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AN ANALYTIC EXPRESSION FOR THE LUMINOSITY FUNCTION FOR GALAXIES*

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ABSTRACT

A new analytic approximation for the luminosity function for galaxies is proposed, which shows good agreement with both a luminosity distribution for bright nearby galaxies and a composite luminosity distribution for cluster galaxies. The analytic expression is proportional to $L^{-5/4}e^{-L/L^*}$, where L^* is a characteristic luminosity corresponding to a characteristic absolute magnitude $M^*_{B(0)} = -20.6$. For an individual cluster, the characteristic magnitude may be determined with an accuracy of ~ 0.25 mag, suggesting its use as a standard candle. The analytic expression is used to compute an expected richness–absolute magnitude correlation for first ranked cluster galaxies and an expected dispersion, which are compared with the data of Sandage and Hardy.

Subject headings: galaxies: clusters of — galaxies: photometry

I. INTRODUCTION

For a wide range of extragalactic problems one needs to know the luminosity function for galaxies. For example, the spatial covariance function for galaxies can be obtained from the projected angular covariance function only if one specifies the luminosity function (Peebles and Hauser 1974). Likewise, it is necessary to the determination of evolutionary and cosmological corrections to the number-magnitude relation for galaxies (Brown and Tinsley 1974). If one specifies the dependence of mass on luminosity, one can use it to determine the local mass density (Shapiro 1971), or the mean binding energy in pairs of galaxies in a magnitude-limited sample (Geller and Peebles 1973). The luminosity function allows one to estimate the frequency of absorption lines in QSOs due to intervening galaxies (Bahcall 1975) and to estimate the available parent population for exotic objects such as Markarian (Huchra and Sargent 1972) or radio galaxies (Schmidt 1966). It may be used to extrapolate observed luminosities in clusters of galaxies to total luminosities (Oemler 1974) and can be used to determine the distances to clusters (Abell 1962; Schechter and Press 1975). Most of these problems require integration of the luminosity function over a range of volumes and luminosities. While such integration can always be carried out numerically by using the observed luminosity function, the calculation is time consuming, and one frequently adopts an analytic expression which is taken to be a reasonable approximation to the luminosity function.

We propose here a new analytic approximation for the luminosity function for galaxies. Letting $\varphi(L)dL$ be number of galaxies per unit volume in the luminosity interval from L to $L + dL$, we investigate the expression

$$\varphi(L)dL = \varphi^*(L/L^*)^\alpha \exp(-L/L^*)d(L/L^*) \qquad (1)$$

where φ^*, L^*, and α are parameters to be determined from the data. The parameter φ^* is a number per unit volume, and L^* is a "characteristic luminosity" (with an equivalent "characteristic absolute magnitude," M^*) at which the luminosity function exhibits a rapid change in the slope in the ($\log \varphi$, $\log L$)-plane. The existence of such a characteristic magnitude has long been stressed by Abell (1962, 1965), and his notation M^* has been pirated for the present discussion. The dimensionless parameter α gives the slope of the luminosity function in the ($\log \varphi$, $\log L$)-plane when $L \ll L^*$.

The proposed representation derives from a self-similar stochastic model for the origin of galaxies (Press and Schechter 1974) but differs in that we allow ourselves the latitude of adjusting the "faint-end slope parameter" α to fit the available data. The thrust of this work is not therefore to argue the merits of the stochastic model, but only the merits of the expression as a good approximation to the luminosity function.

Equation (1) differs from previous analytic expressions proposed by Zwicky (1957), Kiang (1961), Abell (1962, 1965), and Arakelyan and Kalloglyan (1968) in that the luminosity appears in an exponential as well as in a power law. Expressed in terms of absolute magnitude, equation (1) exhibits a *double* exponential. This rapid cutoff at brighter absolute magnitudes is consistent with the statistical interpretation (Peebles 1968) of the narrow range of absolute magnitudes found for the brightest members of rich clusters of galaxies (e.g., Sandage 1968).

Our first job is to demonstrate that the proposed expression is, in fact, a good approximation to the luminosity function for galaxies. To this end, we construct, in § II, a general luminosity distribution for galaxies using data taken primarily from the *Reference Catalogue of Bright Galaxies* (de Vaucouleurs and de Vaucouleurs 1964). A composite luminosity distribution of cluster galaxies is also constructed, using

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Oemler's (1974) data. In § III we fit the proposed expression to these data, showing that equation (1) fits both luminosity distributions well in a sense made precise in § II*d*. We find that the parameters α and L^* for field and cluster galaxies show no significant difference. In § IV, we test the proposed expression against data for individual clusters and examine the accuracy with which L^* can be determined for a rich cluster of galaxies. The expected correlation of the absolute magnitude of brightest cluster member with richness is treated in § V.

II. OBSERVED LUMINOSITY DISTRIBUTIONS

a) *Luminosity Distributions and Luminosity Functions*

It is helpful to distinguish between the terms "luminosity distribution" and "luminosity function." For the present discussion a luminosity distribution $n_S(L)$ has the units of number of galaxies per unit luminosity and refers to a specific sample of galaxies S. Thus if one has a sample S

$$n_S(L)\Delta L \equiv \begin{pmatrix} \text{the number of galaxies contained} \\ \text{in } S \text{ in the luminosity interval of} \\ \text{width } \Delta L \text{ centered on } L \end{pmatrix}. \quad (2)$$

