KONSTANTINOS PAGANOPOULOS 01769789

OPTIMISATION AND DECISION MODELS

ASSIGNMENT 2 22/11/2019

Introduction

In this assignment solutions for homework 3 and 4 of "Optimisation and Decision Models MSc Business Analytics 2019/20" Exercises 2 are provided. The solutions include hand written calculations, as well as using AMPL program for solving linear problems.

Question 3

We know that the dual problem of

$$\begin{aligned} &maximise\ c^T x \\ &subject\ to\ Ax \le b \\ &\qquad x \ge 0 \end{aligned} \quad \text{is} \quad \begin{aligned} &minimise\ b^T y \\ &subject\ to\ A^T y \ge c \\ &\qquad y \ge 0 \end{aligned}$$

Let's symbolise the above definition with (1).

We now take the dual problem and try to bring it into the form of the primal. As a result, we have the following:

$$\begin{aligned} &minimise\ b^T y \\ &subject\ to\ A^T y \ge c \\ &\qquad y \ge 0 \end{aligned} \quad \text{Then} \quad \begin{aligned} &maximise\ -b^T y \\ &subject\ to\ -A^T y \le -c \\ &\qquad y \ge 0 \end{aligned}$$

Now by using the definition (1) we have the following dual problem:

$$\begin{aligned} &minimise\ -c^T x \\ &subject\ to\ -Ax \le -b \\ &\qquad x \ge 0 \end{aligned} \quad = \quad \begin{aligned} &maximise\ c^T x \\ &subject\ to\ Ax \le b \\ &\qquad x \ge 0 \end{aligned}$$

Which is equivalent to the primal problem of (1).

Question 4

(a)

The 1-norm regression problem is the following:

$$Chance = \beta_0 + \beta_1 GRE + \beta_2 TOEFL + \beta_3 Univ + \beta_4 SOP + \beta_5 LOR + \beta_6 CGPA + \beta_7 Res$$

We also know that k = 7 = number of independent variables and n = 100 = number of records

As a result, we can rewrite the 1-norm regression problem as:

$$Chance_i = \beta_0 + \beta_1 GRE_{i1} + \dots + \beta_7 Res_{i7}$$

We would like to solve:

$$minimise \parallel Y - X\beta \parallel_1$$

$$where \quad Y = \begin{bmatrix} Chance_1 \\ Chance_2 \\ . \\ . \\ . \\ Chance_{100} \end{bmatrix}, \quad X = \begin{bmatrix} 1 & GRE_1 & TOEFL_1 & Univ_1 & SOP_1 & LOR_1 & CGPA_1 & Res_1 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 1 & GRE_{100} & TOEFL_{100} & Univ_{100} & SOP_{100} & LOR_{100} & CGPA_{100} & Res_{100} \end{bmatrix}, \quad \begin{bmatrix} \beta_0 \\ \beta_1 \\ . \\ . \\ . \\ \beta_7 \end{bmatrix}$$

Which is equal to $\begin{array}{c} minimise \parallel Y - X\beta \parallel_1 \\ s.t.\ \beta \,\epsilon\, R^8 \end{array}$

As a result, we solve the following:

$$\begin{aligned} minimise &\parallel Chance_1 - (\beta_0 + \beta_1 GRE_1 + \beta_2 TOEFL_1 + \dots + \beta_7 Res_1) \parallel + \\ &\quad . \\ &\quad . \\ &\quad . \\ &\parallel Chance_{100} - (\beta_0 + \beta_1 GRE_{100} + \beta_2 TOEFL_{100} + \dots + \beta_7 Res_{100}) \parallel \\ &s.t.\ \beta_0, \beta_1, \beta_2, \dots, \beta_7 \,\epsilon\, \mathbb{R} \end{aligned}$$

We now move the non-linearities to the constraints:

$$\begin{aligned} &minimise\ \theta_1 + \theta_2 + \dots + \theta_{100} \\ &\quad s.t.\ \theta_1 = \parallel Chance_1 - (\beta_0 + \beta_1 GRE_1 + \beta_2 TOEFL_1 + \dots + \beta_7 Res_1) \parallel \\ &\quad . \\ &\quad . \\ &\quad . \\ &\quad \theta_{100} = \parallel Chance_{100} - (\beta_0 + \beta_1 GRE_{100} + \beta_2 TOEFL_{100} + \dots + \beta_7 Res_{100}) \parallel \\ &\quad \beta_0, \beta_1, \beta_2, \dots \beta_7, \theta_1, \theta_2 \dots, \theta_{100} \,\epsilon\, \mathbb{R} \end{aligned}$$

