First-order PDE:-

$\ast$ $au_x + bu_y = 0$

$\nabla u(x,y)\cdot(a,b) = 0 \Rightarrow$ Directional derivative of u in the direction (a,b).

$$D_v f(x) = \lim_{t\to 0} \frac{f(x+tv)-f(x)}{t}.$$

Let a line along the direction be $bx - ay = k \rightarrow$ characteristic curve.

Then $u(x,y) = f(bx-ay) = f(k) \longrightarrow$ gen. solution

e.g:- $4u_x - 3u_y = 0$ s.t $u(0,y) = y^3$

$$u(x,y) = f(bx-ay)$$

$$u(x,y) = f(-3x-4y)$$

$$u(0,y) = f(-4y)$$

$$y^3 = f(-4y)$$

$$f(y) = -\frac{y^3}{64}$$

$$\Rightarrow u(x,y) = \frac{(3x+4y)^3}{64} \quad ///$$

$\ast$ $u_x + yu_y = 0$

$\nabla u(x,y)\cdot(1,y) = 0$

Along C, u is constant.

$$\frac{dy}{dx} = \frac{y}{1}$$

$$\ln y = x + c$$

$C: y = ke^x \Rightarrow$ characteristic curve

Along C, u is constant $C: y = ke^x$ → characteristic curve

$\Rightarrow u(x, ce^x)$ is a constant.

$$u(x,y) = u(0, ce^0) = u(0,c) = f(ye^{-x})$$

e.g.: $u_x + yu_y = 0$; $u(0,y) = y^3$

$$u(x,y) = f(ye^{-x})$$
$$u(0,y) = f(y) = y^3$$

$$\Rightarrow u(x,y) = y^3 e^{-3x} \quad //$$

* $a(x,y)u_x + b(x,y)u_y = 0$

$$\nabla u(x,y) \cdot (a(x,y), b(x,y)) = 0$$

$\Rightarrow$ Dir. derivative is const. along $a(x,y)$ and $b(x,y)$

The characteristic curve will have tangent vector $a(x,y)$ and $b(x,y)$. It is obtained by

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$$

e.g.:- $u_x + 2xy^2u_y = 0$

$$\Rightarrow \frac{dy}{dx} = 2xy^2$$

$$\frac{-1}{y} = x^2 + c$$

$$C: y = \frac{1}{c - x^2}$$

$$u(x,y) = u\left(x, \frac{1}{c-x^2}\right) = u\left(0, \frac{1}{c}\right) = f(c) = f\left(\frac{yx^2+1}{y}\right)$$

Second order PDE's :-

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

$\Rightarrow$ They are classified into 3 types

(i) Elliptic — $B^2 - 4AC < 0$
(ii) Parabolic — $B^2 - 4AC = 0$
(iii) Hyperbolic — $B^2 - 4AC > 0$

e.g: ① Laplace Equation $\Rightarrow u_{xx} + u_{yy} = 0 \Rightarrow$ Elliptic

② Wave Equation $\Rightarrow u_{xx} - u_{tt} = 0 \Rightarrow$ Hyperbolic

③ Heat Equation $\Rightarrow u_t = c^2 u_{xx} \Rightarrow$ Parabolic

Wave Equation:-

(a) Consider $u_{tt} = c^2 u_{xx}$; $-\infty < x < \infty$

$$u_{tt} - c^2 u_{xx} = 0$$

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right) = 0$$

let $v = u_t + cu_x$

$\Rightarrow \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) v = 0$

$$\rightarrow \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) v = 0$$

$$v_t - c v_x = 0$$

Solving $\left.\begin{array}{l} v_t - c v_x = 0 \\ u_t + c u_x = v. \end{array}\right\}$ same as solving the given eqⁿ.

First we solve for v.

$$\nabla v(t,x) \cdot (1,-c) = 0$$

$$v(t,x) = h(ct+x)$$, where h is any function.

