Singular Points:-

A singular point x_0 is called a regular singular point if $(x-x_0)p$ and $(x-x_0)^2 q$ are analytic at x_0 in

$$y'' + p(x)y' + q(x)y = r(x).$$

e.g: [Bessel's eqn]

$$x^2y'' + xy' + (x^2 - p^2) = 0$$

$$y'' + \frac{1}{x}y' + \left(1 - \frac{p^2}{x^2}\right) = 0$$

$x = 0$ is a singular point.

$$\left.\begin{aligned} xp(x) &= x \cdot \frac{1}{x} = 1 \\ x^2 q(x) &= x^2\left(1 - \frac{p^2}{x^2}\right) = x^2 - p^2. \end{aligned}\right\} \text{analytic at } x_0 = 0.$$

$\Rightarrow$ $$y'' + p(x)y' + q(x)y = 0$$

$$y'' + \frac{xp(x)}{x}y' + \frac{x^2 q(x)}{x^2}y = 0$$

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n$$

$$x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n.$$

$$y'' + \frac{(p_0 + p_1 x + \ldots)}{x}y' + \frac{(q_0 + q_1 x + \ldots)}{x^2}y = 0$$

$\rightarrow$ So, a solution will be of the form;

$$y = x^m(a_0 + a_1 x + \ldots\ldots).$$

(By comparing with solutions to Cauchy-Euler eqn.).

$$\boxed{y = x^m \sum_{n=0}^{\infty} a_n x^n}, m \in \mathbb{R}$$

[Frobenius series solution]

→ The second L.I solution may be another Frobenius series solution of the form

$$y = x^{m'} \sum_{n=0}^{\infty} b_n x^n.$$

(or)

$$y = x^m \ln x \sum_{n=0}^{\infty} a_n x^n.$$

→ We try to put the solution into the equation to produce a recurrance relation.

$$y = \sum_{n=0}^{\infty} a_n x^{m+n}$$

$$y' = \sum_{n=0}^{\infty} a_n (m+n) x^{m+n-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (m+n)(m+n-1) x^{m+n-2}$$

$$p(x) = \frac{1}{x} \sum_{n=0}^{\infty} p_n x^n$$

$$q(x) = \frac{1}{x^2} \sum_{n=0}^{\infty} q_n x^n$$

$$\Rightarrow p(x) y' = \frac{x^{m-1}}{x} \sum_{n=0}^{\infty} p_n x^n . \sum_{n=0}^{\infty} (m+n) a_n x^n$$

$$= x^{m-2} \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} p_{n-k} a_k (m+k) \right] x^n$$

$$\sum^{\infty}$$

$$= x^{m-2} \sum_{n=0} \left[\sum_{k=0}^{n-1} p_{n-k}\, a_k (m+k) + p_0 a_n (m+n) \right] x^n .$$

$$\Rightarrow q(x) y = x^{m-2} \sum q_n x^n \sum a_n x^n$$

$$= x^{m-2} \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} q_{n-k}\, a_k \right] x^n$$

$$= x^{m-2} \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n-1} q_{n-k}\, a_k + q_0 a_n \right] x^n .$$

Substituting in the eq^n.

$$\Rightarrow x^{m-2} \sum_{n=0}^{\infty} \left[a_n (m+n)(m+n-1) + \left[\sum_{k=0}^{n-1} q_{n-k} a_k + q_0 a_n \right] + \left[\sum_{k=0}^{n-1} p_{n-k}\, a_k (m+k) + p_0 a_n (m+n) \right] \right] x^n$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[a_n \left[(m+n)(m+n-1) + p_0 (m+n) + q_0 \right] + \sum_{k=0}^{n-1} a_k \left(q_{n-k} + p_{n-k} (m+k) \right) \right] x^n$$

Comparing coeffs:-

$n=0 \Rightarrow \quad a_0 \left[m(m-1) + m p_0 + q_0 \right] = 0$ [Indicial eq^n]

$\hookrightarrow$ (solve to get m).