We now relax the equalities to inequalities:

$$minimise\ \theta_1 + \theta_2 + \dots + \theta_{100}$$

$$s.t.\ \theta_1 \geq \|\ Chance_1 - (\beta_0 + \beta_1 GRE_1 + \beta_2 TOEFL_1 + \ldots + \beta_7 Res_1)\ \|$$

$$\vdots$$

$$\theta_{100} \geq\ \|\ Chance_{100} - (\beta_0 + \beta_1 GRE_{100} + \beta_2 TOEFL_{100} + \ldots + \beta_7 Res_{100})\ \|$$

$$\beta_0, \beta_1, \beta_2, \ldots \beta_7, \theta_1\ , \theta_2 \ldots , \theta_{100}\ \epsilon\ \mathbb{R}$$

We now rewrite the ‖ ‖ - terms:

It is a mathematical fact that if we take any $a\ \epsilon\ \mathbb{R}$ then $\|\ a\ \| = max\{a, -a\}$

In other words, $x \geq \|\ y\ \|$ is the same as writing $x \geq\ max\{y, -y\}$, which in turn holds exactly when $x \geq y\ \ and\ \ x \geq -y$.

Hence, we want to solve the following:

$$minimise\ \ \theta_1 + \theta_2 + \ldots + \theta_{100}$$

$$\begin{aligned} s.t.\ \theta_1 &\geq\ Chance_1 - (\beta_0 + \beta_1 GRE_1 + \beta_2 TOEFL_1 + \ldots + \beta_7 Res_1) \\ \theta_1 &\geq\ (\beta_0 + \beta_1 GRE_1 + \beta_2 TOEFL_1 + \ldots + \beta_7 Res_1) - Chance_1 \end{aligned}$$

$$\vdots$$

$$\begin{aligned} \theta_{100} &\geq\ Chance_{100} - (\beta_0 + \beta_1 GRE_{100} + \beta_2 TOEFL_{100} + \ldots + \beta_7 Res_{100}) \\ \theta_{100} &\geq\ (\beta_0 + \beta_1 GRE_{100} + \beta_2 TOEFL_{100} + \ldots + \beta_7 Res_{100}) - Chance_{100} \end{aligned}$$

$$\beta_0, \beta_1, \beta_2, \ldots \beta_7, \theta_1\ , \theta_2 \ldots , \theta_{100}\ \epsilon\ \mathbb{R}$$

(b)

We now construct an AMPL file that solves the 1-norm regression problem using “CPLEX” as a solver (“option solver ‘cplex’;”). The AMPL model could look like this:

```
set NUM ordered; # candidate number

param GRE {NUM}; # GRE Score
param TOEFL {NUM}; # TOEFL Score
param Univ {NUM}; # University Rating
param SOP {NUM}; # Statement of Purpose Strength
param LOR {NUM}; # Letter of Recommend. Strength
param CGPA {NUM}; # Undergraduate GPA
param Res {NUM}; # Research Experience
param Chance {NUM}; # Chance of Admission

# import the data set
data Graduate_Admissions.dat;
```

```
# define the rest variables
var x {NUM};
var b0;
var b1;
var b2;
var b3;
var b4;
var b5;
var b6;
var b7;

# define the objective function
minimize error: sum {i in NUM} x[i];

# define the constraints
subject to constr1 {i in NUM}: x[i] >= Chance[i] - (b0 + b1*GRE[i] + b2*TOEFL[i] +
b3*Univ[i] + b4*SOP[i] + b5*LOR[i] + b6*CGPA[i] + b7*Res[i]);
subject to constr2 {i in NUM}: x[i] >= (b0 + b1*GRE[i] + b2*TOEFL[i] + b3*Univ[i] +
b4*SOP[i] + b5*LOR[i] + b6*CGPA[i] + b7*Res[i]) - Chance[i];
```

The optimal value and the optimal coefficients are the following:

```
ampl: model Konstantinos_Paganopoulos_Assignment_2a.mod;
ampl: option solver cplex;
ampl: solve;
CPLEX 12.9.0.0: optimal solution; objective 5.840174093
187 dual simplex iterations (125 in phase I)
ampl: display b0, b1, b2, b3, b4, b5, b6, b7;
b0 = -1.33337
b1 = 0.00332792
b2 = 0.00359076
b3 = 0.0377193
b4 = 0.00475836
b5 = 0.0418593
b6 = 0.0349495
b7 = 0.0102293
```

Please find attached the solution in the AMPL file ("Konstantinos_Paganopoulos_Assignment_2a.mod") for more details.

