1.

$$B = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$$

First, I need to find eigenvectors and eigenvalues.
I know that λ eigenvalue for B if and only if

$$|\lambda I - B| = 0$$

$$|\lambda I - B| = \left| \begin{bmatrix} \lambda-2 & -3 \\ -3 & \lambda-2 \end{bmatrix} \right| = (\lambda-2)^2 - 9 = \lambda^2 - 4\lambda - 5$$

$$\Delta = 16 + 20 = 36 \quad |\sqrt{\Delta}| = 6$$

$$\lambda_1 = \frac{4-6}{2} = -1$$

$$\lambda_2 = \frac{4+6}{2} = 5$$

• Calculating eigenvectors, they are all the vectors such that $\begin{bmatrix} \lambda-2 & -3 \\ -3 & \lambda-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or just $N(\lambda I - B)$

$\lambda_1 = -1$, v_1

$$\left[\begin{array}{cc|c} -3 & -3 & 0 \\ -3 & -3 & 0 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{cc|c} -3 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow -\frac{1}{3}R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$y = t$, $t \in \mathbb{R}$
$x = -t$
$v_1 = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$

$\lambda_2 = 5$, v_2

$$\left[\begin{array}{cc|c} 3 & -3 & 0 \\ -3 & 3 & 0 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 + R_1} \left[\begin{array}{cc|c} 3 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow \frac{1}{3}R_1} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$y = s$, $s \in \mathbb{R}$
$x = s$
$v_2 = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $s \in \mathbb{R}$

• Diagonalisation – I know eigenvectors, so the diagonal matrix is $\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$. Also, B is symmetric, so $v_1 \perp v_2$, but not normal.

$\|v_1\| = \|v_2\| = \sqrt{2}$, so, to normalize eigenvectors

$v_1' = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $v_2' = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

Then $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ is orthogonal $\Leftrightarrow Q^T = Q^{-1}$

- Checks:

1) $Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ $\qquad Q^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$Q^{-1} = -1 \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

so, indeed $Q^{-1} = Q^T$ ✓

2) $Q A Q^T = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 5 & 5 \end{bmatrix} =$

$= \frac{1}{2} \begin{bmatrix} 4 & 6 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} = B$ ✓

Writing $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ as a linear combination of eigenvectors:

$\begin{bmatrix} 2 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\boxed{\begin{aligned} &{}^*\begin{cases} -c_1 + c_2 = 2 \rightarrow c_2 = 2 + c_1 \\ c_1 + c_2 = 0 \end{cases} \\ &2c_1 + 2 = 0 \\ &\underline{c_1 = -1} \\ &\underline{\underline{c_2 = 2 - 1 = 1}} \end{aligned}}$$

$\begin{bmatrix} 2 \\ 0 \end{bmatrix} = -1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

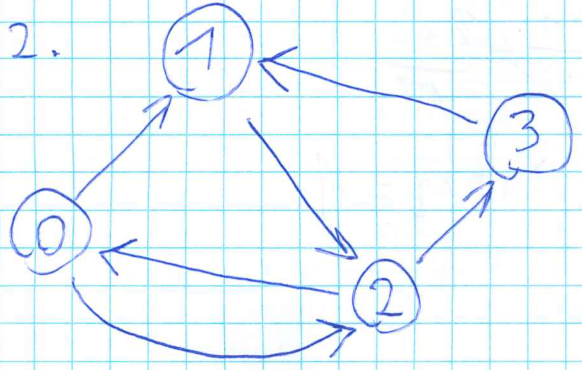
$B^n \begin{bmatrix} 2 \\ 0 \end{bmatrix} = B^n \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) =$

$= B^n \begin{bmatrix} 1 \\ -1 \end{bmatrix} + B^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1^n \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \lambda_2^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (-1)^n \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 5^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$

$= \begin{bmatrix} (-1)^n + 5^n \\ -(-1)^n + 5^n \end{bmatrix}$

the closed formula

2.



• A general formula for matrix M is

$M=(1-m)(A+D)+mS$ where m is an unknown damping factor, S is a matrix filled with $1/n$ (here $n=4$), and D is a matrix with dangling columns filled with $1/n$. There are no dangling nodes in the example, so it is a zero matrix. Let's call A the „probability matrix", and it will be calculated later.

