Simulation & Modeling

Estimation of Uncertain Parameters

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Abstract

This report discusses the **Modeling** and **Simulation** of two dynamic systems, presenting all the analysis entailed in the conducted experiments and the results gathered by the simulations. It also provides extensive reasoning and mathematical explanations for the applied formulations on each system's differential equation, in order to reach the necessary form, for the computational methods to succeed.

1 Introduction

The subject of both analysis, is to approximate uncertain parameters of the dynamic system, given input-output measurements. Our knowledge about the system is limited and only regards its differential equation. This is classified as a **Grey Box** system, since the its parameters are unknown, but nevertheless, we are aware of the system's mathematical description. The method chosen to perform the approximation of the parameters, is the **Least Squares** method, which minimizes the mean square error.

2 Dynamic System 1

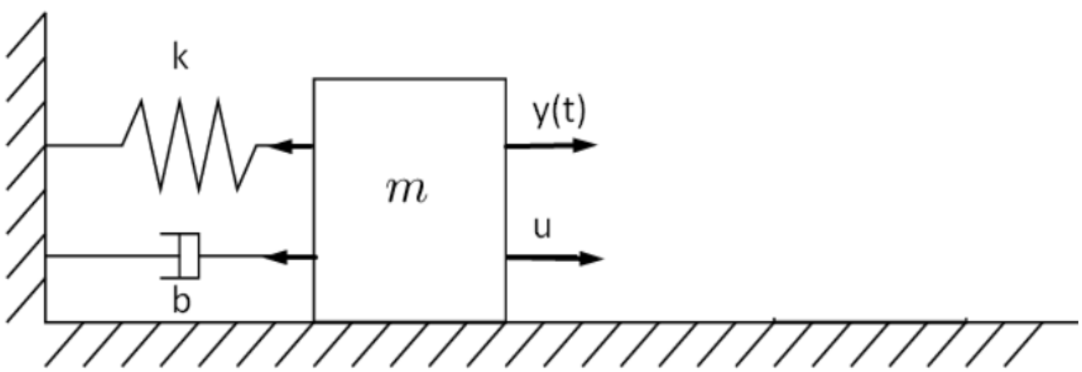

Figure 1: Mass-spring-damper system

2.1 description

In order to approximate the uncertain parameters using **Least Squares**, we first have to formulate the mathematical description of the system such that, there is a linear relation between the input, and the output. Our goal is to convert the differential equation of the system, to its **Linearly Parameterized** form:

$$Y = \theta_{\lambda}^{T}\zeta$$

where θ_{λ}^{T} is the parameter vector, and ζ is our new input vector, resulted from the applied transformations. The first step includes the identification of the system, and the laws of physics that it entails.

For the subject system, the following mathematical description is easily acquired using **Newton's second law**:

$$\ddot{y} = (1/m)(-b\dot{y} - ky + u)$$

2.2 theoretical analysis

From the second order ODE, the system state variables can be immediately obtained as:

$$\dot{x_1} = x_2$$

$$\dot{x_2} = (1/m)(-b\dot{x_2} - kx_1 + u)$$

by setting $x_1 = y$ and $x_2 = \dot{y}$. The parameter vector is obviously:

$$\theta^* = \begin{bmatrix} \frac{b}{m} & \frac{k}{m} & \frac{1}{m} \end{bmatrix}^T \quad (1.1)$$

Also,

$$\zeta = \begin{bmatrix} -\frac{\Delta_n^T(s)}{\Lambda(s)} y & \frac{\Delta_m^T(s)}{\Lambda(s)} u \end{bmatrix}^T \quad (1.2)$$

with

$$\Delta_i(s) = \begin{bmatrix} s^i & s^{i-1} & \ldots & 1 \end{bmatrix}^T$$

Specifically in our example, observing the differential equation, we gather $n = 2$ and $m = 0$ so (2). In addition, we arbitrarily choose a second degree stable filter:

$$\Lambda(s) = s^2 + (p_1 + p_2)s + p_1 p_2 \quad (1.3)$$

p_1 and p_2 being the chosen poles. Them being positive is mandatory in order to continue the analysis. Finally, using (1.3), we consider

$$\lambda = \begin{bmatrix} p_1 + p_2 & p_1 p_2 \end{bmatrix} \quad (1.4)$$

and also

$$\theta_\lambda = \begin{bmatrix} \theta_1^* - \lambda^T & \theta_2^* \end{bmatrix}^T \quad (1.5)$$

with

$$\theta_1^* = \begin{bmatrix} \frac{b}{m} & \frac{k}{m} \end{bmatrix}^T, \quad \theta_2^* = \begin{bmatrix} \frac{1}{m} \end{bmatrix} \quad (1.6)$$

