Proof of damped CLT

We will follow the proof from Levin et al paper appropriately modified to our case, i.e., for the damped omnibus matrix.

1 Concentration Bound

Theorem 1. *(Matrix Bernstein)*
Consider independent random Hermitian matrices $\mathbf{H}^{(1)}, \mathbf{H}^{(2)}, \cdots, \mathbf{H}^{(k)} \in \mathbb{R}^{n\times n}$ with $\mathbb{E}\,\mathbf{H}^{(i)} = \mathbf{0}$ and $\|\mathbf{H}^{(i)}\| \leq L$ with probability 1 for all i for some fixed $L > 0$. Define $\mathbf{H} = \sum_{i=1}^{k} \mathbf{H}^{(i)}$, and let $\nu(\mathbf{H}) = \|\mathbb{E}\,\mathbf{H}^2\|$. Then, for all $t \geq 0$,

$$\mathbb{P}[\|\mathbf{H}\| \geq t] \leq 2n \exp\Big\{ -\frac{t^2/2}{\nu(\mathbf{H}) + Lt/3} \Big\}.$$

Let the damped omnibus matrix $M \in \mathbb{R}^{mn\times mn}$ defined as follows,

$$M = \begin{bmatrix} A^{(1)} & \frac{A^{(1)}+A^{(2)}}{2} & \frac{A^{(1)}+2A^{(3)}}{3} & \frac{A^{(1)}+3A^{(4)}}{4} & \cdots & \frac{A^{(1)}+(m-1)A^{(m)}}{m} \\ \frac{A^{(1)}+A^{(2)}}{2} & A^{(2)} & \frac{A^{(2)}+2A^{(3)}}{3} & \frac{A^{(2)}+3A^{(4)}}{4} & \cdots & \frac{A^{(2)}+(m-1)A^{(m)}}{m} \\ \frac{A^{(1)}+2A^{(3)}}{3} & \frac{A^{(2)}+2A^{(3)}}{3} & A^{(3)} & \frac{A^{(3)}+3A^{(4)}}{4} & \cdots & \vdots \\ \frac{A^{(1)}+3A^{(4)}}{4} & \frac{A^{(2)}+3A^{(4)}}{4} & \frac{A^{(3)}+3A^{(4)}}{4} & A^{(4)} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{A^{(1)}+(m-1)A^{(m)}}{m} & \frac{A^{(2)}+(m-1)A^{(m)}}{m} & \frac{A^{(3)}+(m-1)A^{(m)}}{m} & & \cdots & A^{(m)} \end{bmatrix}.$$

Lemma 1. With high probability, $\|M - \mathbb{E}\,M\| \leq Cmn^{\frac{1}{2}} \log^{\frac{1}{2}} mn$.

Proof. Condition on some $P = XX^T$, so that

$$\mathbb{E}\,M = \tilde{P} = \begin{bmatrix} P & P & \cdots & P \\ P & P & \cdots & P \\ \vdots & \vdots & \ddots & \vdots \\ P & P & \cdots & P \end{bmatrix}.$$

For all $q \in [m]$ and $i, j \in [n]$, we define an auxillary block matrix $E_{q,i,j}$ which will help us express the difference $M - \mathbb{E}\,M$ into a sum of independent Hermitian matrices, so that we will be able to apply Bernstein's matrix bound. Let $e^{ij} = e_i e_j^T + e_j e_i^T$ where e_i is a vector in $\mathbb{R}^n$ with all its entries zero but one in the i-th entry. Then, the block matrix $E_{q,i,j}$ is defined as,

$$E_{q,i,j} \begin{bmatrix} 0 & \cdots & 0 & \frac{q-1}{q}e^{ij} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{q-1}{q}e^{ij} & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{q-1}{q}e^{ij} & 0 & \cdots & 0 \\ \frac{q-1}{q}e^{ij} & \cdots & \frac{q-1}{q}e^{ij} & e^{ij} & \frac{e^{ij}}{q+1} & \cdots & \frac{e^{ij}}{m} \\ 0 & \cdots & 0 & \frac{e^{ij}}{q+1} & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{e^{ij}}{m} & 0 & \cdots & 0 \end{bmatrix}.$$

where the nonnegative blocks appear in the q-th row and q-th column. Now, we can write $M - \mathbb{E}\, M$ as

$$M - \mathbb{E}\, M = \sum_{q=1}^{m} \sum_{i<j} (A_{ij}^{(q)} - P_{ij}) E_{q,i,j},$$

which is a sum of $m\binom{n}{2}$ independent zero-mean matrices, with $\|(A_{ij}^{(q)} - P_{ij})E_{q,i,j}\| \leq \sqrt{m}$ for all $q \in [m]$ and $i, j \in [n]$.
It remains to compute the variance term $\nu(M - \mathbb{E}\, M)$. Letting $D_{ij} = e^{ij}e^{ij} = e_i e_i^T + e_j e_j^T \in \mathbb{R}^{n \times n}$, we have

