CS2020A Discrete Mathematics

Tutorial 05 | 08/Sep/2025

1. Convert the following statements in first order predicate logic to English and identify the famous theorem or conjecture they refer to. Domain of discourse is $\mathbb{N}^+$, the set of positive natural numbers, P denotes the unary predicate `is prime`, $+$ denoted the binary function `addition` (in infix notation) and $>$ denotes the binary predicate `is greater than` (also in infix notation).

   (a) $\forall x \forall y \neg(x^2 = 2y^2)$
   (b) $\forall x \exists y((y > x) \wedge P(y))$
   (c) $\forall x \exists y((y > x) \wedge (y \leq 2x) \wedge P(y))$
   (d) $\forall x \exists y((y > x) \wedge P(y) \wedge P(y + 2))$
   (e) $\forall (x > 1) \exists y \exists z((y + z = 2x) \wedge P(y) \wedge P(z))$
   $\equiv \forall x[(x > 1) \rightarrow \exists y \exists z((y + z = 2x) \wedge P(y) \wedge P(z))]$

2. This is a "loose" definition of an algebraic structure called field.

   A *field* is a set S with two operations (binary functions) '$+$' (called addition) and '$\times$' (called multiplication) which satisfies the following properties.

   (a) Addition is commutative and associative.
   (b) There exists an additive identity in S.
   (c) Every member of S has an additive inverse.
   (d) Multiplication is commutative and associative.
   (e) There exists a multiplicative identity in S which is different from the additive identity.
   (f) Every member of S, which is not the additive identity, has a multiplicative inverse.
   (g) Multiplication distributes over addition.

   Write a first order formula for each of the seven properties above.

3. Write a first order formula for the following predicates over natural numbers. You can use the predicates defined earlier in the later ones.

   (a) x divides y. (*Ans:* $D(x, y) = \exists k(y = kx)$)
   (b) x is prime. (*Ans:* $P(x) = \forall y(D(y, x) \rightarrow (y = 1) \vee (y = x))$.
   (c) x is even.
   (d) x is a perfect square.
   (e) x has exactly three distinct prime factors.
   (f) x and y are relatively prime.
   (g) x and y have the same set of prime factors.

4. Convert the following statements into predicate logic formulae.

   (a) If a prime number divides the product of two integers, then it divides at least one of them.

(b) Fermat's Last Theorem: No three positive integers a, b, and c satisfy the equation $a^n + b^n = c^n$ for any integer value of n greater than 2.
(c) Every non-empty subset of natural numbers has a least element.
(d) Principle of Mathematical Induction

5. Let the domain of discourse $D = \{1, \ldots, 10\}$. Let P be a 3-ary predicate defined on D. P can be represented as a binary 3D-Matrix M (a tensor), where $M(i, j, k) = 1$ if $P(i, j, k) = True$ and $M(i, j, k) = 0$ if $P(i, j, k) = False$. The statement *"M has an all-ones plane perpendicular to its first axis"* is the first order sentence $\exists x \forall y \forall z P(x, y, z)$ (where x, y, z are domain variables). Write down similar first order sentences for the following statements.

   (a) *M has an all-zeros plane perpendicular to its second axis*
   (b) *M has an all-ones line perpendicular to two of its three axes*
   (c) *M has a zero in each plane perpendicular to its third axis*
   (d) *M has at least two ones*
   (e) *M has at least two ones in each plane perpendicular to its third axis*
   (f) *Each plane perpendicular to the first axis in M represents a symmetric binary relation.*