Variable Seperable:-

$$y' = f(x,y) = f(x)\cdot f(y)$$

$$\Rightarrow \frac{dy}{dx} = f(x)\cdot f(y)$$

$$\Rightarrow \frac{dy}{f(y)} = f(x)\,dx$$

$$\Rightarrow \int \frac{dy}{f(y)} = \int f(x)\,dx$$

e.g:- $xy' = y + y^2$

$$\int \frac{dy}{y+y^2} = \int \frac{dx}{x} \Rightarrow \int \frac{1}{y} - \frac{1}{1+y}\,dy = \int \frac{1}{x}\,dx$$

$$\ln\left(\frac{y}{1+y}\right) = \ln(kx)$$

$$\boxed{\frac{y}{1+y} = kx}$$

Homogenous DE:-

A function, $f(x,y)$ is a homogenous eq^n of degree 'n' if $f(tx,ty) = t^n f(x,y)$.

* A DE $y' = f(x,y)$ ① is said to be Homogenous if f is o

* A DE $y' = f(x,y)$ is said to be homogenous if f is a homogenous fn. of deg 0

* A DE of the form $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be homogenous, if M and N are homogenous functions of a degree (fixed).

* The procedure for solving (1) relies on the fact that a suitable substitution can be made in (1). so that (1) turns out to be a DE in variable seperable form.

$\Rightarrow$ Since f is a homogenous fn of deg. 0,

$$f(tx, ty) = f(x,y)$$

Let $t = \frac{1}{x}$, $f\left(1, \frac{y}{x}\right) = f(x,y)$.

Let $y = vx$, $f(1,v) = f(x, vx)$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$ Substitute here.

e.g.

$$(x+y)\,dx - (x-y)\,dy = 0$$

$$\frac{dy}{dx} = \frac{x+y}{x-y}$$

Put $y = vx$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1+v}{1-v}$$

$$x\frac{dv}{dx} = \frac{1+v^2}{\ldots}$$

$$\int \frac{1-v}{1+v^2}\,dv = \int \frac{1}{x}\,dx$$

$$\int \frac{1}{1+v^2}\,dv - \frac{1}{2}\int \frac{2v}{1+v^2}\,dv = \ln(x)+c$$

$$\tan^{-1}(v) - \frac{1}{2}\ln(1+v^2)\,dv = \ln(x)+c$$

$$\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(\frac{x^2+y^2}{x^2}\right) = \ln(x)+c$$

$$\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\ln(x^2+y^2) + \cancel{\ln(x)} = \cancel{\ln(x)}+c$$

$$\boxed{\tan^{-1}\left(\frac{y}{x}\right) = c + \ln\left(\sqrt{x^2+y^2}\right).}$$

Reduction to Variable Seperable:-

* $y' = f(ax+by+c)$ — ①

Note that if $b=0$, ① is already in V.S form
Assume $b \neq 0$. Put $ax+by+c = z$.

$$\frac{dz}{dx} = a + b\frac{dy}{dx} = a + bf(x,y)$$

$$\frac{dz}{a+bf(x,y)} = dx$$

e.g:- $\quad y' = (x+y)^2$

$$z = x+y \Rightarrow \frac{dz}{dx} = 1 + y' = 1+z^2$$

$$\frac{dz}{1+z^2} = dx$$

$$\boxed{\tan^{-1}(z) = x + c}$$

Reduction to Homogenous Eqn:

$$\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1} \quad (c \neq 0 \text{ or } c_1 \neq 0) \quad -(1)$$

$$x = X+h \quad , \quad y = Y+k.$$

$$a(X+h)+b(Y+k)+c = aX+bY+\underbrace{(ah+bk+c)}_{=0}$$

$$a_1(X+h)+b_1(Y+k)+c_1 = a_1X+b_1Y+\underbrace{(a_1h+b_1k+c_1)}_{=0.}$$

Taking the constant terms to be zero, find h and k.