Combining all (1.1) – (1.6) we get:

$$\zeta = \begin{bmatrix} \frac{\begin{bmatrix} -s & -1 \end{bmatrix}}{\Lambda(s)} y & \frac{1}{\Lambda(s)} u \end{bmatrix} \Longleftrightarrow$$

$$\zeta = \begin{bmatrix} \frac{-s}{\Lambda(s)} y & \frac{-1}{\Lambda(s)} y & \frac{1}{\Lambda(s)} u \end{bmatrix} \quad (1.7)$$

So at this point we have concluded that

$$y = \theta_\lambda^T \zeta$$

and the vector ζ contains values that can be calculated by solving the ODE on MATLAB. We now give ζ as input to the system, and we get the output y, for every t inside the requested time span. The last part is to use these measurements and estimate the parameters of the model, according to the means square error:

$$e = y - \hat{y} \longrightarrow e = y - \theta_\lambda^T \zeta$$

The optimal estimation of the parameters, θ_0, is directly concluded to be:

$$\theta_0 = \underset{\theta}{\operatorname{argmin}} \frac{1}{N} \sum_{i=1}^{N} \frac{e^2(i)}{2}$$

thus the solution that minimizes the error between the model and the real system. Here, N denotes the number of measurements that were taken when solving the ODE. By equating the derivative of the objective function to zero, we get

$$\theta_0 = \frac{\sum_{i=1}^{N} y(i)\zeta(i)}{\sum_{i=1}^{N} \zeta^2(i)}$$

which at this point, given the input-output measurements, is completely solvable and should return a $3x1$ vector containing the estimated parameters.

2.3 simulation

The preceded theoretic approach suggests that the estimation should be precise and fairly accurate. But what does the practical application indicate?

