Semi-Formal Proof of Minimum Information for Calculating Key Transposition in Western Music

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1 Definition of Pertinent Concepts in Western Music Notation

1.1 Written music notation is based off notes on a staff line. Every note can be represented as (location , degree, frequency) where:

1. location corresponds to an ordered set representing location on a staff

   $location = \{C, D, E, F, G, A, B\}$

2. degree corresponds to an ordered set representing numerical representation within Western 12-tone scale

   $degree = \{d_0, d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, d_9, d_{10}, d_{11}\}$, where each degree has the absolute difference of a semitone

3. frequency corresponds to absolute pitch of note that we hear. The difference between each pitch is set as 1 (semitone) to keep consistent units with degree. For this exercises the size of the set will equal 12, corresponding to 1 octave

   $frequency = \{f_0, f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}\}$, where the the value of f_0 is known as the fundamental 'pitch' of the sheet music. For example, all oboe music is in "concert pitch" ($f_0 = C$) and music written for Bflat clarinet is pitched at Bflat ($f_0 = Bflat$)

4. Define $FULL$ as set containing all 12 degrees or frequencies

1.2 Only 7 out of the 12 members of the set have a coordinate for location that can notated on a staff without additional operations (see operation types in 1.3 and 1.4). This subset V corresponds to notes in the key of C and is as follows:

$$\{(C, d_0, f_0), (D, d_2, f_2), (E, d_4, f_4), (F, d_5, f_5), (G, d_7, f_7), (A, d_9, f_9), (B, d_{12}, f_{12})\}$$

1.3 Notes in $V_{location}$ are used as 'characters' to create strings. To represent other members of $FULL$ within a string, one can apply local operators (pre-operators) to the members of V. The operators can be:

1. $sharp$, where $sharp(x) : d(x) = x + 1$ and $f(x) = x + 1$
2. $flat$, where $flat(x) : d(x) = x - 1$ and $f(x) = x - 1$
3. $natural$, where $natural(x)$: $d(x)$ and $f(x)$ are the degree and frequency specified in V

For example, to represent in a string the note (F, d_6, f_6) , also known as F sharp, one must notate it as $sharp($ $F, d_5, f_5)$ because (F, d_5, f_5) $\in V$

1.4 In addition, global operators can also be applied to the entire string S (aka key signature). Key signatures allows for the creation of a shifted V, V', where the "tonic" scale degrees (d_0) does not have to coincide with the fundamental pitch (f_0).

- There is one possible key signature for every $f, f \in V_{frequency}$ and two valid global operations every $a, a \notin V_{frequency}$ because one can represent any a in reference to the visually representable note above or below it(e.g. the notes Gflat and Fsharp represent the same pitch but have different associated key signature).
- Global operators are defined:

  Given key is a subset of notes from $V_{location}$, O = {sharp, flat } ,op $\in O$, and s is a string:

  $op_{key}($ $s)$ $= s'$, s' is s where for all $k \in key$, any instance of k in s is replaced with $op($ $k)$
- For example, the set of notes making up the key of G (in concert pitch)

  $$\{(G, d_0, f_7), (A, d_2, f_9), (B, d_4, f_{11}), (C, d_5, f_0), (D, d_7, f_2), (E, d_9, f_4), (Fsharp, d_{12}, f_6)\}$$

  can be mapped from V shifting d0 so it corresponds to G and by applying a global sharp to the position of F. Therefore, for the key of G, $op_{key}()$ is $flat_{B,E}()$
- To be a valid global operator, the key is constrained in that if every member of key's degree shifts to its post-operated value, the degrees in the set consisting of $op(key) + (V - key)$ will still map to V via addition or subtraction

- For example: If $key = \{B, E\}$ and $op =< flat >$, the degree of B and E in V is (B, d_{12}) and (E, d_5) in V would now shift to

  $B = flat(B) = B - 1 = d_{12} - 1 = d_{11}$
  $E = flat(E) = E - 1 = d_5 - 1 = d_4$