(c)

In this sub question we will follow the same steps with Question 4a. However, the infinite norm regression problem is equal to:

$$minimise \ \| Y - X\beta \|_{\infty}$$

$$where \quad Y = \begin{bmatrix} Chance_1 \\ Chance_2 \\ . \\ . \\ . \\ Chance_{100} \end{bmatrix}, \quad X = \begin{bmatrix} 1 & GRE_1 & TOEFL_1 & Univ_1 & SOP_1 & LOR_1 & CGPA_1 & Res_1 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 1 & GRE_{100} & TOEFL_{100} & Univ_{100} & SOP_{100} & LOR_{100} & CGPA_{100} & Res_{100} \end{bmatrix}, \quad \begin{bmatrix} \beta_0 \\ \beta_1 \\ . \\ . \\ . \\ \beta_7 \end{bmatrix}$$

Which is equal to $\quad minimise \; \| \, Y - X\beta \, \|_\infty$
$\quad s.t. \; \beta \, \epsilon \, R^8$

As a result, we solve the following:

$$minimise \;\; max \, \{ \| \, Chance_1 - (\beta_0 + \beta_1 GRE_1 + \beta_2 TOEFL_1 + \beta_3 Univ_1 + \beta_4 SOP_1 + \beta_5 LOR_1 + \beta_6 CGPA_1 + \beta_7 Res_1) \, \| \}$$
$$\vdots$$
$$max \, \{ \| \, Chance_{100} - (\beta_0 + \beta_1 GRE_{100} + \beta_2 TOEFL_{100} + \beta_3 Univ_{100} + \beta_4 SOP_{100} + \beta_5 LOR_{100} + \beta_6 CGPA_{100} + \beta_7 Res_{100}) \, \| \}$$
$$s.t. \; \beta_0, \beta_1, \beta_2, \dots, \beta_7 \, \epsilon \, \mathbb{R}$$

We now move the non-linearities to the constraints:

$$minimise \;\; \theta$$
$$s.t. \;\; \theta = max \, \{ \| \, Chance_1 - (\beta_0 + \beta_1 GRE_1 + \beta_2 TOEFL_1 + \beta_3 Univ_1 + \beta_4 SOP_1 + \beta_5 LOR_1 + \beta_6 CGPA_1 + \beta_7 Res_1) \, \| \}$$
$$\vdots$$
$$\theta = max \, \{ \| \, Chance_{100} - (\beta_0 + \beta_1 GRE_{100} + \beta_2 TOEFL_{100} + \beta_3 Univ_{100} + \beta_4 SOP_{100} + \beta_5 LOR_{100} + \beta_6 CGPA_{100} + \beta_7 Res_{100}) \, \| \}$$
$$\beta_0, \beta_1, \dots, \beta_7, \theta \, \epsilon \, \mathbb{R}$$

We now relax the equalities to inequalities:

$$minimise \;\; \theta$$
$$s.t. \; \theta \geq max \, \{ \| \, Chance_1 - (\beta_0 + \beta_1 GRE_1 + \beta_2 TOEFL_1 + \beta_3 Univ_1 + \beta_4 SOP_1 + \beta_5 LOR_1 + \beta_6 CGPA_1 + \beta_7 Res_1) \, \| \}$$
$$\vdots$$
$$\theta \geq max \, \{ \| \, Chance_{100} - (\beta_0 + \beta_1 GRE_{100} + \beta_2 TOEFL_{100} + \beta_3 Univ_{100} + \beta_4 SOP_{100} + \beta_5 LOR_{100} + \beta_6 CGPA_{100} + \beta_7 Res_{100}) \, \| \}$$
$$\beta_0, \beta_1, \dots, \beta_7, \theta \, \epsilon \, \mathbb{R}$$

