2.3 Synchrotron motion in action-angle variables

Although synchrotron motion of particles inside a bunched beam is non-linear, it is integrable. It is directly equivalent to the pendulum problem [**?**], one of the most studied non-linear problems.

Above transition energy and without acceleration, the synchronous phase is equal to π and the Hamiltonian H describing the synchrotron motion is:

$$H = \frac{qV_{RF}}{2\pi f_{RF} P_0 C_0} \cos\left(\frac{2\pi f_{RF}}{c}\tau\right) - \frac{1}{2}\eta_p p_\tau^2, \tag{2.121}$$

where q is the charge of the particle, V_{RF} is the amplitude of the voltage powering the Radio-Frequency (RF) cavity, f_{RF} is the frequency of the RF cavity, C_0 is the accelerator's circumference and η_p is the phase slip factor equal to:

$$\eta_p = \alpha_p - \frac{1}{\gamma_0^2}, \tag{2.122}$$

with α_p being the momentum companction factor defined by the beam optics. To make notation easier, the following variables are introduced:

$$A = \frac{qV_{RF}}{2\pi f_{RF} P_0 C_0}, \tag{2.123}$$

$$B = \frac{2\pi f_{RF}}{c}, \tag{2.124}$$

$$C = \frac{1}{2}\eta_p. \tag{2.125}$$

Using Eqs. (2.123), (2.124) and (2.125) the hamiltonian becomes

$$H = A\cos(B\tau) - Cp_\tau^2. \tag{2.126}$$

For the "libration" case, i.e. for stable motion inside the "RF bucket", the value of H ranges in the interval $[0, A]$ with $H = 0$ corresponding to the separatrix separating "libration" and "rotation" of the pendulum and with $H = A$ corresponding the synchronous particle. Using the trigonometric identity

$$\cos(\theta) = 1 - 2\sin^2\left(\frac{\theta}{2}\right), \tag{2.127}$$

Eq. (2.126) is rewritten to

$$H = A\left(1 - 2\sin^2\left(\frac{B}{2}\tau\right)\right) - Cp_\tau^2, \tag{2.128}$$

$$-H + A = 2A\sin^2\left(\frac{B}{2}\tau\right) + Cp_\tau^2, \tag{2.129}$$

$$\frac{-H + A}{2A} = \sin^2\left(\frac{B}{2}\tau\right) + \frac{C}{2A}p_\tau^2. \tag{2.130}$$

Introducing the variable

$$m = \frac{-H + A}{2A}, \tag{2.131}$$

Eq. (2.130) becomes

$$m = \sin^2\left(\frac{B}{2}\tau\right) + \frac{C}{2A}p_\tau^2. \tag{2.132}$$

The value of m now ranges in the interval $[0, 1]$ with $m = 0$ corresponding to the synchronous particle and with $m = 1$ corresponding to the separatrix separating "libration" and "rotation" of the pendulum (captured and, respectively, uncaptured motion by the RF system). The stable fixed point is located at $\tau = 0$ while the unstable fixed points are located at $\tau = \pm\frac{\pi}{B}$

The action variable is defined as:

$$\begin{aligned} J &= \frac{1}{2\pi}\oint p_\tau \mathrm{d}\tau && (2.133)\\ &= \frac{1}{2\pi}4\int_0^{\frac{\pi}{B}} p_\tau \mathrm{d}\tau && (2.134)\\ &= \frac{2\sqrt{2A}}{\pi\sqrt{C}}\int_0^{\frac{\pi}{B}}\sqrt{m - \sin^2\left(\frac{B}{2}\tau\right)}\mathrm{d}\tau, && (2.135)\end{aligned}$$

Through a change of variables with

$$\sqrt{m}\sin\phi = \sin\left(\frac{B}{2}\tau\right) \tag{2.136}$$

and

$$\frac{2\sqrt{m}}{B}\frac{\cos\phi}{\sqrt{1 - m\sin^2\phi}}\mathrm{d}\phi = \mathrm{d}\tau \tag{2.137}$$