$$E_{q,i,j}E_{q,i,j} = \begin{bmatrix} \frac{(q-1)^2}{q^2}D_{ij} & \cdots & \frac{(q-1)^2}{q^2}D_{ij} & \frac{q-1}{q}D^{ij} & \frac{q-1}{q(q+1)}D_{ij} & \cdots & \frac{q-1}{qm}D_{ij} \\ \vdots & \ddots & & \vdots & & \ddots & \vdots \\ \frac{(q-1)^2}{q^2}D_{ij} & \cdots & \frac{(q-1)^2}{q^2}D_{ij} & \frac{q-1}{q}D^{ij} & \frac{q-1}{q(q+1)}D_{ij} & \cdots & \frac{q-1}{qm}D_{ij} \\ \frac{q-1}{q}D_{ij} & \cdots & \frac{q-1}{q}D_{ij} & D_{ij}^{\star} & \frac{1}{q+1}D^{ij} & \cdots & \frac{1}{m}D^{ij} \\ \frac{q-1}{q(q+1)}D_{ij} & \cdots & \frac{q-1}{q(q+1)}D_{ij} & \frac{1}{q+1}D_{ij} & \frac{1}{(q+1)^2}D_{ij} & \cdots & \frac{1}{(q+1)m}D_{ij} \\ \vdots & \ddots & & \vdots & & \ddots & \vdots \\ \frac{q-1}{qm}D_{ij} & \cdots & \frac{q-1}{qm}D_{ij} & \frac{1}{m}D_{ij} & \frac{1}{(q+1)m}D_{ij} & \cdots & \frac{1}{m^2}D_{ij} \end{bmatrix}.$$

where $D_{ij}^{\star} = \left((q-1)\frac{(q-1)^2}{q^2} + 1 + \frac{1}{(q+1)^2} + \cdots + \frac{1}{m^2}\right)D_{ij}$ and it appears in the q-th row and q-th column. Hence,

$$\nu(M - \mathbb{E}\, M) = \left\| \sum_{q=1}^{m} \sum_{i<j} (A_{ij}^{(q)} - P_{ij})^2 E_{q,i,j}E_{q,i,j} \right\| \leq 2m^2(n-1).$$

To apply Bernstein's matrix bound, let $L = \sqrt{m}$ and $t = 2\sqrt{m(n-1)\log mn}$. Then,

$$\mathbb{P}[\|M - \mathbb{E}\, M\| \geq 2\sqrt{m(n-1)\log mn}] \leq 2m^{-3}n^{-2}.$$

Integrating over all X yields the result.

2 Additional theory from Keith et al.

Observation 1. Let $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_d > 0$ denote the top d eigenvalues of $P = XX^T$, and let $\tilde{P} = J \otimes P$, where $J \in \mathbb{R}^{m \times m}$ is the matrix with all of its entries equal to one. Then, $\sigma(P) = \{m\lambda_1, m\lambda_2, \cdots, m\lambda_d, 0, 0, \cdots, 0\}$.

Proof. trace($\tilde{P}$) =trace(J)trace(P)=$m \sum_{i=1}^{d} \lambda_i = \sum_{i=1}^{d} \lambda_i^{new}$, where $\lambda_i^{new} = m\lambda_i$ for any i.

Observation 2. Let F be an inner product distribution on $\mathbb{R}^d$ with random vectors $X_1, X_2, \cdots, X_n, Y \overset{i.i.d}{\sim} F$. With probability at least $1 - \frac{d^2}{n^2}$, it holds for all $i \in [d]$ that $|\lambda_i(P) - n\lambda_i(\mathbb{E}YY^T)| \leq 2d\sqrt{n \log n}$. Further, we have for all $i \in [d]$, $\lambda_i(\tilde{P}) \geq Cnm\delta$ w.h.p.

Proof. Follows from Keith et al.

Weyl's Inequality: If $\|M - N\|_2 \leq \epsilon$, then $|\mu_i - \nu_i| \leq \epsilon$ for all i, where μ_i, ν_i are the corresponding eigenvalues of M, N respectively.

Lemma 2. *Let $\tilde{P} = U_{\tilde{P}} S_{\tilde{P}} U_{\tilde{P}}^T$ be the eigendecomposition of $\tilde{P}$, where $U_{\tilde{P}} \in \mathbb{R}^{mn \times d}$ has orthonormal columns and $S_{\tilde{P}} \in \mathbb{R}^{d \times d}$ is diagonal and invertible. Let $S_M \in \mathbb{R}^{d \times d}$ be the diagonal matrix of the top d eigenvalues of M and $U_M \in \mathbb{R}^{mn \times d}$ be the matrix with orthonormal columns containing the top d corresponding eigenvectors, so that $U_M S_M U_M^T$ is our estimate of $\tilde{P}$. Let $V_1 \Sigma V_2^T$ be the SVD of $U_{\tilde{P}}^T U_M$. Then,*

$$\|U_{\tilde{P}}^T U_M - V_1 V_2^T\|_F \leq \frac{C \log mn}{n} w.h.p.$$

Proof. Let $\sigma_1, \sigma_2, \cdots, \sigma_d$ denote the diagonal entries of Σ. Then, $\sigma_i = \cos(\theta_i)$ where θ_i are the principal angles between the sub-spaces spanned by $U_M, U_{\tilde{P}}$. Implementing the results from Lemma 1 and Observation 2 into Davis-Kahan theorem, we get

$$\|U_M U_M^T - U_{\tilde{P}} U_{\tilde{P}}^T\| = \max_i |\sin \theta_i| \leq \frac{\|M - \tilde{P}\|}{\lambda_d(\tilde{P})} \leq C \frac{\log^{1/2} mn}{n^{1/2}} \quad w.h.p. \tag{1}$$

for sufficiently large mn. Next,

$$\|U_{\tilde{P}}^T U_M - V_1 V_2^T\|_F = \|\Sigma - I\|_F \leq \sum_{i=1}^{d} \sin^2(\theta_i) \leq d\|U_M U_M^T - U_{\tilde{P}} U_{\tilde{P}}^T\|^2 \leq C \frac{\log mn}{n} \quad w.h.p.$$

which is the desired result.

