CS2020A Discrete Mathematics

Problem Set 03 | Nov 2025

Infinite Sets and Cardinality

1. Prove that if you remove a countable set from an uncountable set, the result is still uncountable. That is, if A is uncountable and $B \subset A$ is countable, then $A \setminus B$ is uncountable.

   Note. Hence irrationals are uncountable.

2. Show that if A and B are of the same size, then their power sets $P(A)$ and $P(B)$ are of the same size.

3. Let A is an infinite set and let A^* denote the set of all (finite length) strings over A. Is A^* countable?

4. Show that the set of infinite sequences of natural numbers is uncountable.

5. The *SpeedTest* problem is to decide which among two given python programs P_1 and P_2 runs faster on a given input string x. (If both the programs take the same time, or both enter an infinite loop, SpeedTest should return `Tie`.). Can there exist a program which can solve the SpeedTest problem? Prove your answer.

Probability

6. India and South-Africa are playing a 3-match series. They play until one team has won two games (the third game is played only if each team won India has a win probability 3/5, regardless of the outcomes of previous games.

   Answer the questions below using the four step method. You can use the same tree diagram for all three problems.

   (a) What is the probability that a total of 3 games are played?
   (b) What is the probability that the winner of the series loses the first game?

7. You have the same three six-sided fair dice A, B, C with the following numbers on their faces that you had in Tutorial 12.

$$A = (2, 6, 7, 2, 6, 7)$$
$$B = (1, 5, 9, 1, 5, 9)$$
$$C = (3, 4, 8, 3, 4, 8)$$

   You can choose any one of the dice and then I will choose another. Then we throw the chosen dice **twice** and who ever gets the larger sum (adding the values in both

throws) will win. Which dice will you choose and what is your winning probability (assuming that the second player is your probability teacher)? Compare it with the winning probabilities for a single throw.

8. Let A and B be independent events on the same probability space.

   (a) Are A and $\bar{B}$ independent? Prove your answer rigorously.
   (b) Are $\bar{A}$ and $\bar{B}$ independent? Prove your answer rigorously.

   Note. $\bar{A}$ is the complement of A.

9. Let A, B, C be three mutually independent events on the same probability space.

   (a) Are A and $B \cap C$ independent? Prove your answer rigorously.
   (b) Are A and $B \cup C$ independent? Prove your answer rigorously.

   Will your answer change if A and B where only pair-wise independent? Justify.

   Note. $\bar{A}$ is the complement of A.

10. Give an example of a probability space and a collection of events which are 3-wise independent but not mutually independent.

    Defn. A collection of events is 3-wise independent if every subcollection of three or fewer events is independent.

11. Gabbar Singh places two bullets in random chambers his six-bullet revolver. He gives the bullet chamber a random spin and says "Tera kya hoga Kaaliya?" as he holds the gun against your heart.

    (a) What is the probability that you will get shot if he pulls the trigger?
    (b) Suppose he pulls the trigger and you don't get shot. What is the probability that you will get shot if he pulls the trigger a second time?
    (c) Suppose you noticed that he placed the two shells next to each other in the cylinder. How does this change the answers to the previous two questions?