1. Solve the following optimization problem:

$$\begin{aligned} \text{minimize} \quad & \|x - x_0\| \\ \text{subject to} \quad & \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 4 \end{bmatrix} x = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \end{aligned}$$

   where

$$x_0 = \begin{bmatrix} 0 & -1 & 4 \end{bmatrix}^\top.$$

2. Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, with $m < n$ and rank $A = m$. Show that

$$x^* = A^\top (AA^\top)^{-1} b$$

   is the only vector in $\operatorname{row}(A)$ satisfying $Ax^* = b$.

3. Question 12.24 in textbook[1] :
   "derive a recursive least-squares algorithm where we remove (instead of add) a data point."

4. Let

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 4 & 1 & 0 \\ -1 & -2 & 1 & 3 \end{bmatrix}.$$

   Verify that rank $A = 2$. Write a full rank factorization of A, i.e., find matrices $B \in \mathbb{R}^{3\times 2}$ and $C \in \mathbb{R}^{2\times 4}$ such that

$$A = BC,$$

   where both B and C have full rank.

5. Given the matrix

$$B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix},$$

   compute the generalized inverse $B^\dagger$ of B.

6. Verify that the *least squares matrix* and the *minimum norm matrix* satisfy the requirements of being generalized inverses in the respective cases.

7. Let $A \in \mathbb{R}^{m\times n}$. Show that if a generalized inverse $A^\dagger$ of A exists, then it is unique.

8. Show that the Penrose definition of the generalized inverse is equivalent to the definition given in class in terms of U and V.

[1] Chong, E. K. P., & Zak, S. H. (2013). An Introduction to Optimization (4th ed.). Wiley.