# CS2020A Discrete Mathematics

TUTORIAL 12 SUBMISSION

**Submitted By**


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**Problem 1.** What is the best strategy to win a **5-door Monty Hall problem**? What is the winning probability of your strategy?

**5-Door Monty Hall Problem.** A car is equally likely behind one of the five doors on stage. A contestant is asked to choose one door, following which the host opens one of the other doors behind which there is no car. The contestant now has an option to make a second (final) guess.

**Solution.** The best strategy is to always switch the door.

Assume the contestant picked a door x. The initial probability of winning is 1/5. The probability of the winning door being in any of the other doors is 4/5. Now the host opens a door from the other ones which is not a winning or choosen door. The probability of the winning door to be in the non-choosen doors is still the same. But now the probability is divided between the three unopened doors which is $\frac{4}{5} \cdot \frac{1}{3} = \frac{4}{15}$ for each door.

$$\frac{4}{15} > \frac{1}{5}$$

$\implies$ the unchosen and unopened doors have higher probability of being the winning door.

$\therefore$ The best strategy is to switch door with a winning probability of 4/15.

**Problem 2.** A fair coin is tossed $n$ times. What is the probability that you get a sequence in which **no head follows a tail**?

**Solution.** In this experiment, we are tossing a coin $n$ times and will consider all those coin tosses, where no head follows a tail.

So the total number of possibilities on tossing $n$ coins is $2^n$.
Now, the sample space of interest for us is as follows:

$$\mathcal{E} = \{T^n, HT^{n-1}, H^2T^{n-2} \dots H^{n-1}T, H^n\}$$

The number of elements in this sample space is $n + 1$.

So the probability of no head following a tail is $\frac{n+1}{2^n}$.

Another approach to solve this problem is as follows:
We know the sample space of interest is

$$\mathcal{E} = \{T^n, HT^{n-1}, H^2T^{n-2} \dots H^{n-1}T, H^n\}$$

So now considering the probability of each member of the sample space:

$$P(T^n) = \frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2} = \frac{1}{2^n}$$
$$P(HT^{n-1}) = \frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2} = \frac{1}{2^n}$$
$$\vdots$$
$$P(H^n) = \frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2} = \frac{1}{2^n}$$

So the probability of the sample space of interest :

$$P(\mathcal{E}) = \frac{1}{2^n} + \frac{1}{2^n} + \cdots + \frac{1}{2^n} = \frac{n+1}{2^n}$$

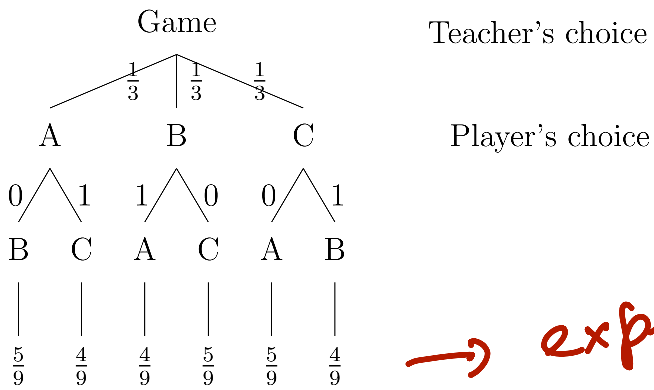
**Problem 3.** You have three six-sided fair dice $A, B, C$ with the following numbers on their faces:

$$A = (2, 6, 7, 2, 6, 7), \qquad B = (1, 5, 9, 1, 5, 9), \qquad C = (3, 4, 8, 3, 4, 8).$$

You can choose any one of the dice and then I will choose another. Then we throw the dice (fairly) and whoever gets the larger number will win.

Which die will you choose and what is your winning probability (assuming that the second player is your probability teacher)?

**Solution.** The following tree gives the probabilities of the player winning for all the possible outcomes of the sample space:




→ explanation required.

Assuming our opponent is our Probability teacher and that he chooses after the player chooses, he would always choose the dice which has a higher probability of him winning(or lower probability of the player winning).

As it can be seen that irrespective of which die the player chooses, the teacher will always choose the die which has a winning probability of $\frac{4}{9}$ for the player.