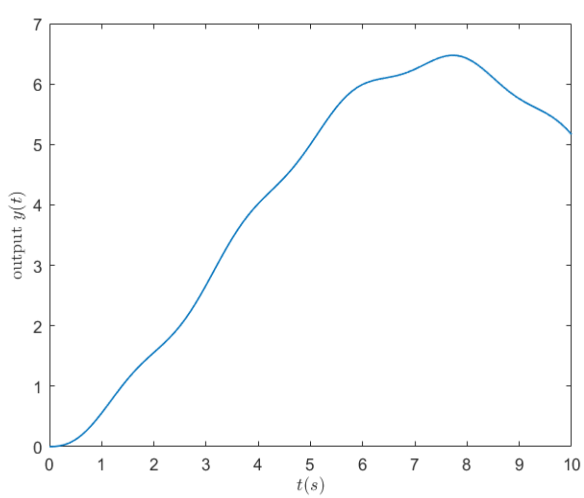

Figure 1.1: System's output, during the time-span

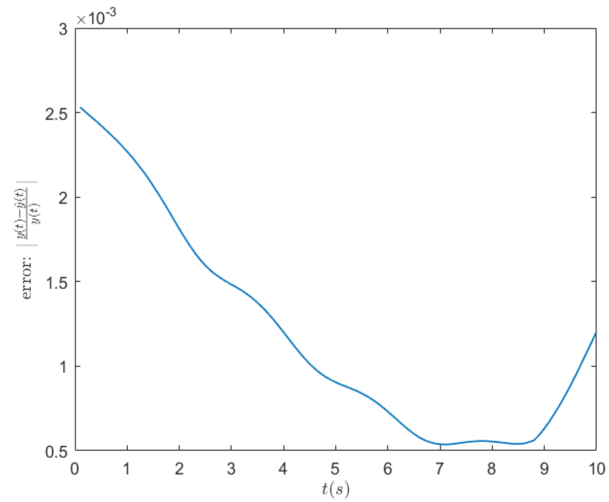

Figure 1.2: Error between model & real system

For the experiments, we set $m = 10kg$, $k = 1.5kg/s^2$, $b = 0.3kg/s$ and $u = 10\sin(3t) + 5N$. *Figure 1.1*, depicts the output, both from the real system, and the model itself. They are completely overlapping as the model does not displays any deviation from the real system, in a visible degree. This observation is confirmed by *Figure 1.2*, where the error, computed by the formula $|\frac{y(t)-\hat{y}(t)}{y(t)}|$, obviously lies inside the range of $[0.5, 2.5]x10^{-3}$. Overall, the results on the first system are absolutely satisfying, and the correspondence between our theoretical analysis and the practical outcome, is highly prominent.

3 Dynamic System 2

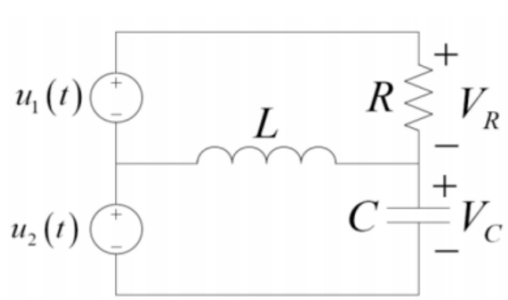

Figure 2: RLC circuit system

3.1 description

Here, our goal is to estimate the transfer function of the RLC circuit, presented in *Figure 2*. In order to extract a solid mathematical description for the circuit, we apply the two **Laws of Kirchhoff**, KVL and KCL, getting the following, second order differential equation. In this description, we consider u_1 and u_2 as the input and $y = V_C$ as the output.

$$\ddot{y} + \frac{1}{RC}\dot{y} + \frac{1}{LC}y = \frac{1}{RC}\dot{u}_2 + \frac{1}{LC}u_2 + \frac{1}{RC}\dot{u}_1 \quad (B)$$

3.2 theoretical analysis

Following a mathematical approach, identical to that of *section 2.2*, we obtain the following:

$$\theta^* = \begin{bmatrix} \frac{1}{RC} & \frac{1}{LC} & \frac{1}{RC} & \frac{1}{LC} & \frac{1}{RC} & 0 \end{bmatrix}^T \quad (2.1)$$

The zero, in the last index of the parameter vector is inserted, because we have to take into consideration the coefficient of the derivative $u_1^{(0)}$, which is not present in the equation, so it's assumed that its coefficient is zero. Similarly,

$$\zeta = \begin{bmatrix} \frac{\begin{bmatrix} -s & -1 \end{bmatrix}}{\Lambda(s)} y & \frac{\begin{bmatrix} s & 1 \end{bmatrix}}{\Lambda(s)} u_2 & \frac{\begin{bmatrix} s & 1 \end{bmatrix}}{\Lambda(s)} u_1 \end{bmatrix} \Longleftrightarrow$$

$$\zeta = \begin{bmatrix} \frac{-s}{\Lambda(s)} y & \frac{-1}{\Lambda(s)} y & \frac{s}{\Lambda(s)} u_2 & \frac{1}{\Lambda(s)} u_2 & \frac{s}{\Lambda(s)} u_1 & \frac{1}{\Lambda(s)} u_1 \end{bmatrix} \quad (2.2)$$

where

$$\Lambda(s) = s^2 + (p_1 + p_2)s + p_1 p_2 \quad (2.3)$$

is again a stable filter with positive poles p_1 and p_2, from which we obtain the vector

$$\lambda = \begin{bmatrix} p_1 + p_2 & p_1 p_2 \end{bmatrix} \quad (2.4)$$

and also

$$\theta_\lambda = \begin{bmatrix} \theta_1^* - \lambda^T & \theta_2^* \end{bmatrix}^T \quad (2.5)$$

with θ_1 and θ_2 being the parameter sub-vectors of θ, corresponding to the coefficients of the input and the output derivatives respectively.

$$\theta_1^* = \begin{bmatrix} \frac{1}{RC} & \frac{1}{LC} \end{bmatrix}^T, \quad \theta_2^* = \begin{bmatrix} \frac{1}{RC} & \frac{1}{LC} & \frac{1}{RC} & 0 \end{bmatrix} \quad (2.6)$$

Eventually, we have again formulated our system into the desired form

$$Y = \theta_\lambda^T \zeta$$

from which we are able to estimate the unknown parameters using the **Least Squares** solver.