Now continue solving the DE.

$$ah+bk = -c$$

$$a_1h+b_1k = -c_1$$

$$\begin{bmatrix} a & b \\ a_1 & b_1 \end{bmatrix}\begin{bmatrix} h \\ k \end{bmatrix} = -\begin{bmatrix} c \\ c_1 \end{bmatrix}$$

Case-I $\quad \begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} \neq 0$, Unique solution exists

Then ① becomes $\dfrac{dy}{dx} = \dfrac{aX+bY}{a_1X+b_1Y}$

$$\frac{dY}{dX} = \frac{aX+bY}{a_1X+b_1Y} \quad \left[\begin{array}{l} \because y = Y+k \\ dy = dY \end{array}\right]$$

After solving substitute back $Y = y-k$, $X = x-h$.

Case-II $\quad \begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} = 0$,

$$\exists\ \alpha \text{ s.t } a_1 = \alpha a,\ b_1 = \alpha b$$

$$① \Rightarrow \frac{dy}{dx} = \frac{ax+by+c}{\alpha(ax+by)+c_1}$$

Put $ax+by = t$

$$b\frac{dy}{dx} = \frac{dt}{dx} - a$$

$$\frac{dy}{dx} = \frac{t+c}{\alpha t+c_1}$$

$$\frac{1}{b}\left(\frac{dt}{dx} - a\right) = \frac{t+c}{\alpha t+c_1}$$

$$\frac{dt}{dx} = a + b\left(\frac{t+c}{\alpha t+c_1}\right)$$

Use variable seperable.

---

e.g. 1

$$\frac{dy}{dx} = \frac{3x-2y+1}{6x-4y+1}$$

Put $3x-2y = z$

$$\frac{dz}{dx} = 3-2\left(\frac{z+1}{2z+1}\right)$$

$$\frac{dz}{dx} = \frac{6z+3-2z-2}{2z+1} = \frac{4z+1}{2z+1}$$

$$\left(\frac{2z+1}{4z+1}\right)dz = dx$$

$$x+C = \int \frac{2z+1}{4z+1}\,dz$$

$$= \int \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{4z+1}\,dz$$

$$x + C = \frac{z}{2} + \frac{1}{2}\log(4z+1)$$

$$\boxed{\frac{x}{2} - y + \frac{1}{2}\log(12x - 8y + 1) = 0}$$

② $$\frac{dy}{dx} = \frac{y - x + 1}{y + x + 5}$$

$$x = X + h, \quad y = Y + k.$$

$$k - h + 1 = 0, \quad k + h + 5 = 0$$

$$k = -3, \quad h = -2.$$

$$\therefore x = X - 2, \quad y = Y - 3$$

$$\frac{dY}{dX} = \frac{Y - X}{Y + X}$$

$$Y = VX$$

$$V + X\frac{dV}{dX} = \frac{V - 1}{V + 1}$$

$$X\frac{dV}{dX} = -(1 + V^2)$$

$$\frac{dx}{} \qquad \frac{}{v+1}$$

$$-\frac{(1+v)}{1+v^2}\,dv = \frac{dx}{x}$$

$$-\tan^{-1}(v) - \frac{1}{2}\log(1+v^2) = \log(x) + c$$

$$-\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\log\left(\frac{x^2+y^2}{x^2}\right) = \log(x) + c$$

$$\tan^{-1}\left(\frac{y+3}{x+2}\right) + \frac{1}{2}\log\left((x+2)^2+(y+3)^2\right) = c$$

---

* Let $\frac{dy}{dx} = F\left(\frac{ax+by+c}{a_1x+b_1y+c_1}\right)$

→ Origin Shift

→ Homogenous . . .