For a specific sample S the volume sampled at luminosity L is given by $V_S(L)$. The "luminosity function" $\varphi_S(L)$ of a sample of galaxies S has the units of number of galaxies per unit luminosity per unit volume and is defined by

$$\varphi_S(L)\Delta L \equiv \frac{n_S(L)\Delta L}{V_S(L)}\,. \quad (3)$$

If one is willing to assume that the Universe is homogeneous on large scales, then in the limit of large, randomly chosen sample volumes, all luminosity functions approach a universal limit $\varphi(L)$ defined by

$$\varphi(L) \equiv \lim_{V_S(L)\to\infty} \varphi_S(L)\,. \quad (4)$$

We shall henceforth assume that the Universe is homogeneous on large scales and refer to this limit as the luminosity function for galaxies. In practice, one can only determine the luminosity function for finite samples, and sample luminosity functions will show deviations from the universal luminosity function which decrease as sample volumes increase. The size of these deviations depends upon the nature of the processes giving rise to the distribution of galaxies in space and luminosity. For a randomly chosen sample volume, the luminosity function yields an expected luminosity distribution

$$n_e(L) \equiv \varphi(L)V_S(L)\,. \quad (5)$$

Luminosity functions (and distributions) may be obtained for any subclass of galaxies which can be identified by criteria other than luminosity. Hence it is possible to find luminosity functions for elliptical galaxies (Shapiro 1971), Markarian galaxies (Huchra and Sargent 1972), and cluster galaxies. The luminosity function for all galaxies will be called the "general" luminosity function.

b) *A General Luminosity Distribution*

We present here a general luminosity distribution, obtained using galaxies listed in de Vaucouleurs and de Vaucouleurs (1964). The sample is much the same as that used by Shapiro (1971) and Christiensen (1975). The redshift data for the present sample are somewhat more complete, and the use of redshift as the sole distance indicator for the present sample (cf. Christiensen) allows easy application of the Eddington correction for the uncertainties in the derived absolute magnitudes.

The sample includes all galaxies brighter than $m_{B(0)\,\mathrm{lim}} = 11.75$ listed in the *Reference Catalogue* with new galactic latitude greater than 30° from the galactic plane, but (following Shapiro) excluding all galaxies within 6° of the center of the Virgo cluster at $\alpha = 12^{\mathrm{h}}27^{\mathrm{m}}$ and $\delta = 13^\circ\!.5$ (de Vaucouleurs 1961). The large velocity dispersion of galaxies in the direction of the Virgo cluster makes redshift a poor distance indicator in this region. Absolute magnitudes were computed as follows:

$$M_{B(0)} = m_{B(0)} - 25 - 5\log(cz/H_0) - A_B \csc b\,. \quad (6)$$

A Hubble constant of 50 km s^{-1} Mpc^{-1} and an absorption coefficient A_B of 0.12 mag (Peterson 1970*a*) were used. The sample volume at $M_{B(0)}$ is given by

$$V[M_{B(0)}] = \tfrac{4}{3}\pi\,\mathrm{dex}[0.6(m_{B(0)\,\mathrm{lim}} - M_{B(0)} - 25)] \times \int_{|b_{\mathrm{lim}}|}^{90^\circ} d(\sin b)\,\mathrm{dex}(-0.6A_B \csc b)\,. \quad (7)$$

After a change of variables the integral on the right-hand side becomes an incomplete gamma function. For $|b_{\mathrm{lim}}| = 30°$ and $A_B = 0.12$ its value is 0.3978. Had we assumed no absorption, its value would have been one-half.

Redshifts were taken from the *Reference Catalogue* with the following exceptions: (1) Neutral hydrogen velocities of Lewis and Robinson (1973) were used for members of the South Polar Group of galaxies; (2) radial velocities published after 1964 were found for 12 of the 20 galaxies in the sample without cataloged velocities (see Table 1); (3) of the remaining eight galaxies, five have been identified as members of groups by de Vaucouleurs (1976) and have been assigned the mean group radial velocities; (4) NGC 5054 was assumed to be associated with NGC 5049 and assigned the corresponding redshift. All velocities were corrected for solar motion according to de Vaucouleurs and de Vaucouleurs (1964).

TABLE 1

REFERENCE CATALOG GALAXIES BRIGHTER THAN $m_{B(0)} = 11.75$ AND WITH $|b^{\mathrm{II}}| > 30°$ WITHOUT CATALOGUED RADIAL VELOCITIES

A.

ID	Velocity	Reference
NGC 1326....	1233	(*a*)
NGC 1532....	1587	(*b*)
NGC 1559....	1284	(*b*)
NGC 1672....	1034	(*b*)
NGC 1792....	1035	(*c*)
NGC 4096....	540	(*a*)
NGC 4145....	1035	(*d*)
NGC 4236....	186	(*e*)
NGC 4651....	685	(*c*)
NGC 4654....	960	(*c*)
NGC 4939....	2862	(*c*)
NGC 5247....	1530	(*f*)

B.

ID	Group Velocity	Group Identification
NGC 1448....	665	G21
NGC 1617....	999	G16
NGC 4395....	342	G3
NGC 5054....	2597	NGC 5049
NGC 7424....	1561	G27
IC 5332.......	142	G1
IC 5201.......	...	None
A58..........	...	Local

REFERENCES.—(*a*) Bottinnelli *et al.* 1970; (*b*) Carranza 1967; (*c*) de Vaucouleurs and de Vaucouleurs 1967; (*d*) Chincarini and Rood 1972; (*e*) Rogstad *et al.* 1967; (*f*) Balkowski *et al.* 1973.