Now we solve for u

$$u_t + c u_x = h(ct+x).$$

General sol. to the corresponding Homogenous eqⁿ is obtained as

$$u_g(x,t) = g(x-ct)$$

Verify that $u_p(x,t) = f(x+ct)$ where $f'(s) = \frac{h(s)}{2c}$ is a particular solution

$$u_t + c u_x = cf' + cf' = 2cf' = \underline{\underline{h}}$$

$$\boxed{u(x,y) = f(x+ct) + g(x-ct).}$$

Now this is an IVP with the initial equations

$$u(x,0) = \phi(x) \, , \quad u_t(x,0) = \varphi(x)$$

The gen. sol is $u(x,t) = f(x+ct) + g(x-ct)$

At $t=0$, $u(x,0) = \boxed{\phi(x) = f(x) + g(x)}$

$u_t(x,0) = \boxed{\varphi(x) = cf'(x) - cg'(x)}$

$$\phi(x) = f + g \qquad \frac{1}{c}\varphi(x) = f' - g'$$

$$\phi'(x) = f' + g' \qquad \frac{1}{c}\varphi(x) = f' - g'$$

$$f' = \frac{1}{2}\left[\phi'(x) + \frac{1}{c}\varphi(x)\right]$$

$$g' = \frac{1}{2}\left[\phi'(x) - \frac{1}{c}\varphi(x)\right]$$

$$f(s) = \frac{1}{2}\int_0^s \phi'(x) + \frac{1}{2c}\int_0^s \varphi(x) \qquad g(s) = \frac{1}{2}\int_0^s \phi'(s) - \frac{1}{2c}\int_0^s \varphi(x)$$

$$f(s) = \frac{1}{2}\phi(s) + \frac{1}{2c}\int_0^s \varphi + A \qquad g(s) = \frac{1}{2}\phi(s) - \frac{1}{2c}\int_0^s \varphi + B$$

But $f(s) + g(s) = \phi(s) \Rightarrow A + B = 0$

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \varphi$$

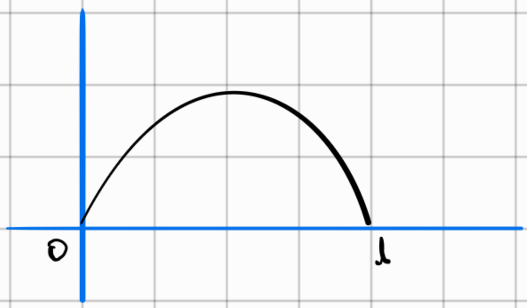
d'Alemberts solution to the IVP

(b) $u_{tt} = c^2 u_{xx}$

Initial cond. $u(x,0) = \phi(x), \quad u_t(x,0) = \varphi(x)$

Bound. cond. $u(0,0) = u(l,0) = 0, \ \forall t$

Dirichlet Condition.



$u(x,t)$ - position of string at pos. x

at time t

[Method of seperation of variables]

Consider $u(x,t) = F(x)G(t)$

$$u_{tt} = c^2 u_{xx}$$

$$F(x)G''(t) = c^2 F''(x)G(t)$$

$$\frac{G''(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = -\lambda \quad \text{(say)}$$

$$\left.\begin{aligned} F'' + \lambda F &= 0 \\ G'' + c^2\lambda G &= 0 \end{aligned}\right\} \text{Solve}$$

Consider $F'' + \lambda F = 0$

$u(0,t) = 0 \Rightarrow F(0)G(t) = 0 \Rightarrow F(0) = 0$

Similarly, $F(l) = 0$

This is an eigen value problem

(i) $\lambda = 0$. then $F'' = 0 \Rightarrow F(x) = Ax + B$

$F(0) = 0 \Rightarrow B = 0$

$F(l) = 0 \Rightarrow A = 0$

$\Rightarrow F = 0$

(ii) $\lambda < 0$

Take $\lambda = -\mu^2$, then $F'' - \mu^2 F = 0$

$F = Ae^{\mu x} + Be^{-\mu x}$

$\Rightarrow A = 0 = B.$

(iii) $\lambda > 0$

Take $\lambda = \beta^2$, the $F'' + \beta^2 F = 0$

$F = A\cos\beta x + B\sin\beta x.$

$F(0) = 0 \Rightarrow A = 0$

$F(l) = 0 \Rightarrow \sin\beta l = 0$

$\Rightarrow \beta l = n\pi$

$\Rightarrow \beta = \frac{n\pi}{l}$

$$\boxed{\lambda_n = \left(\frac{n\pi}{l}\right)^2 \qquad F_n(x) = D\sin\left(\frac{n\pi x}{l}\right)}$$

