A PARTICULAR STOCHASTIC DIFFERENTIAL EQUATION

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1. Introduction

Consider the stochastic process,

$$X(t+dt) = \begin{cases} X(t) + dW(t) & (X(t) < \beta) \\ \alpha & (X(t) = \beta), \end{cases} \tag{1}$$

where $\alpha, \beta \in \mathbb{R}$ and $\alpha < \beta$, with initial conditions $X(0) = X_0$, and the stochastic process $dW(t)$ is defined

$$dW(t) = \xi(t)\sqrt{dt}, \tag{2}$$

where $\xi(t)$ is sampled sampled from a normal distribution with mean $\mu = 0$ and variance $\sigma^2 = 1$. We hope to find discrete stochastic processes X_n which converge strongly with (1). A candidate SDE for equation (1) would be

$$dX = -(\beta - \alpha)\delta[X(t) - \beta]dt + dW(t), \tag{3}$$

where $\delta(x)$ is the Dirac delta function. Let us become familiar with the equation and demonstrate its close association to (1). Suppose $X(\tau) = \beta$ and $t > 0$, then at the limit $t \to \tau$,

$$\begin{aligned} X(\tau) + \lim_{t\to\tau} \int_\tau^t dX &= \beta - (\beta - \alpha) \lim_{t\to\tau} \int_\tau^t \delta(X(s) - \beta) ds + \lim_{t\to\tau} W(t - \tau) \\ &= \alpha + \lim_{t\to\tau} W(t-\tau) \\ &= \alpha, \end{aligned}$$

provided that $\beta \notin \{X(s) : \tau < s \leq t\}$[1]. Recall the Itō-Doeblin formula,

$$d\phi(X) = \phi'(X)(a(X,t)dt + b(X,t)dW) + \frac{1}{2}\phi''(X)b(X,t)^2 dt, \tag{4}$$

has a stochastic generator,

$$\mathcal{L}\phi(x) = \left(a(x)\frac{d}{dx} + \frac{1}{2}b^2(x)\frac{d^2}{dx^2}\right)\phi(x).$$

Therefore, the stochastic generator for (3) is

$$\mathcal{L}\phi(x) = \left((\alpha - \beta)\delta(x - \beta)\frac{d}{dx} + \frac{1}{2}\frac{d^2}{dx^2}\right)\phi(x). \tag{5}$$

Let $\phi(x) = x$, then the expectation

$$\begin{aligned} \mathbb{E}[dX] &= \mathbb{E}[(\alpha - \beta)\delta(X - \beta)]dt \\ &= (\alpha - \beta) \int_{-\infty}^{\beta} \delta(X - \beta) dX dt \\ &= (\alpha - \beta)dt. \end{aligned}$$

[1] The justification only works for $t > \tau$. What about $t \leq \tau$?

Integrating X with respect to t[2] yields $\mathbb{E}[X(t)] = (\alpha - \beta)t$.

2. Euler Maruyama Methods

The Euler-Maruyama method is a first-order numerical approximation to most SDEs, but may not be completely accurate in this case. Nevertheless, it is useful in giving a rough intuition of what our process would look like, and expectations should also be reasonably close, too.

Let $h > 0$ be our timestep of choice and $h\nu = t$. The first scheme is as follows,

$$X_{k+1} = \begin{cases} X_k + \xi_k\sqrt{h} & (X_k < \beta), \\ \alpha & (X_k \geq \beta), \end{cases} \tag{6}$$

where $\xi_k = \xi(kh)$.

[2] This is probably illegal.