$n \geq 1 \Rightarrow$

$$\left[a_n \left[(m+n)(m+n-1) + p_0 (m+n) + q_0 \right] + \sum_{k=0}^{n-1} a_k \left(q_{n-k} + p_{n-k} (m+k) \right) \right] = 0$$

Let $f(m) = m(m-1) + m p_0 + q_0$

Then the recurrance relation becomes

$(n \geq 1)$ $a_n f(m+n) + a_0 (m p_n + q_n) + a_1 ((m+1) p_{n-1} + q_{n-1}) + \dots + a_{n-1} ((m+n-1) p_1 + q_1) = 0$

$(n \geq 1) \quad a_n f(m+n) + a_0 (m p_n + q_n) + a_1 ((m+1) p_{n-1} + q_{n-1}) + \dots + a_{n-1} ((m+n-1) p_1 + q_1) = 0$

* If $f(m+n) = 0$, there is an ambiguity with the relation. So we always take the larger root of the indicial eqn.

→ Let $m_1 > m_2$ be the roots of the indicial equation.
→ If $m_1 - m_2 = k$ is an integer Then,

$$f(m_2 + k) = 0$$

$$\Rightarrow a_k (0) = -\left(a_0 \left[m_2 p_k + q_k \right] + \dots\dots + a_{k-1} \left[(m_2 + k - 1) p_1 + q_1 \right] \right)$$

(1) If RHS $= 0$, then a_k is arbitrary.
Compute other a_n's using the relation.

(2) If RHS $\neq 0$, then no a_k satisfies.
Hence, m_2 will not give a second Frobenius series solution.

* The solutions of the Indicial eq^n will have 3 cases.

Case-1 :- $m_1 - m_2$ is not an integer.

2 F.S.S are

$$y_1 = x^{m_1} \sum_{n=0}^{\infty} a_n x^n$$

$$y_2 = x^{m_2} \sum_{n=0}^{\infty} a_n x^n$$

Case-2 :- $m_1 - m_2 = 0$

Only one FSS.

$$y_1 = x^{m_1}\sum_{n=0}^{\infty} a_n x^n$$

The other L.I sol is

$$y_2 = y_1(x)\ln x + x^{m_1}\sum_{n=0}^{\infty} A_n x^n \qquad (x \geqslant 0)$$

Case-3 :- $m_1 - m_2$ is an integer

$$y_1 = x^{m_1}\sum_{n=0}^{\infty} a_n x^n$$

The other L.I sol is

$$y_2 = k y_1(x)\ln x + x^{m_2}\sum_{n=0}^{\infty} A_n x^n.$$

eg: (1) $2x^2y'' + x(2x+1)y' - y = 0$

$$y'' + \frac{2x+1}{2x}y' - \frac{1}{2x^2}y = 0$$

$$xp(x) = x + 1/2 \quad , x^2q(x) = -\frac{1}{2} \rightarrow \begin{cases} p_0 = 1/2 & ; p_i = 0 \\ p_1 = 1 & \\ q_0 = -1/2 & ; q_i = 0 \end{cases}$$

Sub, $y = x^m \sum_{n=0}^{\infty} a_n x^n$ in DE.

(Indicial eq^n) ($n=0$)
$$a_0\left(m(m-1) + \frac{1}{2}m - \frac{1}{2}\right) = 0$$

$$a_0\left(m^2-\frac{m}{2}-\frac{1}{2}\right)=0$$

$$m=1,\ -1/2$$

For $m=1$

$n=1$ $$a_1\left(\frac{5}{2}\right)+a_0=0$$

$$a_1=-\frac{2}{5}a_0$$

$n=2$ $$a_2(7)+a_1(2\cdot 1)+a_0=0$$

$$a_2=\frac{4}{35}a_0$$

For $m=-1/2$

[Solve later]

