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CS2020A Discrete Mathematics

Tutorial 11 Submission

Submitted By

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Definition 1. A set A is countable if there exists an injective function from A to $\mathbb{N}$.

Prove or give a counterexample to the following.

Theorem 1. The union of any two countable sets is countable.

Solution. As A and B are countable, let f and g be their respective injective funtions to $\mathbb{N}$, i.e;

$$f : A \to \mathbb{N}$$
$$g : B \to \mathbb{N}$$

Now we try to construct a function h from $A \cup B$ to $\mathbb{N}$ that is injective. Consider the function

$$h(x) = \begin{cases} 2^{f(x)+1}, & x \in A, \\ 3^{g(x)+1}, & x \in B. \end{cases}$$

From the Fundamental Theory of Arithmetic, every natural number has a unique prime factorisation, and powers of 2 and 3 are disjoint. Hence numbers of the form 2^k and 3^m are never equal. Also, within each case, h is injective because f and g are injective.

Thus h is an injective function from $A \cup B$ to $\mathbb{N}$.

Therefore, $A \cup B$ is countable.

Theorem 2. The Cartesian product of any two countable sets is countable.

Solution. Let A and B be two countable sets, and let f and g be their respective injective functions to $\mathbb{N}$, i.e.;

$$f : A \to \mathbb{N}$$
$$g : B \to \mathbb{N}$$

Now we try to construct a function h from $A \times B$ to $\mathbb{N}$ that is injective. Consider the function;

$$h((x, y)) = 2^{f(x)} \cdot 3^{g(y)}$$

From the Fundamental Theory of Arithmetic, every natural number has a unique prime factorisation and powers of 2 and 3 are disjoint. Hence numbers of the form $2^k.3^m$ are never equal.

Thus h is an injective function from $A \times B$ to $\mathbb{N}$

$\therefore A \times B$ is countable.

Theorem 3. The union of countably many countable sets is countable. Equivalently, if $A_0, A_1, A_2, \ldots$ is a sequence of countable sets, then $A = \bigcup_{i\in\mathbb{N}} A_i$ is also countable.

Solution. Let $\{A_i\}_{i\in\mathbb{N}}$ be a countable sequence of countable sets, with $A = \bigcup_{i\in\mathbb{N}} A_i$. We want to find an injective function $h : A \to \mathbb{N}$.

Since each A_i is countable, there exists an injective function $f_i : A_i \to \mathbb{N}$ for each $i \in \mathbb{N}$.

The sets A_i may overlap. We first construct a sequence of pairwise disjoint sets B_i such that their union is also A.
Let $B_i = A_i \setminus \left(\bigcup_{j=0}^{i-1} A_j\right)$.
Then $B_i \cap B_j = \emptyset$ for $i \neq j$, and $\bigcup_{i\in\mathbb{N}} B_i = A$.
Each B_i is a subset of the countable set A_i, so B_i is also countable.
Thus, there exists an injective function $g_i : B_i \to \mathbb{N}$ for each i.

Now we define our function $h : A \to \mathbb{N}$. Let p_i be the $(i+1)$-th prime number (so $p_0 = 2, p_1 = 3, p_2 = 5, \ldots$).
Since the sets B_i are disjoint, any $x \in A$ belongs to *exactly one* set B_k.
We define h as:

$$\text{If } x \in B_k, \text{ then } h(x) = p_k^{g_k(x)+1}$$

We must show h is injective. Assume $h(x) = h(y)$ for some $x, y \in A$. Let $x \in B_i$ and $y \in B_j$. Then $h(x) = p_i^{g_i(x)+1}$ and $h(y) = p_j^{g_j(y)+1}$. The equality $p_i^{g_i(x)+1} = p_j^{g_j(y)+1}$ holds.

By the Fundamental Theorem of Arithmetic (unique prime factorization), the prime bases and the exponents must be equal.

a) $p_i = p_j \implies i = j$. This means x and y must belong to the same set B_i.

b) $g_i(x) + 1 = g_j(y) + 1 \implies g_i(x) = g_i(y)$.

Since g_i is an injective function on B_i, $g_i(x) = g_i(y)$ implies $x = y$.

Therefore, $h(x) = h(y) \implies x = y$. We have constructed an injective function $h : A \to \mathbb{N}$, so the union A is countable by the given definition. ■

Theorem 4. The set of all binary strings of finite length is countable.

Solution. Let S be the set of all binary strings of finite length. Let B_n be the set of all binary strings of length n i.e, $B_n = \{0,1\}^n$

$$S = B_1 \cup B_2 \cup B_3 \ldots$$

$$\implies S = \bigcup_{n=0}^{\infty} B_n$$

For a fixed n, the number of possible binary strings is 2^n, i.e, $|B_n| = 2^n$ which is countable. Since S is the union of countable sets, S is countable.

Theorem 5. If there exists an injective function from a set A to a set B, then there exists a surjective function from B to A.

Solution. Let there be an injective function from A to B. So this means we can map every element of A to a unique element in B, say $f(a) = b$ where $a \in A$ and $b \in B$.

Now consider a function from B to A, such that the elements of B which had a pre-image according to the the function f are mapped to the respective elements in A, and the elements of B which did not have a pre-image are mapped to any element of A. So now, the range is same as the co-domain as all the element of A have at least one pre-image in B. Therefore, this new function from B to A is surjective.

Theorem 6. There is no surjective function from $\mathbb{R}$ to $\mathcal{P}(\mathbb{R})$.

Note. Do not assume Cantor's general theorem. This is obviously a special case of that theorem.

Solution. We prove by contradiction that there is no surjective function from $\mathbb{R}$ to $\mathcal{P}(\mathbb{R})$

Assume for the sake of contradiction, that f is a surjection from $\mathbb{R}$ to $\mathcal{P}(\mathbb{R})$. Define the "diagonal" subset of $\mathbb{R}$ by

$$S = \{x \in \mathbb{R} : x \notin f(x)\}$$

Clearly $S \subseteq \mathbb{R}$, hence $S \in \mathcal{P}(\mathbb{R})$.

Since f is assumed surjective, there exists $a \in \mathbb{R}$ such that $f(a) = S$.

Now we ask whether $a \in S$ or not:

$$a \in S \iff a \notin f(a) \iff a \notin S$$

Here, the statement $a \in S$ if and only if $a \notin S$ is a contradiction.

Therefore no surjection $f : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ can exist.