WEEK 2

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Exercise 1. *Derive the infinitesimal generator for the Ornstein-Uhlenbeck process,*

$$d\boldsymbol{p} = -\gamma \boldsymbol{p} dt + \sqrt{2\gamma kT}\boldsymbol{M}^{1/2} d\boldsymbol{W}. \tag{1}$$

Let $\boldsymbol{a}(\boldsymbol{p}, t) = -\gamma \boldsymbol{p}$ and $\boldsymbol{b}(\boldsymbol{p}, t) = \sqrt{2\gamma kT}\boldsymbol{M}^{1/2}$. Then

$$-\gamma \boldsymbol{p} dt + \sqrt{2\gamma kT}\boldsymbol{M}^{1/2} d\boldsymbol{W} = \boldsymbol{a}(\boldsymbol{p}, t) + \boldsymbol{b}(\boldsymbol{p}, t) = d\boldsymbol{p},$$

and the corresponding generator for the OU process is

$$\mathcal{L}\phi(\boldsymbol{p}) = \left(\boldsymbol{a}(\boldsymbol{p}) \cdot \nabla + \frac{1}{2}\boldsymbol{b}^2(\boldsymbol{p})\nabla^2\right)\phi(\boldsymbol{p}) = (-\gamma \boldsymbol{p} \cdot \nabla + \gamma kT \mathrm{tr}(M)\nabla^2)\phi(\boldsymbol{p}). \tag{2}$$

Exercise 2. *Derive the Fokker-Planck operator for the OU process.*

Recall that the operator for a stochastic process is the Kolmogorov operator,

$$\mathcal{L}^\dagger \rho = -(a(x)\rho)_x + \frac{1}{2}(b^2(x)\rho)_{xx}. \tag{3}$$

In the case of the OU process,

$$\mathcal{L}^\dagger \rho = \nabla(\gamma \boldsymbol{p}) \tag{4}$$