Consider $G'' + \lambda c^2 G = 0$

We know $\lambda = \beta^2 \Rightarrow G'' + \beta^2 c^2 G = 0$

$$G(x) = A\cos(\beta c t) + B\sin(\beta c t)$$

$$\boxed{G_n(x) = A\cos\left(\frac{n\pi c t}{l}\right) + B\sin\left(\frac{n\pi c t}{l}\right)}$$

For each n, $\boxed{u_n(x,t) = \sin\left(\frac{n\pi x}{l}\right)\left[A_n\cos\left(\frac{n\pi c t}{l}\right) + B_n\sin\left(\frac{n\pi c t}{l}\right)\right]}$

is a solution to the wave equation satisfying the boundary conditions

* Now we consider

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right) \right] \sin\left(\frac{n\pi x}{l}\right)$$

From initial conditions,

$u(x,0) = \phi(x)$

$$\Rightarrow \phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)$$

$u_t(x,0) = \varphi(x)$

$$\Rightarrow \varphi(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{l} \cdot B_n \sin\left(\frac{n\pi x}{l}\right)$$

These are fourier sine series.

$$\Rightarrow A_n = \frac{2}{l} \int_0^l \phi(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$\Rightarrow \frac{n\pi c}{l} B_n = \frac{2}{l} \int_0^l \phi(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$\Big\}$ ⊛

$\Rightarrow$ If $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right)$$

$\Rightarrow$ If $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$

$$a_0 = \frac{2}{l} \int_0^l f(x)$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right).$$

So, the solution to the wave eqn is

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right) \right] \sin\left(\frac{n\pi x}{l}\right)$$

satifying conditions ⊛.

Heat Equation:-

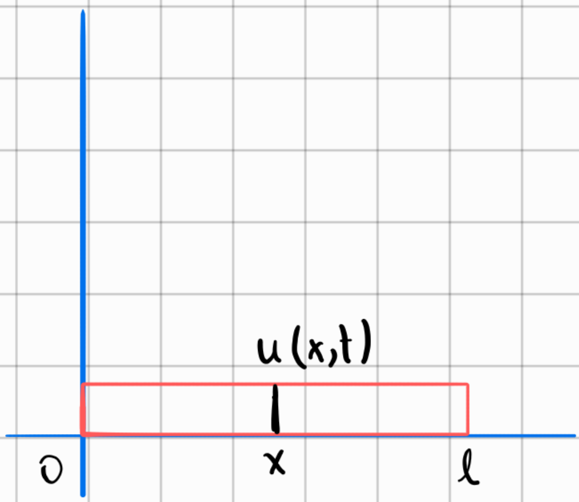
Heat Equation:

$u(x,t)$ – temperature of the iron rod at time t and position x.

$$u_t = c^2 u_{xx} \qquad 0 < x < l.$$

with boundary condition

$$u(0,t) = u(l,t) = 0$$



and initial condition

$$u(x,0) = \phi(x).$$

[Method of seperation of variables]

let $u(x,t) = F(x) \cdot G(t)$

$$u_t = c^2 u_{xx}$$

$$FG' = c^2 F'' G$$

$$\frac{G'}{c^2 G} = \frac{F''}{F} = -\lambda \quad \text{(say)}$$

From boundary conditions, $F(0) = F(l) = 0$

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2$$

$$F_n = D_n \sin\left(\frac{n\pi x}{l}\right)$$

$$G' + \lambda c^2 G = 0$$

$$G_n = c_n e^{-\left(\frac{\cdots}{l}\right)t}$$

$$\boxed{u_n(x,t) = C_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n\pi c}{l}\right)^2 t}}$$

