Aspects of Domain Theory: a Formal Study of Semivaluations

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Abstract

Domain Theory provides the formal foundation for Denotational Semantics of Programming Languages. Quantitative Domain Theory is concerned with metric basic approaches to Domain Theory, with applications to real number computation and complexity analysis. One important concept in this area is the notion of a semivalution. We study this notion, and investigate novel results in this area.

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1 Introduction

Domain Theory, which focuses on finding fix points through Scott-Continuous functions has been historically used as the formal foundation for denotational semantics for programming languages and recently, widely used on real number computation, quantum computing and complexity analysis. Quantitative Domain Theory provides a novel branch in this field which puts a special emphasis on metric approaches and leads to the new concept, *Semivaluations*.

Semivaluations, which non-trivially use the single operation to represent the semi-modular law in a semilattice has been introduced and investigated in [Sch04, Sch03]. We study this notation and discuss novel results that extend Birkhoff's theorems to semi-modular lattices [SYRV14]. Birkhoff's well-known theorems build up two correspondences in lattice theory [Bir84] . One shows that metric lattices are modular, another claims that distributed lattices can equipped with a poset composed by its downsets.

This research, motivated by a counter example we found, generated recently that a non-modular semilattice can be equipped with a semivaluation [SYRV14]. It directly leads to the result that a quasi-metric semilattice can be non-modular which actually extend the Birkhoff's correspondence on non-modular semilattices and shows us that using semivaluations are a suitable way to analyses the non-modular semilattices.

We followed the research on generating typical theorems from the concept of lattices to semilattices by semivaluations. One of the results we obtain is extending the Von Neumann independence on semi-modular semilattices and the corresponding theorem about ranking functions on it [Gra11].

We then begin to apply order dimension theory [Dus41] on semimodular lattices, generate a similar way as used on finite distributed lattices by Marigo [Mar12]. It actually provides us an algorithm to construct a semivaluation on modular lattices by realisers. As for semimodular semilattices, we generate a novel approach to construct such a semivaluation by defining the height function on each linear extension and total height function on the realizer.

2 Definitions

We first recall the basic definitions and formulas about valuation and metric lattice [Bir84] [Gra11]. Then, we introduce the notation of semivaluation and quasi-metric semilattice [Sch04, Sch03] .

2.1 Valuation, Modular and Metric lattice

Definition 1 *A function f on a lattice L, $f : L \to \mathcal{R}_0^+$ is a valuation iff*

$$\forall x, y \in L.\, f(x \sqcap y) + f(x \sqcup y) = f(x) + f(y).$$

Definition 2 *A lattice $L = (P, \sqsubseteq)$ is modular iff $\forall x, y, z \in L.\, x \sqsubseteq z \Rightarrow x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap z$.*

Definition 3 *Every distributive lattice is modular.*

Definition 4 *A lattice L equipped with a strictly increasing valuation $v \colon L \to \mathcal{R}_0^+$ is called a metric lattice. The metric induced by the valuation is defined as follows: $d_B(x, y) = v(x \sqcup y) - v(x \sqcap y)$.*

Theorem 5 *(Birkhoff) Metric lattices are modular.*

2.2 Semivaluation, Semimodular and Quasi-metric semilattice

Definition 6 *If $(X, \preceq)$ is a meet semilattice then a function $f \colon (X, \preceq) \to \mathcal{R}_0^+$ is a meet valuation iff*

$$\forall x, y, z \in X.\, f(x \sqcap z) \geq f(x \sqcap y) + f(y \sqcap z) - f(y)$$

and f is meet co-valuation iff

$$\forall x, y, z \in X.\, f(x \sqcap z) \leq f(x \sqcap y) + f(y \sqcap z) - f(y).$$

Definition 7 *If $(X, \preceq)$ is a join semilattice then a function $f \colon (X, \preceq) \to \mathcal{R}_0^+$ is a join valuation iff*

$$\forall x, y, z \in X.\, f(x \sqcup z) \leq f(x \sqcup y) + f(y \sqcup z) - f(y)$$

and f is join co-valuation iff

$$\forall x, y, z \in X.\, f(x \sqcup z) \geq f(x \sqcup y) + f(y \sqcup z) - f(y).$$