Thus, if $E$ = Event of the player winning
Then,

$$P(E) = \frac{1}{3} \times 0 \times \frac{5}{9} + \frac{1}{3} \times 1 \times \frac{4}{9} + \frac{1}{3} \times 1 \times \frac{4}{9} + \frac{1}{3} \times 0 \times \frac{5}{9} + \frac{1}{3} \times 0 \times \frac{5}{9} + \frac{1}{3} \times 1 \times \frac{4}{9}$$
$$P(E) = \frac{4}{9}$$

**Problem 4. Inclusion–Exclusion Principle.** Prove that for any sequence of $n$ events $A_1, \ldots, A_n$ in some probability space $(\Omega, P)$,

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I|=k}} P(A_I),$$

where the inner sum runs over all subsets $I$ of indices $\{1, \ldots, n\}$ with $|I| = k$, and $A_I := \bigcap_{i \in I} A_i$.

**Proof Strategy 1.** Let $x$ be any member of $\Omega$. Depending on how many sets among $A_1, \ldots, A_n$ contain $x$, calculate the contribution of $P(x)$ on the RHS of the above equation.

**Proof Strategy 2.** Try induction on $n$.

**Solution.** We prove the Inclusion–Exclusion Principle by induction on the number of events $n$.

<u>*Base Case:*</u> For $n = 2$, Let $A_1 = A$ and $A_2 = B$. We can decompose the union $A \cup B$ into three disjoint parts:

$$A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B).$$

Since probability is additive on disjoint events,

$$P(A \cup B) = P(A \setminus B) + P(B \setminus A) + P(A \cap B).$$

Also,

$$P(A) = P(A \setminus B) + P(A \cap B), \qquad P(B) = P(B \setminus A) + P(A \cap B).$$

Adding these gives:

$$P(A) + P(B) = P(A \cup B) + P(A \cap B),$$

which rearranges to

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Thus, the formula holds for $n = 2$.

<u>*Induction Hypothesis:*</u> Assume the formula holds for $n = t$, i.e.,

$$P\left(\bigcup_{i=1}^{t} A_i\right) = \sum_{k=1}^{t} (-1)^{k-1} \sum_{\substack{I \subseteq \{1,\ldots,t\} \\ |I|=k}} P(A_I).$$

<u>*Induction Step:*</u> For $n = t + 1$, consider

$$P\left(\bigcup_{i=1}^{t+1} A_i\right) = P\left(\left(\bigcup_{i=1}^{t} A_i\right) \cup A_{t+1}\right)$$

Using the two-set formula $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$, we get:

$$P\left(\bigcup_{i=1}^{t+1} A_i\right) = P\left(\bigcup_{i=1}^{t} A_i\right) + P(A_{t+1}) - P\left(A_{t+1} \cap \bigcup_{i=1}^{t} A_i\right)$$

Now, the intersection term can be expanded as:

$$P\left(A_{t+1} \cap \bigcup_{i=1}^{t} A_i\right) = P\left(\bigcup_{i=1}^{t} (A_{t+1} \cap A_i)\right).$$

By the induction hypothesis (applied to the $t$ sets $A_1 \cap A_{t+1}, \ldots, A_t \cap A_{t+1}$), we have:

$$P\left(\bigcup_{i=1}^{t} (A_{t+1} \cap A_i)\right) = \sum_{k=1}^{t} (-1)^{k-1} \sum_{\substack{I \subseteq \{1,\ldots,t\} \\ |I|=k}} P(A_{t+1} \cap A_I).$$

Substitute this and the induction hypothesis for $t$ sets $A_1, \ldots, A_t$ into our earlier equation:

$$\begin{aligned} P\left(\bigcup_{i=1}^{t+1} A_i\right) &= \sum_{k=1}^{t} (-1)^{k-1} \sum_{\substack{I \subseteq \{1,\ldots,t\} \\ |I|=k}} P(A_I) + P(A_{t+1}) \\ &\quad - \sum_{k=1}^{t} (-1)^{k-1} \sum_{\substack{I \subseteq \{1,\ldots,t\} \\ |I|=k}} P(A_{t+1} \cap A_I). \end{aligned}$$

Adding $A_{t+1}$ to each subset $I$ forms all subsets $J \subseteq \{1, \ldots, t+1\}$ of size $k+1$ that contain $t+1$. Hence,

$$P\left(\bigcup_{i=1}^{t+1} A_i\right) = \sum_{k=1}^{t+1} (-1)^{k-1} \sum_{\substack{J \subseteq \{1,\ldots,t+1\} \\ |J|=k}} P(A_J).$$

This matches the required formula for $n = t + 1$.

Hence, by induction, the Inclusion–Exclusion Principle holds for all $n \in \mathbb{N}$.