(2) $$xy''+2y'+xy=0$$

$$xp(x)=2\ ;\quad x^2q(x)=x^2$$

$$p_0=2,\quad q_0=0,\ q_2=1$$

$$\sum(m+n)(m+n-1)a_nx^n+2\sum(m+n)a_nx^n+\sum a_{n-2}x^n=0$$

$$a_0\left[m(m-1)+2m\right]=0 \qquad n=0$$

$$a_1\left[(m+1)m+2(m+1)\right]=0 \qquad n=1$$

$$a_n\left[(m+n)(m+n-1)+2(m+n)\right]+a_{n-2}=0 \qquad n\geqslant 2$$

$$a_n = \frac{-a_{n-2}}{(m+n)(m+n+1)}$$

$$m=0,-1.$$

For $m=0$

$$a_n = \frac{-a_{n-2}}{n(n+1)}$$

$$a_1 = 0$$

$$a_2 = \frac{-1}{3!}a_0$$

$$a_3 = 0$$

$$a_4 = \frac{1}{5!}a_0$$

$$y_1 = a_0\left[1-\frac{x^2}{3!}+\frac{x^4}{5!}+\ldots\right] \qquad (\text{Take } a_0=1)$$

$$\boxed{y_1 = \frac{\sin x}{x}}$$

$\Rightarrow$ Check for second frobenious series sol.

$$a_n f(m+n) = -\left[a_0(mp_n+q_n)+\ldots+a_{n-1}((m+n-1)p_1+q_1)\right].$$

As $m_1-m_2=1$, take $n=1$

$$a_1 f(m_1+1) = -\ a_0(m_1 p_1 + q_1)$$

$$a_1 f(m_2) = -a_0 q_1$$

$$a_1 \cdot 0 = 0$$

Hence, a_1 is arbitrary.

For computational purposes, take $a_1 = 0$

For $m = -1$,

$$a_n = \frac{-a_{n-2}}{n(n-1)}$$

$$a_2 = \frac{-a_0}{2!}$$

$$a_4 = \frac{-a_2}{4\cdot 3} = \frac{a_0}{4!}$$

$$\vdots$$

$$y_2 = a_0 \cdot \frac{1}{x}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots\cdots\right] \quad (\text{Put } a_0 = 1)$$

$$\boxed{y_2 = \frac{\cos x}{x}}$$

(3) $4x^2y'' - 8x^2y' + (4x^2+1)y = 0$

$$xp(x) = -2 \ ; \quad x^2 q(x) = x^2 + \frac{1}{4}.$$

⋮

[Solve later]

Bessel's Equation:-

$$x^2y'' + xy' + (x^2 - p^2)y = 0 \quad , \quad p \to \text{nonneg. const.}$$

$$xp(x) = 1, \quad x^2q(x) = x^2 - p^2$$

$$y = x^m \sum a_n x^n$$

$$m(m-1) + m(1) - p^2 = 0$$
$$m^2 - p^2 = 0$$
$$m = \pm p.$$

$$m_1 = p \quad ; \quad m_2 = -p.$$

For $m_1 = p$;

$$p^2 y = p^2 \sum_{n=0}^{\infty} a_n x^{n+p}$$

$$x^2 y = \sum_{n=2}^{\infty} a_{n-2} x^{n+p}$$

$$xy' = \sum_{n=0}^{\infty} (n+p) a_n x^{n+p}$$

$$x^2 y'' = \sum_{n=0}^{\infty} (n+p)(n+p-1) a_n x^{n+p}$$

$$\sum^{\infty} (n+p)(n+p-1) a_n x^n + \sum^{\infty} (n+p) a_n x^n + \sum^{\infty} a_{n-2} x^n - p^2 \sum^{\infty} a_n x^n = 0$$

$n=0$ $\quad n=0$ $\quad n=2$ $\quad n=0$

$\underline{\underline{n=0}}$ $\quad \left[p(p-1)+p-p^2\right]a_0=0$

$\underline{\underline{n=1}}$ $\quad \left[p(p+1)+p+1-p^2\right]a_1=0 \Rightarrow a_1=0$

$\underline{\underline{n\geq 2}}$

$$a_n\left[(n+p)(n+p-1)+(n+p)-p^2\right]+a_{n-2}=0$$

$$\boxed{a_n=\frac{-a_{n-2}}{n(n+2p)}}$$

Hence, we only have even indices in the Frobenius solution.