(Same as b4)

---

e.g.:-

$$\frac{dy}{dx} = \left(\frac{2x+y-1}{x-2}\right)^2.$$

$$x = X+h, \quad y = Y+k.$$

$$2h+k-1=0, \quad h-2=0$$

$$h=2, \quad k=-3$$

$$X = x-2, \quad Y = y+3$$

$$\frac{dY}{dX} = \left(\frac{2X+Y}{X}\right)^2 = \left(2+\frac{Y}{X}\right)^2$$

$$V = \frac{Y}{X}$$

$$V + X\frac{dV}{dX} = (2+V)^2 = V^2+4V+4$$

$$X\frac{dV}{dX} = V^2+3V+4.$$

$$\frac{dV}{V^2+3V+4} = \frac{dX}{X}$$

$$dV \qquad = \qquad dX$$

$$v^2+2\cdot\frac{3}{2}v+\frac{9}{4}+\frac{7}{4} \qquad \frac{\quad}{x}$$

$$\int \frac{dv}{\left(v+\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2} = \int \frac{dx}{x}$$

$$\frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{2v+3}{\sqrt{7}}\right) = \ln x + c$$

$$v = \frac{y+3}{x-2}, \quad x = x-2$$

$$\frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{3x+2y}{\sqrt{7}(x-2)}\right) = \ln(x-2)+c$$

Exact Equations:-

Let $f(x,y)=c$

$$D\,f(x,y)=0$$

$$\underbrace{\frac{\partial f}{\partial x}}_{M(x,y)} \cdot dx + \underbrace{\frac{\partial f}{\partial y}}_{N(x,y)} dy = 0$$

$$M(x,y)\,dx + N(x,y)\,dy = 0.$$

* For a given DE $M(x,y)\,dx + N(x,y)\,dy = 0$, if there exist a fn. $f : D \to \mathbb{R}$ s.t. $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$ then the DE is called an Exact Eqn. (for some $D \subseteq \mathbb{R}^2$)

e.g.: $$y\,dx + (x^2y - x)\,dy = 0$$

$$M(x,y) = y \quad ; \quad N(x,y) = (x^2y - x)$$

We know,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}$$

$$\frac{\partial M}{\partial y} = 1 \quad ; \quad \frac{\partial N}{\partial x} = 2xy - 1$$

As $2xy-1 \neq 1$.

$\therefore$ The given DE is not exact

* If $M(x,y)\,dx + N(x,y)\,dy$ is exact, it becomes $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$.

$\Rightarrow Df = 0$ [Total derivative]

$\Rightarrow f(x,y) = c$ is a one-parameter family of solution.

Theorem: Suppose M and N are continous and have continuous first order partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in a domain $D \subseteq \mathbb{R}^2$:

(1) If $M\,dx + N\,dy = 0$ is exact, then

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

(2) If D is a convex domain in $\mathbb{R}^2$ and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

then $Mdx + Ndy = 0$

Proof :- Let $F = (M(x,y), N(x,y))$

$$\nabla \times F = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}$$

$\nabla \times F = 0$ iff $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$.

$\nabla \times F = 0$ if $F = \nabla \cdot f$ for some f vectorfield.

$$\Rightarrow \underline{\underline{M = \frac{\partial f}{\partial x}}}, \quad N = \frac{\partial f}{\partial y}$$

Proof -2 :-

(1) Suppose $M(x,y)dx + N(x,y)dy = 0$ is exact

$\exists f$ st $\dfrac{\partial f}{\partial x} = M$, $\dfrac{\partial f}{\partial y} = N$

$$\Rightarrow \left.\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial f}{\partial y \partial x} \\ \frac{\partial N}{\partial x} &= \frac{\partial f}{\partial x \partial y} \end{aligned}\right\} \text{They equal}$$

(2) Suppose $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. (To find a f st $\dfrac{\partial f}{\partial x} = M$, $\dfrac{\partial f}{\partial y} = N$)

