ONGOING QUESTIONS

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1. Convergence practice

Let ξ^k be sampled from a distribution with mean 0 and variance 1. Define the Wiener process at time t as

$$W(t) = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{1\le k\le \lfloor nt\rfloor} \xi^k. \tag{1}$$

Consider two Wiener processes W_1, W_2 whose respective, random variables ξ_1^k, ξ_2^k are sampled from two distributions with mean 0 and variance 1. Then, the mean error is

$$\begin{aligned}
\mathbb{E}|W_1(t) - W_2(t)| &= \frac{1}{\sqrt{n}} \cdot \mathbb{E}\Bigg| \sum_{1\le k\le \lfloor nt\rfloor} (\xi_1^k - \xi_2^k)\Bigg| \\
&= \frac{1}{\sqrt{n}} \cdot \int_\Omega \Bigg| \sum_{1\le k\le \lfloor nt\rfloor} x(\rho_1(x) - \rho_2(x))\Bigg| dx \\
&= \frac{1}{\sqrt{n}} \cdot \sum_{1\le k\le \lfloor nt\rfloor} \Big[\int_{\Omega_1} dx - \int_{\Omega_2} dx\Big][x(\rho_1(x) - \rho_2(x))] \\
&= \frac{\lfloor nt\rfloor}{\sqrt{n}} \Big[\int_{\Omega_1} dx - \int_{\Omega_2} dx\Big][x(\rho_1(x) - \rho_2(x))].
\end{aligned}$$

where $\Omega_1 = \{x : x(\rho_1(x) - \rho_2(x)) \ge 0\}$ and $\Omega_2 = \Omega_1^c$. We see that two arbitrary processes do not necessarily converge strongly.

Let ξ_1 sample from -1 and 1 uniformly, while ξ_2 samples from a standard normal distribution, so that $\rho_2 = \psi$. To extend the support of ξ_1 onto the reals, we can use the pdf,

$$\rho_1(x) = \frac{1}{2}(\delta(x+1) + \delta(x-1)). \tag{2}$$

Suppose $x \le 0$ and $x \ne -1$. Since $\rho_1(x) = 0$, then $x(\rho_1(x) - \psi(x)) \ge 0$. If $x = -1$, then $\rho_1(x) = \infty$ and $x(\rho_1(x) - \psi(x)) \ge 0$. Thus, $\Omega_1 = \{x : x \le 0\}$. Likewise, $\Omega_2 = \{x : x > 0\}$. Therefore,

$$\begin{aligned}
\Big[\int_{\Omega_1} dx - \int_{\Omega_2} dx\Big][x(\rho_1(x) - \psi(x))] &= \Big[\int_{x\le 0} dx - \int_{x>0} dx\Big][x(\rho_1(x) - \psi(x))] \\
&= \int_{x\le 0} x\rho_1(x)dx - \int_{x>0} x\rho_1(x)dx \\
&\quad - \int_{x\le 0} x\psi(x)dx + \int_{x>0} x\psi(x)dx \\
&= -1 + 2\sqrt{\frac{2}{\pi}},
\end{aligned}$$

so these two processes don't converge on the mean.

2. What we know

Consider the stochastic process $X(t)$ which satisfies the following,

$$X(t+dt) = \begin{cases} X(t) + dW(t) & X(t) < b \\ a & X(t) = b. \end{cases} \tag{3}$$

where $a, b \in \mathbb{R}$ and $a < b$. The stochastic process $dW(t)$ is defined

$$dW(t) = \xi(t)\sqrt{dt}, \tag{4}$$

where the differential $dW(t)$ satisfies the diffusion limit $dt = dx^2$, and $\xi \sim \mathcal{N}(0,1)$ is our choice of random variable.

Some ideas that could probably work.

- If $X(t) = b$, then for all $\epsilon > 0$, there exists $\delta > 0$ such that $|X(t+\delta) - a| < \epsilon$. This describes how $X(t)$ "jumps" from b to a.
- (Adaptive timestep) If $X_n < b$ and $\xi_n \sim \mathcal{N}(0,1)$, there exists a timestep $\delta t_n > 0$ such that $X_n + \xi_n\sqrt{\delta t_n} \leq b$. Specifically, let $\delta t_n = \left(\frac{b - X_n}{\xi_n}\right)^2$. This method says that if X_n "overshoots" beyond b, then we have a way to temporarily reduce the timestep so that doesn't happen, while respecting the diffusion limit constraint. But the problem with this method is that it may introduce a downwards pressure on overall results.
- (Adaptive distribution) Suppose instead $\delta t_1 = \cdots = \delta t_n > 0$ is fixed. Then, the random variable ξ_n must be sampled from the support
$$-\infty < \xi_n \leq \frac{b - X_n}{\sqrt{\delta t_n}} = M_n,$$
where $M_n = (b - X_n)/\sqrt{\delta t_n}$. The corresponding density function is a truncated normal distribution,
$$\psi_n(x) = \frac{1}{Z_n}\psi(x)[-\infty < x \leq M_n],$$
where ψ is the pdf for the normal distribution, and
$$Z_n = \int_{-\infty}^{M_n} \psi(x)dx$$
is the partition function. The problem with this method is that it unrealistically reduces the chances of X_n evolving to b.

3. Properties of this SDE

Ideally we would want to formalize the behavior of this SDE and its properties before comparing different methods' accuracies.