CS2020A Discrete Mathematics

Problem Set 01 | Sep 2025

1. Prove
   Theorem 1. For every natural number n, $n^7 + n^5 - n^4 + n + 1$ is odd.

2. What are the weakest conditions you can put on integers $a_0, a_1, \ldots, a_k$ so that for every natural number n, $a_0 + a_1 n + a_2 n^2 + \cdots + a_k n^k$ is even?

3. Prove that $\sqrt{3} + \sqrt{2}$ is irrational.

   Hint. $(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2}) = 1$

4. Prove that, for every natural number n, $6^n - 1$ is a multiple of 5.
   Hint. Ordinary Induction

5. Prove that, for every natural number n, $n^3 - n$ is a multiple of 6.
   Hint. Ordinary Induction

6. Show that the maximum number of 3D regions into which the 3D space can be divided by n planes is $\binom{n}{3} + \binom{n}{2} + n + 1$.
   Hint. Ordinary Induction

7. Show that every natural number has a unique base-3 representation. *Hint.* Strong Induction

8. Prove that for any non-empty set of integers $\{a_1, a_2, \ldots, a_n\}$, the gcd of them can be expressed as $\sum_{i=1}^{n} \alpha_i a_i$ for some integers $\alpha_1, \ldots, \alpha_n$.
   Hint. Well Ordering Principle

9. Let n be a non-negative integer. Show that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using triominoes. A *triomino* is a piece which covers three squares of the checkerboard which form an 'L' shape.

10. There are three stacks of cards kept on a table. Each card has a proposition written on its top side. A card is called a *true card* if the proposition written on it is true. A card is called a *false card* if the proposition written on it is false. You know that one of the stacks contains only true cards, one of the stacks contains only false cards and one of them contains both true and false cards. We do not know which is which though.
    You can only see the propositions on the top card of each stack and they are the following.

    - **Stack 1:** Stack 2 contains true and false cards.
    - **Stack 2:** Stack 3 contains true and false cards.

- **Stack 3:** Stack 2 contains true and false cards.

Identify the true, the false and the mixed stacks. Justify your answer.

11. The logical connective $\oplus$ (exclusive or) is defined as follows: For propositions p and q the proposition $p \oplus q$ is true if and only if exactly among p and q is true (not both).
    a) Write the truth table for $\oplus$
    b) Write a logically equivalent formula for $p \oplus q$ using only
       - $\neg$, $\wedge$ and $\vee$
       - $\neg$, $\wedge$ and $\rightarrow$
       - $\neg$ and $\leftrightarrow$
    c) Is it associative? (Prove or disprove using truth tables)

12. These are some important logical equivalences. Convince yourselves that they are all valid using natural reasoning and then verify the same using truth tables.

1.	Commutative Laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \wedge p$
2.	Associative Laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3.	Distributive Laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4.	Identity Laws	$p \wedge T \equiv p$	$p \vee F \equiv p$
5.	Negation Laws	$p \wedge \neg p \equiv F$	$p \vee \neg p \equiv T$
6.	Double Negation	$\neg(\neg p) \equiv p$	
7.	Idempotent laws	$p \wedge p \equiv p$	$p \vee p \equiv p$
8.	Domination laws	$p \wedge F \equiv F$	$p \vee T \equiv T$
9.	De Morgan's laws	$\neg(p \wedge q) \equiv \neg p \vee \neg q$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$
10.	Absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11.	Contrapositive	$p \rightarrow q \equiv \neg q \rightarrow \neg p$	
12.	Condition-Disj	$p \rightarrow q \equiv \neg p \vee q$	$p \vee q \equiv \neg p \rightarrow q$
13.	Condition Merges	$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$	$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
		$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$	$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

13. Fill in the blanks with either "a tautology", "a contradiction", "satisfiable", or "unsatisfiable".
    a) If ϕ is a tautology, then $\neg\phi$ is _____
    b) If ϕ is a tautology, then ϕ is not _____
    c) If ϕ is a contradiction, then $\neg\phi$ is _____
    d) If ϕ is a contradiction, then ϕ is not _____
    e) If ϕ is satisfiable, then $\neg\phi$ is not _____
    f) If ϕ is satisfiable, then ϕ is not _____
    g) ϕ is a tautology if and only if $\neg\phi$ is _____
    h) ϕ is satisfiable if and only if ϕ is not a _____
    i) ϕ is satisfiable if and only if $\neg\phi$ is not a _____

14. Truth tables of three propositional formulae α, β and γ on propositional variables p, q, r, s are given below

p	q	r	s	α	β	γ
T	T	T	T	T	T	T
T	T	T	F	T	T	F
T	T	F	T	T	T	T
T	T	F	F	F	T	T
T	F	T	T	T	T	T
T	F	T	F	T	F	T
T	F	F	T	T	F	T
T	F	F	F	F	F	T
F	T	T	T	T	T	F
F	T	T	F	T	F	F
F	T	F	T	T	F	F
F	T	F	F	F	F	F
F	F	T	T	F	T	F
F	F	T	F	F	F	F
F	F	F	T	F	F	F
F	F	F	F	F	F	F

a) Write down the CNF and DNF for α, β and γ
b) Simplify each one of them to a formula in which no variable repeats

15. We know that if 4 pigeons occupy 3 holes, then at least one hole has more than one pigeon. The task is to capture this fact using the unsatisfiability of a propositional formula involving the following 12 propositional variables:

- For $i \in \{1, 2, 3, 4\}$ and $j \in \{1, 2, 3\}$, the propositional variable p_{ij} denotes the proposition "i-th pigeon is occupying j-th hole."

(a) Write a propostional formula ϕ which is true only if the 4 pigeons occupy the 3 holes such that no hole has more than one pigeon. (Hence pigeonhole principle will be the claim that ϕ is a contradiction)

(b) Convert ϕ to CNF

(c) If there are $n + 1$ pigeons and n holes, how many clauses will the obove type of CNF have?