CS2020A Discrete Mathematics

Tutorial 6 Submission

Submitted By

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Problem 1. Convert the following statements in first order predicate logic to English and identify the famous theorem or conjecture they refer to. Domain of discourse is $\mathbb{N}^+$, the set of positive natural numbers, P denotes the unary predicate "is prime", $+$ denotes the binary function addition (in infix notation) and $>$ denotes the binary predicate "is greater than" (also in infix notation).

a) $\forall x \forall y \neg(x^2 = 2y^2)$

b) $\forall x \exists y((y > x) \wedge P(y))$

c) $\forall x \exists y((y > x) \wedge (y \leq 2x) \wedge P(y))$

d) $\forall x \exists y((y > x) \wedge P(y) \wedge P(y + 2))$

e) $\forall(x > 1) \exists y \exists z((y + z = 2x) \wedge P(y) \wedge P(z))$
$\equiv \forall x[(x > 1) \rightarrow \exists y \exists z((y + z = 2x) \wedge P(y) \wedge P(z))]$

Solution.

a) For all x and y belonging to positive numbers, x^2 cannot be equal to $2y^2$.
Theorem: $\sqrt{2}$ is irrational

b) For all x, there exists some prime number y, such that y is greater than x.
Theorem: There exist infinite prime numbers.

c) For all x, there exists some prime number y, such that y is greater than x, but less than $2x$.
Theorem: Bertrand's Postulate

d) For all x, there exists some prime number y, such that $(y + 2)$ is a prime and y is greater than x.
Theorem: Twin Prime Conjecture

e) For all x greater than 1, there exists some prime numbers y and z, such that $2x$ can be written as a sum of y and z.
Theorem: Goldbach's Conjecture

Problem 2. This is a "loose" definition of an algebraic structure called field.

A field is a set S with two operations (binary functions) '$+$' (called addition) and '$\times$' (called multiplication) which satisfies the following properties.

a) Addition is commutative and associative.

b) There exists an additive identity in S.

c) Every member of S has an additive inverse.

d) Multiplication is commutative and associative.

e) There exists a multiplicative identity in S which is different from the additive identity.

f) Every member of S, which is not the additive identity, has a multiplicative inverse.

g) Multiplication distributes over addition.

Write a first order formula for each of the seven properties above.

Solution.

Here, $\mathbb{D} = S$

a) $\forall x \forall y \forall z[(x + y = y + x) \wedge (x + (y + z) = (x + y) + z)]$

b) $\exists y \forall x(x + y = y + x = x)$

c) $\exists O \forall x \exists y((x + O = O + x = x) \wedge (x + y = y + x = O)$

d) $\forall x \forall y \forall z[(xy = yx) \wedge (x(yz) = (xy)z)]$

e) $\forall x \exists y((xy = yx = x) \wedge \neg(x + y = y + x = x))$

f) $\forall x \exists y \exists z(\neg(x + y = y + x = x) \rightarrow (yz = zy = 1))$ X ? $\forall x\,(x \neq \text{additive identity}) \rightarrow \exists y(xy = 1)$

g) $\forall x \forall y \forall z(x(y + z) = xy + xz))$

Problem 3. Write a first order formula for the following predicates over natural numbers. You can use the predicates defined earlier in the later ones.

a) x divides y. (Ans: $D(x,y) = \exists k(y = kx)$)

b) x is prime. (Ans: $P(x) = \forall y(D(y,x) \rightarrow (y = 1) \vee (y = x))$).

c) x is even.

d) x is a perfect square.

e) x has exactly three distinct prime factors.

f) x and y are relatively prime.

g) x and y have the same set of prime factors.