$m=0$, so $mS=0$, $(1-m)=1$

No dangling nodes, therefore

$N=0$, which gives

$M=A$ in this example

importance scores equations
$$\begin{cases} x_0 = \frac{1}{2}x_2 \\ x_1 = \frac{1}{2}x_0 + x_3 \\ x_2 = 1/2\, x_0 + x_1 \\ x_3 = \frac{1}{2}x_2 \end{cases}$$

so $A=\begin{bmatrix} 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1 \\ 1/2 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \end{bmatrix}$

A is column-stochastic, so it has an eigenvalue $\lambda=1$. We are looking for the corresponding eigenvector (a vector such that
$$\begin{bmatrix} \lambda & 0 & -1/2 & 0 \\ -1/2 & \lambda & 0 & -1 \\ -1/2 & -1 & \lambda & 0 \\ 0 & 0 & -1/2 & \lambda \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
)

• For $\lambda=1$

$$\begin{bmatrix} 1 & 0 & -1/2 & 0 \\ -1/2 & 1 & 0 & -1 \\ -1/2 & -1 & 1 & 0 \\ 0 & 0 & -1/2 & 1 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3+\frac{1}{2}R_1]{R_2 \leftarrow R_2+\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{4} & -1 \\ 0 & -1 & 3/4 & 0 \\ 0 & 0 & -1/2 & 1 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3+R_2} \begin{bmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -1/4 & -1 \\ 0 & 0 & 1/2 & -1 \\ 0 & 0 & -1/2 & 1 \end{bmatrix} \xrightarrow{R_4 \leftarrow R_4+R_3}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -1/4 & -1 \\ 0 & 0 & 1/2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 \leftarrow 2R_3} \begin{bmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -1/4 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_2 \leftarrow R_2+1/4R_3]{R_1 \leftarrow R_1+\frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3/2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The Null space is 1-dimensional (one row of P_3) – seems about right (due to uniqueness quality)

$x_0 = t$
$x_1 = 3/2\, t$
$x_2 = 2t$
$x_3 = t$, $t \in \mathbb{R}$ ← free column

so, $v_{eigen} = t \begin{bmatrix} 1 \\ 3/2 \\ 2 \\ 1 \end{bmatrix}$

But we know that $x_0 + x_1 + x_2 + x_3 = 1$, so

$$t + 3/2\, t + 2t + t = 1$$
$$\frac{11}{2} t = 1$$
$$t = \frac{2}{11}$$

$$\Rightarrow v_{eigen} = \frac{2}{11} \begin{bmatrix} 1 \\ 3/2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/11 \\ 3/11 \\ 4/11 \\ 2/11 \end{bmatrix}$$

• The two highest-ranking pages are 2 (with ranking value $x_2 = 4/11$) and 1 ($x_1 = 3/11$). This indeed makes sense because 1 is pointed at by two lower-quality sites (3, 0), whereas 2 is pointed at by one of those sites (0), and at the same time by a high-tier website 1 itself. It is of a higher-quality because more other sites link to it than ~~[illegible]~~ in the cases of 0 and 3, and what is important to note, 3 points ~~only~~ to 1, which points only to 2, ~~[illegible]~~ transferring its whole traffic to 2. Therefore, 2 is ranked higher than 1.

Code Authors:
Simon Vanting Daneesøe
Malthe Have Musaeus
Krzysztof Parocki

Random Surfer

Random surfer runs for about 13 minutes for 60 000 000 iterations, which is the default number of iterations in the code.

Top 10 most visited nodes are:

- 367 (151019 visits)
- 249 (137810 visits)
- 145 (129372 visits)
- 264 (126265 visits)
- 266 (123305 visits)
- 127 (117535 visits)
- 123 (117310 visits)
- 1317 (116621 visits)
- 5 (116086 visits)
- 122 (115782 visits)

Page Rank

```
PAGE RANK RESULTS
The 10 highest ranked pages are:
-Rank: 0.002387909330860553, page: 367
-Rank: 0.0021844944049990685, page: 249
-Rank: 0.0020551139314652824, page: 145
-Rank: 0.0019989882113544796, page: 264
-Rank: 0.0019636118511579446, page: 266
-Rank: 0.0018635872012099492, page: 123
-Rank: 0.0018606188127957722, page: 127
-Rank: 0.0018534004542623609, page: 122
-Rank: 0.001843726167836632, page: 1317
-Rank: 0.0018312727075293594, page: 5

Converges after 38 iterations
```