CS2020A Discrete Mathematics

Tutorial 10 Submission

Submitted By

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Theorem (Hall's Marriage Theorem, 1935) A bipartite graph G with parts L and R has an L-perfect matching if and only if the following condition (known as Hall's Condition) is satisfied:

$$\forall S \subseteq L,\ |N(S)| \geq |S|.$$

You may use this theorem without proof for Problems 1 and 2.

Theorem 1. Every k-regular bipartite graph with $k \geq 1$ has a perfect matching.

Defn: A graph is k-regular if all its vertices have degree exactly k.

Prove or give a counterexample.

Solution. Let there be a bipartite graph G with the two parts, namely L and R, having n and m vertices respectively.
Then,

$$n.k = m.k \text{ (i.e.the number of edges are equal)}$$
$$n = m$$

prove it

Since $n \leq m \implies$ L-perfect matching and $m \leq n \implies$ R-perfect matching.
So the matching is L-perfect and R-perfect (from Hall's Theorem)
$\therefore$ The matching is a perfect matching.

Theorem 2. Every L-heavy bipartite graph with at least one edge has an L-perfect matching.

Defn: A bipartite graph with parts L and R is called L-*heavy* if the maximum degree in R is at most the minimum degree in L.

Prove or give a counterexample.

Solution. Let S be a subset of L having m vertices and let R have n vertices.
If L and R is L-heavy, then S and R is L-heavy as maximum degree in R is still smaller then min degree of S.
Let min degree of S be p and max degree of R be q.
There is at least one edge $\implies q \geq 1$
No.of edges coming out of S is same as the no.of edges going into R.
No.of edges coming out of $S \geq m * p$
No.of edges going into $R \leq n * q$

$\implies m * p \leq n * q$
$\implies m/n \leq q/p$

By definition of L-heavy bipartite graph, $q \leq p \implies q/p \leq 1$

$\implies m/n \leq 1$
$\implies m \leq n$
m is the number of vertices in the subset of L and n is the number of neighbors of subset of L.

By Hall's Marriage Theorem, this is a L-perfect matching.

Theorem 3. If M is a non-maximum matching in G, then G has an M-augmenting path.

Defn: A path P in a graph G with a matching M is called M-*alternating* if the edges of P alternate between M and $E(G) \setminus M$. Further, P is called M-*augmenting* if it is M-alternating and starts and ends on distinct vertices which are not saturated by M.

Defn: If P is an M-augmenting path, then the symmetric difference $P \triangle M$ is a bigger matching than M.

Prove or give a counterexample.

Solution. Assume M is a non-maximum matching in G. Then there exists another matching M' in G with $|M'| > |M|$.

Consider the graph H whose edge set is the **union**

$$H = M \cup M'$$

Every vertex of H has **degree at most** 2 (because in both M and M' each vertex is incident to at most one matching edge). Hence each connected component of H is either an **even cycle** whose edges alternate between M and M', or **a path whose edges alternate** between M and M'.

Now we count the number of edges from M and M' in each component. In any even alternating cycle the number of edges from M equals the number from M'.

In an alternating path, if a path has its two endpoints not saturated by M (i.e. the path starts and ends with edges of M'), then that path contains one more edge of M' than of M.

Since $|M'| > |M|$, adding up the number of edges over all components we conclude that there must be at least one component of H which is an alternating path that contains strictly more edges from M' than from M.

Such a component is a path whose edges alternate between M' and M, and whose two endpoints are vertices not saturated by M. This is, by definition, an M-augmenting path.

Thus G contains an M-augmenting path whenever M is non-maximum.

Theorem 4. Reachability relation in any DAG is a partial order.

Defn: A binary relation on a set is called a *partial order* if it is reflexive, antisymmetric, and transitive.

Defn: A vertex v is *reachable* from a vertex u in a directed graph G if there is a path starting at u and ending at v.

Prove or give a counterexample.

Solution. To show that the reachability relation in a DAG is a partial order, we need to show that any path from a vertex u to a vertex v of the graph is reflexive, antisymmetric and transitive.

Reflexive: This means that any object is related to itself. In terms of reachability relation, every vertex is related to itself as a vertex can always reach itself via a path of length zero, meaning starting and immediately ending at that vertex without traversing any edge. Therefore, it follows the reflexive property.

Antisymmetric: This means that if any object a is related to b, and b is also related to a, then that means $a = b$. In terms of reachability relation, if there is a path from a vertex a to a vertex b, there cannot exist a path from b to a, since the graph is acyclic and directed. So if there also exist a path from b to a, it must be the same vertex, as this follows the reflexive property. So, vertex a = vertex b and hence it follows the antisymmetric property.

Transitive: This means that if an object a is related to an object b, and b is related to c, then a is also related to c. In terms of reachability relation, if there is a path from vertex a to another vertex b and a path from b to another vertex c. Then by combining these two paths, we have a path from a to c, meaning c is also reachable from a. So it follows the transitive property.

Hence the reachability relation in any DAG is a partial order.

Theorem 5. The chromatic number of a DAG G is at most $l+1$, where l is the length of a longest path in G.

Defn: A proper coloring of a DAG is a proper coloring of its underlying undirected graph.

Prove or give a counterexample.

Solution. Given that l is the length of a longest path in G i.e, it has $l+1$ vertices.

In the case that all the vertices in this path are connected, we would need one unique colour for each of the vertices, thus giving us a total of $l+1$ different colours.

In case all the vertices are not connected, the number of unique colours required would be less than $l+1$.

In case there are other vertices outside this path, we can use one of the colours already used in the previous path as it is not connected to any of the vertices there. l being the length of a longest path ensures that all the other paths would have atmost $l+1$ unique colours.

Thus, the chromatic number of a DAG G is atmost $l+1$, where l is the length of a longest path in G

↳ Try to prove it by defining a colouring function & proving that it is a proper colouring.