  The set Q consisting of $key + (V - key)$ would thus contain the locations $\{flat(B), flat(E)\} + \{C, D, F, G, A\}$ which corresponds to degrees of $\{d_{10}, d_4\} + \{d_1, d_3, d_6, d_8, d10\}$ Ordering the values in N, you get

  $\{d_1, d_3, d_4, d_6, d_8, d_{10}, d_{11}\}$ Adding 2 to each value gives you

  $\{d_3, d_5, d_6, d_8, d_{10}, d_{12}, d_1\} = V$ degree works on a mod 12 number system with size of 12 so 13mod12 is 1

- Thus since N maps to V, $< flat >_{B,E}$ is a valid global operation. (in musical terms, this is the key of B flat and is represented by the Bflat key signature)

1.5 Transpositions are a mapping that shift f_0 ("key of the sheet music") but retains the relative positions of notes in the string

- Define: $TRANSPOSITION(f_0', string)$ where f_0' is the new "pitch" of the sheet music, is a mapping from V to V', $s.t.$

  $V = \{(d_0, f_0), (d_1, f_1)....(d_{12}, f_{12})\}$ and

  $V' = \{(d_0, f_0'), (d_1, f_0' + 1).....(d_{12}, f_0' + 2)\}$

2 Proof

2.1 Posit

Given S is a string, one could apply $TRANSPOSITION(f_0', S)$ s.t. these conditions will be met:

1. The coordinates of $S_{location}$ (position of every note on the staff) retain the same relative positions (i.e. staff line positions could shift while pitch differences between adjacent notes remain constant)
2. No new notes require a local operation, i.e. all notes previously without a local operation will still not possess one
3. Notes with local operations would retain a local operation

2.2 Proposed Transformation

Given $f_0' = n$ and string S on music pitched at k ($f_0 = k$):

1. Decompose n and k into the ordered pair (*basenote* , *operation*). E.g. if k=B flat, k can be decomposed to (B, $flat$). If no operation exists, set *operation* as *natural*, e.g. if n=C, C can be decomposed into (C, *natural*)

2. Set $interval$ equal to the distance between n and k in $V_{location}$ If n occurs closer to the front of the set than k in $V_{location}$, $interval = -(interval)$

3. Set $opsum = k_{operation} + n_{operation}$. E.g. if k=B flat and n=D sharp, $opsum = flat + sharp = (\ -1)\ \ + (\ 1)\ \ = 0$

4. Set $diff = |n - k|$, where the sign of diff (+ or −) is equal to the sign of $interval$

5. $S' = \{$For $s \in S$, $s_{location} = s_{location} + interval$, $s_{degree} = s_{degree} + diff\}$

6. Given $(0, t)$ as the "tonic" note associated with the key signature of S, decompose t into $(\ t_{location}, t_{operation})$

   $t' = (\ t_{location} + interval, t_{operation} + opsum)$

7. Set key signature of S' to new key signature associated with $(d0, t')$, meaning t' is the new tonic

2.3 Proof Transformation Satisfies Conditions

2.3.1 Condition 1

- Step 5 of the of the transformation ensures relative position will not change because the location of every note shifts by the same amount ($interval$)

2.3.2 Condition 2

- From section 1.4, we know the global operation (key signature) associated with a key X allows notes in V_X to sit on the staff without local operations.

- From section 1.4, we also know given S as the set of all possible key signatures, for any $v, s \in S$, V_s maps to V and V maps to V_v, so V_s maps to V_v. Therefore, V_t maps to $V_{t'}$. Since t''s corresponding global operation (key signature associated with t') was applied in Step 7, all notes that mapped to V in S will still map to V in S'. Those notes will thus not require any added accidentals.

2.3.3 Condition 3

- Since the key signature mapping detailed in section 1.4 is one-to-one, all notes previously not in V_t are also not in $V_{t'}$, so they must retain some sort of local operation (flat, sharp or natural)