Consider $f(x,y) = \int M dx + g(y)$

$f(x,y) = \int M\,dx + g(y)$

<u>Aim</u>: To choose g s.t $\frac{\partial f}{\partial y} = N$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\int M\,dx\right) + g'(y)$$

$$N = \frac{\partial}{\partial y}\left(\int M\,dx\right) + g'(y)$$

$$g'(y) = N - \frac{\partial}{\partial y}\left(\int M\,dx\right)$$

Observe,

$$\frac{\partial}{\partial x}\left(N - \frac{\partial}{\partial y}\left(\int M\,dx\right)\right) = 0$$

$$\frac{\partial N}{\partial x} - \frac{\partial}{\partial x\,\partial y}\left(\int M\,dx\right) = 0$$

$$\frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(\int M\,dx\right)\right] = 0$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

$$\frac{\partial M}{} \quad \frac{\partial N}{}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Required function f is

$$\boxed{f(x,y) = \int M\,dx + \int\left(N - \frac{\partial}{\partial y}\int M\,dx\right)dy}$$

e.g:-

① Check $e^y dx + (xe^y + 2y)dy = 0$ for exactment and solve it if it is exact.

Sol:-

$M = e^y \quad , \quad N = xe^y + 2y$

$$\frac{\partial M}{\partial y} = e^y \quad , \quad \frac{\partial N}{\partial x} = e^y$$

$\therefore$ Eq^n is exact.

$$\frac{\partial f}{\partial x} = e^y$$

$$f = xe^y + g(y)$$

$$\frac{\partial f}{\partial y} = xe^y + g'(y) = xe^y + 2y$$

$$g'(y) = 2y$$

$$g'(y) = 2y$$

$$g = \int 2y\,dy = y^2 + C$$

$$\boxed{f(x,y) = xe^y + y^2 = C}$$

② Solve $(y\cos x + 2xe^y)\,dx + (\sin x + x^2e^y - 1)\,dy$

$$\frac{\partial M}{\partial y} = \cos x + 2xe^y \quad , \quad \frac{\partial N}{\partial x} = (\cos x + 2xe^y)$$

$$\frac{\partial f}{\partial x} = y\cos x + 2xe^y$$

$$f = y\sin x + x^2e^y + g(y)$$

$$\frac{\partial f}{\partial y} = \sin x + x^2e^y + g'(y) = N$$

$$g'(y) = -1$$

$$g(y) = -y$$

$$f(x,y) = y\sin x + x^2e^y - y$$

$f(x,y)=c$ is the 1 per. fam. of curves

(3) $\left(\frac{-y}{x^2+y^2}\right)dx + \left(\frac{x}{x^2+y^2}dy\right) = 0$

Sol:- Not a convex domain. $\left(\mathbb{R}^2 - \{(0,0)\}\right)$

$$\frac{\partial M}{\partial y} = \left(\frac{(-1)(x^2+y^2)+y(2y)}{(x^2+y^2)^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x} = \left(\frac{1(x^2+y^2)-x(2x)}{(x^2+y^2)^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2}$$

NOT EXACT on D

Suppose $\exists f$ s.t $\frac{\partial f}{\partial x} = M$, $\frac{\partial f}{\partial y} = N$.

$$F(x,y) = (M(x,y), N(x,y)) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

$$\nabla \times F = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k} \qquad = \nabla f$$

Recall,

$\int \nabla f \, ds = 0$ on any simply closed curve.

c

Let c be a circle $\quad \gamma(t) = (\cos t, \sin t)$

$$\int_{\gamma} \nabla f \, ds = \int_{\gamma} F(\gamma(t))\, \gamma'(t)\, dt$$

$$= \int_0^{2\pi} F(\cos t, \sin t) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t \cdot \cos t)\, dt$$

$$= \int_0^{2\pi} dt = 2\pi \neq 0$$

$\Rightarrow$ F is not a gradient vectorfield.

Integrating factor:

* A function $\mu(x,y)$ is called an Integrating factor for $M dx + N dy = 0$ if $\mu(x,y) M dx + \mu(x,y) N dy = 0$ is exact.

