Understanding a Stochastic Differential Equation

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This is a differential equation I borrowed from Strogatz. The equation

$$dx = \sin x dt \tag{1}$$

has attractors at $\bar{x} = k\pi$ for odd values of k. As a result, solutions with initial values between $-\pi$ and π will tend towards either of these fixed points, depending on their initial sign. This equation imitates logistic behavior when $\pm x \in [0, \pi)$.

The stochastic differential equation

$$dx = \sin x dt + \sin t dW, \tag{2}$$

where W is given by the Wiener process, has an additional time-dependent, random component, which is suppressed whenever $t \to k\pi$ for $k \in \mathbb{Z}$. Using the `ItoProcess` function provided by Mathematica, we computed a solution ensemble for the stochastic process described above.

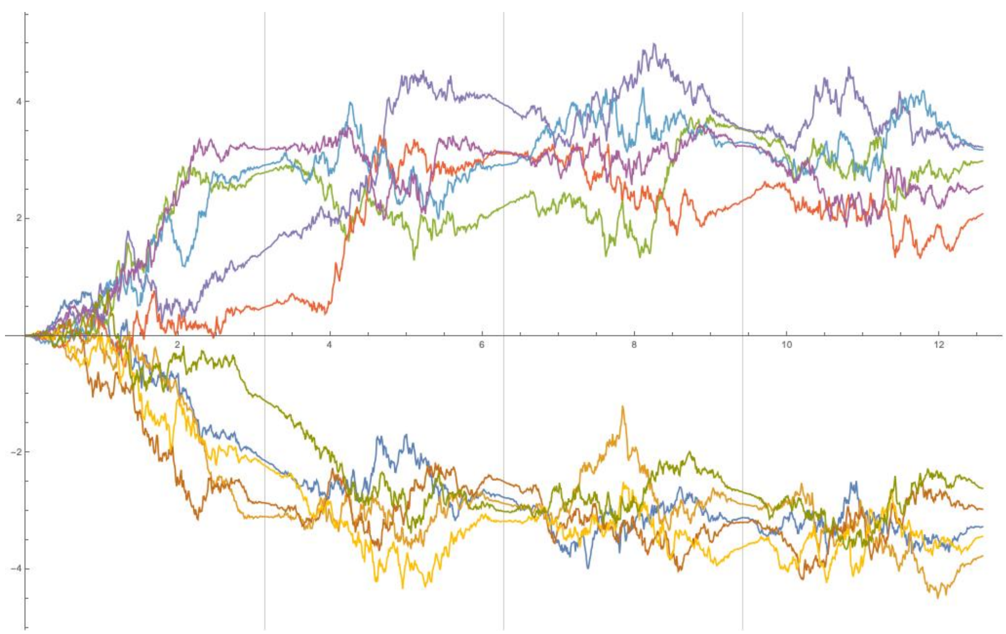

Figure 1: Ensemble of 10 solutions to $dx = \sin x dt + \sin t dW$. Each vertical line represents an integer multiple of π.

We observe the suppression $\sin t$ exerts on dW as t approaches an integer multiple of π, illustrated by a smoothing of trajectories near those points. When $t = k\pi$, the solutions locally follow the phase space described by the ordinary differential equation. Although all solutions originate from 0, a fixed point, the fixed point is unstable, and forces trajectories towards stable fixed points. Occasionally there are trajectories which even pass through the unstable fixed point and switch signs.