CHAPTER 4

DAVID LI

Constraint equations: Consider the (1-dimensional) constraint equation,

$$g(q(t)) = 0. \tag{1}$$

Taking a time derivative and substituting $\dot{q} = p/m$ yields

$$\frac{d}{dt} g(q(t)) = g'(q)p/m = 0,$$

this is the cotangency condition. Taking a second time derivative yields

$$\begin{aligned} \frac{d}{dt}(g'(q)p/m) &= g''(q)\dot{q}p/m + g'(q)\dot{p}/m \\ &= g''(q)p^2/m^2 + g'(q)\dot{p}/m \\ &= 0. \end{aligned}$$

A generalization of this gives equation (4.10),

$$\boldsymbol{g}'(\boldsymbol{q}(t))\boldsymbol{M}^{-1}\dot{\boldsymbol{p}}(t) + \boldsymbol{\phi} = \boldsymbol{0}, \tag{2}$$

where $\phi_k = \boldsymbol{p}^T \boldsymbol{M}^{-1} \nabla^2 \boldsymbol{g} \boldsymbol{M}^{-1} \boldsymbol{p}$.

Symplectic structure: Let $\boldsymbol{Z} = (Z_1(t, \boldsymbol{z}), \cdots, Z_{2N_c}(t, \boldsymbol{z})) = (\boldsymbol{q}(t, \boldsymbol{z}), \boldsymbol{p}(t, \boldsymbol{z}))$, where

$$\begin{aligned} \boldsymbol{q} &= (Z_1, \cdots, Z_{N_c}) \\ \boldsymbol{p} &= (Z_{N_c+1}, \cdots, Z_{2N_c}). \end{aligned}$$

Then, $d\boldsymbol{Z} = (d\boldsymbol{q}, d\boldsymbol{p})$ and $\boldsymbol{J}d\boldsymbol{Z} = (-d\boldsymbol{p}, d\boldsymbol{q})$, and

$$\begin{aligned} d\boldsymbol{Z} \wedge \boldsymbol{J}d\boldsymbol{Z} &= (d\boldsymbol{Z})^T \boldsymbol{J}d\boldsymbol{Z} - (\boldsymbol{J}d\boldsymbol{Z})^T d\boldsymbol{Z} \\ &= (d\boldsymbol{q}, d\boldsymbol{p})^T(-d\boldsymbol{p}, d\boldsymbol{q}) - (-d\boldsymbol{p}, d\boldsymbol{q})^T(d\boldsymbol{q}, d\boldsymbol{p}) \\ &= -d\boldsymbol{q}d\boldsymbol{p} - d\boldsymbol{p}d\boldsymbol{q} + d\boldsymbol{p}d\boldsymbol{q} - d\boldsymbol{q}d\boldsymbol{p} \\ &= -2d\boldsymbol{q}d\boldsymbol{p} \end{aligned}$$