Proposition 1. *With notation as above,*

$$\|U_{\tilde{P}}^T (M - \tilde{P})\|_F \leq C\sqrt{mn \log mn} \quad w.h.p.$$

Proof. First, we give a bound for an arbitrary entry of $U_{\tilde{P}}^T(M - \tilde{P})$ using Hoeffding inequality and then, via a union bound over all the entries we obtain a bound for the Frobenius norm. Let the ij-th entry of $U_{\tilde{P}}^T(M - \tilde{P})$ be of the form $(U_{\tilde{P}}^T(M - \tilde{P}))_{ij} = (U_{\tilde{P}}^T)_{i\cdot}(M - \tilde{P})_{\cdot j} = \sum_{k=1}^{mn} (M_{kj} - \tilde{P}_{kj}) U_{ki}$, where $U_{\cdot i}$ is the i-th column(orthonormal) of $U_{\tilde{P}} \in \mathbb{R}^{mn \times d}$. Conditional on $P = XX^T$, we have mn independent mean-zero random variables, thus by Hoeffding inequality,

$$\mathbb{P}\left(\Big|\sum_{k=1}^{mn}(M_{kj}-\tilde{P}_{kj})U_{ki}\Big| \geq 2\sqrt{\log mn}\right) \leq 2\exp\left(-\frac{8\log mn}{\sum_{k=1}^{mn}(2U_{ki})^2}\right) = 2m^{-2}n^{-2}.$$

Finally, the desired result is given by,

$$\|U_{\tilde{P}}^T(M-\tilde{P})\|_F = \sqrt{\sum_{i=1}^{d}\sum_{j=1}^{mn}\Big|(U_{\tilde{P}}^T(M-\tilde{P}))_{ij}\Big|^2} \leq C\sqrt{mn\log mn} \quad w.h.p.$$

Proposition 2. *With notation as above,*

$$\|(U_{\tilde{P}}^T(M-\tilde{P})U_{\tilde{P}})\|_F \leq C\sqrt{\log mn} \quad w.h.p.$$

Proof. Following the proof of Proposition 1, we first give a bound for the ij-th entry of matrix $U_{\tilde{P}}^T(M-\tilde{P})U_{\tilde{P}}$ using Hoeffding inequality. We have that $(U_{\tilde{P}}^T(M-\tilde{P})U_{\tilde{P}})_{ij} = \sum_{k,l=1}^{mn} U_{ki}U_{lj}(M_{kl}-\tilde{P}_{kl}) = \sum_{k<l} 2U_{ki}U_{lj}(M_{kl}-\tilde{P}_{kl}) - \sum_{k=1}^{mn}\tilde{P}_{kk}U_{ki}U_{kj}$, where $U_{\cdot i}$, $U_{\cdot j}$ correspond to the i-th and j-th columns of $U_{\tilde{P}}$. Note that conditioned on P, the first term is a sum of $\binom{mn}{2}$ zero-mean independent random variables and the second term is bounded by 1, since $U_{\tilde{P}}$ is a matrix with orthonormal columns and $\tilde{P}_{kk} \in [0,1]$ for all k. By Hoeffding's inequality,

$$\mathbb{P}\left[\Big|\sum_{k<l} 2U_{ki}U_{lj}(M_{kl}-\tilde{P}_{kl})\Big| \geq t\right] \leq 2\exp\left\{-\frac{2t^2}{\sum_{k<l}(4U_{ki}U_{lj})^2}\right\} \leq 2e^{-t^2/8}.$$

Hence, with probability at least $1-(mn)^{-2}$, we have $\sum_{k<l} 2U_{ki}U_{lj}(M_{kl}-\tilde{P}_{kl}) < 4\sqrt{\log mn}$. Moreover,

$$\begin{aligned}
\|(U_{\tilde{P}}^T(M-\tilde{P})U_{\tilde{P}})\|_F &= \sqrt{\sum_{i,j=1}^{d}\Big|\sum_{k<l} 2U_{ki}U_{lj}(M_{kl}-\tilde{P}_{kl}) - \sum_{k=1}^{mn}\tilde{P}_{kk}U_{ki}U_{kj}\Big|^2} \\
&\leq \sqrt{\sum_{i,j=1}^{d} 16\log mn + 8\sqrt{\log mn}+1} \\
&\leq C\sqrt{\log mn}.
\end{aligned}$$

Lemma 3. *Let $V = V_1V_2^T$ where V_1, V_2^T are as defined above. Then,*

$$\|VS_M - S_{\tilde{P}}V\|_F \leq Cm\log mn \quad w.h.p. \tag{2}$$

$$\|VS_M^{1/2} - S_{\tilde{P}}^{1/2}V\|_F \leq C\frac{m^{1/2}\log mn}{n^{1/2}} \quad w.h.p. \tag{3}$$

$$\|VS_M^{-1/2} - S_{\tilde{P}}^{-1/2}V\|_F \leq C\frac{\log mn}{mn^{3/2}} \quad w.h.p. \tag{4}$$

Proof.

Lemma 4. *Define*

$$
\begin{aligned}
R_1 &= U_{\tilde{P}} U_{\tilde{P}}^T U_M - U_{\tilde{P}} V \\
R_2 &= V S_M^{1/2} - S_{\tilde{P}}^{1/2} V \\
R_3 &= U_M - U_{\tilde{P}} U_{\tilde{P}}^T U_M + R_1 = U_M - U_{\tilde{P}} V
\end{aligned}
$$