3.3 simulation

For this system, measurements were taken from a voltage generator function of high frequency. *Figure 2.1* illustrates the correlation between the output (voltage of capacitor) and the time in seconds. The remaining graphs present the system's output, for different values for the step picked during the sampling process. After a significant amount of trial-error efforts, i managed to realize that pairs of poles such that $p_1 \in [100, 200]$ and $p_2 \in [200, 400]$, results in the minimum possible error $e = 0.051$.

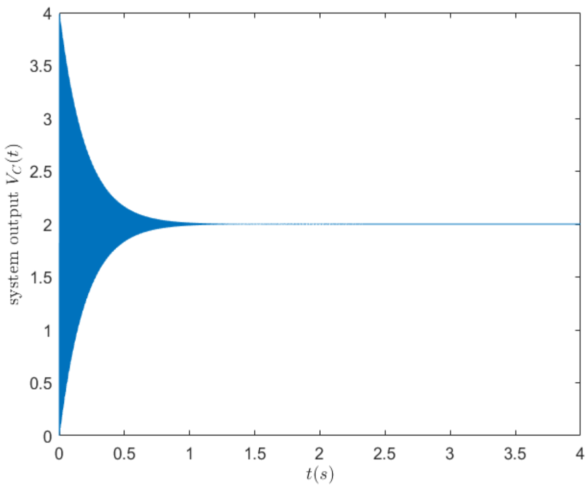

Figure 2.1: System's output, during the time-span

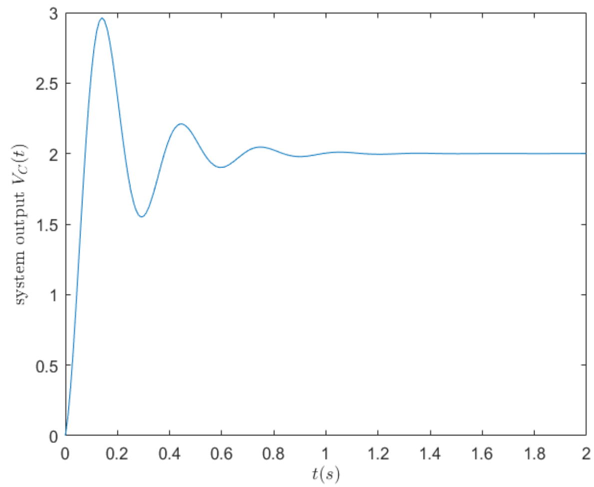

Figure 2.2: Sampling resolution 1e-2

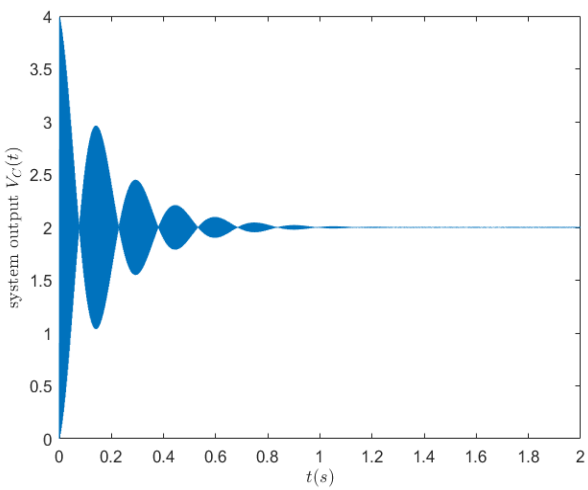

Figure 2.3: Sampling resolution 1e-3

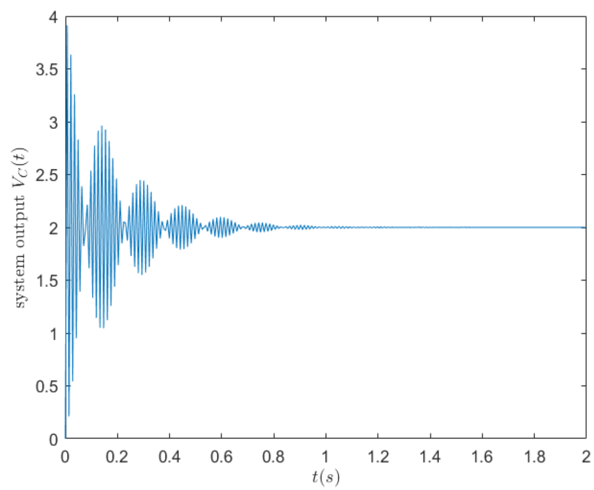

Figure 2.4: Sampling resolution 7x1e-3

Now that we gathered all the necessary knowledge about the system, we are able to compute its transfer function, applying **Laplace** transformations to the differential equation (B).

$$\ddot{y} + \frac{1}{RC}\dot{y} + \frac{1}{LC}y = \frac{1}{RC}\dot{u}_2 + \frac{1}{LC}u_2 + \frac{1}{RC}\dot{u}_1 \iff$$

$$s^2 y + \frac{s}{RC}y + \frac{1}{LC}y = \frac{s}{RC}u_2 + \frac{1}{LC}u_2 + \frac{s}{RC}u_1 \iff$$

$$\left(s^2 + \frac{s}{RC} + \frac{1}{LC}\right)y = \left(\frac{s}{RC} + \frac{1}{LC}\right)u_2 + \left(\frac{s}{RC}\right)u_1$$

Now, considering

$$Q(s) = s^2 + \frac{s}{RC} + \frac{1}{LC}, \quad u = \begin{bmatrix} u1 & u2 \end{bmatrix}^T$$

we obtain, in tabular form,

$$y = \begin{bmatrix} \frac{\frac{1}{RC}s}{Q(s)} & \frac{\frac{1}{RC}s + \frac{1}{LC}}{Q(s)} \end{bmatrix} u \iff$$

$$G(s) = \frac{y}{u} = \begin{bmatrix} \frac{\frac{1}{RC}s}{Q(s)} & \frac{\frac{1}{RC}s + \frac{1}{LC}}{Q(s)} \end{bmatrix} \quad (2.7)$$