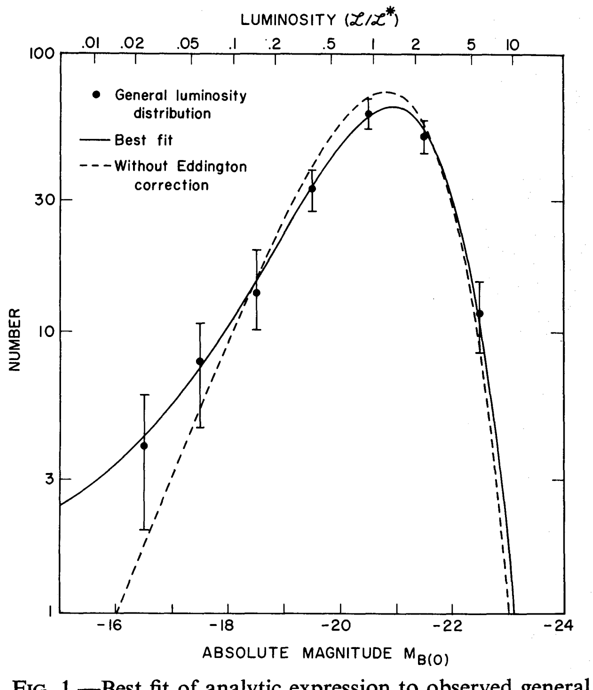

FIG. 1.—Best fit of analytic expression to observed general luminosity distribution. Broken line shows the effect of deleting the Eddington correction.

The resulting luminosity distribution is shown in Figure 1, where, for convenience, we have presented $n(M_{B(0)})$ rather than $n(L)$. In all, 184 galaxies are included in Figure 1. Of the eight galaxies in the sample not shown, two had no redshift, one was fainter than $M_{B(0)} = -16$, and five had blueshifts.

The observation of blueshifts implies the presence of non-Hubble components in the observed radial velocities. These components introduce uncertainties in the computed absolute magnitudes, particularly at the faint end of the distribution. The problem is analogous to that encountered in stellar astronomy in determining luminosity distributions for stars with uncertain parallaxes. The Eddington method can be used to correct the distribution for this effect, but we shall postpone our discussion of the problem until § III.

The sample luminosity function $\varphi_S(L)$ may be calculated from the luminosity distribution of Figure 1. The resulting luminosity function does not differ substantially from that of van den Bergh (1961), Shapiro (1971), or Christiensen (1975). We shall leave the data in the form of a luminosity distribution to facilitate the application of the Eddington method.

c) *A Cluster Galaxy Luminosity Distribution*

A composite luminosity distribution for cluster galaxies is constructed here from Oemler's (1974) individual luminosity distributions. For observations of clusters the volume sampled is the same at all luminosities, and the luminosity distribution will have the same shape as the luminosity function. Indeed the terms are used interchangeably in the literature on clusters of galaxies, but as defined here a luminosity function should have the units of density.

The composite luminosity distribution has been constructed using the luminosity distributions for 13 of the 15 rich clusters studied by Oemler (1974). Two of the clusters, Abell 2670 and Zw Cl 1545.1+2104 have not been included. Since both clusters were studied at extremely faint apparent magnitudes ($m_J > 20$), background corrections were substantially larger for these clusters than for the other clusters studied. At the faintest magnitudes sampled in Abell 2670 the background subtracted was twice as large as the number of cluster galaxies (Oemler 1973). Oemler has noted that the cluster Abell 2670 is substantially flatter at the faint end than the other clusters studied (Oemler 1974). This cluster received a somewhat different treatment from the others studied in two additional respects: (*a*) the core region studied was only 0.4 Abell radii, compared with ~1 Abell radius for the others; (*b*) the seeing was so poor on the plate studied that faint stars and galaxies could not be distinguished (Oemler 1973), and a correction for foreground stars was therefore necessary. *The exclusion of Abell 2670 is nonetheless post hoc.*

The construction of the composite distribution proceeded as follows: The luminosity distributions

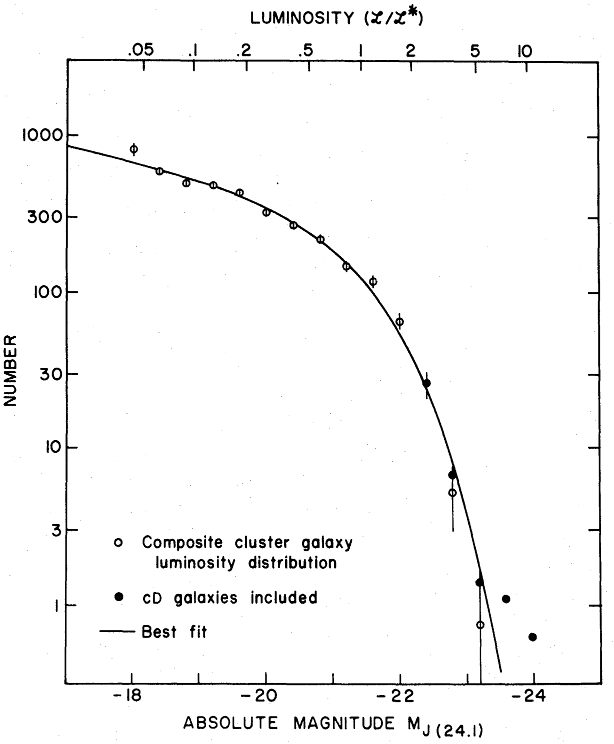

FIG. 2.—Best fit of analytic expression to observed composite cluster galaxy luminosity distribution. Filled circles show the effect of including cD galaxies in composite.

for the 13 clusters were ranked by the depth of the distribution, with the cluster containing the galaxy with the faintest absolute magnitude ranked first. The data for the second ranked cluster was "rebinned" so that the bins matched the bins of the first and were then added to the first. Data in the first cluster beyond the limiting magnitude of the first cluster were scaled by the ratio of the total number of galaxies brighter than this limiting magnitude in both clusters to the corresponding total for the second cluster. The procedure was then repeated, adding the third through thirteenth clusters.