Observe: If $M dx + N dy$ has a general solution, then IF exists

Suppose $f(x,y) = c$ is the general sol.

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

$$\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y} = \frac{-M}{N}$$

$$\frac{\partial f/\partial x}{M} = \frac{\partial f/\partial y}{N} = \underline{\underline{\mu(x,y)}}$$

$\Rightarrow$ $\mu M dx + \mu N dy = 0$ is exact.

How to determine IF?

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

$$\mu\frac{\partial M}{\partial y} + M\frac{\partial \mu}{\partial y} = \mu\frac{\partial N}{\partial x} + N\frac{\partial \mu}{\partial x}$$

$$\boxed{\frac{1}{\mu}\left[N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y}\right] = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}.}$$

If μ is an IF, it satisfies the above DE.

Case-I: μ is a fn of x alone.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{1}{\mu}\frac{\partial \mu}{\partial x} = \frac{1}{\mu}\frac{d\mu}{dx} = g(x)$$

Only a fn of x., say $g(x)$

$\int g(x)\,dx$

$\Rightarrow \quad \mu(x) = e^{\int g(x)\,dx}$

Case-II: μ is a fn of y alone.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} = \frac{1}{\mu}\frac{\partial \mu}{\partial y} = \frac{1}{\mu}\frac{d\mu}{dy} = h(y)$$

Only a fn of y, say $h(y)$

$$\Rightarrow \quad \mu(y) = e^{\int h(y)\,dy}$$

$\Rightarrow$ Solve $(8xy - 9y^2)\,dx + (2x^2 - 6xy)\,dy = 0$

Soln

$$\frac{\partial M}{\partial y} = 8x - 18y \; ; \; \frac{\partial N}{\partial x} = 4x - 6y$$

$$g(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2\cancel{(2x - 6y)}}{x\cancel{(2x - 6y)}} = \frac{2}{x}$$

$$IF = e^{\int \frac{2}{x}dx} = e^{2\ln x} = \underline{\underline{x^2}}$$

$$\Rightarrow x^2(8xy - 9y^2)\,dx + x^2(2x^2 - 6xy)\,dy = 0$$

$$(8x^3y - 9x^2y^2)\,dx + (2x^4 - 6x^3y)\,dy = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} = 8x^3y - 9x^2y^2$$

$$f = 2x^4y - 3x^3y^2 + g(y)$$

$$\frac{\partial f}{\partial y} = 2x^4 - 6x^3y + g'(y) = \mu N$$

$$g'(y) = 0$$

$$g(y) = c$$

$$f(x,y) = 2x^4y - 3x^3y^2$$

The family of solutions is $\underline{\underline{f(x,y) = C}}$

$\Rightarrow$ $y^3dx + (xy^2-1)dy = 0$

$$\frac{\partial M}{\partial y} = 3y^2 \quad , \quad \frac{\partial N}{\partial x} = y^2$$

$$h(y) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} = -\frac{2}{y}$$

$$\mu(y) = e^{\int -\frac{2}{y}dy} = \frac{1}{y^2}$$

$$ydx + \left(x - \frac{1}{y^2}\right)dy = 0$$

$$\mu M = \frac{\partial f}{\partial x} = y$$

$$f = xy + t(y)$$

$$\frac{\partial f}{\partial y} = x + t'(y) = \mu N$$

$$t'(y) = \frac{-1}{y^2}$$

$$t = \frac{1}{y}$$

$$f(x,y) = xy + \frac{1}{y}.$$

$\therefore$ The one parameter family of sol is $\boxed{xy + \frac{1}{y} = c.}$

$\Rightarrow$ $(y + xy^2)dx - xdy = 0$

$$\frac{\partial M}{\partial y} = 1 + 2xy, \quad \frac{\partial N}{\partial x} = -1$$

$$h(y) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} = \frac{2(1+xy)}{-y(1+xy)} = \frac{-2}{y}.$$