Definition 8 *A function is a semivaluation if it is either a join valuation or a meet valuation. A join (meet) valuation space is a join (meet) semilattice equipped with a join (meet) valuation. A semivaluation space is a semilattice equipped with a semivaluation.*

Definition 9 *A join semilattice L is upper semimodular iff $\forall x, y, z \in L.\, x \succ y \Rightarrow x \sqcup z \succeq y \sqcup z$.*

Definition 10 *We define a semilattice equipped with a strictly increasing semivaluation is a quasi-metric semilattice.*

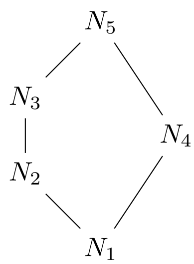

Figure 1: Non-distrobutive pentagon Lattice

3 Semivaluation on non-modular semilattice

We begin the investigation based on the basic example below, which indicate that a typical pentagon non-modular semilattice can be equipped with a semi-valuation and it can be seen as a quasi-metric semilattice.

Consider a five point pentagon join semilattice $\mathcal{L}=\{N_1, N_2, N_3, N_4, N_5\}$, in which we have $N_5 \succ N_3 \succ N_2 \succ N_1$ and $N_5 \succ N_4 \succ N_1$. Let v be a function on $\mathcal{L}$ defined by: $v(N_1) = 0$, $v(N_2) = 2$, $v(N_3) = 3$, $v(N_4) = 4$, $v(N_5) = 5$ as shown in **Figure 1**.

Lemma 1 *The function v on this join semilattice $\mathcal{L}$ is a strictly increasing join semi-valuation.*

Proof: Since $\mathcal{L}$ is a lattice, we proof this lemma based on Proposition 5. It is easy to find this function is increasing that $v(N_1) < v(N_2) < v(N_3 < v(N_4) < v(N_5)$. Then we show it satisfies join-modularity:

$$\forall x, z \in L, v(x \sqcup z) + v(x \sqcap z) \leq v(x) + v(z).$$

We only need to check the situation when x represent N_4 and z represent N_2 or N_3, otherwise those two points would be related in this lattice and the modularity law must be satisfied.

□

Lemma 2 *This join semilattice $\mathcal{L}$ is not (upper) semimodular.*

Proof: We choose N_4, N_2 and N_1 in $\mathcal{L}$ to check. Since we have $N_4 \succ N_1$ and $N_1 \vee N_2 = N_2$ and $N_4 \vee N_2 = N_5$. We find that $N_2 \not\prec N_5$, then we get $N_1 \vee N_2 \not\prec N_4 \vee N_2$ which do not satisfy the semi-modular law.

□

4 Semivaluation on Semi-modular lattice

4.1 Height function on Semimodular Semilattice

Before we can describe our result for semivaluation on semilattice, We first need to introduce the correspondence between height function and semivaluation on semimodular semilattice

Theorem 11 *Let L be a join semilattice and having a minimum. The following are equivalent:*

1) L is upper semimodular

2) The height function h is a semivaluation.

Proof: From(i)to(ii): First, we consider the case when x<y and z<y. Thus we have h(x∨z)<h(x∨y). Obviously, we have h(y)=h(y∨z). So we get the result: h(x∨z)+h(y)<h(x∨y)+h(y∨z).

Then we consider the more general case when x>y or z>y. It suffices to verify it when x>y, then we have h(x∨y)=h(x). As $\mathcal{L}$ is upper semimodular, based on Gratzer's theorem 375 [Gra11], It is trivially the same for semilattice[1] that given C to a maximal chain in [y,x], then D={c∨z | c∈C} is a maximal chain in [y∨z,x∨z]. By the Jordan-Holder Chain Condition, the length of C is h(x)-h(y) which equal to h(x∨y)-h(y). Then the length of D is at most the length of C, which is h(x∨z)-h(y∨z). So we get the result: h(x∨z)-h(y∨z)<h(x∨y)-h(y).

For the case when y is no related with x and z, we can verify it by assuming h(y)>h(x) or h(y)<h(x), respectively. So the processes will be the same as when y>x or y<x as what we do above.