Then, the following terms converge to zero in probability:

$$
\sqrt{n}\Big[(M-\tilde{P})U_{\tilde{P}}(VS_M^{-1/2} - S_{\tilde{P}}^{-1/2}V)\Big]_h \xrightarrow{P} 0, \tag{5}
$$

$$
\sqrt{n}\Big[U_{\tilde{P}} U_{\tilde{P}}^T (M-\tilde{P}) U_{\tilde{P}} V S_M^{-1/2}\Big]_h \xrightarrow{P} 0, \tag{6}
$$

$$
\sqrt{n}\Big[(I - U_{\tilde{P}} U_{\tilde{P}}^T)(M-\tilde{P}) R_3 S_M^{-1/2}\Big]_h \xrightarrow{P} 0, \tag{7}
$$

and

$$
\|R_1 S_M^{1/2} + U_{\tilde{P}} R_2\|_F \leq \frac{C m^{1/2} \log mn}{n^{1/2}} \qquad w.h.p. \tag{8}
$$

Proof. First note that for two matrices A, B with compatible dimensions the following inequality holds:

$$
\|AB\|_F \leq \|A\| \|B\|_F
$$

We prove first Equation (5). Using the above inequality, Lemma 1 and Lemma 3 we get with high probability the following

$$
\begin{aligned}
\sqrt{n}\|(M-\tilde{P})U_{\tilde{P}}(VS_M^{-1/2} - S_{\tilde{P}}^{-1/2}V)\|_F &\leq \sqrt{n}\|M-\tilde{P})U_{\tilde{P}}\| \|VS_M^{-1/2} - S_{\tilde{P}}^{-1/2}V\|_F \\
&\leq C \frac{\log^{3/2} mn}{n^{1/2}},
\end{aligned}
$$

which goes to 0 asymptotically in n.
Next, to prove Equation (6), we will need the following lemma from Cape et al.,

Lemma 5. *For $A \in \mathbb{R}^{p_1 \times p_2}$ and $B \in \mathbb{R}^{p_2 \times p_3}$, then*

$$
\|A\|_{2\to\infty} = \max_{i \in [p_1]} \|A_i\|_2
$$

$$
\|AB\|_{2\to\infty} \leq \|A\|_{2\to\infty} \|B\|_2
$$

Moreover, from Levin et al., we have that $\|U_{\tilde{P}}\|_{2\to\infty} \leq C(mn)^{-1/2}$ w.h.p.

Hence, by Observation 2, Proposition 2 and Lemma 5 we have that

$$
\begin{aligned}
\sqrt{n}\|U_{\tilde{P}} U_{\tilde{P}}^T (M-\tilde{P}) U_{\tilde{P}} V S_M^{-1/2}\|_{2\to\infty} &\leq \sqrt{n}\|U_{\tilde{P}}\|_{2\to\infty} \|U_{\tilde{P}}^T (M-\tilde{P}) U_{\tilde{P}}\| \|S_M^{-1/2}\| \\
&\leq C \frac{\log^{1/2} mn}{mn^{1/2}} \qquad w.h.p.
\end{aligned}
$$

For Equation (8), simply notice that

$$\|R_1 S_M^{1/2} + U_{\tilde{P}} R_2\|_F \leq \|R_1\|_F \|S_M\|^{1/2} + \|R_2\|_F \leq \|U_{\tilde{P}}^T U_M - V\|_F \|S_M\|^{1/2}\| + \|R_2\|_F$$

and by Lemma 2 and Equation (3) we obtain the desired bound.
Finally, to establish (7), the two to infinity norm can be bounded as follows,

$$\begin{aligned}\sqrt{n}\|(I - U_{\tilde{P}} U_{\tilde{P}}^T)(M - \tilde{P}) R_3 S_M^{-1/2}\|_{2\to\infty} &\leq \sqrt{n}\|(I - U_{\tilde{P}} U_{\tilde{P}}^T)(M - \tilde{P})\|_{2\to\infty} \|R_3 S_M^{-1/2}\| \\ &\leq \sqrt{n}\|M - \tilde{P}\| \|R_3\|_{2\to\infty} \|S_M^{-1/2}\| \\ &\leq C\|R_3\|_{2\to\infty} m^{1/2} n^{1/2} \log^{1/2} mn\end{aligned}$$

To bound the two to infinity norm of R_3, we decompose R_3 as follows

$$\begin{aligned}R_3 = U_M - U_{\tilde{P}} V &= (I - U_{\tilde{P}} U_{\tilde{P}}^T)(M - \tilde{P}) U_{\tilde{P}} V S_M^{-1} \\ &+ (I - U_{\tilde{P}} U_{\tilde{P}}^T)(M - \tilde{P})(U_M - U_{\tilde{P}} V) S_M^{-1} \\ &+ (I - U_{\tilde{P}} U_{\tilde{P}}^T)\tilde{P}(U_M - U_{\tilde{P}} U_{\tilde{P}}^T U_M) S_M^{-1} \\ &+ U_{\tilde{P}}(U_{\tilde{P}}^T U_M - V)\end{aligned}$$

As $\tilde{P} = U_{\tilde{P}} S_{\tilde{P}} U_{\tilde{P}}^T$, the term $(I - U_{\tilde{P}} U_{\tilde{P}}^T)\tilde{P}$ vanishes. Also, the two to infinity norm of the term $I - U_{\tilde{P}} U_{\tilde{P}}^T$ is bounded by the spectral norm which in turn, it is bounded by 1. Hence, by Lemma 5 and triangle inequality we obtain the following bound on the two to infinity norm of R_3,

$$\begin{aligned}\|R_3\|_{2\to\infty} &\leq \|(M - \tilde{P}) U_{\tilde{P}} V\|_{2\to\infty} \|S_M^{-1}\| + \|U_{\tilde{P}}\|_{2\to\infty} \|U_{\tilde{P}}^T (M - \tilde{P}) U_{\tilde{P}}\| \|S_M^{-1}\| \\ &+ \|M - \tilde{P}\| \|R_3\| \|S_M^{-1}\| + \|U_{\tilde{P}}\|_{2\to\infty} \|U_{\tilde{P}}^T U_M - V\| \end{aligned} \tag{9}$$