The resulting composite luminosity distribution is plotted in Figure 2. The open circles indicate the composite constructed excluding four possible cD galaxies (two in Coma and one each in Abell 1413 and Abell 2199), and the filled circles show the effect of including these galaxies.

d) *Uncertainties*

In the following section we shall show that the proposed analytic expression fits the observed luminosity distributions "within the observational uncertainties." The data shown in Figures 1 and 2 are uncertain due to uncertainty in the completeness of the sample and, in Figure 2, uncertainty in the amount of subtracted background. In addition, if one assumes that the distribution of galaxies is governed by a stochastic process, there will be "random" uncertainties in the observed numbers of galaxies in each magnitude interval. In particular, if we assume that the distribution is governed by a Poisson process, we expect uncertainties of the order of $\sqrt{N}$ where N is the number of galaxies in a given bin (before scaling). While the adoption of "square root of N" uncertainties appears harmless enough the "Poisson hypothesis" leads inevitably to "Scott effects" (see § V). The error bars in Figures 1 and 2 were computed assuming *only* $\sqrt{N}$ uncertainties. Thus we shall be testing whether the proposed analytic expression fits the observed distributions on the assumption that the number of galaxies per magnitude bin is governed by a Poisson process. Since the process governing galaxy luminosity may not be a Poisson process, it should be noted that the assumption of uncertainties proportional to N° and N^1 yield values for the parameters close to those obtained by assuming $N^{1/2}$ uncertainties.

III. χ^2 FITS OF ANALYTIC EXPRESSION TO GENERAL AND COMPOSITE LUMINOSITY DISTRIBUTIONS

a) *General Luminosity Function*

The proposed analytic expression and the observed local luminosity distribution are related by equations (5) and (7). Substituting, we find

$$n_e(L)dL = \varphi^* V^*(L/L^*)^{\alpha+3/2} \exp(-L/L^*)d(L/L^*) \quad (8)$$

where $V^* = V(L^*)$. This expression may now be corrected for uncertainties in the luminosities due to non-Hubble components in the redshifts. If $\sigma(L)$ is the r.m.s. uncertainty at luminosity L, then according to the modified Eddington method (cf. Trumpler and Weaver 1953),

$$n_{e1}(L) = n_e[1 + \sigma'^2 + \sigma''\sigma] + 2n_e'\sigma'\sigma + n_e''\sigma^2/2 + \cdots, \quad (9)$$

where primes indicate derivatives with respect to luminosity. We must of course specify $\sigma(L)$. If $\langle\Delta v^2\rangle^{1/2}$ is the rms non-Hubble velocity and l is the limiting flux of the sample, then

$$\sigma(L) = (1.08)(\sqrt{3})2\left(\frac{\langle\Delta v^2\rangle lL}{4\pi H_0^2}\right)^{1/2}, \quad (10)$$

where the first factor in parentheses comes from averaging over galactic latitude with $A_B = 0.12$ and the second factor in parentheses comes from averaging over all velocities in a given luminosity bin.

The rms deviation of line-of-sight velocities from the Hubble flow $\langle\Delta v^2\rangle^{1/2}$ is a difficult quantity to determine. Sandage and Tammann (1975) give an upper limit of 50 km s^{-1}, but this quantity refers to the average deviation of groups of galaxies. Geller and Peebles (1973) give rms *differences* for pairs of galaxies ranging from 150 to 300 km s^{-1} depending upon separation and the correction for measurement errors. The corresponding value of $\langle\Delta v^2\rangle^{1/2}$ would be $\sqrt{2}$ smaller. Humason *et al.* (1956) suggest a value of $\langle\Delta v^2\rangle^{1/2}$ of 200–300 km s^{-1} while de Vaucouleurs (1958) suggests an upper limit of 100 km s^{-1}. For the present work we adopt the (perhaps somewhat high) value of 200 km s^{-1}.

The data in Figure 1 have been binned in rather large (1 mag) intervals to assure a substantial number of counts per bin. Since $n_{e1}(M)$ changes substantially over a magnitude, we must correct for such variation according to

$$n_{e2}(M) = n_{e1} + n_{e1}''(\Delta M)^2/24\,, \qquad (11)$$

where primes denote derivatives with respect to absolute magnitude.

Values of the parameters α, L^*, and $\varphi^* V^*$ were obtained by minimizing the quantity χ^2 defined by

$$\chi^2 \equiv \sum_i \frac{[n(M_i) - n_{e2}(M_i)]^2}{\sigma_i^2}\,, \qquad (12)$$

where σ_i is the uncertainty in the ith data bin. The values of the parameters which minimize χ^2 are

$$\varphi^* V^* = 216 \pm 6\,,$$
$$M^*_{B(0)} = -20.60 \pm 0.11\,, \qquad (13)$$
$$\alpha = -1.24 \pm 0.19\,,$$

giving a minimum χ^2 of 0.53 with 4 degrees of freedom. The uncertainties in equation (11) (and all subsequent uncertainties obtained from χ^2 fits) are "unbiased estimates," obtained by scaling the matrix $(\partial^2\chi^2/\partial x_i \partial x_j)^{-1}$ by the quantity χ^2/ν and taking the square roots of the diagonal elements (Wolberg 1967). The poorer the fit, the larger the uncertainties. The correlation coefficients are given by

$$\rho_{\varphi^*V^*, M^*} = 0.240; \qquad \rho_{\varphi^*V^*, \alpha} = 0.231;$$
$$\rho_{M^*, \alpha} = 0.973\,. \qquad (14)$$

The solid line in Figure 1 shows the solution for $n_{e2}(M)$. The broken line plots $n_{e2}(M)$ using the same values for the parameters but letting $\langle \Delta v^2 \rangle^{1/2} = 0$. The difference between the two curves demonstrates the effect of the Eddington correction. A second way to gauge the importance of the correction is to fit the observed luminosity distribution using values for the non-Hubble component other than our assumed value of 200 km s^{-1}. Using values of 150 and 100 km s^{-1}, we obtain values for α of -1.35 and -1.43, respectively. It should be noted that the first-order Eddington correction is barely adequate to our task since the assumed non-Hubble velocity components are of roughly the same size as the mean velocity in the faintest bin.