the action becomes

$$\begin{aligned}
J &= \frac{2\sqrt{2A}}{\pi\sqrt{C}}\int_0^{\frac{\pi}{B}}\sqrt{m-m\sin^2\phi}\mathrm{d}\tau && (2.138)\\
&= \frac{2\sqrt{2A}}{\pi\sqrt{C}}\int_0^{\frac{\pi}{B}}\sqrt{m}\sqrt{1-\sin^2\phi}\mathrm{d}\tau && (2.139)\\
&= \frac{2\sqrt{2A}}{\pi\sqrt{C}}\int_0^{\frac{\pi}{B}}\sqrt{m}\cos\phi\mathrm{d}\tau && (2.140)\\
&= \frac{4\sqrt{2A}}{\pi B\sqrt{C}}\int_0^{\frac{\pi}{2}}\frac{m\cos^2\phi}{\sqrt{1-m\sin^2\phi}}\mathrm{d}\phi && (2.141)\\
&= \frac{4\sqrt{2A}}{\pi B\sqrt{C}}\int_0^{\frac{\pi}{2}}\frac{m\cos^2\phi}{\sqrt{1-m\sin^2\phi}}\mathrm{d}\phi && (2.142)\\
&= \frac{4\sqrt{2A}}{\pi B\sqrt{C}}\int_0^{\frac{\pi}{2}}\frac{m-m\sin^2\phi}{\sqrt{1-m\sin^2\phi}}\mathrm{d}\phi && (2.143)\\
&= \frac{4\sqrt{2A}}{\pi B\sqrt{C}}\int_0^{\frac{\pi}{2}}\frac{m-1+1-m\sin^2\phi}{\sqrt{1-m\sin^2\phi}}\mathrm{d}\phi && (2.144)\\
&= \frac{4\sqrt{2A}}{\pi B\sqrt{C}}\left((m-1)\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{1-m\sin^2\phi}}\mathrm{d}\phi+\int_0^{\frac{\pi}{2}}\frac{1-m\sin^2\phi}{\sqrt{1-m\sin^2\phi}}\mathrm{d}\phi\right) && (2.145)\\
&= \frac{4\sqrt{2A}}{\pi B\sqrt{C}}\left(\int_0^{\frac{\pi}{2}}\sqrt{1-m\sin^2\phi}\,\mathrm{d}\phi-(1-m)\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{1-m\sin^2\phi}}\mathrm{d}\phi\right), && (2.146)
\end{aligned}$$

or

$$J = \frac{4\sqrt{2A}}{\pi B\sqrt{C}}\left(\int_0^{\frac{\pi}{2}}\sqrt{1-m\sin^2\phi}\,\mathrm{d}\phi-(1-m)\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{1-m\sin^2\phi}}\mathrm{d}\phi\right). \tag{2.147}$$

The two integrals define the complete elliptic integrals of the first and second kind. The complete elliptic integral of the first kind is:

$$K(m) = \int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{1-m\sin^2\phi}}\mathrm{d}\phi. \tag{2.148}$$

The complete elliptic integral of the second kind is:

$$E(m) = \int_0^{\frac{\pi}{2}}\sqrt{1-m\sin^2\phi}\,\mathrm{d}\phi. \tag{2.149}$$

By substituting Eqs. (2.148), (2.149) into Eq. (2.147):

$$J = \frac{4\sqrt{2A}}{\pi B\sqrt{C}}\left(E(m)-(1-m)K(m)\right). \tag{2.150}$$

The frequency of the synchrotron oscillations is:

$$\nu(J) = \frac{\partial H}{\partial J} \tag{2.151}$$
$$= \frac{\partial H}{\partial m}\left(\frac{\partial J}{\partial m}\right)^{-1} \tag{2.152}$$
$$= -2Am\left(\frac{\partial J}{\partial m}\right)^{-1} \tag{2.153}$$
$$= -\frac{\pi B\sqrt{2AC}}{4}\left(\frac{\partial}{\partial m}\left(E(m) - (1-m)\,K(m)\right)\right)^{-1} \tag{2.154}$$
$$= -\frac{\pi B\sqrt{2AC}}{4}\left(\frac{2}{K(m)}\right) \tag{2.155}$$
$$= -\frac{\pi B\sqrt{2AC}}{2K(m)} \tag{2.156}$$
$$\tag{2.157}$$

or

$$\nu(J) = -\frac{\pi\sqrt{\pi f_{RF} q V_{RF} \eta_p}}{c\sqrt{P_0 C_0} K(m(J))}, \tag{2.158}$$

where m can be found by inverting Eq. (2.150). The angle variable is then defined as:

$$\phi = \nu(J)t + a, \tag{2.159}$$

with a being an integration constant.

The original variables with respect to the normalized co-ordinates are recovered:

$$\tau = \frac{2}{B}\sin^{-1}\left(\sqrt{m}\cdot sn\left(4K(m)\nu(m)t\,|\,m\right)\right), \tag{2.160}$$
$$p_\tau = \sqrt{\frac{2Am}{C}}\cdot cn\left(4K(m)\nu(m)t\,|\,m\right), \tag{2.161}$$

where the Jacobi elliptic functions sn and cn have been used, which are given by:

$$sn\left(u|\,m\right) = \sin\phi, \tag{2.162}$$
$$cn\left(u|\,m\right) = \cos\phi, \tag{2.163}$$

with ϕ defined as the inverse of the function:

$$u = \int_0^\phi \frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}} \tag{2.164}$$

For a given value in the action variable, the τ and p_τ coordinates can be sampled by choosing a random $\nu(m)t \in (0, 1)$.