From(ii)to(i): Say we have that the height function h is a join semivaluation, then we get h(x∨z)≤h(x∨y)+h(y∨z)-h(y) by definition which is the same as h(x∨z)-h(y∨z)≤h(x∨y)-h(y). Suppose x≻y, so h(x)=h(y)+1. Then, we can say h(x∨y)=h(y)+1 (as h is the height function). Then it is clearly that h(x∨z)-h(y∨z)≤h(x∨y)-h(y)=1 which means h(x∨z)=h(y∨z)+1 and lead to the result that x∨z ≻ y∨z.

□

[1]Gratzer verified it using only the single operation which is exactly what we have for semilattice.

4.2 Von Neumann Independence on Semimodular Semilattice

Definition 12 *Let $\mathcal{L}$ be a semimodular semilattice satisfying DCC having a minimum. A subset $\mathcal{I}_n$ = { a_1, ... ,a_n} is called independent* if*

$$\forall i \in n, \{\bigvee \mathcal{I}_{i-1}, a_i\} \textit{ is a antichain.}$$

Theorem 13 *Let $\mathcal{I}_n = \{a_1, \ldots, a_n\}$ be an independent* set of a semimodular semilattice with a minimum composed by atoms. Then there is:*

$$h(\bigvee \mathcal{I}_n) = n.$$

Proof: We proof the equation by induction. Given an index i, we proof that:

$$h(\bigvee \mathcal{I}_i) = i.$$

Obviously, this is true for i=1 as a_i is an atom. If $h(\bigvee \mathcal{I}_n) = i$, consider the semimodular law for the height function on $\bigvee \mathcal{I}_i$ and a_{i+1}. Then we obtain:

$$h(\bigvee \mathcal{I}_i \vee a_{i+1}) \le h(\bigvee \mathcal{I}_i \vee \perp) + h(\perp \vee a_{i+1}) - h(\perp)$$

As a_{i+1} is also an atom and $\perp$ is the minimum, we obtain h(a_{i+1})=1, h($\perp$)=0. Follow this:

$$h(\bigvee \mathcal{I}_{i+1}) \le h(\bigvee \mathcal{I}_i) + 1$$

In the same time, consider $\mathcal{I}$ is an *independent** set composed all by atoms, trivially we can say that $\bigvee \mathcal{I}_i$ is no related with a_{i+1} and then $\bigvee \mathcal{I}_i < \bigvee \mathcal{I}_{i+1}$. Based on the condition that height function is strictly increasing which means h($\bigvee \mathcal{I}_i$) < h($\bigvee \mathcal{I}_{i+1}$), so we obtain:

$$h(\bigvee \mathcal{I}_{i+1}) = h(\bigvee \mathcal{I}_i) + 1 = i + 1.$$

□

4.3 M-symmetric on Semimodular Semilattice

Definition 14 *Let $\mathcal{L}$ be a join semilattice. The pair of elements (a,b) of $\mathcal{L}$ is modular, in notation, aM^*b:*

$$\forall x \le b, if\ \exists y \le a \ and\ y \le b, and\ the\ map\ \mathcal{P}: [x \vee y, b] \to [x \vee a, a \vee b]\ is\ one\ to\ one$$

We say the triple {a,b,y} is a modular triple and the semilattice $\mathcal{L}$ is called M-symmetric if aM^*b implies that bM^*a for every a,b∈ $\mathcal{L}$.

Corollary 15 *$\forall$ $x \leq b$, if $\exists$ $y \leq a$ and $y \leq b$ and aM^*b in semilattice $\mathcal{L}$, then we obtain the following equivalence for the height function of $\mathcal{L}$:*

1) h(a$\vee$b) - h(x$\vee$a) = h(b) - h(x$\vee$y)

2) h(a$\vee$b) = h(a$\vee$y) + h(y$\vee$b) - h(y)

The equation 2) lead to a result that the join valuation (see the equivalence in Corollary 17) on the modular tripe {a,b,y} is also a join co-valuation. Then we can say that the constraint for modular pair is more strict than the law for semivaluation.

Proof: 1): Note that by definition 18, we obtain the map $\mathcal{P}$: [x $\vee$ y, b] $\rightarrow$ [x $\vee$ a, a $\vee$ b] is one to one which indicate that the length of this two internals linked by map $\mathcal{P}$ is equivalent. In another word, the maximal chains for each of them have the same length which lead to our equation on height function.

