DISCONTINUOUS ODES

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1. A Discontinuous Differential Equation

Before jumping right into SDEs, we consider a way to numerically evaluate the differential equation,

$$x(t+dt) = \begin{cases} x(t)+dt & |x(t)|<1 \\ 0 & |x(t)|=1, \end{cases} \tag{1}$$

with initial condition $x(0) = x_0 \in [0,1]$. A solution to this differential equation is

$$x(t) = t + x_0 \pmod 1.$$

Proof. Let $t + x_0 \pmod 1 \neq 0$, then $dx = dt$. Let $t + x_0 \pmod 1 = 0$, then there exists $\delta t > 0$ such that

$$|x(t+\delta t)| < \epsilon,$$

for all $\epsilon > 0$, therefore $t + x_0 \pmod 1$ is a solution (in the limit). □

2. Adaptive Timesteps

A numerical approach with default timestep $h > 0$ to (1) could be achieved by Euler's method,

$$x_{k+1} = \begin{cases} x_k+h & |x_k+h|<1 \\ 0 & |x_k+h|\geq 1, \end{cases} \tag{2}$$

where $t_k = \sum_{i=1}^{k} \delta t_i$ and $\delta t_k > 0$ is an adaptive timestep $\delta t_{k+1} = \min(h, 1-x_k)$. By using an adaptive timestep, the solution attempts to correct itself at the boundaries. Then, $x_k = x(t_k)$ for all k.

Proof. Assume that $x_i = x(t_i)$ for some arbitrary $i \in \mathbb{N}$. Then by definition of (2),

$$x_{i+1} = \begin{cases} x_i+h & |x_i+h|<1 \\ 0 & |x_i+h|\geq 1. \end{cases}$$

If $x_i + h < 1$, then

$$x_{i+1} = x(t_i) + h = (t_i + x_0) \pmod 1 + h < 1.$$

Using the fact that $h = \delta t_{i+1}$,

$$\begin{aligned} x_{i+1} &= t_i + x_0 + h \pmod 1 \\ &= t_i + x_0 + \delta t_{i+1} \pmod 1 \\ &= t_{i+1} + x_0 \pmod 1 \\ &= x(t_{i+1}). \end{aligned}$$

If $x_i + h \geq 1$, then $x_{i+1} = 0$. Since

$$x(t_i + \delta t_{i+1}) = t_i + \delta t_{i+1} + x_0 \pmod 1 = x_i + 1 - x_i \pmod 1 = 0,$$

then $x(t_{i+1}) = x_{i+1}$, completing the proof by induction. □