The sample and expected luminosity distributions, $n_s(L)$ and $n_e(L)$ will be nearly identical if the Universe is homogeneous on the scales sampled. But Hauser and Peebles (1974) have shown that inhomogeneities exist on scales at least as large as 40 Mpc. Worse yet, our sample volume has not been chosen randomly: our observatories are located in a galaxy, and there is likely to be an excess of galaxies in our vicinity. One might use the covariance function for galaxies (Peebles 1974) to subtract off a mean expected excess, but the fluctuations about this mean may well be large. These effects and the non-Hubble velocities combine to make the faint end slope very uncertain.

b) The Cluster Galaxy Luminosity Distribution

The composite luminosity distribution (without cD galaxies) has been fitted in a similar manner to obtain values of α and L^* for cluster galaxies. Since we have no way of estimating the volume being sampled, we can only obtain the parameter n^* such that the expected luminosity distribution is given by

$$n_e(L)dL = n^*(L/L^*)^\alpha \exp(-L/L^*)d(L/L^*)\,. \quad (15)$$

The data of Figure 2 have been binned twice: once by Oemler in his Figure 5 (Oemler 1974), and again here, in Figure 2. If we assume that the location of the second set of bins is independent of the first, we find (cf. eq. [11])

$$n_{e3}(M) = n_e + n_e''(\Delta M)^2/8\,, \qquad (16)$$

where primes denote derivatives with respect to magnitude.

The best fit was obtained for the following values of the parameters:

$$n^* = 910 \pm 120\,,$$
$$M^*_{J(24.1)} = -21.41 \pm 0.10\,, \qquad (17)$$
$$\alpha = -1.24 \pm 0.05\,,$$

with a χ^2 of 16.4 for 11 degrees of freedom. Since we are unable to estimate uncertainties in the amount of subtracted background, this is a reasonable fit given our estimated uncertainties. The correlation coefficients are

$$\rho_{n^*, M^*} = 0.928; \qquad \rho_{n^*, \alpha} = 0.939;$$
$$\rho_{M^*, \alpha} = 0.823\,. \qquad (18)$$

The solid curve in Figure 2 shows $n_{e3}(M)$ computed using the above values of the parameters. Note that the cD galaxies have luminosities of up to 10 L^* when the above values of the parameters are used. Galaxies as luminous as this are exceedingly improbable if one accepts the proposed analytic expression. The expression therefore gives a good approximation only to the *non-cD* cluster galaxy luminosity function. This presents no problem if cD galaxies can be identified by morphological rather than luminosity criteria (Matthews *et al.* 1964; Morgan and Lesh 1965). *If they cannot, our reasoning is circular: we are throwing them out because they do not fit a curve obtained by excluding them!*

c) Comparison of Parameters for General and Cluster Luminosity Distributions

The apparent magnitudes used to construct the general and cluster luminosity distributions were based on two different photometric systems: the de Vaucouleurs' $B(0)$ system (de Vaucouleurs and de Vaucouleurs 1964) and Oemler's $J(24.1)$ system (Oemler 1974). Ideally to compare the two luminosity distributions, one would transform the individual apparent magnitudes from one system to the other, construct a new luminosity distribution, fit the two distributions, and compare values of the characteristic absolute magnitude and faint-end slope parameter. A cruder comparison is obtained by correcting the parameters after fitting the data, but such a comparison is less reliable. The comparison presented here is yet cruder: we correct the characteristic magnitudes to a common system but not the faint end slopes.

Oemler (1974) gives a correction of roughly 0.20 mag from his $J(24.1)$ magnitudes to total magnitude at $M_{J(24.1)}) \approx -21.5$. Adding to this another 0.11 mag for absorption at the galactic poles, we have

$$M^*_{JT} = M^*_{J(24.1)} - 0.31\,. \tag{19}$$

The de Vaucouleurs's $B(0)$ magnitudes require roughly a 0.5 mag correction to total magnitude (de Vaucouleurs and de Vaucouleurs 1964) and a correction of 0.65 $(B - V)$ to the J band (Oemler 1974). Using the mean $(B - V)$ for our bright-galaxy sample of 0.75,we obtain a correction

$$M^*_{JT} = M^*_{B(0)} - 0.99\,. \tag{20}$$

The curves in Figure 3 show 50 percent confidence ellipses in the (α, M^*_{JT})-plane, obtained using the formal uncertainties and correlation coefficients of the χ^2 fits. The parameters for the two distributions agree fairly well, consistent with the hypothesis that the general luminosity function and the cluster-galaxy luminosity function differ only by a multiplicative factor. This similarity of the two distributions has been noted before by Peebles (1971).

The theoretician's job is made a little easier if he can remember a few round numbers. We do no great injustice to the data if we adopt a working value of

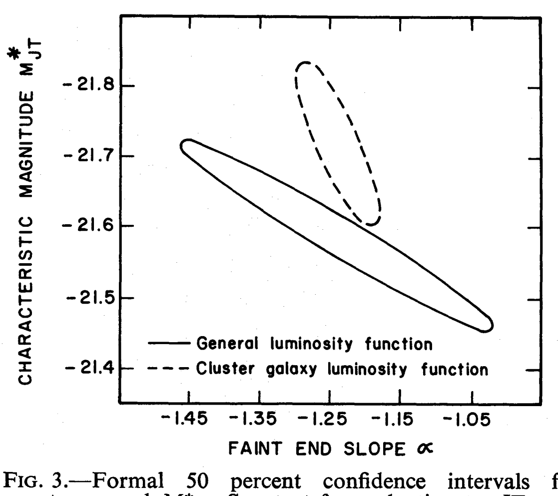

FIG. 3.—Formal 50 percent confidence intervals for parameters α and M^*_{JT}. See text for reduction to JT magnitude system.

