Instructions:

1. Cell phones are not allowed within the exam hall.

2. Please state all the results which you use and justify your answers.

3. Write all the steps as the steps carry partial marks.

1. Solve the following periodic eigenvalue problem

$$y'' + \lambda y = 0$$

   subject to

$$y(-\pi) = y(\pi), \quad y'(-\pi) = y'(\pi).$$

   [5]

2. Let $p \in \mathbb{R}$. Consider the Legendre's equation

$$(1 - x^2)y'' - 2xy' + p(p+1)y = 0 \quad \text{for } |x| < 1.$$

   (a) Determine the values of p for which the Legendre's equation has a polynomial solution. Justify. [6]

   (b) For a fixed value of p as in part (a), prove that any two polynomial solutions of the Legendre's equation with parameter p are linearly dependent.

3. Determine the differentiable function $u(x, y)$ for $x > 0$ and $y > 0$ satisfying the PDE
$$u_x + (2xy^2)u_y = 0$$
   and
$$u(x, 1/x) = (x^2 + x)^2.$$

   [5]

4. Answer all parts of this question in order.   [4+3+3+4 = 14]

   (a) Prove that any second order linear equation

$$y'' + P(x)y' + Q(x)y = 0$$

   can be written in normal form after a change of variable:

$$u'' + Q(x)u = 0.$$

1

(b) Write down the Bessel's equation for any given value of the parameter p and derive its normal form.

(c) Let $p > 1/2$ and y_p denote a non-trivial solution to Bessel's equation with parameter p. Determine with justification whether y_p can have more than one zero in some interval of length π.

(d) Show that any nontrivial solution of the Bessel's equation for $p > 0$ has infinitely many zeros on the positive x-axis.

5. Derive the D'Alembert solution to the wave equation

$$u_{tt} = c^2 u_{xx} \quad \text{for} \ -\infty < x < \infty$$

subject to initial conditions

$$u(x, 0) = 0, \qquad u_t(x, 0) = \cos x.$$

[6]

6. Find one Frobenius series solution around 0 for

$$(x^2 - x)y'' - xy' + y = 0.$$

Check whether a second linearly independent Frobenius series solution exists. Determine a second linearly independent Frobenius series solution, if it exists. Justify your answer. [7]

7. Solve the following Boundary Value Problem associated with the heat equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad u(x, 0) = 20, \quad u(0, t) = 0, \quad u(L, t) = 0.$$

[7]