The simulation performed in MATLAB, estimated the parameters $\hat{a}_1$ and $\hat{a}_2$, where $a_1 = \dfrac{1}{RC}$ and $a_2 = \dfrac{1}{LC}$, to be $\boldsymbol{\hat{a}_1 = 9.96}$ and $\boldsymbol{\hat{a}_2 = 10000944.72}$. Substituting these values in (2.7) results in

$$G(s) = \frac{y}{u} = \begin{bmatrix} \dfrac{\hat{a}_1 s}{Q(s)} & \dfrac{\hat{a}_1 s + \hat{a}_2}{Q(s)} \end{bmatrix}, \quad Q(s) = s^2 + \hat{a}_1 s + \hat{a}_2$$

All parameters are currently known and the transfer function can be computed by the above formula.

3.4 analysis & handling of outliers

In reality, there is a strong possibility that not all measured values are valid, in contrast to our simplistic and independent of external factors approach. In order to observe the effect of outlier values on the estimation result, we add three, arbitrarily picked values, to the already gathered data. The new values are significantly larger than the normal ones, specifically in our testing, they were chosen to be of size class 1e+3 larger than the correct values. The error was calculated based on the formula

$$e_i = \left| \frac{\hat{a}_i - \bar{a}_i}{\hat{a}_i} \right|, \quad i = 1, 2$$

where $\hat{a}_i$ and $\bar{a}_i$ refer to the previous and the current approximation of parameter i respectively. Multiple tests lead to the mean error for each parameter estimation to be $e_1 = 40\% - 65\%$ and $e_2 = 0.75\% - 0.97\%$. These deviations seem to be extremely significant and the fact that they were caused by such a negligible number of outliers in comparison to the number of total measurements, indicates that an outlier-elimination tactic is undisputed mandatory.

The elimination of invalid values could be, for example, a part of a pre-processing, where each value is checked individually and every unreasonable deviation from the norm, results in excluding the corresponding value from the computations.