2): Consider 1), we can use y to replace x in the equation. Based on the condition, we also have h(y$\vee$b)=h(b). Thus, we obtain the equation 2).

□

Theorem 16 *Let $\mathcal{L}$ be a join semilattice. $\mathcal{L}$ is semimodular if it is M-symmetric.*

Proof: Let a,b,c $\in \mathcal{L}$ and let b $\succ$ a. Obviously, We have b$\vee$c$\geq$a$\vee$c. When b$\vee$c = a$\vee$c, it leads to the result that b$\vee$c $\succ$ a$\vee$c directly. So we proof the case when b$\vee$c > a$\vee$c below.

Put d=a$\vee$c, then we need to prove b$\vee$c $\succ$ d. Let x that b$\vee$c $\succ$ x and x $\geq$ d. Then we find there is a one to one map from [a, b]to[x, b$\vee$c]. So for all z < b, it follows that the map $\mathcal{P}_1$:[z$\vee$a, b]$\rightarrow$[z$\vee$x, x$\vee$b] is one to one, then we obtain that xM^*b.

By M-symmetry, we obtain bM^*x simultaneously, which means, by definition, the map $\mathcal{P}_2$: [d$\vee$a, x] $\rightarrow$ [d$\vee$b, b$\vee$x] is one to one (based on the condition that x $\geq$ d and b > a, so c < x and a < x,b). Consider we have b$\vee$c $\succ$ x and b$\vee$c > d, then it suffices to say that b$\vee$c = b$\vee$x = d$\vee$b which means the right side of $\mathcal{P}_2$ contains only one node. This lead to that the left side of $\mathcal{P}_2$ can only contains one node as well. So we obtain the result that x = d$\vee$a = d. Finally, We get b$\vee$c $\succ$ a$\vee$c.

We give the **Figure 2** for easier understanding. A similar result and graph for lattice has given on Gratzer's Theorem 384 and Theorem 385 [Gra11].

□

Note: We do not obtain a trivial injection in Theorem19. We find that a semilattice may not contains any modular pairs when it does not have a $\perp$

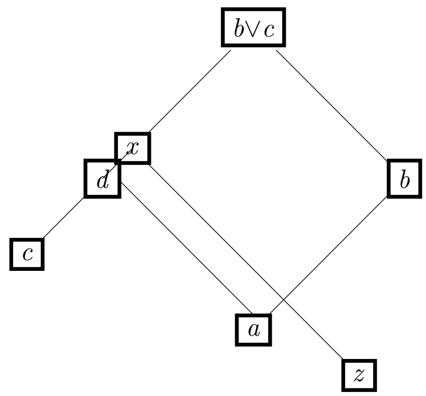

Figure 2: Modular elements on semilattice

(or we say a *strictly* semilattice) because there may not be such a y as in definition19 in general.

Consider a basic semilattice $\mathcal{L}_3$ = {N1,N2,N3} in which N1 $\succ$ N2 and N1 $\succ$ N3, respectively. It is obvious that no pairs $\mathcal{L}_3$ have a meet, then we can say it doesn't contain any modular pairs and it can not be M-symmetric.

5 Semivaluation and Order Dimension

In this section, we first recall the notation, Order Dimension and Realizer [Dus41] on a poset. The connection between Order Dimension theory and Valuation on modular (distributed) lattice through Birkhoff's representation theorem [Bir84] has been indicated by Marigo [Mar12]. Then, We maintain the bottleneck of extending this theorem on semi-modular lattice and non-modular lattice based on the fact that we can not view those kind of lattices as downsets of poset.

While by considering the order dimension on semi-modular lattice directly, we can generate a novel approach to build up the Semivaulation on semimodular lattice by realizer which shown in Theorem 18.

5.1 Order Dimension and Realizer

Definition 17 *(Dushink, Ben; Miller, E. W.) Given a poset $\mathcal{P}$, we define the order dimension of $\mathcal{P} \leq n$ iff there is a set R of linear extensions of $\mathcal{P}$: $R = \{ \bigwedge_1, \bigwedge_2, \ldots, \bigwedge_n \}$ such that:*

1) $\forall x,y \in P$, if $x \leq y$ in P, then $\forall \bigwedge_i \in R$, $x \leq y$ in $\bigwedge_i$

2) $\forall x,y \in P$, if $x \nleq y$ in P, then $\exists \bigwedge_i \in R$, $y \leq x$ in $\bigwedge_i$

We name R the realizer of $\mathcal{P}$, the order dimension is n iff n is the least number of linear extensions in R.