$$a_2=\frac{-a_0}{2(2p+2)}=\frac{-a_0}{2^2(p+1)}$$

$$a_4=\frac{-a_2}{4(4+2p)}=\frac{a_0}{2^2(p+1)}\cdot\frac{1}{2\cdot 2^2(2+p)}=\frac{a_0}{2^4 2!(p+1)(p+2)}$$

$$\boxed{a_{2n}=\frac{(-1)^n a_0}{2^{2n}\cdot n!\cdot\left[\frac{(p+n)!}{p!}\right]}}$$

$$y_1(x)=a_0x^p\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}\cdot n!\cdot\left[\frac{(p+n)!}{p!}\right]}\right)$$

*Def: The Bessel's function of the first kind of order p is denoted by $J_p(x)$ is obtained by taking

$$a_0 = \frac{1}{2^p\, p!} \quad \text{in } y_1$$

$$\boxed{\begin{aligned}\Gamma(t) &= \int_0^\infty e^{-x} x^{t-1}\,dx \\ \Gamma(n+1) &= n!\end{aligned}}$$

$$J_p(x) = \frac{1}{2^p \cancel{p!}} x^p \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} \cdot n! \cdot \left[\frac{(p+n)!}{\cancel{p!}}\right]}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+p}}{2^{2n+p}\, n!\, (n+p)!}$$

$$\boxed{J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+p)!} \left(\frac{x}{2}\right)^{2n+p}}$$

$$\Rightarrow J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n} = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4(2)^2} - \frac{x^6}{2^6(6)^2} + \cdots\cdots$$

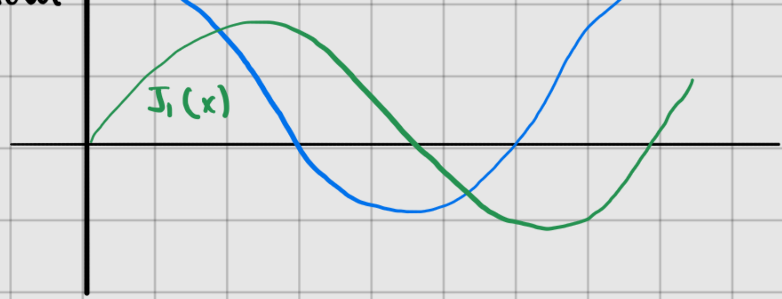
$$\Rightarrow J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+1)!} \left(\frac{x}{2}\right)^{2n+1}$$

$$= \frac{x}{2} - \frac{x^3}{2^3(2!)} + \frac{x^5}{2^5(2!\cdot 3!)}$$

$$\vdots$$

*Observe $J_0'(x) = -J_1(x)$ [Only for J_0 and J_1]

* $J_p(x)$ has an oscillatory behaviour



[Recall] Gamma Function:-

$$\Gamma(p) = \int_0^\infty e^{-t} t^{p-1}\, dt \quad ; \; p > 0.$$

Properties:-

(1) $\Gamma(1) = 1$

(2) $\Gamma(n+1) = n!$

(3) $\Gamma(p+1) = p\,\Gamma(p)$

(4) $\Gamma(1/2) = \sqrt{\pi}$

*Def:- A solution of Bessel's eqn which is L.I with $J_p(x)$ is called a Bessel's eqn of 2^{nd} kind.

We try $m_2 = -p$.

Case-I:- $m_1 - m_2 = 2p$ is not an integer.

2^{nd} FSS exists corresponding to $m_2 = -p$. $[J_{-p}(x)]$

Case-II:- $m_1 - m_2 = 2p = 0$

2^{nd} FSS doesn't exist.

Case-III:- $m_1 - m_2 = 2p$ is an integer.

Case-(a):- p is not an int $\left[\frac{odd}{2} \text{ form}\right]$

Proceed as previously to get the relation

$$a_n = \frac{-a_{n-2}}{n(n-2p)}$$

(Why aren't we checking $a_k f(m+k) = (\ldots)$)

$$J_{-p}(x) = \sum \frac{(-1)^n}{n!(n-p)!} \left(\frac{x}{2}\right)^{2n-p}$$ [make a diff choice of a_0]

As J_{-p} is unbounded at $x=0$, unlike J_p; they are L.I.