$$\mu(y) = \frac{1}{y^2}$$

$$\frac{1}{y^2}dx + x\,dx - \frac{x}{y^2}dy = 0$$

$$\frac{\partial f}{\partial x} = \frac{1}{y} + x$$

$$f = \frac{x}{y} + \frac{x^2}{2} + t(y)$$

$$\frac{\partial f}{\partial y} = \frac{-x}{y^2} + t'(y) = \mu N$$

$$t'(y) = 0$$

$$t(y) = 0$$

$$f(x,y) = \frac{x}{y} + \frac{x^2}{2} = c.$$

First order linear differential equations:

$$\frac{dy}{dx} + p(x)y = q(x)$$

Is this exact?

$$\left.\begin{array}{l} \left[p(x)y - q(x)\right]dx + dy = 0 \\ \dfrac{\partial M}{\partial y} = p(x) \quad ; \quad \dfrac{\partial N}{\partial x} = 0 \end{array}\right\} \text{Not exact.}$$

$$\Rightarrow \frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial y}}{N} = p(x) \text{ is a fn of } x \text{ alone.}$$

$$\boxed{I.F = e^{\int p(x)dx}}$$

$$\Rightarrow \mu(x)\left[p(x)y - q(x)\right]dx + \mu(x)dy = 0 \text{ is exact}$$

$$\frac{\partial f}{\partial y} = \mu(x) \Rightarrow f(x,y) = \mu(x)y + t(x)$$

$$\frac{\partial f}{\partial x} = \mu'(x)y + t'(x)$$

$$\mu(x)\left[p(x)y - q(x)\right] = \mu'(x)y + t'(x)$$

$$\mu'(x) = \mu(x)p(x) \quad ; \quad t'(x) = -q(x)\mu(x)$$

$$\Rightarrow t(x) = -\int q(x)\mu(x)\,dx.$$

$$\boxed{f(x,y) = \mu(x)y - \int q(x)\mu(x)\,dx = c}$$

$$\Rightarrow \quad y = e^{-\int p(x)dx}\left[\int q(x)e^{\int p(x)dx}\,dx + c\right]$$

$$\Rightarrow \quad y' = x + y, \quad y(0) = 0$$

$$p(x) = -1, \; q(x) = x$$

$$\frac{dy}{dx} + (-1)y = x$$

$$y = e^{\int dx}\left[\int x\, e^{\int -dx}\,dx\right]$$

$$= e^{x}\left[\int x e^{-x}\,dx + c\right]$$

$$= e^{x}\left[x e^{-x}(-1) + \int e^{-x} + c\right]$$

$$= e^{x}\left[-x e^{-x} - e^{-x} + c\right]$$

$$= -x - 1 + c e^{x}$$

$$\boxed{\int uv = u\int v - \int\left(u'\int v\right)}$$

$$y = c e^{x} - x - 1$$

$$y(0) = c - 1 = 0 \Rightarrow c = 1$$

$$\boxed{y(x) = e^{x} - x - 1}$$

Bernoulli's Differential Equations:

$$y^{-n}\left[\frac{dy}{dx} + p(x)y = q(x)y^{n}\right]$$

$$y^{-n}\frac{dy}{dx} + p(x)y^{1-n} = q(x)$$

Let $z = y^{1-n}$

$$\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

In general

$$f'(y)\frac{dy}{dx} + f(y)p(x) = q(x)$$

$$y^{-n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dz}{dx}$$

$f(y) = z$

$$\Rightarrow \frac{1}{1-n}\cdot\frac{dz}{dx} + p(x)z = q(x)$$

$$\Rightarrow \frac{dz}{dx} + \underbrace{\left[(1-n)p(x)\right]}_{p(x)}z = \underbrace{(1-n)q(x)}_{q(x)}$$

$p_1(x) = p(x)$

$q_1(x) = q(x)$

Now a lin. ODE

This is now a linear diff. eqn.

