1.

a) $x(t) = t\sin(t)$

$y(t) = t\cos(t)$

tangent vector $= \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle$

$= \langle 1\cdot\sin(t) + \cos(t)\cdot t, \cos(t) - t\sin(t) \rangle$

$= \langle \sin t + t\cos t, \cos t - t\sin t \rangle$

Normal vector $= \langle -\frac{dy}{dt}(t), \frac{dx}{dt} \rangle$

$= \langle -\cos t + t\sin t, \sin t + t\cos t \rangle$

b) $x(t) = t\sin(t)$ $\qquad ax + by + c = 0$

$y(t) = t\cos(t)$

$ax + by + c = 0$

Let $x = \omega$

$\therefore a\omega + by + c = 0$

$\therefore by = -a\omega - c$

$\therefore y = \frac{-a\omega - c}{b}$

Since the lines intersect,

$x(t) = \omega$

$t\sin(t) = \omega$

and

$t\cos(t) = \frac{-a\omega - c}{b}$

$\therefore t\cos(t)b + a\omega + c = 0$

$\therefore t\cos(t)b + a\,t\sin(t) + c = 0$ —— ① general equation

To find a point,

lets take $c = 0, a = 1, b = 1$

$\therefore t\cos(t) + t\sin(t) = 0$

$\therefore (\cos(t) + \sin(t))\,t = 0$

$\therefore \cos(t) + \sin(t) = 0$

$t = \pi n - \frac{\pi}{4} \quad n \in \mathbb{Z}$

If $n = 1$, $t = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\therefore x(\frac{3\pi}{4}) = \frac{3\pi}{4} \times \frac{1}{\sqrt{2}} = \frac{3\pi}{4\sqrt{2}}$

$y(\frac{3\pi}{4}) = \frac{3\pi}{4} \times -\frac{1}{\sqrt{2}}$

$= -\frac{3\pi}{4\sqrt{2}}$

2. Transformations applied : Flip, rotation, scale, translation

① Flip $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

We can see that there is a flip on Y-axis on the right one.

② Rotation : $\begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix}$

After the flip, there seems to be a rotation by $-45°$ ($\pi/4$).

③ Scaling : $\begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$

Each smaller square size on left is 2×2

On the right, the diagonal line seems close to 4.

$\therefore\ 2x^2 = 16$

$\therefore\ x^2 = 8$

$\therefore\ x = 2\sqrt{2}$

Therefore, there seems to be scaling by almost $\sqrt{2}$ on x and y.

④ Translation : $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

The center $(0,0)$ moves to the point $(4,4)$.

$$M = TSR_0R_1$$

$$= \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & 0 \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & 4 \\ -1 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

3.

a) $(0,0) \to (4,2)$

$(0,1) \to (3.5,1)$

$(1,0) \to (3,1.5)$

$(1,1) \to (3,1)$

$$\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \cong \begin{bmatrix} a & b & c \\ d & e & f \\ h & k & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$a x_k + b y_k + c - u_k (h x_k + k y_k + 1) = 0$

$\therefore 0 + 0 + c - 4(1) = 0$

$\therefore \boxed{c = 4}$

$d x_k + e y_k + f - v_k (h x_k + k y_k + 1) = 0$

$\therefore 0 + 0 + f - 2(1) = 0$

$\therefore \boxed{f = 2}$

$$\begin{bmatrix} 3.5 \\ 1 \\ 1 \end{bmatrix} \cong \begin{bmatrix} a & b & 4 \\ d & e & 2 \\ h & k & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$a x_k + b y_k + c - u_k (h x_k + k y_k + 1) = 0$

$\therefore 0 + b + 4 - 3.5(0 + k + 1) = 0$

$\therefore b + 4 - 3.5k - 3.5 = 0$

$\therefore b - 3.5k = -0.5$ ——— ①

$d x_k + e y_k + f - v_k (h x_k + k y_k + 1) = 0$

$\therefore 0 + e + 2 - 1(0 + k + 1) = 0$

$\therefore e + 2 - k - 1 = 0$

$\therefore e - k = -1$ ——— ②

$$\begin{bmatrix} 3 \\ 1.5 \\ 1 \end{bmatrix} \cong \begin{bmatrix} a & b & 4 \\ d & e & 2 \\ h & k & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$a x_k + b y_k + c - u_k (h x_k + k y_k + 1) = 0$

$\therefore a + 0 + 4 - 3(h + 0 + 1) = 0$

$\therefore a + 4 - 3h - 3 = 0$

$\therefore a - 3h = -1$ ——— ③

$$dx_k + ey_k + f - v_k(hx_k + ky_k + 1) = 0$$

$$\therefore d + 2 - 1.5h - 1.5 = 0$$

$$\therefore d - 1.5h = -0.5 \quad \text{———— (4)}$$

$$\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \cong \begin{bmatrix} a & b & 4 \\ d & e & 2 \\ h & k & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$ax_k + by_k + c - u_k(hx_k + ky_k + 1) = 0$$

$$\therefore a + b + 4 - 3(h + k + 1) = 0$$

$$\therefore a + b + 4 - 3h - 3k - 3 = 0$$

$$\therefore a + b - 3h - 3k = -1 \quad \text{———— (5)}$$

$$dx_k + ey_k + f - v_k(hx_k + ky_k + 1) = 0$$

$$\therefore d + e + 2 - 1(h + k + 1) = 0$$

$$d + e + 2 - h - k - 1 = 0$$

$$\therefore d + e - h - k = -1 \quad \text{———— (6)}$$

=> After using online tool to solve system of equations, I get matrix

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

b) $$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 11/4 \\ 1 \\ 1 \end{bmatrix}$$

c) No. Because affine transformations always have last row [0, ... 0 1 1] and we have [1 1 1] as bottom row.

4. a) To do a rotation by θ, we will prove that it is doable by 3 consecutive shears.

To do rotation by θ,

$$M = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We will find shears M_1, M_2, M_3 which will produce M

$$\therefore M = M_3 \cdot M_2 \cdot M_1$$

$$M = M_3 \cdot M_2 \cdot M_1 = \begin{bmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & 0 \\ b & 1+ab & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+bc & a+c+abc & 0 \\ b & 1+ab & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore b = \sin\theta$

$\therefore 1 + a \cdot \sin\theta = \cos\theta$ $\qquad$ $1 + \sin\theta \cdot c = \cos\theta$

$\therefore a\sin\theta = \cos\theta - 1$ $\qquad$ $\therefore c\sin\theta = \cos\theta - 1$

$\therefore a = \dfrac{\cos\theta - 1}{\sin\theta}$ $\qquad$ $\therefore c = \dfrac{\cos\theta - 1}{\sin\theta}$

$\therefore a = -\left(\dfrac{1-\cos\theta}{\sin\theta}\right)$ $\qquad$ $c = -\left(\dfrac{1-\cos\theta}{\sin\theta}\right)$

$\therefore a = -\tan\left(\frac{\theta}{2}\right)$ $\qquad$ $\therefore c = -\tan\left(\frac{\theta}{2}\right)$

$$\therefore M = \begin{bmatrix} 1 & -\tan(\theta/2) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \sin\theta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \cancel{\tan}(\frac{\theta}{2}) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can use shears above to do rotation

b) We can not the above approach for $\theta = 180, 540, \ldots,$ etc

i.e. We can't do flip because $\tan(90), \tan(270) \ldots$ is undefined.