General sol is $c_1 J_p + c_2 J_{-p}$

Standard Bessel's fn of 2nd kind

$$Y_p(x) = \frac{J_p(x)\cos p\pi - J_{-p}(x)}{\sin p\pi}$$

Case-(b):- p is an int

$$J_{-m} = (-1)^m J_m$$

$\Rightarrow J_m, J_{-m}$ are not L.I. Thus J_{-m} is not a Bessel's fn.

Thus, there is no 2nd FSS.

$$y_1 = J_p(x)$$

$$y_2 = J_m(x) \int \frac{1}{J_m(x)^2} e^{-\int \frac{1}{x} dx} dx$$

$$y_2 = J_m(x) \int \frac{1}{x\, J_m(x)^2} dx.$$

Analysis of Solutions:-

Sturm Seperation Theorem:-

If y_1 and y_2 are 2 L.I sols of $y'' + p(x)y' + q(x)y = 0$ then the zeros of these fns are distinct and occur alternatively, i.e, y_1 vanishes exactly once b/w any 2 consecutive roots of y_2 and vice versa.

Proof:- $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$ is a continuous fn. that never vanishes and has a const. sign.

Let x_1, x_2 be consecutive zeros of y_2.

$$W(y_1, y_2)(x_i) = y_1(x_i)\, y_2'(x_i)$$

Here, $y_2'(x_1)$ and $y_2'(x_2)$ have opposite signs. So for the wronskian to maintain sign, $y_1(x_1)$ and $y_1(x_2)$ also have opposite signs.

So, there exists atleast one root for y_1 b/w x_1, x_2.

If there exists more than one root of y_1 b/w x_1 and x_2, then we reverse the roles of y_1 and y_2 and find another root of y_2 b/w any 2 of the aforementioned roots of y_1, contradicting the consecutive choice of x_1, x_2.

Here only one root of y_1 exists b/w x_1, x_2.

Hence proved.

Normal form:-

Standard form:- $y'' + p(x)y' + q(x)y = 0$

Normal form:- $y'' + q_1(x)y = 0$

Take $y(x) = u(x)v(x)$

$$y'(x) = uv' + vu' \quad ; \quad y''(x) = uv'' + 2u'v' + vu''$$

$$\Rightarrow vu'' + \underbrace{(vp(x) + 2v')}_{=0} u' + (vq(x) + v'p(x) + v'')u = 0$$

Chose v s.t

$$\frac{v'}{v} = \frac{-p(x)}{2}.$$

$$\boxed{v(x) = e^{-\frac{1}{2}\int p(x)\,dx}}$$

$$v' = \frac{-p(x)}{2} v(x)$$

$$v'' = -\frac{1}{2}\left[p'(x)v(x) + p(x)v'(x)\right]$$

$$v'' + p(x)v' + q(x)v = -\frac{1}{2}p'v - \frac{1}{2}p\left(-\frac{1}{2}pv\right) + p\left(-\frac{1}{2}pv\right) + qv$$

$$= -\frac{1}{2}p'v - \frac{1}{4}p^2 v + qv$$

$$= v\left(q - \frac{p^2}{4} - \frac{p'}{2}\right)$$

$$\Rightarrow \cancel{v}u'' + \cancel{v}\left(q - \frac{p^2}{4} - \frac{p'}{2}\right)u = 0$$

$$\boxed{u'' + \left[q(x) - \frac{[p(x)]^2}{4} - \frac{p'(x)}{2}\right]u = 0}$$

Zeros of sol. of std. form and the Normal form are the same.

Theorem:- If $q(x) < 0 \ \forall x$ in $u'' + q(x)u$ then any non trivial solution u has atmost one zero.

Theorem:- Let u be a non trivial sol of $u'' + q(x)u = 0$ when $q(x) > 0$ $\forall x > 0$

if $\int_1^\infty q(x)\,dx = \infty$, then u has infinitly many zeros on the +ve x axis.

e.g.:- $y'' + 4y = 0 \Rightarrow y = \sin 2x \rightarrow$ Oscillates faster. Larger q.