Claim 1:

$$\|R_3\| = O\Big(\frac{\log^{1/2} mn}{n^{1/2}}\Big)$$

Proof. Following the proof of Lemma 6.8 from Cape et al., we add and substract $U_{\tilde{P}} U_{\tilde{P}}^T U_M$ and by triangle inequality,

$$\|R_3\| = \|U_M - U_{\tilde{P}} V\| \leq \|U_M - U_{\tilde{P}} U_{\tilde{P}}^T U_M\| + \|U_{\tilde{P}}(U_{\tilde{P}}^T U_M - V)\|.$$

The first term can be rewritten as follows,

$$\|U_M - U_{\tilde{P}} U_{\tilde{P}}^T U_M\| = \|U_M U_M^T - U_{\tilde{P}} U_{\tilde{P}}^T U_M U_M^T\| = \|(I - U_{\tilde{P}} U_{\tilde{P}}^T) U_M U_M^T\| = \|\sin\Theta(U_M, U_{\tilde{P}})\|$$

By Davis-Kahan,

$$\|U_M - U_{\tilde{P}} U_{\tilde{P}}^T U_M\| \leq \frac{\|M - \tilde{P}\|}{\lambda_d(\tilde{P})} \leq C\frac{\log^{1/2} mn}{n^{1/2}}$$

Further, we can bound the second term from Lemma 2 which leads us to the desired result.

Claim 2:

$$\|(M - \tilde{P}) U_{\tilde{P}} V\|_{2\to\infty} \leq \|(M - \tilde{P}) U_{\tilde{P}}\|_{2\to\infty} \|V\| \leq \|(M - \tilde{P}) U_{\tilde{P}}\|_{2\to\infty} = O_d(m\sqrt{\log mn}).$$

Proof. We note first that

$$\frac{1}{\sqrt{d}}\|(M-\tilde{P})U_{\tilde{P}}\|_{2\to\infty}\leq \|(M-\tilde{P})U_{\tilde{P}}\|_{\max}=\max_{i\in[mn],j\in[d]}|\langle (M-\tilde{P})U_{\cdot j},e_i\rangle|,$$

where $e_i \in \mathbb{R}^{mn}$ is the unit vector with all of its entries equal to 0, except for the i-th entry.

Let $l\in[m]$ arbitrary. For all $1\leq k\leq n$, $U_{k,j}=U_{k+n,j}=\cdots=U_{(m-1)k+n,j}$, and for each $(l-1)n+1\leq i\leq ln$ and $1\leq j\leq d$,

$$\langle (M-\tilde{P})U_{\cdot j},e_i\rangle = e_i^T(M-\tilde{P})U_{\cdot j}=\sum_{\tilde{k}=1}^{mn}(M_{i,\tilde{k}}-\tilde{P}_{i,\tilde{k}})U_{\tilde{k},j}=$$

$$=\sum_{k=1}^{n}\Big[\frac{A_{ik}^{(1)}+\cdots+A_{ik}^{(l-1)}}{l}+\Big((l-1)\frac{l-1}{l}+1+\frac{1}{l+1}+\cdots+\frac{1}{m}\Big)A_{ik}^{(l)}+$$

$$\frac{l}{l+1}A_{ik}^{(l+1)}+\cdots+\frac{m-1}{m}A_{ik}^{(m)}-mP_{ik}\Big]U_{k,j}.$$

For all $l\in[m]$, for any $(l-1)n+1\leq i\leq ln$ and $1\leq j\leq d$, the above expansion is a sum of independent, bounded, mean zero random variables taking values in $[-mU_{k,j},mU_{k,j}]$. Hence, by Hoeffding inequality, with probability at least $1-\frac{2}{(mn)^2}$,

$$\mathbb{P}\Big(\Big|\sum_{\tilde{k}=1}^{mn}(M_{i,\tilde{k}}-\tilde{P}_{i,\tilde{k}})U_{\tilde{k},j}\Big|\geq 2m\sqrt{\log mn}\Big)\leq 2\exp\left(-\frac{8m^2\log mn}{\sum_{k=1}^{n}(2mU_{k,j})^2}\right)\leq 2(mn)^{-2}.$$

Therefore, $\|(M-\tilde{P})U_{\tilde{P}}\|_{2\to\infty}=O_d(m\sqrt{\log mn})$ holds.

Using the above claims, the Equation 9 becomes

$$\|R_3\|_{2\to\infty}=O\Big(\frac{\sqrt{\log mn}}{n}\Big)+O\Big(\frac{\log mn}{(mn)^{3/2}}\Big)+O\Big(\frac{\log mn}{n}\Big)+O\Big(\frac{\log mn}{m^{1/2}n^{3/2}}\Big),$$

and therefore, $\sqrt{n}\|(I-U_{\tilde{P}}U_{\tilde{P}}^T)(M-\tilde{P})R_3S_M^{-1/2}\|_{2\to\infty}$ goes to 0 as n goes to infinity.

By the definition of the JRDPG, the latent positions of the expected omnibus matrix $\mathbb{E}\,M=\tilde{P}=U_{\tilde{P}}S_{\tilde{P}}U_{\tilde{P}}^T$ are given by

$$Z^\star=\begin{bmatrix}X^\star\\X^\star\\\vdots\\X^\star\end{bmatrix}=U_{\tilde{P}}S_{\tilde{P}}^{1/2}\in\mathbb{R}^{mn\times d}$$

Recall that the matrix of the true latent positions is denoted by $Z=[X^TX^T\cdots X^T]^T\in\mathbb{R}^{mn\times d}$, so that $Z=Z^\star W$ for some suitable-chosen orthogonal matrix W.

