Semi-Formal Proof of Minimum Information for Calculating Key Transposition in Western Music

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1 Definition of Pertinent Concepts in Western Music Notation

1.1 Written music notation is based off notes on a staff line. Every note can be represented as (location, degree, frequency) where:

1. location corresponds to an ordered set representing location on a staff

   $location = \{C, D, E, F, G, A, B\}$

2. degree corresponds to an ordered set representing numerical representation within Western 12-tone scale

   $degree = \{d_0, d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, d9, d_{10}, d_[11\}$, where each degree has the absolute difference of a semitone

3. frequency corresponds to the ordered set representing frequency in hz of note that we hear. The difference between each pitch is set as 1 (semitone) to keep consistent units with degree. For this exercises the size of the set will equal 12, corresponding to 1 octave

   $frequency = \{f_0, f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}\}$, where the the value of f0 is known as the 'key' of the sheet music

4. Define $FULL$ as set containing all 12 degrees or frequencies

1.2 Only 7 out of the 12 members of the set have a coordinate for location. This subset V corresponds to notes in the key of C and is as follows:

$\{(C, d_0, f_0), (D, d_2, f_2), (E, d_4, f_4), (F, d_5, f_5), (G, d_7, f_7), (A, d_9, f_9), (B, d_{12}, f_{12})\}$

1.3 $V_{location}$ is used as an alphabet to create strings. To represent other members of $FULL$ within a string, one can apply local operators (pre-operators) to the members of V. The operators can be:

1. $sharp$, where $sharp(x) : d(x) = x + 1$ and $f(x) = x + 1$
2. $flat$, where $flat(x) : d(x) = x - 1$ and $f(x) = x - 1$

3. *natural*, where $natural(x) : d(x)$ and $f(x)$ are the degree and frequency specified in V

Thus, a music string consists of a list of locations and operators on certain locations

1.4 In addition, global operators can also be applied to the entire string (aka key signature). Key signatures allows for the creation of set $V_{keysignature}$ where the scale degrees (d_i) does not need to be based on the fundamental frequency (f_0) in order to directly be represented as a location in the staff (L).

- For example, a set of notes making up the key of G :

  $$\{(G, d_0, f_8), (A, d_2, f_{10}), (B, d_4, f_{12}), (C, d_5, f_0), (D, d_7, f_2), (E, d_9, f_4), (Fsharp, d_{12}, f_6)\}$$

  can be mapped from V shifting d0 so it corresponds to G and by applying a global sharp to the position of F

- There are two valid global operations (one with sharps and one with flats) corresponding to every note in $FULL$. Global operators are defined:

  $op_{key}(s), / = ss.t.ifa/inkey, a/ = op(a)$ where $op/inO, key/inV_location$, and $s$is a string

- To be a valid global operator, the key is constrained in that if every member of key's degree shifts to its post-operated value, the degrees in the set consisting of $op(key) + (V - key)$ will still map to V via addition or subtraction

- For example: If $key = \{B, E\}$ and $op =< flat >$, the degree of B and E in V is (B, d_{12}) and (E, d_5) in V would now shift to

  $$B = flat(B) = B - 1 = d_{12} - 1 = d_{11}$$
  $$E = flat(E) = E - 1 = d_5 - 1 = d_4$$

  The set Q consisting of $key + (V - key)$ would thus contain the locations $\{flat(B), flat(E)\} + \{C, D, F, G, A\}$ which corresponds to degrees of $\{d_{10}, d_4\} + \{d_1, d_3, d_6, d_8, d10\}$ Ordering the values in N, you get

  $\{d_1, d_3, d_4, d_6, d_8, d_{10}, d_{11}\}$ Adding 2 to each value gives you

  $\{d_3, d_5, d_6, d_8, d_{10}, d_{12}, d_1\} = V$ degree works on a mod 12 number system with size of 12 so 13mod12 is 1

- Thus since N maps to V, $< flat >_{B,E}$ is a valid global operation. (in musical terms, this is the key of B flat and is represented by the Bflat key signature)

1.5 Transpositions are a mapping that shift f_0 ("key of the sheet music") but retains the relative positions of notes in the string

- Define: $TRANSPOSITION(f_0', string)$ is a mapping from V to V' $s.t.$

  $V = \{(d_0, f_0), (d_1, f_1)....(d_{12}, f_{12})\}$ and

  $V' = \{(d_0, f_0'), (d_1, f_0' + 1).....(d_{12}, f_0' + 2)\}$

  where f_0' is the new "key" of the sheet music

2 Proof

2.1 a. Posit: One could apply $TRANSPOSITION(f_0', string)$ s.t. 1) the coordinates of L (locations of note on staff) in V retain the same relative positions (i.e. staff line positions could shift while pitch differences between adjacent notes remain constant) 2) notes with local operations would retain a local operation, and 3) no new local operations would need to be added

2.2 Process applying transformations s.t. the above conditions are met

Given string s on music in the key of k ($f_0 = k$) and $f_0' = n$:

1. $diff = n - k$
2. Move s upwards $diff$ positions on the staff

   This sets condition 1) because spacing between note locations is preserved
3. Replace any global operation applying to s with global operation corresponding with n to the string

   Since the global operation (key signature) associated with a key allows notes in V of that key to sit on the staff without local operations (see section 1.4), this step ensures condition 3) by mapping all notes previous in V to $V_{keysignature}$
4. For notes with local operations, apply flats or sharps until the original relative positions are achieved

   Since the key signature mapping detailed in section 1.4 is one-to-one, all notes previously not in V are not in $V_{keysignature}$, so they will retain some sort of local operation, ensuring 2)

Since step (3) ensured the relative positions of the non-accidental notes are preserved and step (4) ensured the relative positions of the accidental notes are preserved and [accidentaled notes] and [non-accidentaled notes] are complementary, all notes have maintained their relative position after f_0 shifted to n in step (3)