$u'' + u = 0 \Rightarrow u = \sin x \rightarrow$ Oscillates slower. Smaller q.

Sturm Comparision Theorem:-

Let y and z be non trivial solution of $y'' + q(x)y = 0$ and $z'' + r(x)z = 0$ s.t $q, r > 0$, $q(x) > r(x)$. Then y vanishes atleast once between 2 succ. zeros of z.

* Consider $y'' + q(x)y = 0$ where $q(x) > k^2 \ \forall x$. Let y be non-triv. sol
Compare with $z'' + k^2 z = 0$.

$z = \sin k(x-x_0)$. is a solution

Observe that in any interval of length $\frac{\pi}{k}$, there is a zero of y.

* Consider Bessel's eqn.

$$x^2y'' + xy' + (x^2-p^2)y = 0$$

$$u'' + q(x)u = 0$$

where $q(x) = \left[1 - \frac{p^2}{x^2}\right] - \frac{1}{2}\left(\frac{-1}{x^2}\right) - \frac{1}{4}\frac{1}{x^2} = 1 - \frac{p^2}{x^2} - \frac{1}{4}\frac{1}{x^2}$.

$$= 1 + \left(\frac{1-4p^2}{4x^2}\right)$$

$$u'' + \left[1 + \left(\frac{1-4p^2}{4x^2}\right)\right]u = 0$$

Compare with $y'' + y = 0 \Rightarrow y = \sin x$ is a non-trivial sol.

Let u be a nontrivial sol of Bessel's eqn on the +ve x-axis

Sturm Comp Thm.

$\Rightarrow$ If $0 < p < 1/2$, then every interval of length π contains atleast one zero of u.

$\Rightarrow$ If $p = 1/2$, then the dist. b/w 2 zeros of u is exactly π.

$\Rightarrow$ If $p > 1/2$, then every interval of length π contains atmost one zero of u.

[y become more freq. than u].

Boundary Value Problems

Boundary Value Problem:-

$$y'' + p(x)y' + q(x)y = 0$$

Subject to $y(a) = y_1$ and $y(b) = y_2$.

Eigen Value Problem:-

e.g:- $y'' + \lambda y = 0$ s.to $y(0) = 0$, $y(\pi) = 0$ — ⊛

Problem: What are those real number λ for which ⊛ has a non-trivial solution.

Compare the eqn given with $L(y) = y''$

So, we try to find λ s.t $L(y) = -\lambda y$

* The λ for which ⊛ has a non trivial solution is called an eigenvalue and the corresponding non-zero solution is called the eigenfunction corresponding to λ.

Case-I:- $\lambda = 0$

$\Rightarrow y'' = 0$

$\Rightarrow y = Ax + B$

$$\left.\begin{array}{l} \Rightarrow y(0) = 0 \Rightarrow B = 0 \\ \quad y(\pi) = 0 \Rightarrow A = 0 \end{array}\right\} \underbrace{y = 0}_{\text{trivial sol.}} . \text{ Hence } 0 \text{ is not an eigenvalue}$$

Case-II:- $\lambda < 0$

$\lambda = -\mu^2$ [As $-\lambda$ is +ve]
($\mu > 0$)

$\Rightarrow y'' = \mu^2 y$

$m^2 - \mu^2 = 0 \Rightarrow m = \pm\mu$

$y(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$

$$\left.\begin{aligned} &\Rightarrow y(0) = 0 \Rightarrow c_1 + c_2 = 0 \\ &\Rightarrow y(\pi) = 0 \Rightarrow c_1 e^{\mu\pi} + c_2 e^{-\mu\pi} = 0 \end{aligned}\right\} \; c_1 = c_2 = 0$$

Case-III :- $\lambda > 0$

$\lambda = \mu^2$

$\Rightarrow y'' + \mu^2 y = 0$

$y(x) = c_1 \cos\mu x + c_2 \sin\mu x$

$\Rightarrow y(0) = 0 \Rightarrow c_1 = 0$

$\Rightarrow y(\pi) = 0 \Rightarrow c_2 \sin\mu\pi = 0$

So, for a non-trivial solution, $c_2 \neq 0$; $\mu = n, n \neq 0$

$\lambda = n^2$; $\lambda = 1^2, 2^2, 3^2, \ldots$

* Eigenvalues are $\lambda = n^2$; $n = 1, 2, \ldots$ and corresponding eigenfunctions are $y = \sin(nx)$