Lemma 6. *Fix some $i\in[n]$ and some $s\in[m]$ and let $h=n(s-1)+i$. Conditional on $X_i=x_i\in\mathbb{R}^d$, there exists a sequence of d-by-d orthogonal matrices $\{W_n\}_n$ such that*

$$n^{1/2}W_n^T\Big[(M-\tilde{P})U_{\tilde{P}}S_{\tilde{P}}^{-1/2}\Big]_h\xrightarrow{\mathcal{L}}\mathcal{N}(0,\Sigma(x_i)),$$

where $\Sigma(x_i)$is the covariance matrix that depends on x_i.

Proof. Following the proof of Lemma 6 in Levin et al, for each $n = 1, 2, \cdots$, choose orthogonal $W_n \in \mathbb{R}^{d\times d}$ so that $X = X^\star W_n = U_P S_P W_n$. We rewrite the term $n^{1/2} W_n^T \Big[(M-\tilde{P})U_{\tilde{P}} S_{\tilde{P}}^{-1/2}\Big]_h$ as follows

$$
\begin{aligned}
n^{1/2} W_n^T \Big[(M-\tilde{P})U_{\tilde{P}} S_{\tilde{P}}^{-1/2}\Big]_h &= n^{1/2} W_n^T \Big[MZ^\star S_{\tilde{P}}^{-1} - \tilde{P} Z^\star S_{\tilde{P}}^{-1}\Big]_h \\
&= n W_n^T S_P^{-1} W_n \frac{n^{-1/2}}{m} \Big[MZ - \tilde{P}Z\Big]_h
\end{aligned} \tag{10}
$$

The h-th scaled row of the matrix difference $MZ - \tilde{P}Z$ can be further expanded as

$$
\begin{aligned}
\frac{n^{-1/2}}{m}[MZ - \tilde{P}Z]_h = \frac{n^{-1/2}}{m} \sum_{j=1}^{n} \Bigg(& \frac{A_{ij}^{(1)} + \cdots + A_{ij}^{(s-1)}}{s} + \Big(\frac{(s-1)^2}{s} + 1 + \frac{1}{s+1} + \cdots + \frac{1}{m}\Big) A_{ij}^{(s)} \\
& + \frac{s}{s+1} A_{ij}^{(s+1)} + \cdots + \frac{m-1}{m} A_{ij}^{(s)} - mP_{ij} \Bigg) X_j \\
= \frac{n^{-1/2}}{m} \sum_{j\neq i} & \Bigg(\sum_{q<s} \frac{1}{s}(A_{ij}^{(q)} - P_{ij}) + \phi(s)(A_{ij}^{(s)} - P_{ij}) + \sum_{q>s} \frac{q-1}{q}(A_{ij}^{(q)} - P_{ij}) \Bigg) X_j \\
- n^{-1/2} P_{ii} X_i, &
\end{aligned}
$$

where $\phi(s) = \frac{(s-1)^2}{s} + 1 + \frac{1}{s+1} + \cdots + \frac{1}{m} = (s-1) + \sum_{q\geq s} \frac{1}{q}$.

Conditioning on $X_i = x_i \in \mathbb{R}^d$, we first observe that

$$
\frac{P_{ii}}{n^{1/2}} X_i = \frac{x_i^T x_i}{n^{1/2}} x_i \xrightarrow{a.s.} 0,
$$

moreover, the scaled sum becomes

$$
n^{-1/2} \sum_{j\neq i} \frac{1}{m} \Bigg(\sum_{q<s} \frac{1}{s}(A_{ij}^{(q)} - x_i^T X_j) + \phi(s)(A_{ij}^{(s)} - x_i^T X_j) + \sum_{q>s} \frac{q-1}{q}(A_{ij}^{(q)} - x_i^T X_j) \Bigg) X_j,
$$

which is a sum of $n-1$ independent 0-mean random variables, each with covariance matrix denoted by $\tilde{\Sigma}(x_i)$, computed as follows

$$
\begin{aligned}
\tilde{\Sigma}(x_i) &= \frac{1}{m^2} Cov\Big(\Big(\sum_{q<s} \frac{1}{s}(A_{ij}^{(q)} - x_i^T X_j) + \phi(s)(A_{ij}^{(s)} - x_i^T X_j) + \sum_{q>s} \frac{q-1}{q}(A_{ij}^{(q)} - x_i^T X_j)\Big) X_j\Big) \\
&= \frac{1}{m^2} \mathbb{E}[\mathbb{E}\Big[\Big(\Big(\sum_{q<s} \frac{1}{s}(A_{ij}^{(q)} - x_i^T X_j) + \phi(s)(A_{ij}^{(s)} - x_i^T X_j) + \sum_{q>s} \frac{q-1}{q}(A_{ij}^{(q)} - x_i^T X_j)\Big) X_j\Big)^2 \Big| X_j = x_j\Big]] \\
&= \frac{1}{m^2} \Big(\frac{s-1}{s^2} + \phi(s)^2 + \sum_{q>s} \Big(\frac{q-1}{q}\Big)^2\Big) \mathbb{E}[(x_i^T X_j - (x_i^T X_j)^2) X_j X_j^T] \\
&= \frac{1}{m^2} \Phi(s) \, \mathbb{E}[(x_i^T X_j - (x_i^T X_j)^2) X_j X_j^T],
\end{aligned}
$$

where

$$\Phi(s) = (s-1)^2 + (m-s) + 2(s-2)\sum_{q\geq s}\frac{1}{q} + \frac{3s-1}{s^2} + \sum_{q>s}\frac{1}{q^2} + \Big(\sum_{q\geq s}\frac{1}{q}\Big)^2.$$