Solution.

a) $D(x,y) = \forall \mathrm{x}\, \forall \mathrm{y}\, \exists \mathrm{k}\ (y = kx)$

b) $P(x) = \forall \mathrm{x}\, \forall \mathrm{y}\ (D(x,y) \rightarrow ((y = 1) \vee (y = x)))$

c) $E(x) = \forall \mathrm{x}\ (D(2,x))$

d) $S(x) = \forall \mathrm{x}\, \exists \mathrm{y}\ (x = y^2)$

e) $T(x) = \exists \mathrm{p}\, \exists \mathrm{q}\, \exists \mathrm{r}\, \forall \mathrm{s}\ [(P(p) \wedge P(q) \wedge P(r) \wedge P(s)) \wedge (D(p,x) \wedge D(q,x) \wedge D(r,x)) \wedge \neg(s = p) \wedge \neg(s = q) \wedge \neg(s = r)] \rightarrow (\neg D(s,x))$

f) $R(x,y) = \forall \mathrm{x}\, \forall \mathrm{y}\, \exists \mathrm{k}\ [(P(x) \wedge P(y)) \vee \underline{(D(k,y) \wedge D(k,x) \rightarrow (k = 1))}]$

only this is sufficient.

g) $F(x) = \forall \mathrm{p}\, \forall \mathrm{x}\, \forall \mathrm{y}\ [P(p)\ \cancel{\wedge}\ (D(p,x) \leftrightarrow D(p,y))]$

$\rightarrow$

Problem 4. Convert the following statements into predicate logic formulae.

a) If a prime number divides the product of two integers, then it divides at least one of them.

b) Fermat's Last Theorem: No three positive integers a, b, c satisfy the equation $a^n + b^n = c^n$ for any integer $n > 2$.

c) Every non-empty subset of natural numbers has a least element.

d) Principle of Mathematical Induction

Solution.

a) $\forall x\, \forall y\, \forall p\, \big(P(p) \wedge D(p, xy) \rightarrow (D(p, x) \vee D(p, y))\big)$

b) $\forall a\, \forall b\, \forall c\, \forall n\, \big((n > 2) \rightarrow \neg(a^n + b^n = c^n)\big)$

c) $\forall S\, \big(S \subseteq \mathbb{N} \wedge S \neq \varnothing \rightarrow \exists m \in S\, \forall x \in S\; m \leq x\big)$

d) $\forall P\, \big(P(0) \wedge \forall n \in \mathbb{N}\, (P(n) \rightarrow P(n+1)) \rightarrow \forall n \in \mathbb{N}\; P(n)\big)$

Problem 5. Let the domain of discourse $D = \{1, \ldots, 10\}$. Let P be a 3-ary predicate defined on D. P can be represented as a binary 3D-Matrix M (a tensor), where $M(i,j,k) = 1$ if $P(i,j,k) =$ True and $M(i,j,k) = 0$ if $P(i,j,k) =$ False. The statement "M has an all-ones plane perpendicular to its first axis" is the first order sentence $\exists x \forall y \forall z\, P(x,y,z)$ (where x, y, z are domain variables). Write down similar first order sentences for the following statements.

a) M has an all-zeros plane perpendicular to its second axis

b) M has an all-ones line perpendicular to two of its three axes

c) M has a zero in each plane perpendicular to its third axis

d) M has at least two ones

e) M has at least two ones in each plane perpendicular to its third axis

f) Each plane perpendicular to the first axis in M represents a symmetric binary relation.

Solution.

a) $\exists y\ \forall x\ \forall z\ \neg P(x,y,z)$

b) $\exists x\ \exists y\ \forall z\ P(x,y,z) \quad \vee \quad \exists x\ \exists z\ \forall y\ P(x,y,z) \quad \vee \quad \exists y\ \exists z\ \forall x\ P(x,y,z)$

c) $\forall z\ \exists x\ \exists y\ \neg P(x,y,z)$

d) $\exists x_1 \exists y_1 \exists z_1 \exists x_2 \exists y_2 \exists z_2 \left(P(x_1,y_1,z_1) \wedge P(x_2,y_2,z_2) \wedge \neg(x_1 = x_2 \wedge y_1 = y_2 \wedge z_1 = z_2) \right)$

e) $\forall z\ \exists x_1 \exists y_1 \exists x_2 \exists y_2 \left(P(x_1,y_1,z) \wedge P(x_2,y_2,z) \wedge \neg(x_1 = x_2 \wedge y_1 = y_2) \right)$

f) $\forall x\ \forall y\ \forall z\ \left(P(x,y,z) \leftrightarrow P(x,z,y) \right)$