⇒ Observe that

(1) λ_n increases as λ_n increases to ∞

(2) $\int \sin(nx)\sin(mx)\,dx = 0$ [Eigenfunctions corresponding to distinct eigenvalues are orthogonal]

(3) Eigenspace corresponding to an eigenvalue λ is 1-dim

Sturm-Liouville Problem :-

For continuous real valued functions p, q, r on $[a,b]$ such that p' exists and is continuous on $[a,b]$. Consider the DE

$$\frac{d}{dx}\left[p\frac{dy}{dx}\right] + \left[\lambda q(x) + r(x)\right]y = 0 \qquad -①$$

together with the boundary conditions

$$c_1 y(a) + c_2 y'(a) = 0 \qquad -② \;;\; c_1 \text{ or } c_2 \text{ is non-zero}$$
$$d_1 y(b) + d_2 y'(b) = 0 \qquad -③ \;;\; d_1 \text{ or } d_2 \text{ is non-zero.}$$

This problem is called the Sturm Liouville problem. (SLP)

* The λ for which there exists a non-trivial fn y that satisfies ①, ② and ③ is called an eigenvalue and the corresponding non zero solution is called the eigenfunction corresponding to λ.

Let $$L(y) = \frac{d}{dx}\left(p\frac{dy}{dx}\right) + r(x)y = 0$$

Then ① is $L(y) + \lambda q(x) y = 0$.

* If $p>0$ and $q>0$ in ①, then

(1) Eigenvalues of SLP are real and form an increasing sequence

$\lambda_1 < \lambda_2 < \lambda_3 < \ldots\ldots$

(2) If $\lambda_n \neq \lambda_m$ are eigenvalues of ① and y_n, y_m are their corresponding eigenfunctions then,

$$\int_0^{\pi} q(x)\, y_n(x)\, y_m(x)\, dx = 0$$

i.e; eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to weight function $q(x)$.

(3) If y_1 and y_2 are eigenfunctions corresponding to the same eigenvalue λ, then they are L.D.

eg:- ① Solve $y'' + \lambda y = 0$ subject to $\left.\begin{array}{l} y(-\pi) = y(\pi) \\ y'(-\pi) = y'(\pi) \end{array}\right\}$ Not SLP

Periodic B.C.

Case-I :- $\lambda > 0$ $[\lambda = \beta^2]$

$y'' + \beta^2 y = 0$

$y(x) = c_1 \cos \beta x + c_2 \sin \beta x$

$\Rightarrow y(-\pi) = y(\pi) \Rightarrow c_1 \cos(-\beta\pi) + c_2 \sin(-\beta\pi) = c_1 \cos(\beta\pi) + c_2 \sin(\beta\pi)$

$c_2 \sin(\beta\pi) = 0$

$\Rightarrow c_2 = 0$ or $\beta = n$

$\Rightarrow y'(-\pi) = y'(\pi) \Rightarrow c_1 \beta \sin(\beta x) = 0$

$\Rightarrow c_1 = 0$ or $\cancel{\beta = 0}$ or $\beta = n$.

As $\beta \neq 0$.

Chose $\beta = n$

$\therefore \quad \lambda = n^2 ; \quad y = c_1 \cos(nx) + c_2 \sin(nx)$ ///

Case-II :- $\lambda = 0$

$y'' = 0$

$y = Ax + B$

$\Rightarrow y(-\pi) = y(\pi) \Rightarrow A = 0$

$\Rightarrow y'(-\pi) = y'(\pi) \Rightarrow A = A$ ✓

$y = B$ is a solution for $B \neq 0$.

$\therefore \lambda = 0 ; \ y =$ non-zero const. fn.

Case-III :- $\lambda < 0$

Proceed sim, $\lambda < 0$ not an eigenvalue.