Eclipse, Kliezl P. Exercise 2.2 – 3

Q: For each of the following functions, indicate the class Θ (g(n)) the function belongs to. (Use the simplest g(n) possible in your answers.) Prove your assertions.

a. $(n^2 + 1)^{10}$

b. $\sqrt{10n^2 + 7n + 3}$

c. $2n \lg(n + 2)^2 + (n + 2)^2 \lg \frac{n}{2}$

d. $2^{n+1} + 3^{n-1}$

e. $\lfloor \log_2 n \rfloor$

A:

a. $(n^2 + 1)^{10}$
$\equiv (n^2)^{10}$
$= n^{20}$
$\rightarrow \Theta(n^{20})$

Proof:

$$\lim_{n\to\infty} \frac{(n^2+1)^{10}}{n^{20}} = \lim_{n\to\infty} \left(\frac{n^2+1}{n^2}\right)^{10} = \lim_{n\to\infty} \left(1+\frac{1}{n^2}\right)^{10} = \lim_{n\to\infty} 1^{10} + \lim_{n\to\infty} \left(\frac{1}{n^2}\right)^{10} = 1+0 = 1$$

$\rightarrow$ *Same order of growth*

$$\boxed{\therefore\ (n^2 + 1)^{10} \in \Theta(n^{20})}$$

b. $\sqrt{10n^2 + 7n + 3}$
$= (10n^2 + 7n + 3)^{\frac{1}{2}}$
$= 10n^{\frac{2}{2}} + 7n^{\frac{1}{2}} + 3^{\frac{1}{2}}$
$= 10n + 7n^{\frac{1}{2}} + 3^{\frac{1}{2}}$
$\equiv n$
$\rightarrow \Theta(n)$

Proof:

$$\lim_{n\to\infty} \frac{10n + 7n^{\frac{1}{2}} + 3^{\frac{1}{2}}}{n} = \lim_{n\to\infty} 10 + \lim_{n\to\infty} \frac{10}{n^{\frac{1}{2}}} = 10 + 0 = 10$$

$\rightarrow$ *Same order of growth*

$$\boxed{\therefore\ \sqrt{10n^2 + 7n + 3} \in \Theta(n)}$$

c. $2n\lg(n+2)^2 + (n+2)^2 \lg\frac{n}{2}$

$= 2n(2\lg(n+2)) + (n+2)^2(\lg n - \lg 2)$

$= 4n(\lg(n+2)) + (n+2)^2(\lg n - 1)$

$\in \Theta(n\ log\ n) + \Theta(n^2\ log\ n)$

$\boxed{\rightarrow \Theta(n^2\ log\ n)}$

d. $2^{n+1} + 3^{n-1}$

$\in \Theta(2^n) + \Theta(3^n)$

$\boxed{\rightarrow \Theta(3^n)}$

Proof:

$$\lim_{n\to\infty} \frac{2^{n+1} + 3^{n-1}}{3^n} = \lim_{n\to\infty} \frac{2^{n+1}}{3^n} + \lim_{n\to\infty} \frac{3^{n-1}}{3^n}$$

$$= \lim_{n\to\infty} \frac{2^n \cdot 2^1}{3^n} + \lim_{n\to\infty} \frac{1}{3} = 2\left(\lim_{n\to\infty} \left(\frac{2}{3}\right)^n\right) + \frac{1}{3} = 2(0) + \frac{1}{3} = \frac{1}{3}$$

$\rightarrow$ *Same order of growth*

e. $\lfloor \log_2 n \rfloor$

$(\log_2 n) - 1 < \lfloor \log_2 n \rfloor \leq \log_2 n$

$\Rightarrow (\log_2 n) - \left(\frac{1}{2}\log_2 n\ \ (where\ n \geq 4)\right) < \lfloor \log_2 n \rfloor \leq \log_2 n$

$\boxed{\rightarrow \Theta(\log n)}$