$\alpha = -5/4$ and use values of $M^*_{B(0)}$, $M^*_{J(24.1)}$ and φ^* appropriate to this value:

$$\begin{aligned} \alpha &= -5/4\,, \\ M^*_{B(0)} &= -20.6 + 5\log(\mathrm{H_0}/50)\,, \\ M^*_{J(24.1)} &= -21.4 + 5\log(\mathrm{H_0}/50)\,, \\ \varphi^* &= 0.005(\mathrm{H_0}/50)^3\ \mathrm{Mpc}^{-3}\,. \end{aligned} \tag{21}$$

The uncertainties in these values are of the same order as those given in equations (13) and (17). By way of comparison, we note that for a magnitude-limited sample of galaxies, the luminosity distribution peaks at $5L^*/4$ (see Fig. 1). For rich clusters of galaxies, the first ranked cluster member is likely to be a magnitude brighter than M^* (see §V). If we adopt a distance modulus for M31 of $(m - M) = 24.72$ (Sandage and Tammann 1971) and an apparent magnitude $m_{B(0)} = 4.61$ (de Vaucouleurs and de Vaucouleurs 1964), we obtain an absolute magnitude of $M_{B(0)} = -20.11$. This gives M31 a luminosity of roughly $\frac{2}{3}L^*$. Figure 2 shows that cD galaxies have luminosities of the order of 5–10 L^*.

IV. LUMINOSITY DISTRIBUTIONS FOR INDIVIDUAL CLUSTERS

a) *Test of the Proposed Expression*

It remains to be seen whether the proposed analytic expression gives a good approximation to the observed luminosity distributions for individual clusters of galaxies, again assuming that "square root of N" uncertainties dominate the uncertainties. The data for individual clusters are so uncertain as to guarantee that we can obtain a good fit for *some* values of α and L^*. More interesting is a test of whether universal values of α and L^* are valid for all clusters. If universal values are appropriate, then only n^* will vary from cluster to cluster.

We have fitted the proposed expression to Oemler's (1974) luminosity distributions for individual clusters, fixing α and L^* at the working values and letting n^* vary. The results of these fits are shown in Table 2. While the expression does fit many of the clusters well, it fits others rather poorly, as evidenced by values of χ^2 per degree of freedom ν much larger than unity. The quality of these fits might have been better had it been possible to include estimates of uncertainties due to background subtraction. It should be noted that the two clusters giving the worst fits are those with flat faint ends noted by Oemler (1974): Abell 2670 and Abell 665. For the remaining discussion we shall adopt the hypothesis that universal values of α and L^* apply to *all* individual clusters, although further careful observational work is required to check this important assumption.

b) *n^* as a Measure of Cluster Richness* (*one-parameter fits*)

If equation (15) does give a good approximation to the luminosity distributions for clusters of galaxies, and if α and L^* are given by the working values, then

TABLE 2

PARAMETERS FOR FOURTEEN CLUSTERS ($\alpha = -5/4$)

	One-Parameter Fits ($M^* = 21.43$)			Two-Parameter Fits	
Cluster	n^*	$\sigma(n^*)$	χ^2/ν	M^*	$\sigma(M^*)$
A194........	19	3	1.8	−22.32	0.64
A400........	33	6	0.4	−21.41	0.33
A539........	41	3	0.7	−21.26	0.22
A665........	109	25	6.2	−22.52	0.66
A1228.......	39	4	0.7	−21.32	0.25
A1314.......	45	2	0.2	−21.41	0.11
A1367.......	46	5	1.8	−21.30	0.37
A1413.......	113	11	1.0	−21.36	0.24
A1656.......	107	8	2.8	−21.26	0.20
A1904.......	78	6	0.4	−21.50	0.17
A2151.......	60	11	0.5	−21.52	0.17
A2197.......	46	11	3.1	−20.63	0.44
A2199.......	75	6	1.3	−21.06	0.16
A2670.......	55	7	4.5	−22.09	0.53

n^* will be proportional to the cluster luminosity, and is a measure of cluster richness. Integrating over all luminosities, we obtain

$$L_{\text{clus}} = \int_0^\infty Ln(L)dL = n^*\Gamma(\alpha + 2)L^* \,. \quad (22)$$

For the working value $\alpha = -5/4$, the gamma function takes the value $\Gamma(\frac{3}{4}) = 1.225$. Using the working value $M^*_{J(24.1)} = -21.4$, we obtain from equation (22) the total luminosity *inside the J(24.1) isophote.* Listed in Table 2 are values of n^* obtained from one-parameter fits to Oemler's data using the working values of α and L^*. The uncertainties given are unbiased estimates: clusters for which the working values fit the data poorly have larger uncertainties.

The poorest of the clusters studied by Oemler have a value of n^* of order 20 while the richest three clusters all have values of order 115. If Oemler's sample is typical of rich clusters, a reasonable range of values of n^* appears to be

$$20 \lesssim n^* \lesssim 115 \,. \quad (23)$$

The reader is cautioned that the value of n^* depends upon how one determines the limits of a cluster. Oemler counted galaxies out to two or three Abell radii (Abell 1958). One would therefore expect smaller values of n^* within one Abell radius.

c) L^* as a Standard Candle (two-parameter fits)

If the working values of α and L^* are universally appropriate to individual clusters, one can use M^* as a synthetic standard candle to determine distance moduli to clusters. The potential of such a synthetic standard candle has been stressed by Abell (1962, 1965). Abell found that the integrated luminosity distributions $N(\geq L)$ for individual clusters could be approximated by two power laws of luminosity. If we let L_A^* be Abell's characteristic luminosity, then

$$\begin{aligned} N(\geq L) &= N^*(L/L_A^*)^{1+\beta_A} \quad (L > L_A^*) \\ &= N^*(L/L_A^*)^{1+\alpha_A} \quad (L < L_A^*) \,, \end{aligned} \quad (24)$$

with α_A and β_A taking the values -1.62 and -2.95, respectively (Abell 1965). Differentiating equation (24) with respect to luminosity and fitting to the data of Figure 2, we obtain values $\alpha_A = -1.72$, $\beta_A = -2.84$, and $M_A^* = -20.45$ with a χ^2 of 120 for 10 degrees of freedom.