By the multivariate central limit theorem we have that

$$n^{-1/2}\sum_{j\neq i}\frac{1}{m}\left(\sum_{q<s}\frac{1}{s}(A_{ij}^{(q)} - x_i^T X_j) + \phi(s)(A_{ij}^{(s)} - x_i^T X_j) + \sum_{q>s}\frac{q-1}{q}(A_{ij}^{(q)} - x_i^T X_j)\right)X_j \xrightarrow{\mathcal{L}} \mathcal{N}(0, \tilde{\Sigma}(x_i)). \tag{11}$$

Recall $\Delta = \mathbb{E}[X_1 X_1^T] \in \mathbb{R}^{d\times d}$. For the asymptotic behavior of the term $nW_n^T S_P^{-1} W_n$ in Equation 10, observe first that by the law of large numbers,

$$\frac{X^T X}{n} - \Delta \xrightarrow{a.s.} 0.$$

Further, note that $(X^\star)^T X^\star = S_P^{1/2} U_P^T U_P S_P^{1/2} = S_P$ and also

$$\frac{1}{n}(X^\star)^T X^\star - W_n \Delta W_n^T = W_n\Big(\frac{X^T X}{n} - \Delta\Big)W_n^T \xrightarrow{a.s.} 0,$$

Thus,

$$\frac{S_P}{n} - W_n \Delta W_n^T \xrightarrow{a.s.} 0.$$

Since all matrices above are $d\times d$, with d fixed in n, and both matrices S_P/n, Δ are diagonal, it holds that $S_P/n \to \Delta$, which implies $nW_n^T S_P^{-1} W_n \xrightarrow{a.s.} \Delta^{-1}$. Finally, combinging this with Equation 11, the multivariate version of Slutsky's theorem yields the desired result.

Proof of Theorem 1. We are now ready to provide a proof of the main theorem. Let $h = n(s-1)+i$, with $s\in[m], i\in[n]$ and recall $V = V_1 V_2^T$, where $V_1 \Sigma V_2^T$ is the SVD of $U_{\tilde{P}}^T U_M$ as shown in Lemma 2. Following the reasoning of Keith et al., we decompose the term $U_M S_M^{1/2} - U_{\tilde{P}} S_{\tilde{P}}^{1/2} V$ as follows

$$\begin{aligned} U_M S_M^{1/2} - U_{\tilde{P}} S_{\tilde{P}}^{1/2} V &= (M-\tilde{P})U_{\tilde{P}} S_{\tilde{P}}^{-1/2} V + (M-\tilde{P})U_{\tilde{P}}(V S_M^{-1/2} - S_{\tilde{P}}^{-1/2} V) \\ &\quad - U_{\tilde{P}} U_{\tilde{P}}^T (M-\tilde{P}) U_{\tilde{P}} S_M^{-1/2} + (I - U_{\tilde{P}} U_{\tilde{P}}^T)(M-\tilde{P}) R_3 S_M^{-1/2} \\ &\quad + R_1 S_M^{1/2} + U_{\tilde{P}} R_2, \end{aligned}$$

where $R_1.R_2.R_3$ as in Lemma 4.

Applying Lemma 6 and integrating over X_i, we have that there exists a sequence of orthogonal matrices $\{W_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(n^{1/2} W_n^T [(M-\tilde{P})U_{\tilde{P}} S_{\tilde{P}}^{-1/2}]_h \leq x\right) = \int_{supp(F)} \Phi(x, \Sigma(y)) dF(y). \tag{12}$$

Setting $N = (M - \tilde{P})U_{\tilde{P}}S_{\tilde{P}}^{-1/2}$ and

$$
\begin{aligned}
H &= (M - \tilde{P})U_{\tilde{P}}(VS_M^{-1/2} - S_{\tilde{P}}^{-1/2}V) - U_{\tilde{P}}U_{\tilde{P}}^T(M - \tilde{P})U_{\tilde{P}}S_M^{-1/2} \\
&\quad + (I - U_{\tilde{P}}U_{\tilde{P}}^T)(M - \tilde{P})R_3S_M^{-1/2} + R_1S_M^{1/2} + U_{\tilde{P}}R_2
\end{aligned}
$$

we can write the term $\sqrt{n}(U_MS_M^{1/2}V^T - U_{\tilde{P}}S_{\tilde{P}}^{1/2})W_n$ as $\sqrt{n}(NW_n + HV^TW_n)$.

If we show that the h-th row of $\sqrt{n}HV^TW_n$ converges to zero in probability, then by Equation 12 and by Slutsky's theorem we obtain the desired result.
Since V^TW_n unitary, it suffices to show that the h-th row of $\sqrt{n}H$ goes to 0 in probability, which in turn by Lemma 4, it suffices to show that $\sqrt{n}[R_1S_M^{1/2} + U_{\tilde{P}}R_2]_h \xrightarrow{P} 0$. By Lemma 5,

$$
\|[R_1S_M^{1/2} + U_{\tilde{P}}R_2]_h\| \leq \|R_1S_M^{1/2} + U_{\tilde{P}}R_2\|_{2\to\infty} \leq \|R_1\|_{2\to\infty}\|S_M\|^{1/2} + \|U_{\tilde{P}}\|_{2\to\infty}\|R_2\|
$$

Recall that $\|U_{\tilde{P}}\|_{2\to\infty} \leq C(mn)^{-1/2}$ w.h.p. Hence, by Lemma 2 we have that

$$
\|R_1\|_{2\to\infty} \leq \|U_{\tilde{P}}\|_{2\to\infty}\|U_{\tilde{P}}^TU_M - V\| = O\Big(\frac{\log mn}{m^{1/2}n}\Big) \text{ w.h.p.,}
$$

Finally, combining Observation 2, Lemma 3 and the above bound yield

$$
\|[R_1S_M^{1/2} + U_{\tilde{P}}R_2]_h\| = O\Big(\frac{\log mn}{n}\Big) \text{ w.h.p.}
$$

and the proof is complete.