Now consider,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n\pi c}{l}\right)^2 t}$$

From initial condition,

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\phi(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right)$$

$$c_n = \frac{2}{l}\int_0^l \phi(x) \sin\left(\frac{n\pi x}{l}\right) \quad - \ (*)$$

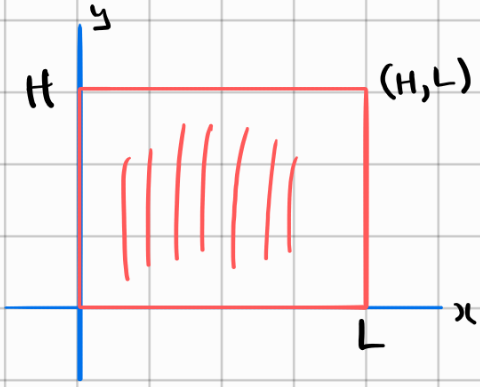
The solution to the heat eqn is given by

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n\pi c}{l}\right)^2 t}$$

satisfying condition $(*)$.

Laplace Equation:

$$u_{xx}+u_{yy}=0$$



$$u(0,y)=0 \quad \forall y$$
$$u(L,y)=0 \quad \forall y$$
$$u(x,0)=f(x) \quad \forall x$$
$$u(x,H)=0 \quad \forall x.$$

Consider $u(x,y)=F(x)G(y)$

$$F''G+G''F=0$$

$$-\frac{F''}{F}=\frac{G''}{G}=\lambda$$

$$F''+\lambda F=0$$
$$G''-\lambda G=0$$

From boundary conditions, $F(0)=F(L)=0$

$$\Rightarrow \lambda_n=\left(\frac{n\pi}{\ell}\right)^2$$

$$F_n=D_n\sin\left(\frac{n\pi x}{\ell}\right)$$

Then

$$G''-\left(\frac{n\pi}{\ell}\right)^2G=0$$

$$G_n=A_ne^{\left(\frac{n\pi}{\ell}\right)y}+B_ne^{-\left(\frac{n\pi}{\ell}\right)y}$$

From boundary conditions

$$G(H)=0 \Rightarrow \quad B_n = -A_n e^{\left(\frac{2n\pi}{\ell}\right)H}$$

Then $$G_n = A_n\left[e^{\left(\frac{n\pi}{\ell}\right)y} - e^{\left(\frac{n\pi}{\ell}\right)(2H-y)}\right]$$

$$= 2A_n e^{\left(\frac{n\pi}{\ell}\right)H}\left[\frac{e^{-\left(\frac{n\pi}{\ell}\right)(H-y)} - e^{\left(\frac{n\pi}{\ell}\right)(H-y)}}{2}\right]$$

$$G(y) = -2A_n e^{\left(\frac{2n\pi}{\ell}\right)H} \sinh\left(\frac{n\pi}{\ell}(H-y)\right)$$

$$G(y) = a_n \sinh\left(\frac{n\pi}{\ell}(H-y)\right) \quad ; \text{ where } a_n = -2A_n e^{\left(\frac{2n\pi H}{\ell}\right)}$$

$$u_n(x,y) = a_n \sinh\left(\frac{n\pi}{\ell}(H-y)\right)\sin\left(\frac{n\pi x}{L}\right).$$

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi}{\ell}(H-y)\right)\sin\left(\frac{n\pi x}{L}\right).$$

From boundary conditions,

$$u(x,0) = f(x)$$

$$f(x) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi}{\ell}H\right)\sin\left(\frac{n\pi x}{L}\right).$$

$$a_n \sinh\left(\frac{n\pi H}{\ell}\right) = \frac{1}{2L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad - \quad (*)$$

So, the solution to Laplace Eqn is

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi}{L}(H-y)\right) \sin\left(\frac{n\pi x}{L}\right).$$

satisfying (*)