The measures of a standard candle are the intrinsic dispersion in its absolute magnitude and the accuracy with which its apparent magnitude can be measured. To judge M^* as a standard candle, we fix α at the working value and fit individual clusters for m^* and n^*. (Similarly, one can fix α_A and β_A and fit for m_A^* and N^*.) We then use the known redshifts to obtain observed values of M^*. The smaller the dispersion in these observed values, the better M^* is as a standard candle.

The results of such two-parameter fits of equation (15) to Oemler's data are shown in Table 2. The rms dispersion in M^* about the mean value $\langle M^*\rangle$ is given by

$$\langle (M^* - \langle M^*\rangle)^2\rangle^{1/2} = 0.50 \text{ mag} \,. \quad (25)$$

By comparison the rms dispersion in M_A^* obtained for similar fits to the same data yield

$$\langle (M_A^* - \langle M_A^*\rangle)^2\rangle^{1/2} = 0.42 \text{ mag} \quad (26)$$

(cf. Bautz and Abell 1973).

Root-mean-squared dispersion makes no use of the uncertainty in the value of m^* obtained from a χ^2 fit. For some purposes (such as determination of second order corrections to the Hubble law) these uncertainties may be used to weight individual points. Those clusters which fit equation (15) poorly have larger uncertainties and hence lower weights. By way of example, the weighted mean value $\overline{M}^*$ obtained from the results of Table 2 is -21.36 ± 0.06 mag. Comparable accuracy would be obtained from 14 unweighted values only if the rms dispersion were as small as 0.22 mag.

Much of the dispersion in equation (23) is due to low-weight clusters. If we ignore the three clusters for which the values of M^* are most uncertain, we obtain an rms dispersion of only 0.25 mag. It should be noted that if one is willing to sacrifice the goodness-of-fit information which χ^2 fitting provides, one can obtain m^* using a simple maximum-likelihood procedure which requires only that one average the luminosity of the N brightest cluster members (Schechter and Press 1975).

V. EXPECTED ABSOLUTE MAGNITUDES OF THE BRIGHTEST CLUSTER MEMBERS

There has been considerable discussion as to whether the absolute magnitudes of the first ranked members are governed by a special process or whether they are determined by a stochastic process which also governs the absolute magnitudes of the second through Nth ranked cluster members (Scott 1957; Peebles 1968, 1969; Peach 1969; Peterson 1970*b*, *c*; Sandage 1972; Sandage and Hardy 1973; Geller 1974). By taking the uncertainties in the luminosity distributions of § II to be given by the square root of the number of galaxies in a given bin, we have implicitly adopted a "Poisson hypothesis." We shall investigate some of the implications of the Poisson hypothesis under the assumption that the proposed analytic expression is a good approximation to the cluster-galaxy luminosity function. For a more extensive treatment (using an Abell-type luminosity function) the reader is referred to Geller's (1974) work.

We assume that the individual luminosity distributions for clusters are fair samples of a universal luminosity function. Clusters differ only by the richness parameter n^*, which indicates how large a sample of this universal distribution has been taken. We adopt the working value of α. Letting $N_e(\geq L)$ be the expected number of galaxies brighter than L, then

$$N_e(\geq L) = \int_L^\infty n_e(L')dL' = \Gamma(\alpha + 1, L/L^*)n^* , \tag{27}$$

where $\Gamma(\beta, \lambda)$ is the incomplete gamma function. The probability that the jth brightest cluster member has absolute magnitude M is given by

$$P_j(M)dM = \frac{[N_e(\leq M)]^{j-1}}{(j-1)!} \exp\left[-N_e(\leq M)\right]n_e(M)dM \tag{28}$$

(Peebles 1968). A most probable value of the absolute magnitude M_j of the jth brightest galaxy is found by setting the first derivative of the natural logarithm of P_j equal to zero. The inverse of the second derivative of the natural log gives a good estimate of the variance of M_j about its most probable value. The use of most probable rather than mean value makes a difference of only a few hundredths of a magnitude and simplifies computation. Since N_e is a function of n^*, the most probable values of M_j will be functions of n^*.

The solid, dashed, and dotted curves in Figure 4 present the differences $M_j - M_1$ (using most probable values) for values of n^* of 25, 50, and 100, respectively. The points plotted are the mean values of $m_j - m_1$ found by Humason *et al.* (1956) for 18 rich clusters of galaxies. The data agree well with the curve obtained for $n^* = 25$. While the results of Table 2 indicate that larger values of n^* are appropriate to the clusters studied by Oemler, perhaps a smaller fraction of each cluster was sampled by Humason *et al.*

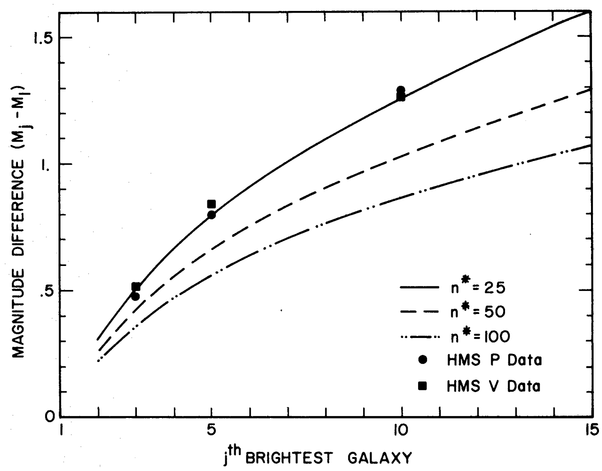

FIG. 4.—Expected magnitude differences between first ranked and jth ranked cluster galaxies computed from analytic expression. Points show mean differences observed by Humason *et al.* (1956).