Theorem 18 *(Marigo) There is a bijection between finite distributed lattice with complete valuations and poset whose order dimension is no more than two.*

5.2 Generate Semivaluation from Realizer on modular lattice

We can use a similar way based on the theorem above as indicated by Marigo [Mar12] to generate a semivaluation from realiser through the Birkhoff's representation theorem on modular lattice.

Given a modular lattice $\mathcal{L}$, view it as composed by downsets by inclusion of a corresponding poset $\mathcal{P}$ whose order dimension is no more than two, with a realiser R:$\{\bigwedge_1,\bigwedge_2\}$, we first inverse the order in $\bigwedge_2$ to create a complementary realiser R': $\{\bigwedge_1,\bigwedge_2'\}$ which indicate a complementary poset of $\mathcal{P}$.

Then we use the weight function ω(x) on this complementary poset which counting chains from each node. After this, we can define the semivaluation (which is actually as valuation) on $\mathcal{L}$ as below:

$$\forall a \in \mathcal{L}, v(a) = \sum_{x \in a_{\downarrow}} \omega(x)$$

We refer to [Mar12] for the proof of the correctness of this process as it is the similar way indicated in there.

5.3 Bottleneck of Birkhoff representation on semimodular and non-modular lattice

5.3.1 On semimodular lattice

Consider the typical semimodular lattice below,

5.3.2 On non-modular lattice

5.4 Generate Semivaluation from Realizer on semimodular semilattice

Instead of viewing the semimodular semilattice as downsets of a poset, we consider the realiser of the semilattice as a constraint poset and build up the semivaluation on this semilattice directly. Before describe the Semivaluation, first we need define the height function on linear extension and realiser.

Definition 19 *Given a poset $\mathcal{P}$, we define the height function h on a linear extension L of $\mathcal{P}$ as follow:*

1) $\forall x_i \in L$*, if* $x_0 \leq x_i$*,* $h_l(x_0) = 1$

2) $\forall x_i, x_j \in L$*, if* $x_i \succ x_j$*,* $h_l(x_i) = h_l(x_j) + 1$

Definition 20 *Given a poset* $\mathcal{P}$*, we define the Total height function* H *on a realiser* R *of* $\mathcal{P}$ *as follow:*

$H_R(x) = \sum_{l \in R} h_l(x)$

Note: We loosely refer $H_R(x)$ on poset $\mathcal{P}$ with realiser R as $H(x)$ based on the trivial symmetry on different realisers for a certain poset.

Now we can generate the semivaluation directly from the realiser on semilattice which indicated by the following theorem.

Theorem 21 *Total height function H(x) on a semimodular semilattice L is a semivaluation.*

Proof:

Given a semimodular semilattice L with realiser R, We proof it by induction on the value n of order dimension (the number of linear extensions in R).

We first consider the case when n=1 which means L is an linear order. Then trivially we obtain $H(x)=h(x)$ and the height function on the linear extension is actual the height function on the lattice, based on Theorem 11 above, the result is easily shown.

Then we suppose the result has been proved when n=k-1:

$$H(x \sqcup z) \leq H(x \sqcup y) + H(y \sqcup z) - H(y)$$

we want to present it also satisfy the law when n=k which means

$$H(x \sqcup z) + h_k(x \sqcup z) \leq H(x \sqcup y) + h_k(x \sqcup y) + H(y \sqcup z) + h_k(y \sqcup z) - (H(y) + h_k(y))$$

then we only need to proof:

$$h_k(x \sqcup z) \leq h_k(x \sqcup y) + h_k(y \sqcup z) - h_k(y)$$

Due to the fact that k is a new linear extension belong to R, so x, y must be comparable in k. Based on the symmetry on the notation of semivaulation, we only need to proof one case such as x > z. Then h_k(x $\sqcup$ z)= h_k(x). Consider h_k(x)$\leq$$h_k$(x $\sqcup$ y) and h_k(y)$\leq$$h_k$(y $\sqcup$ z), we verified the result.

$\square$

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