3 Covariance of the difference between two estimated latent positions

Let $h_1 = n(s_1 - 1) + i$ and $h_2 = n(s_2 - 1) + i$, where $s_1, s_2 \in [m]$, $s_1 < s_2$ and $i \in [n]$. The analogous scaled sum for the difference between h_1-th and h_2-th rows of M is given by

$$
\begin{aligned}
&n^{-1/2}\sum_{j\neq i}\frac{1}{m}\Bigg(\frac{1}{s_1s_2}\Big((s_2 - s_1)\sum_{q<s_1}A_{ij}^{(q)} - s_1\sum_{s_1\leq q<s_2}A_{ij}^{(q)}\Big) + \Big(\phi(s_1)A_{ij}^{(s_1)} - \phi(s_2)A_{ij}^{(s_2)}\Big) \\
&+ \sum_{s_2\geq q>s_1}\frac{q-1}{q}A_{ij}^{(q)}\Bigg)X_j \\
&= n^{-1/2}\sum_{j\neq i}\frac{1}{m}\Bigg(\frac{1}{s_1s_2}\Big((s_2 - s_1)\sum_{q\leq s_1}A_{ij}^{(q)} - s_1\sum_{s_1\leq q<s_2}A_{ij}^{(q)}\Big) \\
&+ \Big(\sum_{s_2\geq q>s_1}A_{ij}^{(q)} + (s_1 - 1)A_{ij}^{(s_1)} - (s_2 - 1)A_{ij}^{(s_2)}\Big) \\
&+ \Big(\sum_{q\geq s_2}\frac{1}{q}(A_{ij}^{(s_1)} - A_{ij}^{(s_2)}) + \sum_{s_2\geq q>s_1}\frac{1}{q}(A_{ij}^{(s_1)} - A_{ij}^{(q)})\Big)\Bigg)X_j
\end{aligned}
$$

Conditioning on $X_i = x_i$, the above term is a sum of $n-1$ independent 0-mean random variables with covariance matrix $\tilde{\Sigma}(x_i)$.

The covariance matrix is shown below,

$$\tilde{\Sigma}(x_i) = \frac{1}{m^2}\Bigg[(s_2 - s_1)^2 + 2s_1s_2 - 3(s_1 + s_2 - 2) + \frac{5}{s_2} - \frac{1}{s_1} + 2(s_1 - 2)\sum_{q>s_1}\frac{1}{q}$$
$$+ 2\Big(s_2 - \frac{2}{s_2}\Big)\sum_{q\geq s_2}\frac{1}{q} + 2\Big(\sum_{q>s_1}\frac{1}{q^2} + \sum_{l>q\geq s_1}\frac{1}{ql} + \sum_{l>q\geq s_2}\frac{1}{ql}\Big)\Bigg]\mathbb{E}[X_jX_j^T(x_i^TX_j - (x_i^TX_j)^2)]$$

3.1 Forward omni case

With the same setup as above, the covariance matrix of the difference between two estimated latent positions using forward omnibus matrix is

$$\tilde{\Sigma}(x_i) = \frac{1}{m^2}\Bigg[2\Big[\binom{m-s_1}{2} + \binom{m-s_2}{2}\Big]$$
$$+ \sum_{q<s_1}\frac{2(m-s_1-1)}{s_1-q+1} - \frac{2(m-s_2)}{s_2-q+1} + \frac{4(s_2-s_1)+2}{(s_2-s_1+1)(s_1-q+1)(s_2-q+1)} + \frac{2(s_2-s_1)^2}{(s_1-q+1)^2(s_2-q+1}$$
$$+ \sum_{s_1<q<s_2}\frac{2(m-s_2-1)}{s_2-q+1} - \frac{2(m-s_1)}{q-s_1+1}$$
$$+ 2\Big(\frac{1}{(q-s_1+1)^2} + \frac{1}{(s_2-q+1)^2} - \frac{1}{(q-s_1+1)(s_2-q+1)(s_2-s_1+1)}\Big)$$
$$- \sum_{q>s_2}\frac{2(m-s_1-1)}{q-s_1+1} + \frac{2(m-s_2)}{q-s_2+1} - \frac{4(s_2-s_1)+2}{(s_2-s_1+1)(q-s_1+1)(q-s_2+1)} - \frac{2(s_2-s_1)^2}{(q-s_1+1)^2(q-s_2+1}$$
$$+ 2\Big[\sum_{l<q<s_1}\frac{1}{(s_1-q+1)(s_1-l+1} + \sum_{l<q<s_2}\frac{1}{(s_2-q+1)(s_2-l+1)}$$
$$+ \sum_{l>q>s_2}\frac{1}{(q-s_2+1)(l-s_2+1)} + \sum_{l>q>s_1}\frac{1}{(q-s_1+1)(l-s_1+1)}$$
$$- \sum_{q<s_1}\frac{1}{s_1-q+1}\sum_{q>s_1}\frac{1}{q-s_1+1} - \sum_{q<s_2}\frac{1}{s_2-q+1}\sum_{q>s_2}\frac{1}{q-s_2+1}\Big]\Bigg]\mathbb{E}[X_jX_j^T(x_i^TX_j - (x_i^TX_j)^2)]$$