$$\Rightarrow \quad y' - y/x = y^3$$

$$y^{-3}\frac{dy}{dx} + \left(\frac{-1}{x}\right)\left(\frac{1}{y^2}\right) = 1$$

$$z = \frac{1}{y^2}$$

$$\frac{dz}{dx} = \frac{-2}{y^3}\frac{dy}{dx}$$

$$\frac{1}{-2}\frac{dz}{dx} + \left(\frac{-1}{x}\right)z = 1$$

$$\frac{dz}{dx} + \left(\frac{2}{x}\right) z = -2.$$

$$p(x) = \frac{2}{x} \quad ; \quad q(x) = -2$$

$$y = e^{-\int \frac{2}{x} dx} \left[\int -2\, e^{\int \frac{2}{x} dx}\, dx + c \right]$$

$$= \frac{1}{x^2} \left[\int -2x^2\, dx + c \right]$$

$$= \frac{-1}{x^2} \cdot \frac{2}{3} x^3 + \frac{c}{x^2} = -\frac{2x}{3} + \frac{c}{x^2}$$

$$z = -\frac{2x}{3} + \frac{c}{x^2}$$

$$\frac{1}{y^2} = -\frac{2x}{3} + \frac{c}{x^2}$$

$$\frac{x^2}{y^2} + \frac{2x^3}{3} = c.$$

$$\Rightarrow \cos y \frac{dy}{dx} + \frac{1}{x} \sin y = 1$$

I am enjoying :)

$$z = \sin y \qquad \frac{dz}{dx} = \cos y \frac{dy}{dx}$$

$$\frac{dz}{dx} + \left(\frac{1}{x}\right) z = 1$$

$$z = e^{-\int \frac{1}{x} dx}\left[\int e^{\int \frac{1}{x} dx} dx + c\right]$$

$$z = \frac{1}{x}\left[\frac{x^2}{2} + c\right]$$

$$\sin y = \frac{x}{2} + \frac{c}{x}$$

Orthogonal trajectories:

Let Γ and F be 2 family of curves if each curve in each family is orthogonal to every curve in the other family, then we can say F is orthogonal to Γ.

Let $\Gamma : f(x,y,c) = 0$

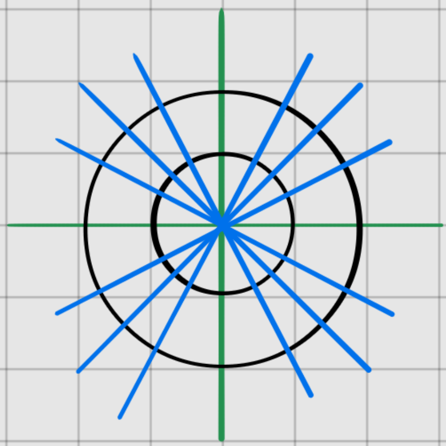
Let $\frac{dy}{dx} = g(x,y)$ — (1)

Now look at the DE, $\frac{dy}{dx} = \frac{-1}{g}$.

The sol to this will give F, the orthogonal trajectory to Γ

e.g: $\Gamma: x^2+y^2=c^2$

$$\Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-x}{y}.$$



Consider $\frac{dy}{dx} = \frac{y}{x}$.

$\boxed{y = mx}$ Any line through the origin.

$\Rightarrow$ Find O.T for $y^2 = cx^3$

$$y^2 = cx^3$$

$$2yy' = 3cx^2$$

$$y' = \frac{3x^2}{2y} \cdot \frac{y^2}{x^3} = \frac{3}{2} \cdot \frac{y}{x}.$$

Now,

$$\Rightarrow \frac{dy}{dx} = -\frac{2x}{3y}$$

$$\frac{3y^2}{2} = -x^2 + k.$$

$$\boxed{\frac{x^2}{3} + \frac{y^2}{2} = k.}$$