CS2020A Discrete Mathematics

Tutorial 10 | 27/Oct/2025

You can use the following theorem without proof for questions 1 and 2 in this tutorial.

Theorem (Hall's Marriage Theorem, 1935). A bipartite graph G with parts L and R has an L-perfect matching if and only if the following condition (known as Hall's Condition) is satisfied.

$$\forall S \subseteq L, \ |N(S)| \geq |S|.$$

Prove or give a counterexample to the following.

1. **Theorem 1.** Every k-regular bipartite graph with $k \geq 1$ has a perfect matching.

   Defn. A graph is *k-regular* if all its vertices have degree exactly k.

2. **Theorem 2.** Every L-heavy bipartite graph with at least one edge has an L-perfect matching.

   Defin. A bipartite grpah with parts L and R is called L-heavy if the maximum degree in R is at most the minimum degree in L.

3. **Theorem 3.** If M is a non-maximum matching in G then G has an M-augmenting path.

   Defn. A path P in a graph G with a matching M is called *M-alternating* if the edges of P alternate between M and $E(G) \setminus M$. Further, P is called M-augmenting if it is M-alternating and starts and ends on distinct vertices which are not saturated by M.

   Note. If P is an M-augmenting path, then their symmetric difference $P \triangle M$ is a bigger matching than M.

4. **Theorem 4.** Reachability relation in any DAG is a partial order.

   Defn. A binary relation on a set is called a *partial order* if it is reflexive, anti-symmetric and transitive.

   Defn. A vertex v is *reachable* from a vertex u in a directed graph G if their is a path starting at u and ending at v.

5. **Theorem 5.** Chromatic number of a DAG G is at most $l + 1$, where l is the length of a longest path in G.

   Defn. A proper coloring of a DAG is a proper coloring of the underlying undirected graph.