We now rewrite the $\| \;\; \|$ - terms:

$$minimise \;\; \theta$$
$$s.t. \; \theta \geq max \, \{ Chance_1 - (\beta_0 + \beta_1 GRE_1 + \beta_2 TOEFL_1 + \beta_3 Univ_1 + \beta_4 SOP_1 + \beta_5 LOR_1 + \beta_6 CGPA_1 + \beta_7 Res_1) \}$$
$$\theta \geq max \, \{ (\beta_0 + \beta_1 GRE_1 + \beta_2 TOEFL_1 + \beta_3 Univ_1 + \beta_4 SOP_1 + \beta_5 LOR_1 + \beta_6 CGPA_1 + \beta_7 Res_1) - Chance_1 \}$$
$$\vdots$$

$$\theta \geq max\,\{Chance_{100} - (\beta_0 + \beta_1 GRE_{100} + \beta_2 TOEFL_{100} + \beta_3 Univ_{100} + \beta_4 SOP_{100} + \beta_5 LOR_{100} + \beta_6 CGPA_{100} + \beta_7 Res_{100})\}$$
$$\theta \geq max\,\{(\beta_0 + \beta_1 GRE_{100} + \beta_2 TOEFL_{100} + \beta_3 Univ_{100} + \beta_4 SOP_{100} + \beta_5 LOR_{100} + \beta_6 CGPA_{100} + \beta_7 Res_{100}) - Chance_{100}\}$$
$$\beta_0, \beta_1, \dots, \beta_7, \theta \in \mathbb{R}$$

The interpretation is the same with that of Question 4 (a), however instead of using the ℓ_1 norm: $\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$ definition it uses the ℓ_∞ norm: $\|\mathbf{x}\|_\infty = \max_{1\leq i\leq n} |x_i|$.

In other words, we now find the maximum error term and try to minimise that, whereas in the Question 4 (a) we are minimising not the maximum error term, but the sum of the errors. For the above reasons, our model gives the correct solution.

If we try to go a bit beyond the scope of the assignment, we can say that this illustration is not perfect. That is, because we choose only one "θ", so our model might be a little biased and provide not a very good estimate.

(d)

We now construct an AMPL file that solves the 1-norm regression problem using "CPLEX" as a solver ("option solver 'cplex';"). The AMPL model could look like this:

```
set NUM ordered; # candidate number

param GRE {NUM}; # GRE Score
param TOEFL {NUM}; # TOEFL Score
param Univ {NUM}; # University Rating
param SOP {NUM}; # Statement of Purpose Strength
param LOR {NUM}; # Letter of Recommend. Strength
param CGPA {NUM}; # Undergraduate GPA
param Res {NUM}; # Research Experience
param Chance {NUM}; # Chance of Admission

# import the data set
data Graduate_Admissions.dat;

# define the rest variables
var x;
var b0;
var b1;
var b2;
var b3;
var b4;
var b5;
var b6;
var b7;

# define the objective function
minimize error: x;
```

```
# define the constraints
subject to constr1 {i in NUM}: x >= Chance[i] - (b0 + b1*GRE[i] + b2*TOEFL[i] +
b3*Univ[i] + b4*SOP[i] + b5*LOR[i] + b6*CGPA[i] + b7*Res[i]);
subject to constr2 {i in NUM}: x >= (b0 + b1*GRE[i] + b2*TOEFL[i] + b3*Univ[i] +
b4*SOP[i] + b5*LOR[i] + b6*CGPA[i] + b7*Res[i]) - Chance[i];
```

The optimal value and the optimal coefficients are the following:

```
ampl: model Konstantinos_Paganopoulos_Assignment_2b.mod;
ampl: option solver cplex;
ampl: solve;
CPLEX 12.9.0.0: optimal solution; objective 0.1492485949
20 dual simplex iterations (0 in phase I) on the dual problem
ampl: display b0, b1, b2, b3, b4, b5, b6, b7;
b0 = -0.471969
b1 = -0.000741659
b2 = 0.00233573
b3 = 0.0360209
b4 = -0.0184988
b5 = 0.037078
b6 = 0.105112
b7 = 0.0572708
```

Please find attached the solution in the AMPL file (“Konstantinos_Paganopoulos_Assignment_2b.mod”) for more details.

CONCLUSION

In the provided coursework assignment, solutions to 2 linear problems through various ways are provided. The AMPL files are attached separately.