We can use equation (28) to compute a richness appropriate to any value of n^*. Abell (1958) defines the richness of a cluster to be the number of galaxies in the two magnitude interval following the third brightest galaxy in a circle of radius $1.72z^{-1}$ arcmin. We can therefore compute a richness estimate appropriate to n^* by computing a most probable value of M_3 and then integrating the luminosity distribution from that value of M_3 to $M_3 + 2$. In a similar manner, one can estimate the "population" of a cluster N_c^{48}, defined by Sandage and Hardy (1973) to be the number of galaxies in the $2\frac{1}{2}$ magnitude interval following the third brightest galaxy in a circle of diameter 137 $(1 + z)^2z^{-1}$ arcsec.

An inevitable consequence of the statistical hypothesis is an expected correlation of the absolute magnitude of the brightest cluster galaxy with n^* (or Abell richness or Sandage-Hardy population). This expected correlation, combined with a bias toward selecting bright galaxies and rich clusters at great distance, has come to be known as the Scott effect (Scott 1957). We have computed most probable absolute magnitudes for brightest cluster galaxies as a function of n^* using the proposed analytic expression. We expect a correlation of M_1 with n^* as shown by the solid line in Figure 5.

The available data show little or no such correlation. Sandage and Hardy (1973) present standard metric absolute magnitudes and population estimates for a great number of rich clusters. The filled circles of Figure 5 show absolute magnitudes and populations

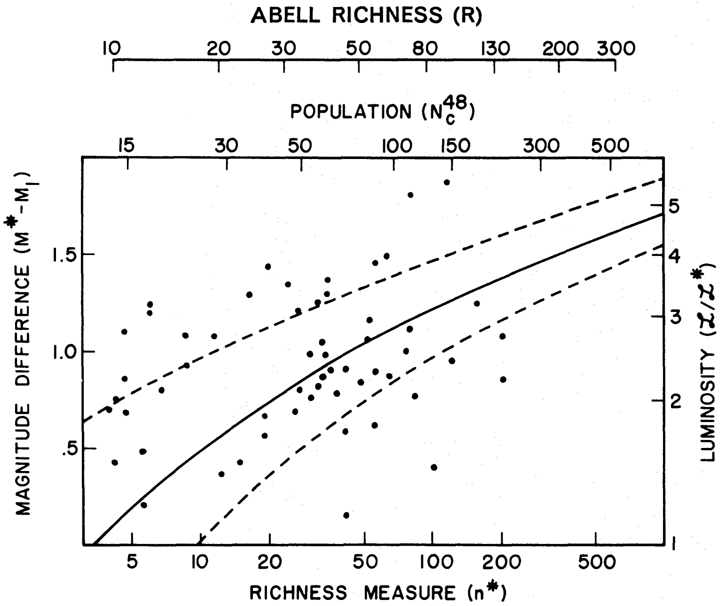

FIG. 5.—Expected correlation of absolute magnitude of brightest cluster galaxy with cluster richness. Broken lines show expected rms dispersion. Filled circles show data of Sandage and Hardy (1973).

for a homogeneous subset of their data; Peterson's (1970*a*) data have not been plotted since a substantially different aperture was used in his photometry (see Sandage and Hardy for a similar plot including these data). The absolute magnitude scale was calibrated by assuming $M^* - \langle M_1 \rangle = 1$, a value appropriate to a cluster with $n^* \approx 44$. While there appears to be some correlation for poorer clusters ($n^* \lesssim 20$), there is little correlation for richer clusters.

The absence of a richness–absolute magnitude correlation is extremely surprising. Geller (1974) has offered a possible explanation: the richer clusters in the Sandage-Hardy sample lie at greater redshifts than the poorer clusters. The amplitude and direction of the richness-magnitude correlation therefore depends upon the value of the deceleration parameter used in the analysis.

Another possible explanation lies in the fact that the value α used in predicting the richness-magnitude correlation was obtained using isophotal magnitudes. But the prediction is compared with Sandage and Hardy's standard metric magnitudes. Only under very special circumstances will the amplitude of the correlation be the same in both magnitude systems. In general, standard metric magnitudes may vary either more or less rapidly than isophotal magnitudes. In this respect it is interesting to note that while Sandage's data, obtained with an 86 kpc aperture, exhibit a marginal richness-magnitude correlation, Peterson's data, using a 41 kpc aperture, show a marginal anti-correlation (see Table 6 in Sandage 1972). A precise understanding of the relation between isophototal and standard metric magnitudes is required before the Sandage-Hardy data rule out the use of the present analytic expression at the extreme bright end.

Peebles (1968) has noted that the narrow dispersion in absolute magnitudes of the first ranked cluster galaxies can be understood in large part on the basis of the statistical hypothesis. The broken lines in Figure 5 show the expected rms fluctuations of M_1 about its most probable value. The data of Sandage and Hardy appear to show reasonable agreement with this expected dispersion. Geller (1974) has found that for a reasonable distribution of richness, one can also reproduce the narrow dispersion using an Abell-type analytic expression.

VI. CONCLUSIONS AND SHORTCOMINGS

The proposed analytic expression gives a good approximation to the general luminosity function and the cluster-galaxy luminosity function over a range of 6 magnitudes. Moreover, the two luminosity functions are identical, except for a multiplicative constant. However, the prediction of a weak correlation of absolute magnitude of first ranked cluster galaxies with cluster richness is not borne out by the data of Sandage and Hardy. The proposed expression may therefore fail at the extreme bright end of the luminosity function, but the agreement is otherwise excellent.

The apparent magnitude m^* may be determined with an accuracy of order 0.25 mag for most clusters if one assumes that the parameter α has a universal value. The data for most clusters suggest that α and M^* are constant from cluster to cluster, although at least two clusters appear to have substantially different luminosity distributions. Further observations would be helpful in confirming or resolving the apparent discrepancy.

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